

Sampling Colorings and Independent Sets of Random Regular Bipartite Graphs in the Non-Uniqueness Region*

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Abstract

We give an FPRAS for counting q -colorings for even $q = O(\frac{\Delta}{\log \Delta})$ on almost every Δ -regular bipartite graph. This improves significantly upon the previous best bound of $q = O(\frac{\sqrt{\Delta}}{(\log \Delta)^2})$ by Janssen, Keevash, and Perkins (SODA'19). Analogously, for the hard-core model on independent sets weighted by $\lambda > 0$, we present an FPRAS for estimating the partition function when $\lambda = \Omega(\frac{\log \Delta}{\Delta})$, which improves upon previous results by an $\Omega(\log \Delta)$ factor. Our results for the colorings and hard-core models follow from a general result that applies to arbitrary spin systems. Our main contribution is to show how to elevate probabilistic/analytic bounds on the marginal probabilities for the typical structure of phases on random bipartite regular graphs into efficient algorithms, using the polymer method. We further show evidence that our results for colorings and independent sets are within a constant factor of best possible using current polymer-method approaches.

1 Introduction

Polymer models have recently been used to obtain algorithms for spin systems in regimes where standard algorithmic tools (such as correlation-decay algorithms or Gibbs sampling/Glauber dynamics) are inefficient. The prototypical class of graphs where polymer models have been applied to are classes of expander and random regular graphs [25, 11, 22, 4, 28, 8, 16], see also [23, 3, 24] for applications on the grid.

Random bipartite regular graphs are particularly tantalizing [25, 28, 16], since on the one hand there is a somewhat standard probabilistic framework to obtain rough analysis estimates for arbitrary spin systems on them (using first/second moment arguments [19]), but on the other hand the corresponding algorithmic framework, and in particular the development of efficient sampling/counting algorithms, is lacking.

This paper will focus on finding the algorithmic limits of the polymer method for the two canonical models of interest, the q -colorings and the hard-core model (weighted independent sets), though our results apply much more generally as we will detail later. One of the main contributions of this work is to elevate the rough guarantees obtained by analytic/probabilistic methods into efficient approximate sampling/counting algorithms.

We begin with the colorings problem: given an integer $q \geq 3$ and a graph $G = (V, E)$ of maximum degree Δ , the goal is to approximate the number of proper q -colorings of G , and sample a proper q -coloring uniformly at random. For general graphs there is an intriguing computational phase transition that is conjectured to occur at the statistical physics phase transition for uniqueness/non-uniqueness of the Gibbs measure on the infinite Δ -regular tree. When $q \geq \Delta + 2$ it is conjectured that the simple single-site update Markov chain known as the Glauber dynamics is rapidly mixing on any graph of maximum degree Δ (rapid mixing refers to a convergence rate which is polynomial in $n = |V|$). In contrast when $q \leq \Delta$ it is believed that the problem is intractable.

Current bounds are far from resolving this conjecture but have made considerable progress. On the algorithmic side, recent results establish $O(n \log n)$ mixing time of the Glauber dynamics on an n -vertex graph of maximum degree Δ when $q > (11/6 - \varepsilon_0)\Delta$ for a positive constant $\varepsilon_0 \approx 10^{-5}$ [2, 26, 6] and on triangle-free graphs when $q > 1.764\Delta$ [10, 14, 9]. On the negative side, it was shown in [19] that for *even* $q < \Delta$ it is NP-hard to

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approximate the number of q -colorings. The restriction that q is even in this hardness result is rather technical and is a byproduct of a certain maximisation which was carried out in [19] for even q .

The above results address the problem on worst-case graphs; in this paper we address the behavior on *typical/random* graphs. In this vein, random regular *bipartite* graphs are particularly interesting as they manifest the phase transition of regular trees, and consequently they serve as the key gadget in hardness results [30, 31, 20, 19, 12, 5]. However, standard approximate counting techniques, such as Markov Chain Monte Carlo (MCMC), fail in the non-uniqueness region; e.g., the Glauber dynamics is exponentially slow to converge, with high probability over the choice of the random regular bipartite graph, for even $q < \Delta$ [19].

Even though optimization problems are typically easy on bipartite graphs, the corresponding counting/sampling problems appear to be computationally intractable. In particular, designing an algorithm for approximately counting colorings on bipartite graphs (see [21]) and a large variety of other counting problems (see [13, 7, 17, 15]) are $\#$ BIS-hard where $\#$ BIS is the problem of approximately counting independent sets in bipartite graphs. Moreover, in [21] it was established that it is $\#$ BIS-hard to approximate the number of q -colorings of maximum degree Δ graphs when $q < \Delta/2 \ln \Delta$, which is within a constant factor of our colorings result stated below.

For bipartite graphs, such as random regular bipartite graphs or grid graphs, it is common to conjecture that the Glauber dynamics is rapidly mixing in the non-uniqueness region when restricted to a particular maximal phase (namely, mostly even or odd independent sets for the case of the hard-core model or the $\binom{q}{q/2}$ partitions for q -colorings with even q); however establishing such results even for the ferromagnetic Ising model is a well-known open problem, see [29].

For a graph $G = (V, E)$ and an integer $q \geq 3$, the partition function Z_G is the number of q -colorings of G . We say that an algorithm $\mathcal{A}$ is an FPRAS for the partition function on almost all Δ -regular bipartite graphs if, with probability $1 - o(1)$ over a graph G chosen u.a.r. from n -vertex Δ -regular bipartite graphs, given G , an accuracy $\epsilon > 0$, and a tolerance $\delta > 0$, the algorithm $\mathcal{A}$ produces in time $\text{poly}(n, 1/\epsilon, \log(1/\delta))$ an estimate $\hat{Z}$ of the partition function Z_G satisfying $(1 - \epsilon)Z_G \leq \hat{Z} \leq (1 + \epsilon)Z_G$ with probability $\geq 1 - \delta$. The algorithm is an FPTAS if it is deterministic and achieves the same approximation guarantees in $\text{poly}(n, 1/\epsilon)$.

Intriguing algorithmic results for the non-uniqueness region of $q \ll \Delta$ on random bipartite graphs were devised using the recently introduced polymer method of [25] and [23]. Jenssen, Keevash, and Perkins [25] presented an FPTAS (see below for definitions) for almost every regular bipartite graph when $q \leq C \frac{\sqrt{\Delta}}{(\log \Delta)^2}$ for a constant $C > 0$ (see also the independent result of Liao, Lin, Lu, and Mao [28]). The running time of these algorithms was improved to $O(n^2(\log n)^3)$ in [8] using a randomized method, see Remark 1.1 below.

Here we present an FPRAS for q -colorings on almost every regular bipartite graph for even $q = O(\frac{\Delta}{\log \Delta})$. This improves significantly over the best previous known bound of $q = O(\frac{\sqrt{\Delta}}{(\log \Delta)^2})$ given in [25], and is within only an $O(\log \Delta)$ -factor from the uniqueness/hardness threshold. In fact, we also provide strong evidence that this is the limit of the polymer method up to the implicit constants in the given bounds, see the upcoming Lemma 2.4 for details.

THEOREM 1.1. *For all even $q \geq 4$ and all $\Delta \geq 100q \log q$, there is an FPRAS for the number of q -colorings on almost all Δ -regular bipartite graphs.*

We provide analogous results for the hard-core model on weighted independent sets. For a graph $G = (V, E)$, let Ω_G denote the collection of independent sets of G . For a parameter $\lambda > 0$, let independent set $\sigma \in \Omega_G$ have weight $w(\sigma) = \lambda^{|\sigma|}$. The partition function for the hard-core model on graph G at fugacity λ is defined as $Z_G = \sum_{\sigma \in \Omega} w(\sigma)$ and the Gibbs distribution is $\mu(\sigma) = w(\sigma)/Z_G$. The hard-core model on the infinite Δ -regular tree undergoes a phase transition corresponding to uniqueness vs. non-uniqueness of the infinite-volume Gibbs measure at $\lambda_c(\Delta) = \frac{(\Delta-1)^{\Delta-1}}{(\Delta-2)^\Delta} \sim \frac{e}{\Delta}$. For any graph G of maximum degree Δ , for all $\lambda < \lambda_c(\Delta)$, the Glauber dynamics mixes in $O(n \log n)$ time [10]. On the other side, when $\lambda > \lambda_c(\Delta)$, the problem of approximating the partition function is NP-hard on Δ -regular graphs [30, 31, 20]. Moreover, for *bipartite* graphs of maximum degree Δ , the problem is $\#$ BIS-hard for any $\lambda > \lambda_c(\Delta)$ [7].

For random Δ -regular bipartite graphs, [25] presented an FPTAS for $\lambda > 50 \frac{(\log \Delta)^2}{\Delta}$ when Δ is sufficiently large, and [28] for $\lambda \geq 1$ and $\Delta \geq 53$, see also [11, 4, 27] for related results on bipartite graphs. We get an improved range of $\lambda = \Omega(\frac{\log \Delta}{\Delta})$, which is again within an $O(\log \Delta)$ -factor from the uniqueness/hardness threshold.

THEOREM 1.2. For all $\Delta \geq 53$ and all $\lambda > 100 \frac{\log \Delta}{\Delta}$, there is an FPRAS for the partition function of the hard-core model with parameter λ on almost all Δ -regular bipartite graphs.

REMARK 1.1. In Theorems 1.1 and 1.2, we can also obtain deterministic approximation schemes (FPTAS) by applying the interpolation method, analogously to [25]. Here, we follow the Markov-chain framework of [8], which provides substantially stronger running time guarantees than those we state for convenience here. In particular, the FPRASes in Theorems 1.1 and 1.2 run in time $O((\frac{n}{\epsilon})^2 \log^3(\frac{n}{\epsilon}))$ when the desired accuracy error is not exponentially small (i.e., $\epsilon \geq e^{-\Omega(n)}$). Moreover, in the same range of the parameters, we obtain in addition approximate samplers from the Gibbs distribution with analogous running-time guarantees.

We remark that the condition that q is even in Theorem 1.1 is for the same technical reasons that the earlier stated hardness results of [19] were obtained for q even. We conjecture that the result can be extended to odd q and our proof approach extends verbatim (once one has the analogue of the upcoming Lemma 2.1). Interestingly, the only difficulty in extending to odd q is analyzing a first moment calculation. The previous results of [25, 28] did apply to odd q .

In fact, Theorems 1.1 and 1.2 will be proved as special cases of a general algorithmic result that applies to arbitrary spin systems on random bipartite regular graphs. We first introduce general spin systems following the framework of [16]. Note that the techniques in [16] were targeted to obtain bounds for general spin system and do not yield tight results, e.g., for colorings the bound obtained therein is roughly $q = O(\Delta^{1/4})$, cf. with the bound on q in Theorem 1.1. Also, to obtain the result of the hard-core model in Theorem 1.2 we will also need to explicitly account for the presence of external fields, as detailed in the next section.

2 Proof outline

2.1 Preliminaries: general spin systems and bicliques Let $q \geq 2$ be an integer. A general q -spin system $(\mathbf{B}, \boldsymbol{\lambda})$ consists of a symmetric interaction matrix $\mathbf{B} = \{B_{ij}\}_{i,j \in [q]}$, whose entries are between 0 and 1, and an activity vector $\boldsymbol{\lambda} = \{\lambda_i\}_{i \in [q]}$ with strictly positive entries which are ≤ 1 . For a graph $G = (V, E)$, an assignment $\sigma : V \rightarrow [q]$ has weight

$$w_G(\sigma) = \prod_{v \in V} \lambda_{\sigma(v)} \prod_{(u,v) \in E} B_{\sigma(u), \sigma(v)}.$$

The Gibbs distribution is given by $\mu_G(\sigma) = w_G(\sigma)/Z_G$, where $Z_G = \sum_{\sigma: V \rightarrow [q]} w_G(\sigma)$ is the partition function. Note, that up to normalising, we may assume that $\mathbf{B}$ and $\boldsymbol{\lambda}$ have each at least one entry equal to 1. We let Σ_G be the set of all spin assignments $\sigma : V \rightarrow [q]$.

For a spin system on a bipartite graph G , the following notion of *bicliques* is relevant [16, 25, 15, 18].

DEFINITION 2.1. (BICLIQUE) For a q -spin system with interaction matrix $\mathbf{B}$, we say that a pair (S, T) where $S, T \subseteq [q]$ is a biclique if $B_{ij} = 1$ for all $i \in S, j \in T$. A biclique (S, T) is maximal if there is no other biclique $(S', T') \neq (S, T)$ satisfying $S \subseteq S' \subseteq [q]$ and $T \subseteq T' \subseteq [q]$.

Note that bicliques are defined using only the interaction matrix $\mathbf{B}$ and do not depend on $\boldsymbol{\lambda}$.

Example. For the q -colorings model, we have that $\mathbf{B}$ is the $q \times q$ matrix with all ones except on the diagonal where the entries are zero (and $\boldsymbol{\lambda}$ is the all-ones vector). The bicliques (S, T) are given by pairs of disjoint sets $S, T \subseteq [q]$, whereas maximal bicliques by pairs of $S, T \subseteq [q]$ that form a partition of $[q]$. For the hard-core model, we have $\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\boldsymbol{\lambda} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Indexing the rows/columns of $\mathbf{B}$ with $\{0, 1\}$ (instead of $\{1, 2\}$), the bicliques are $\{(0, 0), (0, 1), (1, 0), (0, 01), (01, 0)\}$ and the maximal bicliques are $\{(0, 01), (01, 0)\}$. $\square$

2.2 Our approach: phase vectors and phase maximality Let $(\mathbf{B}, \boldsymbol{\lambda})$ be an arbitrary spin system and G be a Δ -regular bipartite graph, whose vertex set V is partitioned as (L, R) with $|L| = |R| = n$. Our approach to obtain approximation algorithms is to consider the likely frequencies of the spins on each side of the graph in the Gibbs distribution of G . Adapting methods from [16, 25], we show that we can obtain efficient approximation schemes for those spin systems where the “likely” frequency vectors are captured by maximal bicliques (S, T) , see the upcoming Definition 2.3. The main new ingredient in our work is to give a tight method to study when this condition is satisfied for general spin systems, which ultimately yields Theorems 1.1 and 1.2 as special cases.

To formalise the above, for q -dimensional probability vectors $\alpha = \{\alpha_i\}_{i \in [q]}, \beta = \{\beta_i\}_{i \in [q]}$, we let

$$(2.1) \quad \Sigma_G^{\alpha, \beta} = \{\sigma : V \rightarrow [q] \mid |\sigma^{-1}(i) \cap L| = n\alpha_i, |\sigma^{-1}(i) \cap R| = n\beta_i\}$$

be the set of spin assignments where exactly $n\alpha_i, n\beta_i$ vertices are assigned the spin $i \in [q]$ on L, R , respectively. Denote by $Z_G^{\alpha, \beta}$ the contribution to the partition function from configurations in $\Sigma_G^{\alpha, \beta}$, i.e.,

$$Z_G^{\alpha, \beta} = \sum_{\sigma \in \Sigma_G^{\alpha, \beta}} w_G(\sigma).$$

We will be interested in those pairs (α, β) that contribute significantly to the partition function, as detailed below.

DEFINITION 2.2. (PHASE VECTORS) *Let $\eta > 0$. For a q -spin system on an n -vertex regular bipartite graph G , we say that a pair (α, β) of q -dimensional probability vectors is an η -phase vector of G if $Z_G^{\alpha, \beta}/Z_G \geq e^{-\eta n}$.*

Understanding the phase vectors is in general a hard task. For random bipartite regular graphs, these have been identified to lie among the set of fixpoints $(\mathbf{r}, \mathbf{c})$ of the following so-called tree recursions on the Δ -regular tree [19]:

$$(2.2) \quad \begin{aligned} r_i &\propto \lambda_i \left(\sum_{j \in [q]} B_{ij} c_j \right)^{\Delta-1} \quad \text{for } i \in [q]; \\ c_j &\propto \lambda_j \left(\sum_{i \in [q]} B_{ij} r_i \right)^{\Delta-1} \quad \text{for } j \in [q]. \end{aligned}$$

The underpinning principle here leading to this correspondence is that the neighbourhood structure of a random Δ -regular bipartite graph is similar to a Δ -regular tree. Nevertheless, identifying the actual phase vectors, even among the finite-set of fixpoints in (2.2), has turned out to be rather challenging. Even in the canonical case of q -colorings, the current best known analysis works for even q and is a result of technically intense arguments.

Before looking into this in more detail, we first explain how to convert the information about phase vectors into algorithms. Adapting methods from [16, 25], we show that this is feasible when the phase vectors correspond to maximal bicliques. More precisely, for a non-empty set $S \subseteq [q]$, define the q -dimensional probability vector $\mathbf{g}_S$ whose i -th entry is given by $\frac{\lambda_i}{\sum_{j \in S} \lambda_j}$ for $i \in S$, and 0 otherwise. The following notion of “phase maximality” will be important in what follows.

DEFINITION 2.3. (PHASE MAXIMALITY) *Let $(\mathbf{B}, \boldsymbol{\lambda})$ be a q -spin system and $\Delta \geq 3$. For $\rho > 0$ and a set of maximal bicliques $\mathcal{B}_\Delta$, we say that the spin system is ρ -maximal with respect to $\mathcal{B}_\Delta$ if there is $\eta > 0$ such that, for almost all Δ -regular bipartite graphs, every η -phase vector (α, β) satisfies $\|(\alpha, \beta) - (\mathbf{g}_S, \mathbf{g}_T)\|_\infty \leq \rho$ for some maximal biclique $(S, T) \in \mathcal{B}_\Delta$.*

The key new ingredient to prove Theorems 1.1 and 1.2 is to establish maximality for the colorings and hard-core models in the corresponding parameter regimes, as detailed in the following theorems.

LEMMA 2.1. *For even $q \geq 4$ and $\Delta \geq 8q \log \Delta$, the q -colorings model is $\frac{1}{12\Delta q}$ -maximal with respect to the set of bicliques $\mathcal{B}_\Delta = \{(S, [q] \setminus S) \mid |S| = \frac{q}{2}\}$.*

LEMMA 2.2. *For $\Delta \geq 50$ and $\lambda \geq \frac{50}{\Delta}$, the hard-core model with fugacity λ is $\frac{1}{24\Delta}$ -maximal with respect to the set of bicliques $\mathcal{B}_\Delta = \{(0, 01), (01, 0)\}$.*

Previous approaches in [25, 16, 28] to establish the analogues of Lemmas 2.1 and 2.2 used expansion properties of random Δ -regular bipartite graphs which do not however give tight results in terms of the range of the parameters that they apply. Instead, we follow a more direct analytical approach, using the tree-recursions view mentioned in (2.2), further details are given in the full version. These more precise bounds allow us to push significantly further the applicability of the polymer method, see also the beginning of Section 3 for further explanation.

Indeed, we show that ρ -maximality yields approximation algorithms on random Δ -regular bipartite graphs, provided that ρ is sufficiently small and that the weight of configurations corresponding to maximal bicliques is

sufficiently big relatively to other type of configurations. To capture the latter condition, recall that the entries of $\mathbf{B}, \boldsymbol{\lambda}$ are between 0 and 1, and each of them includes at least one entry equal to 1. We say that $\mathbf{B}$ is a δ -matrix for some $\delta \in [0, 1)$ if the second largest entry of $\mathbf{B}$ is $\leq \delta$, and we denote by $\min(\boldsymbol{\lambda})$ the minimum entry in $\boldsymbol{\lambda}$ (note that this is strictly bigger than 0).

Applying the notion of phase maximality together with second moment calculations for general spin systems, and then using the algorithmic approach of [25, 16] we obtain the following general result (see Section 3).

LEMMA 2.3. *Let $(\mathbf{B}, \boldsymbol{\lambda})$ be a q -spin system, $\Delta \geq 3$ be an integer, and $\rho = \frac{1}{12\Delta q}$. Suppose further that $\mathbf{B}$ is a δ -matrix for some $\delta \in [0, 1)$ and that $\Delta(1 - \delta) \min(\boldsymbol{\lambda}) \geq 7q(5 + \log \frac{(q-1)\Delta^3}{\min(\boldsymbol{\lambda})})$.*

If the spin system is ρ -maximal, then there is an FPRAS for the partition function for almost all Δ -regular bipartite graphs. In fact, for almost all Δ -regular bipartite graphs, for $\epsilon = \exp(-\Omega(n))$, the algorithm produces an ϵ -estimate for the partition function and an ϵ -sample from the Gibbs distribution in time $O((\frac{n}{\epsilon})^2 (\log \frac{n}{\epsilon})^3)$.

Using the above ingredients, we can prove our main Theorems 1.1 and 1.2.

Proof. [Proof of Theorems 1.1 and 1.2] We first prove the result for colorings, Theorem 1.1. We just need to combine Lemmas 2.1 and 2.3. In the setting of Lemma 2.3 and Example 2.1, we have that the interaction matrix for colorings is a δ -matrix for $\delta = 0$ and $\min(\boldsymbol{\lambda}) = 1$. Hence, for $\Delta \geq 100q \log q$, we have that $\Delta(1 - \delta) \min(\boldsymbol{\lambda}) \geq 7q(5 + \log \frac{(q-1)\Delta^3}{\min(\boldsymbol{\lambda})})$ as needed. Moreover, Lemma 2.1 establishes the required ρ -maximality that is further needed. Therefore, the conclusion of Lemma 2.3 applies and we obtain the Theorem 1.1.

The proof for independent sets, Theorem 1.2, is analogous, by now combining Lemmas 2.2 and 2.3. We may assume that $\lambda < 1$, otherwise the result follows from the FPRAS for $\Delta \geq 53$ in [28, Theorem 1]. In the setting of Example 2.1, we have that $q = 2$, $\delta = 0$ and $\min(\boldsymbol{\lambda}) = \lambda$. Then, for $\lambda > 100 \frac{\log \Delta}{\Delta}$, we have that

$$\Delta(1 - \delta) \min(\boldsymbol{\lambda}) \geq 7q(5 + \log \frac{(q-1)\Delta^3}{\min(\boldsymbol{\lambda})}),$$

and the result follows analogously to above. $\square$

Finally, as mentioned in the introduction, we give evidence that the bounds on q in Theorem 1.1 capture the limit of the polymer method for colorings, by showing that maximality fails when we go beyond the relevant range (note, some form of maximality is either implicitly or explicitly shown in all previous works on the problems).

LEMMA 2.4. *For all even $q \geq 4$ and $\Delta = O(q \log q)$, for the q -colorings model, $O(\frac{1}{\Delta q})$ -maximality fails with respect to any set of bichiques on almost all Δ -regular bipartite graphs.*

We note that Lemma 2.4 does not exclude the possibility of some exotic polymer model that can perhaps break the barrier therein. It does show however that the current approach cannot go substantially beyond the guarantee in Theorem 1.1, and at the very least some major refinement of the framework will be needed. We conjecture that a similar barrier applies for the result of Theorem 1.2, though here the bottleneck is in Lemma 2.3. More precisely, for $\lambda = O(\frac{\log \Delta}{\Delta})$ in the non-uniqueness region, it appears that polymers can be of size $n^{\Omega(1)}$, which is in contrast to what happens when the polymer method applies (where the size of polymers turns out to be logarithmic in n). The corresponding phenomenon on the $(\Delta - 1)$ -ary tree is easier to establish, and it can be shown that the size of 2-connected polymers is supercritical when $\lambda = O(\frac{\log \Delta}{\Delta})$ (which corresponds to the occupation probability on even levels being $> \frac{1}{\Delta^2}$). In turn, this implies existence of polymers of size $n^{\Theta(1)}$ in a $(\Delta - 1)$ -ary tree of depth $\Theta(\log n)$ for $\lambda = O(\frac{\log \Delta}{\Delta})$ in the non-uniqueness region, which suggests that the corresponding phenomenon should occur in random Δ -regular bipartite graphs for the same range of λ .

REMARK 2.1. *The algorithms presented in Theorems 1.1 and 1.2 (and Remark 1.1) satisfy the guarantees of an FPRAS (and FPTAS, respectively) with probability $1 - o(1)$ over the choice of the random bipartite graph. This is the same guarantee provided in the earlier works of [25, 28] for colorings and the hard-core model. In the specific case of the ferromagnetic Potts model, [25] was able to establish an additional guarantee: they can verify whether a given graph satisfies the spectral expansion conditions that are required by their algorithm, see [25, Remark 1] for more details.*

3 Algorithms from maximality: Proof of Lemma 2.3

Let $\Delta \geq 3$ be an integer, and $(\mathbf{B}, \boldsymbol{\lambda})$ be a q -spin system, which is ρ -maximal for $\rho = \frac{1}{12\Delta q}$. Consider also a bipartite graph $G = (V, E)$ with vertex bipartition (L, R) and $|L| = |R| = n$. The following expansion property of sets $U \subseteq V$ in random regular bipartite graphs relaxes the previous expansion properties that were used in [25, 28] which needed to consider bigger sets U ; instead, whenever the spin system is $\frac{1}{12\Delta q}$ -maximal, we only need to consider sets U with size roughly $\frac{1}{\Delta}|V|$, whose expansion is $\Omega(\Delta)$. For a set $U \subseteq V$, we use ∂U to denote the vertices in G which have a neighbor in U but do not belong to U , and by U^+ the set $U \cup \partial U$.

LEMMA 3.1. *Let $\Delta \geq 3$ be an integer. For almost all Δ -regular bipartite graphs $G = (V, E)$ with bipartition (L, R) , the following expansion properties hold:*

1. every set $U \subseteq V$ with $|U \cap L| \leq \frac{1}{3\Delta}|L|$ and $|U \cap R| \leq \frac{1}{3\Delta}|R|$ satisfies $|U^+| \geq \frac{\Delta-1}{2}|U|$.
2. every set $U \subseteq V$ with $|U \cap L| \leq \frac{1}{6\Delta}|L|$ and $|U \cap R| \leq \frac{1}{6\Delta}|R|$ satisfies $|\partial U| \geq \frac{\Delta}{7}|U|$.

Proof. For the first item, consider a subset $U \subseteq V$ with $|U \cap L| \leq \frac{1}{3\Delta}|L|$ and $|U \cap R| \leq \frac{1}{3\Delta}|R|$. We will show that

$$(3.3) \quad |\partial(U \cap L)| \geq \frac{\Delta-1}{2}|U \cap L| \text{ and } |\partial(U \cap R)| \geq \frac{\Delta-1}{2}|U \cap R|.$$

From this, we obtain that

$$|U^+| = |U \cup \partial U| \geq |\partial(U \cap L)| + |\partial(U \cap R)| \geq \frac{\Delta-1}{2}|U|.$$

To verify (3.3), we use a sufficient condition due to Bassalygo [1], see also [25, Theorem 22]. Namely, for $a = \frac{1}{3\Delta}$, $b = \frac{\Delta-1}{2}$ and $H(x) = -x \log_2(x) - (1-x) \log_2(1-x)$, we check that

$$\Delta > \frac{H(a) + H(ab)}{H(a) - abH(1/b)},$$

which indeed holds for all $\Delta \geq 3$.

The proof of the second item is analogous. Consider a subset $U \subseteq V$ with $|U \cap L| \leq \frac{1}{6\Delta}|L|$ and $|U \cap R| \leq \frac{1}{6\Delta}|R|$. We will show that

$$(3.4) \quad |\partial(U \cap L)| \geq \left(\frac{\Delta}{7} + 1\right)|U \cap L| \text{ and } |\partial(U \cap R)| \geq \left(\frac{\Delta}{7} + 1\right)|U \cap R|.$$

From this, we obtain that

$$|\partial U| \geq |\partial(U \cap L)| + |\partial(U \cap R)| - |U| \geq \frac{\Delta}{7}|U|.$$

For the proof of (3.4), we verify the same condition as above, now for the values $a = \frac{1}{6\Delta}$ and $b = \frac{\Delta}{7} + 1$. $\square$

Following [16], we will define a polymer model corresponding to a biclique (S, T) of the spin system. Let G^3 be the graph on vertex set V where two vertices u, v are adjacent iff $\text{dist}(u, v) \leq 2$. A subset $U \subseteq V$ of vertices is said to be G^3 -connected if the induced subgraph $G^3[U]$ is connected. A polymer $\gamma = (V_\gamma, \sigma_\gamma)$ consists of a subset of vertices of G , V_γ , which is G^3 connected, and a spin assignment on V_γ , $\sigma_\gamma : V_\gamma \rightarrow [q]$, such that every vertex in $V_\gamma \cap L$ gets a spin in $[q] \setminus S$ and every vertex in $V_\gamma \cap R$ gets a spin in $[q] \setminus T$. Two polymers γ_1, γ_2 are compatible (written as $\gamma_1 \sim \gamma_2$) if and only if $\text{dist}(\gamma_1, \gamma_2) > 3$, i.e., $\gamma_1 \cup \gamma_2$ is not G^3 -connected.

The size of a polymer γ , denoted by $|\gamma|$, is the number of vertices it contains. We use E_γ to denote the edges of G whose both endpoints lie in γ , ∂V_γ to denote the vertices in G which have a neighbor in V_γ but do not belong to V_γ , and by V_γ^+ the set $V_\gamma \cup \partial V_\gamma$. For a polymer γ , the weight $w_G^{S,T}(\gamma)$ of the polymer is given by

$$(3.5) \quad w_G^{S,T}(\gamma) = \frac{\prod_{u \in V_\gamma} \lambda_{\sigma_\gamma(u)} \prod_{(u,v) \in E_\gamma} B_{\sigma_\gamma(u), \sigma_\gamma(v)} \prod_{u \in \partial V_\gamma} F_u}{\left(\sum_{i \in S} \lambda_i\right)^{|V_\gamma^+ \cap L|} \left(\sum_{j \in T} \lambda_j\right)^{|V_\gamma^+ \cap R|}},$$

where

$$(3.6) \quad \begin{aligned} F_u &= \sum_{i \in S} \lambda_i \prod_{v \in V_\gamma \cap \partial u} B_{i, \sigma_\gamma(v)} \text{ if } u \in \partial V_\gamma \cap L, \\ F_u &= \sum_{j \in T} \lambda_j \prod_{v \in V_\gamma \cap \partial u} B_{j, \sigma_\gamma(v)} \text{ if } u \in \partial V_\gamma \cap R. \end{aligned}$$

Let $\mathcal{P}_G^{S,T}$ be the set of all polymers $\gamma = (V_\gamma, \sigma_\gamma)$ with $|V_\gamma| \leq 2qpn = \frac{n}{6\Delta}$. A configuration $\Gamma = (V_\Gamma, \sigma_\Gamma)$ of polymers is a collection of mutually compatible polymers $\gamma_1, \dots, \gamma_k \in \mathcal{P}_G^{S,T}$ with $V_\Gamma = \cup_{t \in [k]} V_{\gamma_t}$ and σ_Γ the spin assignment on V_Γ which agrees with σ_{γ_t} on V_{γ_t} for each $t \in [k]$. Let $\Omega_G^{S,T}$ be the set of all possible configurations Γ . The size of a configuration is $|\Gamma| = \sum_{\gamma \in \Gamma} |V_\gamma|$.

LEMMA 3.2. *Every configuration Γ satisfies $|V_\Gamma| \leq 12n/\Delta$.*

Proof. Suppose that there exists a configuration Γ with $|V_\Gamma| > 12n/\Delta$. Then, we can extract greedily disjoint configurations $\Gamma_1, \dots, \Gamma_{36} \subseteq \Gamma$ (which are a collection of polymers belonging to Γ) such that $\frac{n}{6\Delta} < |\Gamma_i| \leq \frac{n}{3\Delta}$. By Lemma 3.1, we have that $|V_{\Gamma_i}^+| \geq \frac{\Delta-1}{2} |\Gamma_i| > \frac{n}{6\Delta} \frac{\Delta-1}{2}$ and therefore

$$\sum_{i=1}^{36} |V_{\Gamma_i}^+| > \frac{6n}{\Delta} \frac{\Delta-1}{2} \geq 2n.$$

Therefore, since G has $2n$ vertices, the sets $V_{\Gamma_1}^+, \dots, V_{\Gamma_{36}}^+$ cannot be pairwise disjoint, contradicting the fact that the configuration Γ consists of pairwise compatible polymers. $\square$

The weight $w_G^{S,T}(\Gamma)$ of a configuration Γ is given by the product of the weights of the polymers that Γ consists of. We define the partition function of the polymer model as

$$Z_G^{S,T} = \sum_{\Gamma \in \Omega_G^{S,T}} w_G^{S,T}(\Gamma),$$

and its Gibbs distribution by $\mu_G^{S,T}(\Gamma) = w_G^{S,T}(\Gamma)/Z_G^{S,T}$ for $\Gamma \in \Omega_G^{S,T}$. Finally, we let

$$Z_G^{\text{pmer}} = \sum_{(S,T) \in \mathcal{B}_\Delta} \left(\sum_{i \in S} \lambda_i \right)^n \left(\sum_{j \in T} \lambda_j \right)^n Z_G^{S,T}.$$

LEMMA 3.3. *Let $\Delta \geq 3$ be an integer, and $(\mathbf{B}, \boldsymbol{\lambda})$ be a q -spin system which is $\frac{1}{12\Delta q}$ -maximal with respect to a set of maximal bicliques $\mathcal{B}_\Delta$. Suppose further that $\Delta \min(\boldsymbol{\lambda}) \geq 15q$. Then, there is $\epsilon = e^{-\Omega(n)}$ such that, for almost all Δ -regular bipartite graphs G with n vertices on each part, it holds that $(1 - \epsilon)Z_G \leq Z_G^{\text{pmer}} \leq (1 + \epsilon)Z_G$.*

Proof. By the $\frac{1}{12\Delta q}$ -maximality of the spin system with respect to $\mathcal{B}_\Delta$ (cf. Definition 2.3), there is an $\eta > 0$ such that for almost all Δ -regular graphs G , every η -phase vector $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ of G belongs to

$$\mathcal{F}_\Delta := \left\{ (\boldsymbol{\alpha}, \boldsymbol{\beta}) \mid \|(\boldsymbol{\alpha}, \boldsymbol{\beta}) - (\mathbf{g}_S, \mathbf{g}_T)\|_\infty \leq \frac{1}{12\Delta q} \text{ for some maximal biclique } (S, T) \in \mathcal{B}_\Delta \right\}.$$

Let

$$\Sigma_G^{\max} = \{\sigma \mid \sigma \in \Sigma_G^{\boldsymbol{\alpha}, \boldsymbol{\beta}} \text{ for some } (\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{F}_\Delta\}$$

where, recall from (2.1), that $\Sigma_G^{\boldsymbol{\alpha}, \boldsymbol{\beta}}$ is the set of spin assignments where exactly $n\alpha_i, n\beta_i$ vertices are assigned the spin $i \in [q]$ on L, R , respectively.

We first show the lower bound on Z_G^{pmer} . Consider the polymer model corresponding to a maximal biclique $(S, T) \in \mathcal{B}_\Delta$. Every configuration $\Gamma \in \Omega_G^{S,T}$ maps to a set of spin assignments

$$\Sigma_G^{S,T}(\Gamma) = \{\sigma : V \rightarrow [q] \mid \sigma(V_\Gamma) = \sigma_\Gamma, \sigma(L \setminus V_\Gamma) \subseteq S, \sigma(R \setminus V_\Gamma) \subseteq T\},$$

where recall that σ_Γ is a spin assignment on V_Γ that satisfies $\sigma_\Gamma(V_\Gamma \cap L) \subseteq [q] \setminus S$ and $\sigma_\Gamma(V_\Gamma \cap R) \subseteq [q] \setminus T$. Therefore, for distinct $\Gamma, \Gamma' \in \Omega_G^{S,T}$ we have that the sets $\Sigma_G^{S,T}(\Gamma)$ and $\Sigma_G^{S,T}(\Gamma')$ are disjoint. Let $\Sigma_G^{S,T} = \bigcup_{\Gamma \in \Omega_G^{S,T}} \Sigma_G^{S,T}(\Gamma)$. Using that configurations Γ consist of disjoint G^3 -connected sets, we obtain that (see for example [16, Lemma 17])

$$\sum_{\sigma \in \Sigma_G^{S,T}(\Gamma)} w_G(\sigma) = \left(\sum_{i \in S} \lambda_i \right)^n \left(\sum_{j \in T} \lambda_j \right)^n w_G^{S,T}(\Gamma),$$

and therefore

$$Z_G^{S,T} = \sum_{\sigma \in \Sigma_G^{S,T}} w_G(\sigma).$$

Moreover, note that for (α, β) with $\|(\alpha, \beta) - (\mathbf{g}_S, \mathbf{g}_T)\|_\infty \leq \frac{1}{12\Delta q}$, the number of vertices in L that do not get a spin in S is at most $\frac{n}{12\Delta}$, and similarly for vertices in R that do not get a spin in T , for a total of $\frac{n}{6\Delta}$ vertices, giving that $\Sigma_G^{\alpha, \beta} \subseteq \Sigma_G^{S,T}$. Observe now that

$$\text{every } (\alpha, \beta) \notin \mathcal{F}_\Delta \text{ is not an } \eta\text{-phase vector and therefore } Z_G^{\alpha, \beta} \leq e^{-\eta n} Z_G.$$

There are at most n^{2q} such pairs with $n\alpha, n\beta \in \mathbb{Z}^q$ and therefore, combining the above, it follows that

$$Z_G - Z_G^{\text{pmer}} \leq \sum_{(\alpha, \beta) \notin \mathcal{F}_\Delta} Z_G^{\alpha, \beta} \leq n^{2q} e^{-\eta n} Z_G \leq e^{-\Omega(n)} Z_G,$$

showing that

$$Z_G^{\text{pmer}} \geq (1 - e^{-\Omega(n)}) Z_G.$$

We next show the upper bound on Z_G^{pmer} . Consider

$$\Sigma_G^{\text{overlap}} = \bigcup_{(S,T) \neq (S',T') \in \mathcal{B}_\Delta} (\Sigma_G^{S,T} \cap \Sigma_G^{S',T'}).$$

We will show shortly that $\Sigma_G^{\text{overlap}} \subseteq \Sigma_G \setminus \Sigma_G^{\text{max}}$. Assuming this for the moment, we conclude the proof by noting first that for (α, β) which is not an η -phase vector it holds that $Z_G^{\alpha, \beta} / Z_G < e^{-\eta n}$. Therefore we obtain that the aggregate weight of spin assignments in $\Sigma_G^{\text{overlap}}$ is at most $n^{2q} e^{-\eta n} Z_G = e^{-\Omega(n)} Z_G$, yielding that

$$Z_G \geq (1 - e^{-\Omega(n)}) Z_G^{\text{pmer}}.$$

It remains to prove that $\Sigma_G^{\text{overlap}} \subseteq \Sigma_G \setminus \Sigma_G^{\text{max}}$. For the sake of contradiction, suppose otherwise. Then there exists a spin assignment σ , distinct bicliques $(S, T), (S', T') \in \mathcal{B}_\Delta$, and a biclique $(S^*, T^*) \in \mathcal{B}_\Delta$ such that $\sigma \in \Sigma_G^{S,T} \cap \Sigma_G^{S',T'}$ and $\sigma \in \Sigma_G^{\alpha, \beta}$ for some α, β satisfying

$$\|(\alpha, \beta) - (\mathbf{g}_{S^*}, \mathbf{g}_{T^*})\|_\infty \leq \frac{1}{12\Delta q}.$$

Since (S, T) and (S', T') are distinct and maximal, we may assume w.l.o.g. have that $S \neq S^*$ and $T \neq T^*$. Since (S^*, T^*) is maximal, it cannot be the case that $S^* \subseteq S$ and $T^* \subseteq T$, so assume w.l.o.g. that $i \in S^* \setminus S$. Let n_i be the vertices in L that have the spin i under σ . Since $\sigma \in \Sigma_G^{S,T}(\Gamma)$ for some $\Gamma \in \Omega_G^{S,T}$ and $i \notin S$, from Lemma 3.2 we have that $n_i \leq |V_\Gamma| \leq 12n/\Delta$. Then, using the assumption $\Delta \min(\lambda) \geq 15q$ and the fact that the entries of λ are ≤ 1 , we have the crude bound

$$\frac{\lambda_i}{\sum_{i' \in S^*} \lambda_{i'}} \geq \min(\lambda)/q \geq 15/\Delta,$$

and therefore

$$\left| \frac{\lambda_i}{\sum_{i' \in S^*} \lambda_{i'}} - \frac{n_i}{n} \right| \geq \frac{3}{\Delta} > \frac{1}{12\Delta q}$$

contradicting the choice of (S^*, T^*) . $\square$

With Lemma 3.3 at hand, the proof of Lemma 2.3 follows the Markov chain approach for studying polymer models in [8], as employed for general spin systems in [16], the details are given in the full version. There, the proofs of Lemmas 2.1 and 2.2 are given, which as we show in Section 2.2 were used in conjunction with Lemma 2.3 to derive Theorems 1.1 and 1.2.

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