Structuring and Connecting 2D and 3D Space Using Linear Combinations

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Developing a rich understanding of linear combinations is key to understanding linear algebra. In this paper, I explore the rich connections students make between the geometric and numeric representations of linear combinations through playing and analyzing a video game. I look at population of students who have never taken linear algebra before and analyze how they structure space using the video game, Vector Unknown, as a realistic starting point. I detail and analyze this activity including the activities that transition them from 2D to 3D space.

Keywords: Linear Algebra, Linear Combinations, Game-based Learning

Linear combinations are the heart of linear algebra and involve understanding variety of different representations and connections between those representations (Hillel, 2000; Sierpinska, 2000; Larson & Zandieh, 2013). Video games that are well-designed can help students engage with a variety of representations (Ke & Clark, 2019) and build connections between these representations resulting learning complex systems. (Gee, 2003). The game *Vector Unknown* (Author, 2021; Author, 2019; Zandieh et al., 2018) is a well- designed game created to connect different representations of linear combinations by combining the ideas of Realistic Mathematics Education (RME), Inquiry-Oriented Instruction (IOI), and Game-Based Learning (GBL). In this paper, I will look at how the game can be used as a realistic basis for establishing a student structuring of space built upon taking linear combinations and explore how students connect 2D and 3D space.

Literature Review

The Magic Carpet Ride task (Wawro et al., 2013) was the basis for the game *Vector Unknown*. The task begins with a sequence that initially introduces linear combinations and then develops the ideas of span and linear independence. The task leverages the idea of the travel metaphor, which is one of the keyways students understand linear combinations (Plaxco & Wawro, 2015). Dogan-Dunlap (2010) explored linear combinations using Sierpinska's modes of reasoning (2000) to analyze student reasoning about linear independence and found that students' geometric reasoning could provide a gateway to analytic thinking about linear independence indicating the importance of developing geometric reasoning about linear combinations. Paraguez and Oktac (2010) found that a lack of geometric representation resulted in the students having difficulty forming a schema of vector spaces, a keyway of organizing space. Zandieh et al. (2019) analyzed student metaphors of basis and found that when students changed their metaphors when describing basis in 2D and 3D space indicating a shift in understanding. This indicates a need to explore both how students connect geometric representations of space with more analytic elements and how they transition from 2D to 3D.

The Game Vector Unknown and Prior Game Research

Vector Unknown (https://tinyurl.com/linearbunny) is a game developed by capstone students at Arizona State University (Author, 2022; Author, 2021, Author, 2019). Participants are presented with four vectors in 2D, which they can drag into a vector equation and scale. The goal of the game is to reach the goal location, which is represented by a basket on a geometric grid. The game has three levels that were used for this interview. During the first level, the game provides the player with a geometric representation of the linear combination of two selected scaled vectors. The second level removes the geometric representation to force them to reflect more deeply upon the vector equation. Finally, the third level requires the player to visit several keys before reaching the goal. This forces the player to look at the geometric aspects of the game as the coordinates of the lock are not provided to the students.

Author (2021) found four core strategies that players used to play the game. The first strategy, quadrant-based reasoning, focuses on matching the quadrant of the vectors in the linear combination with the quadrant of the goal. The second strategy focuses on one vector involved and the player selects a vector and scales it to be as close to the goal as possible and then uses another vector to reach the goal. Thus, the player focuses on one coordinate involved, chooses either the *x*- or *y*-coordinate of the goal vector, and adjusts the scalars to match that one coordinate. The final strategy, button-pushing, involves players choosing vectors and rapidly adjusting the scalars to find a path to the goal. Each of these four strategies had a more geometric version in which players focused on the graphical display and a more numeric version where the focus was more on the vector equation.

Theoretical Framing

In this study, mathematical learning has been theorized as student's mathematical activity (Freudenthal, 1971, as cited by Gravemeijer & Terwell, 2000) that relates to the topic of linear combinations. The goal of this study is to describe the emergent mathematical activity relevant to topics in linear algebra such as linear combinations, span, and linear dependence developed from playing *Vector Unknown,* analyzing the gameplay, and designing a new 3D game based on the core choices made in *Vector Unknown*. I call this emerging activity structuring space. The sequence was designed to have students engage in at least three levels of activity from Realistic Mathematics Education (RME) (Gravemeijer, 1999). First, students engage in the situation of the game, which involves taking linear combinations. Next, they have to consider how other players might play the game referring to their own experiences. Finally, they have to develop design ideas for the 3D game which is intended to move them towards making generalizations about the linear combinations that are relevant beyond the 2D game. This paper presents an overview of the student's structuring of space from as situated in the dissertation work (Author, 2022) while providing examples to the core research questions:

1. How did students who have never taken linear algebra structure two-dimensional space with respect to linear combinations in relation to the game *Vector Unknown*?
2. How did students adapt their structure of two-dimensional space to a three-dimensional setting when designing a three-dimensional game based upon *Vector Unknown*?

Structure of the Interview

The interview was conducted over the course of four days. During the first day, the participants played the game individually. For the second day, I paired the participants into three groups according to their gaming experience (Gamer/Gamer (GG), Gamer/Non-Gamer (GNG), Non-Gamer/Non-Gamer(NGNG)). This was done because, at the time, it was thought that gaming experience might impact their activity with the game. First, each pair was asked to describe all possible goal locations that could be obtained from a set of vectors. Second, they were asked to create a set of easy, medium, and hard vectors (since the game had an easy, medium, and hard level) to reach a fixed goal of <-3,5>. Each group was shown how to use

GeoGebra during the second day for geometric exploration of all possible goals. On the third day, each pair was asked to design a 3D game by either creating easy, medium, and hard vectors and illustrating all the available vectors or creating easy, medium, or hard vectors to reach the goal <3,-4,5>. During the third day, students were again provided access to GeoGebra 3D to explore the geometric aspects of the linear combinations in 3D. During the final day, each participant played the game individually. This was done to explore if the participants' strategies for playing the game changed from the initial playthrough. This paper focuses on days 2 and 3 of the interview to analyze how they structure space.

Participants, Data Sources, and Methods of Analysis

For this paper, I looked at six students recruited from Calculus 1 and Calculus 2 classes from a large Southern University. None of the students had taken linear algebra before participating in the interview. The groups consisted of Alex(G)/Angel(G), Marti(G)/Betty(NG), and Gabby(NG)/Dee(NG) (demographic information provided in Table 1). The gaming experience was determined by the number of hours students spent per week gaming, with gamers being those who played 5 or more hours per week.

Table 1. Participant Demographic Information

Pseudonym	Year	Major	Gaming Experience	Gender Identity	Hispanic or Latino	Race	English Primary Language
Marti	Soph	Environment al science	G	Gender Variant/Non-conforming	Yes	White	No
Betty	Soph	Biology and Education	NG	Woman	Yes	White	Yes
Alex	Fresh	Civil Engineering	G	Man	No	White	Yes
Angel	Fresh	Mechanical Engineering	G	Man	Yes	White	Yes
Gabby	Fresh	Mechanical Engineering	NG	Woman	No	White	Yes
Dee	Junior	Mathematics and Education	NG	Woman	No	Other	Yes

I conducted the interviews over Zoom. They were recorded on a secure server and auto-transcribed using the Zoom software. The videos and transcripts were reviewed for instances of structuring space for each group and open coded (Strauss & Corbin, 1990). These codes were then condensed across all three groups to produce categories of structuring space. Finally, a round of axial coding was conducted to find themes of structuring space across the categories: structuring by design, structuring by generating, and structuring across dimensions.

Findings

In this section, I present four different types of structuring that are meant to capture the student's activity with examples of each from the student's work. Emergent categories are bolded throughout this section, and the emerging themes are the types of structuring. Structuring by design contains different design choices that were made by participants as they created different elements of the game that were based upon the participant's assumptions about the game and/or its difficulty. Structuring by generating captures process by which students generated example vectors which they utilized to generalize. Structuring all possible goals presents how students describe their structuring of all possible goal locations (or linear combinations). Structuring across dimensions describes how students made transitions and connections between 2D and 3D space. Together these four themes are meant to capture how students structure space in the context of analyzing the 2D game and designing a 3D game about linear combinations of vectors.

Structuring by Generating

This section presents different methods the participants utilized to create new vectors either to reach a provided goal or to list all possible goals given a set of vectors. **Pairing** vectors refers to selecting two vectors from a set and adding them together to generate new vectors. **Scaling** refers to scaling one or more vectors in pairs to generate a new goal location. Pairing and scaling [Figure 1] were used systematically for all three groups to generate goal locations spanned by a set of vectors. Here Betty pairs <-3,2> with each of the vectors <-2,4>, <1,2>, <-6,4>. She then pairs <-2,-4> with the remaining vectors <1,2> and <-6,4>, and finally <1,2> and <-6,4>. She then begins to scale the first vector in the first pair she generated. This progression of scaling was also seen in GeoGebra when students scaled according to difficulty. Gabby describes the activity of generating vectors to reach the goal of <-3,5> by **adding up/subtracting down** as looking "for two numbers that would add up to –3, –2, and –1. Fairly simple numbers, because it's medium difficulty, and then two other numbers that would add up to 5." Here she focused on the x-coordinates first and found two numbers that *add up* to -3, and then looking at the y-coordinate and she found two numbers that add up to 5. Adding up/subtracting down is the process of finding pairs that add up or subtract to goal x- or y-coordinate.

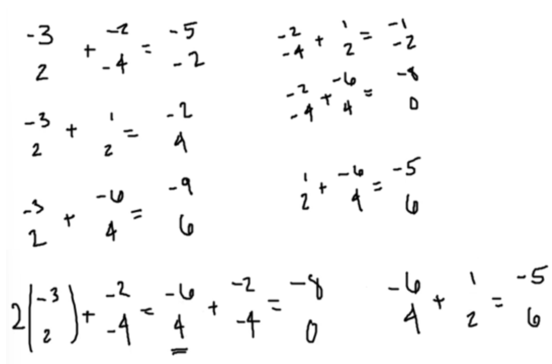


Figure 1. Example of pairing and scaling by Betty to generate possible goal locations

Structuring Across Dimensions

In this section, I investigate activities that students engaged in both 2D and 3D or activities that allowed them to connect the 2D and 3D worlds. Betty illustrates several of these strategies while looking at the linear combination of $v(-2, -4, -3) + o(3,0,0)$ and trying to determine all the possible goal locations [linear combinations] of these two vectors. Betty stated that she

> looked at where each of the combinations fell in the different planes. First of all, the 2D plane and then the 3D plane. This combination here [points to $v(-2, -4, -3) + o(3,0,0)$] has at least the capacity to go through all four 2D planes. Oh I mean quadrants [she manipulates the sliders to illustrate how it can go into all four planes from a top down view in Figure 2a]…and it can go into the upper level of the *z*-plane [she adjusts her view, Figure 2b, and adjusts the scalars to illustrate the point moving up and down along the blue axis].

First, Betty adjusted the camera to obtain a top-down viewpoint [Figure 2a]. From this viewpoint, she adjusted the scalars in the linear combinations to illustrate that the linear combination $v(-2, -4, -3) + o(3,0,0)$ visited all four quadrants in *xy*-plane, indicating that **reasoning about quadrants**, describing vectors using the signs or quadrants that they are located. After this, she rotated the camera and adjusted the scalars to illustrate that the point also went to the "upper level of the z-plane." I call this **traveling along an axis**, which is using checking to see if a point travels outside a particular. Essential to Betty's reasoning is here **adjusting the viewpoint,** which consists of rotating the camera to obtain an optimal view for checking a particular property or technique.

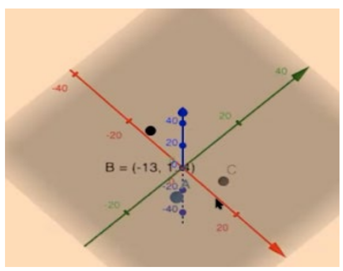
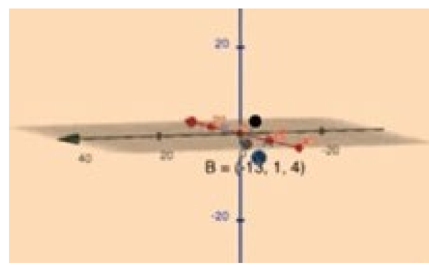
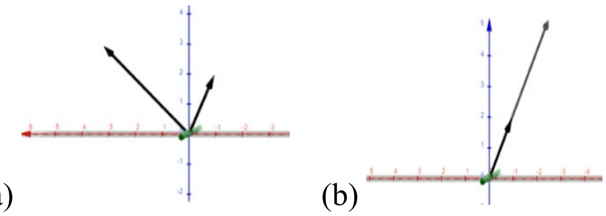
(a) 
(b) 

Figure 2. Betty's analysis of the linear $v(-2, -4, -3) + o(3,0,0)$]

Marti, Betty's partner, also employed adjusting the viewpoint to see connections between the 2D and 3D worlds. In 2D, Marti was looking at the linear combination $f<-2,-4> + g<1,2>$ and noticed that it does "not matter how you multiply it seems to be moving along the same line. You can move it along both of the sliders, and it still moves in the same direction." This was in contrast to the linear combination $a<-3,2> + b<1,2>$ which "seems to cover the entire 360 [degrees] of the graph depending on what you multiply them by." Marti then began to classify other linear combinations according to whether or not they were in the **same line** or **rotated around 360 degrees**. This became an important design consideration when Marti started to design her 3D game. When creating vectors to reach a goal she wanted vectors that "had as much of a reach around the graph as possible." Marti tested linear combinations by rotating the camera to viewpoint that projected onto a plane and adjusted the scalars to ensure they traveled around 360 degrees [Figure 3a]. When two linear combinations aligned she changed the vectors in one of the linear combinations [Figure 3b]. This indicates that Marti was reflecting upon Marti's 2D work when creating vectors in the 3D game, in particular the delineation between linear combinations that traveled in the **same line** or rotated **360 degrees**.



(a) (b)

Figure 3. Marti looking for linear combinations that are in the same line or rotate 360 degrees

Structuring across dimensions is important because it captures activities that participants thought worked in both 2D and 3D dimensions, such as quadrants and same line/rotations. Also, adjusting the camera view was an important activity that allowed participants to build connections between 2D and 3D. Betty was able to see the connections using different viewpoints, and coordinate these viewpoints which occurred. She coordinated a planar projection viewpoint with an isometric viewpoint to deduce that the goal locations traveled off the projected plane. This category answers the research question 2, how students structure space across dimensions.

Structuring by Design

This section presents design choices students made while creating their own vectors for different difficulties in the 2D and 3D game. First, students would **attend to coordinate properties** which encompasses designing vectors by selecting coordinates with specific numerical properties such as being big numbers (such as 6 through 9), small numbers (such as 1 through 3), or zeroes for individual x-, y-, and z-coordinates. Since the game only allowed x- and y-coordinates that were between -20 and 20, several groups only considered possible goals locations that Marti described as "**within the grid**." This category was important because it provided an activity for students like Betty to generate goal locations and noting that <−3,2> + 3<−6,4> = <−21, 14> and "found it would be going off the 20 and it would be going off the mark so it wouldn't count" as a possible goal location. While designing vectors to reach the goal of <-3,5>, Alex and Angel decided that they wanted to design a "**bluff vector**" which was intentionally designed to not help the player reach the goal. For example, they used the vector <5,-3> as a bluff vector. Finally, scaling **vectors according to difficulty** refers to the design decision to have more vectors to be scaled with the difficulty of the game. For example, Gabby and Dee present easy goal locations as (2,0,1) + (-1,2,3), medium difficulty as b(2,0,1) + (-1,2,3) or (2,0,1) + b(-1,2,3), and hard difficulty as b(2,0,1) + c(-1,2,3). Structuring by design was important because it presented goals created by students in the design of the game that generated activity allowing them to notice patterns.

Discussion and Limitations

I initially set out to categorize the different types of activities that students engage in after playing, reflecting, and designing a 3D version of the game *Vector Unknown* in regards to structuring space and linear combinations. Now that I have presented these results, I want to reflect on the ways in which my findings can inform a more general definition of structuring space. Examining my data, I found that students engaged in the following activities:

1. Worked with a variety of objects, including coordinate vector representations such as $\begin{bmatrix}1\\2\end{bmatrix}$; points and arrows on a 2D or 3D plane; symbolic representations of vectors; representations of paths in 3D space.

2. Took actions such as **pairing** and **scaling** vectors both on paper and in GeoGebra (through adjusting the sliders), **changing the viewpoint**, and creating vectors using **adding up/subtracting down**.
3. Set design goals that shaped their structuring activity such as structuring by design with **fitting the grid**, **having bluff vectors**, or wanting vectors with specific **coordinate properties**.
4. Made observations such as recognizing if **vectors span all quadrants** or lie along the same line in **structuring across dimensions** and observing **coordinate properties** about individual vectors.

Taking actions, setting goals, creating objects, and making observations form the skeleton of structuring as its key components. The act of structuring involves doing one or more of these activities and re-evaluating the connections between the remaining elements. In this way, introducing an action like changing the viewpoint might have the students set new goals such as finding the best viewpoint, work with different objects such as projections down to 2D space, and make new observations such as being able to classify 3D vectors as in the same line or a rotation. Now the foundation of a particular structuring is the starting activity that sets these parts in motion with example actions, goals, objects, and observations. Playing the game *Vector Unknown* provides the initial set of actions, goals, objects, and observations for structuring space under linear combinations for this study. Here the actions are the moves by the player (scaling and choosing vectors), the goal is to reach the basket, the initial objects are the vectors and their geometric representation, and the observations are highlighted by feedback provided to the player from the game. Playing the game and asking questions about the game introduce or modify each of these components resulting in a student structuring of space.

I observed that the idea of linear independence was observed through the idea of same line/visiting all quadrants/rotating 360 degrees. Quadrants appearing in both this analysis and the video game strategies (Author, 2020) indicate that they might be a fruitful starting point for understanding linear dependence. Additionally, I found that understanding viewpoint and, in particular, coordinating viewpoints was crucial to the participants transitioning between 2D and 3D space. This is important because looking at viewpoints is a paradigm that is often relegated to dynamic geometry software (such as GeoGebra) and video games and may not be covered in classes without these tools. Future work will investigate this data more thoroughly doing an in-depth analysis of the different viewpoints utilized and linking them to the mathematical activity for each viewpoint. I see view this as important to developing the geometric intuition that is considered valuable by Dogan-Dunlap (2010) and Paraguez and Oktac (2010). Finally, I want to stress the importance of the game in the student's structuring by contributing elements such as fitting the grid and designing bluff vectors. These design ideas from the game provided students with goals and a reason to generate vectors which they then used to make generalizations and explore advanced mathematical concepts creating a complex network of ideas (Gee, 2003).

The project was conducted with only three groups of students and thus does not capture the entirety of structuring space. Additionally, the students were introduced to GeoGebra at different times during the interview and in different versions. Some students utilized a GeoGebra applet in 3D, while others chose to use the default program. Because of this, their structuring when working in GeoGebra was likely impacted by the tool and when it was introduced.

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