

A PIE Representation of Scalar Quadratic PDEs and Global Stability Analysis Using SDP

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Abstract—It has recently been shown that the evolution of a linear Partial Differential Equation (PDE) can be more conveniently represented in terms of the evolution of a higher spatial derivative of the state. This higher spatial derivative (termed the ‘fundamental state’) lies in L_2 - requiring no auxiliary boundary conditions or continuity constraints. Such a representation (termed a Partial Integral Equation or PIE) is then defined in terms of an algebra of bounded integral operators (termed Partial Integral (PI) operators) and is constructed by identifying a unitary map from the fundamental state to the state of the original PDE. Unfortunately, when the PDE is nonlinear, the dynamics of the associated fundamental state are no longer parameterized in terms of PI operators. However, in this paper we show that such dynamics can be compactly represented using a new tensor algebra of partial integral operators acting on the tensor product of the fundamental state. We further show that this tensor product of the fundamental state forms a natural distributed equivalent of the monomial basis used in representation of polynomials on a finite-dimensional space. This new representation is then used to provide a simple SDP-based Lyapunov test of stability of quadratic PDEs. The test is applied to three illustrative examples of quadratic PDEs.

I. INTRODUCTION

In this paper, we consider the problem of representation and stability analysis of quadratic Partial Differential Equations (PDEs). Quadratic PDEs are frequently used to model physical processes, including fluid dynamics (e.g. Navier-Stokes), population growth (e.g. Fisher’s equation), and wave propagation (e.g. Korteweg-de Vries). However, certain aspects of the standard PDE representation of such nonlinear systems present difficulties when applied to the problems of analysis and simulation. For example, consider a modified version of Burger’s equation with Dirichlet Boundary Conditions (BCs),

$$\begin{aligned} u_t(t, s) &= u_{ss}(t, s) + ru(t, s) - u(t, s)u_s(t, s), \quad s \in [0, 1], \\ u(t, 0) &= 0, \quad u(t, 1) = 0, \end{aligned} \quad (1)$$

wherein we added a reaction term $ru(t, s)$ to the dynamics. To verify stability of this system, we can use the candidate Lyapunov Functional (LF) $V(u) = \frac{1}{2}\|u\|_{L_2}^2 \geq 0$. Using integration by parts, and invoking the BCs, it can be shown that the derivative of this LF along solutions to the PDE is $\dot{V}(u) = r\|u\|_{L_2}^2 - \|u_s\|_{L_2}^2$. Then, for any $r < \pi^2$, it can be proven that this derivative satisfies $\dot{V}(u) \leq 0$, thus certifying stability of the system [1]. However, verifying that $\dot{V}(u) \leq 0$ for $r \in (0, \pi^2)$ is not trivial, as it necessitates deriving some upper bound on the norm of u in terms of the norm of its derivative u_s , invoking e.g. the Poincaré inequality. In

this manner, the representation of the system dynamics as a (polynomial) function of u , u_s , and u_{ss} complicates the task of verifying suitability of even a simple, fixed LF candidate $V(u) = \frac{1}{2}\|u\|_{L_2}^2$.

Similarly, suppose that we adjust the BCs, imposing e.g. a Neumann condition $u_s(t, 1) = 0$. In this case, despite the fact that neither the gradient $\nabla V(u)$ nor the expression for $\dot{u}(t)$ is changed, the derivative $\dot{V}(u) = \nabla V(u)\dot{u}$ no longer satisfies $\dot{V}(u) = r\|u\|_{L_2}^2 - \|u_s\|_{L_2}^2$ along solutions to the system. More generally, it is unclear how the BCs affect stability properties of the system, and how we can account for them in testing fitness of any LF candidate.

Because of these difficulties associated with verifying suitability of LFs for nonlinear PDEs, most prior work focuses only on limited classes of PDEs with specific BCs, proving results only for the particular system under consideration. For example, extensive research has been done deriving stability conditions for Navier-Stokes equations [2]–[5], commonly expanding solutions using e.g. a Galerkin basis, and proving decay of a LF using Sum-Of-Squares (SOS) techniques. Similarly, stability of the Kuramoto-Sivanshinsky Equation was studied in [6], assuming periodic BCs, and verifying negativity $\dot{V}(u) \leq 0$ of a quadratic LF using discretization. However, these results apply only to specific PDEs, offering limited insight into how to test stability of other nonlinear systems.

Prior work studying more general systems includes [7], deriving a Linear Matrix Inequality (LMI) stability test for a class of wave equations $u_{tt} = u_{ss} + f(u, s, t)$, assuming a bound $f(u, s, t) < g$ on the nonlinear term. Similarly, stability of classes of 2nd order, parabolic PDEs is analysed in [8]–[11], e.g. deriving polynomial positivity conditions for verifying stability of such systems. However, these results too are limited in their applications, and the proposed stability conditions may be challenging to enforce in practice.

Alternatively, the authors of [12]–[14] propose the use of moment methods for analysis of nonlinear PDEs, testing bounds on e.g. energy functionals $V(u) = \|u\|_{L_2}^2$ using Semidefinite Programming (SDP). In order to obtain such bounds, these works study a dual formulation of the PDE system, attempting to use properties of this dual representation to derive properties of the original PDE. However, an intuitive algorithm applying these techniques to test stability of general PDEs has not yet been developed.

In order to construct a comprehensive framework for testing stability of nonlinear PDEs, in this paper, we illustrate how the main obstacle prohibiting the construction of such a framework is the PDE representation itself. In particular, as

noted, the representation of the PDE dynamics in terms of a polynomial function of not only the state u but also its derivatives ($u_s, u_{ss}, \dots$) makes it difficult to verify $\dot{V}(u) \leq 0$ for any candidate LF $V(u)$, in general. Moreover, the presence of BCs in the PDE representation further complicates such analysis, as any LF need only satisfy $\dot{V}(u) \leq 0$ for solutions u satisfying the BCs. A more suitable representation of distributed-state systems, then, should ideally satisfy the following two properties:

1. The system dynamics should be represented as a polynomial function of only a single distributed state $v(t) \in L_2$.
2. The representation should not impose any auxiliary constraints on the state $v(t)$, i.e. should be free of BCs and continuity constraints.

In this paper, we show how, for scalar-valued, quadratic, 1D PDEs, we may derive an equivalent representation satisfying both of these conditions, termed the quadratic Partial Integral Equation (PIE) representation.

II. PROBLEM FORMULATION

A. Notation

For a given domain $\Omega \subset \mathbb{R}^d$, let $L_2^n[\Omega]$ and $L_\infty[\Omega]$ denote the set of all $\mathbb{R}^n$ -valued square-integrable and bounded functions on Ω , respectively, where we omit the domain when clear from context. For $k \in \mathbb{N}$, define Sobolev subspaces $H_k^n[\Omega]$ of $L_2^n[\Omega]$ for $\Omega \subseteq \mathbb{R}$ as

$$H_k^n[\Omega] = \{ \mathbf{v} \in L_2^n[\Omega] \mid \partial_s^\alpha \mathbf{v} \in L_2^n[\Omega], \forall \alpha \in \mathbb{N} : \alpha \leq k \},$$

where we write $\partial_s^\alpha \mathbf{v} = \frac{\partial^\alpha}{\partial s^\alpha} \mathbf{v}$, and where we similarly omit the domain when clear from context.

B. Objectives and Approach

We consider the problem of representation of nonlinear PDEs and parametrization of quadratic (and non-quadratic) forms on a distributed state-space. In particular, suppose we are given a scalar 2nd order quadratic PDE of the form

$$\text{PDE:} \quad u_t(t, s) = c(s)^T \begin{bmatrix} u \\ u_s \\ u_{ss} \\ u^2 \\ uu_s \\ u_s^2 \end{bmatrix} (t, s), \quad s \in [a, b], \quad (2)$$

with linear boundary conditions of the form

$$u(t) \in X_B[a, b] := \left\{ \mathbf{u} \in H_2[a, b] \mid B \begin{bmatrix} \mathbf{u}^{(a)} \\ \mathbf{u}_s^{(a)} \\ \mathbf{u}^{(b)} \\ \mathbf{u}_s^{(b)} \end{bmatrix} = 0 \right\}, \quad (3)$$

parameterized by $\{c, B\} \in L_2^6[a, b] \times \mathbb{R}^{2 \times 4}$.

The goal of this paper is to create a unified representation of this class of PDEs (extending to higher-order PDEs) and to verify the existence of quadratic Lyapunov functionals which prove stability of such systems. To create such a unified representation and parametrization, in Section III, we will first use the BCs to express the distributed state, $u \in X_B$, in terms of its highest order derivative, $\mathbf{v} := u_{ss} \in L_2$ – constructing the linear map, $\mathcal{T} : \mathbf{v} \mapsto u$ so that $u = \mathcal{T} \partial_s^2 u$ and $\mathbf{v} = \partial_s^2 \mathcal{T} \mathbf{v}$. This mapping is referred to as the Partial

Integral (PI) transformation, and has already been studied extensively in the context of linear PDEs [15].

Equipped with this PI transformation, we will show in Section V that any such quadratic PDE (including higher-order PDEs) can be represented as

$$\partial_t \mathcal{T} \mathbf{v}(t) = \mathcal{A} \mathbf{v}(t) + \mathcal{B}[\mathbf{v}(t) \otimes \mathbf{v}(t)]. \quad (4)$$

where $[\mathbf{v}(t) \otimes \mathbf{v}(t)](\theta, \eta) = \mathbf{v}(t, \theta) \mathbf{v}(t, \eta)$ is the tensor product of $\mathbf{v}(t) \in L_2[a, b]$ with itself and defines a distributed monomial basis on $L_2[a, b]$. Although the focus of this paper is on quadratic PDEs, this representation can also be extended to cubic or higher degree PDEs by inclusion of higher degree monomials such as $\mathbf{v}(t) \otimes \mathbf{v}(t) \otimes \mathbf{v}(t)$. The operator $\mathcal{A}$, meanwhile, is a PI operator, which for simplicity can take the form

$$(\mathcal{A} \mathbf{v})(t, s) = \int_a^s A_1(s, \theta) \mathbf{v}(t, \theta) d\theta + \int_s^b A_2(s, \theta) \mathbf{v}(t, \theta) d\theta.$$

Accordingly, the operator $\mathcal{B}$ is a tensor product of PI operators, taking the form

$$\begin{aligned} (\mathcal{B} \mathbf{w})(t, s) = & \int_a^s \int_a^\theta Q_1(s, \theta) R_1(s, \eta) \mathbf{w}(t, \theta, \eta) d\eta d\theta \\ & + \int_s^b \int_a^s Q_2(s, \theta) R_2(s, \eta) \mathbf{w}(t, \theta, \eta) d\eta d\theta \\ & + \int_s^b \int_s^\theta Q_3(s, \theta) R_3(s, \eta) \mathbf{w}(t, \theta, \eta) d\eta d\theta. \end{aligned}$$

Next, in Section VI, we show that for a quadratic Lyapunov functional of the form $V(\mathbf{v}) = \langle \mathcal{T} \mathbf{v}, \mathcal{P} \mathcal{T} \mathbf{v} \rangle_{L_2}$ (where $\mathcal{P}$ is a PI operator decision variable and $\mathbf{v} \in L_2[a, b]$) the derivative of this LF can be represented as

$$\dot{V}(\mathbf{v}) = \left\langle \begin{bmatrix} \mathbf{v} \\ \mathbf{v} \otimes \mathbf{v} \end{bmatrix}, \begin{bmatrix} \mathcal{A}^* \mathcal{P} \mathcal{T} + \mathcal{T}^* \mathcal{P} \mathcal{A} & \mathcal{T}^* \mathcal{P} \mathcal{B} \\ \mathcal{B}^* \mathcal{P} \mathcal{T} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{v} \otimes \mathbf{v} \end{bmatrix} \right\rangle_{L_2}.$$

Exploiting the algebraic closure of our class of tensor products of PI operators, we show how to take a quadratic form such as $\langle \mathbf{v}, \mathcal{T}^* \mathcal{P} \mathcal{B}[\mathbf{v} \otimes \mathbf{v}] \rangle$ and convert it to a corresponding linear representation so that

$$\langle \mathbf{v}, \mathcal{T}^* \mathcal{P} \mathcal{B}[\mathbf{v} \otimes \mathbf{v}] \rangle_{L_2} = \mathcal{K}[\mathbf{v} \otimes \mathbf{v} \otimes \mathbf{v}].$$

Finally, to test stability of a PDE, in Theorem 12, we enforce negativity of $\mathcal{A}^* \mathcal{P} \mathcal{T} + \mathcal{T}^* \mathcal{P} \mathcal{A}$ along with the constraint $\mathcal{K} = 0$, thus ensuring $\dot{V}(\mathbf{v}) \leq 0$ for all $\mathbf{v} \in L_2$. This approach is applied to verify stability of three commonly encountered scalar quadratic PDEs in Section VII.

III. A MAP FROM FUNDAMENTAL STATE TO PDE STATE

In this section, we show how a suitably well-posed quadratic PDE of the form given in (2) and defined in terms of spatial derivatives of the variable $u(t, s)$ can be equivalently represented in terms of spatial integrals of the variable $u_{ss}(t, s)$. The advantage of such a representation will be that it is more compact, defined only using bounded integral operators, and is free of auxiliary constraints in the form of boundary conditions and continuity constraints (i.e. $u_{ss}(t) \in L_2$ whereas $u(t) \in X_B$). To begin, we recall the standard definition of a Partial Integral (PI) operator [16].

Definition 1 (Algebras of PI Operators: Π_3, Π_2): For a given domain, $[a, b]$, we define the parameter space

$$\mathcal{N}_3 := L_\infty[a, b] \times L_2[[a, b]^2] \times L_2[[a, b]^2],$$

and its subspace

$$\mathcal{N}_2 := L_2[[a, b]^2] \times L_2[[a, b]^2].$$

Then, for any $R := \{R_0, R_1, R_2\} \in \mathcal{N}_3$, we define the associated operator $\mathcal{R} := \mathcal{P}_{\{R_0, R_1, R_2\}}$ for $\mathbf{u} \in L_2[a, b]$ as

$$(\mathcal{R}\mathbf{u})(s) = R_0(s)\mathbf{u}(s) + \int_a^s R_1(s, \theta)\mathbf{u}(\theta)d\theta + \int_s^b R_2(s, \theta)\mathbf{u}(\theta)d\theta.$$

We say $\mathcal{R} \in \Pi_3$ if $\mathcal{R} = \mathcal{P}_{\{R_0, R_1, R_2\}}$ for some $\{R_0, R_1, R_2\} \in \mathcal{N}_3$ and $\mathcal{R} \in \Pi_2$ if $\mathcal{R} = \mathcal{P}_{\{R_1, R_2\}}$ for some $\{R_1, R_2\} \in \mathcal{N}_2$. For convenience, we say that $\mathcal{R}$ is a 3-PI operator if $\mathcal{R} \in \Pi_3$ and a 2-PI operator if $\mathcal{R} \in \Pi_2$.

Defining the 3-PI and 2-PI operators in this manner, it has been shown that both Π_2 and Π_3 (and by extension $\mathcal{N}_2, \mathcal{N}_3$) form $*$ -algebras, being closed under summation, composition, scalar multiplication, and adjoint with respect to L_2 . We refer to [15] for explicit parameter maps defining these operations.

Consider now solutions u of a second order quadratic PDE of the form in (2). Since, at any time $t \geq 0$, $u(t) \in X_B \subset H_2$ is only twice spatially differentiable, the second-order spatial derivative of this solution need only satisfy $u_{ss}(t) \in L_2$ and hence is free of any continuity constraints or boundary conditions. We refer to this state $\mathbf{v}(t) := u_{ss}(t)$ as the *fundamental state* associated to the PDE, defining a map from the PDE state space X_B to the fundamental state space L_2 by the differential operator $\partial_s^2 : X_B \rightarrow L_2$. Then, assuming the boundary conditions to be sufficiently well-posed (e.g. excluding periodic boundary conditions), we can define an inverse map $\mathcal{T} : L_2 \rightarrow X_B$ by a 2-PI operator $\mathcal{T}$. In particular, we recall the following result from e.g. [15].

Lemma 2: Let X_B be as defined in Eqn. (3) for some $B \in \mathbb{R}^{2 \times 4}$ which satisfies the well-posedness conditions as given in Defn. 9 in [15]. If $\mathcal{T}, \mathcal{R} \in \Pi_2$ are as defined in [15] then for any $\mathbf{u} \in X_B$,

$$\mathbf{u} = \mathcal{T}\mathbf{u}_{ss}, \quad \text{and} \quad \mathbf{u}_s = \mathcal{R}\mathbf{u}_{ss}.$$

Proof: We refer to Thm. 10 and Thm. 12 in [15] for a proof, as well as for explicit formulae mapping the matrix B to operators $\mathcal{T}$ and $\mathcal{R}$. ■

Using this lemma, we can express both the PDE state $u(t)$ and its derivative $u_s(t)$ in terms of the fundamental state $\mathbf{v}(t) := u_{ss}(t)$ as $u(t) = \mathcal{T}\mathbf{v}(t)$ and $u_s(t) = \mathcal{R}\mathbf{v}(t)$. Substituting these relations into the PDE (2), we find that if u satisfies the PDE, then $\mathbf{v}$ satisfies

$$\partial_t(\mathcal{T}\mathbf{v})(t, s) = c(s)^T \begin{bmatrix} (\mathcal{T}\mathbf{v}) \\ (\mathcal{R}\mathbf{v}) \\ (\mathcal{T}\mathbf{v})(\mathcal{T}\mathbf{v}) \\ (\mathcal{T}\mathbf{v})(\mathcal{R}\mathbf{v}) \\ (\mathcal{R}\mathbf{v})(\mathcal{R}\mathbf{v}) \end{bmatrix} (t, s), \quad s \in [a, b], \quad (5)$$

where $\mathbf{v}(t) \in L_2$ at each $t \geq 0$ is free of boundary conditions and continuity constraints. Moreover, in this representation,

the dynamics are expressed as a quadratic function of only the fundamental state $\mathbf{v}(t)$ and PI operators applied to this state. In the following section, we show how this quadratic function can be expressed more compactly in a linear format $\mathcal{C}Z_2(\mathbf{v}(t))$, by defining a suitable basis of monomials $Z_2(\mathbf{v}(t))$ in the state $\mathbf{v}(t)$, and an associated class of operators $\mathcal{C}$ acting on this monomial basis.

Example: Suppose that $u(t, s)$ for $s \in [0, 1]$ satisfies Burgers' equation with Dirichlet boundary conditions, so that

$$\text{PDE:} \quad u_t(t, s) = \nu u_{ss}(t, s) - u(t, s)u_s(t, s), \quad (6)$$

$$\text{BCs:} \quad u(t, 0) = 0, \quad u(t, 1) = 0.$$

For any $\mathbf{v} \in L_2[0, 1]$, let $\mathcal{T}, \mathcal{R} \in \Pi_2$ be defined as

$$\begin{aligned} (\mathcal{T}\mathbf{v})(s) &:= \int_0^s [s-1]\theta\mathbf{v}(\theta)d\theta + \int_s^1 s[\theta-1]\mathbf{v}(\theta)d\theta, \\ (\mathcal{R}\mathbf{v})(s) &:= \int_0^s \theta\mathbf{v}(\theta)d\theta + \int_s^1 [\theta-1]\mathbf{v}(\theta)d\theta. \end{aligned} \quad (7)$$

Now, if we define the fundamental state as $\mathbf{v}(t) := u_{ss}(t)$, then $u(t) = \mathcal{T}\mathbf{v}(t)$ and $u_s(t) = \mathcal{R}\mathbf{v}(t)$. Furthermore, $\mathbf{v}$ satisfies

$$\partial_t(\mathcal{T}\mathbf{v}(t)) = \nu\mathbf{v}(t) - (\mathcal{T}\mathbf{v}(t))(\mathcal{R}\mathbf{v}(t)). \quad (8)$$

Conversely, if $\mathbf{v}(t)$ satisfies (8) and $u(t) = \mathcal{T}\mathbf{v}(t)$, then $u(t, s)$ satisfies Eqn. (6) for all $s \in [a, b]$.

IV. POLYNOMIAL REPRESENTATION ON A DISTRIBUTED STATE

In the previous section, we showed how a quadratic, 2nd order PDE of form as in (2), defined using state $u(t)$ can be equivalently represented in terms of Eqn. (5), defined using the 'fundamental' state $\mathbf{v}(t) = u_{ss}(t)$. However, the expression for the dynamics of $\mathbf{v}$ in Eqn. (5) is not suitable for the purpose of computational stability analysis – being defined using terms such as $(\mathcal{T}\mathbf{v})(\mathcal{R}\mathbf{v})$.

To illustrate the difficulty posed by terms such as $(\mathcal{T}\mathbf{v})(\mathcal{R}\mathbf{v})$ in testing stability of systems as in (5), let us briefly recall the problem of computational stability analysis of nonlinear ordinary differential equations (ODEs). When the vector field, f , is polynomial of degree d , the ODE may be uniquely represented as $\dot{x} = f(x) = AZ_d(x)$, where $Z_d(x)$ is the vector of monomials in state $x \in \mathbb{R}^n$ of degree d or less. Additionally, positive quadratic Lyapunov functions may be parameterized as $V(x) = \frac{1}{2}x^T Px$, where $P \geq 0$ is a positive matrix variable. The time-derivative of this Lyapunov function is then $\dot{V}(x) = x^T PAZ_d(x)$ for which we can enforce negativity using the equality constraint

$$x^T PAZ_d(x) = -Z_q(x)^T QZ_q(x), \quad (9)$$

where Q is likewise a positive matrix variable, and $q = \frac{d+1}{2}$. However, enforcing this constraint is made problematic due to the fact that expressions of the form $x^T PAZ_d(x)$ and $Z_q(x)^T QZ_q(x)$ are *not* unique – i.e. there exist $Q \neq 0$ such that $Z_q(x)^T QZ_q(x) = 0$. Therefore, in order to enforce the constraint in (9), we must first convert this expression to the linear representation $\mathcal{C}Z_{d+1}(x) = 0$, which *is* unique.

Returning now to the distributed-state system in (5), we likewise want to derive a linear representation $\mathcal{C}Z_d(\mathbf{v})$ of terms such as $(\mathcal{T}\mathbf{v})(\mathcal{R}\mathbf{v})$. In order to define this linear representation, however, we first have to define a suitable basis of monomials $Z_d(\mathbf{v})$ on distributed states $\mathbf{v} \in L_2$, which we do in the following subsection. In the next subsection, we then define a suitable class of operators Π_{2^2} and a map $\mathcal{T} \times \mathcal{R} \mapsto \mathcal{C} \in \Pi_{2^2}$ so that $(\mathcal{T}\mathbf{v})(\mathcal{R}\mathbf{v}) = \mathcal{C}Z_2(\mathbf{v})$, allowing us to express each of the quadratic terms in (5) in a linear format.

A. Monomial Basis on a Distributed State

In order to define a suitable monomial basis for polynomials on a distributed state $\mathbf{v} \in L_2[a, b]$, let us first consider what such a basis looks like for a discretized state $v = [v_1 \dots v_n]^T = [\mathbf{v}(s_1) \dots \mathbf{v}(s_n)]^T \in \mathbb{R}^n$. For this vector $v \in \mathbb{R}^n$, the basis of monomials $Z_2(v)$ of degree 2 in v consists of each of the independent variables v_i , as well as any product $v_i v_j$ of these variables for $i, j \in \{1, \dots, n\}$ – i.e. the unique terms in the Kronecker product $v \otimes v$. Extending the vector $v \in \mathbb{R}^n$ to a function $\mathbf{v} \in L_2[a, b]$, we now have that $\mathbf{v}(s_i)$ and $\mathbf{v}(s_j)$ are independent variables for any $s_i, s_j \in [a, b]$. Thus whereas in finite dimensions, the independent variables are indexed by $i \in \{1, \dots, n\}$, for functions on $L_2[a, b]$, every point $s \in [a, b]$ defines an independent variable. Accordingly, the degree one monomials in infinite dimensions would simply be the function $\mathbf{v}(s)$, where we have made the continuum extension $v_i \mapsto \mathbf{v}(s)$. Likewise, for the monomials of degree 2, the products $v_i v_j$ may be naturally extended to infinite dimensions as $\mathbf{v}(s)\mathbf{v}(\theta)$, for $s, \theta \in [a, b]$. We may denote these monomials compactly using the tensor product $(\mathbf{v} \otimes \mathbf{v})(s, \theta) = \mathbf{v}(s)\mathbf{v}(\theta)$, so that this tensor product $\mathbf{v} \otimes \mathbf{v} \in L_2[[a, b]^2]$ is the functional equivalent of the homogeneous degree 2 monomial basis. Generalizing this to arbitrary degrees, the functional equivalent of the homogeneous degree $d \in \mathbb{N}$ monomial basis becomes $\underbrace{\mathbf{v} \otimes \dots \otimes \mathbf{v}}_{d \text{ factors}} \in L_2[[a, b]^d]$, and any distributed polynomial of degree d in $\mathbf{v}$ may be expressed in terms of the basis

$$Z_d(\mathbf{v}) = \begin{bmatrix} \mathbf{v} \\ \mathbf{v} \otimes \mathbf{v} \\ \vdots \\ \underbrace{\mathbf{v} \otimes \dots \otimes \mathbf{v}}_{d \text{ factors}} \end{bmatrix}.$$

We refer to this basis as the degree d distributed monomial basis in $\mathbf{v}$. In the following subsection, we show how we may represent quadratic polynomials $(\mathcal{T}\mathbf{v})(\mathcal{R}\mathbf{v})$ in a format $\mathcal{Q}(\mathbf{v} \otimes \mathbf{v})$ that is linear in the degree 2 distributed monomial.

B. Tensor Products of 2-PI Operators

Having defined a monomial basis for distributed states in terms of tensor products of this state, we now show how we may similarly define a tensor product of 2-PI operators, so that we may express e.g. $(\mathcal{T}\mathbf{v})(\mathcal{R}\mathbf{v})$ in the linear format $(\mathcal{T} \otimes \mathcal{R})(\mathbf{v} \otimes \mathbf{v})$. To motivate this class of operators, let us again recall the linear representation of scalar polynomials on a finite dimensional state, $x \in \mathbb{R}^n$. In particular, suppose we

are given two such polynomials, $t(x) = c_t^T Z_d(x)$ and $r(x) = c_r^T Z_d(x)$, and want to construct a linear representation of $t(x)r(x)$. Such a linear representation is readily computed using the Kronecker (i.e. tensor) product of c_t and c_r as

$$t(x)r(x) = (c_t \otimes c_r)(Z_d(x) \otimes Z_d(x)).$$

The goal of this subsection is to define the distributed equivalent of this formula, where now t and r are the distributed polynomials (albeit linear) $\mathcal{T}\mathbf{v}$ and $\mathcal{R}\mathbf{v}$, and where the 2-PI operators $\mathcal{T}$ and $\mathcal{R}$ are the distributed equivalent of the vectors c_r and c_t . Specifically, we define a space Π_{2^2} , consisting of the tensor product of 2-PI operators so that $\mathcal{Q} = \mathcal{T} \otimes \mathcal{R}$ for some $\mathcal{T}, \mathcal{R} \in \Pi_2$ implies $\mathcal{Q} \in \Pi_{2^2}$.

To illustrate the notion of tensor product for 2-PI operators, consider $\mathcal{T} = \mathcal{P}_{\{0, T_1, T_2\}} \in \Pi_2$ and $\mathcal{R} = \mathcal{P}_{\{0, R_1, R_2\}} \in \Pi_2$. Then we can express the product $(\mathcal{T}\mathbf{v})(\mathcal{R}\mathbf{v})$ in terms of $[\mathbf{v} \otimes \mathbf{v}](\theta, \eta) = \mathbf{v}(\theta)\mathbf{v}(\eta)$ as

$$\begin{aligned} (\mathcal{T}\mathbf{v})(s)(\mathcal{R}\mathbf{v})(s) &= \int_a^s \int_a^s T_1(s, \theta) R_1(s, \eta) \mathbf{v}(\theta) \mathbf{v}(\eta) d\eta d\theta \\ &+ \int_s^b \int_a^s T_2(s, \theta) R_1(s, \eta) \mathbf{v}(\theta) \mathbf{v}(\eta) d\eta d\theta \\ &+ \int_a^s \int_s^b T_1(s, \theta) R_2(s, \eta) \mathbf{v}(\theta) \mathbf{v}(\eta) d\eta d\theta \\ &+ \int_s^b \int_s^b T_2(s, \theta) R_2(s, \eta) \mathbf{v}(\theta) \mathbf{v}(\eta) d\eta d\theta = (\mathcal{Q}[\mathbf{v} \otimes \mathbf{v}])(s), \end{aligned} \quad (10)$$

where $\mathcal{Q}$ is again a partial integral operator. Clearly, however, this new PI operator $\mathcal{Q} \in \Pi_{2^2}$ (defined on $\mathbf{v} \otimes \mathbf{v} \in L_2[[a, b]^2]$) has a structure which is distinct from that in Π_2 and Π_3 . Instead, $\mathcal{Q}$ belongs to a class of “tensor product PI operators”, which we define as follows.

Definition 3 (Class of Tensor Product PI Operators, Π_{2^2}): For a given domain, $[a, b]$, we define the parameter space

$$\mathcal{N}_{2^2}[a, b] = L_2[[a, b]^3] \times L_2[[a, b]^3] \times L_2[[a, b]^3].$$

Then, for any $Q := [Q_1, Q_2, Q_3] \in \mathcal{N}_{2^2}$, we define the associated PI operator $\mathcal{Q} := \mathcal{P}[Q]$ for $\mathbf{w} \in L_2[[a, b]^2]$ as

$$\begin{aligned} (\mathcal{Q}\mathbf{w})(s) &= \int_a^s \int_a^\theta Q_1(s, \theta, \eta) \mathbf{w}(\theta, \eta) d\eta d\theta \\ &+ \int_s^b \int_a^s Q_2(s, \theta, \eta) \mathbf{w}(\theta, \eta) d\eta d\theta + \int_s^b \int_s^\theta Q_3(s, \theta, \eta) \mathbf{w}(\theta, \eta) d\eta d\theta. \end{aligned}$$

Finally, we say that $\mathcal{Q} \in \Pi_{2^2}$ if $\mathcal{Q} = \mathcal{P}[Q]$ for some $Q := [Q_1, Q_2, Q_3] \in \mathcal{N}_{2^2}$.

We remark that the structure in this definition is slightly different from that of the operator $\mathcal{Q}$ defining the product $\mathcal{Q}[\mathbf{v} \otimes \mathbf{v}] = (\mathcal{T}\mathbf{v})(\mathcal{R}\mathbf{v})$ in (10). However, we can convert the representation in (10) to one of the form in Defn. 3, defining a map $\mathcal{T} \otimes \mathcal{R} \rightarrow \mathcal{Q} \in \Pi_{2^2}$ such that $(\mathcal{T}\mathbf{v})(\mathcal{R}\mathbf{v}) = \mathcal{Q}[\mathbf{v} \otimes \mathbf{v}]$.

Definition 4 (Tensor Product of PI Operators: $\mathcal{T} \otimes \mathcal{R}$): For any $\mathcal{T} = \mathcal{P}_{\{T_i\}} \in \Pi_2$ and $\mathcal{R} = \mathcal{P}_{\{R_i\}} \in \Pi_2$ where $T := \{T_1, T_2\} \in \mathcal{N}_2$ and $R := \{R_1, R_2\} \in \mathcal{N}_2$, we say that $\mathcal{Q} = \mathcal{T} \otimes \mathcal{R}$ if $\mathcal{Q} = \mathcal{P}[Q]$, where $Q := [Q_1, Q_2, Q_3] \in \mathcal{N}_{2^2}$ is such that for all $s, \theta, \eta \in [a, b]$,

$$\begin{aligned} Q_1(s, \theta, \eta) &:= T_1(s, \theta) R_1(s, \eta) + T_1(s, \eta) R_1(s, \theta), \\ Q_2(s, \theta, \eta) &:= T_2(s, \theta) R_1(s, \eta) + T_1(s, \eta) R_2(s, \theta), \\ Q_3(s, \theta, \eta) &:= T_2(s, \theta) R_2(s, \eta) + T_2(s, \eta) R_2(s, \theta). \end{aligned}$$

Proposition 5: For any $\mathcal{T}, \mathcal{R} \in \Pi_2$ and $\mathbf{v} \in L_2[a, b]$, we have $(\mathcal{T}\mathbf{v})(\mathcal{R}\mathbf{v}) = ((\mathcal{T} \otimes \mathcal{R})[\mathbf{v} \otimes \mathbf{v}])$.

Proof: The proof follows by splitting e.g.

$$\begin{aligned} (\mathcal{P}_{\{T_1, 0\}}\mathbf{v})(s)(\mathcal{P}_{\{R_1, 0\}}\mathbf{v})(s) &= \int_a^s \int_a^\theta T_1(s, \theta) R_1(s, \eta) \mathbf{v}(\theta) \mathbf{v}(\eta) d\eta d\theta \\ &= \int_a^s \int_a^\theta T_1(s, \theta) R_1(s, \eta) \mathbf{v}(\theta) \mathbf{v}(\eta) d\eta d\theta \\ &\quad + \int_a^s \int_\theta^s T_1(s, \theta) R_1(s, \eta) \mathbf{v}(\theta) \mathbf{v}(\eta) d\eta d\theta, \end{aligned}$$

and invoking the identity

$$\int_a^s \int_\theta^s F(s, \theta, \eta) d\eta d\theta = \int_a^s \int_a^\theta F(s, \eta, \theta) d\eta d\theta. \quad (11)$$

A full proof is given in Appx. A of the extended version of this paper [17]. ■

V. A PIE REPRESENTATION OF QUADRATIC PDES

We now return to the quadratic representation of the evolution of $\mathbf{v} := u_{ss}$ given in (5), of solutions u of the PDE in (2). Using the distributed monomials and tensor products of PI operators defined in the previous section, we can now represent each of the quadratic terms in the PDE in terms of the monomials of $\mathbf{v}$ – i.e. $\mathbf{v} \otimes \mathbf{v}$. Specifically, suppose that $c = [c_0 \ c_1 \ \dots \ c_5]^T \in L_2^6[a, b]$, and for each c_i , denote the associated multiplier operator as $\mathcal{M}_{c_i}$, so that $(\mathcal{M}_{c_i}\mathbf{v})(s) = c_i(s)\mathbf{v}(s)$. The goal of this section is to show that the evolution of $\mathbf{v} := u_{ss}$ is governed by a quadratic PIE of the form

$$\textbf{PIE: } \partial_t \mathcal{T}\mathbf{v}(t) = \mathcal{A}\mathbf{v}(t) + \mathcal{B}[\mathbf{v}(t) \otimes \mathbf{v}(t)], \quad (12)$$

where $\mathcal{A} \in \Pi_3$ and $\mathcal{B} \in \Pi_{22}$ are given by

$$\begin{aligned} \mathcal{A} &= \mathcal{M}_{c_0}\mathcal{T} + \mathcal{M}_{c_1}\mathcal{R} + \mathcal{M}_{c_2}, \\ \mathcal{B} &= \mathcal{M}_{c_3}[\mathcal{T} \otimes \mathcal{T}] + \mathcal{M}_{c_4}[\mathcal{T} \otimes \mathcal{R}] + \mathcal{M}_{c_5}[\mathcal{R} \otimes \mathcal{R}]. \end{aligned} \quad (13)$$

We note here that the fact that Π_{22} is closed under composition with multiplier operators is relatively clear, but is also stated formally in Prop. 9 in the next section.

To show that $\mathbf{v} := u_{ss}$ satisfies the quadratic PIE in Eqn. (12), we first give a definition of solution of both quadratic PDEs and quadratic PIEs. Then, in Lemma 8, we show that u is a solution of the quadratic PDE if and only if $\mathbf{v} := u_{ss}$ is a solution of the quadratic PIE, and that the mapping between solutions is bijective.

Definition 6 (Classical Solution to the Quadratic PDE):

For a given initial state $\mathbf{u}_0 \in X_B$, we say that $\mathbf{u}$ is a classical solution to the quadratic PDE defined by $\{B, c\}$ if $\mathbf{u}$ is Frechét differentiable, $\mathbf{u}(0) = \mathbf{u}_0$, and for all $t \geq 0$, $\mathbf{u}(t) \in X_B$, and $\mathbf{u}(t)$ satisfies (2).

Definition 7 (Classical Solution to the Quadratic PIE):

For a given initial state $\mathbf{v}_0 \in L_2$, we say that $\mathbf{v}$ is a classical solution to the quadratic PIE defined by $\{\mathcal{T}, [\mathcal{A}, \mathcal{B}]\}$ if $\mathbf{v}$ is Frechét differentiable, $\mathbf{v}(0) = \mathbf{v}_0$, and for all $t \geq 0$, $\mathbf{v}(t)$ satisfies (12).

The following lemma proves that there exists an invertible map between classical solutions to the PDE (2), and classical solutions to the associated PIE (12).

Lemma 8: Suppose that $c = [c_0 \ \dots \ c_5]^T \in L_2^6$ and $B \in \mathbb{R}^{2 \times 4}$ satisfies the well-posedness conditions of Defn. 9 in [15], and let the associated operators $\mathcal{T}, \mathcal{R} \in \Pi_2$ be as defined in Lemma 2. Define the operators $\mathcal{A} \in \Pi_3$ and $\mathcal{B} \in \Pi_{22}$ as in (13). Then, $\mathbf{v}$ is a classical solution to the quadratic PIE defined by $\{\mathcal{T}, [\mathcal{A}, \mathcal{B}]\}$ with initial state $\mathbf{v}_0$ if and only if $\mathbf{u} := \mathcal{T}\mathbf{v}$ is a classical solution to the quadratic PDE defined by $\{B, c\}$ with initial state $\mathbf{u}_0 := \mathcal{T}\mathbf{v}_0$. Conversely, $\mathbf{u}$ is a classical solution to the quadratic PDE defined by $\{B, c\}$ with initial state $\mathbf{u}_0$ if and only if $\mathbf{v} := \partial_s^2 \mathbf{u}$ is a classical solution to the quadratic PIE defined by $\{\mathcal{T}, [\mathcal{A}, \mathcal{B}]\}$ with initial state $\mathbf{v}_0 := \partial_s^2 \mathbf{u}_0$.

Proof: The proof follows by substitution of the relations $(u, u_s, u_{ss}) = (\mathcal{T}\mathbf{v}, \mathcal{R}\mathbf{v}, \mathbf{v})$ into the PDE representation in (2), and applying Prop. 5 to express e.g. $c_4(\mathcal{T}\mathbf{v})(\mathcal{R}\mathbf{v}) = \mathcal{M}_{c_4}[\mathcal{T} \otimes \mathcal{R}][\mathbf{v} \otimes \mathbf{v}]$. A full proof is given in the extended version of this paper [17]. ■

We remark that, although the results here apply only to second order PDEs, the methodology naturally generalizes to N th order quadratic PDEs as well. In particular, in this case, we may define a fundamental state as $\mathbf{v}(t) = \partial_s^N u(t)$, and prove that there exists PI operators $\mathcal{R}_j \in \Pi_2$ such that $\partial_s^j u = \mathcal{R}_j \mathbf{v}$ for all $j \in \{0, \dots, N-1\}$. Substituting these relations into the PDE, we can then use the tensor products to express $(\mathcal{R}_i \mathbf{v})(\mathcal{R}_j \mathbf{v}) = (\mathcal{R}_i \otimes \mathcal{R}_j)(\mathbf{v} \otimes \mathbf{v})$, combining the different terms to obtain a PIE representation as in (12).

Example: To illustrate the quadratic PIE representation, suppose that u is a solution of Burgers' equation as defined in Eqn. (6). Define $\mathcal{T}, \mathcal{R} \in \Pi_2$ as in (7), and define $\mathcal{B} \in \Pi_{22}$ for $\mathbf{w} \in L_2[[0, 1]^2]$ as

$$\begin{aligned} (\mathcal{B}\mathbf{w})(s) &= -((\mathcal{T} \otimes \mathcal{R})\mathbf{w})(s) \\ &= - \int_0^s \int_0^\theta 2[s-1]\theta \eta \mathbf{w}(\theta, \eta) d\eta d\theta \\ &\quad - \int_s^1 \int_0^s [2s-1][\theta-1] \eta \mathbf{w}(\theta, \eta) d\eta d\theta \\ &\quad - \int_s^1 \int_s^\theta 2s[\theta-1][\eta-1] \mathbf{w}(\theta, \eta) d\eta d\theta. \end{aligned}$$

Then $\mathbf{v} := u_{ss}$ is a solution of the quadratic PIE given by

$$\textbf{PIE: } \partial_t \mathcal{T}\mathbf{v}(t) = \nu \mathbf{v}(t) + \mathcal{B}[\mathbf{v}(t) \otimes \mathbf{v}(t)].$$

VI. A STABILITY TEST FOR QUADRATIC PDES

In the previous sections, we showed that by constructing a basis of distributed monomials on L_2 and by representing the dynamics of the PDE using the fundamental state (which lies in L_2), we may propose a compact and universal representation of a class of quadratic PDEs using two PI operators, $\mathcal{A}$ and $\mathcal{B}$, where $\mathcal{A} \in \Pi_3$ is a 3-PI operator and $\mathcal{B} \in \Pi_{22}$ is a linear combination of tensors products of 2-PI operators. Based on this representation, in this section, we propose a simple stability test using only quadratic storage functions. While such a stability test is necessarily conservative, it is sufficient to verify stability of many common quadratic PDEs and may be later extended to non-quadratic storage functions.

Specifically, we would like to verify the existence of a quadratic Lyapunov functional of the form $V(\mathbf{u}) = \langle \mathbf{u}, \mathcal{P}\mathbf{u} \rangle_{L_2} = \langle \mathcal{T}\mathbf{v}, \mathcal{P}\mathcal{T}\mathbf{v} \rangle_{L_2}$, where $\mathcal{P} = \mathcal{P}^* \in \Pi_3$ is positive with respect to L_2 . If $\mathbf{v}(t)$ satisfies the dynamics of the quadratic PIE in (12), then

$$\begin{aligned} \dot{V}(\mathcal{T}\mathbf{v}(t)) &= \langle \partial_t \mathcal{T}\mathbf{v}(t), \mathcal{P}\mathcal{T}\mathbf{v}(t) \rangle_{L_2} + \langle \mathcal{T}\mathbf{v}(t), \mathcal{P}(\partial_t \mathcal{T}\mathbf{v}(t)) \rangle_{L_2} \\ &= \left\langle \begin{bmatrix} \mathbf{v}(t) \\ \mathbf{v}(t) \otimes \mathbf{v}(t) \end{bmatrix}, \begin{bmatrix} \mathcal{A}^* \mathcal{P} \mathcal{T} + \mathcal{T}^* \mathcal{P} \mathcal{A} & \mathcal{T}^* \mathcal{P} \mathcal{B} \\ \mathcal{B}^* \mathcal{P} \mathcal{T} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v}(t) \\ \mathbf{v}(t) \otimes \mathbf{v}(t) \end{bmatrix} \right\rangle. \end{aligned}$$

Since it can be shown that $\mathbf{v}$ and $\mathbf{v} \otimes \mathbf{v}$ are linearly independent, we have that $\dot{V}(\mathcal{T}\mathbf{v}) \leq 0$ for all $\mathbf{v} \in L_2$ if and only if $\mathcal{Q} := [\mathcal{A}^* \mathcal{P} \mathcal{T} + \mathcal{T}^* \mathcal{P} \mathcal{A}] \preceq 0$ and $\langle \mathbf{v}, \mathcal{T}^* \mathcal{P} \mathcal{B} [\mathbf{v} \otimes \mathbf{v}] \rangle_{L_2} = 0$ for all $\mathbf{v} \in L_2$ (see the extended version of this paper for a proof [17]). Since $\mathcal{A}, \mathcal{P}, \mathcal{T}$ are standard PI operators, the first constraint $\mathcal{Q} \preceq 0$ may be enforced using existing functionality of the PIETOOLS software suite [18].

Concentrating on the condition $\langle \mathbf{v}, \mathcal{T}^* \mathcal{P} \mathcal{B} [\mathbf{v} \otimes \mathbf{v}] \rangle_{L_2} = 0$, then, we first show that the set Π_{22} of tensor products of PI operators is closed under composition with a 2-PI operator – so that $\mathcal{T}^*, \mathcal{P} \in \Pi_3$ and $\mathcal{B} \in \Pi_{22}$ implies $\mathcal{T}^* \mathcal{P} \mathcal{B} \in \Pi_{22}$.

Proposition 9: Let $\mathcal{Q} = \mathcal{P}_{\{Q_i\}} \in \Pi_3$ and $\mathcal{B} = \mathcal{P}[B] \in \Pi_{22}$ be defined by parameters $Q = \{Q_0, Q_1, Q_2\} \in \mathcal{N}_3$ and $B = [B_1, B_2, B_3] \in \mathcal{N}_{22}$. Define $\mathcal{G} := \mathcal{P}[G] \in \Pi_{22}$, where $G := [G_1, G_2, G_3] \in \mathcal{N}_{22}$ is given by

$$\begin{aligned} G_1(s, \theta, \eta) &:= \int_a^\eta R_{13}(s, \zeta, \theta, \eta) d\zeta + \int_\eta^\theta R_{12}(s, \zeta, \theta, \eta) d\zeta \\ &+ \int_\theta^s R_{11}(s, \zeta, \theta, \eta) d\zeta + \int_s^b R_{21}(s, \zeta, \theta, \eta) d\zeta + Q_0(s) B_1(s, \theta, \eta), \\ G_2(s, \theta, \eta) &:= \int_a^\eta R_{13}(s, \zeta, \theta, \eta) d\zeta + \int_\eta^s R_{12}(s, \zeta, \theta, \eta) d\zeta \\ &+ \int_s^\theta R_{22}(s, \zeta, \theta, \eta) d\zeta + \int_\theta^b R_{21}(s, \zeta, \theta, \eta) d\zeta + Q_0(s) B_2(s, \theta, \eta), \\ G_3(s, \theta, \eta) &:= \int_a^\eta R_{13}(s, \zeta, \theta, \eta) d\zeta + \int_s^\eta R_{23}(s, \zeta, \theta, \eta) d\zeta \\ &+ \int_\eta^\theta R_{22}(s, \zeta, \theta, \eta) d\zeta + \int_\theta^b R_{21}(s, \zeta, \theta, \eta) d\zeta + Q_0(s) B_3(s, \theta, \eta), \end{aligned}$$

with $R_{ij}(s, \zeta, \theta, \eta) := Q_i(s, \zeta) B_j(\zeta, \theta, \eta)$. Then, for any $\mathbf{v} \in L_2$,

$$(\mathcal{Q}(\mathcal{B}\mathbf{v}))(s) = (\mathcal{G}\mathbf{v})(s).$$

Proof: The proof is given in Appx. B of the extended version of this paper [17]. ■

Applying this result, we can define an operator $\mathcal{G} := \mathcal{T}^* \mathcal{P} \mathcal{B} \in \Pi_{22}$, so that we may express $\langle \mathbf{v}, \mathcal{T}^* \mathcal{P} \mathcal{B} [\mathbf{v} \otimes \mathbf{v}] \rangle_{L_2} = \langle \mathbf{v}, \mathcal{G}[\mathbf{v} \otimes \mathbf{v}] \rangle_{L_2}$.

Now, in order to enforce $\langle \mathbf{v}, \mathcal{G}[\mathbf{v} \otimes \mathbf{v}] \rangle_{L_2} \equiv 0$, we note that, in general, the quadratic representation of a polynomial is not unique, and hence it would be conservative to simply enforce the constraint that $\mathcal{G} = 0$. To resolve this issue, in Prop. 11, we show that for any $\mathcal{G} \in \Pi_{22}$, the quadratic representation of $\langle \mathbf{v}, \mathcal{G}[\mathbf{v} \otimes \mathbf{v}] \rangle_{L_2}$ can be converted to a linear representation of the form $\mathcal{K}_{\text{lin}}[\mathcal{G}](\mathbf{v} \otimes \mathbf{v} \otimes \mathbf{v}) = \langle \mathbf{v}, \mathcal{G}[\mathbf{v} \otimes \mathbf{v}] \rangle_{L_2}$, where the transformation $\mathcal{G} \rightarrow \mathcal{K}_{\text{lin}}[\mathcal{G}]$ is given in Defn. 10. Since the linear representations are uniquely defined, we may then enforce the constraint $\langle \mathbf{v}, \mathcal{G}[\mathbf{v} \otimes \mathbf{v}] \rangle_{L_2} \equiv 0$ without conservatism by requiring $\mathcal{K}_{\text{lin}}[\mathcal{G}] = 0$.

Definition 10: For any $\mathcal{G} \in \Pi_{22}$ where $\mathcal{G} = \mathcal{P}[G]$ for $G = [G_1, G_2, G_3] \in \mathcal{N}_{22}$, we define the operator $\mathcal{K}_{\text{lin}}[\mathcal{G}] : L_2[[a, b]^3] \rightarrow \mathbb{R}$ as

$$\mathcal{K}_{\text{lin}}[\mathcal{G}]\mathbf{w} = \int_a^b \int_a^s \int_a^\theta K(s, \theta, \eta) \mathbf{w}(s, \theta, \eta) d\eta d\theta ds,$$

where

$$K(s, \theta, \eta) := G_1(s, \theta, \eta) + G_2(\theta, s, \eta) + G_3(\eta, s, \theta).$$

Proposition 11: For any $\mathcal{G} \in \Pi_{22}$, we have

$$\langle \mathbf{v}, \mathcal{G}[\mathbf{v} \otimes \mathbf{v}] \rangle_{L_2} = \mathcal{K}_{\text{lin}}[\mathcal{G}][\mathbf{v} \otimes \mathbf{v} \otimes \mathbf{v}].$$

for any $\mathbf{v} \in L_2[a, b]$,

Proof: The proof follows by expanding

$$\begin{aligned} \langle \mathbf{v}, \mathcal{G}[\mathbf{v} \otimes \mathbf{v}] \rangle_{L_2} &= \int_a^b \int_a^s \int_a^\theta G_1(s, \theta, \eta) \mathbf{v}(s) \mathbf{v}(\theta) \mathbf{v}(\eta) d\eta d\theta ds \\ &+ \int_a^b \int_s^\theta \int_a^\eta G_2(s, \theta, \eta) \mathbf{v}(s) \mathbf{v}(\theta) \mathbf{v}(\eta) d\eta d\theta ds \\ &+ \int_a^b \int_s^\theta \int_\eta^\theta G_3(s, \theta, \eta) \mathbf{v}(s) \mathbf{v}(\theta) \mathbf{v}(\eta) d\eta d\theta ds, \end{aligned}$$

and invoking e.g. the identity in (11) to express all terms using integrals $\int_a^b \int_a^s \int_a^\theta$. A full proof is given in Appx. C of the extended version of this paper [17]. ■

Applying these result, we can now declare an optimization program for testing stability of a quadratic PDE as follows.

Theorem 12: Let $\{B, c\}$ define a quadratic PDE as in (2), and let associated PI operators $\{\mathcal{T}, [\mathcal{A}, \mathcal{B}]\}$ be as defined in Lem. 8. Suppose that there exist $\epsilon, \delta > 0$ and $\mathcal{P} = \mathcal{P}^* \in \Pi_3$ such that

$$\begin{aligned} \mathcal{P} &\succeq \epsilon I, \\ \mathcal{Q} &:= [\mathcal{A}^* \mathcal{P} \mathcal{T} + \mathcal{T}^* \mathcal{P} \mathcal{A}] \preceq -\delta \mathcal{T}^* \mathcal{T}, \\ \mathcal{K}_{\text{lin}}[\mathcal{T}^* \mathcal{P} \mathcal{B}] &= 0. \end{aligned} \tag{14}$$

Finally, let $\mu = \|\mathcal{P}\|_{\mathcal{L}_{L_2}}$. Then, any solution $\mathbf{u}(t)$ to the PDE defined by $\{B, c\}$ satisfies

$$\|\mathbf{u}(t)\|_{L_2}^2 \leq \frac{\mu}{\epsilon} \|\mathbf{u}(0)\|_{L_2}^2 e^{-\frac{\delta}{\mu} t}.$$

Proof: Consider the functional $V : L_2 \rightarrow \mathbb{R}$ defined for $\mathbf{v} \in L_2$ as

$$V(\mathbf{v}) = \langle \mathcal{T}\mathbf{v}, \mathcal{P}\mathcal{T}\mathbf{v} \rangle_{L_2} \geq \epsilon \|\mathcal{T}\mathbf{v}\|_{L_2}^2.$$

Since $\|\mathcal{P}\|_{\mathcal{L}_{L_2}} = \mu$, this function is bounded from above as

$$V(\mathbf{v}) = \langle \mathcal{T}\mathbf{v}, \mathcal{P}\mathcal{T}\mathbf{v} \rangle_{L_2} \leq \mu \|\mathcal{T}\mathbf{v}\|_{L_2}^2.$$

Now, let $\mathbf{u}$ be an arbitrary solution to the PDE defined by $\{B, c\}$, and fix $\mathbf{v} := \partial_s^2 \mathbf{u}$. Then, by Lemma 8, $\mathbf{u} = \mathcal{T}\mathbf{v}$, and $\mathbf{v}$ is a solution to the quadratic PIE defined by $\{\mathcal{T}, [\mathcal{A}, \mathcal{B}]\}$. As such, the temporal derivative of V along $\mathbf{v}$ satisfies

$$\begin{aligned} \dot{V}(\mathbf{v}(t)) &= \langle \partial_t \mathcal{T}\mathbf{v}(t), \mathcal{P}\mathcal{T}\mathbf{v}(t) \rangle_{L_2} + \langle \mathcal{T}\mathbf{v}(t), \mathcal{P}(\partial_t \mathcal{T}\mathbf{v}(t)) \rangle_{L_2} \\ &= \left\langle \begin{bmatrix} \mathcal{A} & \mathcal{B} \end{bmatrix} \begin{bmatrix} \mathbf{v}(t) \\ \mathbf{v}(t) \otimes \mathbf{v}(t) \end{bmatrix}, \mathcal{P}\mathcal{T}\mathbf{v}(t) \right\rangle_{L_2} \\ &\quad + \left\langle \mathcal{T}\mathbf{v}(t), \mathcal{P} \begin{bmatrix} \mathcal{A} & \mathcal{B} \end{bmatrix} \begin{bmatrix} \mathbf{v}(t) \\ \mathbf{v}(t) \otimes \mathbf{v}(t) \end{bmatrix} \right\rangle_{L_2} \\ &= \left\langle \begin{bmatrix} \mathbf{v}(t) \\ \mathbf{v}(t) \otimes \mathbf{v}(t) \end{bmatrix}, \begin{bmatrix} \mathcal{A}^* \mathcal{P} \mathcal{T} + \mathcal{T}^* \mathcal{P} \mathcal{A} & \mathcal{T}^* \mathcal{P} \mathcal{B} \\ \mathcal{B}^* \mathcal{P} \mathcal{T} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v}(t) \\ \mathbf{v}(t) \otimes \mathbf{v}(t) \end{bmatrix} \right\rangle_{L_2} \\ &= \langle \mathbf{v}(t), \mathcal{Q}\mathbf{v}(t) \rangle_{L_2} + 2 \langle \mathbf{v}(t), \mathcal{T}^* \mathcal{P} \mathcal{B} [\mathbf{v}(t) \otimes \mathbf{v}(t)] \rangle_{L_2}. \end{aligned}$$

Here, since $\mathcal{K}_{\text{lin}}[\mathcal{T}^*\mathcal{P}\mathcal{B}] = 0$, by Proposition 11 we have

$$\langle \mathbf{v}(t), \mathcal{T}^*\mathcal{P}\mathcal{B}[\mathbf{v}(t) \otimes \mathbf{v}(t)] \rangle_{L_2} = \mathcal{K}_{\text{lin}}[\mathcal{T}^*\mathcal{P}\mathcal{B}][\mathbf{v}(t) \otimes \mathbf{v}(t) \otimes \mathbf{v}(t)] = 0.$$

Since also $Q := [\mathcal{A}^*\mathcal{P}\mathcal{T} + \mathcal{T}^*\mathcal{P}\mathcal{A}] \preceq -\delta\mathcal{T}^*\mathcal{T}$, it follows that

$$\dot{V}(\mathbf{v}(t)) = \langle \mathbf{v}(t), Q\mathbf{v}(t) \rangle_{L_2} \leq -\delta \|\mathcal{T}\mathbf{v}(t)\|_{L_2}^2 \leq -\frac{\delta}{\mu} V(\mathbf{v}(t)).$$

Applying the Grönwall-Bellman inequality, we find that

$$V(\mathbf{v}(t)) \leq V(\mathbf{v}(0))e^{-\frac{\delta}{\mu}t},$$

and therefore

$$\|\mathcal{T}\mathbf{v}(t)\|_{L_2}^2 \leq \frac{\mu}{\epsilon} \|\mathcal{T}\mathbf{v}(0)\|_{L_2}^2 e^{-\frac{\delta}{\mu}t}.$$

Finally, since $\mathbf{u} = \mathcal{T}\mathbf{v}$, we conclude that

$$\|\mathbf{u}(t)\|_{L_2}^2 \leq \frac{\mu}{\epsilon} \|\mathbf{u}(0)\|_{L_2}^2 e^{-\frac{\delta}{\mu}t}.$$

■

VII. NUMERICAL EXAMPLES

Implementation of the stability test in Theorem 12 requires certain functionality not implemented in PIETOOLS 2022 [18] (the current release). Specifically, PIETOOLS 2022 does not support PI operators of the class Π_{2^2} – i.e. the operator $\mathcal{B}$. As a result, construction of the tensor PIE representation and use of tensor PI equality constraints requires significant expertise on the part of the user. While we expect such functionality to be included in a future release, for the purposes of this paper, we have created a CodeOcean capsule which allows the user to declare a limited class of quadratic PDEs and then automates the process of construction of the tensor PI representation and Semidefinite Programming (SDP)-based stability test [19]. An early version of this software was used to produce the results in the following subsections.

A. Burgers' Equation

Consider Burgers' equation on $s \in [0, 1]$, with an added reaction term $ru(t, s)$, and Dirichlet boundary conditions:

$$\text{PDE: } \dot{u}(t, s) = u_{ss}(t, s) + ru(t, s) - u(t, s)u_s(t, s),$$

$$\text{BCs: } u(t, 0) = 0, \quad u(t, 1) = 0.$$

Let $\mathbf{v}(t) := u_{ss}(t)$ be the fundamental state, and define PI operators $\mathcal{T}, \mathcal{R} \in \Pi_2$ as in (7). Then, $u(t) = \mathcal{T}\mathbf{v}(t)$ and $u_s(t) = \mathcal{R}\mathbf{v}(t)$. We obtain an equivalent PIE representation

$$\text{PIE: } \partial_t \mathcal{T}\mathbf{v}(t) = \left[\underbrace{1 + r\mathcal{T}}_{\mathcal{A}} \quad \underbrace{-(\mathcal{T} \otimes \mathcal{R})}_{\mathcal{B}} \right] \begin{bmatrix} \mathbf{v}(t) \\ \mathbf{v}(t) \otimes \mathbf{v}(t) \end{bmatrix}.$$

Applying the conditions of Theorem 12 with $\epsilon = 10^{-4}$, $\delta = 10^{-6}$, the proposed algorithm is able to find a Lyapunov stability proof for any $r \leq 9.8696 \approx \pi^2$, which corresponds precisely to the bound obtained in [16] for the linearization of Burger's equation – i.e. where we neglect the uu_s term.

Before moving on to the next example, we note that it is well known that the nonlinear term uu_s vanishes when taking the derivative of a candidate Lyapunov functional of the form $V(u) = \|u\|_{L_2}^2 = \|\mathcal{T}\mathbf{v}\|_{L_2}^2$, i.e. letting $\mathcal{P} = 1$ in (14). To

illustrate this phenomenon in the tensor PIE representation, note that, for $r = 0$, the functional $V(\mathbf{v}) = \|\mathcal{T}\mathbf{v}\|_{L_2}^2$ satisfies

$$\dot{V}(\mathbf{v}) = \langle \mathbf{v}, [\mathcal{T}^* + \mathcal{T}]\mathbf{v} \rangle_{L_2} - \langle \mathbf{v}, \mathcal{T}^*[\mathcal{T} \otimes \mathcal{R}][\mathbf{v} \otimes \mathbf{v}] \rangle_{L_2}.$$

Then, defining $\mathcal{K}_{\text{lin}}$ as in Proposition 11, we find that $\mathcal{K}_{\text{lin}}(\mathcal{T}^*[\mathcal{T} \otimes \mathcal{R}]) = 0$, and thus $\dot{V}(\mathbf{v}) = \langle \mathbf{v}, [\mathcal{T}^* + \mathcal{T}]\mathbf{v} \rangle_{L_2}$. Finally, a simple calculation yields $\mathcal{T}^* = \mathcal{T} = -\mathcal{R}^*\mathcal{R} < 0$, so that $\dot{V}(\mathbf{v}) = -2\langle \mathcal{R}\mathbf{v}, \mathcal{R}\mathbf{v} \rangle_{L_2} = -2\|u_s\|_{L_2}^2 \leq 0$. Of course, one could derive a similar result combining integration by parts with the boundary conditions. However, the advantage of the PIE framework is that such ad hoc manipulations are unnecessary due to the fact that the boundary conditions are embedded in the operators $\mathcal{T}$ and $\mathcal{R}$.

B. Kortweg-De Vries Equation

As mentioned in the previous example, the nonlinear term uu_s is well-known to vanish in the derivative of the Lyapunov functional candidate $V(u) = \|u\|_{L_2}^2$ – as can be proved using integration by parts. For this reason, we now consider a modified version of the Kortweg-De Vries (KdV) equation with a different type of quadratic term $-u(s)^2$. The equation is defined on $s \in [0, 1]$ with Dirichlet-Neumann boundary conditions,

$$\text{PDE: } \dot{u}(t, s) = -u_{sss}(t, s) + u(t, s)[ru(t, s) + 6u_s(t, s)],$$

$$\text{BCs: } u(t, 0) = 0, \quad u(t, 1) = 0, \quad u_s(t, 1) = 0.$$

Define the PI operators $\mathcal{T}, \mathcal{R} \in \Pi_2$ for $\mathbf{v} \in L_2[0, 1]$ as

$$(\mathcal{T}\mathbf{v})(s) := \int_0^1 \frac{1}{2}[s-1]^2 \theta^2 \mathbf{v}(\theta) d\theta - \int_s^1 \frac{1}{2}[s-\theta]^2 \mathbf{v}(\theta) d\theta,$$

$$(\mathcal{R}\mathbf{v})(s) := \int_0^1 [s-1] \theta^2 \mathbf{v}(\theta) d\theta - \int_s^1 [s-\theta] \mathbf{v}(\theta) d\theta.$$

Then, defining fundamental state $\mathbf{v}(t) := u_{sss}(t)$ we have $u(t) = \mathcal{T}\mathbf{v}(t)$ and $u_s(t) = \mathcal{R}\mathbf{v}(t)$. Imposing this relation in the PDE, the system can be equivalently represented as

$$\text{PIE: } \partial_t \mathcal{T}\mathbf{v}(t) = \left[\underbrace{-1}_{\mathcal{A}} \quad \underbrace{\mathcal{T} \otimes (r\mathcal{T} + 6\mathcal{R})}_{\mathcal{B}} \right] \begin{bmatrix} \mathbf{v}(t) \\ \mathbf{v}(t) \otimes \mathbf{v}(t) \end{bmatrix}.$$

Applying the conditions of Theorem 12 with $\epsilon = 10^{-4}$, $\delta = 10^{-6}$, the proposed algorithm is able to find a global Lyapunov stability proof for any $0 \leq r \leq 4.6491$.

Although the authors are unaware of any analytic results for stability of the modified KdV equation, the numerical algorithm for each value of r appears to converge to a Lyapunov functional of the form

$$V(u) = C_0 \int_0^1 e^{\frac{r}{2}s} u(s)^2 ds + C_1 \int_0^1 \int_0^1 e^{\frac{r}{3}(s+\theta)} u(s) u(\theta) d\theta ds,$$

for constants $C_0, C_1 > 0$. Considering the simpler Lyapunov functional candidate $V(u) = \int_0^1 e^{\frac{r}{2}s} u(s)^2 ds$, it can be shown that, along solutions to the PDE, the temporal derivative of this functional is given by

$$\dot{V}(u) = \int_0^1 e^{\frac{r}{2}s} \left[\frac{r^3}{16} u(s)^2 - \frac{3r}{4} u_s(s)^2 \right] ds - \frac{1}{2} u_s(0)^2.$$

Invoking the Poincaré inequality, stability can then be verified for $r \in [0, 4.0017]$, though it is unclear how conservative this bound is (see the extended version for more details [17]).

C. Kuramoto-Sivashinsky Equation

We now consider the Kuramoto-Sivashinsky Equation (KSE). As in the previous example, we add a quadratic term ru^2 to the dynamics, introducing a nonlinear term which does not vanish trivially for quadratic Lyapunov functional candidates. In particular, we consider a system of the form

$$\begin{aligned} \dot{u}(t, s) &= -u_{ssss}(t, s) - u_{ss}(t, s) - u(t, s)[ru(t, s) + u_s(t, s)], \\ u(t, 0) &= u(t, 1) = u_s(t, 0) = u_s(t, 1) = 0. \end{aligned}$$

Define PI operators $\mathcal{T}, \mathcal{R}_1, \mathcal{R}_2 \in \Pi_2$ for $\mathbf{v} \in L_2[0, 1]$ as

$$\begin{aligned} (\mathcal{T}\mathbf{v})(s) &:= -\int_0^s \frac{1}{6}[s-1]^2\theta^2[2s\theta-3s+\theta]\mathbf{v}(\theta)d\theta \\ &\quad - \int_s^1 \frac{1}{6}[\theta-1]^2s^2[2s\theta-3\theta+s]\mathbf{v}(\theta)d\theta, \\ (\mathcal{R}_1\mathbf{v})(s) &:= -\int_0^s \frac{1}{2}[s-1]\theta^2[2s\theta-3s+1]\mathbf{v}(\theta)d\theta \\ &\quad - \int_s^1 \frac{1}{2}s[\theta-1]^2[2s\theta+s-2\theta]\mathbf{v}(\theta)d\theta, \\ (\mathcal{R}_2\mathbf{v})(s) &:= -\int_0^s \theta^2[2s\theta-3s-\theta+2]\mathbf{v}(\theta)d\theta \\ &\quad - \int_s^1 [\theta-1]^2[2s\theta+s-\theta]\mathbf{v}(\theta)d\theta. \end{aligned}$$

Then, defining fundamental state $\mathbf{v}(t) := u_{ssss}(t)$, we have $u(t) = \mathcal{T}\mathbf{v}(t)$, $u_s(t) = \mathcal{R}_1\mathbf{v}(t)$, and $u_{ss}(t) = \mathcal{R}_2\mathbf{v}(t)$. We obtain an equivalent PIE representation as

$$\partial_t \mathcal{T}\mathbf{v}(t) = \underbrace{\begin{bmatrix} -\mathcal{T} & -\mathcal{R}_2 \end{bmatrix}}_{\mathcal{A}} \underbrace{\begin{bmatrix} -\mathcal{T} \otimes (r\mathcal{T} + \mathcal{R}_1) \end{bmatrix}}_{\mathcal{B}} \begin{bmatrix} \mathbf{v}(t) \\ \mathbf{v}(t) \otimes \mathbf{v}(t) \end{bmatrix}.$$

Applying the conditions of Theorem 12 with $\epsilon = 10^{-4}$, $\delta = 10^{-6}$, the proposed algorithm is able to find a global Lyapunov stability proof for any $r \in [-0.6599, 0.6599]$. Again, although we have no analytic stability for this modified KSE, we note that for the Lyapunov functional candidate

$$V(u) = \int_0^1 e^{3rs} u(s)^2 ds,$$

the temporal derivative along solutions to the PDE satisfies

$$\dot{V}(u) = \int_0^1 e^{3rs} [-81r^4 + 9r^2]u(s)^2 + [36r^2 + 2]u_s(s)^2 - 2u_{ss}(s)^2 ds.$$

Invoking the Poincaré inequality, this derivative can be proven to be nonpositive whenever $|r| \leq 0.3608$, though again, it is unclear how conservative this bound is.

VIII. CONCLUSION

In this paper, we have proposed a new, compact PIE representation of scalar-valued quadratic PDEs, expressed in terms of PI operators on states $\mathbf{v}$ and $\mathbf{v} \otimes \mathbf{v}$. In order to derive this representation, we first defined a new class of PI operators Π_{2^2} , acting on states $\mathbf{v} \otimes \mathbf{v}$. We derived formulae for computing the tensor product $\mathcal{Q} := \mathcal{T} \otimes \mathcal{R}$ of standard PI operators $\mathcal{T}, \mathcal{R} \in \Pi_2$, proving that the resulting operator $\mathcal{Q}$ belongs to the newly defined class Π_{2^2} . Using this tensor product, we then derived expressions for operators $\mathcal{A} \in \Pi_3$ and $\mathcal{B} \in \Pi_{2^2}$ defining the PIE representation associated to

a particular quadratic PDE. Finally, using this PIE representation, we proposed a method for testing existence of a quadratic Lyapunov functional certifying stability of the PDE, posing this test as an optimization problem that can be solved with semidefinite programming. While currently limited to scalar quadratic PDEs, the results of this paper may be extended to higher-degree polynomial PDEs and higher-degree Lyapunov functionals.

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