“The Shape Will Have No Volume”: Relationships Students Observed about Determinants in a Dynamic Geometric Applet

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We discuss two Dynamic Geometry Software applets designed as part of an Inquiry-Oriented instructional unit on determinants and share students' generalizations based on using the applet. Using the instructional design theory of Realistic Mathematics Education, our team developed a task sequence supporting students' guided reinvention of determinants. This unit leverages students' understanding of matrix transformations as distortion of space to meaningfully connect determinants to the transformation as the signed multiplicative change in area that objects in the domain undergo from the linear transformation. The applets are intended to provide students with feedback to help connect changes in the matrix to changes in the visualization of the linear transformation and, so, to changes in the determinant. Critically, the materials ask students to make generalizations while reflecting on their experiences using the applets. We discuss patterns among these generalizations and implications they have on the applets' design.

Keywords: Determinants, dynamic geometry software, inquiry, student reasoning

Matrix determinants are often taught formulaically, obscuring geometric connections between determinants and linear transformations. The Inquiry-Oriented Linear Algebra (IOLA) curricular materials (Wawro et al., 2013) build from experientially real task settings that allow for active student engagement in the guided reinvention of mathematics through student and instructor inquiry (Gravemeijer, 1999). The IOLA determinants task sequence uses distortion of space as an experientially real starting point to support students’ development of determinant as a measure of (signed) multiplicative change in the area. The sequence uses GeoGebra applets that allow students to actively explore the geometric effects of changing 2x2 and 3x3 matrix transformations and note their effect on the determinant. Through exploration of the applet, students observe links between the determinant and concepts such as linear (in)dependence, span, and invertibility, as well as how changes in matrix entries impact the determinant. In this study, we investigate the research question: Based on interacting with our curriculum's dynamic geometry applets, what relationships do students observe between matrices, determinants, and geometric objects transformed by the associated linear transformation?

Theoretical Background and Literature Review

According to Axler (1995), “determinants are difficult, non-intuitive, and often defined without motivation” (p. 139). Aygor and Ozdag (2012) found that students have difficulty identifying what happens to the determinant when switching rows, multiplying a column by *k*, row-reducing a matrix, and performing column operations. This might be because of difficulties relating the numeric, algebraic, and geometric aspects of the determinant (Durkaya et al., 2011) or the reliance of the determinant on bilinearity (Donevska-Todorova, 2016). These difficulties would be mitigated by connecting to students’ existing reasoning and understanding, as is consistent with the curriculum design theory of Realistic Mathematics Education (RME; Freudenthal, 1991). Adopting this theoretical framing for our curriculum design, we identified students’ understanding of matrix transformations as distortions of space as an experientially real starting point for developing notions of determinant. We sought to design an activity that could support students in establishing an understanding of determinants as a measure of how matrix transformations distort space and then leverage these connections toward more general properties of determinants. We situate this work within our research group's design research (NSF DUE 1915156, 1914841, 1914793), referred to as a Design Research Spiral, in which the design team iteratively drafts, implements, reflects, and refines the task sequence as guided by multiple design research theories at each cycle of refinement (Wawro et al., 2022). This paper presents materials and results from the penultimate cycle of refinement for this unit.

In order to support students' generalizing from their situated activity, we turned to Dynamic Geometric Software (DGS), which synchronously display dynamic representations of numeric, geometric, and algebraic aspects of the determinant. This makes DGS a suitable tool for supporting students' exploration of determinants and connecting this to geometric and algebraic knowledge. research has shown that DGS applets allow students to investigate, visualize, make predictions, calculate, simulate, and generalize certain situations, all of which are critical practices for inquiry (Gol Tabaghi & Sinclair, 2013; Gol Tabaghi, 2014; Greefrath et al., 2018; Hollenbrands, 2007; Paoletti et al., 2020; Zandieh et al., 2018). In linear algebra, applets, games, and simulations have been used to explore key topics such as linear combinations (Mauntel et al., 2021), eigentheory (Gol Tabaghi & Sinclair, 2013; Gol Tabaghi, 2014), and determinants (Donevska-Todorova, 2012, 2016; Donevska-Todorova & Turgut, 2022).

Donevska-Todorova (2012) used DGS to support students' exploration of determinants in a teaching experiment setting and found that DGS can help build meaningful connections between the determinant and conceptions of area in 2D. The author attributes the DGS's ability to simultaneously display numeric and geometric feedback as important for connecting to students' prior geometric understanding. However, Donevska-Todorova's setting is more focused on direct instruction and approaches the determinant as a way to calculate the area of a parallelogram at the origin whose sides correspond to the column vectors of the matrix, rather than connecting the geometry to the linear transformation defined by the matrix. In our materials, and consistent with the above cited literature, we designed the materials so that students contextualize the DGS relative to their existing notions of linear transformations. The goal is that this additional context will provide students with a foundation for the connections they make while using our applet.

The Determinants Task Sequence

The main goal of this unit is to build from students' knowledge of matrices as representations of linear transformations towards a conceptualization of the determinant of a 2x2 or 3x3 matrix as a measure of (signed) multiplicative change in area or volume, respectively. Task 1 is designed to support students toward suggesting *change in area* as a way of quantifying distortion that objects undergo from specific matrix transformations. In Task 2, students reinvent the 2x2 determinant formula so that by the end of Task 2, when the term *determinant* is introduced, the class has developed the notion of the determinant of a 2x2 matrix A as: (1) the signed area scaling factor of the linear transformation defined by A [i.e., the multiplicative change in area]; (2) the signed area of the unit square's image under A; (3) the signed area of the parallelogram created by the columns of A; and (4) $det(A) = ad - bc$, for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The main goal of Task 3 is for students to reinvent $det(A^{-1}) = 1/det(A)$ for invertible matrices by coordinating their knowledge of invertible linear transformations with their developing understanding of determinants as a measure of change in area. Tasks 1 and 3 provide space for students to use their understanding of composition of functions to reinvent $det(AB) = det(A)det(B)$ for matrices A and B and justify them via change in area lines of reasoning.

In Task 4, students further explore the geometric interpretation of matrix transformations and their determinants via GeoGebra applets for 2x2 and 3x3 matrices, which can be found at https://www.geogebra.org/u/iolalinearalgebra. Each applet consists of a matrix, sliders to control each entry in the matrix, a real-time calculation of the matrix determinant, and a real-time dynamic parallelogram or parallelepiped showing the image of the unit square or cube under the matrix transformation. As shown in Figure 1, both applets have sliders (a) & (h) that students use to change the matrix entries. The applets also calculate the determinant of the matrix in real-time as the user changes the entries of the matrix (b) & (i). In the 2x2 applet, users can see (and alter) a geometric representation of a preimage object (c) composed of a yellow vector, blue vector, the parallelogram they form, and the inscribed ellipse (defaulted to the standard basis vectors, unit square, and its inscribed circle). Users can change this shape by moving the terminal endpoints of the yellow and blue vectors (d). These color-coded vectors correspond with their images under the transformation (e). In the 3x3 applet, the preimage is fixed as the unit cube (defined by the standard basis vectors) and is not shown or alterable because of the complexity of programming such an object in GeoGebra. In both applets, the image of the respective image is shown on the right side of the screen (f) & (j) and changes color from green to red when the matrix determinant is negative. Users can reset the applet (g) & (k), which will revert the matrix back to the identity matrix and, in the 2x2 applet, will reset the preimage to the unit square.

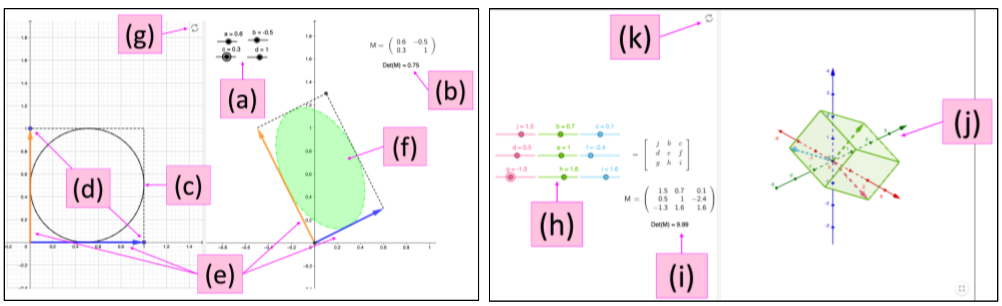

Figure 1. Screenshots of the two determinant applets, with markers (a) - (k) to help explain the applet components.

With the applet components, students are able to create any matrix within the constraints of the parameters (between -4 and 4, in increments of 0.1) and see how the changes affect the distortion of the image of the unit square or cube. Consistent with Rumack and Huinker (2019), Task 4 explicitly champions the role of mathematical curiosity as students explore, observe, conjecture, and justify; this allows students' mathematics to be what is leveraged and furthered. Adopting the RME design heuristic of guided reinvention, the instructor's role includes facilitating students' open-ended activity toward more organized and focused statements or generalizations connecting changes in matrices to changes in the distortion of the space and how they relate to determinants.

Methods

The data for this study come from in an in-person introductory linear algebra class at a large, public, research university in the Mid-Atlantic region of the US. There were 27 students in the course, 13 of whom gave consent and completed the assignment analyzed here. In the university system, eight of these students chose he/him/his pronouns (pseudonyms begin with "M"), four chose she/her/hers pronouns (pseudonyms begin with "W"), and one did not choose pronouns (pseudonym P1). Most were second-year students by credit hours and were general engineering majors. The prerequisite was a B or higher in Calculus I or a passing grade in Calculus II.

The data analyzed and present herein are student responses to a written reflection. After most class sessions, students were asked to complete a reflection by the end of the day and submit their work via an online learning management system. Students were asked to spend 5-10 minutes on a reflection, for which full credit was awarded based on effort rather than correctness. The following is the reflection prompt analyzed in this paper: "We covered a lot of ground in class today! Let's focus on the applets and what they helped you understand. Give at least two observations you made from using either/both applet(s). State it in words / mathematical symbols and illustrate with a snapshot from the applet(s) to help convey it. If you have started to think about "why" for the observation, go ahead and include that, too." The prompt also included the URLs for the applets and a reminder that the class slides were on the course cloud-based folder.

To analyze the data in service of our research question, we aimed to understand what observations students made from their interactions with the determinant applets. Extending our previous preliminary analysis with a similar data set (Kerrigan et al., 2012), we first focused on three aspects of students' responses that align with features of the applets that users attend to: a user can notice (changes in) the entries of the matrix, (changes in) the value of the determinant of the matrix, or (changes in) the graphical appearance of the transformed object. For brevity, we label these as pertaining to the matrix [M], the determinant value [D], or the graphical imagery [G]. For each student response, we coded phrases as M, D, G, or "other." Distinctions between phrases were indicated by transition words such as "if," "then," "and," and "because," as well as by punctuation use. To establish viability of the coding scheme, two of the authors independently analyzed about half of the data according to these codes, and obtained a high level of agreement. These authors explained their codes and rationales to the third author and then all authors coded the remainder of the data set, again reaching near-total agreement. We expected and thought it natural that an observation from the applet would likely involve at least two phrases (e.g., "when I change this entry in the matrix" [M], "the determinant stays the same" [D]). Thus, we examined the data for relationships between phrases, looking for any causality or logical structure in the relationships (e.g., the aforementioned example has an implied if-then structure, so it would be coded as [M⇒D]). We referred to the phrases paired with their logical structure and additional supportive justifications (if they existed), as observations. Such an analysis is important to us as instructors and curriculum designers because the structure in these initial student observations is foundational in setting the stage for the co-development of key theorems (such as "Suppose A is a nxn matrix; A is not invertible $\Leftrightarrow det(A) = 0$"), and they grow out of students inquiring into the mathematics and the instructor inquiring into student reasoning (Rasmussen & Kwon, 2007).

Results

From the phrases within the 13 student responses, we found 43 phrases pertaining to the matrix [M], 41 phrases pertaining to the determinant [D], and 24 phrases pertaining to the graphical appearance of the transformed object [G]. Overall, we found there to be 42 observations. We organize the remainder of this section according to summaries of each of the three main phrase categories and then a summary of the various student applet observations.

Summary of results at the individual phrase level

First, 43 of the phrases pertained to aspects that students noticed regarding the matrix [M] in the 2x2 or 3x3 applet. 16 of these phrases related to the columns of the matrix forming a linearly dependent set. Although only 5 of the phrases say linear dependence explicitly, the other phrases give specific examples of linear dependence: the matrix entries being the same number (2), at least two columns being scalar multiples (2), and the matrix containing a column of zeros (7). As researchers, we grouped the latter three as types of linear dependence, but we cannot know if the students who wrote them were considering that concept as they wrote their phrases. The bulk of the remaining phrases coded with [M] had the common trait of focusing on changes in the matrix entry. Students wrote phrases about individual matrix entries or multiple entries (e.g., the values of the diagonal components), as well as about interchanging two columns, and multiplying row(s) or column(s) by a constant. As previously mentioned, the identity matrix is what first appears when opening or refreshing the applet, and the students are able to manipulate each individual matrix component. To reason about the ideas related to linear dependence and the ones related to multiplying a row, column, or the whole matrix by a constant, students had to set the entries to be the values they desired. On the other hand, for phrases that focus on "as an entry changes," the phrase has a more dynamic character. These are further detailed in the next section.

Second, 41 of the phrases pertained to observations the students made regarding the value of the determinant [D] that appeared in either the 2x2 or 3x3 applet or its computation. 15 of the phrases focused on a zero determinant value, 6 phrases focused on noticing a negative determinant value or a change in its sign, and 2 phrases highlighted an unchanging determinant value. Third, 24 phrases pertained to the graphical appearance of the transformed object [G] in either applet. Of these, 8 mention the resulting parallelepiped's volume (one also mentioned the resulting parallelogram's area in the 2x2 applet); more specifically, all but one mention a volume (or area) of zero. We group another 7 of the phrases together because they relate to the object losing dimension (e.g., the vectors are on the same line), and 5 phrases were other observations about the resulting image (such as flipping orientation). Finally, 2 students included snapshots of the 2x2 applet, 7 included snapshots of the 3x3 applet, and 4 did not include any snapshots. We did not code the images with [M], [D], or [G] unless a student explicitly referenced them in their written explanation. This occurred 4 times (twice for two students), all of which were coded [G].

Summary of results at the observation level

The prompt that students responded to stated "Give at least two observations you made from using either/both applet(s)." We referred to the phrases paired with their logical structure and any supportive justifications as observations. Consider student M1's response as an exemplar:

> "If a column has all zeros, the determinant is 0, and if the determinant is zero, the shape has no volume in the third dimension. Thinking about the determinant as (new volume)/(old volume), we can algebraically see that the numerator must be zero for the quotient to be zero. Also, if one of the column vectors is the zero vector, the shape does not span all three dimensions so it cannot possibly contain a volume"

We coded "If a column has all zeros" as [M], "the determinant is zero" as [D], and "the shape has no volume in the third dimension" as [G]. In the first part of M1's response, we coded two observations with these three phrases: [M⇒D] and [D⇒G]. The middle portion of his response, in which he brings in knowledge about the determinant as "(new volume)/(old volume)" and "the numerator must be zero for the quotient to be zero" was coded as "other" because it is something that the student could not directly observe in the applet itself. We coded "one of the column vectors is the zero vector" as [M], "the shape does not span all three dimensions" as [G], "it cannot possibly contain a volume" as [G], and paired those as observations [M⇒G] and [G⇒G].

Overall, our analysis shows that a vast majority of the observations that the students made began with a matrix statement [M]. In fact, only 5 of the 42 observations started with a statement about the graph [G] or the determinant [D] (see Figure 2). Relating back to the previous section, the [G] phrases mentioned were about the shape or the volume of the resulting parallelepiped.

Observation		
Phrase One	*implies*	*Phrase Two*
The shape does not span all three dimensions [G]		It cannot possibly contain a volume [G]
At least 1 dimension of the image would be 0 [G]		To the whole area/volume to be 0 [G]
When a matrix creates just a line or a plane [G]		The determinant would be zero [D]
The determinant is zero [D]		The shape has no volume in the third dimension. [G]
The determinant is zero [D]		Image was either flat or a straight line [G]

Figure 2. The 5 observations that started with a graphical statement [G] or determinant statement [D].

The most common type of observation was what we refer to as [MDG], by which we mean the student observation was of the form "[M] implies [D], which makes sense because of [G]." For example, M8 wrote, "If any of the columns in the 3x3 are LD with another, the det(M) will be zero due to there being no volume, only area or a line." He noticed when a certain matrix characteristic [M] occurred that a certain determinant value occurred [D], which he explained as sensible because of the corresponding graphical imagery [G]. We posit this likely corresponds to the nature of student activity while interacting with the applet: first manipulate the matrix, notice things about the determinant, and explain that in terms of what you see geometrically. Another example of an [MDG] is W4's observation (coding in the data): "One observation I made was that if the set is linearly dependent [M] or has two or more vectors that are scalar multiples [M], the determinant will be zero [D]. I found out after our discussion that this is because the vectors will be all on the same line [G] and therefore the shape will have no volume [G]."

As stated above, some observations conveyed a dynamic character, meaning that the student was most likely actively changing something in the applet while noticing the associated impact of that dynamic action. For example, M3 stated, "When observing the 3x3 applet in class today, I noticed that entries a_{11}, a_{22}, and a_{33}, as well as sometimes a_{13} and a_{31} were the only entries that changes the volume of the image," and M9 stated, "when moving the last column the 'footprint' didn't move, it just wobbled around like jelly." Both these observations are of the type [M⇒G]. We hypothesize that, for instance, M3 was dragging the slide for entry a_{11} and noticing the simultaneous change in the parallelepiped's volume. Or that M9 was, for instance, dragging the slider for entry a_{13} (called "c" in the applet) and noticing the simultaneous movement in the "top" of the parallelepiped. We note that the applet allows only one slider to be dragged at a time. As previously mentioned, 7 phrases were about a matrix with a column of zeros; each of these were part of an observation about the resulting determinant being zero. Because the default matrix in the applet is the identity matrix, it is likely that the observation was able to be made by only dragging one slider because each column of the identity matrix only has one nonzero entry.

In comparison to the dynamically oriented observations, some required multiple changes in the matrix entries and thus may have required many separate actions within the applet before an observation was made. For example, M6 wrote, "When I change two columns in the 3x3 applet, the determinant flips sign. It works the same on the 2x2 applet. I think it's because it flips the shape," which we coded as [MDG]. W6 wrote, "Another observation from the applet is that if you multiply one row by a constant the determine [sic] increases by that constant, and if you multiply the entire matrix by a constant, the determinant is multiplied by k to the power of n. For example, in a 2x2 matrix the determinant would be multiplied by k^2" which we coded as three instances of [M⇒D]. The actions needed to carry out both M6's and W6's second observation – switching one column with another and multiplying a matrix by a constant – require dragging multiple applet sliders *and* comparing an existing determinant with a resulting determinant. W6's first observation – multiplying a row by a constant increases the determinant by a factor of that constant – may have been made from only changing a row of the identity matrix or other diagonal matrix, in which only dragging one slider would have been needed for the observation.

Discussion

As design researchers, part of our work involves reflecting on student observations with respect to our curriculum design goals. Within the design research spiral (Wawro et al., 2022), considering student reasoning during tasks in light of the overall learning goals informs revisions that we make to the task sequence and supplementary materials, such as the applets. The first overarching learning goal include of this task sequence is (1) having students deepen their understanding of the geometric interpretation of linear transformations in $\mathbb{R}^2$ and in $\mathbb{R}^3$ through observing the impact that varying the matrix entries has on the determinant and on the transformed image. The second overarching learning goal is (2) for students to actively engage in the guided reinvention of key properties of nxn matrices A and B, such as: ($2a$) $det(A) = 0 \Leftrightarrow$ columns of A are linearly dependent, ($2b$) $det(A)$ is positive or negative according to whether the linear transformation defined by A preserves or reverses the orientation of the transformed objects $\Leftrightarrow$If B is obtained by interchanging two rows (or two columns) of A, then $det(B) = -det(A)$, ($2c$) If B is obtained by multiplying a row (column) of A by scalar k, then $det(B) = kdet(A)$, and ($2d$) If $B = kA$ for a scalar k, then $det(B) = k^n det(A)$. Through reflection on data collected through various parts of the design research cycle, including the data analyzed in this paper, we feel confident that the applets have strong potential in helping an instructor achieve (1) and ($2a$) with their students. In particular, the results here show ($2a$) as a strong connection students make while working with the applet. These results also indicate that, when students are provided specific ideas to explore in the applet (namely, switching two columns or rows, multiplying a row by a constant k, or multiplying the whole matrix by a constant k), they can use the applet to make conjectures equivalent to ($2b$) - ($2d$). We would, however, like to consider ways that refining the applet might facilitate students' exploration of ($2b$) - ($2d$). For example, including features that can swap columns, swap rows, scale columns, or scale rows (rather than requiring individual entries to change one at a time), might help students develop ($2b$) - ($2d$) entirely on their own, rather than requiring targeted exploration prompts.

Our results showed that most student observations began with statements about the matrix [M] in either the 2x2 or 3x3 applet. We believe this is sensible, given the current design of the applets. The current aspect that is the most interactive is adjusting the matrix component values. For instance, students can rotate or zoom in on the transformed parallelepiped in the 3x3 applet but cannot manipulate, for instance, the image of the basis vectors that define the parallelepiped; if they could, we posit there may be more instances of students making [G⇒M] or [G⇒D] observations. One reason this is important to us as curriculum designers is the biconditionality of determinant-related properties and theorems. This informs applet revisions to better facilitate student understanding of and role in reinventing the biconditionality of the generalizations.

One limitation of the analysis presented here is that we analyzed post-hoc data of students' generalizations after interacting with the applet in class. We did not collect data of the students exploring the applet and conjecturing in real time. Analyzing real-time interactions would allow us to make stronger inferences about students' generalizations and the examples they explored to support them. We also had limited access to student thinking about why their observations were true because it was not required that they share justifications. Finally, students were informed all semester to spend 5-10 min on reflections and that they were graded on effort. Thus, we cannot know if students would have answered differently in a different setting. We look forward to our future work revising the applets, such as designing aspects to further facilitate specific learning goals. This work will also allow us to explore and theorize the alignment between specific aspects of RME, such as emergent models, and student exploration and engagement in DGS.

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