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ON THE CONNECTEDNESS OF THE TOPOLOGY OPTIMIZATION PREDICTORS

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ABSTRACT

Deep learning-based topology optimization predictors have been shown to be effective in generating optimal designs. However, these predictors are prone to topological errors, particularly for high-resolution domains. Although various methods have been developed to enhance the accuracy of predicted structures, such as using large training datasets, complex networks, and physics-based loss functions, they do not include topological metrics in the deep learning models. Similar issues arise in other applications, such as blood vessels, neurons, or road segmentation from images, and several modifications to typical loss functions have been proposed to improve the topological validity of the predictions. In this study, we evaluate and compare four distinct topological loss functions to explore their influence on the performance of deep learning-based topology optimization predictors. Our findings offer insights into the advantages and limitations of these modified loss functions and provide a basis for future research and development aimed at improving the accuracy and efficiency of deep learning predictors in topology optimization.

Keywords: Topology Optimization, Topological loss functions, Deep learning

1. INTRODUCTION

Various deep learning algorithms have been used in topology optimization (TO) to efficiently predict optimal designs. These approaches have successfully generated exemplary designs through gradient-based TO and have shown some level of generalizability in exploring the design space. However, a major drawback of these methods is that the predicted structures often suffer from topological errors, resulting in weaker structural performance and decreased manufacturability compared to the ground truths obtained from gradient-based optimization methods [1]. To address this challenge, different approaches have been proposed, such as using large training datasets [2], employing complex networks [3], and incorporating a physical loss function into the training process [4]. Although these methods show promise in improving the quality of predicted structures, they are computationally inefficient and do not integrate topological metrics into the deep learning model. The premise of this work is that by integrating topological information into the model, topologically correct structures can be generated without requiring large amounts of training data or complex networks.

Of course, topological errors are not confined to topology optimization (TO) predictors. Segmentation algorithms, such as those used for road and blood vessel segmentation, are also susceptible to this type of errors. Various methods have been developed to address this issue in segmentation models, including the use of topological loss functions [5–7] and of complex networks [8, 9]. For further information, please refer to [10]. Despite extensive efforts to integrate topological features into deep learning models in the segmentation literature, this issue has not received much attention in the TO community. Our recent work [11] demonstrates that the incorporation of topological loss terms improves the connectivity of the predicted optimal structures. Specifically, in [11] we have shown that using persistence homology-based loss functions can generate structures with twice better connectivity compared to the baseline predictions.

The existing loss functions are essentially global metrics that assess the overall fit for a domain, typically represented as a uniform grid in 2D or 3D. Despite incorporating topology-aware terms, these loss functions generally fail to account for localized geometric changes, rendering them ineffective in penalizing the unwanted "artifacts" in the predictions produced by the generative models. Developing metrics that capture local geometric changes, which in turn induce desired topological properties, remains a challenge.

This paper aims to take a step in addressing this limitation by investigating the effectiveness of various topological loss functions in enhancing the accuracy of deep learning-based topology optimization predictors and by assessing the corresponding advantages and limitations. Specifically, a convolutional neu-

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ral network similar to the one used in [12] is trained with four different topological loss functions: clDice [5], the TopoLoss [6], the Wasserstein loss [13], and the Bottleneck loss function [11]. Moreover, we also explore the limitations of each loss function, such as the associated computational complexity, and evaluate their effectiveness in generating accurate and well-connected structures. Overall, this study contributes to the ongoing efforts to improve the accuracy and efficiency of deep learning-based topology optimization predictors, and its findings can be used to guide future research and development in this area.

2. BACKGROUND

Gradient-based topology optimization algorithms are computationally expensive, which limits their ability to explore the design space efficiently. In recent years, deep learning-based topology optimization predictors have emerged as a potential alternative because, once the models are trained, they can efficiently generate optimal designs [14–17]. However, one major challenge of all these methods is that the predicted structures often suffer from topological errors, which in turn leads to weaker structural performance and decreased manufacturability compared to the optimal solutions predicted by gradient-based optimization methods.

Various approaches have been proposed to improve the topological quality of the structures predicted by deep learning models. For example, the authors in [2] showed that by using 296,000 training cases for their CNN model, they could obtain the structures with better pixel-wise accuracy compared to their benchmark method [18] where the authors used 80,000 cases for training. As expected, using a larger training dataset leads to better accuracy. However, generating this type of data is a very expensive process because every solution requires often hundreds of Finite Element simulations, and the cost increases exponentially with the resolution and the dimension of the space. Moreover, these models are trained to predict material occupancy for each cell of the grid (pixel in 2D or voxel in 3D), and do not include any topological information in the development of the model. Some other studies used more complex models, e.g., Generative Adversarial Networks, to obtain better structures [19, 20]. Although these models have shown a better performance and generalizability compared to CNNs, these models are often difficult to train, need a large amount of data, and often produce structures with disconnected members. At the same time, the severity of these errors increases rapidly as the boundary conditions move away from those included in the training distribution. Physical constraints have been added to the loss function in [4], and these constraints are based on a compliance error defined through the pixel-wise loss function in the training process of the deep learning model. Despite some improvements in the quality of the predicted structures, the proposed method estimates the compliance at every step through a finite element analysis each time the loss function is calculate, which is time-consuming. Although these approaches can improve the quality of the predicted structures, they do not incorporate topological information into the training process.

The inclusion of topological information into the training process did not receive significant attention in the TO community. Nevertheless, numerous studies have focused on the topological connectedness of segmentation algorithms, which encounter similar challenges. Topological errors in segmentation can lead to inaccurate identification of regions, thereby affecting the accuracy of subsequent analysis. Consequently, several methods have been developed to address this problem, and the vast majority of the existing methods included topology-aware terms into their loss functions. For example, in one of the early attempts to use topological loss functions, [6] proposed a modified Wasserstein distance based on persistent homology [21, 22]. This loss function enforced the segmented image to exhibit the same Betti error as the ground truth. Results from this study demonstrated that the loss function effectively reduced Betti errors on various 2D datasets containing natural and biomedical images. Another work by [23] introduced a topological loss function that utilizes persistent homology barcodes. They showed that their loss function improved the Betti error by a factor of 10 in the multi-class segmentation of 2D short axis and 3D whole heart cardiac magnetic resonance (CMR) images compared to a baseline U-Net model. In a different approach, [5] proposed a novel connectivity-aware similarity measure called center-lineDice (clDice) for image segmentation problems. Their loss function involved intersecting the segmentation masks with morphological skeletons. Results presented in their study, encompassing five public 2D and 3D datasets, demonstrated that clDice effectively improved topological errors in the segmented images. Moreover, [13] employed the Wasserstein distance in the loss function to minimize topological discrepancies in 3D reconstruction. They revealed that incorporating the topological loss function alongside the pixel-wise loss function improved the quality of the reconstructed images. Numerous other studies have also integrated topological information into segmentation problems, such as those conducted in [24–30]. Interested readers can explore these studies for more detailed information.

It's important to consider that while many of the existing topological loss functions have proven to be versatile in their ability to handle both 2D and 3D images, they often come with a significant computational burden, especially when dealing with 3D images. Specifically, approaches based on persistence homology may be too computationally expensive for high-resolution images, which can limit their usefulness in certain scenarios. Therefore, it is crucial to continue exploring alternative approaches or implementations for integrating topological features into deep learning methods' training process. This includes investigating topology optimization predictors, with a specific emphasis on enhancing their scalability and computational efficiency when working with 3D image datasets.

3. CONTRIBUTIONS AND OUTLINE

While the importance of topological connectedness in topology optimization problems cannot be overstated, only one study [11] has examined the utilization of topological loss functions during the training phase. However, this previous work did not comprehensively evaluate the significance and limitations of different topological loss functions. To address this gap, the present paper investigates the impact of various existing topological loss functions, including those proposed by [5, 6, 11, 13], on the performance of deep learning-based topology optimization pre-

dictors. Our investigation includes a detailed analysis of the advantages and limitations of each loss function, taking into account factors such as computational complexity. Additionally, we assess the effectiveness of each loss function in generating accurate and connected structures. This research study contributes to ongoing efforts aimed at enhancing the accuracy and efficiency of deep learning-based topology optimization predictors by exploring the application of diverse topological loss functions. The findings of this study can provide valuable guidance for future research and development endeavors in this field.

4. METHOD

4.1 Topological loss functions

A CNN model similar to the one employed in [12] has been trained using $2,250$ structures at a resolution of 120×240, as described in [11], and all model hyperparameters are similar to those used in [12]. The model underwent an initial training of 100 epochs, followed by fine-tuning for another 150 epochs using four different topological loss functions: TopoLoss, clDice, Wasserstein distance, and Bottleneck distance. During the fine-tuning process of all but the clDice loss function, the topological losses were incorporated as an additional term with an appropriate coefficient into the binary cross-entropy loss. The clDice loss function uses the Dice score instead of binary cross-entropy as its primary loss function. In the following section, we provide a detailed explanation of each topological loss function.

4.1.1 TopoLoss [6]. Persistent homology is a fundamental concept in the field of topological data analysis, which aims to capture the essential topological properties of an object. It achieves this by examining how the homology groups of a space change as a parameter, such as scale or distance, varies. Let's consider a continuous image domain denoted as $\Omega \subset R^2$, along with a predicted image function $f : \Omega \rightarrow R$ and a segmentation image $X \subset R$ obtained by applying a threshold to the predicted image. The d-dimensional topological structure of X represents a class of d-manifolds that can be transformed into each other within X. Simply put, 0-dimensional structures correspond to connected components, while 1-dimensional structures correspond to handles. To capture the complete range of topological information within f, persistent homology involves applying different thresholds to the predicted image f. As the threshold decreases, the topology of f undergoes changes, leading to the "birth" of new topological structures and the elimination of existing ones. This information is often visualized using a persistence diagram, denoted as $Dgm(f)$, which provides a graphical representation of the birth and death of topological features across different parameter values [21, 22].

TopoLoss [6] describes a modified version of the Wasserstein distance that is specifically designed for comparing persistence diagrams. Given the persistence diagrams of an image prediction ($Dgm(f)$) and the ground truth ($Dgm(g)$), TopoLoss aims to find the optimal one-to-one mapping between the points in $Dgm(f)$ and $Dgm(g)$, and then calculate the total squared distance between the corresponding point sets. The matching algorithm proceeds as follows. The points from $Dgm(g)$ are placed at the upper-left corner $p_{ul} = (0, 1)$, with $\text{birth}(p_{ul}) = 1$ and $\text{death}(p_{ul}) = 0$. Then, TopoLoss identifies the points in $Dgm(f)$ that are closest to p_{ul} and maps them to the corresponding points in $Dgm(g)$. The algorithm computes the squared distances from all points in $Dgm(f)$ to p_{ul}, and then sorts these distances. In summary, TopoLoss can be formulated as:

$$L_{topo}(f, g) = \sum_{p \in dgm(f)} [birth(p) - birth(\gamma^*(p)]^2 + [death(p) - death(\gamma^*(p)]^2 \quad (1)$$

Where γ^* is the optimal matching between two different point sets. For further information about TopoLoss, we direct the reader to the original paper [6].

4.1.2 Center-LineDice (clDice) [5]. The clDice loss function utilizes morphological thinning, achieved through min- and max-pooling as a substitute for morphological erosion and dilation, on both the predicted and ground truth images. Given the ground truth (G) and the predicted image (P) as a binary image and their corresponding skeletons S_G and S_P, the clDice loss function can be obtained from the following formula:

$$clDice(G, P) = 2 \times \frac{T_{prec}(S_P, G) \times T_{sens}(S_G, P)}{T_{prec}(S_P, G) + T_{sens}(S_G, P)} \quad (2)$$

Where T_{prec} and T_{sens} are the Topology Precision and Topology Sensitivity, respectively, that can be obtained from:

$$T_{prec}(S_P, G) = \frac{|S_P \cap G|}{|S_P|} \quad (3)$$

$$T_{sens}(S_G, P) = \frac{|S_G \cap P|}{|S_G|} \quad (4)$$

The crucial step to calculate the clDice loss is to find the skeleton of the image. Importantly, the process of obtaining the skeleton should be differentiable in order to be used in the loss function. The authors in [5] proposed an iterative process based on min- and max-pooling that are used as a proxy for morphological erosion and dilation to calculate the skeleton of the image. The proposed skeletonization algorithms allow the use of clDice as a fully differentiable, real-valued, optimizable measure. Finally, the loss function used in the training process is described by:

$$L_{clDice} = (1 - \alpha)(1 - Dice) + \alpha(1 - clDice) \quad (5)$$

Where $\alpha \in [0, 0.5]$.

4.1.3 Wasserstein distance [13]. The q-Wasserstein distance measures the similarity between two persistence diagrams, which is defined as the minimum value achieved by a perfect matching between the points of the two diagrams. If $Dgm(f)$ and $Dgm(g)$ are the persistence diagrams of the predicted and the ground truth images, respectively, the q-Wasserstein distance can be obtained from:

$$W_q(Dgm(f), Dgm(g)) = \Big[\inf_{\eta} \sum_{p \in Dgm(f)} ||p - \eta(p)||_\infty^q \Big]^{\frac{1}{q}} \quad (6)$$

Where η is a perfect matching between the diagrams. The Wasserstein distance can be obtained by solving the optimal transport algorithms [31]. For the purpose of this study, the value of q is assumed to be 1.

4.1.4 Bottleneck distance [11]. The Bottleneck distance is the fourth metric that can be employed to assess the similarity between two persistence diagrams. More precisely, it determines the minimum distance required to achieve a perfect matching between the points in the two diagrams. Alternatively, the Bottleneck distance can be computed as the maximum weight in the solution to the minimum weight perfect matching problem. Similar to the Wasserstein distance, the Bottleneck distance can be defined as:

$$W_\infty(Dgm(f), Dgm(g)) = \inf_{\eta} \sup_{p \in Dgm(f)} ||p - \eta(p)||_\infty \quad (7)$$

4.2 Evaluation Metrics

We utilize multiple evaluation metrics to assess the performance of each loss function. These metrics include the pixel-wise accuracy, which is defined as the proportion of correctly classified pixels, as well as the Betti number errors, which directly measure the topological difference (e.g., the number of handles) between the predicted and ground truth images. The 0^{th} Betti number is calculated by counting the number of connected components in the image, while the 1^{st} Betti number is obtained by counting the number of holes in the structure. The Betti number error is calculated as follows:

$$i^{th} Betti\ error = \frac{B^i(p) - B^i(g)}{B^i(g)} \quad (8)$$

where B^i is the i^{th} Betti number, p is short for predicted structure and g stands for ground truth.

Two other metrics that we used to measure the performance of the loss functions are the Compliance Error (CE), which compares the performance of the structures in terms of structural compliance, and the Volume Fraction Error (VFE), which compares the amount of material used in the prediction an the ground truth. The compliance error can be calculated from the following:

$$CE = \frac{|C(p) - C(g)|}{C(g)} \quad (9)$$

where $C(p)$ and $C(g)$ are the compliance of the predicted and the ground truth structure, respectively. Similarly, VFE is obtained from:

$$VF\ error = \frac{|VF(p) - VF(g)|}{VF(g)} \quad (10)$$

where $VF(p)$ and $VF(g)$ are the volume fraction of the prediction and the ground truth, respectively.

5. RESULTS AND DISCUSSION

5.1 Overall Results

We started by conducting an initial training session for our convolutional neural network (CNN). This session involved a maximum of 100 epochs, utilizing the binary cross-entropy loss function and early stopping to identify the best-performing network. Afterward, we proceeded to fine-tune the network by incorporating various topological loss functions. During the fine-tuning process, the model underwent training for a maximum of 150 epochs, with early stopping used to select the model that achieved the lowest topological loss value. All model parameters were considered trainable throughout the fine-tuning stage. The topological loss function was integrated into the binary cross-entropy loss, except for clDice, as described above. Each term in the loss function was assigned an appropriate weight, determined through grid search to optimize performance. To assess the effectiveness of these different approaches, we present a comparison of the original CNN prediction with structures generated using the fine-tuned network employing various topological loss functions (refer to Figure 1). Additionally, we collected evaluation data using performance metrics outlined in Section 4.2 and summarized our findings in Table 1. It's worth noting that both the models trained with and without topological loss terms were capable of making predictions in real-time, with minimal disparity in the prediction time.

The performance analysis indicates that the utilization of different topological loss functions has the potential to improve the connectivity of predicted structures. Specifically, when employing the TopoLoss function, both the 0^{th} and 1^{st} Betti errors showed notable enhancements, decreasing from 95.78% to 77.1% and from 50.1% to 27.2%, respectively. This improvement in Betti errors also resulted in lower compliance errors, indicating better structural performance, along with slightly better accuracy. Comparing the predicted structures using TopoLoss (shown in Figure 1) with the original predictions clearly demonstrates the significant improvement achieved through the use of TopoLoss. Similarly, the clDice function also improved the 0^{th} and 1^{st} Betti errors, reducing them from 95.78% to 46.82% and from 50.1% to 41.25%, respectively. From figure 1, it can be seen that in some cases clDice filled the holes in the structure, which contributed to higher 1^{st} Betti error, compliance error, volume fraction error, and lower accuracy compared to other topological loss functions.

Both the Wasserstein and Bottleneck distances demonstrated significant improvements in Betti errors, structural performance, and accuracy. Although the Wasserstein distance did not substantially improve the 0^{th} Betti error, it outperformed other loss functions in terms of 1^{st} Betti error, compliance error, and volume fraction error. The better compliance error and volume fraction error, despite having nearly the highest 0^{th} Betti error, can be attributed to the Wasserstein distance's ability to connect floating material in the original prediction to the main body through the correct path, albeit at the cost of introducing additional floating members in the structure. A comparison of the results shown in the last row of Figure 1 for the Wasserstein distance with the original prediction and ground truth highlights this trade-off. The Bottleneck distance outperformed the Wasserstein distance in terms of 0^{th} Betti error, but it exhibited higher 1^{st} Betti error, compliance error, and volume fraction error. The high volume fraction error in the Bottleneck distance could be attributed to the addition of extra material that connect the unwanted part to the main body, as well as the failure to remove unwanted material from the original prediction. For instance, comparing the structures shown in the fourth row of Figure 1 for the Bottleneck distance with those for the Wasserstein distance and TopoLoss reveals that the latter two losses attempted to remove the extra material from the original prediction, while the Bottleneck loss connected that part to the main body. This difference explains the

TABLE 1: THE QUALITATIVE COMPARISON BETWEEN THE PREDICTED STRUCTURES WITH DIFFERENT TOPOLOGICAL LOSS FUNCTIONS AND THE ORIGINAL MODEL WITHOUT ANY TOPOLOGICAL LOSS FUNCTION. THE DOMAINS HAVE A RESOLUTION OF 120 × 240 AND THE SEEN BOUNDARY CONDITIONS REPORTED IN [11].

Metrics	Baseline	TopoLoss	clDice	Wasserstein distance	Bottleneck distance
0^{th} Betti error	95.78 %	77.1 %	46.82 %	94.2 %	86.5 %
1^{st} Betti error	50.1 %	27.2 %	41.25 %	25.5 %	26.9 %
Compliance Error	13.96 %	11.30 %	14.50 %	10.5 %	11.2 %
Volume Fraction Error	0.41 %	0.45 %	1.67 %	0.28 %	1.01 %
Accuracy	94.34 &	94.74 %	94.25 %	94.65 %	94.36 %

TABLE 2: THE COMPARISON BETWEEN THE TRAINING TIME REQUIRED BY EACH LOSS FUNCTION. (EPOCH/HOUR)

Time	BCE	TopoLoss	clDice	Wasserstein distance	Bottleneck distance
Training time	40	1.8	30	0.41	0.41

higher volume fraction error observed in the Bottleneck distance.

Upon examining the data presented in table 1, it is evident that achieving a superior 0^{th} Betti error does not necessarily lead to improved structural performance. Nevertheless, the findings strongly indicate a significant correlation between the 1^{th} Betti error and compliance error since structures with lower 1^{th} Betti error exhibit correspondingly lower compliance errors. The 1^{th} Betti error compares the main body of the predicted structure (which contributes to compliance error) with the ground truth, disregarding completely disconnected members. Consequently, a lower 1^{th} Betti error suggests that the predicted structure's main body aligns more closely with the ground truth, resulting in decreased compliance error. Hence, the 1^{th} Betti error is a more critical factor in determining the best loss function. In this regard, the Wasserstein distance outperforms other loss functions, although all loss functions can improve the quality of predicted structures by TO predictors.

5.2 Computational Cost

Table 2 provides a breakdown of the training time for each loss function utilizing an NVIDIA GeForce RTX 2080Ti GPU on the UConn HPC. The results reveal that the topological losses employing persistence homology are computationally expensive. Specifically, the training for TopoLoss is almost 20 times slower than binary cross-entropy and 15 times slower than clDice. However, TopoLoss is more than 4 times faster than the Wasserstein and Bottleneck distance, as the latter requires solving the optimal matching problem to find the optimal matching between the points in the diagrams of the ground truth and the predicted structures in addition to obtaining the persistence diagram, which is a time-consuming process. The computational cost of clDice comes from the iterative skeletonization process, which uses min-max pooling operations that can be applied to higher-resolution images. However, the process may require more iterations for higher-resolution domains, slightly increasing its training time. On the other hand, the persistence-based loss functions face challenges when working with high-resolution domains, as the cost of calculating the persistence diagram increases significantly as the image size grows.

5.3 Limitations

The application of persistence homology-based loss functions to enhance the accuracy of deep learning models is constrained by various factors. One significant limitation is their high computational cost, rendering them impractical for handling high-resolution images. As image size grows, the computation cost associated with calculating the persistence diagram also escalates considerably, thereby restricting the use of these loss functions in numerous real-world scenarios. Additionally, fine-tuning their contribution in the training process is necessary for different networks and datasets, leading to time-consuming procedures demanding substantial time and computational resources.

One limitation of all topological loss functions is their lack of generalizability, which means they cannot be directly employed with diverse networks and tasks without undergoing training. Thus, given their computational demands, their application may be unfeasible in many contexts. In addition, this study has also revealed that the utilization of topological loss functions may introduce noise into predicted structures, requiring subsequent post-processing. Figure 2 provides an illustration of the noise generated by TopoLoss and the Bottleneck loss function integrated with the relatively simple CNN model described above. We note here that the results presented in section 5.1 have undergone noise removal procedures, and that the extent of noise created by the loss functions varies depending on their weight during the training process. At the same time, the network's architecture itself may impact the presence of noise, as evidenced by the fact that the GAN used in [11] did not exhibit noise in the generated structures. Nevertheless, additional work is required to better understand the source of noise, although this work is outside the scope of this paper.

6. CONCLUSION

This study investigates the effectiveness of various topological loss functions in improving the quality of predicted structures generated by topology optimization (TO) predictors. The performance of the Wasserstein distance, Bottleneck distance, clDice, and TopoLoss was evaluated based on their impact on the 0^{th} and 1^{st} Betti errors, compliance error, volume fraction error, and accuracy. It should be noted that the CNN model used in this research was deliberately selected to be simple, and that the training was conducted with a relatively small dataset in order to manage the computational costs. Nonetheless, these loss functions can be applied effectively to more complex models like GANs,

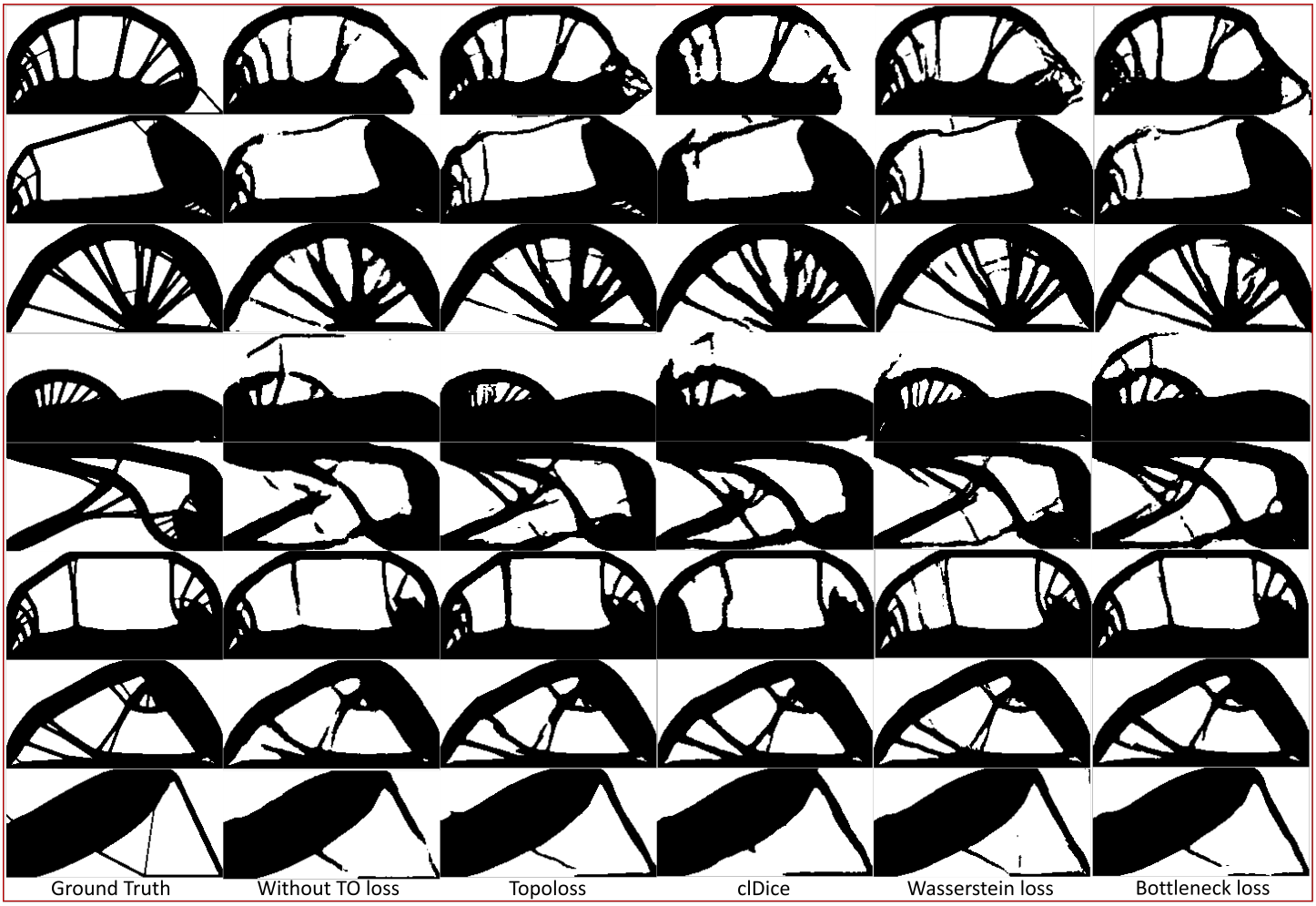

FIGURE 1: COMPARISON BETWEEN THE GROUND TRUTH (SIMP OPTIMIZED) 2D STRUCTURES, THE PREDICTED STRUCTURES BY CNN WITHOUT TOPOLOGICAL LOSS, AND THE CNN PREDICTIONS WITH DIFFERENT TOPOLOGICAL LOSSES.

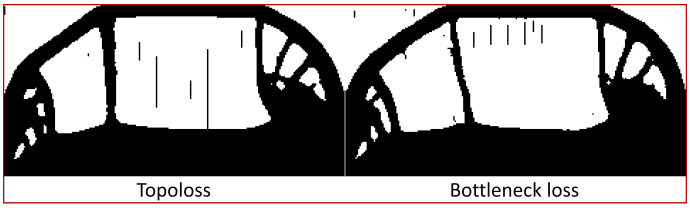

FIGURE 2: EXAMPLE OF NOISE GENERATED BY TOPOLOSS AND BOTTLENECK LOSS FUNCTIONS USING THE CNN. OBSERVE THAT THIS TYPE OF NOISE FOR THE BOTTLENECK DISTANCE WAS NOT SEEN PREVIOUSLY IN [11].

as demonstrated in [11], with larger training datasets to achieve improved prediction accuracy.

The results indicate that employing topological loss functions can significantly enhance the connectivity of predicted structures, resulting in better Betti errors, compliance errors, volume fraction errors, and accuracy. TopoLoss and the Wasserstein distance were identified as the most effective, with the latter being particularly effective in improving the 1^{st} Betti error, compliance error, and volume fraction error. However, the Bottleneck distance outperformed the Wasserstein distance in terms of the 0^{th} Betti error, albeit with higher values for the 1^{st} Betti error, compliance error, and volume fraction error.

A key advantage of topological loss functions is their ability to be integrated into the training process of any deep learning model, such as CNNs and GANs, as demonstrated in [11], enabling improvements in the topological connectivity of images across various applications. However, it should be noted that these loss functions do not guarantee the connectivity of the resulting predictions. One significant drawback of persistence homology-based loss functions is the computational cost associated with persistent homology, which increases significantly with larger image sizes. Consequently, such loss functions are currently impractical for many real-world applications that require high-resolution 2D or 3D images. Considering that topological loss functions represent global measures of connectivity, an alternative approach for future research could involve leveraging more local constructs like connectivity graphs.

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