# Robust Decoupling, Disturbance Rejection and Linearization of Unknown Nonlinear Systems

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**Abstract.** Multi-input multi-output (MIMO) systems are frequently encountered in energy systems, robotics, communications, cybersecurity and control engineering. They are difficult to control because of the existence of couplings between the inputs and outputs. These couplings are often nonlinear and unknown or unmodelled. In many such systems it is desirable to be able to independently control the outputs in a linear manner. Also there often are disturbance inputs from which the outputs must be protected, that is decoupled or made uncontrollable. This has led to a substantial amount of research effort directed at the decoupling, linearization and disturbance rejection problems for linear and nonlinear systems. Typically, most decoupling and linearization methods available in the literature require knowledge of a model of the system which is often unavailable. Moreover exact implementation of decoupling and disturbance controllers are fragile and therefore of limited applicability. In this paper we show how high-gain feedback can effectively and robustly reject disturbances, linearize and decouple an unknown nonlinear static multivariable system under fairly mild conditions. These conditions are that there should be at least as many inputs as outputs to be controlled, and that the unknown nonlinear system be left invertible, although knowledge of the left inverse is not needed. An illustrative example of a circuit where two coupled output voltages are to be controlled independently and linearly in the presence of sources which are disturbances, is solved using PSpice. Future research problems are discussed.

**Keywords:** High-gain feedback · Robust disturbance rejection · Nonlinear systems

## 1 Introduction

Multivariable systems having several interacting inputs and outputs are ubiquitious in process control, power and energy systems, motion control, communication systems, aerospace systems and myriad other engineering applications.


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Often, in such systems, there are nonlinearities present that are not known precisely, and disturbance inputs which are unknown and "random" in nature, such as wind disturbances. These facts make the control of multivariable systems a challenging problem. Considerable effort has been devoted in the control literature to decoupling (see [1,2] and references therein), disturbance rejection (see [3–6] and references therein) and linearizing (see [7,8] and references therein) of such systems. However most results on disturbance rejection and decoupling are restricted to linear systems with known models. Similarly, linearization also requires knowledge of the nonlinearities.

In this paper, we develop a simple high-gain feedback solution to the problem of decoupling and linearization of an unknown multivariable nonlinear algebraic (non-dynamic) system. Under the assumption that the plant has at least as many inputs as outputs and is "left invertible" we show that the outputs of the plant can be independently controlled by high-gain feedback and the resulting feedback system effectively "linearized" with high and controllable accuracy despite the presence of unknown disturbances. This useful result is illustrated by simple examples including a circuit example wherein two voltages in a DC circuit must be independently controlled linearly and in the presence of unknown disturbance sources. Future research problems are briefly discussed.

## 2 Robust Feedback Linearization

In many instances, a physical element may have a nonlinear characteristic $y = f(u)$ (see Fig. 1) and it may be desirable, for design, calibration or measurement purposes, to have a linear input–output relationship.

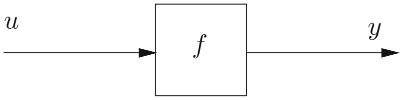


**Fig. 1.** A nonlinear element

This could be achieved by the high-gain feedback system shown in Fig. 2 without detailed knowledge of the characteristics of the nonlinear element.

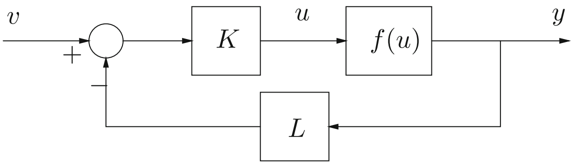


**Fig. 2.** A high-gain "linear" feedback system

Since

$$v = Ly + \frac{1}{K} f^{-1}(y) \tag{1}$$

it follows that as $K$ approaches $\infty$, $v$ approaches $Ly$.

*Example 1.* Consider a nonlinear system with

$$f(u) = 10u^3 = y. \tag{2}$$

Choose $L = 10$ and $K = 100$ in Fig. 2. Then

$$v = 10y + \frac{1}{100} \sqrt[3]{\frac{y}{10}}. \tag{3}$$

If $y$ varies between 0 and 10, we have $v$ varying between 0 and 100 linearly with $y$ with an error less than 1%.

This example shows clearly that a nonlinear system can be rendered as "linear" as desired by closing the loop by high-gain feedback. Knowledge of the nonlinear gain is not needed.

*Remark 1.* Although the example above has no dynamics we might mention that a similar phenomenon works in servomechanisms using say a Proportional Integral Derivative (PID) controller. Thus an integral feedback controller (high-gain at $s = 0$) driving a nonlinear system will also exhibit "linear" behavior in the closed loop.

## 3 Main Results

For simplicity in the presentation of the main idea, we assume that the plant is represented by algebraic relations connecting the output $\mathbf{y}$ (an $m$ vector) to the input $\mathbf{u}$ (an $r$ vector). Thus, the plant model is:

$$\mathbf{y} = P(\mathbf{u}) \tag{4}$$

where $P(\mathbf{u})$ is in general an unknown nonlinear function.

To control the components $y_i$ for $i \in \mathbf{m}$ independently, it is intuitively clear there must exist an input $\mathbf{u}$ for each output $\mathbf{y}$. Here, we will formalize this by introducing the following useful *standing assumptions*.

**Assumption 1**

*(i)* $r \geq m$
*(ii)* $P^{-1}(\mathbf{y})$ *exists for each* $\mathbf{y}$.

This Assumption 1.ii is referred to as "left invertibility" of the plant, in the rest of the paper. It will be seen that the above assumptions will facilitate a solution to the problem of robust decoupling, disturbance rejection and linearization.

Now consider the nonlinear feedback system in Fig. 3

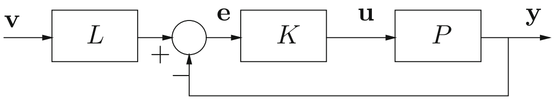


**Fig. 3.** A nonlinear feedback system

where $L$ is diagonal $(m \times m)$

$$L = \begin{bmatrix} l_1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & \cdots & l_m \end{bmatrix} \tag{5}$$

and $K$ is $r \times m$ with $\mathrm{rank}[K] = m$:

$$K = \begin{bmatrix} k_{11} & \cdots & \cdots & k_{1m} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ k_{r1} & \cdots & \cdots & k_{rm} \end{bmatrix} . \tag{6}$$

**Theorem 1.** *As $\|K\|$ approaches $\infty$, the nonlinear system of Fig. 3 tends to the linear system in Fig. 4.*

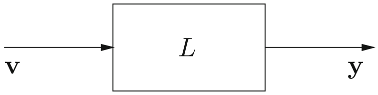


**Fig. 4.** Equivalent "Linear" system

*Proof.* From Fig. 3,

$$K(L\mathbf{v} - \mathbf{y}) = \mathbf{u}. \tag{7}$$

By left invertibility of $P$, and Assumption i,

$$L\mathbf{v} - \mathbf{y} = \left(K^T K\right)^{-1} K^T \mathbf{u} \tag{8a}$$

$$= \left(K^T K\right)^{-1} K^T P^{-1}(\mathbf{y}). \tag{8b}$$

Thus, as $\|K\|$ approaches $\infty$,

$$\mathbf{y} = L\mathbf{v} + \epsilon \tag{9}$$

with $\epsilon$ tending to zero.

# 4 Examples

*Example 2 (Decoupling and Linearization of an Unknown Nonlinear System).* Consider the nonlinear system or plant shown in Fig. 5

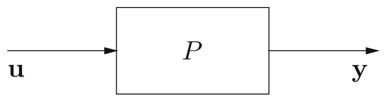


**Fig. 5.** A nonlinear plant

operating in steady-state. Let

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{10}$$

and even though our method does not require knowledge of the plant model, for the sake of developing the example, suppose that we know the nonlinear model of the plant:

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} u_1^2 + u_2^2 \\ u_1^2 + 5u_2 \end{bmatrix} = P(\mathbf{u}). \tag{11}$$

Suppose we know that $\mathbf{u}$ varies in the set $\mathcal{U}$ as shown in Fig. 6:

$$u_i \in [1, 10], \quad \text{for} \quad i = 1, 2. \tag{12}$$

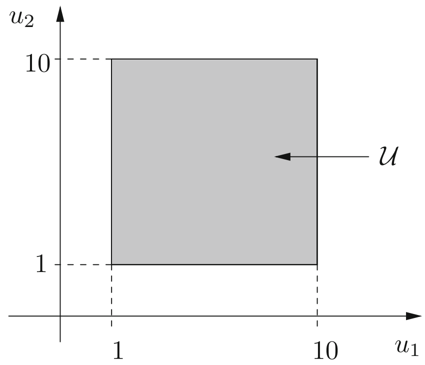


**Fig. 6.** The set $\mathcal{U}$

To linearize and decouple the system, consider the feedback system in Fig. 3. Let

$$L = \begin{bmatrix} l_1 & 0 \\ 0 & l_2 \end{bmatrix}, \qquad K = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} = \begin{bmatrix} 10^3 & 0 \\ 0 & 10^3 \end{bmatrix}, \tag{13}$$

for example, for arbitrary $l_i$, $i = 1, 2$.

From Fig. 3, assuming left invertibility of the plant:

$$K(L\mathbf{v} - \mathbf{y}) = \mathbf{u} \tag{14a}$$

$$= P^{-1}(\mathbf{y}). \quad \text{(left invertible)} \tag{14b}$$

Thus,

$$y_1 = l_1 v_1 + \epsilon_1 \tag{15a}$$

$$y_2 = l_2 v_2 + \epsilon_2 \tag{15b}$$

with

$$|\epsilon_i| := \left|\frac{u_i}{k_i}\right| \in [0.001, 0.01], \quad i = 1, 2 \tag{16}$$

for all $\mathbf{u} \in \mathcal{U}$. Since

$$y_1 \in [2, 200] \tag{17a}$$

$$y_2 \in [6, 150] \tag{17b}$$

for $\mathbf{u} \in \mathcal{U}$, we have

$$y_i = l_i v_i, \quad \text{for} \quad i = 1, 2 \tag{18}$$

with an error of less than 0.005%. Clearly, as $\|K\|$ increases, $\|\epsilon\|$ decreases.

*Example 3 (Left Invertibility Fails).* Suppose that $P(\mathbf{u})$ is:

$$y_1 = u^2 \tag{19}$$

$$y_2 = u + \frac{1}{u}. \tag{20}$$

We must have

$$y_2 = \sqrt{y_1} + \frac{1}{\sqrt{y_1}} \tag{21}$$

and so decoupling must fail. This example shows why the number of inputs must be greater than or equal to the number of outputs.

## 5 Disturbance Rejection

Most processes and systems are subject to unknown and unmeasurable disturbances. There is a large amount of literature on the so-called *disturbance rejection problem* (see [9,10] and references therein). In the majority of these references, the system is assumed to be linear and the model is *known* exactly. In many situations, these assumptions are unrealistic.

In the present section, we give an approach to disturbance rejection for an *unknown* nonlinear static system, using high-gain feedback. It is shown that under mild conditions (the number of control inputs is greater than or equal to the number of outputs and left invertibility of the control to output map), disturbance rejection with arbitrary accuracy can be achieved without knowledge of the model of the system or knowledge of the disturbance.

Consider the nonlinear plant in Fig. 7

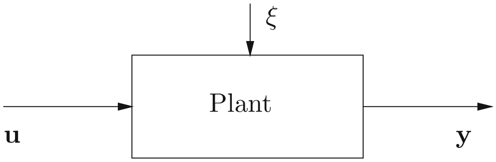


**Fig. 7.** A nonlinear plant with disturbances

where as before $\mathbf{u}$ is an $r$-dimensional control *input*, $\mathbf{y}$ is an $m$-dimensional *output* and $\xi$ is a *disturbance*. We assume that

$$\mathbf{y} = P(\mathbf{u}) + Q(\xi) \tag{22}$$

where $P$ and $Q$ are unknown nonlinear operators.

Now consider the feedback system in Fig. 8.

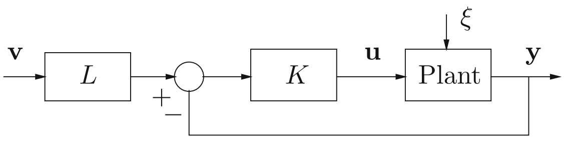


**Fig. 8.** A disturbance decoupled feedback system

From Fig. 8, assuming $P(\cdot)$ is left invertible

$$K(L\mathbf{v} - \mathbf{y}) = \mathbf{u} \tag{23a}$$

$$= P^{-1}(\mathbf{y} - Q(\xi)). \tag{23b}$$

Then assuming that $r \geq m$ and we have $\text{Rank}[K] = m$,

$$\mathbf{y} - L\mathbf{v} = -\underbrace{\left(K^T K\right)^{-1} K^T P^{-1}(\mathbf{y} - Q(\xi))}_{\epsilon}. \tag{24}$$

Thus,

$$\mathbf{y} = L\mathbf{v} - \epsilon \tag{25}$$

where $\epsilon$ goes to zero as $\|K\|$ goes to infinity. Thus, under high-gain feedback, $\mathbf{y}$ is unaffected by $\xi$, and indeed the error $\epsilon$ can be made arbitrarily small.

By way of illustration we consider below a DC circuit with unknown model, where independent control over two output voltages is required and they are to be unaffected by two disturbance sources.

*Example 4.* Consider the DC circuit in Fig. 9.

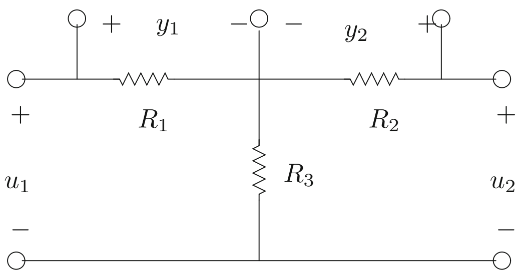


**Fig. 9.** A DC circuit

Let

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}}_{\mathbf{G}} \underbrace{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}}_{\mathbf{u}} \tag{26}$$

where

$$\mathbf{G} = \begin{bmatrix} \dfrac{R_1 (R_2 + R_3)}{(R_1 + R_3)(R_2 + R_3) - R_3^2} & \dfrac{-R_1 R_3}{(R_1 + R_3)(R_2 + R_3) - R_3^2} \\ \dfrac{R_2 R_3}{(R_1 + R_3)(R_2 + R_3) - R_3^2} & \dfrac{-R_2 (R_1 + R_3)}{(R_1 + R_3)(R_2 + R_3) - R_3^2} \end{bmatrix} \tag{27}$$

which is invertible for all positive values of the resistances.

The *design objective* is to robustly decouple $y_1$ and $y_2$, and control them independently with inputs $v_1$ and $v_2$ as shown in Fig. 10. Subsequently we will also extend this example to the case of disturbance rejection.

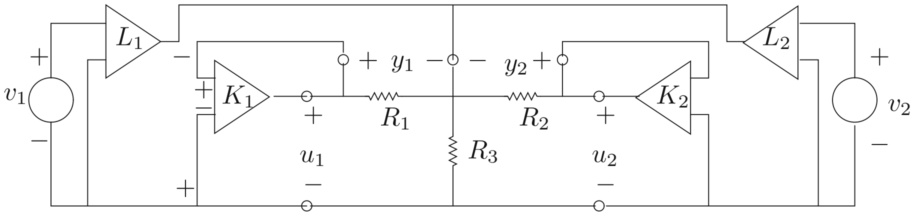


**Fig. 10.** A solution to the decoupling control problem

Note that $L_1$ and $L_2$ are arbitrary linear amplifier gains and $K_1$ and $K_2$ are high-gain operational amplifiers.

$$y_1 - L_1 v_1 = \frac{1}{K_1} u_1 \tag{28}$$

$$y_2 - L_2 v_2 = \frac{1}{K_2} u_2 \tag{29}$$

and

$$\mathbf{y} - \mathbf{L}\mathbf{v} = \begin{bmatrix} \frac{1}{K_1} & 0 \\ 0 & \frac{1}{K_2} \end{bmatrix} \mathbf{G}^{-1}\mathbf{y}. \tag{30}$$

As $K_1$ and $K_2$ approach infinity, $y_i$ approaches $L_i v_i$ for $i = 1, 2$.

An LTSpice simulation model for Fig. 10 is built and shown in Fig. 11. The resistors $R_1$, $R_2$, $R_3$ are set to 100 $\Omega$, 200 $\Omega$, 300 $\Omega$, respectively. The high-gain blocks $K_1$ and $K_2$ consist of several operational amplifiers cascaded and the resistor values are set such that $K_1$ and $K_2$ are both 1000. The resistors for the nominal gain blocks $L_1$ and $L_2$ are set such that the gains are precisely set to be desired gain values of the circuit. The simulation is run for different values of gains $\mathbf{L}$ and also for different values of reference inputs $\mathbf{v}$. The results are summarized in Table 1. It verifies the fact that the output $\mathbf{y}$ closely follows $\mathbf{Lv}$.

**Table 1.** Comparison of $\mathbf{Lv}$ and $\mathbf{y}$ for different values of $\mathbf{v}$ and $\mathbf{L}$.

| $v_1$ (V) | $v_2$ (V) | $L_1$ | $L_2$ | $L_1 v_1$ (V) | $y_1$ (V) | $L_2 v_2$ (V) | $y_2$ (V) |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 2.9999926 | 3.0036776 | 7.9999762 | 8.0086718 |
| 3 | 4 | 3 | 4 | 8.9999771 | 9.0106449 | 15.999952 | 16.017633 |
| 1 | 2 | 5 | 6 | 4.9999924 | 5.0060697 | 11.999972 | 12.013063 |
| 3 | 4 | 5 | 6 | 14.999977 | 15.017627 | 23.999945 | 24.026611 |

*Example 5.* In this example, we consider the disturbance rejection problem where disturbance sources are present in the circuit shown in Fig. 10. The modified circuit is shown in Fig. 12.

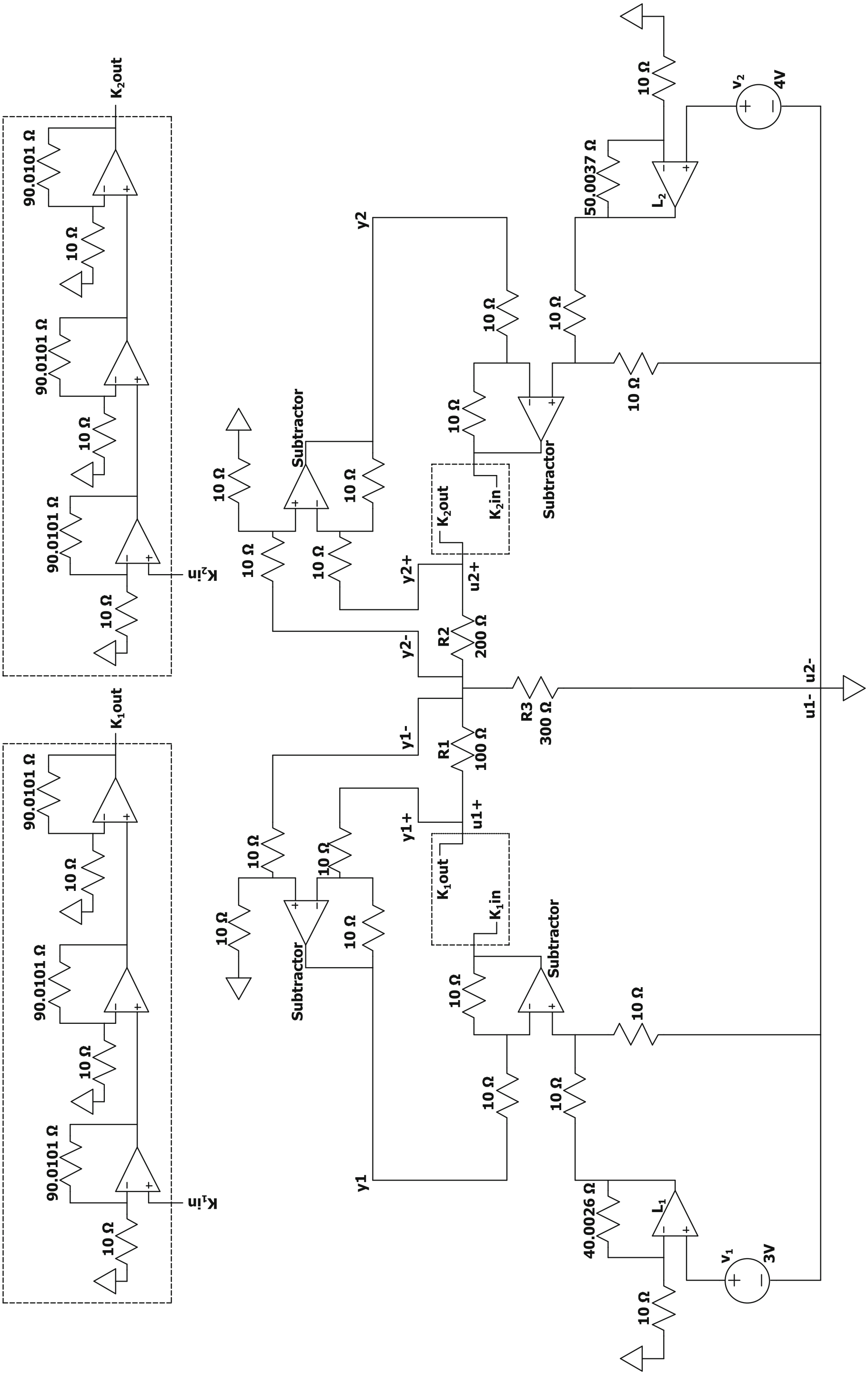


**Fig. 11.** Simulation circuit for the solution

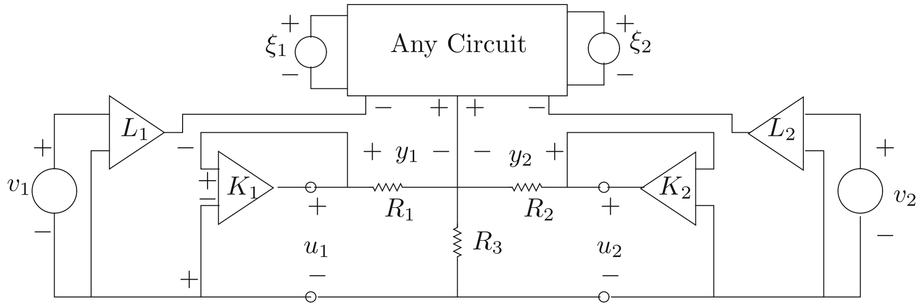


**Fig. 12.** A solution to the disturbance rejection problem

The simulation circuit for Fig. 12 is drawn in Fig. 13. The LTSpice model with disturbances is built in a similar way. The simulation is run for different values of $\mathbf{L}$, $\mathbf{v}$, and $\xi$. The results are summarized in Table 2. It is again verified that $\mathbf{y}$ closely follows $\mathbf{Lv}$ despite the presence of arbitrary disturbances.

**Table 2.** Comparison of $\mathbf{Lv}$ and $\mathbf{y}$ for different values of $\mathbf{v}$, $\mathbf{L}$ and $\xi$.

| $v_1$ (V) | $v_2$ (V) | $L_1$ | $L_2$ | $\xi_1$ (V) | $\xi_2$ (V) | $L_1v_1$ (V) | $y_1$ (V) | $L_2v_2$ (V) | $y_2$ (V) |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 3 | 4 | 2.9999926 | 3.0071039 | 7.9999762 | 8.0130987 |
| 1 | 2 | 3 | 4 | 10 | 20 | 2.9999926 | 3.0153463 | 7.9999762 | 8.0303507 |
| 3 | 4 | 5 | 6 | 3 | 4 | 14.999977 | 15.021054 | 23.999945 | 24.03104 |
| 3 | 4 | 5 | 6 | 10 | 20 | 14.999977 | 15.029297 | 23.999945 | 24.048292 |

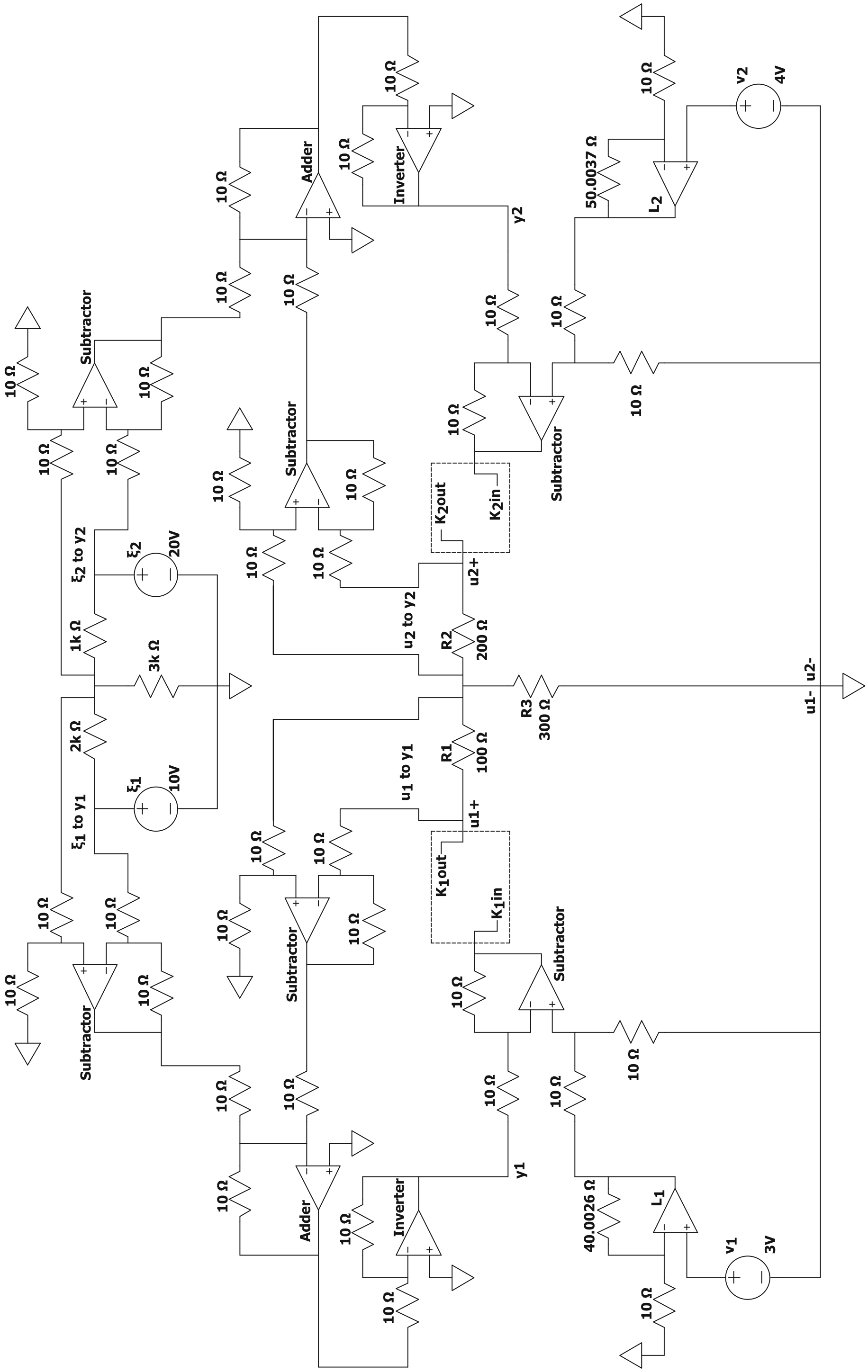


**Fig. 13.** Simulation circuit with disturbance

## 6 Concluding Remarks

We assumed, for simplicity, that the plant model is algebraic; equivalently that the input and outputs are related by a time invariant non-dynamic model. This analysis applies to systems in steady state and therefore applies, for example to DC circuits or AC circuits operating in steady state with inputs containing a single frequency. In future work this will be extended to dynamic models. It is worth pointing out that high-gain controllers are ubiquitious in the control field. For example the PID controller has infinite gain at zero frequency. Thus, the present results may have useful applications in decentralized PID-based robust control of nonlinear systems (see [11–13] and references therein). We also anticipate applications to many other areas of power, energy, communications and cybersecurity.

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