How Do We Disentangle Equality from Equivalence? Well, It Depends

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In this paper, we report on interviews with mathematicians exploring the ways in which they think about the relationship between equality and equivalence. Given sometimes unclear and conflicting presentations of equality and equivalence in the literature, we are motivated to understand subtleties about how these constructs interact; doing so can have pedagogical implications, which we explore in this paper. We present three major themes that emerged from our analysis of the mathematicians' discussions of equality: 1) that equality represents a well-specified equivalence relation, 2) that audience matters when specifying equivalence and equality, and 3) that context is imperative when discussing equivalence and equality.

Keywords: equivalence, equality, equivalence relations, mathematicians

In this paper, we unpack the nuances of multiple mathematicians' understandings and uses of equivalence and equality, and the potential implications of these nuances for K-16 pedagogy. Rather than bringing new mathematical insights to light (see Mazur (2008)), we seek to consider pedagogical implications pertaining to relationships among and distinctions between equivalence and equality as expressed by mathematicians. We suggest that such insights are valuable for researchers who want to better understand the nature of equivalence, and for instructors who seek general principles and guidelines for supporting students in productive ways of reasoning about both equivalence and equality. We address the following research question: *How do mathematicians view the relationship between equivalence and equality?*

Relevant Literature and Theoretical Perspectives

In this section, we introduce relevant literature and situate our work within theoretical perspectives. This includes framing our working definitions of equivalence and some mathematical discussion about equivalence and equivalence relations.

Equivalence and Equality

There has been a considerable amount of research exploring equivalence and equality; much of this work has been conducted in K-12 settings (e.g., Knuth et al., 2006; McNeil & Alibali, 2005; Solares & Kieran, 2013), although some has explored these ideas with older students as well (e.g., Chin & Tall, 2001; Godfrey & Thomas, 2008; Stephens, 2006). We focus here on work that has explored the relationship between these two foundational mathematical notions, which are broadly synonymous but subtly different: while the equivalence concept is well-specified (via the definition of an equivalence relation), we observe that explicating its relationship to and distinction from equality is important but not yet well articulated or understood. Despite its ubiquity, for instance, equality has been referred to as "slippery" (Mazur, 2008, p. 222) and "precarious" (Saldanha & Kieran, 2005, p. 5). Hersch (1997) encapsulated this idea well when he noted that equality "is used freely, from kindergarten to postgraduate. It's never defined or explained" (p. 50).

The K-16 curriculum provides a multitude of examples that highlight the potential benefits of being able to disentangle equality from equivalence. For example, Saldanha & Kieran (2005) found that "many students' emerging thinking about equivalence was rightly bound up – indeed, arguably confounded – with notions of numerical equality" (p. 5). We infer that, here, *equality* refers to the equivalence of numerical expressions, whereas *equivalence* refers to the equivalence of algebraic expressions. Note that, in this situation, it would not be inappropriate to use equality to refer to *both* the equivalence of numerical expressions and the equivalence of algebraic expressions, as is in fact normative.

Also consider the notion of equation solving, which not only involves "a grasp of the notion that [the] right and left sides of the equation are equivalent expressions, but also that each equation can be replaced by an equivalent equation (i.e., one having the same solution set)" (Kieran, 1981, p. 323). That is, two forms of equivalence are at play. We observe that, in such instances, the algebraic expressions in question could be reasonably said to be equal (and thus also equivalent). Though while the equations are equivalent because they share the same solution set, they are *not* considered to be equal (we consider it normative to say, for example, that $2x + 1 = 5$ is *equivalent* to $2x = 4$, rather than to say that $2x + 1 = 5$ is *equal* to $2x = 4$). Thus, in contrast to the first example we discussed, equality applies to one instance (expressions) but not the other (equations).

The topics featured in these examples – the equivalence of expressions and equations – are foundational in the mathematics curriculum, yet equality does not uniformly apply in each situation. Indeed, these peculiarities illustrate that "[m]ath lingo sometimes says 'equal,' sometimes 'equivalent'" (Hersch, 1997, p. 50). We observe that the rules governing *when* one might use one or the other are implicit and underspecified. This spurs us to ask: in what situations might we use equality, and why?

The literature does contain some attempts to answer this question. Mazur (2008) conducted a deep and mathematically rigorous exploration, using category theory to explicate the relationship between equivalence and equality. However, we note that his argument – while undoubtedly useful to a research mathematician and generalizable across many context – is far beyond the scope of mathematics with which K-16 students (and most K-16 teachers) are familiar. What is needed, we propose, is an explanation of the relationship between these ideas that, in addition to being plausible and generalizable, is also accessible to K-16 students and instructors. In this vein, McNeil and colleagues (2012), for example, explained that numerical equivalence is "formally represented by the equal sign" (p. 1109). Similarly, Fyfe and Brown (2018) framed numerical equivalence as "the relation indicating that two quantities are equal and interchangeable" (p. 158). The example of equivalent equations, for instance, illustrates that equivalence is not always "formally represented by the equal sign" and certainly can – but need not always – indicate that "two quantities are equal." While these descriptions of the equivalence-equality relationship are certainly appropriate for the contexts from which they were drawn, we note that they do not frame the relationship in a way that generalizes to other contexts, and thus are of limited use for our purposes here.

Others have highlighted equivalence as a more general and sophisticated version of equality, but have pointed out that equality is nevertheless used differently (e.g., Rupnow & Sassman, 2022). Kieran's (1981) seminal paper on equivalence begins with Gattegno's (1974) observation that, with respect to equality, "equivalence is concerned with a wider relationship" (p. 83). Additionally, Fischbein (1999) argued that "the concept of equivalence is more general and much more subtle than the concept of equality. What seems to be trivial for equality, is not necessarily trivial for equivalence" (p. 23). Our initial impression is that framing the well-specified notion of equivalence as more general than the "slippery" and "precarious" notion of equality is both useful as a starting point for our analysis and consistent across mathematical contexts. A contribution of our work, therefore, is to further explore and elaborate these conceptual statements regarding the specificity of equality and the generality of equivalence as a result of analyzing the perspectives of multiple mathematicians.

Framing Equivalence

Here we define and characterize equivalence, which connects to our interview design and our analysis. We broadly draw on Cook, Reed, and Lockwood's (2022) framework, which employed conceptual analysis to identify three interpretations of equivalence taken on by students across mathematical domains and scholastic settings (these included *common characteristic*, *descriptive*, and *transformational* interpretations). We note that each of these focus on the features of students' possible interpretations of the equivalence of specific objects; that is, these interpretations involve what Hamdan (2006) calls a *local*, element-wise view of equivalence.

While we inferred the presence of these *local* interpretations when analyzing our interviews with mathematicians, we determined that the mathematicians' perspectives on the relationship between equivalence and equality can be better characterized in terms of a *global* view of equivalence, which "requires going beyond the element-wise conception of a relation" (Hamdan, 2006, pp. 130-131) in order to focus on the structure of a set in terms of the equivalence classes based on the relation that partitions the set. One key global interpretation of equivalence taken on by the mathematicians, which theoretically grounds our discussion of equivalence and equality here, is that objects "are identified as 'the same' if they lie in the same equivalence class" (Rupnow & Sassman, 2022, p. 119). A key aspect of this interpretation is that it is by nature a stipulated attribution of equivalence. Indeed, given any set, an extreme use of this interpretation might entail an arbitrary partitioning of the set and then attribution of equivalence to the members of each individual partition. While employing this interpretation is usually much more purposeful (and less arbitrary), we find that the key subtleties of attributing equality or equivalence to mathematical objects are grounded in one's ability to assign equivalence based on membership in an equivalence class. In fact, our exploration of when to use equality largely centers on the conditions by which the mathematicians might not just attribute equivalence to members of each class, but instead might determine it appropriate to ignore differences amongst elements in the same class altogether.

Methods

Data Collection

The data on which we report is taken from a larger study examining the ways that both mathematicians and students view equivalence across undergraduate mathematical domains. Early data collection targeted research mathematicians to, among other goals, generate hypotheses about how students might productively understand equivalence at high levels of mathematics. One outcome of these interviews was an early awareness that the relationship between equality and equivalence was relevant to these mathematicians' understandings and uses of equivalence, and that there were many subtleties to the mathematicians' views on equality that have key implications for mathematics instruction in general.

As part of the larger study, we interviewed research mathematicians specializing either in abstract algebra or in combinatorics. Our primary objective was to build an initial theory that accounted for the key ways in which algebraists and combinatorists might think about equivalence. This report is based upon individual semi-structured interviews (Fylan, 2005) we conducted with eight mathematicians, all of whom were tenured mathematics faculty at large universities across the United States. The primary data for this report were the video records of the 90-minute interviews conducted with the mathematicians (referred to here by the pseudonyms Dr. A, Dr. B, etc.). Though we asked a variety of questions about equivalence and how it manifests in their research and instruction, here we focus primarily on their responses to the following question: "Do you think that there is a difference between equivalence and equality? If so, please explain in what kinds of situations you would use equality and in what kinds of situations you would use equivalence. If not, please explain why."

Data Analysis

The video records were transcribed in full, and the transcripts were initially analyzed according to Cook et al.'s (2022) framework for interpreting local, element-to-element instances of equivalence. After inferring the various ways in which the mathematicians were interpreting equivalence, and determining that a global, equivalence-class based perspective on equivalence (Hamdan, 2006) would be more productive for analyzing the mathematicians' conceptions of equality, we isolated the segments of the interviews in which the mathematicians specifically discussed the relationship between equivalence and equality. We then employed thematic analysis (Braun & Clarke, 2012) on this subset of the data. Thematic analysis was useful for our purposes because it provided a mechanism by which we could identify themes related to the mathematicians' views of the relationship between equivalence and equality and also make sense of these themes with respect to our research question. Our analysis was primarily data-driven and exploratory, though we did use Hamdan's (2006) global perspective on equivalence and equivalence classes to guide our identification and description of themes. We followed Braun and Clarke's (2012) stages of thematic analysis (familiarizing oneself with the data, developing initial codes, searching for themes, reviewing themes, and characterizing and naming themes), then met to compare and negotiate our interpretations. From this process we identified the three themes that we discuss below.

Results

We now detail the overarching themes we identified regarding the nature of the relationship between equality and equivalence. Generally, the mathematicians we interviewed framed equality as a relationship between objects that results from declaring and imposing a well-specified equivalence relation. Their criteria for when they would comfortably use equality (instead of the more general notion of equivalence) to describe the relationship depended on our two major themes of *context* and *audience*. Though we observed instances of each of these themes across a multitude of excerpts and interviews in our data set, due to space constraints we present each theme only with prototypical, illustrative responses from our data.

Equality Indicates a Well-Specified Equivalence Relation

Our first theme is that equality indicates a well-specified equivalence relation. That is, equality is used to relate elements in the same equivalence class to each other once an equivalence relation has been clearly defined and enacted on a given set. To illustrate this point, we focus on responses from Dr. A, a combinatorist who explained the nuances of the equality-equivalence relationship by pointing out the differences in the role of equality before and after a well-specified equivalence relation is imposed. For Dr. A, before an equivalence relation is imposed, equality is “kind of the trivial case where every block of the partition is a single element.” In such cases, he said, “there’s one element that’s equal, you know, the equivalence classes have size one. There’s no variation.” In other words, prior to the implementation of a well-specified equivalence relation, equality means *identity*. That is, equality only applies to elements that are represented in exactly the same way with no variation in representation (e.g., prior to establishing what it means for two numerical expressions to be equivalent, 3 is only equal to 3 and is not, for example, equal to 1+2). Dr. A then explained equality in the context of using modular congruence as an equivalence relation:

> *Dr. A*: But if I’m willing to just say, “No, I’m not interested in all the integers anymore, I now just want one element from these classes,” I feel, like, okay committing to that. Then it’s okay. That's sort of my, that's what it means to me to now say, “Okay, when I say equals … I'm not thinking of three as three, I’m thinking of three as a … representative of this set [of integers congruent to 3 modulo 7],” or whatever. But it’s kind of like you’re changing the idea of what [the equal sign] even means. And now the definition of equality, I guess, is okay in my mind now […]. It’s just that that equality has to be defined.

We highlight two key features of this theme that we infer from this comment. First, Dr. A emphasizes that “equality has to be defined.” He made a similar comment at another point in the interview, noting that “I’m okay with [equality] once you declare it and clean it up.” We therefore infer that, for Dr. A, an essential element of declaring objects to be equal is that equality is well-specified. First, notice that Dr. A’s language distinguishes the meaning of equality *before* and *after* the introduction of modular congruence as an equivalence relation. Before, equality meant only identity (e.g., 3 is only equal to 3). After, the introduction of the equivalence relation and collapsing of the integers into equivalence class representatives establishes a new form of equality (e.g., 3 can now be considered equal to 10, 17, etc.). Bringing these two features together, we infer that equality indicates a relationship between elements that results after one declares and imposes a well-specified equivalence relation.

Audience Matters When Specifying Equivalence and Equality

Our second theme is that one’s use of equality is also conditioned by one’s image of the audience. For example, Dr. A noted that equality “has to be either explicitly there or at least understood by all the people in the conversation as an okay use of equality.” Also consider the following comment from Dr. B, who had previously indicated the importance of specifying what equality means:

> *Dr. B*: So, what’s probably going on in my head, the rule of thumb is if I, as the speaker, I’m confident that everybody in the conversation really understands what’s going on, then we can get careless. In [an introductory abstract algebra class], I don’t think it’s ever appropriate to have that confidence. […] So yeah, if I’m doing modular arithmetic in [class] $\overline{1} = \overline{8}$ instead of $1 = 8$. And I'm gonna be very clear that I'm always using bars […] because with that collection of students, I don’t know, there’s probably a significant fraction where the notational abuse is fine and honestly helpful, but there's another bunch

who are just gonna be so completely lost if you start to do it that I wouldn't try it. On the other hand, if I'm teaching […] the master's level class, yeah, I'm not putting those bars on after the first day that I introduced the quotient definition.

Dr. C offered a very similar explanation:

> *Dr. C*: The students may not be comfortable with equivalence and equality being the same, like, and this probably comes up […] in modular. You know, if I'm in Z_5 and I'm saying, "Oh, six is the same as one," […] I think for mathematicians that's, we're very casual with our language, you know? So, […] that could be okay. But maybe for an undergrad that would be not so much okay to say 6=1.

Overall, the mathematicians we interviewed indicated that, with respect to a particular topic, equality might be allowable when in discourse with some populations (e.g., graduate students and mathematicians) but not advisable when in discourse about the same topic in other populations (e.g., undergraduates in their first abstract algebra course).

The Mathematical Context Is Imperative When Discussing Equivalence And Equality

Our final theme is that nuances related to the mathematical contexts under discussion also influence whether or not equality is used. For example, above we discussed a comment from Dr. B in which he discussed how he would feel free to write statements like $1 = 8$ to connote the equivalence of elements in the quotient Z_7 while amongst mathematicians or in a graduate course. He went on to add, however, that this might not be the case if he has "got multiple quotients interacting or something." We infer here that, in such situations, the mathematical context at hand might shape one's use of equality. Also consider the following comment from Dr. D:

> *Dr. D*: So, equality is an equivalence relation, […] so if I'm looking at a quotient ring – so I'm defining, you know, ring mod ideal – then $Q = R$ mod i. Then I can really say that $a + i$ really *equals* [his emphasis] $b + i$ in Q because we have defined equality to be that way. […] So, in terms of discrete math, I will *define* $\{a, b, c, d\}$ to *equal* $\{c, b, a, d\}$. I will define those to be equal if I put the curly brackets on and I understand that these things are sets. But I will say (a, b, c, d) is not equal to (c, b, a, d). […] equality depends on the context that we've agreed upon.

Similarly, Dr. C explained that, to him, consideration of mathematical context also influences when equality might be most appropriately used:

> *Dr. C*: It's very situational to the problem being asked. […] I might be going off on a tangent now, but I'm just thinking when I say equals it depends on where I'm working, you know? An integer, if I'm saying six and one are integers, then no, they can't be equal. They're different. But if I'm working in Z_5, then yeah, they're totally equal.

We view these considerations of the mathematical context as complementary (and not disjoint from) considerations of the audience.

Discussion

In response to our research question (*How do mathematicians view the relationship between equivalence and equality?*), our analysis indicates that – in alignment with the literature – equivalence is indeed the more general, comprehensive notion, and that mathematicians' use of equality is conditioned by (1) whether or not the form of equivalence in question has been well-specified, (2) the mathematical context, and (3) their images of the intended audience for the mathematical discourse in question.

A major goal of this report is to make a case that more care is needed when conceptualizing equality in educational contexts, and that there are subtle aspects of how we use equality in both research and teaching contexts that can have important implications for research and instruction. Returning to the literature, when it is said that equivalence is "formally represented by the equal sign" (McNeil et al., 2012, p. 1109), our findings suggest that there is more specificity needed to situate *when* we use equality to formally represent equivalence relations. This functionally brings clarity to the theme in the literature about how equivalence is more general than equality: for the mathematicians, equality represented a well-specified equivalence relation that was appropriate for the intended audience and mathematical context.

We believe our analysis builds upon other pieces of the discussion of equivalence and equality in the literature as well. Particularly, we propose that a reason that equality has been called "slippery" and "precarious" is perhaps because its use depends not solely on mathematical rigor and precision, but on the user's image of the audience and mathematical context. That is, to a certain extent, this is an *epistemological* – and not solely a *mathematical* – distinction. These themes support the idea that there is not just one definition of equality, underscoring the importance of our first theme (as well as many calls by researchers and curriculum designers): it is critical to specify exactly the scope of situations in a given context to which we apply notions of equality. As a practical takeaway, then, our findings highlight that we should be much more explicit about equivalence in our teaching of mathematics across the spectrum. Currently, there is a danger that equivalence is being brushed under the rug, under the guise of equality, and there are not opportunities for broader conversation about how and why we attribute equivalence. Further, it may be useful to emphasize that our use of the equal sign expands in complexity and layering as we progress throughout the K-12 curriculum (and then into the undergraduate curriculum). Our findings may raise additional suggestions for certain populations; for example, for pre-service secondary teachers this is an opportunity to draw potentially meaningful connections between advanced mathematics (e.g., equivalence relations and equivalence classes) and the secondary mathematics they will soon be teaching (e.g., equivalence in the context of rational numbers, expressions, and equations).

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