USING ABSTRACTION AS A LENS TO ANALYZE INSTRUCTIONAL MATERIALS

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Over the past few decades, researchers have adopted forms of abstraction introduced by Piaget to build explanatory models of student and teacher knowledge. Although Piaget's forms of abstraction have proved productive for developing models of knowledge, their broader applicability to mathematics education remains an open question. In this brief report, we extend these forms of abstraction in order to analyze hypothetical outcomes of teachers' enactment of instructional materials.

Keywords: Cognition, Learning Theory, Curriculum

Piaget's (1970, 2001) genetic epistemology has played a critical role in mathematics education via researchers adopting his theory to develop models of students' mathematics, models of teachers' mathematics, and models of student-teacher interactions. Researchers carrying out this work have provided important insights into those meanings that prove productive for students' mathematical development, as well as those meanings that constrain students' mathematical development (Steffe & Olive, 2010; Thompson, 2013). Furthermore, these researchers have provided useful ways to characterize teaching in terms of teacher knowledge necessary to build upon students' ways of thinking (Liang, 2021; Tallman, 2015).

An important construct spanning these contributions is that of abstraction. Stated generally, abstraction is the process by which an individual develops stable, generalized knowledge structures. To Piaget, abstraction provided a vehicle to developing precise accounts of knowledge development while also articulating generalized differentiated characteristics of knowledge structures. Piaget proposed several forms of abstraction including empirical, pseudo-empirical, reflecting, and reflected abstraction (Montangero & Maurice-Naville, 1997; Piaget, 2001). Mathematics educators have adopted these forms to provide differentiated accounts of student and teacher knowledge in numerous contexts (Ellis et al., in preparation; Tallman & O'Bryan, in preparation; Thompson, 1994).

Given the usefulness of Piaget's forms of abstraction for developing accounts of student and teacher knowledge, it is plausible that the forms of abstraction are productive for analyzing other aspects contributing to the teaching and learning of mathematics. In this brief report, we extend Piaget's forms of abstraction in order to analyze instructional materials. Specifically, we analyze two secondary teachers' instructional materials for teaching quadratic growth in order to develop hypotheses of the knowledge students may abstract from engaging in those materials. Because this is a brief report, we close with potential implications of this work and future directions building on this preliminary analysis.

Background

Ellis et al. (in preparation) and Tallman and O'Bryan (in preparation) synthesized Piaget's forms of abstraction and described how mathematics education researchers have adapted these forms of abstraction to be viable in their areas of research. Empirical abstractions primarily concern observables and foreground sensory-motor experience, and reflected abstractions rest on a subject's consciousness of their ways of operating. These two forms of abstraction are critical aspects of knowledge development, but they are less relevant to the analysis of secondary mathematics instructional materials when compared to the two forms of reflective abstraction that are pseudo-empirical abstraction and reflecting abstraction.

Speaking on pseudo-empirical abstraction, Piaget (1977) explained, "When the object has been modified by the subject's actions and enriched by the properties drawn from their coordinations…the abstraction bearing upon these properties is called 'pseudo- empirical' because…the facts it reveals concern, in reality, the products of the coordination of the subject's actions…" (p. 303). To Piaget, a critical aspect of pseudo-empirical abstraction is that such an abstraction requires the presence of perceptual material or observables and foregrounds actions on that available material. Drawing on the work of Moore (2014), Ellis et al. (in preparation) argued for extending Piaget's construct of pseudo-empirical abstraction so that "perceptual material" or "observables" includes the products of activity, even if these products of activity are purely cognitive. As they illustrated, such an extension of pseudo-empirical abstraction is productive for developing viable models of students' mathematics at numerous levels.

Piaget's distinction between pseudo-empirical abstraction and reflecting abstraction rested on the extent perceptual material or observables are required. Ellis et al. (in preparation) noted that the broader interpretation of pseudo-empirical abstraction provided above requires a more restrictive framing of reflecting abstraction. A primary difference between these two forms of abstraction is that while the source material for pseudo-empirical abstractions is perceptual material or the result of actions, the source material for reflecting abstractions is the coordination of a subject's actions themselves. Reflecting abstractions thus involve differentiating an action from the effect of an action so that the actions themselves can be projected to a level of representation and taken as objects of thought (Ellis et al., in preparation; Tallman & O'Bryan, in preparation; Thompson, 1994). As we illustrate with our task analysis, these differences in the source material for a subject's abstractions have important implications for their learning.

Project Setting and Methods

The current work is situated in a multi-year project investigating students' generalizing including the ways in which teachers support generalizing in their teaching (Ellis et al., in press; Ellis et al., 2017). Our approach to generalization is cognitive, drawing on an actor-oriented perspective as detailed by Ellis et al. (in press). The project's guiding research questions are: What are the opportunities for generalizing in classroom settings? Specifically, what types of instructional moves, student engagement, and enacted tasks support classroom generalizing? The current paper addresses these questions by investigating the abstractions potentially supported during the implementation of instructional materials.

The project involves two high school teachers and two middle school teachers. We concentrate this paper on the two high school teachers' instructional materials in order to restrict our focus to one content area. We analyzed the instructional materials using conceptual analysis (Thompson, 2008) with a guiding framework of the forms of abstraction identified above. At its most general level, conceptual analysis involves answering the question, "What mental operations must be carried out to see the presented situation in the particular way one is seeing it?" (von Glasersfeld, 1995, p. 78). With respect to analyzing curricular materials, conceptual analysis involves developing hypothetical accounts of *realized curriculum* (Kilpatrick, 2011) or *conveyed meanings (Tallman & Frank, 2020)*. This is accomplished via generating and interpreting "typical" solutions to the instructional materials using the lens of abstraction in combination with ways of reasoning held by secondary mathematics students as suggested by research (Ellis, 2011; Ellis & Grinstead, 2008; Fonger et al., 2020; Moore et al., 2019).

Tasks and Task Analysis

The two secondary teachers' instructional activity focused on quadratic growth. The instructional activity explored a sequence of discretely growing shapes (see one example in Figure 1) with the intention that students identify patterns in quantities' values including their first- and second-differences. The primary goal and generalization of the activity was to identify that for successive equal increases in Q_A of a situation (e.g., sail size), Q_B (e.g., sail area) increases by constantly increasing amounts, and that such a covariational relationship is modeled by a quadratic relationship. We discuss hypothetical pseudo-empirical and reflecting abstractions against the backdrop of the aforementioned goal. Underscoring that the forms of abstraction are cognitive constructs, we discuss each form of abstraction using a typical solution that involves a student generating a table of values, first-differences, second-differences, and a formula.

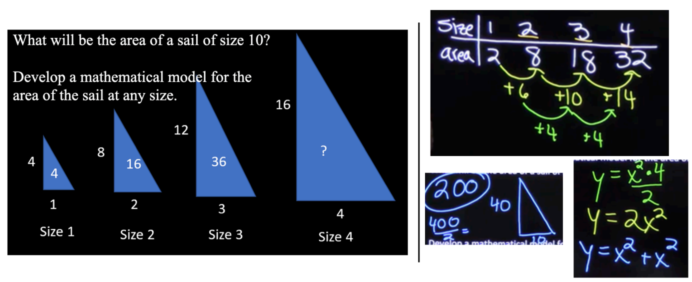

Figure 1: Example Activity (left) and a "Typical" Student Solution (right)

Pseudo-Empirical Abstraction

After working a series of activities like that presented in Figure 1, a student might observe that each time they obtain constant second-differences in a quantity, a quadratic formula models the situation. Recall that pseudo-empirical abstractions are those abstractions that foreground "perceptual material" or "observables" including the products of activity. In the case of the example activity (Figure 1), the products of activity include a table of values and a quadratic formula. Thus, the observation of the student would be a pseudo-empirical abstraction if their association is strictly based on noticing that constant second-differences were accompanied by a quadratic formula. The abstraction consists of an indexical association between constant-second differences and a quadratic formula with no logico-mathematical operations forming the basis for that association. The actions that produced the table of values and formula are inconsequential to the abstraction except in that they yielded an outcome or product to act as source material for the student's abstraction. We contrast this with a reflecting abstraction in the next section.

Reflecting Abstraction

A limitation of pseudo-empirical abstractions stems from the abstraction foregrounding the products of actions rather than the actions themselves. For instance, and based on our experiences with students, the abstraction described in the previous section often results in the student associating a quadratic formula with constant second-differences regardless of how the other quantity's values are ordered in a table (e.g., non-constant first-differences in Q_A that produce constant second-differences in Q_B). In the case of the example activity and solution in Figure 1, a reflecting abstraction that foregrounds the coordination of actions and their results would involve a student reflecting upon both the quantitative referents of their tabular activity, as well as how their relationship necessitates a quadratic formula.

With respect to the tabular activity, this would involve the student conceiving first-differences as the amount by which a quantity increases (or decreases) and second-differences as the amount by which a quantity's increase increases (or decreases) as shown in Figure 2. Furthermore, because a reflecting abstraction foregrounds the coordination of actions as opposed to the products of actions, the student's abstraction would include awareness that the constant "+1" increases in size are intrinsic to the constantly increasing increase in area. With respect to a quadratic formula model, a reflecting abstraction involves understanding how the aforementioned quantitative relationship necessitates a second-degree polynomial. Although the connection between the two is not trivial, researchers (Ellis, 2011; Ellis & Grinstead, 2008; Fonger et al., 2020) have illustrated its feasibility for students including those in middle grades.

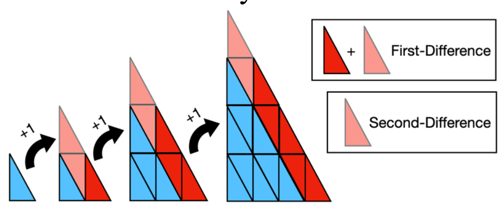

Figure 2: Conceiving first- and second-differences quantitatively

Discussion and Future Work

Although we are not aware of studies that have used the aforementioned forms of abstraction to develop hypothetical accounts of student activity in the context of teachers' instructional materials, mathematics education researchers and teachers have been sensitive to the role of abstraction in instructional design. For example, it is impossible to read the collective works of Steffe or Thompson and not sense the forms of abstraction directly informing their work even when not explicitly mentioned. As another example, Oehrtman (2008) provided a more general description of how Piaget's notion of abstraction can inform a layered sequence of activities so that students have the opportunity to reflect upon and identify common structures in their actions across a variety of contexts. In each of these cases, researchers leveraged abstraction in the context of their own research-based work and design. We find it important to include a complementary focus on teachers' extant instructional materials, as those materials play a significant role in students' educational experiences.

We illustrated that an abstraction framing provides a way to analyze instructional materials and produce differentiated accounts of knowledge development. Moving forward, we envision several productive avenues to continue investigating the viability of this framing. First, the current report is limited to one content area. Future work should look to extend the framing to other content areas including those not within secondary mathematics. Second, our analysis of the instructional materials consists of *hypotheses*. A more holistic account should include a focus on students' realized abstractions, as well as the role of teacher knowledge and instructional moves in students' construction of those abstractions. Third, we envision that an abstraction framing can provide a cognitive-focused approach to modifying instructional materials and their implementation. For instance, based on analysis like that provided here and then investigations into students' realized abstractions and aspects of instruction contributing to the construction of those abstraction, researchers and teachers can look to modify instructional materials to better support students' reflecting (and reflective) abstraction of productive meanings.

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