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ABSTRACT

This paper is concerned with the stability of periodic traveling waves of dnoidal type, of the Zakharov system. This problem was considered in a study of Angulo and Brango [Nonlinearity **24**, 2913 (2011)]. In particular, it was shown that under a technical condition on the perturbation, such waves are orbitally stable, with respect to perturbations of the same period. Our main result fills up the gap created by the aforementioned technical condition. More precisely, we show that for all natural values of the parameters, the periodic dnoidal waves are spectrally stable.

I. INTRODUCTION

We consider the Zakharov system, which is the following system of coupled nonlinear partial differential equations:

$$\begin{cases} v_{tt} - v_{xx} = \frac{1}{2}(|u|^2)_{xx}, \\ iu_t + u_{xx} - uv = 0. \end{cases} \tag{1.1}$$

In particular, v is a real-valued function, while u is a complex-valued function. Problem (1.1) was introduced in Ref. 1 to describe the Langmuir turbulence in plasma.

The problem of the stability of solitary waves for nonlinear dispersive equations goes back to the studies of Benjamin[2] and Bona[3] (see also Refs. 4–7). A general approach for investigating the stability of solitary waves for nonlinear equations having a group of symmetries was proposed in Ref. 8. The well-posedness theory for the Zakharov system in the periodic setting was investigated in Ref. 9. In Refs. 10 and 11, the existence and stability of smooth solitary wave solutions were considered; in fact, we state for reference purposes the precise stability results of Ref. 10 below.

The goal of this paper is to consider the spectral stability of periodic traveling wave solutions of the form

$$\begin{cases} v(t,x) = \psi(x-ct), \\ u(t,x) = e^{-i\omega t} e^{i\frac{c}{2}(x-ct)} \phi(x-ct), \end{cases} \tag{1.2}$$

where $\psi,\ \phi : \mathbb{R} \to \mathbb{R}$ are smooth,[12–15] periodic functions with a fixed period $2T$ and $\omega, c \in \mathbb{R}$. In order to ensure that the traveling wave u above is $2T$ periodic, we will require that there is an integer l so that

$$c = \frac{2\pi l}{T}. \tag{1.3}$$

We now construct such waves.

A. Construction of the periodic waves for the Zakharov system

Substituting (1.2) in (1.1), we obtain

$$\begin{cases} (c^2-1)\psi'' = \frac{1}{2}(\phi^2)'', \\ \phi'' + \left(w + \frac{c^2}{4}\right)\phi = \phi\psi. \end{cases} \tag{1.4}$$

Integrating the first equation in (1.4), we get

$$\psi = -\frac{\phi^2}{2(1-c^2)} + a_0 + b_0 x.$$

By the periodicity of ϕ, ψ, we immediately conclude that $b_0 = 0$. For the rest, we also consider that $a_0 = 0$, as the other cases easily reduce, without loss of generality, to this one by a simple change of parameters; see the defining equations(1.1). That is,

$$\psi = -\frac{\phi^2}{2(1-c^2)}. \tag{1.5}$$

Using relation (1.5) in the second equation of (1.4), we get the following equation for ϕ:

$$-\phi'' + \sigma\phi - \frac{\phi^3}{2(1-c^2)} = 0, \tag{1.6}$$

where we have introduced the new parameter $\sigma := -\omega - \frac{c^2}{4}$. Multiplying by ϕ and integrating once, we get

$$\phi'^2 = \frac{1}{4(1-c^2)}\left[-\phi^4 + 4\sigma(1-c^2)\phi^2 + a_1\right], \tag{1.7}$$

where a_1 is a constant of integration. This is Newton's equation, which is well studied in the literature. In fact, one can construct several different types of solutions in terms of elliptic functions, including dnoidal, cnoidal, and even snoidal solutions. Unfortunately, our preliminary results for the cnoidal and snoidal type waves are far from definitive, so we will restrict our attention to the dnoidal waves. The cnoidal and snoidal waves will be a subject of a future publication.

Next, we present the construction of the dnoidal waves. Later, we will state some relevant spectral properties of the corresponding linearized operator, as they will be essential for our considerations in the sequel.

Let $1-c^2 > 0$ and $\sigma > 0$. Assume that the quadratic equation $r^2 - 4\sigma(1-c^2)r - a_1 = 0$ has two positive roots $r_0 > r_1 > 0$, and set $\phi_0 = \sqrt{r_0} > \phi_1 = \sqrt{r_1} > 0$. Clearly, there is an even and decreasing function in $[0,T]$ periodic solution of (1.7), with

$$\phi(0) = \max_{0<x<T}\phi(x) = \phi_0, \phi(T) = \min_{0<x<L}\phi(x) = \phi_1.$$

These are explicitly given, up to a translation, as follows.

Proposition 1 (Existence of dnoidal solutions).

Let $1-c^2>0, \sigma>0$. Assume that the quadratic equation $r^2 - 4\sigma(1-c^2)r - a_1 = 0$ has *two positive roots, denoted by $\phi_0^2 > \phi_1^2$. Then, the solution to (1.7) is given by*

$$\phi(x) = \phi_0 dn(\alpha x, \kappa), \tag{1.8}$$

where

$$\kappa^2 = \frac{\phi_0^2 - \phi_1^2}{\phi_0^2} = \frac{2\phi_0^2 - 4\sigma(1-c^2)}{\phi_0^2}, \qquad \alpha^2 = \frac{1}{4(1-c^2)}\phi_0^2 = \frac{\sigma}{2-\kappa^2}. \tag{1.9}$$

In addition, the fundamental period of ϕ is

$$2T = \frac{2K(k)}{\alpha}.$$

We now turn our attention to the spectral stability of such solutions, in the context of the Zakharov system (1.1).

B. The linearized problem

For the purposes of linearization, we rewrite system (1.1) as a first order in time system, in the form

$$\begin{cases} v_t = -V_x, \\ V_t = -\left(v + \frac{1}{2}|u|^2\right)_x, \\ iu_t + u_{xx} = uv. \end{cases} \tag{1.10}$$

Note that we enforce uniqueness by adding the condition $\int_{-T}^{T} V(t,x)dx = 0$. Consider the perturbations in the form

$$\begin{aligned} u(t,x) &= e^{-iwt} e^{i\frac{c}{2}(x-ct)}[\phi(x-ct) + p(t,x-ct)], \\ v(t,x) &= \psi(x-ct) + q(t,x-ct), \\ V(t,x) &= \varphi(x-ct) + h(t,x-ct), \end{aligned}$$

where q and r are real-valued functions and p is a complex-valued function. Here, φ may be identified as the unique mean-value zero function, i.e., $\int_{-T}^{T}\varphi(x)dx = 0$, satisfying

$$\varphi' = c\psi' = \frac{1}{c}\left(\psi + \frac{\phi^2}{2}\right)'. \tag{1.11}$$

Note that (1.11) is consistent with the zero order terms in (1.10) and (1.5). Accordingly, as V is mean-value zero, we must require that the perturbation h is mean-value zero as well, $\int_{-T}^{T} h(t,x)dx = 0$.

Plugging the above-mentioned expression into (1.10) and ignoring the quadratic and higher order terms, we get the following linear system:

$$\begin{cases} q_t = cq_x - h_x, \\ r_t = cr_x - q_x - (\phi\Re p)_x, \\ ip_t = -p_{xx} - \left(w + \frac{c^2}{4}\right)p + \psi p + \phi q. \end{cases} \tag{1.12}$$

Furthermore, by letting $p = p_1 + ip_2$, system (1.12) takes the form

$$\begin{cases} q_t = cq_x - h_x, \\ r_t = ch_x - q_x - (\phi\Re p)_x, \\ p_{1t} = -p_{2xx} - \left(w + \frac{c^2}{4}\right)p_2 + \psi p_2, \\ -p_{2t} = -p_{1xx} - \left(w + \frac{c^2}{4}\right)p_1 + \psi p_1 + \phi q. \end{cases} \tag{1.13}$$

For $\vec{U} = (p_2, p_1, q, h)$, the above system can be written in the form

$$\vec{U}_t = \mathcal{J}\mathcal{H}\vec{U}, \tag{1.14}$$

where

$$\mathcal{J} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\partial_x \\ 0 & 0 & -\partial_x & 0 \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} \mathcal{L}_- & 0 & 0 & 0 \\ 0 & \mathcal{L}_- & \phi & 0 \\ 0 & \phi & 1 & -c \\ 0 & 0 & -c & 1 \end{pmatrix}, \tag{1.15}$$

$$\mathcal{L}_- = -\partial_x^2 + \sigma + \psi = -\partial_x^2 + \sigma - \frac{\phi^2}{2(1-c^2)}. \tag{1.16}$$

Clearly, $\mathcal{J}^* = -\mathcal{J}$, whereas $\mathcal{H}^* = \mathcal{H}$, where we associate with the operators $\mathcal{J}, \mathcal{H}$ the following domains on the periodic functions:

$$D(\mathcal{J}) = (L^2[-T,T])^2 \oplus (H^1[-T,T])^2,$$
$$D(\mathcal{H}) = (H^2[-T,T])^2 \oplus L^2[-T,T] \oplus L_0^2[-T,T].$$

Note that the last component of the domain of $\mathcal{H}$ is $L_0^2[-T,T] = \{f \in L^2[-T,T] : \int_{-T}^{T} f(x)dx = 0\}$ per our earlier requirement that the last component $h \in L_0^2[-T,T]$.

Transforming the time-dependent linearized problem (1.14) into an eigenvalue problem, through the transformation $\vec{U} \to e^{\lambda t}\vec{U}$, yields

$$\mathcal{J}\mathcal{H}\vec{U} = \lambda\vec{U}. \tag{1.17}$$

As we are in the periodic context, it is well known that all essential spectrum is empty, thus reducing the spectrum to a pure point spectrum, that is, isolated eigenvalues with finite multiplicities. The standard notion of stability is given next.

Definition 1. We say that the traveling wave solution described in (1.2), (1.5), and (1.7) is spectrally stable, if the eigenvalue problem (1.17) does not have non-trivial solutions with $\Re\lambda > 0$. *That is,*

$$(\lambda, \vec{U}) : \Re\lambda > 0, \vec{U} \neq 0, \vec{U} \in D(\mathcal{J}\mathcal{H}) = (H^2[-T,T])^2 \oplus (H^1[-T,T] \oplus H_0^1[-T,T].$$

Otherwise, if there are such solutions, we refer to the family in (1.2) as spectrally unstable.

For orbital stability, we refer to the standard formulation; see (1.18) and (1.19).

C. Main result

The following is the main result of this work.

Theorem 1. *Periodic traveling waves of dnoidal type of (1.1) are spectrally stable for all natural values of the parameters.*

More specifically, the periodic dnoidal waves constructed in Proposition 1, *with the respective speed, subject to (1.3), are spectrally stable solutions of (1.10).*

Remark. Recall that condition (1.20) is necessary to guarantee the periodicity of such waves.

It is worth noting that in Ref. 10, the orbital stability of periodic waves of dnoidal type for system (1.1) was proved. Angulo and Brango[10] showed that for the equivalent system (1.10), there is orbital stability, if one asks for an additional technical condition; see (1.20) below. More precisely, they proved that for all $\varepsilon > 0$, there exists $\delta > 0$ such that for any initial data $(v_0, V_0, u_0) \in L^2[-T,T] \times L_0^2[-T,T] \times H^1[-T,T]$ satisfying

$$\|v_0 - \psi\|_{L^2[-T,T]} < \delta, \quad \|V_0 - \varphi\|_{L^2[-T,T]} < \delta, \quad \|u_0 - \phi\|_{H^1[-T,T]} < \delta, \tag{1.18}$$

$$\begin{cases} \inf_{y\in\mathbf{R}} \|v(\cdot+y,t) - \psi\|_{L^2[-T,T]} < \varepsilon, \quad \inf_{y\in\mathbf{R}} \|V(\cdot+y,t) - \psi\|_{L^2[-T,T]} < \varepsilon, \\ \inf_{(\theta,y)\in[0,2\pi)\times\mathbf{R}} \|e^{i\theta}u(\cdot+y,t) - \phi\|_{H^1[-T,T]} < \varepsilon \end{cases} \tag{1.19}$$

if

$$\int_0^T v_0(x)dx \leq \int_0^T \psi(x)dx. \tag{1.20}$$

This result is established by adapting the results in Refs. 2, 3, and 6 to the periodic case.

Our work is structured as follows: In Sec. II, we provide some basic and preliminary results—about the instability index counting theory and the relation of the linearized operators $\mathcal{L}_\pm$ to the classical Schrödinger operators arising in the elliptic function theory. In Sec. III, we develop the spectral theory for the self-adjoint matrix linearized operator $\mathcal{H}$ and the full linearized operator $\mathcal{J}\mathcal{H}$. In particular, we describe the kernels and the generalized kernels in full detail. We also show that the Morse index $\mathcal{H} \leq 1$. This is later upgraded in Sec. IV to $n(\mathcal{H}) = 1$. Section IV also contains the proof of the main result, namely the spectral and orbital stability of the dnoidal waves. The spectral stability is achieved via the instability index count, while the orbital stability is obtained as a consequence of an abstract result, Theorem 5.2.11, p. 143,[16] which relates the two notions.

II. PRELIMINARIES

We start with some facts about the instability index counting theory for eigenvalue problems of the type described in (1.17).

A. Instability index counting

We will give some results about the instability index count theories developed in Ref. 17. These allow us to count the number of unstable eigenvalues for eigenvalue problems of the form (2.1) based on the information about the spectrum of various self-adjoint operators, both scalar and matrix, and some specific estimates,

$$\mathcal{I}\mathcal{L}z = \lambda z. \tag{2.1}$$

Here, our standing assumption is that for appropriate Hilbert space X, $\mathcal{L} : X \to X^*$ is bounded and symmetric, and in addition, $\mathcal{L} = \mathcal{L}^*$ on the appropriately defined Hilbert space $X \subset H \subset X^*$ and domain $D(\mathcal{L})$. In addition, assume that $\mathcal{L}$ also has a finite number of negative eigenvalues, $n(\mathcal{L})$, a quantity referred to as the Morse index of the operator $\mathcal{L}$. In addition, $\mathcal{I}^* = -\mathcal{I}$.

Let k_r be the sum of algebraic multiplicities of positive eigenvalues of the spectral problem (2.1) (i.e., the number of real instabilities or real modes), k_c be the sum of algebraic multiplicities of quadruplets of eigenvalues with non-zero real and imaginary parts, and $k_i^- = k_i^{\leq 0}$, the number of pairs of purely imaginary eigenvalues with non-positive Krein signature. For a simple pair of imaginary eigenvalues $\pm i\mu$, $\mu \neq 0$, and the corresponding eigenvector $\vec{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, the Krein signature, either ± 1 or 0, is the quantity $\operatorname{sgn}(\langle \mathcal{L}\vec{z}, \vec{z}\rangle)$.

Also of importance in this theory is a finite-dimensional matrix $\mathcal{D}$, which is obtained from the adjoint eigenvectors for (2.1). More specifically, consider the generalized kernel of $\mathcal{I}\mathcal{L}$,

$$gKer(\mathcal{I}\mathcal{L}) = span[(Ker[(\mathcal{I}\mathcal{L})^l], \quad l = 1, 2, \ldots].$$

Assume that the dimension of the space $gKer(\mathcal{I}\mathcal{L}) \ominus Ker(\mathcal{L})$ is finite. Here, for a subspace $A \subset X$, where X is a fixed Banach space, write $B = X \ominus A$, whenever $X = A \oplus B$. Select a basis in

$$gKer(\mathcal{I}\mathcal{L}) \ominus Ker(\mathcal{L}) = span[\eta_j, j = 1, \ldots, N].$$

Then, $\mathcal{D} \in \mathcal{M}_{N\times N}$ is defined via

$$\mathcal{D} := \{\mathcal{D}_{ij}\}_{i,j=1}^N : \mathcal{D}_{ij} = \langle \mathcal{L}\eta_i, \eta_j\rangle.$$

Then, according to Ref. 17, we have the following formula, relating the number of "instabilities" or Hamiltonian index of the eigenvalue problem (2.1)and the Morse indices of $\mathcal{L}$ and $\mathcal{D}$:

$$k_{Ham} := k_r + 2k_c + 2k_i^- = n(\mathcal{L}) - n_0(\mathcal{D}), \tag{2.2}$$

where $n_0(\mathcal{D}) = \#\{\lambda \leq 0 : \lambda \in \sigma(\mathcal{D})\}$ is the number of non-positive eigenvalues of $\mathcal{D}$.

Remark. As an easy corollary, if $n(\mathcal{L}) = 1$, it follows from (2.2) that $k_c = k_i^- = 0$ and

$$k_r = 1 - n(\mathcal{D}). \tag{2.3}$$

Thus, in the case $n(\mathcal{L}) = 1$, instability occurs exactly when $n(\mathcal{D}) = 0$, while stability occurs whenever $n(\mathcal{D}) = 1$.

B. The linearized operators $\mathcal{L}_\pm$ in terms of the standard Hill operators

We start by introducing another classical linearized Schrödinger operator, associated with the wave ϕ, namely

$$\mathcal{L}_+ := -\partial_x^2 + \sigma - \frac{3}{2(1-c^2)}\phi^2.$$

We now consider two concrete classical Hill operators, which are related to the linearized operators $\mathcal{L}_\partial$, along with some relevant spectral properties. These will allow us to accurately determine the negative spectrum and the kernel of the scalar Schrödinger operators, which will be of use in the sequel.

More specifically, the Schrödinger operator

$$\Lambda_1 = -\partial_y^2 + 6k^2 sn^2(y,k),$$

with periodic boundary conditions on $[0, 4K(k)]$, has eigenvalues that are all simple. The first few eigenvalues and the corresponding eigenfunctions are given in the following list:

$$\begin{aligned} \nu_0 &= 2 + 2k^2 - 2\sqrt{1-k^2+k^4}, \quad \phi_0(y) = 1 - (1 + k^2 - \sqrt{1-k^2+k^4})sn^2(y,k), \\ \nu_1 &= 1 + k^2, \quad \phi_1(y) = cn(y,k)dn(y,k) = sn'(y,k), \\ \nu_2 &= 1 + 4k^2, \quad \phi_2(y) = sn(y,k)dn(y,k) = -cn'(y,k), \\ \nu_3 &= 4 + k^2, \quad \phi_1(y) = sn(y,k)cn(y,k) = -k^{-2}dn'(y,k). \end{aligned}$$

Similarly, for the operator

$$\Lambda_2 = -\partial_y^2 + 2\kappa^2 sn^2(y,\kappa),$$

with periodic boundary conditions on $[0, 4K(k)]$, the eigenvalues are all simple. The first three eigenvalues and the corresponding eigenfunctions are

$$\begin{cases} \epsilon_0 = k^2, & \theta_0(y) = dn(y,k), \\ \epsilon_1 = 1, & \theta_1(y) = cn(y,k), \\ \epsilon_2 = 1 + k^2, & \theta_2(y) = sn(y,k). \end{cases}$$

We now relate the Schrödinger operators $\mathscr{L}_\pm$ to Λ_1, Λ_2. We start with the dnoidal case.

1. The operators $\mathscr{L}_+, \mathscr{L}_-$ in terms of Λ_1, Λ_2

An elementary and classical calculation shows that

$$\mathscr{L}_+ = \alpha^2[\Lambda_1 - (4 + \kappa^2)], \tag{2.4}$$

$$\mathscr{L}_- = \alpha^2[\Lambda_2 - \kappa^2]. \tag{2.5}$$

Based on formulas (1.8) and (1.9), we can formulate the following useful spectral properties.

Proposition 2. The linearized operators $\mathscr{L}_\pm$ have the following spectral properties:

- $n(\mathscr{L}_+) = 1$, $\ker(\mathscr{L}_+) = span[\phi']$.
- $\mathscr{L}_- \geq 0$, $\ker(\mathscr{L}_-) = span[\phi]$.

III. SPECTRAL THEORY FOR $\mathscr{H}$ AND $\mathscr{J}\mathscr{H}$

We now discuss some elementary spectral properties of the scalar Schrödinger operators $\mathscr{L}_\pm$.

A. Elementary properties of $\mathscr{L}_\pm$

Note that $\mathscr{L}_-$ and $\mathscr{L}_+$ have some generic properties, which can be gleaned directly from the defining equation (1.6). More precisely, we have the following two relations:

$$\mathscr{L}_-[\phi] = 0, \qquad \mathscr{L}_+[\phi'] = 0. \tag{3.1}$$

Indeed, the formula $\mathscr{L}_-[\phi] = 0$ is nothing but (1.6), while $\mathscr{L}_+[\phi'] = 0$ is obtained from (1.6) by differentiation in the spatial variable. We have, thus, identified at least one element in each $Ker(\mathscr{L}_-), Ker(\mathscr{L}_+)$. Clearly, per the standard Sturm–Liouville theory for Schrödinger operators acting on periodic functions, it is possible that there might be up to one additional element in each of $Ker(\mathscr{L}_-), Ker(\mathscr{L}_+)$. In our example of the dnoidal waves, this does not happen and, indeed, it turns out that $\ker(\mathscr{L}_-) = span[\phi], \ker(\mathscr{L}_+) = span[\phi']$; see Sec. II B. Interestingly, and based on this information only, we can identify the kernel of the self-adjoint matrix operator $\mathscr{H}$.

B. Determination of $Ker(\mathscr{H})$

Before we start with our analysis, let us recall that the domain of the operator $\mathscr{H}$ is so that the last component is mean-zero. We have the following result.

Proposition 3. The kernel of $\mathscr{H}$ is two-dimensional. In fact, $\ker(\mathscr{H}) = span[\Psi_1, \Psi_2]$, *where*

$$\Psi_1 = \begin{pmatrix} \phi \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \Psi_2 = \begin{pmatrix} 0 \\ \phi' \\ \frac{1}{c^2-1}\phi\phi' \\ \frac{c}{c^2-1}\phi\phi' \end{pmatrix}.$$

Remark. Note that since the fourth components of both Ψ_1, Ψ_2 are mean-zero functions, they do belong to the domain of $\mathscr{H}$, as stated.

Proof. Let $\mathscr{H}\vec{f} = 0$, where $\vec{f} = (f_1, f_2, f_3, f_4)$. Then, we have the following system:

$$\begin{cases} \mathscr{L}_- f_1 = 0, \\ \mathscr{L}_- f_2 + \phi f_3 = 0, \\ \phi f_2 + f_3 - c f_4 = 0, \\ -c f_3 + f_4 = 0. \end{cases} \tag{3.2}$$

Obviously, we have

$$\Psi_1 = \begin{pmatrix} \phi \\ 0 \\ 0 \\ 0 \end{pmatrix} \in \ker \mathscr{H}.$$

From the last three equations of system (3.2), we have $f_4 = cf_3$ and $\phi f_2 = (c^2-1)f_3$. Plugging them into the second equation of system (3.2), we get

$$\left(\mathscr{L}_- + \frac{\phi^2}{c^2-1}\right) f_2 = 0.$$

Noting that $\mathscr{L}_+ = \mathscr{L}_- + \frac{\phi^2}{c^2-1}$, the last equation means that $\mathscr{L}_+ f_2 = 0$. Hence, and up to a constant, $f_2 = \phi'$, $f_3 = \frac{1}{c^2-1}\phi\phi'$, and $f_4 = \frac{c}{c^2-1}\phi\phi'$, and with this, we get

$$\Psi_2 = \begin{pmatrix} 0 \\ \phi' \\ \frac{1}{c^2-1}\phi\phi' \\ \frac{c}{c^2-1}\phi\phi' \end{pmatrix} \in \ker \mathscr{H}.$$

Clearly, this describes all the linearly independent elements in $\ker(\mathscr{H})$, and Proposition 3 is established in full. □

Our next task is to identify $gKer(\mathscr{J}\mathscr{H}) \ominus \ker(\mathscr{H})$, as any basis of this subspace is relevant in our stability calculation, i.e., the matrix $\mathscr{D}$; see Sec. II A.

C. Identifying $\ker(\mathscr{J}\mathscr{H})$

We start with the elements in $\ker(\mathscr{J}\mathscr{H}) \ominus \ker(\mathscr{H})$. That is, we would like to find elements $\vec{\eta}$ so that

$$\mathscr{H}\vec{\eta} \in \ker(\mathscr{J}) = span[(0,0,1,0), (0,0,0,1)]. \tag{3.3}$$

Let $\mathscr{H}\vec{\eta}_1 = (0,0,1,0)$, where $\vec{\eta}_1 = (\eta_{11}, \eta_{12}, \eta_{13}, \eta_{14})$. We obtain the following system:

$$\begin{cases} \mathscr{L}_-\eta_{11} = 0, \\ \mathscr{L}_-\eta_{12} + \phi\eta_{13} = 0, \\ \phi\eta_{12} + \eta_{13} - c\eta_{14} = 1, \\ -c\eta_{13} + \eta_{14} = 0, \end{cases}$$

which, apart from $\ker(\mathscr{L}_-)$, has the following solution:

$$\vec{\eta}_1 = \begin{pmatrix} 0 \\ -\frac{1}{1-c^2}\mathscr{L}_+^{-1}\phi \\ \frac{1}{1-c^2} + \frac{1}{(1-c^2)^2}\phi\mathscr{L}_+^{-1}\phi \\ \frac{c}{1-c^2} + \frac{c}{(1-c^2)^2}\phi\mathscr{L}_+^{-1}\phi \end{pmatrix}.$$

Similarly, let $\mathscr{H}\vec{\eta}_2 = (0,0,0,1)$, where $\vec{\eta}_2 = (\eta_{21}, \eta_{22}, \eta_{23}, \eta_{24})$. That is, we need to solve the following system:

$$\begin{cases} \mathscr{L}_-\eta_{21} = 0, \\ \mathscr{L}_-\eta_{22} + \phi\eta_{23} = 0, \\ \phi\eta_{22} + \eta_{23} - c\eta_{24} = 0, \\ -c\eta_{23} + \eta_{24} = 1. \end{cases}$$

We have the following solution:

$$\vec{\eta}_2 = \begin{pmatrix} 0 \\ -\frac{c}{1-c^2}\mathscr{L}_+^{-1}\phi \\ \frac{c}{1-c^2} + \frac{c}{(1-c^2)^2}\phi\mathscr{L}_+^{-1}\phi \\ \frac{1}{1-c^2} + \frac{c^2}{(1-c^2)^2}\phi\mathscr{L}_+^{-1}\phi \end{pmatrix}.$$

We have found two linearly independent solutions of (3.3). However, it is clear that the fourth components of both cannot be mean-value zero. In fact, in the generic case, we need to take a specific linear combination so that we can achieve mean-value zero in the last component.

To this end, we take the following linear combination:

$$\vec{\tilde{\eta}}_1 := -\vec{\eta}_1\left(\langle 1,1\rangle + \frac{c^2}{1-c^2}\langle \mathscr{L}_+^{-1}\phi, \phi\rangle\right) + \vec{\eta}_2\left(c\langle 1,1\rangle + \frac{c}{1-c^2}\langle \mathscr{L}_+^{-1}\phi, \phi\rangle\right) = -\vec{\eta}_1\left(2T + \frac{c^2}{1-c^2}\langle \mathscr{L}_+^{-1}\phi, \phi\rangle\right) + \vec{\eta}_2\left(2Tc + \frac{c}{1-c^2}\langle \mathscr{L}_+^{-1}\phi, \phi\rangle\right), \quad (3.4)$$

which clearly has the property $\int_{-T}^{T}\vec{\tilde{\eta}}_{14}(x)dx = 0$ and, as a linear combination of $\vec{\eta}_1, \vec{\eta}_2$, belongs to the set described in (3.3), i.e., $\vec{\tilde{\eta}}_1 \in \ker(\mathscr{J}\mathscr{H}) \ominus \ker(\mathscr{H})$. We have thus established the following proposition.Our next task is to describe the generalized kernel of the linearized operator.

Proposition 4. Under the assumption that $\ker(\mathscr{L}_-) = span[\phi]$, $\ker(\mathscr{L}_+) = span[\phi']$, *the kernel* $\ker(\mathscr{J}\mathscr{H})$ *is three-dimensional. More specifically,*

$$\ker(\mathscr{J}\mathscr{H}) = \ker(\mathscr{H}) \oplus (\ker(\mathscr{J}\mathscr{H}) \ominus \ker(\mathscr{H})) = span[\Psi_1, \Psi_2] \oplus span[\vec{\tilde{\eta}}_1].$$

D. Structure of $gKer(\mathscr{J}\mathscr{H})$

Since we have determined $\ker(\mathscr{J}\mathscr{H})$ in Proposition 4, it remains to find the generalized eigenvectors associated with this system.

As a first step, we show that a linear combination of $\vec{\eta}_1, \vec{\eta}_2$ does not give rise to any adjoint eigenvectors.

Proposition 5. Assume that $\langle \mathscr{L}_+^{-1}\phi, \phi\rangle \neq 0$. *Then, the equation*

$$\mathscr{J}\mathscr{H}\vec{f} = \gamma_1\vec{\eta}_1 + \gamma_2\vec{\eta}_2 \quad (3.5)$$

does not have any solutions, unless $\gamma_1 = \gamma_2 = 0$.

Proof. Note that

$$\gamma_1\vec{\eta}_1 + \gamma_2\vec{\eta}_2 = (\gamma_1 + c\gamma_2)\vec{\eta}_1 + \gamma_2\begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

Assuming that (3.5) has a solution f, we test it first against the vector $\vec{g} := \begin{pmatrix}0\\\phi\\0\\0\end{pmatrix}$. We obtain

$$-\frac{\gamma_1 + c\gamma_2}{1-c^2}\langle \mathscr{L}_+^{-1}\phi, \phi\rangle = \langle \mathscr{J}\mathscr{H}\vec{f}, \vec{g}\rangle = \langle \mathscr{H}\vec{f}, \mathscr{J}^*\vec{g}\rangle = \langle \mathscr{L}_- f_1, \phi\rangle = \langle f_1, \mathscr{L}_-\phi\rangle = 0.$$

This yields that $\gamma_1 + c\gamma_2 = 0$. However, then, (3.5) turns into

$$\mathscr{J}\mathscr{H}\vec{f} = \gamma_2\begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

As the last component on the left-hand side is an exact derivative, this clearly does not have any solutions, unless $\gamma_2 = 0$. □

Next, we show that the equations $\mathscr{J}\mathscr{H}f = \Psi_1$ and $\mathscr{J}\mathscr{H}f = \Psi_2$ do have solutions, and we describe them in detail. To this end, let $\mathscr{J}\mathscr{H}\vec{\eta}_3 = \Psi_1$, where $\vec{\eta}_3 = (\eta_{31}, \eta_{32}, \eta_{33}, \eta_{34})$. We have the following system:

$$\begin{cases} \mathscr{L}_-\eta_{31} = 0,\\ \mathscr{L}_-\eta_{32} + \phi\eta_{33} = -\phi,\\ -(-c\eta_{33} + \eta_{34})_x = 0,\\ -(\phi\eta_{32} + \eta_{33} - c\eta_{34})_x = 0. \end{cases}$$

Note that the first equation solves to $\eta_{31} = const.\phi$. However, this will not contribute anything to $gKer(\mathscr{J}\mathscr{H})$ as this solution is already accounted for in $\ker(\mathscr{H})$. Thus, we take $\eta_{31} = 0$.

Integrating the last two equations yields new integration constants,

$$\begin{cases} \mathscr{L}_-\eta_{32} + \phi\eta_{33} = -\phi,\\ c\eta_{33} - \eta_{34} = d_1,\\ -\phi\eta_{32} - \eta_{33} + c\eta_{34} = d_2. \end{cases}$$

From these three equations, we determine

$$\vec{\eta}_3 = \begin{pmatrix} 0\\ -k\mathscr{L}_+^{-1}\phi\\ \frac{k}{1-c^2}\phi\mathscr{L}_+^{-1}\phi + k_1\\ \frac{ck}{1-c^2}\phi\mathscr{L}_+^{-1}\phi + k_2 \end{pmatrix},$$

where the particular form of the constants k, k_1, k_2 depends on c, d_1, d_2 but is otherwise unimportant in our analysis. Note that

$$\vec{\eta}_3 = k(1-c^2)\vec{\eta}_1 + \begin{pmatrix}0\\0\\\tilde{k}_1\\\tilde{k}_2\end{pmatrix} = k(1-c^2)\vec{\eta}_1 + \tilde{k}_2\vec{\eta}_2 + \tilde{k}_1\begin{pmatrix}0\\0\\1\\0\end{pmatrix}.$$

where again $\tilde{k}_1, \tilde{k}_2$ are two constants. As $\vec{\eta}_1, \vec{\eta}_2 \in \ker(\mathcal{J}\mathcal{H})$, we can clearly take $\vec{\eta}_3 := \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$. In fact, a direct verification confirms that $\mathcal{J}\mathcal{H}\vec{\eta}_3 = -\Psi_1$. Note that as $\vec{\eta}_{34} = 0$, it is mean-value zero in the last component. So, it is an acceptable vector in our analysis.

Finally, we consider the equation $\mathcal{J}\mathcal{H}\eta_4 = \Psi_2$. We have the system

$$\begin{cases} \mathcal{L}_- \eta_{41} = \phi', \\ \mathcal{L}_- \eta_{32} + \phi\eta_{33} = 0, \\ -(\phi\eta_{42} + \eta_{43} - c\eta_{44})_x = -\dfrac{c}{2(1-c^2)}(\phi^2)_x, \\ -(-c\eta_{43} + \eta_{44})_x = -\dfrac{1}{2(1-c^2)}(\phi^2)_x. \end{cases}$$

Note that due to the presence of the vectors $\vec{\eta}_2, \vec{\eta}_3$ in our system, we can integrate the last two equations with constants of integration zero—if non-zero, their contribution can be written in terms of *span*$[\vec{\eta}_2, \vec{\eta}_3]$ and, hence, safely ignored. Thus, we find the solution

$$\vec{\eta}_4 = \begin{pmatrix} \mathcal{L}_-^{-1}[\phi'] \\ -\dfrac{c}{(1-c^2)^2}\mathcal{L}_+^{-1}[\phi^3] \\ \dfrac{c}{(1-c^2)^2}\phi^2 + \dfrac{c}{(1-c^2)^3}\phi\mathcal{L}_+^{-1}[\phi^3] \\ \dfrac{c^2+1}{2(1-c^2)^2}\phi^2 + \dfrac{c^2}{(1-c^2)^3}\phi\mathcal{L}_+^{-1}[\phi^3] \end{pmatrix}.$$

Using that $\mathcal{L}_+\phi = --\frac{1}{1-c^2}\phi^3$, we get $\mathcal{L}_+^{-1}\phi^3 = -(1-c^2)\phi$. This leads to the following representation:

$$\vec{\eta}_4 = \begin{pmatrix} \mathcal{L}_-^{-1}[\phi'] \\ \dfrac{c}{1-c^2}\phi \\ 0 \\ \dfrac{1}{2(1-c^2)}\phi^2 \end{pmatrix}.$$

Here, it becomes clear that $\vec{\eta}_{44}$ does not have mean-value zero; hence, the solution $\vec{\eta}_4$ does not belong to the required $D(\mathcal{J}\mathcal{H})$. We formulate our result in the following proposition.

Proposition 6. Assume that $\langle \mathcal{L}_+^{-1}\phi, \phi\rangle \neq 0$. *Then, there is only one linearly independent first generation adjoint eigenvector, namely* $\vec{\eta}_3$. *That is,*

$$\ker((\mathcal{J}\mathcal{H})^2) \ominus \ker(\mathcal{J}\mathcal{H}) = span[\vec{\eta}_3].$$

Proof. Our strategy here is as follows: we will show that

$$\ker((\mathcal{J}\mathcal{H})^2) \ominus \ker(\mathcal{J}\mathcal{H}) = span[\vec{\eta}_3, \vec{\eta}_4],$$

if we do not restrict with the condition that $\int_{-T}^{T}\vec{\eta}_{44}(x)dx = 0$. Then, due to this restriction, $\vec{\eta}_4$ is excluded, whence our claim follows.

We have, so far, proved that $\ker((\mathcal{J}\mathcal{H})^2) \ominus \ker(\mathcal{J}\mathcal{H}) \supseteq span[\vec{\eta}_3, \vec{\eta}_4]$. In order to show the other direction, set the equation $\mathcal{J}\mathcal{H}f \in \ker(\mathcal{J}\mathcal{H})$ or

$$\mathcal{J}\mathcal{H}f = \lambda_1\Psi_1 + \lambda_2\Psi_2 + \lambda_3\eta_1 + \lambda_4\eta_2.$$

As $\mathcal{J}\mathcal{H}\vec{\eta}_3 = -\Psi_1, \mathcal{J}\mathcal{H}\vec{\eta}_4 = \Psi_2$, we have

$$\mathcal{J}\mathcal{H}[f + \lambda_1\vec{\eta}_3 - \lambda_2\vec{\eta}_4] = \lambda_3\eta_1 + \lambda_4\eta_2.$$

We now apply Proposition 5 to conclude that $\lambda_3 = \lambda_4 = 0$. It follows that $f - \lambda_1\vec{\eta}_3 - \lambda_2\vec{\eta}_4 \in \ker\mathcal{J}\mathcal{H}$, which is the claim. □

Finally, we consider the possibility of second adjoint eigenvectors. Expectedly, it turns out that there are not any.We collect all the findings of this section in the following proposition.

Proposition 7. We have

$$\ker((\mathcal{J}\mathcal{H})^3) \ominus \ker((\mathcal{J}\mathcal{H})^2) = \{0\}.$$

Proof. Consider the subspace $\ker((\mathcal{J}\mathcal{H})^3) \ominus \ker((\mathcal{J}\mathcal{H})^2)$. That is, set up an equation

$$\mathcal{J}\mathcal{H}f = \mu\vec{\eta}_3$$

for some scalar μ. We test it against $\vec{\eta}_3$. We obtain

$$\mu = \langle \mathcal{J}\mathcal{H}f, \vec{\eta}_3\rangle = -\langle \mathcal{H}f, \mathcal{J}\vec{\eta}_3\rangle = 0,$$

as $\mathcal{J}\vec{\eta}_3 = 0$. This implies $\mu = 0$, which establishes the claim. □

Proposition 8. Suppose that $\langle \mathcal{L}_+^{-1}\phi, \phi\rangle \neq 0$. Then,

$$gKer(\mathcal{J}\mathcal{H}) \ominus \ker(\mathcal{H}) = span[\vec{\eta}_1, \vec{\eta}_3].$$

E. The Morse index of $\mathcal{H}$

We have the following proposition.

Proposition 9. For the solution ϕ given by (1.8), we have that $n(\mathcal{H}) \leq 1$.

Remark: We later easily establish that, in fact, $n(\mathcal{H}) = 1$, as a consequence of the index counting formula (2.2). However, and again from the same formula, we have the spectral stability of the waves in the case when $n(\mathcal{H}) = 0$.

Proof. We begin with the observation that due to the tensorial structure of $\mathcal{H} = \mathcal{L}_- \otimes \mathcal{H}_1$, where

$$\mathcal{H}_1 = \begin{pmatrix} \mathcal{L}_- & \phi & 0 \\ \phi & 1 & -c \\ 0 & -c & 1 \end{pmatrix}.$$

Clearly, we have that $n(\mathcal{H}) = n(\mathcal{H}_1) + n(\mathcal{L}_-)$. By Proposition 2, we have that $\mathcal{L}_- \geq 0$. Thus, it remains to prove that $n(\mathcal{H}_1) \leq 1$. To this end, consider the quadratic form associated with $\mathcal{H}_1$, namely

$$q_1(u, v, w) = \langle \mathcal{L}_- u, u\rangle + 2\langle \phi u, v\rangle + \langle v, v\rangle - 2c\langle v, w\rangle + \langle w, w\rangle.$$

By the Cauchy–Schwarz inequality,

$$\langle v, v\rangle - 2c\langle v, w\rangle + \langle w, w\rangle \geq (1 - c^2)\langle v, v\rangle, \tag{3.6}$$

whence $q_1(u, v, w) \geq q_2(v, w) = \langle \mathcal{L}_- u, u\rangle + 2\langle \phi u, v\rangle + (1 - c^2)\langle v, v\rangle$. We further estimate q_2 as follows:

$$\begin{aligned} q_2(u, v) &= \langle \mathcal{L}_- u, u\rangle + 2\langle \phi u, v\rangle + (1 - c^2)\langle v, v\rangle \\ &= \langle \mathcal{L}_- u, u\rangle + \int_{-T}^{T} \left(\sqrt{1 - c^2}v + \frac{\phi}{\sqrt{1 - c^2}}u\right)^2 dx - \frac{1}{1 - c^2}\int_{-T}^{T} \phi^2 u^2 dx \\ &\geq \langle \left(\mathcal{L}_- - \frac{\phi^2}{1 - c^2}\right)u, u\rangle = \langle \mathcal{L}_+ u, u\rangle. \end{aligned} \tag{3.7}$$

It is now easy to conclude that $n(\mathcal{H}_1) \leq 1$. Indeed, based on the fact that $n(\mathcal{L}_+) = 1$, taking u orthogonal to the ground state of $\mathcal{L}_+$ guarantees that $\langle \mathcal{L}_+ u, u\rangle \geq 0$. Thus,

$$\inf_{u \perp \text{ground state of } \mathcal{L}_+} q_1(u, v, w) \geq \inf_{u \perp \text{ground state of } \mathcal{L}_+} q_2(u, v) \geq \inf_{u \perp \text{ground state of } \mathcal{L}_+} \langle \mathcal{L}_+ u, u\rangle \geq 0.$$

This establishes $n(\mathcal{H}_1) \leq 1$. □

IV. PROOF OF THEOREM 1

We apply the instability index theory, as developed in Sec. II. To this end, we need to identify first the spaces X, X^*, H. To this end, we introduce $X = (H^1_{per.}[-T,T])^2 \times (L^2[-T,T])^2$, with $X^* = (H^{-1}_{per.}[-T,T])^2 \times (L^2[-T,T])^2$, while clearly, we use the base Hilbert space $H = (L^2[-T,T])^4$. Clearly, $\mathcal{H} : X \to X^*$ is bounded and symmetric, while $\mathcal{J} : D(\mathcal{J}) = (H^1[-T,T])^4 \to X$ is a closed operator.

By Proposition 8, $\mathcal{D} \in \mathcal{M}_{2\times 2}$, and we take on computing its elements. A direct calculation yields

$$\mathcal{H}\vec{\eta}_3 = \begin{pmatrix} 0 \\ \phi \\ 1 \\ -c \end{pmatrix}.$$

Next, we compute $\mathcal{H}\vec{\eta}_1$, by using formula (3.4), and the values $\mathcal{H}\vec{\eta}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \mathcal{H}\vec{\eta}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, which we have by construction. We obtain

$$\mathcal{H}\vec{\eta}_1 = \begin{pmatrix} 0 \\ 0 \\ -\left(\langle 1,1\rangle + \frac{c^2}{1-c^2}\langle \mathcal{L}_+^{-1}\phi, \phi\rangle\right) \\ \left(c\langle 1,1\rangle + \frac{c}{1-c^2}\langle \mathcal{L}_+^{-1}\phi, \phi\rangle\right) \end{pmatrix}.$$

After some algebraic manipulations (note that $\langle 1,1\rangle = 2T$), we obtain the matrix $\mathcal{D}$ in the following form:

$$\begin{aligned}
\mathcal{D}_{11} &= \langle \mathcal{H}\vec{\eta}_1, \vec{\eta}_1\rangle = 2T\left(2T + \frac{c^2}{1-c^2}\langle \mathcal{L}_+^{-1}\phi, \phi\rangle\right)\left(2T + \frac{1}{1-c^2}\langle \mathcal{L}_+^{-1}\phi, \phi\rangle\right), \\
\mathcal{D}_{12} = \mathcal{D}_{21} &= \langle \mathcal{H}\vec{\eta}_1, \vec{\eta}_3\rangle = -2T\left(2T + \frac{c^2}{1-c^2}\langle \mathcal{L}_+^{-1}\phi, \phi\rangle\right), \\
\mathcal{D}_{22} &= \langle \mathcal{H}\vec{\eta}_3, \vec{\eta}_3\rangle = 2T.
\end{aligned}$$

It follows that

$$\det(\mathcal{D}) = 4T^2\langle \mathcal{L}_+^{-1}\phi, \phi\rangle\left(2T + \frac{c^2}{1-c^2}\langle \mathcal{L}_+^{-1}\phi, \phi\rangle\right).$$

Recall that we have established that $n(\mathcal{H}) \le 1$; see Proposition 9. Due to formula (2.3), it must be that $n(\mathcal{D}) \le 1$). Thus, the spectral stability is equivalent to the property $n(\mathcal{D}) = 1$. In fact, due to formula (2.2), it follows that $n(\mathcal{H}) \ge n(\mathcal{D})$, so establishing $n(\mathcal{D}) = 1$ implies $n(\mathcal{H}) = 1$ as well, as announced earlier.

Next, in order to prove $n(\mathcal{D}) = 1$, we shall need to show that $\det(\mathcal{D}) < 0$, as this guarantees that $\mathcal{D}$ has a negative eigenvalue. To this end, recall that we have $\mathcal{L}_+\phi' = 0$. The function

$$\Phi(x) = \phi'(x)\int^x \frac{1}{\phi'^2(s)}ds, \qquad \begin{vmatrix} \phi' & \Phi \\ \phi'' & \Phi' \end{vmatrix} = 1,$$

is also a solution of $L\Phi = 0$. Formally, since Φ' has zeros, using the identities

$$\frac{1}{sn^2(y,\kappa)} = -\frac{1}{dn(y,\kappa)}\frac{\partial}{\partial_y}\frac{cn(x,\kappa)}{sn(y,\kappa)}$$

and integrating by parts, we get

$$\Phi(x) = \frac{1}{\alpha^2\kappa^2\varphi_0}\left[\frac{1-2sn^2(\alpha x,\kappa)}{dn(\alpha x,\kappa)} - \alpha\kappa^2 sn(\alpha x,\kappa)cn(\alpha x,\kappa)\int_0^x \frac{1-2sn^2(\alpha s,\kappa)}{dn^2(\alpha s,\kappa)}ds\right].$$

Thus, we may construct the Green function

$$\mathscr{L}_+^{-1} f = \phi' \int_0^x \Phi(s) f(s) ds - \Phi(s) \int_0^x \phi'(s) f(s) s + C_f \Phi(x),$$

where C_f is chosen such that $\mathscr{L}_+^{-1} f$ is periodic with the same period as $\phi(x)$. Integrating by parts, we get

$$\langle \mathscr{L}_+^{-1}\phi, \phi\rangle = -\langle \phi^3, \Phi\rangle + \phi^2(T)\langle \phi, \Phi\rangle - \frac{\phi''(T)}{2\Phi'(T)}\langle \phi, \Phi\rangle^2.$$

Similarly as in Ref. 18, using that $\phi^3 = -(1-c^2)\mathscr{L}\phi$ and $\langle \Phi'', \phi\rangle = 2\phi(T)\Phi'(T) + \langle \Phi, \phi''\rangle$, we have the following result:

$$\langle \mathscr{L}_+^{-1}\phi, \phi\rangle = 2(1-c^2)\phi(T)\Phi'(T) + \phi^2(T)\langle \phi, \Phi\rangle - \frac{\phi''(T)}{2\Phi'(T)}\langle \phi, \Phi\rangle^2.$$

Integrating by parts, we get

$$\Phi'(T)\phi(T) = \frac{1}{\alpha\kappa^2}[2(1-\kappa^2)K - (2-\kappa^2)E],$$

$$\langle \phi, \Phi\rangle = \frac{1}{\alpha^3\kappa^2}[E(\kappa) - K(\kappa)],$$

$$\frac{\phi''(T)}{2\Phi'(T)}\langle \phi, \Phi\rangle = \frac{\phi_0^2\kappa^2(1-\kappa^2)[E-K]}{2[2(1-\kappa^2)K - (2-\kappa^2)E]},$$

which leads to the following result:

$$\langle \mathscr{L}_+^{-1}\phi, \phi\rangle = -\frac{2(1-c^2)}{\alpha}\frac{E^2(\kappa) - (1-\kappa^2)K^2(\kappa)}{(2-\kappa^2)E(\kappa) - 2(1-\kappa^2)K(\kappa)} < 0. \tag{4.1}$$

For the other term, we have

$$\begin{aligned} 2T + \frac{c^2}{1-c^2}\langle \mathscr{L}_+^{-1}\phi, \phi\rangle &= \frac{2}{\alpha}\left[K(\kappa) - c^2\frac{E^2(\kappa) - (1-\kappa^2)K^2(\kappa)}{(2-\kappa^2)E(\kappa) - 2(1-\kappa^2)K(\kappa)}\right] \\ &= \frac{2}{\alpha}\frac{E^2(\kappa) - (1-\kappa^2)K^2(\kappa)}{(2-\kappa^2)E(\kappa) - 2(1-\kappa^2)K(\kappa)}\left[\frac{(2-\kappa^2)E(\kappa)K(\kappa) - 2(1-\kappa^2)K^2(\kappa)}{E^2(\kappa) - (1-\kappa^2)K^2(\kappa)} - c^2\right] > 0, \end{aligned}$$

since

$$\frac{(2-\kappa^2)E(\kappa)K(\kappa) - 2(1-\kappa^2)K^2(\kappa)}{E^2(\kappa) - (1-\kappa^2)K^2(\kappa)} > 1 > c^2.$$

Altogether, $\det(\mathscr{D}) < 0$ and the spectral stability is established.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Sevdzhan Hakkaev: Conceptualization (equal); Formal analysis (equal); Investigation (equal). **Milena Stanislavova**: Conceptualization (equal); Formal analysis (equal); Investigation (equal). **Atanas Stefanov**: Conceptualization (equal); Formal analysis (equal); Investigation (equal); Writing – review & editing (equal).

DATA AVAILABILITY

The data that support the findings of this study are available within the article.

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