

DAVID EXTENSION OF CIRCLE HOMEOMORPHISMS, WELDING, MATING, AND REMOVABILITY

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ABSTRACT. We provide a David extension result for circle homeomorphisms conjugating two dynamical systems such that parabolic periodic points go to parabolic periodic points, but hyperbolic points can go to parabolics as well. We use this result, in particular, to prove the existence of a new class of welding homeomorphisms, to establish an explicit dynamical connection between critically fixed anti-rational maps and kissing reflection groups, to show conformal removability of the Julia sets of geometrically finite polynomials and of the limit sets of necklace reflection groups, to produce matings of anti-polynomials and necklace reflection groups, and to give a new proof of the existence of Suf-fridge polynomials (extremal points in certain spaces of univalent maps).

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1. INTRODUCTION

The nicest and best understood classes of dynamical systems are the ones with uniform hyperbolicity. In rational dynamics, it means that a map admits a conformal metric in a neighborhood of its Julia set with respect to which it is uniformly expanding. The next simplest class of rational maps are *parabolic* maps, they have parabolic cycles but no critical points on their Julia sets. Due to the existence of parabolic cycles, such maps are not uniformly hyperbolic, but still enjoy a weaker property of expansiveness on their Julia sets. In the parallel field of Kleinian groups, the concept of hyperbolicity translates into compactness of a certain manifold (or orbifold) with boundary (the “convex core”), and Kleinian groups satisfying this condition are called *convex co-compact*. The analogue of parabolic rational maps in this world are given by *geometrically finite* Kleinian groups. Such groups may also have parabolic fixed points. Also, recently a new area of Schwarz reflection dynamics has been developed which combines features of both rational maps and Kleinian groups in a single dynamical plane, and the notion of geometric finiteness has a natural meaning in this setting as well (see [LLMM18a, LLMM19]).

In general, many topological, dynamical, and analytic properties of hyperbolic dynamical systems are shared by their parabolic counterparts. However, in the absence of uniform expansion, one often needs to develop more elaborate techniques to study parabolic systems. The current paper grew out of concrete problems in conformal dynamics that necessitated direct passage from hyperbolic conformal dynamical systems to parabolic ones in a sufficiently regular manner. Such a surgery procedure first appeared in the work of Haïssinsky in the context of complex polynomials [Hai98, Hai00, BF14]. It was shown there that hyperbolic polynomial Julia sets can be turned into parabolic Julia sets via *David* homeomorphisms, which are generalizations of quasiconformal homeomorphisms. In the present work, we carry this philosophy further by developing new analytic tools, namely, a David extension theorem for dynamical circle homeomorphisms (based upon results of J. Chen, Z. Chen and He [CCH96], and Zakeri [Zak08]) and a surgery technique using the David Integrability Theorem, that gives rise to a unified approach to turn hyperbolic (anti-)rational maps to parabolic (anti-)rational maps, Kleinian reflection groups, and *Schwarz reflection maps* that are matings of anti-polynomials and reflection groups.

At the technical heart of the paper lies a theorem that guarantees that a general class of dynamically defined circle homeomorphisms can be extended as David homeomorphisms of the unit disk. More precisely, let f and g be two expansive covering maps of the unit circle that have the same orientation, have equivalent Markov partitions, are analytic on the defining intervals of the corresponding Markov partitions, and admit piecewise conformal extensions satisfying certain natural conditions; see conditions (4.1) and (4.2) below. Let h be an orientation-preserving homeomorphism of the circle that conjugates f to g . Then, our main result, Theorem 4.9, states that h has a David extension to the unit disk, provided it takes parabolic periodic points of f to parabolic periodic points of g , while hyperbolic (i.e., repelling) periodic points of f can go to either hyperbolic or to parabolic periodic points of g . The following result is a special case of Theorem 4.9.

Theorem A (Blaschke product–circle reflections). Consider ordered points $a_0 = a_{d+1}, a_1, \dots, a_d$ on the circle $\mathbb{S}^1$, where $d \geq 2$. For $k \in \{0, \dots, d\}$ let $f|_{\widehat{(a_k, a_{k+1})}}$

be the reflection along the circle C_k that is orthogonal to the unit circle at the points a_k and a_{k+1} . Moreover, let B be an anti-holomorphic Blaschke product of degree d with an attracting fixed point in $\mathbb{D}$. Then there exists a homeomorphism $h: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ that conjugates $B|_{\mathbb{S}^1}$ to f , and has a David extension in $\mathbb{D}$.

A particular case of the above theorem, where $B(z) = \bar{z}^2$ and the points a_0, a_1, a_2 are the third roots of unity, was proved in a recent work [LLMM19, Lemma 9.4]. The proof given in [LLMM19] relies upon arithmetic properties of the conjugacy h . In contrast, the proof of Theorem A is purely dynamical, and can be applied to a broad range of situations. The piecewise analytic (rather than globally analytic) nature of the map f and the matching of hyperbolic periodic points of B to parabolic points of f pose the main complications in the proof that cannot rely on the standard quasiconformal machinery.

A combination of Theorem A together with a surgery technique devised in Section 7 permit us to construct conformal dynamical systems by replacing the attracting dynamics of an anti-rational map on an invariant Fatou component by piecewise defined circular reflections. Implementing this strategy, we are able to convert anti-rational maps to Kleinian reflection groups and to Schwarz reflection maps in a dynamically natural way. A precursor to the birth of this general machinery is a concrete construction carried out in [LLMM19], where the Julia set of a cubic critically fixed anti-rational map (an anti-rational map with all critical points fixed) was shown to be homeomorphic to the classical Apollonian gasket (the limit set of a Kleinian reflection group) and to the limit set of the Schwarz reflection map associated with a deltoid and an inscribed circle. More generally, a dynamical correspondence between Apollonian-like gaskets (associated to triangulations of the Riemann sphere) and a class of critically fixed anti-rational maps was set up in the same paper. A further generalization appeared in [LLM20], where a bijective correspondence between kissing reflection groups (groups generated by reflections in the circles of finite circle packings) with connected limit sets and critically fixed anti-rational maps was established using Thurston's topological characterization of rational functions. Using the surgery technique designed in the current paper, one can transform each critically fixed anti-rational map into a kissing reflection group such that the Julia set of the former is homeomorphic to the limit set of the latter under a global David homeomorphism. The next theorem presents a direct construction of a map from the class of critically fixed anti-rational maps to the class of kissing reflection groups.

Theorem B (From anti-rational maps to kissing groups). *Let R be a critically fixed anti-rational map. Then, there exists a kissing reflection group Γ such that the Julia set $\mathcal{J}(R)$ is homeomorphic to the limit set $\Lambda(\Gamma)$ via a dynamically natural David homeomorphism of $\widehat{\mathbb{C}}$.*

(See Theorem 8.1 for a precise version of Theorem B.)

As another interesting application of the main extension Theorem 4.9, we produce a new class of *welding* homeomorphisms. A welding homeomorphism is a homeomorphism of the circle that arises as the composition of the conformal map from the unit disk onto the interior region of a Jordan curve with a conformal map from the exterior of this Jordan curve onto the exterior of the unit disk. Previously known examples of welding homeomorphisms include quasisymmetric homeomorphisms [LV60, Pf60], circle maps that extend to David homeomorphisms of

the sphere [Dav88], weakenings of quasimetric homeomorphisms that are sufficiently regular [Ham91], and some quite “wild” homeomorphisms [Bis07]. We prove here that a circle homeomorphism that conjugates two expansive piecewise analytic maps (under conditions (4.1) and (4.2)), each of whose periodic points is either *symmetrically* parabolic or hyperbolic, is a welding homeomorphism (see Section 5 for the definitions). These homeomorphisms are composed of David homeomorphisms and their inverse; see Theorem 5.1.

Let us now elaborate on the background and history of the various problems that motivated the development of the David Extension Theorem 4.9. Along the way, we will also give informal statements of some of the main results.

Conformal removability of certain Julia sets was known from the works of P. Jones [Jon91] and P. Jones–S. Smirnov [JS00]. In [Jon91] it is shown, in particular, that the boundaries of John domains are conformally removable. As a consequence, Jones obtained that the Julia sets of subhyperbolic polynomials are conformally removable. In [JS00], the authors extended these removability results to the boundaries of Hölder domains, and, in particular to the Julia sets of Collet–Eckmann polynomials. However, these techniques do not yield removability of polynomial Julia sets with parabolic periodic points as these Julia sets have cusps.

A Kleinian reflection group is called a *necklace* group if it is generated by reflections in the circles of a circle packing with 2-connected, outerplanar contact graph (see Remark 6.8 for the definitions and Figure 2 for an example). Similarly to Julia sets of parabolic polynomials, limit sets of necklace groups have intricate geometric structure, see, e.g., Figure 2; indeed, limit sets of necklace groups also have cusps. Nevertheless, we are able to use our main result, Theorem 4.9, to show that connected Julia sets of geometrically finite polynomials as well as limit sets of necklace groups are conformally removable.

Theorem C (Removability of limit and Julia sets).

- Limit sets of necklace reflection groups are conformally removable.
- Connected Julia sets of geometrically finite polynomials are conformally removable.

Moreover, we produce an example of a conformally removable Julia set, called the *pine tree*, which is a Jordan curve with both inward and outward cusps; see Figure 1. Examples of this kind seem to be rare as the techniques of [Jon91] and [JS00] do not apply directly to such sets. If, instead, a Jordan curve has only inward cusps (in a precise sense), then the region bounded by the curve is a John domain so [Jon91] would imply the conformal removability of the curve.

The proof of Theorem C is given in Section 9, more precisely in Theorems 9.1 and 9.2, respectively. In fact, the proofs of the two removability statements have a common philosophy. We topologically realize the limit set of a necklace group or the Julia set of a geometrically finite polynomial as a (sub-)hyperbolic polynomial Julia set (which are known to be $W^{1,1}$ -removable), and then construct a global David homeomorphism that carries this (sub-)hyperbolic polynomial Julia set to the limit or Julia set in question. This, combined with Theorem 2.7, which states that the image of a $W^{1,1}$ -removable compact set under a global David homeomorphism is conformally removable, leads to the desired conclusion. In fact, we are unaware of an intrinsic proof of Theorem C. It is also worth mentioning that these removability results contrast with recent works by the fourth author [Nta19, Nta20b], where he

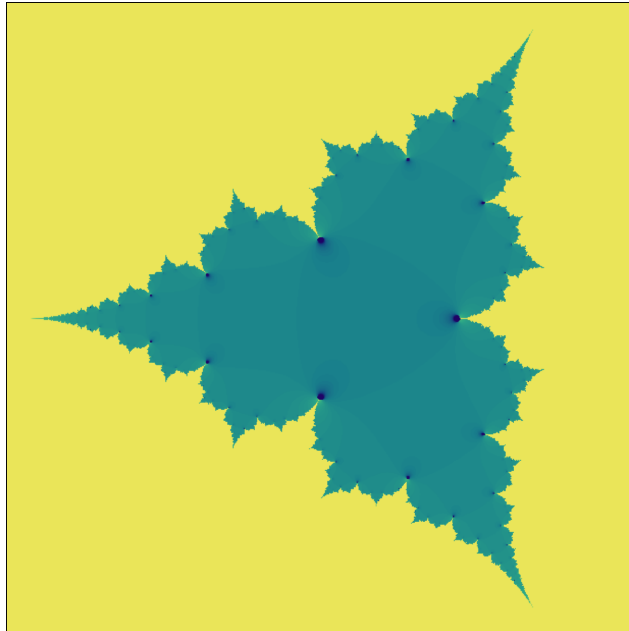


FIGURE 1. The pine tree Julia set. The Julia set of $R(z) = \frac{4z^3+8-3(1-\sqrt{3})}{(1-\sqrt{3})z^3+8+4\sqrt{3}}$ is a conformally removable curve with both inward and outward cusps.

shows that the standard Sierpiński gasket as well as all planar Sierpiński carpets are not removable for conformal maps. The classical Apollonian gasket (which is the limit set of a Kleinian reflection group) contains a homeomorphic copy of the Sierpiński gasket. This suggests that the Apollonian gasket is non-removable as well, but it still remains an open question.

We now turn our attention to the next application of our David extension and surgery theory. The concept of mating of two conformal dynamical systems goes back to *Klein's Combination Theorem*, and was brought to a new level in the seminal work of Bers on the *simultaneous uniformization* of two Riemann surfaces, which allows one to mate two Fuchsian groups to obtain a quasi-Fuchsian group [Ber60]. A significantly more difficult theorem of W. Thurston, called the *Double Limit Theorem*, provides a sufficient condition to ‘mate’ two projective measured laminations (or, two groups on the boundary of the corresponding Teichmüller space) to obtain a Kleinian group [Thu98, Ota01]. Later on, Douady and Hubbard introduced the notion of mating of two polynomials to produce a rational map, which can be seen as a philosophical parallel to the combination theorems in the world of rational dynamics [Dou83]. Each of the aforementioned mating constructions attempts to combine two similar conformal dynamical systems to produce a richer conformal dynamical system. Often it is not hard to mate two conformal dynamical systems topologically, but uniformizing the topological mating (i.e., endowing it with a complex structure) lies at the core of the problem.

In [BP94] (see also [BF14, §7.8]), S. Bullett and C. Penrose used iterated algebraic correspondences to introduce a notion of mating of the modular group

$\mathrm{PSL}(2, \mathbb{Z})$ and certain quadratic polynomials. Also, in [BH00] S. Bullett and W. Harvey used quasiconformal surgery to construct a holomorphic correspondence between any quadratic polynomial and certain representations of the free product $C_2 * C_3$ of cyclic groups (of orders 2 and 3) in $\mathrm{PSL}(2, \mathbb{C})$.

Recently, novel perspectives and techniques were introduced to bind together the actions of two different types of conformal dynamical systems; namely, an anti-rational map and a Kleinian reflection group, in a single conformal dynamical system [LLMM18a, LLMM18b, LLMM19, LMM20]. In particular, various examples of matings of Nielsen maps of Kleinian reflection groups and anti-rational maps were discovered, and these matings were realized as Schwarz reflection maps associated with suitable quadrature domains (see Section 10 for precise definitions). In this paper we introduce a unified framework for these ‘hybrid’ matings, prove a general theorem that ensures the existence of matings for a large class of Kleinian reflection groups and anti-polynomials, and illustrate the mating phenomena with a few explicit examples.

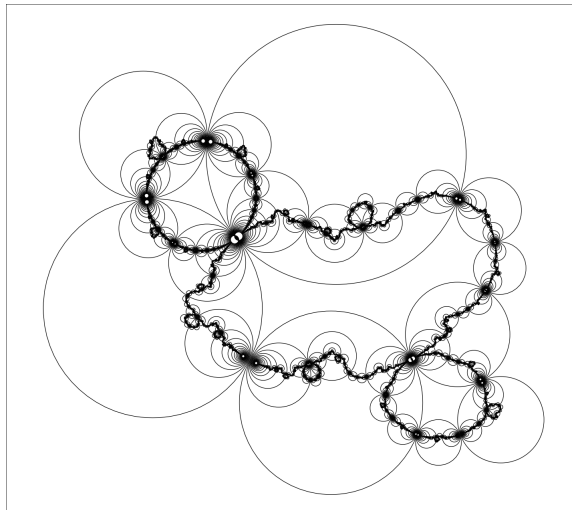


FIGURE 2. The limit set of a necklace group on the boundary of the Bers slice of the regular ideal 9-gon reflection group.

Theorem D (Mating reflection groups with anti-polynomials). A postcritically finite, hyperbolic anti-polynomial of degree d and a necklace group of rank $d + 1$ are conformally mateable if and only if they are topologically mateable.

All previously known examples of conformal matings of postcritically finite anti-polynomials with necklace groups, namely, matings of quadratic postcritically finite anti-polynomials with the ideal triangle reflection group [LLMM18a, LLMM18b], and matings of $\bar{z}^d$ with all necklace groups of rank $d + 1$ [LMM20], are easily recovered from Theorem D (see Subsection 11.1). To illustrate the power of this theorem, we construct first examples of conformal matings of anti-polynomials and necklace groups, neither of which has a Jordan curve Julia/limit set (see Subsections 11.2 and 11.3).

Since the topological mating of a necklace group Γ and a postcritically finite anti-polynomial P is not defined on the whole 2-sphere, one needs to build a novel realization theory to prove Theorem D. The principal difficulty in this theory arises from the mismatch between the quantitative behavior of the Julia dynamics of a hyperbolic anti-polynomial and the limit set dynamics of the Nielsen map of a reflection group. Indeed, the former has only hyperbolic fixed points on its Julia set, while the latter has parabolic fixed points on its limit set, which renders purely quasiconformal tools inapplicable to this setting. This problem can be tackled in two steps. One can use *W. Thurston's topological characterization theorem* to construct an anti-rational map R that is a conformal mating of P and another anti-polynomial P_Γ such that the Julia dynamics of P_Γ is topologically conjugate to the limit set dynamics of the Nielsen map of Γ . The existence of such an anti-polynomial P_Γ follows from [LMM20] or [LLM20] (making use of Poirier's realization of Hubbard trees by anti-polynomials [Poi10, Poi13]), while conformal mateability of P and P_Γ follows from a general mateability criterion proved in [LLM20]. Subsequently, we apply Theorem A and our David surgery tools to conformally glue the reflection group dynamics into suitable invariant Fatou components of R . This produces the desired conformal mating of Γ and P , which turns out to be the Schwarz reflection map associated with finitely many disjoint quadrature domains. The proof of Theorem D is carried out in Section 10, we refer the reader to Theorem 10.20 for a precise statement. The mating construction is illustrated with various explicit examples in Section 11, see Figures 9, 10, 12, and 14.

While the Extension Theorem A is sufficient for the purpose of mating anti-polynomials with necklace reflection groups, the general extension Theorem 4.9 yields a mating result for partially defined conformal dynamical systems on the closed unit disk. More precisely, we show that any two expansive piecewise analytic covering maps of the unit circle admitting conformal extensions satisfying conditions (4.1) and (4.2) can be mated conformally, as long as each of their periodic points is symmetrically parabolic or hyperbolic. (We allow ourselves to match parabolic and hyperbolic points in an arbitrary way). This is stated as Theorem 5.2.

The final application of our hyperbolic-parabolic surgery theory concerns certain extremal problems in spaces of univalent functions. It is well-known that extremal points of the classically studied space Σ of (suitably normalized) schlicht functions on the exterior disk $\mathbb{D}^* := \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ are the so-called *full mappings*; i.e., f is an extremal point of Σ if and only if $\mathbb{C} \setminus f(\mathbb{D}^*)$ has zero area [Dur83, §9.6]. These extremal points play an important role in the study of coefficient bounds for the space Σ (the analogue of the Bieberbach Conjecture/De Brange's Theorem is still open for class Σ), but the purely transcendental nature of these extremal maps add to their complexity. For each $d \geq 2$, the compact subspace $\Sigma_d^* \subset \Sigma$ consists of maps in Σ that extend as degree $d + 1$ rational maps of $\widehat{\mathbb{C}}$ with the maximal number of critical points on $\mathbb{S}^1$. The union of these subspaces is dense in Σ . The extremal points f of Σ_d^* correspond to maps with the maximal number of singular points on $f(\mathbb{S}^1)$: $d + 1$ cusps and $d - 2$ double points. This allows one to describe the 'shape' of extremal maps of Σ_d^* in terms of certain combinatorial trees with angle data, which we call *bi-angled trees*. Using the tools developed in this paper, we give a new proof of the following result that recently appeared in [LMM19].

Theorem E (Extremal points for schlicht functions). Extremal points of Σ_d^* are classified by bi-angled trees with $d - 1$ vertices.

We refer the readers to Theorem 12.8 for a precise formulation of Theorem E. Let us also remark that while the problem of classifying the extremal points of Σ_d^* is a non-dynamical one, the proof of Theorem E makes essential use of the dynamics of Schwarz reflection maps associated with the members of Σ_d^* . With this perspective, the classification problem becomes a problem of realization and rigidity of certain Schwarz reflection dynamical systems. In fact, the desired Schwarz reflection maps are realized using David surgery techniques developed in the paper. On the other hand, rigidity of such Schwarz reflection maps is a consequence of conformal removability of their limit sets.

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2. PRELIMINARIES ON DAVID HOMEOMORPHISMS

2.1. David Integrability Theorem. Recall that if $U \subset \mathbb{C}$ is an open set and $p \in [1, \infty]$, a measurable function $f: U \rightarrow \mathbb{C}$ lies in the Sobolev space $W_{\text{loc}}^{1,p}(U)$ if $f \in L_{\text{loc}}^p(U)$ and f has distributional derivatives lying in $L_{\text{loc}}^p(U)$. We say that a function $f: U \rightarrow \mathbb{C}$ lies in $W^{1,p}(U)$ if $f \in W_{\text{loc}}^{1,p}(U)$ and the quantity

$$\|f\|_{W^{1,p}(U)} := \|f\|_{L^p(U)} + \|Df\|_{L^p(U)}$$

is finite. Here, Df denotes the differential matrix of f and $\|Df\|_{L^p(U)}$ denotes the L^p norm of the operator norm of Df . Moreover, the L^p spaces are with respect to the Lebesgue measure of $\mathbb{C}$.

Consider sets U_i , $i \in \{1, 2\}$, and suppose that each of them is an open subset of either $\mathbb{C}$ or $\widehat{\mathbb{C}}$. We say that a measurable function $f: U_1 \rightarrow U_2$ lies in $W_{\text{loc}}^{1,p}(U_1 \rightarrow U_2)$ if it lies in $W_{\text{loc}}^{1,p}(U_1)$ in local coordinates. Consider the derivative $D_{(d_1, d_2)}f$ of f , regarded as a map between the Riemannian manifolds (U_1, d_1, μ_1) and (U_2, d_2, μ_2) ; here d_i, μ_i are the Euclidean metric and measure if the set U_i is a subset of the plane and d_i, μ_i are the spherical metric and measure, if U_i is a subset of $\widehat{\mathbb{C}}$. Note that in planar coordinates the derivative $D_{(d_1, d_2)}f$ is just a multiple of the Euclidean differential Df . We say that $f \in W^{1,p}(U_1 \rightarrow U_2)$ if $d_2(f, 0) \in L^p(U_1; \mu_1)$ and $D_{(d_1, d_2)}f \in L^p(U_1; \mu_1)$. We note that if the point at ∞ does not lie in the closure of any of the sets U_1 and U_2 then the space $W^{1,p}(U_1 \rightarrow U_2)$ coincides as a set with $W^{1,p}(U_1)$, since the Euclidean and spherical metrics of U_1 and U_2 are comparable. In all cases we will be using the simplified notation $W_{\text{loc}}^{1,p}(U_1)$ and $W^{1,p}(U_1)$, whenever it does not lead to a confusion.

An orientation-preserving homeomorphism $H: U \rightarrow V$ between domains in the Riemann sphere $\widehat{\mathbb{C}}$ is called a *David homeomorphism* if it lies in the Sobolev class $W_{\text{loc}}^{1,1}(U)$ and there exist constants $C, \alpha, K_0 > 0$ with

$$(2.1) \quad \sigma(\{z \in U : K_H(z) \geq K\}) \leq Ce^{-\alpha K}, \quad K \geq K_0.$$

Here σ is the spherical measure and K_H is the distortion function of H , given by

$$K_H(z) = \frac{1 + |\mu_H|}{1 - |\mu_H|},$$

where

$$\mu_H = \frac{\partial H / \partial \bar{z}}{\partial H / \partial z}$$

is the Beltrami coefficient of H . By condition (2.1), K_H is finite a.e. and μ_H takes values in $\mathbb{D}$ a.e. Condition (2.1) is equivalent to

$$(2.2) \quad \sigma(\{z \in U : |\mu_H(z)| \geq 1 - \varepsilon\}) \leq C' e^{-\alpha'/\varepsilon}, \quad \varepsilon \leq \varepsilon_0,$$

where $C', \alpha', \varepsilon_0$ depend only on C, α, K_0 . Moreover, another condition equivalent to (2.1) is the exponential integrability of K_H :

$$\int_U \exp(pK_H) d\sigma < \infty$$

for some $p > 0$, where p and α are related to each other. Hence, David homeomorphisms are also called homeomorphisms of *exponentially integrable distortion*. The theory of such mappings has been developed to a great extent over the past decades. We direct the reader to [AIM09, Chapter 20] for more background.

We list some properties of David homeomorphisms. Let $H: U \rightarrow V$ be such a map. First, H and H^{-1} are *absolutely continuous in measure*:

$$\sigma(H(E)) = 0 \quad \text{if and only if} \quad \sigma(E) = 0$$

for all measurable sets $E \subset U$; see [AIM09, Theorem 20.4.21]. Second, H satisfies the change of coordinates formula

$$(2.3) \quad \int_V g d\sigma = \int_U g \circ H \cdot J_H^\sigma d\sigma$$

for all non-negative Borel measurable functions g on V , where J_H^σ is the spherical Jacobian of H and J_H is the usual Jacobian determinant. This result is due to Federer [Fed69, Theorem 3.2.5, p. 244] and holds in much more general settings. In particular, (2.3) implies that $J_H^\sigma \neq 0$ a.e.

The main result in the theory of David homeomorphisms is the following integrability theorem. If U is an open subset of $\widehat{\mathbb{C}}$ and $\mu: U \rightarrow \mathbb{D}$ is a measurable function such that $(1 + |\mu|)/(1 - |\mu|)$ is exponentially integrable in U , then we say that μ is a *David coefficient in U* or just a *David coefficient* if U is implicitly understood.

Theorem 2.1 (David Integrability Theorem, [Dav88], [AIM09, Theorem 20.6.2, p. 578]). *Let $\mu: \widehat{\mathbb{C}} \rightarrow \mathbb{D}$ be a David coefficient. Then there exists a homeomorphism $H: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ of class $\in W^{1,1}(\widehat{\mathbb{C}})$ that solves the Beltrami equation*

$$\frac{\partial H}{\partial \bar{z}} = \mu \frac{\partial H}{\partial z}.$$

Moreover, H is unique up to postcomposition with Möbius transformations.

The David Integrability Theorem is a generalization of the Measurable Riemann Mapping Theorem [AIM09, Theorem 5.3.4, p. 170], which states that if $\|\mu\|_\infty < 1$, then there exists a quasiconformal homeomorphism $H: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ (thus, of class $W^{1,2}(\widehat{\mathbb{C}})$) that solves the Beltrami equation

$$\frac{\partial H}{\partial \bar{z}} = \mu \frac{\partial H}{\partial z}.$$

The uniqueness part in Theorem 2.1 also holds locally and we have the following factorization theorem.

Theorem 2.2 ([AIM09, Theorem 20.4.19, p. 565]). *Let $\Omega \subset \widehat{\mathbb{C}}$ be an open set and $f, g: \Omega \rightarrow \widehat{\mathbb{C}}$ be David embeddings with*

$$\mu_f = \mu_g$$

almost everywhere. Then $f \circ g^{-1}$ is a conformal map on $g(\Omega)$.

We note that, unlike the theory of quasiconformal mappings, the inverse of a David homeomorphism is not necessarily a David map. The following simple example is given in [Zak04, p. 123]. The homeomorphism $\phi: \mathbb{D} \rightarrow \mathbb{D}$ defined by

$$\phi(re^{i\theta}) = \frac{1}{\log(1/r) + 1} e^{i\theta}$$

is a David map but its inverse is not a David map. Indeed $K_\phi(re^{i\theta}) \simeq \log(1/r)$, which is exponentially integrable near 0, but $K_{\phi^{-1}}(re^{i\theta}) \simeq \log(1/r)^2$, which is not exponentially integrable near 0.

2.2. David extensions of circle homeomorphisms. We will need an extension result for David maps, which is a generalization of the well-known extension of Beurling and Ahlfors [BA56]. Let $h: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be an orientation-preserving homeomorphism. We define the *distortion function* of h to be

$$\rho_h(z, t) = \max \left\{ \frac{|h(e^{2\pi i t} z) - h(z)|}{|h(e^{-2\pi i t} z) - h(z)|}, \frac{|h(e^{-2\pi i t} z) - h(z)|}{|h(e^{2\pi i t} z) - h(z)|} \right\},$$

where $z \in \mathbb{S}^1$ and $0 < t < 1/2$. We define the *scalewise distortion function* of h to be

$$\rho_h(t) = \max_{z \in \mathbb{S}^1} \rho_h(z, t),$$

where $0 < t < 1/2$. If $\rho_h(t)$ is bounded above, then the function h is a quasimetry and has a quasiconformal extension on $\mathbb{D}$, by the theorem of Beurling and Ahlfors. Zakeri observed in [Zak08, Theorem 3.1], by applying a result of [CCH96], that there is a growth condition on $\rho_h(t)$ that is sufficient for a homeomorphism h of the circle to have a David extension in the disk.

Theorem 2.3 ([CCH96, Theorem 3], [Zak08, Theorem 3.1]). *Let $h: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be an orientation-preserving homeomorphism and suppose that*

$$\rho_h(t) = O(\log(1/t)) \quad \text{as } t \rightarrow 0.$$

Then h has an extension to a David homeomorphism $\tilde{h}: \mathbb{D} \rightarrow \mathbb{D}$.

In fact, Zakeri discusses the extensions of homeomorphisms H of $\mathbb{R}$ that commute with the function $z \mapsto z + 1$. These homeomorphisms arise as lifts of circle homeomorphisms h under the universal covering map $z \mapsto e^{2\pi i z}$. Then he simply takes the Beurling–Ahlfors extension $\tilde{H}$ of H to the upper half-plane and observes that [CCH96, Theorem 3] implies that under the condition of Theorem 2.3, translated to the homeomorphism H of $\mathbb{R}$, the map $\tilde{H}$ is a David map of the upper half-plane. Finally, the extension $\tilde{H}$ descends to a David extension $\tilde{h}$ of h in $\mathbb{D}$, as it is pointed out in [Zak08, p. 243].

Remark 2.4. If $h: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is an orientation-preserving homeomorphism that is real-symmetric, i.e., $h(\bar{z}) = \overline{h(z)}$ for $z \in \mathbb{S}^1$, then the David extension of h that is given by Theorem 2.3 has the same property. Indeed, let H denote the lift of h to the real line, under the universal cover $z \mapsto e^{2\pi iz}$. The real-symmetry of h is equivalent to the condition $H(1-z) = 1-H(z)$ for all $z \in \mathbb{R}$. Let $\tilde{H}$ be the Beurling-Ahlfors extension of H . The Beurling-Ahlfors extension operator is equivariant under precomposition and postcomposition with linear maps of the form $z \mapsto az + b$, $a > 0$, $b \in \mathbb{R}$. That is, if T_1 and T_2 are such linear maps, then the Beurling-Ahlfors extension of $T_2 \circ H \circ T_1$ is the map $T_2 \circ \tilde{H} \circ T_1$; see [Zak08, p. 246]. In our case, this implies that $\tilde{H}(1-z) = 1-\tilde{H}(z)$ for all z in the upper half-plane. Equivalently, the extension $\tilde{h}$ of h to the disk $\mathbb{D}$ is real-symmetric, as claimed.

2.3. Composition with quasiconformal maps. We also discuss the invariance of the David property under quasiconformal maps. Recall that a *quasidisk* is the image of the unit disk under a quasiconformal map of $\hat{\mathbb{C}}$. A domain $\Omega \subset \hat{\mathbb{C}}$ is a *John domain* if for each base point $z_0 \in \Omega$ there exists a constant $c > 0$ such that for each point $z_1 \in \Omega$ there exists a simple path γ joining z_0 to z_1 in Ω with the property that for each point z on the path γ we have

$$\text{dist}_\sigma(z, \partial\Omega) \geq c \text{length}_\sigma(\gamma|_{[z, z_1]}),$$

where $\gamma|_{[z, z_1]}$ denotes the subpath of γ whose endpoints are z and z_1 . Here we are using the spherical distance and spherical length. Intuitively, John domains can have inward cusps but not outward cusps and they are a generalization of a quasidisks, which cannot have any cusps at all.

Proposition 2.5. *Let $f: U \rightarrow V$ be a David homeomorphism between open sets $U, V \subset \hat{\mathbb{C}}$.*

- (i) *If $g: V \rightarrow \hat{\mathbb{C}}$ is a quasiconformal embedding then $g \circ f$ is a David map.*
- (ii) *If $W \subset \hat{\mathbb{C}}$ is an open set and $g: W \rightarrow U$ is a quasiconformal homeomorphism that extends to a quasiconformal homeomorphism of an open neighborhood of $\overline{W}$ onto an open neighborhood of $\overline{U}$, then $f \circ g$ is a David map.*
- (iii) *If $W \subset \hat{\mathbb{C}}$ is an open set, and $g: W \rightarrow U$ is a non-constant quasiregular map that extends to a quasiregular map in an open neighborhood of $\overline{W}$, then the function*

$$\mu_{f \circ g} = \frac{\partial(f \circ g)/\partial \bar{z}}{\partial(f \circ g)/\partial z}$$

is a David coefficient in W .

- (iv) *If U is a quasidisk, W is a John domain, and $g: W \rightarrow U$ is a quasiconformal homeomorphism, then $f \circ g$ is a David map.*

Note that (ii) and (iii) imply that if μ is a David coefficient and g is a quasiconformal or quasiregular map, then the pullback $g^*\mu$ is also a David coefficient. The proposition actually requires a David map f with $\mu_f = \mu$; however, there is always such a map f given by Theorem 2.1.

The proof relies partially on [HK14, Theorem 5.13], which concerns the composition of Sobolev functions with Sobolev homeomorphisms. We quote below some special instances of the theorem that we will use.

Theorem 2.6 ([HK14, Theorem 5.13]). *Let Ω_1, Ω_2 be open subsets of $\mathbb{C}$ and let $F: \Omega_1 \rightarrow \Omega_2$ be a homeomorphism with $F \in W_{\text{loc}}^{1,1}(\Omega_1)$ and $J_F \neq 0$ a.e. We set $K_F = \|DF\|^2/J_F$ a.e. in Ω_1 .*

- (i) *If $K_F \in L_{\text{loc}}^1(\Omega_1)$ and $g \in W_{\text{loc}}^{1,2}(\Omega_2)$, then $g \circ F \in W_{\text{loc}}^{1,1}(\Omega_1)$.*
- (ii) *If $K_F \cdot \|DF\|^{-\varepsilon} \in L_{\text{loc}}^{1/(1-\varepsilon)}(\Omega_1)$ for some $\varepsilon \in (0, 1)$ and $g \in W_{\text{loc}}^{1,2-\varepsilon}(\Omega_2)$, then $g \circ F \in W_{\text{loc}}^{1,1}(\Omega_1)$.*

Moreover, we have the usual chain rule:

$$D(g \circ F)(z) = Dg(F(z)) \circ DF(z)$$

for a.e. $z \in \Omega_1$.

Recall that $\|Df\|$ denotes the operator norm of the differential Df of f and the L^p spaces are with respect to the Lebesgue measure of $\mathbb{C}$.

Proof of Proposition 2.5. (i) First, we reduce the assertion to the case that U and V are planar sets. If $U = V = \widehat{\mathbb{C}}$, we precompose f and we postcompose g with suitable isometries of $\widehat{\mathbb{C}}$ so that f and g fix the point at ∞ . Thus, we may assume that $f: \mathbb{C} \rightarrow \mathbb{C}$ is a David map and $g: \mathbb{C} \rightarrow \mathbb{C}$ is a quasiconformal map. If U, V are strict subsets of $\widehat{\mathbb{C}}$, by precomposing and postcomposing f and g with suitable isometries of $\widehat{\mathbb{C}}$, we can assume that $U, V \subset \mathbb{C}$.

We note that if $f: U \rightarrow V$ is a David homeomorphism, then $\exp(pK_f) \in L_{\text{loc}}^1(U)$, so $K_f \in L_{\text{loc}}^1(U)$. By Theorem 2.6 (i), we obtain that if $g \in W_{\text{loc}}^{1,2}(V)$, then $g \circ f \in W_{\text{loc}}^{1,1}(U)$ and

$$D(g \circ f)(z) = Dg(f(z)) \circ Df(z)$$

for a.e. $z \in U$.

Suppose that g is K -quasiconformal as in the statement, so that $g \in W_{\text{loc}}^{1,2}(V)$ and

$$\|Dg(w)\|^2 \leq KJ_g(w)$$

for a.e. $w \in V$. Note that $Df(z)$ exists for a.e. $z \in U$ and $Dg(w)$ exists for a.e. $w \in V$. Since f^{-1} is absolutely continuous, it follows that the set of $z \in U$ such that $Dg(f(z))$ does not exist has measure zero. Hence, for a.e. $z \in U$ we have

$$\begin{aligned} \|D(g \circ f)(z)\|^2 &= \|Dg(f(z)) \circ Df(z)\|^2 \leq \|Dg(f(z))\|^2 \|Df(z)\|^2 \\ &\leq KK_f(z)J_g(f(z))J_f(z) = KK_f(z)J_{g \circ f}(z). \end{aligned}$$

It follows that $K_{g \circ f} \leq KK_f$ a.e. and thus $K_{g \circ f}$ is exponentially integrable, as desired.

(ii) As in (i), by suitable compositions with isometries of $\widehat{\mathbb{C}}$, we may assume that U, V, W are planar sets. In this case, if g is K -quasiconformal, then

$$\frac{\|Dg\|^{2-\varepsilon}}{J_g} \leq K\|Dg\|^{-\varepsilon} \leq KJ_g^{-\varepsilon/2}.$$

It is known that $J_g^{-\delta}$ is locally integrable for a small $\delta > 0$; see e.g. [AIM09, Theorem 13.4.2, p. 345]. Hence, $\|Dg\|^{2-\varepsilon}/J_g \in L_{\text{loc}}^{1/(1-\varepsilon)}(W)$. On the other hand, if f is a David homeomorphism, then $f \in W_{\text{loc}}^{1,2-\varepsilon}(U)$ for all $\varepsilon > 0$ by [AIM09, (20.69), p. 557]. By Theorem 2.6 (ii), it follows that $f \circ g \in W_{\text{loc}}^{1,1}(W)$.

Next, we wish to show that

$$\int_W \exp(pK_{f \circ g}) d\sigma < \infty$$

for some $p > 0$. Arguing as in (i), we have $K_{f \circ g} \leq KK_f(g(w))$ for a.e. $w \in W$. By changing coordinates, we have

$$\begin{aligned} \int_W \exp(pK_{f \circ g}) d\sigma &\leq \int_W \exp(pK \cdot K_f \circ g) d\sigma \\ &= \int_W \exp(pK \cdot K_f(g(w))) J_g^\sigma(w) J_{g^{-1}}^\sigma(g(w)) d\sigma(w) \\ &= \int_U \exp(pK \cdot K_f(z)) J_{g^{-1}}^\sigma(z) d\sigma(z) \\ &\leq \left(\int_U \exp(qpK \cdot K_f(z)) d\sigma(z) \right)^{1/q} \left(\int_U J_{g^{-1}}^\sigma(z)^{q'} d\sigma(z) \right)^{1/q'}, \end{aligned}$$

where $1/q + 1/q' = 1$. The second factor is finite if $q' > 1$ is sufficiently close to 1, since g^{-1} is quasiconformal in a neighborhood of $\bar{U}$ by assumption; see [AIM09, Theorem 13.4.2, p. 345]. The first factor is also finite for a sufficiently small $p > 0$, since f is a David homeomorphism.

(iii) We show, first, that the Beltrami coefficient

$$\mu_{f \circ g} = \frac{\partial(f \circ g)/\partial \bar{z}}{\partial(f \circ g)/\partial z}$$

is defined a.e. in W and takes values in $\mathbb{D}$. If $w \in W$ is not a critical point of g , then there exists a neighborhood O of w in which g is quasiconformal. By (ii) we conclude that $f \circ g$ is a David map in O . Therefore, $\mu_{f \circ g}$ is defined in O and takes values in $\mathbb{D}$. Since g is quasiregular and non-constant, it has at most countably many critical points in W . It follows that $\mu_{f \circ g}$ and $K_{f \circ g} = (1 + |\mu_{f \circ g}|)/(1 - |\mu_{f \circ g}|)$ are defined a.e.

Next, we will show that $K_{f \circ g}$ is exponentially integrable. First, we reduce to the case that g is holomorphic. Suppose that g is K -quasiregular in a neighborhood Z of $\bar{W}$. By the measurable Riemann mapping theorem, there exists a K -quasiconformal homeomorphism $\tilde{g}$ of $\hat{\mathbb{C}}$ such that $\mu_{\tilde{g}} = \mu_g \cdot \chi_Z$. Moreover, the map $h = g \circ (\tilde{g})^{-1}$ is holomorphic in $\tilde{Z} = \tilde{g}(Z)$ and we set $\tilde{W} = \tilde{g}(W)$. We note that $K_{f \circ g} = K_{f \circ h \circ \tilde{g}} \leq KK_{f \circ h}(\tilde{g}(w))$ as in (ii). The exact same computation of (ii) implies that since

$$\int_{\tilde{W}} J_{\tilde{g}^{-1}}^\sigma(z)^{q'} d\sigma(z) < \infty$$

for some $q' > 1$ by the quasiconformality of $\tilde{g}$, it suffices to show that

$$\int_{\tilde{W}} \exp(pK_{f \circ h}) d\sigma < \infty$$

for some $p > 0$. Thus, we have reduced to the case that $g: W \rightarrow U$ is holomorphic and has a holomorphic extension to a neighborhood Z of $\bar{W}$.

Let $w_0 \in Z$ and let $O(w_0) \subset \overline{O(w_0)} \subset Z$ be an open neighborhood of w_0 . As in (ii) we have

$$\begin{aligned} \int_{O(w_0) \cap W} \exp(pK_{f \circ g}) d\sigma &\leq \int_{O(w_0) \cap W} \exp(pK_f(g(w))) J_g^\sigma(w) J_g^\sigma(w)^{-1} d\sigma(w) \\ &\leq \left(\int_{O(w_0) \cap W} \exp(qpK_f(g(w))) J_g^\sigma(w) d\sigma(w) \right)^{1/q} \\ &\quad \cdot \left(\int_{O(w_0)} J_g^\sigma(w)^{1-q'} d\sigma(w) \right)^{1/q'} \\ &= \left(\int_{g(O(w_0)) \cap U} \exp(qpK_f(z)) d\sigma(z) \right)^{1/q} \\ &\quad \cdot \left(\int_{O(w_0)} J_g^\sigma(w)^{1-q'} d\sigma(w) \right)^{1/q'}, \end{aligned}$$

where $1/q + 1/q' = 1$. For each choice of q' there exists a $p > 0$ so that the first factor is finite, since f is a David map on U . We claim that the second factor is finite if the neighborhood $O(w_0)$ of w_0 is sufficiently small and q' is sufficiently close to 1.

By using an isometry of $\widehat{\mathbb{C}}$ we assume that $w_0 = 0$ and we set $O = O(w_0)$. If O is a sufficiently small neighborhood of the origin then the spherical Jacobian and the spherical measure are comparable to the Euclidean ones. Therefore, it suffices to show that

$$\int_O J_g(w)^{-\delta} dw < \infty$$

for a sufficiently small $\delta > 0$ and a sufficiently small neighborhood O of the origin. Suppose that g is m -to-1 at the origin for some $m \geq 1$. Then there exists a neighborhood O of the origin and a conformal map ϕ on O with $\phi(0) = 0$ such that $g(w) = \phi(w)^m$ for all $w \in O$. We have $J_g = |g'|^2 = m^2 |\phi|^{2m-2} |\phi'|^2$. By shrinking the neighborhood O , we may have $|\phi'| \simeq 1$ and $|\phi(w)| \simeq |w|$ for all $w \in O$. Hence $J_g(w) \simeq |w|^{2m-2}$ for $w \in O$. Now, we have

$$\int_O J_g(w)^{-\delta} dw \simeq \int_O |w|^{-\delta(2m-2)} dw,$$

which is finite for all small $\delta > 0$, as desired.

Summarizing, we have proved that each point $w_0 \in Z$ has a neighborhood $O(w_0)$ in Z such that

$$\int_{O(w_0) \cap W} \exp(pK_{f \circ g}) d\sigma < \infty.$$

Since $\overline{W}$ is a compact subset of Z , we can cover it by finitely many neighborhoods $O(w_0)$ of points $w_0 \in \overline{W}$. We conclude that

$$\int_W \exp(pK_{f \circ g}) d\sigma < \infty,$$

as desired.

(iv) As in (ii), it follows that $f \circ g \in W_{\text{loc}}^{1,1}(W)$. In order to obtain the exponential integrability of $K_{f \circ g}$, we only have to justify that

$$(2.4) \quad \int_U J_{g^{-1}}^\sigma(z)^{q'} d\sigma(z) < \infty$$

for some $q' > 1$, close to 1.

In order to prove this, we need the notion of a *uniform domain*. An open set $\Omega \subset \mathbb{C}$ is a uniform domain if there exists a constant $c > 0$ such that for any two points $a, b \in \Omega$ there exists a continuum $E \subset \Omega$ that connects them with $\text{diam}(E) \leq c|a - b|$ and the set

$$\bigcup_{x \in E} B(x, c^{-1} \min\{|x - a|, |x - b|\}),$$

which is called a c -cigar, is contained in Ω . A bounded uniform domain is also a John domain. Other uniform domains include quasidisks and annuli bounded by two quasicircles. We direct the reader to the work of Martio and Väisälä [MV88] and the references therein for more background on uniform and John domains.

The desired condition (2.4) follows from a result of Martio and Väisälä [MV88, Theorem 2.16], which implies that a quasiconformal map from a *bounded* uniform domain onto a John domain has Jacobian lying in $L^{1+\varepsilon}$ for some small $\varepsilon > 0$. The quoted result does not apply immediately to g^{-1} , since the quasidisk U , which is a uniform domain, is not necessarily bounded as a subset of the plane.

In order to apply the result, we first precompose and postcompose g with suitable isometries of $\widehat{\mathbb{C}}$ so that the point at ∞ lies in W and U , and g fixes ∞ . Next, we remove from W a closed ball $B \subset W$ that contains ∞ . The set $W \setminus B$ is still a John domain, with a constant that is possibly different from the constant of W . We note that $W \setminus B$ is a John domain even if we use the Euclidean rather than the spherical metric in the definition of a John domain. Moreover, $U \setminus g(B)$ is a uniform domain since it is an annulus bounded by two quasicircles. Now the map $g^{-1}: U \setminus g(B) \rightarrow W \setminus B$ satisfies the assumptions of [MV88, Theorem 2.16], so $J_{g^{-1}} \in L^{1+\varepsilon}(U \setminus g(B))$ for some $\varepsilon > 0$. This implies that

$$\int_{U \setminus g(B)} J_{g^{-1}}^\sigma(z)^{1+\varepsilon} d\sigma(z) < \infty$$

since the Euclidean and spherical measures are comparable in the bounded set $U \setminus g(B)$. Finally, we also have

$$\int_{g(B)} J_{g^{-1}}^\sigma(z)^{1+\varepsilon} d\sigma(z) < \infty$$

for a possibly smaller $\varepsilon > 0$ by the local regularity of the Jacobian of a quasiconformal map; see [AIM09, Theorem 13.4.2, p. 345]. $\square$

2.4. David maps and removability. A compact set $E \subset \widehat{\mathbb{C}}$ is *conformally removable* if every homeomorphism $f: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ that is conformal on $\widehat{\mathbb{C}} \setminus E$ is a Möbius transformation. We also say that E is *locally conformally removable* if for every open set $U \subset \widehat{\mathbb{C}}$, *not necessarily containing* E , every homeomorphic embedding $f: U \rightarrow \widehat{\mathbb{C}}$ that is conformal on $U \setminus E$ is conformal on U . It is an open problem whether every removable set is locally removable.

While quasiconformal maps preserve the quality of conformal removability, it is not clear whether David homeomorphisms do so. However, they interact with a stronger notion of removability; namely, *removability for the Sobolev space $W^{1,1}$* . A compact set $E \subset \widehat{\mathbb{C}}$ is removable for $W^{1,1}$ functions if every continuous function $f: \widehat{\mathbb{C}} \rightarrow \mathbb{R}$ that lies in $W^{1,1}(\widehat{\mathbb{C}} \setminus E)$ lies actually in $W^{1,1}(\widehat{\mathbb{C}})$. Equivalently, E is removable for $W^{1,1}$ functions if for every open set U , *not necessarily containing E* , every continuous function $f: U \rightarrow \mathbb{R}$ that lies in $W^{1,1}(U \setminus E)$ lies in $W^{1,1}(U)$. The last assertion is discussed before Definition 1.2 in [Nta20a]. Hence, in the case of removability for Sobolev spaces the notions of removability and local removability agree.

Theorem 2.7 (Conformal removability). *Suppose that $E \subset \widehat{\mathbb{C}}$ is a compact set that is removable for $W^{1,1}$ functions and $f: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ is a David homeomorphism. Then $f(E)$ is locally conformally removable.*

We will prove this using the following auxiliary result.

Lemma 2.8 (David removability). *Suppose that $E \subset \widehat{\mathbb{C}}$ is a compact set that is removable for $W^{1,1}$ functions, $U \subset \widehat{\mathbb{C}}$ is an open set, and $g: U \rightarrow \widehat{\mathbb{C}}$ is a homeomorphic embedding that is a David map in $U \setminus E$. Then g is a David map in U .*

Note that the set E is not necessarily contained in U .

Proof. By precomposing and postcomposing g with suitable isometries of $\widehat{\mathbb{C}}$, we may assume that E and $g(E)$ do not contain the point at ∞ and $g(\infty) = \infty$. Since g is a David map on $U \setminus E$, by definition, we have

$$\int_{U \setminus E} \exp(pK_g) d\sigma < \infty$$

for some $p > 0$. By [Nta19, Theorem 1.3], the set E must have measure zero, since it is removable for $W^{1,1}$ functions. Hence, we have

$$\int_U \exp(pK_g) d\sigma < \infty.$$

It suffices to show that $g \in W_{\text{loc}}^{1,1}(U)$.

We claim that if V is a bounded open subset of U , then $\|Dg\| \in L^1(V \setminus E)$. We have

$$\|Dg(z)\| = K_g(z)^{1/2} J_g(z)^{1/2}$$

for a.e. $z \in V \setminus E$. Note that $K_g \in L^1(V \setminus E)$ since $\exp(pK_g) \in L^1(V)$. Moreover, $J_g \in L^1(V \setminus E)$ since by the change of coordinates formula (2.3) (expressed in Euclidean coordinates) we have

$$\int_{V \setminus E} J_g(z) dz = \text{Area}(g(V \setminus E)) < \infty.$$

Here we used the normalization $g(\infty) = \infty$. It follows that $\|Dg\| \in L^1(V \setminus E)$ as desired.

Summarizing, if V is a bounded open subset of U containing $U \cap E$, then the map g lies in $W^{1,1}(V \setminus E)$ and is continuous on V . Since E is removable for the space $W^{1,1}$ by assumption, we have $g \in W^{1,1}(V)$. We conclude that $g \in W_{\text{loc}}^{1,1}(U)$, as desired. $\square$

Remark 2.9. In the proof we used [Nta19, Theorem 1.3], which asserts that if a compact set E is removable for $W^{1,p}$ functions, where $1 \leq p < \infty$, then it must have measure zero. We note that if a set is $W^{1,1}$ -removable then it is also $W^{1,2}$ -removable and if a set is $W^{1,2}$ -removable and has *measure zero*, then it is also conformally removable (by Weyl's lemma). However, the non-removability of sets of positive measure for conformal maps (which can be proved using the Measurable Riemann Mapping Theorem) does not imply in a straightforward way the non-removability of sets of positive measure for $W^{1,2}$ and $W^{1,1}$ functions. Hence, we employ [Nta19, Theorem 1.3].

Proof of Theorem 2.7. Let $U \subset \widehat{\mathbb{C}}$ be an open set and $h: U \rightarrow \widehat{\mathbb{C}}$ be a homeomorphic embedding that is conformal in $U \setminus f(E)$. Our goal is to show that h is conformal on U . Consider the map $g = h \circ f$. Since f is a David map and h is conformal on $U \setminus f(E)$, by Proposition 2.5 (i) it follows that g is a David map on $f^{-1}(U) \setminus E$. By Lemma 2.8 we conclude that g is a David map on $f^{-1}(U)$.

Note $\mu_g = \mu_f$ on $f^{-1}(U) \setminus E$ because h is conformal. Since E is removable for $W^{1,1}$ functions, it must have measure zero by [Nta19, Theorem 1.3]. Therefore, $\mu_g = \mu_f$ a.e. Finally, the factorization Theorem 2.2 implies that $h = g \circ f^{-1}$ is conformal, as desired. $\square$

Boundaries of John domains are removable for $W^{1,1}$ functions. This was proved by Jones and Smirnov [JS00, Theorem 4]. In fact, something stronger is true:

Theorem 2.10. *Let $\{\Omega_i\}_{i \in I}$ be a collection of finitely many, disjoint John domains in $\widehat{\mathbb{C}}$. Then $\bigcup_{i \in I} \partial\Omega_i$ is removable for $W^{1,1}$ functions.*

A similar statement is discussed without proof in [JS00, p. 265]. It can be proved using [Nta20a, Proposition 5.3] as follows.

Proof. Proposition 5.3 from [Nta20a] implies that for each $i \in I$, if f is a continuous function on $\overline{\Omega_i}$ that lies in $W^{1,1}(\Omega_i)$, then $f|L$ is absolutely continuous on $L \cap \overline{\Omega_i}$ for almost every line L , in the sense that f maps sets of linear measure zero to sets of linear measure zero. Hence, if f is continuous in $\widehat{\mathbb{C}}$ and $f \in W^{1,1}(\widehat{\mathbb{C}} \setminus \bigcup_{i \in I} \partial\Omega_i)$, then f is absolutely continuous on almost every line. It follows that $f \in W^{1,1}(\widehat{\mathbb{C}})$, since boundaries of John domains have measure zero. $\square$

By combining [JS00, Theorem 4] with Theorem 2.7, we have the following result.

Theorem 2.11. *Suppose that $\Omega \subset \widehat{\mathbb{C}}$ is a John domain and $f: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ is a David homeomorphism. Then, $f(\partial\Omega)$ is conformally removable.*

3. EXPANSIVE COVERING MAPS OF THE CIRCLE

Let $f, g: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be covering maps of degree $d \geq 2$, having the same orientation. Then under some expansion assumptions there exists an orientation-preserving homeomorphism $h: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ that conjugates the map f to g . Our goal, under suitable assumptions on the covering maps f and g , is to extend the conjugating homeomorphism h to a David self-map of the unit disk. This is the content of Section 4. The conditions on f and g will be in terms of a Markov partition associated to f and g . In particular, we will give conditions in the special, but very interesting case that one of the maps is $z \mapsto z^d$ or $z \mapsto \bar{z}^d$. In this section we discuss

the notion of an expansive covering map of the circle and the notion of a Markov partition associated to such a map.

If $a, b \in \mathbb{S}^1$, we denote by $\widehat{[a, b]}$ and $\widehat{(a, b)}$ the closed and open arcs, respectively, from a to b in the positive orientation. The arc $\widehat{(b, a)}$, for example, is the complementary arc of $\widehat{[a, b]}$. We also denote the arc $\widehat{(a, b)}$ by $\text{int } \widehat{[a, b]}$. We say that two non-overlapping arcs $I, J \subset \mathbb{S}^1$ are *adjacent* if they share an endpoint.

Definition 3.1. A *Markov partition* associated to a covering map $f: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is a covering of the unit circle by closed arcs $A_k = \widehat{[a_k, a_{k+1}]}$, $k \in \{0, \dots, r\}$, $r \geq 1$, that have disjoint interiors and satisfy the following conditions.

- (i) The map $f_k = f|_{\text{int } A_k}$ is injective for $k \in \{0, \dots, r\}$.
- (ii) If $f(\text{int } A_k) \cap \text{int } A_j \neq \emptyset$ for some $k, j \in \{0, \dots, r\}$, then $\text{int } A_j \subset f(\text{int } A_k)$.
- (iii) The set $\{a_0, \dots, a_r\}$ is invariant under f .

We denote the above Markov partition by $\mathcal{P}(f; \{a_0, \dots, a_r\})$.

Note that by definition the points $a_0, \dots, a_r$ are ordered in the positive orientation if $r \geq 2$; if $r = 1$, there is no natural order. Moreover, (ii) and (iii) are equivalent under condition (i).

We provide some more definitions. Let $f: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be a covering map and consider a Markov partition $\mathcal{P} = \mathcal{P}(f; \{a_0, \dots, a_r\})$. We can associate a matrix $B = (b_{kj})_{k,j=0}^r$ to $\mathcal{P}$ so that $b_{kj} = 1$ if $f_k(A_k) \supset A_j$ and $b_{kj} = 0$ otherwise. In the case $b_{kj} = 1$ we define $A_{kj} = f_k^{-1}(A_j)$. If $w = (j_1, \dots, j_n) \in \{0, \dots, r\}^n$, $n \in \mathbb{N}$, $k \in \{0, \dots, r\}$, and once A_w has been defined, we define $A_{kw} = f_k^{-1}(A_w)$ whenever $b_{kj_1} = 1$. A *word* $w = (j_1, \dots, j_n) \in \{0, \dots, r\}^n$, $n \in \mathbb{N}$, is *admissible (for the Markov partition $\mathcal{P}$)* if $b_{j_1 j_2} = \dots = b_{j_{n-1} j_n} = 1$. We also define $A_w = \emptyset$ if w is not admissible. The *length* of a word $w = (j_1, \dots, j_n) \in \{0, \dots, r\}^n$ is defined to be $|w| = n$. It follows from properties (i) and (ii) that for each $n \in \mathbb{N}$ the arcs A_w , where $|w| = n$, have disjoint interiors and their union is equal to $\mathbb{S}^1$. Inductively, we have $A_{wj} \subset A_w$ for all admissible words w and $j \in \{0, \dots, r\}$. If A_{wj} is non-empty, we say that A_{wj} is a *child* of A_w and A_w is the *parent* of A_{wj} . Thus, A_w has at most $r + 1$ children. We direct the reader to [Lyu20, Section 19.14] for more background on Markov partitions.

Definition 3.2. A continuous map $f: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is called *expansive* if there exists a constant $\delta > 0$ such that for any $a, b \in \mathbb{S}^1$ with $a \neq b$ we have $|f^{on}(a) - f^{on}(b)| > \delta$ for some $n \in \mathbb{N}$.

We now list some important properties of expansive maps of $\mathbb{S}^1$. Let $f: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be an expansive covering map.

- (E1) For all $n \in \mathbb{N}$ the map f^{on} has finitely many fixed points.
- (E2) Let $\mathcal{P}(f; \{a_0, \dots, a_r\})$ be a Markov partition. Then

$$\lim_{n \rightarrow \infty} \max\{\text{diam } A_w : |w| = n, w \text{ admissible}\} = 0.$$

- (E3) Suppose that the degree of f is $d \geq 2$. Then there exists an orientation-preserving homeomorphism $h: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ that conjugates f to either the map $g(z) = \bar{z}^d$ or the map $g(z) = z^d$, depending on whether f is orientation-reversing or orientation-preserving, respectively. Moreover, h is unique up to rotation by a $(d + 1)$ -st root of unity if $g(z) = \bar{z}^d$ or a $(d - 1)$ -st root of unity if $g(z) = z^d$.

Property (E2) can be proved easily using [PU11, Theorem 3.6.1, p. 143]. Property (E3) is a consequence of (E2). A more general statement for expansive self-maps of a compact manifold can be found in [CR80, Property (2'), p. 99]. A refined version of property (E3) that we will need for our considerations is the following lemma. Its proof is straightforward, based on property (E2), and is omitted.

Lemma 3.3. *Let $f, g: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be expansive covering maps with the same orientation and $\mathcal{P}(f; \{a_0, \dots, a_r\})$, $\mathcal{P}(g; \{b_0, \dots, b_r\})$ be Markov partitions. Consider the map $h: \{a_0, \dots, a_r\} \rightarrow \{b_0, \dots, b_r\}$ defined by $h(a_k) = b_k$ for $k \in \{0, \dots, r\}$ and suppose that h conjugates the map f to g on the set $\{a_0, \dots, a_r\}$, i.e.,*

$$h(f(a_k)) = g(b_k)$$

for $k \in \{0, \dots, r\}$. Then h has an extension to an orientation-preserving homeomorphism of $\mathbb{S}^1$ that conjugates f to g on $\mathbb{S}^1$.

Suppose that a is a *periodic point* of a covering map $f: \mathbb{S}^1 \rightarrow \mathbb{S}^1$. That is, there exists a minimal $n \in \mathbb{N}$ such that $f^{on}(a) = a$. The number n is called the *period* of a . We note that if f^{on} is orientation-preserving, then it maps an arc of the form $\overline{(z_1, a)}$ to an arc of the form $\overline{(z_2, a)}$. We say that f^{on} is the *first orientation-preserving return map to the periodic point a* and n is the *orientation-preserving period* of a . If f^{on} is orientation-reversing, then it maps an arc of the form $\overline{(z_1, a)}$ to an arc of the form $\overline{(a, z_2)}$. In the latter case, f^{o2n} is orientation-preserving, it maps an arc of the form $\overline{(z_1, a)}$ to an arc of the form $\overline{(z_3, a)}$ fixing a , and $2n$ is the smallest integer with that property. In this case f^{o2n} is the *first orientation-preserving return map to the periodic point a* and $2n$ is the *orientation-preserving period* of a . We denote by f_a the first orientation-preserving return map to a . In what follows we suppress the term “orientation-preserving” and the term “first return map” always refers to the first orientation-preserving return map.

We define the *one-sided multipliers* $\lambda(a^+), \lambda(a^-)$ of f at a to be the one-sided derivatives of the first return maps, if they exist:

$$\begin{aligned} \lambda(a^+) &= f'_a(a^+) = \lim_{\substack{z \rightarrow a \\ z \in (a, z_0)}} \frac{f_a(z) - a}{z - a} = \lim_{z \rightarrow a^+} \frac{f_a(z) - a}{z - a} \quad \text{and} \\ \lambda(a^-) &= f'_a(a^-) = \lim_{\substack{z \rightarrow a \\ z \in (z_0, a)}} \frac{f_a(z) - a}{z - a} = \lim_{z \rightarrow a^-} \frac{f_a(z) - a}{z - a}, \end{aligned}$$

where $z_0 \neq a$ is any point on $\mathbb{S}^1$. Observe that $\lambda(a^+), \lambda(a^-)$ are always non-negative real numbers, since f_a maps the circle to itself with positive orientation.

We extend this definition to preperiodic points. If a is a preperiodic point of f , then there exists a minimal $m \in \mathbb{N}$ such that $f^{om}(a)$ is periodic. We define

$$\lambda(a^\pm) = \lambda(f^{om}(a)^\pm)$$

if f^{om} is orientation-preserving and

$$\lambda(a^\pm) = \lambda(f^{om}(a)^\mp)$$

if f^{om} is orientation-reversing. We also define the orientation-preserving period of the preperiodic point a to be equal to the orientation-preserving period of the periodic point $f^{om}(a)$.

If f is expansive, we obtain some extra information about the one-sided multipliers.

(E4) Suppose that a is a periodic point of f . If the one-sided multiplier $\lambda(a^\pm)$ exists, then $\lambda(a^\pm) \geq 1$.

Indeed, if the conclusion of the statement were not true, then some orbits would be attracted to the periodic point a and this would contradict the expansivity of f .

Next, we give two general classes of expansive maps of the circle. We denote by $\mathbb{D}^*$ the complement of the closed unit disk $\overline{\mathbb{D}}$ in the Riemann sphere $\widehat{\mathbb{C}}$. For a Euclidean circle C in the plane, the bounded complementary component of C will be denoted by $\text{Int } C$. This is not to be confused with the interior of an arc I of $\mathbb{S}^1$, which is denoted by $\text{int } I$. The distinction will be clear from the context.

Example 3.4 (Blaschke products). Consider a Blaschke product

$$B(z) = e^{i\theta} \prod_{i=1}^d \frac{z - c_i}{1 - \overline{c_i}z},$$

where $\theta \in \mathbb{R}$, $d \geq 2$, and $c_1, \dots, c_d \in \mathbb{D}$. Then, $B: \mathbb{D} \rightarrow \mathbb{D}$ and $B: \mathbb{D}^* \rightarrow \mathbb{D}^*$ are branched coverings of degree d , and $B: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is a degree d covering.

Suppose further that B has a parabolic fixed point at $1 \in \mathbb{S}^1$. By the theory of parabolic fixed points [Mil06, §10], there exists a basin of attraction of the point 1, i.e., an open set of points whose iterates under B converge to 1. Since the action of B in $\mathbb{D}$ is conjugate under the map $z \mapsto 1/\overline{z}$ to the action of B in $\mathbb{D}^*$, it follows that the basin of attraction is symmetric with respect to the unit circle, and in particular it intersects both $\mathbb{D}$ and $\mathbb{D}^*$. Then, by the Denjoy–Wolff theorem [Mil06, Theorem 5.4], we conclude that the successive iterates B^{on} converge, uniformly on compact subsets of $\mathbb{D}$ (respectively, on $\mathbb{D}^*$), to the constant function $z \mapsto 1$. Note that if $B''(1) = 0$, then there are at least two attracting directions to the parabolic fixed point 1, and hence at least two immediate basins of attraction of 1. Hence, if $B''(1) = 0$, then the Fatou set of B must have at least two components, so its Julia set must be equal to $\mathbb{S}^1$. We remark that if $B''(1) \neq 0$, then the Julia set of B would be a Cantor set contained in $\mathbb{S}^1$.

According to the Riemann–Hurwitz formula, B has $(d-1)$ critical points (counted with multiplicity) in each of $\mathbb{D}$ and $\mathbb{D}^*$. Thus, B has no critical point on the Julia set. By [DU91, Theorem 4], it follows that if $B''(1) = 0$, then the map $B: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is expansive.

A particular example of such a Blaschke product is

$$B(z) = \frac{(d+1)z^d + (d-1)}{(d-1)z^d + (d+1)}, \quad d \geq 2.$$

The point 1 is a parabolic fixed point of B and $B''(1) = 0$.

Example 3.5 (Circle reflections). Consider ordered points $a_0 = a_{d+1}, a_1, \dots, a_d$, $d \geq 2$, on the circle $\mathbb{S}^1$. Let f_k be the reflection along the circle C_k that is orthogonal to the unit circle at the points a_k and a_{k+1} . Set $A_k = \widehat{[a_k, a_{k+1}]} \subset \mathbb{S}^1$, $k \in \{0, \dots, d\}$. We will assume that the points $a_0, \dots, a_d$ satisfy the following condition.

$$(\star) \quad \text{length}(\widehat{(a_k, a_{k+1})}) < \pi \quad \text{for } k \in \{0, \dots, d\}.$$

We now define a map $f: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ by $z \mapsto f_k(z)$ for $z \in A_k$. Since $f(A_k) = \bigcup_{k' \neq k} A_{k'}$, it follows that f is an orientation-reversing covering map of degree d with fixed point set $\{a_0, \dots, a_d\}$. Moreover, condition $(\star)$ guarantees that $|f'| \geq 1$

on $\mathbb{S}^1$ with equality precisely at the fixed points a_k . It will follow from Lemma 3.7 below that f is an expansive map.

Example 3.6 (Hybrid). Let $d \in \mathbb{N}$, $d \geq 2$, and $a_j = e^{\frac{2\pi j}{d+1}}$, $j \in \{0, 1, \dots, d\}$, be the fixed points of the map $z \mapsto \bar{z}^d$. We select some adjacent pairs

$$\{a_{j_1}, a_{j_1+1}\}, \dots, \{a_{j_m}, a_{j_m+1}\},$$

where the indices are taken modulo $d+1$. For each selected pair $\{a_{j_k}, a_{j_k+1}\}$, $k \in \{1, \dots, m\}$, let C_k be the circle that is orthogonal to the unit circle at the points a_{j_k} and a_{j_k+1} . If $A_j = \overline{[a_j, a_{j+1}]} \subset \mathbb{S}^1$, $j \in \{0, 1, \dots, d\}$, we can define a map $f: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ by setting it to be the reflection on A_{j_k} , $k \in \{1, \dots, m\}$, in the circle C_k , and on the remaining arcs A_j to be the map $z \mapsto \bar{z}^d$. The map f is an orientation-reversing covering map of degree d whose fixed point set is $\{a_0, \dots, a_d\}$. Moreover, we have that the map f is piecewise C^1 , and its set of non-differentiability is contained in $\{a_{j_1}, a_{j_1+1}, \dots, a_{j_m}, a_{j_m+1}\}$. The one-sided derivatives of f exist at each point of non-differentiability and the map f is C^1 on each closed arc A_j , $j \in \{0, 1, \dots, d\}$. Moreover, $|f'| \geq 1$ on A_j , $j \in \{0, 1, \dots, d\}$, with equality only at the endpoints of A_{j_k} , $k \in \{1, \dots, m\}$. It will follow from Lemma 3.7 below that f is an expansive map.

Lemma 3.7. *Let $f: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be a piecewise C^1 covering map that has one-sided derivatives at points of non-differentiability, it is C^1 on the closure of each complementary arc of the non-differentiability set, and such that*

- (i) *f has finitely many fixed points, and each non-differentiability point is fixed,*
- (ii) *$|f'| \geq 1$ on $\mathbb{S}^1$, where at points of non-differentiability we assume that the magnitudes of both one-sided derivatives are ≥ 1 , and*
- (iii) *if $\zeta \in \mathbb{S}^1$ is not a fixed point of f , then $|f'(\zeta)| > 1$.*

Then, f is expansive.

Proof. We only provide a sketch of the proof, since it is elementary. We denote by $d: \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow [0, \pi]$ the length metric in $\mathbb{S}^1$ and by X the finite set of fixed points of f . First, using the assumption (iii), one can show that there exists $\delta > 0$ such that if $z_0 \in \mathbb{S}^1$ and $d(f^{\circ n}(z_0), \zeta) < \delta$ for all $n \geq 0$, and some $\zeta \in X$, then $z_0 = \zeta$. In other words, an orbit of a point z_0 that is not a fixed point cannot stay for all times near a fixed point. Now, consider two points $z_1, z_2 \in \mathbb{S}^1$ such that $d(f^{\circ n}(z_1), f^{\circ n}(z_2))$ remains small, say less than η , for all $n \geq 0$. There are two main cases.

If $f^{\circ n_0}(z_1)$ is a fixed point of f for some $n_0 \geq 0$, then $f^{\circ n}(z_2)$ remains near a fixed point of f for all times $n \geq n_0$. By the previous, we must have $f^{\circ n_0}(z_1) = f^{\circ n_0}(z_2)$. Without loss of generality $n_0 \geq 1$. By assumption, $d(f^{\circ(n_0-1)}(z_1), f^{\circ(n_0-1)}(z_2)) < \eta$. If η is chosen suitably, then by (ii) f is injective on an arc containing $f^{\circ(n_0-1)}(z_1)$ and $f^{\circ(n_0-1)}(z_2)$. Hence, $f^{\circ(n_0-1)}(z_1) = f^{\circ(n_0-1)}(z_2)$. Inductively, $z_1 = z_2$. The same conclusion holds if $f^{\circ n_0}(z_2)$ is a fixed point for some n_0 .

The other case is that $f^{\circ n}(z_1)$ and $f^{\circ n}(z_2)$ are not fixed points for all $n \geq 0$. Then, they actually have to stay away from the δ -neighborhood of the fixed point set X infinitely often, by the first paragraph. Property (iii) implies that $d(f^{\circ(n+1)}(z_1), f^{\circ(n+1)}(z_2)) \geq \lambda d(f^{\circ n}(z_1), f^{\circ n}(z_2))$ for some definite factor $\lambda > 1$, whenever $f^{\circ n}(z_1)$ and $f^{\circ n}(z_2)$ are not in the δ -neighborhood of X . If either $f^{\circ n}(z_1)$ or $f^{\circ n}(z_2)$ is δ -close to X then we still have $d(f^{\circ(n+1)}(z_1), f^{\circ(n+1)}(z_2)) \geq d(f^{\circ n}(z_1), f^{\circ n}(z_2))$ by (ii). Since the first alternative occurs infinitely often, we obtain a contradiction, unless $z_1 = z_2$. $\square$

Remark 3.8. For piecewise C^2 maps, assumptions of Lemma 3.7 can probably be relaxed. Namely, it should be sufficient to assume that all periodic points are topologically repelling on both sides (compare [Man85]). Moreover, if the map has negative Schwarzian derivative (for piecewise C^3 maps) this assumption is satisfied automatically by Singer's Theorem (see [CE80]), except at points of non-differentiability, and the generalization of the lemma to this setting looks straightforward.

We end this section with a known result for hyperbolic Blaschke products that we will use later.

Lemma 3.9. *Let B be a holomorphic (resp. anti-holomorphic) Blaschke product of degree $d \geq 2$ that has an attracting fixed point in $\mathbb{D}$. Then there exists a quasimetric map $h: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ that conjugates the map B to the map $f(z) = z^d$ (resp. $f(z) = \bar{z}^d$).*

Here, an anti-holomorphic Blaschke product is the complex conjugate of a Blaschke product.

Proof. The proof is based on the fact that a conjugacy h between two expanding maps of $\mathbb{S}^1$ that are of class C^2 is quasimetric; see [Lyu20, Proposition 19.64].

Since the Blaschke product B has an attracting fixed point in $\mathbb{D}$, it follows that B is hyperbolic, so $B^{\circ m}$ is expanding for some $m \in \mathbb{N}$ (see [CG93, Lemma 2.1]). By property (E3), there exists an orientation-preserving conjugacy h between B and f . Note that h also conjugates $B^{\circ m}$ and $f^{\circ m}$. By the above fact, the conjugacy between $B^{\circ m}$ and $f^{\circ m}$ is quasimetric. $\square$

4. DAVID EXTENSIONS OF DYNAMICAL HOMEOMORPHISMS OF THE CIRCLE

Let $f, g: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be covering maps of degree $d \geq 2$ having the same orientation and consider Markov partitions $\mathcal{P}(f; \{a_0, \dots, a_r\})$ and $\mathcal{P}(g; \{b_0, \dots, b_r\})$. Let $h: \{a_0, \dots, a_r\} \rightarrow \{b_0, \dots, b_r\}$ be the map defined by $h(a_k) = b_k$ for $k \in \{0, \dots, r\}$ and suppose that h conjugates the map f to g on the set $\{a_0, \dots, a_r\}$. If the maps f and g are expansive, then we know from Lemma 3.3 that h extends to an orientation-preserving homeomorphism $h: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ that conjugates the map f to g . That is, $h(f(z)) = g(h(z))$ for all $z \in \mathbb{S}^1$. In this section, under some assumptions, we will extend the map h to a homeomorphism of the unit disk $\mathbb{D}$ that is a quasiconformal or a David map. Our main extension result is Theorem 4.9 and its proof will be given in Subsection 4.2.

To illustrate the technical difficulties of the proof of Theorem 4.9, we consider the following example. Let $C_1, \dots, C_4$ be circles that bound disjoint disks and are orthogonal to the unit circle. Define f to be a degree 3 covering map of $\mathbb{S}^1$ that is piecewise the reflection on the circles C_i , $i \in \{1, \dots, 4\}$. Similarly, define another covering map g , corresponding to some other circles $C'_1, \dots, C'_4$ with the same properties. The maps f and g are conjugate to each other and to $z \mapsto \bar{z}^3$. The conjugacy between f and g can be constructed implicitly as follows. Consider a quasiconformal map h from the ideal 4-gon defined by the circles C_i and contained in $\mathbb{D}$ onto the ideal 4-gon defined by the circles C'_i , such that h preserves the cusps. Then h can be extended by reflections to a quasiconformal map of $\mathbb{D}$. It follows from the theory of quasiconformal maps that h extends to a homeomorphism of $\mathbb{S}^1$ that is quasimetric and conjugates f to g . Note that h takes the parabolic fixed points

of f to the parabolic fixed points of g . However, if one wishes to study the conjugacy between $\bar{z}^3$ and f , then such a reflection argument cannot be implemented. Instead, we study directly the conjugating homeomorphism $h: \mathbb{S}^1 \rightarrow \mathbb{S}^1$, and prove certain bounds for its scalewise distortion (a quantity defined in Subsection 2.2). These bounds imply, by Theorem 2.3, that there exists a David extension of h in $\mathbb{D}$. The bounds for h are proved by studying carefully the local behavior of f and $z \mapsto \bar{z}^3$ near parabolic and hyperbolic points and then use dynamics to spread around the estimates to all points of the circle.

Before formulating Theorem 4.9, we describe the assumptions on the maps f, g and the Markov partitions $\mathcal{P}(f; \{a_0, \dots, a_r\})$, $\mathcal{P}(g; \{b_0, \dots, b_r\})$. We will give the definitions using the notation associated to the map f .

4.1. Markov partitions of dynamical covering maps. Define $A_k = \overline{[a_k, a_{k+1}]}$ for $k \in \{0, \dots, r\}$ and recall that $f_k = f|_{\text{int } A_k}$ is injective by the definition of a Markov partition. We assume, that f_k is analytic and that there exist open neighborhoods U_k of $\text{int } A_k$ and V_k of $f_k(\text{int } A_k)$ in the plane such that f_k has a conformal extension from U_k onto V_k . We still denote the extension by f_k . We impose the condition that

$$(4.1) \quad \bigcup_{\substack{0 \leq j \leq r \\ (k,j) \text{ admissible}}} U_j \subset V_k$$

for all $k \in \{0, \dots, r\}$. We also require that

$$(4.2) \quad f_k \text{ extends holomorphically to neighborhoods of } a_k \text{ and } a_{k+1}$$

for each $k \in \{0, \dots, r\}$.

Example 4.1 (Power map). Let $f: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be the map $z \mapsto z^d$ or $z \mapsto \bar{z}^d$. Then for any Markov partition $\mathcal{P}(f; \{a_0, \dots, a_r\})$ both conditions (4.1) and (4.2) are satisfied. Indeed, if $f(z) = z^d$ on $\mathbb{S}^1$, then f has an analytic extension to $\mathbb{C}$. If $f(z) = \bar{z}^d$ on $\mathbb{S}^1$, observe that $f(z) = 1/\bar{z}^d$, so f has an analytic extension to $\mathbb{C} \setminus \{0\}$. We denote the extension by f . Condition (4.2) holds trivially. For $k \in \{0, \dots, r\}$ let U_k an open sector with vertex 0 and angle subtended by the arc A_k . Since f is injective on $\text{int } A_k$ by the definition of a Markov partition, it follows that f is conformal on U_k . We set $V_k = f(U_k)$. If $A_j \subset f(A_k)$, then $U_j \subset V_k$. This shows that (4.1) is satisfied.

Example 4.2 (Blaschke product). The map

$$B(z) = \frac{(d+1)z^d + (d-1)}{(d-1)z^d + (d+1)}$$

has a Markov partition that satisfies both conditions (4.1) and (4.2). Such a Markov partition is $\mathcal{P}(B; \{a_0, \dots, a_{2d-1}\})$, where $a_0, \dots, a_{2d-1}$ are d -th roots of 1 and -1 with $a_0 = 1$. Here, in condition (4.1), the set U_k , $k \in \{0, 1, \dots, 2d-1\}$, is an open sector with vertex 0 and angle π/d , whose boundary contains the points a_k and a_{k+1} , indices taken modulo $2d$; the sets V_k are the upper half-plane for even k and the lower half-plane for odd k . Condition (4.2) is trivially satisfied.

Example 4.3 (Circle reflections). Consider ordered points $a_0, \dots, a_d$ on the circle $\mathbb{S}^1$ such that $\text{length}(\overline{(a_k, a_{k+1})}) < \pi$ for $k \in \{0, \dots, d\}$. If f_k is the reflection along the circle C_k that is orthogonal to the unit circle at the points a_k and a_{k+1} , then the

conformal extension of f_k is *not* the reflection along C_k (which is anti-conformal). Instead, it is the reflection along C_k , composed with a reflection in the unit circle, so that the resulting map is conformal. (Note that reflections along circles that intersect orthogonally commute.) The map f_k is a Möbius transformation, so (4.2) is trivially satisfied.

Suppose now that f_l is the reflection along C_l for all $l \in \{0, \dots, d\}$. In this way, we define a covering map f of $\mathbb{S}^1$ of degree d , as in Example 3.5. Note that the circles C_l bound disjoint disks and the points $a_0, \dots, a_d$ are fixed points of f . We fix $k \in \{0, \dots, d\}$. The map f_k maps $\text{Int } C_k$ conformally to its exterior. We consider a large open ball B_k containing all circles C_l , $l \in \{0, \dots, d\}$, and we define V_k to be the intersection of the exterior of C_k with B_k . It is important that V_k is planar and $\infty \notin V_k$. Now, we simply define $U_k = f_k^{-1}(V_k)$. It is immediate now that (4.1) is satisfied.

Example 4.4 (Hybrid). Consider the hybrid map f from Example 3.6. This map satisfies both conditions (4.1) and (4.2). Indeed, reflections in circles orthogonal to the unit circle can be treated similarly to the previous Example 4.3. For the map $z \mapsto \bar{z}^d$, the conformal extension is $z \mapsto 1/z^d$. If A_k is an arc on which the map f is $z \mapsto \bar{z}^d$, the set U_k in condition (4.1) is the smallest open sector with vertex 0 that contains $\text{int}(A_k)$, intersected with an annulus $\{1/R < |z| < R\}$ with sufficiently large $R > 1$.

Definition 4.5. Let $a \in \{a_0, \dots, a_r\}$. We say that a is *parabolic on the right* (resp., *on the left*) if $\lambda(a^+) = 1$ (resp., $\lambda(a^-) = 1$) and a is an isolated fixed point of f_a . Likewise, a is *hyperbolic on the right* (resp., *on the left*) if $\lambda(a^+) > 1$ (resp., $\lambda(a^-) > 1$).

For convenience, we say that a^+ is *parabolic* (resp., *hyperbolic*) if a is parabolic (resp., hyperbolic) on the right. Similarly, we say that a^- is *parabolic* (resp., *hyperbolic*) if a is parabolic (resp., hyperbolic) on the left.

We observe that if f is expansive, then by property (E4) and property (E1)

$$(4.3) \quad \text{each of } a^+, a^- \text{ is parabolic or hyperbolic}$$

for all $a \in \{a_0, \dots, a_r\}$. Equivalently, $\lambda(a^\pm) \geq 1$ and f_a is not the identity map for all $a \in \{a_0, \dots, a_r\}$.

In the case of parabolic points there is a further distinction. If $a = a_k$, $k \in \{0, \dots, r\}$, and a^+ is parabolic, then condition (4.2) implies that the first return map f_a has a holomorphic extension valid in a complex neighborhood of $\widehat{[a, z_0]}$ for some point $z_0 \in \mathbb{S}^1$. We denote the extension by f_a^+ . The map f_a^+ is not the identity map, so there exists a Taylor expansion

$$f_a^+(z) = z + c(z - a)^{N+1} + O((z - a)^{N+2})$$

where $c \neq 0$ and N is a non-negative integer. The number $N + 1$ is called the *multiplicity* of f_a^+ at the parabolic point a^+ , and it is invariant under conjugation. Abusing terminology, we also say that $N + 1$ is the multiplicity of a^+ . We denote the multiplicity by $N(a^+) + 1$. Similarly, we define the multiplicity $N(a^-) + 1$ of a^- , in the case that a^- is parabolic.

From the theory of parabolic fixed points (see e.g. [Mil06, §10]) there are $N(a^+)$ attracting directions and $N(a^+)$ repelling directions for the holomorphic germ f_a^+ . Note that f_a^+ maps the arc $\widehat{[a, z_0]}$ into $\mathbb{S}^1$, since it is an extension of f_a . This implies

that either all points of $\widehat{[a, z_0]}$ are attracted to the point a under iteration of f_a , or they are repelled away from a . We say that $\widehat{[a, z_0]}$ defines an attracting or repelling direction for f_a , respectively. Similarly, $\widehat{[z_0, a]}$ defines either an attracting or a repelling direction for f_a , which has a holomorphic extension f_a^- in a neighborhood of $\widehat{[z_0, a]}$. A trivial consequence of the expansivity of f is that

$$(4.4) \quad \begin{aligned} &\text{if } a^+ \text{ is parabolic then } \widehat{[a, z_0]} \text{ defines a repelling direction for } f_a \text{ and} \\ &\text{if } a^- \text{ is parabolic then } \widehat{[z_0, a]} \text{ defines a repelling direction for } f_a. \end{aligned}$$

Definition 4.6. Let $a \in \{a_0, \dots, a_r\}$. We say that a is symmetrically parabolic if a^+ and a^- are parabolic with $N(a^+) = N(a^-)$. In this case we denote this common number by $N(a)$. We say that a is symmetrically hyperbolic if a^+ and a^- are hyperbolic with $\lambda(a^+) = \lambda(a^-)$. In this case we denote by $\lambda(a)$ the common multiplier.

Remark 4.7. If f is orientation-reversing and $a \in \{a_0, \dots, a_r\}$ is a periodic point with odd period, then it is automatic from condition (4.2) and from condition (4.3) that a is symmetrically hyperbolic or parabolic. Indeed, suppose that the period of a is an odd number $n \in \mathbb{N}$. The first return map f_a is $f^{\circ 2n}$ in this case. Since $f^{\circ n}$ is orientation-reversing, by the chain rule we have:

$$\lambda(a^+) = (f^{\circ 2n})'(a^+) = (f^{\circ n})'(a^-) \cdot (f^{\circ n})'(a^+) = (f^{\circ 2n})'(a^-) = \lambda(a^-).$$

Moreover, for points z lying in an arc $\widehat{[a, z_1]}$ we have

$$f^{\circ n} \circ f_a^+(z) = f_a^- \circ f^{\circ n}(z).$$

By condition (4.2), $f^{\circ n}$ has a holomorphic extension to a complex neighborhood U of a possibly smaller arc $\widehat{[a, z_1]}$ that we denote by $(f^{\circ n})^+$. The uniqueness of analytic maps implies that

$$(f^{\circ n})^+ \circ f_a^+(z) = f_a^- \circ (f^{\circ n})^+(z)$$

for all $z \in U$, assuming that U is sufficiently small so that the holomorphic maps involved are defined on U . The map $(f^{\circ n})^+$ is injective near a since its derivative at a is non-zero by condition (4.3). It follows that $(f^{\circ n})^+$ conjugates f_a^+ to f_a^- near a . Therefore a^+ is parabolic (resp. hyperbolic) if and only if a^- is parabolic (resp. hyperbolic) and the multipliers $\lambda(a^+), \lambda(a^-)$ are necessarily equal to each other by the conjugation. Moreover, if a^+ is parabolic, then the multiplicity of a^+ is a conjugation invariant, so it is equal to the multiplicity of a^- . Our claim follows.

Example 4.8. The Blaschke product $B(z) = (2z^3 + 1)/(z^3 + 2)$, which is a special case of Example 4.2, has two fixed points on $\mathbb{S}^1$, which are 1 and -1 . The point 1 is symmetrically parabolic, while the point -1 is symmetrically hyperbolic with multiplier 9.

If $f(z) = \bar{z}^d$, which is equal to $1/z^d$ on $\mathbb{S}^1$, and $\omega_k, k \in \{0, \dots, d\}$, are the fixed points of f , then $f'(\omega_k) = -d$ for $k = 0, \dots, d$. Since f is orientation-reversing, for each fixed point the first return map in this case is $f^{\circ 2}$. Thus all multipliers of the first return map at the points ω_k are equal to d^2 . It follows that all fixed points of f are symmetrically hyperbolic.

If, instead, $a_0, \dots, a_d, d \geq 2$, are ordered points on $\mathbb{S}^1$ and for each $k \in \{0, \dots, d\}$ the map $f_k = f|_{\widehat{(a_k, a_{k+1})}}$ is the reflection along the circle that is orthogonal to $\mathbb{S}^1$

at a_k and a_{k+1} , then $f'(a_k^+) = f'(a_{k+1}^-) = -1$. Indeed, f_k is conjugate to the reflection of the top half of the unit circle along the real line, i.e., f_k is conjugate to $\bar{z}$. Restricted to the circle, this map is equal to the map $z \mapsto 1/\bar{z}$, so the derivative at the fixed points ± 1 is -1 . For all fixed points the first return map is again $f^{\circ 2}$, so all multipliers are equal to 1. Moreover, since $d \geq 2$, the map $f^{\circ 2}$ cannot be the identity map on any arc (a_k, a_{k+1}) . Therefore, all points a_k , $k \in \{0, \dots, d\}$, are symmetrically parabolic, in view of Remark 4.7.

In the hybrid case of Example 3.6, if a_k , $k \in \{0, 1, \dots, d\}$, is a fixed point such that on one side of it the map f is the reflection in the circle orthogonal to the unit circle and on the other side it is $z \mapsto \bar{z}^d$, then a_k is symmetrically hyperbolic. Indeed, it is easy to see that the first return map is $f^{\circ 2}$ and the multiplier is $d \geq 2$.

We now state the main extension theorem.

Theorem 4.9. *Let $f, g: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be expansive covering maps with the same orientation and $\mathcal{P}(f; \{a_0, \dots, a_r\})$, $\mathcal{P}(g; \{b_0, \dots, b_r\})$ be Markov partitions satisfying conditions (4.1) and (4.2). Suppose that the map $h: \{a_0, \dots, a_r\} \rightarrow \{b_0, \dots, b_r\}$ defined by $h(a_k) = b_k$, $k \in \{0, \dots, r\}$, conjugates f to g on the set $\{a_0, \dots, a_r\}$ and assume that for each periodic point $a \in \{a_0, \dots, a_r\}$ of f and for $b = h(a)$ one of the following alternatives occur.*

(H/P $\rightarrow$ H/P) *There exists $\mu > 0$ such that if $a^\pm$ is parabolic then $b^\pm$ is parabolic with $\mu^{-1}N(a^\pm) = N(b^\pm)$, and if $a^\pm$ is hyperbolic then $b^\pm$ is hyperbolic with $\lambda(a^\pm)^\mu = \lambda(b^\pm)$.*

(H $\rightarrow$ P) *a^+ and a^- are hyperbolic and b is symmetrically parabolic.*

Then the map h extends to a homeomorphism $\tilde{h}$ of $\mathbb{D}$ such that $\tilde{h}|_{\mathbb{S}^1}$ conjugates f to g and $\tilde{h}|_{\mathbb{D}}$ is a David map. Moreover, if the alternative (H $\rightarrow$ P) does not occur, then $\tilde{h}|_{\mathbb{D}}$ is a quasiconformal map and $\tilde{h}|_{\mathbb{S}^1}$ is a quasiconformal map.

Remark 4.10. We note that the alternative (H/P $\rightarrow$ H/P) allows a^+ (resp. b^+) to be hyperbolic and a^- (resp. b^-) to be parabolic and vice versa. In fact, (H/P $\rightarrow$ H/P) covers the following cases:

- a^- is hyperbolic and a^+ is hyperbolic
- a^- is hyperbolic and a^+ is parabolic
- a^- is parabolic and a^+ is hyperbolic
- a^- is parabolic and a^+ is parabolic

The only restriction is that the multiplicities and multipliers of the points $a^\pm$ and $b^\pm$ have to be related by the same number μ , which depends only on the point a .

Remark 4.11. It is conceivable that by means of the smooth distortion techniques (see [dMvS93]) Theorem 4.9 can be extended to the piecewise C^2 setting described in Remark 3.8. Moreover, this looks straightforward under the negative Schwarzian derivative assumption.

The essential assumption in Theorem 4.9 is that parabolic periodic points of f must be mapped to parabolic points of g . On the other hand, hyperbolic points can be mapped to either hyperbolic or parabolic points. We remark that not all points need to be checked, but only the boundary points $\{a_0, \dots, a_r\}$ of the Markov pieces. The theorem holds if one strengthens the two alternatives to one of the following

symmetric alternatives, in which the left and right multipliers and multiplicities are the same:

- (S-I) a is symmetrically hyperbolic and b is symmetrically hyperbolic.
- (S-II) a is symmetrically parabolic and b is symmetrically parabolic.
- (S-III) a is symmetrically hyperbolic and b is symmetrically parabolic.

A special case of Theorem 4.9 is the following statement.

Theorem 4.12 (Power map). *Let $f: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be an expansive covering map of degree $d \geq 2$ and let $\mathcal{P}(f; \{a_0, \dots, a_r\})$ be a Markov partition satisfying conditions (4.1) and (4.2), and with the property that a_k is either symmetrically hyperbolic or symmetrically parabolic for each $k \in \{0, \dots, r\}$. Then there exists an orientation-preserving homeomorphism $h: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ that conjugates the map $z \mapsto z^d$ or $z \mapsto \bar{z}^d$ to f and has a David extension in $\mathbb{D}$.*

Proof. Using property (E3), we obtain an orientation-preserving homeomorphism $h: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ that conjugates the map $z \mapsto z^d$ or $z \mapsto \bar{z}^d$ to f . Consider the induced Markov partition $\{h^{-1}(a_0), \dots, h^{-1}(a_r)\}$. By Example 4.1, this Markov partition satisfies (4.1) and (4.2). We now apply Theorem 4.9 to the map h . $\square$

As an application of Theorem 4.12 we have the following theorem.

Theorem 4.13 (Blaschke product–Circle reflections). *Consider ordered points $a_0 = a_{d+1}, a_1, \dots, a_d$ on the circle $\mathbb{S}^1$, where $d \geq 2$. For $k \in \{0, \dots, d\}$ let $f|_{\widehat{(a_k, a_{k+1})}}$ be the reflection along the circle C_k that is orthogonal to the unit circle at the points a_k and a_{k+1} . Moreover, let B be an anti-holomorphic Blaschke product of degree d with an attracting fixed point in $\mathbb{D}$. Then there exists a homeomorphism $h: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ that conjugates $B|_{\mathbb{S}^1}$ to f and has a David extension in $\mathbb{D}$.*

Proof of Theorem 4.13. We first treat the special case $B(z) = \bar{z}^d$. Suppose first that we have $\text{length}(\widehat{(a_k, a_{k+1})}) < \pi$ for all $k \in \{0, \dots, d\}$. In Example 3.5 we proved that f is expansive. Consider the Markov partition $\mathcal{P}(f; \{a_0, \dots, a_d\})$. In Example 4.3 we verified that conditions (4.1) and (4.2) hold. Finally, in Example 4.8 we proved that if $d \geq 2$, then the fixed points a_k , $k \in \{0, \dots, d\}$, are all symmetrically parabolic. Therefore, Theorem 4.12 gives the desired conclusion.

Suppose now that $\text{length}(\widehat{(a_i, a_{i+1})}) > \pi$ for some $i \in \{0, \dots, d\}$. Note that this can only be the case for one value of i . Then there exists a Möbius transformation M of $\mathbb{D}$ such that $\text{length}(\widehat{(M(a_k), M(a_{k+1})))} < \pi$ for all $k \in \{0, \dots, d\}$. Consider the map $F = M \circ f \circ M^{-1}$. Then $F|_{\widehat{(M(a_k), M(a_{k+1}))}}$ is the reflection along the circle $M(C_k)$ that is orthogonal to the unit circle at the points $M(a_k)$ and $M(a_{k+1})$. By the previous case, the map F has a David extension in $\mathbb{D}$, that we still denote by F . By Proposition 2.5 (i) and (ii) with $U = V = W = \mathbb{D}$, the map $M^{-1} \circ F \circ M$ is a David map that extends f , as desired.

Next, let B be a general anti-holomorphic Blaschke product of degree d with an attracting fixed point in $\mathbb{D}$. By Lemma 3.9 there exists a quasisymmetric map $h_1: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ that conjugates B to $z \mapsto \bar{z}^d$. We extend this map to a quasiconformal map of $\mathbb{D}$ that we still denote by h_1 . By the previous, there exists a homeomorphism $h_2: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ that conjugates $z \mapsto \bar{z}^d$ to f and has a David extension in $\mathbb{D}$. We also denote the extension by h_2 . Then $h_2 \circ h_1$ conjugates on $\mathbb{S}^1$ the map B to f . Moreover, by Proposition 2.5 (ii) $h_2 \circ h_1$ is David map on $\mathbb{D}$. $\square$

Another special case of Theorem 4.9 is the following theorem.

Theorem 4.14 (Hybrid–Circle reflections). *Let $a_0 = a_{d+1}, a_1, \dots, a_d$ be fixed points on the circle $\mathbb{S}^1$ of the map $z \mapsto \bar{z}^d$, where $d \geq 2$, and f be a hybrid map as in Example 3.6. Moreover, for $k \in \{0, \dots, d\}$ let $g|_{\overline{(a_k, a_{k+1})}}$ be the reflection along the circle C_k that is orthogonal to the unit circle at the points a_k and a_{k+1} . Then there exists a homeomorphism $h: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ that conjugates f to g and has a David extension in $\mathbb{D}$.*

Proof. In Examples 3.6 and 3.5 we proved that f and g are expansive. The Markov partitions $\mathcal{P}(f; \{a_0, \dots, a_d\})$ and $\mathcal{P}(g; \{a_0, \dots, a_r\})$ satisfy (4.1) and (4.2) by Examples 4.4 and 4.3. Finally, in Example 4.8 we saw that each point a_k is symmetrically hyperbolic or parabolic for f and symmetrically parabolic for g . Therefore, by Theorem 4.9, the identity map on the set $\{a_0, \dots, a_d\}$ extends to a homeomorphism $h: \mathbb{D} \rightarrow \mathbb{D}$ that conjugates f to g on $\mathbb{S}^1$ and is a David map in $\mathbb{D}$. $\square$

4.2. Distortion estimates and proof of Theorem 4.9. We will need some preparation in order to prove Theorem 4.9. We will formulate most of the statements in this subsection using the notation associated to the map f and the Markov partition $\mathcal{P}(f; \{a_0, \dots, a_r\})$. The map f is assumed, throughout, to be an expansive covering map of $\mathbb{S}^1$, satisfying conditions (4.1) and (4.2). As already remarked, conditions (4.3) and (4.4) follow automatically from expansivity, so they will be assumed. The analogous results hold, of course, for the map g , which satisfies the same assumptions. Moreover, the expansivity of f and g and Lemma 3.3 imply that the map $h: \{a_0, \dots, a_r\} \rightarrow \{b_0, \dots, b_r\}$ from the statement of Theorem 4.9 extends to an orientation-preserving homeomorphism of $\mathbb{S}^1$ that conjugates f to g . It is also implicitly assumed that if $a^\pm$ is parabolic, then $b^\pm = h(a^\pm)$ is also parabolic, as in the assumptions of Theorem 4.9.

Using condition (4.1), we can define $U_{kj} = f_k^{-1}(U_j)$ for $k, j \in \{0, \dots, r\}$, whenever (k, j) is admissible; recall the definitions given after Definition 3.1. Note that $U_{kj} \subset U_k$ and f_k maps conformally U_{kj} onto U_j . Inductively, for each admissible word w we can find open regions U_w with the following properties:

- (i) $U_{wj} \subset U_w$, if (w, j) is admissible, and
- (ii) f_k maps conformally U_{kw} onto U_w , if (k, w) is admissible.

If $w = (k_1, \dots, k_n)$ is admissible, we define $f_w = f_{k_n} \circ \dots \circ f_{k_1}$ on the set $\bigcup \{U_{wj} : 0 \leq j \leq r, (w, j) \text{ admissible}\}$. It follows that f_w maps conformally U_{wj} onto U_j . Observe that for each admissible word w the open arc $\text{int } A_w$ is contained in U_w and is a preimage of one of the arcs $\text{int } A_k$, $k \in \{0, \dots, r\}$, under some iterate of f . Note that $f_w|_{\text{int } A_{wj}}$, where (w, j) is admissible, is just a restriction of the n -th iterate $f^{\circ n}$.

We let F_n be the preimages of $F_1 = \{a_0, \dots, a_r\}$ under $n-1$ iterations of f and $F_0 = \emptyset$. Observe that $F_n \supset F_{n-1}$ for each $n \in \mathbb{N}$. Indeed, $F_1 \supset f(F_1)$ by condition (iii) in Definition 3.1, hence $F_2 = f^{-1}(F_1) \supset f^{-1}(f(F_1)) \supset F_1$, so the conclusion follows by induction. We define the *level* of a point $c \in \bigcup_{n \geq 1} F_n$ to be the unique $n \in \mathbb{N}$ such that $c \in F_n \setminus F_{n-1}$.

We say that a finite collection of non-overlapping arcs $I_1, \dots, I_m \subset \mathbb{S}^1$, $m \in \mathbb{N}$, consists of *consecutive* arcs if every arc I_i shares at least one endpoint with another

arc I_j , $j \neq i$. If all arcs in the family are open, after renumbering, we may assume that $I_i = \overline{(x_{i-1}, x_i)}$ for some points $x_j \in \mathbb{S}^1$, $j = 0, 1, \dots, m$.

For each $n \in \mathbb{N}$, the set $\mathbb{S}^1 \setminus F_n$ consists of consecutive open arcs. For practical purposes we will use the terminology *complementary arcs of F_n* to indicate the family of the *closures* of the components of $\mathbb{S}^1 \setminus F_n$. Hence, all complementary arcs of F_n are closed arcs. Note that the complementary arcs of F_n are the arcs A_w , where w is an admissible word with $|w| = n$; see the comments after Definition 3.1.

Let X, Y be metric spaces. We say that a homeomorphism $\phi: X \rightarrow Y$ has *bounded relative distortion* if there exists $M \geq 1$ such that for any sets $A, B \subset X$ we have

$$M^{-1} \frac{\text{diam } A}{\text{diam } B} \leq \frac{\text{diam } \phi(A)}{\text{diam } \phi(B)} \leq M \frac{\text{diam } A}{\text{diam } B}.$$

In this case we say that ϕ has relative distortion bounded by M . Note that this implies that ϕ is an η -quasisymmetry with $\eta(t) = Mt$. The inverse ϕ^{-1} of a homeomorphism ϕ of bounded relative distortion also has bounded relative distortion. In what follows, if we say that a map ϕ has bounded relative distortion, it is implicitly understood that ϕ is a homeomorphism.

An important consequence of expansivity is the following lemma. Recall the definition of the orientation-preserving period of a periodic point $a \in \{a_0, \dots, a_r\}$, given in Section 3.

Lemma 4.15. *Let $a \in F_1$ be a periodic point with orientation-preserving period equal to q , $n \in \mathbb{N}$, and A be a complementary arc of F_n having the point a as an endpoint. Then A contains at least two and at most $(r+1)^q$ complementary arcs of F_{n+q} .*

Proof. We first treat the upper bound. By the definition of a Markov partition, if A is a complementary arc of F_n , then A contains at most $r+1$ complementary arcs of F_{n+1} , or else, A has at most $r+1$ children; see the comments after Definition 3.1. Therefore, A contains at most $(r+1)^q$ complementary arcs of F_{n+q} .

Next, we prove the lower bound. There exists an admissible word $(j_1, \dots, j_n)$ such that $A_{j_1} \supset A_{j_1 j_2} \supset \dots \supset A_{j_1 \dots j_n} = A$. Consider the corresponding regions $U_{j_1} \supset U_{j_1 j_2} \supset \dots \supset U_{j_1 \dots j_n}$.

We argue by contradiction. Suppose that the arc A does not contain two complementary arcs of F_{n+q} . Then A is itself a complementary arc of F_{n+q} . It follows that there exist $j_{n+1}, \dots, j_{n+q} \in \{0, \dots, r\}$ such that $A = A_{j_1 \dots j_n} = \dots = A_{j_1 \dots j_{n+q}}$. By applying $f^{\circ(n-1)}$, we see that $A_{j_n} = A_{j_n j_{n+1}} = \dots = A_{j_n \dots j_{n+q}} =: A'$ and that A' is a complementary arc of F_1 . We define $a' = f^{\circ(n-1)}(a)$, and note that a' has orientation-preserving period equal to q , since it is contained in the orbit of the point a .

By the definition of the regions U_w , where w is an admissible word, it follows that the map $f^{\circ q}$ maps $U_{j_n \dots j_{n+q}}$ conformally onto $U_{j_{n+q}}$ and $A_{j_n \dots j_{n+q}}$ onto $A_{j_{n+q}}$. However, the orientation-preserving period of a' , which is an endpoint of $A_{j_n \dots j_{n+q}}$ is q . This implies that $A_{j_n \dots j_{n+q}} \subset A_{j_{n+q}}$, and therefore $j_n = j_{n+q}$ and $A_{j_n \dots j_{n+q}} = A_{j_n} = A_{j_{n+q}}$. It follows that $U_{j_n \dots j_{n+q}} \subset U_{j_{n+q}}$ and that the orientation-preserving period of the other endpoint b' of A' is a multiple of q .

Summarizing, the map $F = (f^{\circ q}|_{U_{j_{n+q}}})^{-1}$ maps conformally $U := U_{j_{n+q}}$ onto $V := U_{j_n \dots j_{n+q}}$, which is a subset of U . Moreover, F has two fixed points a' and b' .

in $\overline{U}$ and F extends holomorphically to a neighborhood of a' and b' , by condition (4.2). Each of a' and b' is either attracting or parabolic, by condition (4.3).

We note that U is necessarily a hyperbolic subset of $\widehat{\mathbb{C}}$. Indeed, otherwise, the conformal map F would have to be a loxodromic Möbius transformation with only two fixed points, one of which is attracting and the other is repelling. This is a contradiction. Since $U \supset V$ and U is hyperbolic, it follows by Montel's theorem that the sequence of conformal maps $F^{\circ m}$, $m \in \mathbb{N}$, has a subsequence that converges locally uniformly to a holomorphic map F_∞ on U .

If a' is an attracting fixed point, then all points near a' are attracted under iterates of F to a' . If a' is a parabolic fixed point, then by condition (4.4) the arc A' defines an attracting direction for F , so all points of A' near a' are attracted to a' under iteration of F . It follows that F_∞ is identically the constant function $z \mapsto a'$. However, the same conclusions hold for the fixed point b' , so F_∞ has to be the function $z \mapsto b'$. We have arrived at a contradiction, because $a' \neq b'$. $\square$

4.2.1. Local estimates near hyperbolic and parabolic points. We are first going to establish local estimates near hyperbolic preperiodic points. Recall that all points of $F_1 = \{a_0, \dots, a_r\}$ are preperiodic by condition (iii) in Definition 3.1. Before proceeding to the next lemma, recall the definition of the one-sided multipliers $\lambda(a^\pm)$ and the orientation-preserving period of a preperiodic point $a \in \{a_0, \dots, a_r\}$, given in Section 3.

Lemma 4.16 (Hyperbolic estimates). *Suppose that $a \in F_1$, a^+ is hyperbolic, and let $q \in \mathbb{N}$ be the orientation-preserving period of a . For each $p \in \mathbb{N}$, $z_0 \in \mathbb{S}^1$ with $z_0 \neq a$, and for all sufficiently large $N_0 \in \mathbb{N}$ there exists a constant $L \geq 1$ such that the following is true. If $I_1, \dots, I_p \subset [a, z_0]$ are consecutive complementary arcs of F_n , $n \geq 1$, and a is an endpoint of I_1 , then for each $i \in \{1, \dots, p\}$ we have*

$$(4.5) \quad L^{-1} \lambda(a^+)^{-n/q} \leq \text{diam } I_i \leq L \lambda(a^+)^{-n/q}.$$

Moreover, if $n \geq N_0$, the map $f^{\circ(n-N_0)}$ has relative distortion bounded by L on $\bigcup_{i=1}^p I_i$. The analogous statements hold if a^- is hyperbolic and $I_1, \dots, I_p \subset [z_0, a]$.

Intuitively, the last statement of the lemma says that using dynamics one can blow up with bounded distortion complementary intervals of F_n near a hyperbolic point to complementary intervals of F_{N_0} .

Proof. For any $N_1 \in \mathbb{N}$ there exists a constant $L \geq 1$ so that (4.5) holds for $n \leq N_1$. This is trivial since there is only a finite number of complementary arcs of F_n , $n \leq N_1$. Hence, we will find bounds only when n is sufficiently large.

Suppose first that a is a periodic point. Consider the first return map $f_a = f^{\circ q}$. By condition (4.2) f_a has a holomorphic extension valid in a complex neighborhood of $[a, z_1] \subset [a, z_0]$ for some point $z_1 \in \mathbb{S}^1$, $z_1 \neq a$. We denote this extension by f_a^+ . Since a^+ is hyperbolic, we have $\lambda(a^+) > 1$.

By the Kœnigs linearization theorem (see [Mil06, Theorem 8.2]), there exists a conformal map ϕ defined in a neighborhood U of $[a, z_1]$ such that $\phi(a) = 0$ and

$$(4.6) \quad \phi(f_a^+(z)) = \lambda(a^+) \phi(z)$$

for all $z \in U$. Later, we are going to shrink the arc $[a, z_1]$ and the neighborhood U appropriately.

We let $N_0 \in \mathbb{N}$ be an integer such that there are p consecutive complementary arcs of F_{N_0} that are contained in $\widehat{[a, z_1]}$ and one of which has a as an endpoint. The existence of N_0 follows from the expansivity of f and property (E2). For each $j \in \{1, \dots, q-1\}$ we also consider the p consecutive complementary arcs of F_{N_0+j} that are contained in $\widehat{[a, z_1]}$ and one of which has a as an endpoint. In total, we fixed qp arcs J_l , $l \in \{1, \dots, qp\}$. Suppose that $n \geq N_0$.

Consider the arcs $I_1, \dots, I_p$ as in the statement of the lemma, which are complementary arcs of F_n . For each $i \in \{1, \dots, p\}$ we have $I_i \subset \bigcup_{l=1}^{qp} J_l \subset \widehat{[a, z_1]}$. Hence, $(f_a^+)^{\circ m}(I_i) = J_l$ for some $l \in \{1, \dots, qp\}$, where $m = \lfloor (n - N_0)/q \rfloor$. Note that ϕ is bi-Lipschitz on J_l and on I_i with a uniform constant, since all arcs are contained in a fixed compact subset of U . In combination with the conjugation relation (4.6), we obtain

$$\begin{aligned} \text{diam } J_l &= \text{diam } (f_a^+)^{\circ m}(I_i) \simeq \text{diam } \phi((f_a^+)^{\circ m}(I_i)) \\ &\simeq \lambda(a^+)^m \text{diam } \phi(I_i) \simeq \lambda(a^+)^{(n-N_0)/q} \text{diam } I_i. \end{aligned}$$

It follows that we have

$$\text{diam } I_i \simeq \lambda(a^+)^{-n/q},$$

since the arcs J_l , $l \in \{1, \dots, qp\}$, and the integer N_0 are fixed.

Now, if $I \subset \bigcup_{i=1}^p I_i$ is any arc, then the preceding argument shows that

$$\text{diam } f^{\circ qm}(I) = \text{diam } (f_a^+)^{\circ m}(I) \simeq \lambda(a^+)^{n/q} \text{diam } I.$$

Since $n \geq N_0$, there exists $k \in \{0, 1, \dots, q-1\}$ such that $n - N_0 = qm + k$. By condition (4.2), the maps $f, f^{\circ 2}, \dots, f^{\circ(q-1)}$ have analytic extensions in a neighborhood of the arc $\widehat{[a, z_1]}$, after possibly shrinking the arc. Moreover, the derivatives of these extensions are non-zero at a since $\lambda(a^+) \neq 0$. It follows that $f, f^{\circ 2}, \dots, f^{\circ(q-1)}$ are bi-Lipschitz in $\widehat{[a, z_1]}$. Since $f^{\circ qm}(I) \subset \bigcup_{l=1}^{qp} J_l \subset \widehat{[a, z_1]}$, we have

$$\text{diam } f^{\circ(n-N_0)}(I) = \text{diam } f^{\circ k}(f^{\circ qm}(I)) \simeq \text{diam } f^{\circ qm}(I) \simeq \lambda(a^+)^{n/q} \text{diam } I.$$

Therefore, if $I, J \subset \bigcup_{i=1}^p I_i$, we have

$$\frac{\text{diam } f^{\circ(n-N_0)}(I)}{\text{diam } f^{\circ(n-N_0)}(J)} \simeq \frac{\text{diam } I}{\text{diam } J},$$

showing that $f^{\circ(n-N_0)}$ has bounded relative distortion. This completes the proof in the case that a is periodic. Note that the constants in all above inequalities depend on N_0 .

If $a \in F_1$ is a preperiodic point and a^+ is hyperbolic, then there exists a smallest $k \in \{0, \dots, r\}$ such that $f^{\circ k}(a)$ is periodic and $f^{\circ k}(a^+)$ is hyperbolic. Observe that the map $f^{\circ k}$ is bi-Lipschitz in an arc $\widehat{[a, z_1]}$. The map $f^{\circ k}$ takes the complementary arcs $I_1, \dots, I_p \subset \widehat{[a, z_1]}$ of F_n to complementary arcs of F_{n-k} , as long as $n \geq r+1 \geq k+1$. It follows by the previous case of periodic points that

$$\text{diam } I_i \simeq \text{diam } f^{\circ k}(I_i) \simeq \lambda(f^{\circ k}(a^+))^{-(n-k)/q} \simeq \lambda(a^+)^{-n/q}.$$

This also holds trivially for $n < r+1$, since there are only finitely many complementary intervals of F_n for $n < r+1$.

Let N_0 be a sufficiently large integer corresponding to the relative distortion bounds near the periodic point $f^{\circ k}(a)$. The bounds of the relative distortion for the point a follow in the same way, if one observes that for any $I \subset \bigcup_{i=1}^p I_i$ we have

$$\text{diam } f^{\circ(n-N_0)}(I) \simeq \text{diam } f^{\circ(n+k-N_0)}(I) = \text{diam } f^{\circ(n-N_0)}(f^{\circ k}(I))$$

and then uses the relative distortion bounds near the periodic point $f^{\circ k}(a)$. $\square$

We prove analogous estimates near parabolic points.

Lemma 4.17 (Parabolic estimates). *Suppose that $a \in F_1$ and a^+ is parabolic. For each $p \in \mathbb{N}$, $z_0 \in \mathbb{S}^1$ with $z_0 \neq a$, and for all sufficiently large $N_0 \in \mathbb{N}$ there exists a constant $L \geq 1$ such that the following is true. If $I_1, \dots, I_p \subset \widehat{[a, z_0]}$ are consecutive complementary arcs of F_n , $n \geq 1$, and a is an endpoint of I_1 , then for each $i \in \{1, \dots, p\}$ we have*

$$(4.7) \quad L^{-1}n^{-1/N(a^+)} \leq \text{diam } I_1 \leq Ln^{-1/N(a^+)}$$

and

$$(4.8) \quad L^{-1}n^{-1/N(a^+)-1} \leq \text{diam } I_i \leq Ln^{-1/N(a^+)-1}$$

for $i \in \{2, \dots, p\}$. Moreover, if $n \geq N_0$, the map $f^{\circ(n-N_0)}$ has relative distortion bounded by L on $\bigcup_{i=2}^p I_i$. The analogous statements hold if a^- is parabolic and $I_1, \dots, I_p \subset \widehat{[z_0, a]}$.

Recall that $N(a^\pm) + 1$ is the multiplicity of a parabolic point $a^\pm$. Also note that we only obtain relative distortion bounds for the arcs $I_2, \dots, I_p$, but not for the arc I_1 . This contrasts the hyperbolic case in Lemma 4.16, in which we also had relative distortion bounds for I_1 .

Proof. As in the proof of Lemma 4.16, we will only find bounds for sufficiently large n . Also, the case of preperiodic points is treated exactly in the same way, so we will only focus on periodic points here. Suppose that a is periodic and a^+ is parabolic.

We consider the map f_a^+ , which is a holomorphic extension of f_a in a complex neighborhood of some arc $\widehat{[a, z_1]} \subset \widehat{[a, z_0]}$. Since $(f_a^+)'(a) = 1$ and a is an isolated fixed point of f_a^+ by condition (4.3), it follows that near a we have

$$(4.9) \quad f_a^+(z) = a + (z - a) + c(z - a)^{N(a^+)+1} + O((z - a)^{N(a^+)+2}),$$

where $c \neq 0$ and $N(a^+) + 1$ is the multiplicity of a^+ . By [Mil06, Lemma 10.1] we deduce that there exists a constant $M \geq 1$ such that if a backwards orbit $w_0 \mapsto w_1 \mapsto w_2 \mapsto \dots$ under $(f_a^+)^{-1}$ converges to a , then for all sufficiently large $m \in \mathbb{N}$ we have

$$(4.10) \quad M^{-1} \frac{1}{m^{1/N(a^+)}} \leq |w_m - a| \leq M \frac{1}{m^{1/N(a^+)}}.$$

Recall that by condition (4.4) the arc $\widehat{[a, z_1]}$ defines a repelling direction of the parabolic point a . Equivalently, the inverse orbit of the point $w \in \widehat{[a, z_1]}$ under $(f_a^+)^{-1}$ converges to a . Therefore, (4.10) holds for all inverse orbits of points in $\widehat{[a, z_1]}$.

Let $N_0 \in \mathbb{N}$ be an integer such that there are p consecutive complementary arcs of F_{N_0} that are contained in $\widehat{[a, z_1]}$ and one of which has a as an endpoint. As in Lemma 4.16, the existence of N_0 follows from the expansivity of f . Let

$n \in \mathbb{N}$, $n \geq N_0$. Suppose that $I_1 = \widehat{[a, w_n]}$ is a complementary arc of F_n as in the statement of the lemma. If $n - N_0 = qm + k$, $k \in \{0, \dots, q-1\}$, $m \in \mathbb{N} \cup \{0\}$, from (4.10) we have

$$|w_n - a| \simeq \frac{1}{((n - N_0)/q)^{1/N(a^+)}} \simeq \frac{1}{n^{1/N(a^+)}}.$$

Note that there are only q possible backward orbits w_n , $n \in \mathbb{N}$, under f_a^+ , where w_n is an endpoint of a complementary arc $\widehat{[a, w_n]}$ of F_n , and hence we may have (4.10) for all $n \in \mathbb{N}$. This already proves the first inequality in the statement of the lemma.

The second inequality is more subtle and follows from the relative distortion bounds on $\bigcup_{i=2}^p I_i$ that we claim below.

Claim. Let $s \in \mathbb{N}$ and suppose that $K_1, \dots, K_s \subset \widehat{[a, z_1]}$ are complementary arcs of F_n and a is an endpoint of K_1 . If N_0 is sufficiently large, then for $n \geq N_0$ the map $f^{\circ(n-N_0)}$ has bounded relative distortion on $\bigcup_{i=2}^s K_i$ with constants independent of n .

We postpone the proof of the claim for the moment. The claim implies that

$$\frac{\text{diam } I_i}{\text{diam } I_j} \simeq \frac{\text{diam } f^{\circ(n-N_0)}(I_i)}{\text{diam } f^{\circ(n-N_0)}(I_j)}$$

for $i, j \in \{2, \dots, p\}$. The arcs $f^{\circ(n-N_0)}(I_i), f^{\circ(n-N_0)}(I_j)$ are complementary arcs of F_{N_0} , so the ratio of their diameters is comparable to 1, with constants depending on N_0 . Therefore, $\text{diam } I_i \simeq \text{diam } I_j$ for $i, j \in \{2, \dots, p\}$.

Let $I_2 = \widehat{[w_n, b]}$, $w_n, b \in F_n$, and let $w_{n-1} = f_a^+(w_n)$. Consider the arc $\widehat{[a, w_{n-1}]}$, which is a complementary arc of F_{n-q} . By Lemma 4.15 the arc $\widehat{[a, w_{n-1}]}$ contains at least two complementary arcs of F_n , one of which is $\widehat{[a, w_n]}$. Therefore, I_2 is contained in $\widehat{[w_n, w_{n-1}]}$, which contains at most $(r+1)^q - 1$ complementary arcs of F_n , by Lemma 4.15. Since $\widehat{[w_n, w_{n-1}]}$ contains a uniformly bounded number of complementary arcs of F_n , including I_2 , we have $\text{diam } I_2 \simeq \text{diam } \widehat{[w_n, w_{n-1}]}$ by the Claim. It follows that $\text{diam } I_i \simeq \text{diam } \widehat{[w_n, w_{n-1}]}$ for $i \in \{2, \dots, p\}$. If n is sufficiently large, then $\text{diam } \widehat{[w_n, w_{n-1}]} = |w_n - w_{n-1}|$. Therefore, by the Taylor expansion of f_a^+ at a in (4.9) and by (4.10), we have

$$\text{diam } I_i \simeq |w_n - w_{n-1}| = |w_n - f_a^+(w_n)| \simeq |w_n - a|^{N(a^+)+1} \simeq \frac{1}{n^{1/N(a^+)+1}}$$

for $i \in \{2, \dots, p\}$. □

Proof of Claim. Consider the map f_a^+ that is holomorphic in a neighborhood of its parabolic fixed point a . For this map there exists a *repelling petal* $\mathcal{P}$, which is an open set in the plane, that contains a neighborhood of a inside the arc $\widehat{(a, z_1]}$; this is because all points of $\widehat{(a, z_1]}$ near a are attracted to a under iterations of $(f_a^+)^{-1}$ by condition (4.4).

By the parabolic linearization theorem [Mil06, Theorem 10.9] there exists a conformal embedding $\alpha: \mathcal{P} \rightarrow \mathbb{C}$, unique up to a translation of $\mathbb{C}$, such that

$$\alpha(f_a^+(z)) = 1 + \alpha(z)$$

for all $z \in \mathcal{P} \cap (f_a^+)^{-1}(\mathcal{P})$. Moreover, the domain $\alpha(\mathcal{P})$ contains a left half-plane $\mathbb{H}$. By shrinking $\widehat{(a, z_1]}$, we assume that the above conjugation holds in $\widehat{(a, z_1]}$ and that $\alpha(\widehat{(a, z_1]}) \subset \mathbb{H}$. Since f_a^+ maps $\widehat{(a, z_1]} \subset \mathbb{S}^1$ to an arc of the circle $\mathbb{S}^1$, by symmetry, we have that $\alpha(\widehat{(a, z_1]})$ is an interval on the negative real axis, contained in $\mathbb{H}$. In particular, the distance between $\alpha(\widehat{(a, z_1]})$ and $\partial\mathbb{H}$ is positive.

We let $N_0 \in \mathbb{N}$ be an integer such that there are s consecutive complementary arcs of F_{N_0} that are contained in $\widehat{(a, z_1]}$ and one of which has a as an endpoint. The existence of N_0 follows from the expansivity of f and property (E2). For each $j \in \{1, \dots, q-1\}$ we also consider the s consecutive complementary arcs of F_{N_0+j} that are contained in $\widehat{(a, z_1]}$ and one of which has a as an endpoint. We discard from these collections all arcs that have a as an endpoint. In total, we have $q(s-1)$ fixed arcs J_l , $l \in \{1, \dots, q(s-1)\}$. Suppose that $n \geq N_0$.

Consider the arcs $K_1, \dots, K_s$ as in the statement of the claim, which are complementary arcs of F_n . For each $i \in \{2, \dots, s\}$ we have $K_i \subset \widehat{(a, z_1]}$. Hence, $(f_a^+)^{\circ m}(K_i) = J_l$ for some $l \in \{1, \dots, q(s-1)\}$, where $m = \lfloor (n - N_0)/q \rfloor$. Consider the corresponding arcs $K'_i = \alpha(K_i)$ and $J'_l = \alpha(J_l)$. We then have $J'_l = K'_i + m$ by the conjugation.

Note that $\bigcup_{l=1}^{q(s-1)} J'_l$ is contained in a fixed ball $B(x, R)$ with $B(x, tR) \subset \mathbb{H}$ for some fixed $t > 1$, since the distance between $\bigcup_{l=1}^{q(s-1)} J'_l$ and $\partial\mathbb{H}$ is positive and we have discarded all unbounded arcs. Using Koebe's distortion theorem, we see that α^{-1} has bounded relative distortion on $\bigcup_{l=1}^{q(s-1)} J'_l$. Now observe that $\bigcup_{i=2}^s K'_i$ is contained in $B(x - m, R)$ and $B(x - m, tR)$ is contained in $\mathbb{H}$. Koebe's distortion theorem implies in this case that α^{-1} has bounded relative distortion on $\bigcup_{i=2}^s K'_i$. Since the inverse of a map of bounded relative distortion has the same property, we conclude that α has bounded relative distortion on $\bigcup_{i=2}^s K_i$.

By the conjugation, the map $(f_a^+)^{\circ m}$, restricted to $\bigcup_{i=2}^s K_i$, is the composition of α^{-1} , the translation to the right by m , and α . Since all maps involved have bounded relative distortion, we conclude that $(f_a^+)^{\circ m} = f^{\circ qm}$ has bounded relative distortion on $\bigcup_{i=2}^s K_i$.

Since $n \geq N_0$, there exists $k \in \{0, 1, \dots, q-1\}$ such that $n - N_0 = qm + k$. By condition (4.2), the maps $f, f^{\circ 2}, \dots, f^{\circ(q-1)}$ have analytic extensions in a neighborhood of the arc $\widehat{(a, z_1]}$, after possibly shrinking the arc. Moreover, the derivatives of these extensions are non-zero at a since $\lambda(a^+) \neq 0$. It follows that $f, f^{\circ 2}, \dots, f^{\circ(q-1)}$ are bi-Lipschitz in $\widehat{(a, z_1]}$. Since $f^{\circ qm}(\bigcup_{i=2}^s K_i)$ is contained in $\widehat{(a, z_1]}$, we conclude that $f^{\circ(n-N_0)} = f^{\circ k} \circ f^{\circ qm}$ has bounded relative distortion on $\bigcup_{i=2}^s K_i$. $\square$

Corollary 4.18. *Suppose that $a \in F_1$. For each $p \in \mathbb{N}$, $z_0 \in \mathbb{S}^1$ with $z_0 \neq a$, there exists a constant $L \geq 1$ such that the following is true. If $I_1, \dots, I_p \subset \widehat{(a, z_0]}$ are consecutive complementary arcs of F_n , $n \geq 1$, and a is an endpoint of I_1 , then we have*

$$\text{diam } I_1 \geq L^{-1} \text{diam } I_i \quad \text{and} \quad L^{-1} \text{diam } I_j \leq \text{diam } I_i \leq L \text{diam } I_j.$$

for $i, j \in \{2, \dots, p\}$.

So far, we have established diameter bounds for *dynamical* arcs, i.e., complementary arcs of F_n . In the next very technical lemma we prove estimates for the

diameter of a *non-dynamical* arc I , located near a point $a \in F_1$. Recall that $h: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is an orientation-preserving homeomorphism that conjugates f to g , by Lemma 3.3.

Lemma 4.19 (One-sided estimates). *Suppose that $a \in F_1$, q is the orientation-preserving period of a , and $b = h(a)$. For each $p \in \mathbb{N}$ with $p \geq 2$, $z_0 \in \mathbb{S}^1$ with $z_0 \neq a$, and $M \geq 1$ there exists $L \geq 1$ such that the following is true. If $I_1, \dots, I_p \subset \widehat{[a, z_0]}$ are consecutive complementary arcs of F_n , $n \geq 1$, and a is an endpoint of I_1 , then for each closed arc $I \subset \mathbb{S}^1$ with*

$$I \subset \bigcup_{i=1}^p I_i \quad \text{and} \quad \text{diam } I \geq M^{-1} \text{diam } I_2$$

the following alternatives occur. We let $k \in \mathbb{N} \cup \{0\}$ be the smallest integer such that a complementary arc of F_{n+k} not having a as an endpoint intersects I and $l \in \mathbb{N} \cup \{0\}$ be the smallest integer, if there exists one, such that there exists a complementary arc of F_{n+k+l} having a as an endpoint and not intersecting I . If no such l exists, it is set to be ∞ .

- *If a^+ is parabolic, then*

$$L^{-1}(n+k)^{-\alpha-1} \leq \frac{\text{diam } I}{k + \min\{l+1, n\}} \leq L(n+k)^{-\alpha-1},$$

where $\alpha = 1/N(a^+)$. If, in addition, b^+ is parabolic, the same estimates hold for the arc $h(I)$, if we replace α by $\beta = 1/N(b^+)$. Namely,

$$L^{-1}(n+k)^{-\beta-1} \leq \frac{\text{diam } h(I)}{k + \min\{l+1, n\}} \leq L(n+k)^{-\beta-1}.$$

In particular, if a^+ and b^+ are both parabolic, then

$$L^{-2}(n+k)^{\alpha-\beta} \leq \frac{\text{diam } h(I)}{\text{diam } I} \leq L^2(n+k)^{\alpha-\beta}.$$

- *If a^+ is hyperbolic, then*

$$L^{-1}\lambda(a^+)^{-n/q} \leq \text{diam } I \leq L\lambda(a^+)^{-n/q}$$

and $k \leq L$. If, in addition, b^+ is hyperbolic, then

$$L^{-1}\lambda(b^+)^{-n/q} \leq \text{diam } h(I) \leq L\lambda(b^+)^{-n/q}$$

and if b^+ is parabolic, then

$$L^{-1}(n+k)^{-\beta-1} \leq \frac{\text{diam } h(I)}{k + \min\{l+1, n\}} \leq L(n+k)^{-\beta-1},$$

where $\beta = 1/N(b^+)$.

The corresponding estimates hold for a^- and b^- .

Proof. First, we note that it suffices to prove the statements for sufficiently large n . Indeed, if $n < N_0$ for some $N_0 \in \mathbb{N}$, then $\text{diam } I_i \simeq 1$ and $\text{diam } h(I_i) \simeq 1$ for $i \in \{1, \dots, p\}$, as there are only finitely many complementary arcs of F_n , $n < N_0$. Since $\text{diam } I \gtrsim \text{diam } I_2$, we have $\text{diam } I \simeq 1$. By the continuity of the homeomorphism h , we have $\text{diam } h(I) \simeq 1$. This shows that the estimates in the case that a^+ or b^+ is hyperbolic hold with constants depending on N_0 . Moreover, if k_0 is a sufficiently large integer, then the complementary arcs of F_{n+k_0} are small enough by property (E2), so that the arc I , whose diameter is comparable to 1, is not contained in any

complementary arc of F_{n+k_0} . Hence, I intersects at least two complementary arcs of F_{n+k_0} and $k \leq k_0$, where k is the integer that is defined in the statement of the lemma. Since k and n are bounded, we also obtain the desired estimates in case a^+ or b^+ is a parabolic point.

Another reduction is to assume that a is a periodic point of f . Otherwise, if a is strictly preperiodic, one can obtain the desired estimate using the fact that an iterate of f that maps a to a periodic point is bi-Lipschitz on $\bigcup_{i=1}^p I_i$, if n is sufficiently large, as in the proof of Lemma 4.16. We will split the proof into the two basic cases.

Case 1. a^+ is parabolic. We let $k \in \mathbb{N} \cup \{0\}$ be the smallest integer such that a complementary arc of F_{n+k} not having a as an endpoint intersects I . We will reduce to the case that $k = 0$. If $k \geq 1$, there exists a complementary arc J of F_{n+k-1} that has a as an endpoint and contains I . By passing to the next level, we obtain consecutive complementary arcs $J_1, \dots, J_{p'}$ of F_{n+k} , which are the children of J , so $p' \leq r+1$ by Lemma 4.15, such that I is not contained in J_1 . By assumption, we have $\text{diam } I \gtrsim \text{diam } I_2$. Inequality (4.8) from Lemma 4.17 implies that $\text{diam } I_2 \simeq n^{-\alpha-1}$ and $\text{diam } J_2 \simeq (n+k)^{-\alpha-1}$. Since $n^{-\alpha-1} \geq (n+k)^{-\alpha-1}$, it follows that $\text{diam } I \gtrsim \text{diam } J_2$. Therefore, we have reduced to the case that

$$(4.11) \quad I \subset \bigcup_{i=1}^p I_i, \quad \text{diam } I \gtrsim \text{diam } I_2, \quad I \cap \bigcup_{i=2}^p I_i \neq \emptyset, \quad \text{and} \quad k = 0.$$

We let $l \in \mathbb{N} \cup \{0\}$ be the smallest integer such that there exists a complementary arc of F_{n+l} that has a as an endpoint does not intersect I . If such an integer does not exist, we set $l = \infty$. In the latter case we have $I \supset I_1$, since $I \cap \bigcup_{i=2}^p I_i \neq \emptyset$. It follows from (4.7) and (4.8) in Lemma 4.17 that

$$\begin{aligned} n^{-\alpha} &\simeq \text{diam } I_1 \lesssim \text{diam } I \\ &\lesssim \text{diam } I_1 + \sum_{i=2}^p \text{diam } I_i \lesssim n^{-\alpha} + (p-1)n^{-\alpha-1} \simeq n^{-\alpha}. \end{aligned}$$

Hence $\text{diam } I \simeq n^{-\alpha-1} \cdot \min\{\infty, n\}$. Similarly, if $l = 0$, then $I_1 \cap I = \emptyset$ and $I \subset \bigcup_{i=2}^p I_i$, so

$$n^{-\alpha-1} \simeq \text{diam } I_2 \lesssim \text{diam } I \lesssim \sum_{i=2}^p \text{diam } I_i \simeq n^{-\alpha-1}.$$

Hence, $\text{diam } I \simeq n^{-\alpha-1} \cdot \min\{1, n\}$.

Suppose now that $0 < l < \infty$. Recall that q is the orientation-preserving period of a . Consider $m \in \mathbb{N}$ such that $l = q(m-1) + s$, $s \in \{1, \dots, q\}$. Note that $q(m-1) < l \leq qm$. Set $J_0 = I_1$ and let $J_1 \subset J_0$ be the complementary arc of F_{n+q} that has a as an endpoint. Then, by Lemma 4.15, $J_0 \setminus J_1$ contains at least one complementary arc of F_{n+q} and is contained in the union of at most $(r+1)^q - 1$ complementary arcs of F_{n+q} . It follows from (4.8) in Lemma 4.17 that $\text{diam}(J_0 \setminus J_1) \simeq (n+q)^{-\alpha-1}$. Inductively, we let $J_j \subset J_{j-1}$ be the complementary arc of F_{n+jq} that has a as an endpoint. Again from Lemma 4.17 we have $\text{diam}(J_{j-1} \setminus J_j) \simeq (n+jq)^{-\alpha-1}$. We

note that

$$\bigcup_{j=1}^{m-1} (J_{j-1} \setminus J_j) \subset I \subset \bigcup_{j=1}^m (J_{j-1} \setminus J_j) \bigcup_{i=2}^p I_i.$$

Since $\text{diam } I \geq M^{-1} \text{diam } I_2$, we have

$$\sum_{j=1}^{m-1} \text{diam}(J_{j-1} \setminus J_j) + \text{diam } I_2 \lesssim \text{diam } I \leq \sum_{j=1}^m \text{diam}(J_{j-1} \setminus J_j) + \sum_{i=2}^p \text{diam } I_i.$$

It follows that

$$\begin{aligned} \text{diam } I &\simeq \sum_{j=0}^m (n + jq)^{-\alpha-1} \\ &\simeq \int_n^{n+l+1} x^{-\alpha-1} dx \simeq n^{-\alpha} - (n+l+1)^{-\alpha} \simeq n^{-\alpha} (1 - (1 + (l+1)/n)^{-\alpha}) \\ &\simeq n^{-\alpha} \cdot \begin{cases} (l+1)/n, & \text{if } l+1 \leq n \\ 1, & \text{if } l+1 > n \end{cases}. \end{aligned}$$

The conclusion now follows.

Next, we show that if b^+ is parabolic, it satisfies the same estimate. Note that by (4.11) we have $h(I) \subset \bigcup_{i=1}^p h(I)$ and $h(I) \cap \bigcup_{i=2}^p h(I_i) \neq \emptyset$. If we had that

$$(4.12) \quad \text{diam } h(I) \gtrsim \text{diam } h(I_2),$$

then we would have the desired estimate (for $k = 0$) by following the above argument. Hence, our goal will be to establish inequality (4.12).

We consider the family of complementary arcs of F_{n+q} that are contained in I_i , $i \in \{1, \dots, p\}$. We denote these arcs by K_i , $i \in \{1, \dots, p''\}$. Note that each one of I_i contains at most $(r+1)^q$ such arcs by Lemma 4.15, so $p'' \leq p(r+1)^q$. Moreover, by Lemma 4.15 I_1 contains at least two complementary arcs of F_{n+q} . Let $K_1 \subset I_1$ be the complementary arc of F_{n+q} that has a as an endpoint and $K_2 \subset I_1$ be its adjacent arc. Lemma 4.17 implies that

$$\text{diam } I_2 \simeq n^{-\alpha-1} \simeq (n+q)^{-\alpha-1} \simeq \text{diam } K_2 \simeq \text{diam } K_i$$

for all $i \in \{2, \dots, p''\}$. If $I \cap K_1 \neq \emptyset$, then we necessarily have $K_2 \subset I$, since $I \cap \bigcup_{i=2}^p I_i \neq \emptyset$. Otherwise, if $I \cap K_1 = \emptyset$, we have $I \subset \bigcup_{i=2}^{p''} K_i$. Therefore, in both of the above cases, since $\text{diam } I \gtrsim \text{diam } K_2$, we have

$$\text{diam } K \simeq \text{diam } K_2, \quad \text{where } K = I \cap \bigcup_{i=2}^{p''} K_i \subset \bigcup_{i=2}^{p''} K_i.$$

The importance of these conditions is that on $\bigcup_{i=2}^{p''} K_i$ and on $\bigcup_{i=2}^{p''} h(K_i)$ we have relative distortion bounds for the maps $f^{\circ(n-N_0)}$ and $g^{\circ(n-N_0)}$, respectively, by Lemma 4.17. These distortion bounds are not available on $\bigcup_{i=1}^{p''} K_i$ and $\bigcup_{i=1}^{p''} h(K_i)$. By Lemma 4.17, for all sufficiently large $N_0 \in \mathbb{N}$, if $n \geq N_0$, we have

$$\text{diam } f^{(n-N_0)}(K) \simeq \text{diam } f^{(n-N_0)}(K_2).$$

Since $f^{(n-N_0)}(K_2)$ is a complementary arc of $N_0 + q$, we may assume that

$$\text{diam } f^{(n-N_0)}(K) \simeq \text{diam } f^{(n-N_0)}(K_2) \simeq 1$$

with constants depending on N_0 . Using the continuity of h and the conjugation, we obtain

$$\text{diam } g^{(n-N_0)}(h(K)) \simeq \text{diam } g^{(n-N_0)}h((K_2)) \simeq 1.$$

Using again the distortion bounds from Lemma 4.17 and enlarging N_0 if necessary, we conclude that

$$\text{diam } h(K) \simeq \text{diam } h(K_2).$$

Since $I \supset K$, we have $\text{diam } h(I) \geq \text{diam } h(K_2)$. Finally, by Lemma 4.17, we have

$$\text{diam } h(K_2) \simeq (n+q)^{-\beta-1} \simeq n^{-\beta-1} \simeq \text{diam } h(I_2).$$

Altogether, we have $\text{diam } h(I) \gtrsim \text{diam } h(I_2)$, i.e., (4.12) holds, assuming that $n \geq N_0$. This completes the proof of Case 1.

Case 2. a^+ is hyperbolic. We have $\text{diam } I_i \simeq \lambda(a^+)^{-n/q}$ for $i \in \{1, \dots, p\}$ by Lemma 4.16. Since

$$M^{-1} \text{diam } I_2 \leq \text{diam } I \leq \sum_{i=1}^p \text{diam } I_i,$$

it follows that $\text{diam } I \simeq \lambda(a^+)^{-n/q}$ with constants depending on p and M .

Let $k \in \mathbb{N} \cup \{0\}$ be the smallest integer such that a complementary arc of F_{n+k} not having a as an endpoint intersects I . If $k \geq 1$, there exists a complementary arc J of F_{n+k-1} that has a as an endpoint and contains I . By passing to a further level, we obtain consecutive complementary arcs $J_1, \dots, J_{p'}$ of F_{n+k} , which are the children of J , so $p' \leq r+1$, such that I is not contained in J_1 and $I \subset \bigcup_{i=1}^{p'} J_i$. Inequality (4.5) from Lemma 4.16 implies that $\text{diam } J_i \simeq \lambda(a^+)^{-(n+k)/q}$, so $\text{diam } I \lesssim \lambda(a^+)^{-(n+k)/q}$. However, $\text{diam } I \simeq \lambda(a^+)^{-n/q}$. Therefore, $k \leq C$ for some uniform constant $C > 0$.

Suppose that b^+ is also hyperbolic. By Lemma 4.16, for all sufficiently large $N_0 \in \mathbb{N}$, if $n \geq N_0$ then $f^{o(n-N_0)}$ has bounded distortion on $\bigcup_{i=1}^p I_i$. It follows that

$$\text{diam } f^{o(n-N_0)}(I) \gtrsim \text{diam } f^{o(n-N_0)}(I_2).$$

Note that $f^{o(n-N_0)}(I_2)$ is a complementary arc of F_{N_0} . If N_0 is fixed, then we may assume that $\text{diam } f^{o(n-N_0)}(I_2) \simeq 1$. By continuity, this implies that

$$\text{diam } h(f^{o(n-N_0)}(I)) \simeq 1,$$

or equivalently

$$\text{diam } g^{o(n-N_0)}(h(I)) \simeq 1 \simeq \text{diam } g^{o(n-N_0)}(h(I_2)).$$

Since b^+ is hyperbolic, using Lemma 4.16 and enlarging N_0 , we may also have that $g^{o(n-N_0)}$ has bounded distortion on $\bigcup_{i=1}^p h(I_i)$. Therefore, we obtain

$$\text{diam } h(I) \simeq \text{diam } h(I_2).$$

By (4.5) we have

$$\text{diam } h(I_2) \simeq \lambda(b^+)^{-n/q}.$$

Hence, $\text{diam } h(I) \simeq \lambda(b^+)^{-n/q}$, as desired.

Suppose now that b^+ is parabolic. In this case, we cannot apply distortion estimates near b^+ so we will first pass to some further subdivisions. Consider the consecutive complementary arcs $J_1, \dots, J_{p'}$ of F_{n+k} , such that I is not contained in J_1 and $I \subset \bigcup_{i=1}^{p'} J_i$. Since k is uniformly bounded, we have $\text{diam } J_2 \simeq \lambda(a^+)^{-n/q} \simeq$

$\text{diam } I$ by (4.5) and $\text{diam } h(J_2) \simeq n^{-\beta-1} \simeq \text{diam } h(I_2)$ by (4.8). If we replace n with $n+k$ and the arcs I_i with the arcs J_i , we have $I \subset \bigcup_{i=1}^p I_i$, $\text{diam } I \simeq \text{diam } I_2$, and $I \cap \bigcup_{i=2}^p I_i \neq \emptyset$. By arguing as in Case 1 (see (4.12)), it suffices to prove that

$$\text{diam } h(I) \gtrsim \text{diam } h(I_2).$$

We can now proceed as in Case 1. We consider the family of complementary arcs of F_{n+q} that are contained in I_i , $i \in \{1, \dots, p\}$. We denote these arcs by K_i , $i \in \{1, \dots, p''\}$, where $p'' \leq p(r+1)^q$. Let $K_1 \subset I_1$ be the complementary arc of F_{n+q} that has a as an endpoint and $K_2 \subset I_1$ be its adjacent arc. By Lemma 4.16, we have $\text{diam } I_2 \simeq \text{diam } K_2 \simeq \text{diam } K_i$ for all $i \in \{1, \dots, p''\}$. Since $\text{diam } I \simeq \text{diam } I_2 \simeq \text{diam } K_2$, we have

$$\text{diam } K \simeq \text{diam } K_2, \quad \text{where} \quad K = I \cap \bigcup_{i=2}^{p''} K_i \subset \bigcup_{i=2}^{p''} K_i.$$

The argument now continues exactly as in Case 1, using distortion bounds to blow up the arcs K_2 and K to arcs of large diameter. On $\bigcup_{i=1}^{p''} K_i$ the distortion bounds come from Lemma 4.16 and on $\bigcup_{i=2}^{p''} h(K_i)$ we use Lemma 4.17. The proof is complete. $\square$

4.2.2. Conformal elevator and completion of proof of Theorem 4.9. Our first goal here is to start with an arbitrary arc $I \subset \mathbb{S}^1$ and map it conformally and with bounded relative distortion, by applying a suitable iterate of f , to an arc I' that is located “near” a point $a \in F_1$. This procedure is referred to as the *conformal elevator* and is described more precisely in Lemma 4.20. Once the arc I is blown up to the arc I' that is near $a \in F_1$, then one can apply the diameter estimates from Lemma 4.19. However, there is a basic dichotomy. Either I' contains the point a , or I' lies only on one side of a . Each of these cases is treated separately in Lemma 4.21 and Lemma 4.22, respectively. Finally, using the latter two lemmas, we conclude the proof of the main Theorem 4.9.

Lemma 4.20 (Conformal elevator). *There exists $M \geq 1$ such that for any non-degenerate closed arc $I \subset \mathbb{S}^1$ there exist $n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$, $a \in F_1$, and $z_0 \in \mathbb{S}^1$ with $z_0 \neq a$ such that one of the following alternatives holds.*

- (i) *The arc $I' = f^{\circ m}(I)$ contains the point $a \in F_1$.*
- (ii) *There exist consecutive complementary arcs $I_1, \dots, I_p$, $p \leq M$, of F_{n+m} with*

$$I \subset \bigcup_{i=1}^p I_i \quad \text{and} \quad I \cap \bigcup_{i=2}^p I_i \neq \emptyset$$

such that the arcs $I'_1 = f^{\circ m}(I_1), \dots, I'_p = f^{\circ m}(I_p)$ are consecutive complementary arcs of F_n that contain the arc $I' = f^{\circ m}(I)$, the point a is an endpoint of I'_1 ,

$$\bigcup_{i=1}^p I'_i \subset [a, z_0] \quad \text{or} \quad \bigcup_{i=1}^p I'_i \subset [z_0, a], \quad \text{and} \\ \text{diam } I' \geq M^{-1} \text{diam } I'_2.$$

Moreover, in both cases $f^{\circ m}$ and $g^{\circ m}$ have relative distortion bounded by M on I and $h(I)$, respectively, and

$$\text{diam } I' \geq M^{-1} \text{diam } I.$$

Proof. Recall that A_k , $k \in \{0, \dots, r\}$, are the complementary arcs of F_1 . For each $k \in \{0, \dots, r\}$ we define $\widetilde{A}_k \subset \subset A_k$ to be a closed arc with the property that each of its endpoints is separated from the corresponding endpoint of A_k by one arc of F_q . Here, $q \in \mathbb{N}$ is chosen so that $\widetilde{A}_k$ contains all points of F_2 that lie in $\text{int } A_k$ and moreover each point of F_2 in $\widetilde{A}_k$ is separated from the endpoints of $\widetilde{A}_k$ by at least one complementary arc of F_q . The existence of such a q follows from property (E2). Indeed, as $q \rightarrow \infty$, the endpoints of the arc $\widetilde{A}_k$ converge to the endpoints of the arc A_k , since the diameters of the arcs A_w , $|w| = q$, tend to 0. Hence, we can achieve that $\widetilde{A}_k$ contains in its interior all points of F_2 that are contained in $\text{int } A_k$. Moreover, any point of $z \in F_2 \cap \widetilde{A}_k$ has positive distance from the endpoints of $\widetilde{A}_k$. Using again the fact that the diameters of A_w , $|w| = q$, tend to 0 as $q \rightarrow \infty$, we may achieve that there are arbitrarily many consecutive arcs A_w , $|w| = q$, separating z from the endpoints of $\widetilde{A}_k$.

If I contains a point of F_1 , there is nothing to show, since we are already in alternative (i) of the lemma. Hence, we assume that I does not intersect F_1 . Then, by property (E2) there exists a largest integer $l \geq 1$ such that the arc I is contained in a nested sequence of arcs $A_{j_1} \supset A_{j_1 j_2} \supset \dots \supset A_{j_1 \dots j_l} =: A$. By the choice of l , the arc I must contain a point $c \in F_{l+1}$. Consider the corresponding regions $U_{j_1} \supset U_{j_1 j_2} \supset \dots \supset U_{j_1 \dots j_l} =: U$ and the map $\phi = f^{\circ(l-1)} = f_{j_1 \dots j_{l-1}}$, which maps U conformally onto U_{j_l} . We denote by $\widetilde{A}$ the preimage of $\widetilde{A}_{j_l}$ under ϕ and observe that $\phi(c) \in F_2$. The basic dichotomy arises from whether $I \subset \widetilde{A}$ or not.

Case 1. $I \subset \widetilde{A}$. Then $\phi(I) \subset \widetilde{A}_{j_l}$. Since $\widetilde{A} \subset \subset A \subset U$, by applying Koebe's distortion theorem to the conformal map $\phi^{-1}|_{\widetilde{A}_{j_l}}$, we see that ϕ has uniformly bounded relative distortion on I . Moreover, since $\phi(I) \subset \widetilde{A} \subset U_{j_l}$, we can apply again Koebe's distortion theorem to $f|_{U_{j_l}}$ and conclude that $f \circ \phi = f^{\circ l}$ has bounded relative distortion on I . The point $a = f^{\circ l}(c)$ lies in F_1 , so we have arrived to alternative (i), with $m = l$. Using the conjugation between f and g , we can perform the same combinatorial analysis and show that $g^{\circ m}$ has bounded relative distortion on $h(I)$.

Case 2. I is not contained in $\widetilde{A}$. By the choice of q , there exists at least one complementary arc of F_q that separates $\phi(c) \in F_2$ from the endpoints of $\widetilde{A}_{j_l}$. Since $\phi(I)$ is not contained in $\widetilde{A}_{j_l}$, we conclude that there exists a complementary arc of F_q that is contained in $\phi(I) \cap \widetilde{A}_{j_l}$.

Suppose that $A = [z_1, z_2]$ and without loss of generality z_1 has the smallest level among the two endpoints, equal to $m+1 \in \mathbb{N}$. We note that $m+1 \leq l$, since A is a complementary arc of F_l . The arc $A_{j_l} = \phi(A)$ consists of a uniformly bounded number of complementary arcs of F_q . Hence, A is the union of a uniformly bounded number of consecutive complementary arcs of F_{q+l-1} that we denote by $I_1, \dots, I_p$. We number them so that I_1 has z_1 as an endpoint. Note that by the previous discussion, there exists $i_0 \in \{2, \dots, p\}$ such that $I_{i_0} \subset I$. In particular,

$I \cap \bigcup_{i=2}^p I_i \neq \emptyset$. We set $n = q + l - 1 - m \geq q$ and we claim that alternative (ii) holds with the defined m, n and the arcs I_i .

If $m = 0$, then the level of z_1 is 1, thus $a = z_1 \in F_1$, and there is nothing to be proved. So we assume that $m \geq 1$. Consider the arc $A_{j_1 \dots j_m}$, which is a complementary arc of F_m and its endpoints have level at most m . It follows that $A_{j_1 \dots j_m}$ contains the arc A and the point z_1 in its interior. Let $\psi = f^{\circ(m-1)}$ and note that ψ maps conformally $U_{j_1 \dots j_m}$ onto U_{j_m} . Since $\psi(z_1) \in F_2$ and $\psi(A)$ is a complementary arc of F_{l-m+1} , the other endpoint $\psi(z_2)$ of $\psi(A)$ has level k , where $2 \leq k \leq l - m + 1$. Note also that $\psi(z_1) \in F_2 \cap \text{int } A_{j_m}$, so $\psi(z_1) \in \widetilde{A_{j_m}}$, by the definition of $\widetilde{A_{j_m}}$.

If $k \leq q$, then $\psi(z_2)$ cannot be contained in the interior of a complementary arc of F_q that separates the endpoints of $\widetilde{A_{j_m}}$ from the endpoints of A_{j_m} . Therefore, $\psi(z_2) \in \widetilde{A_{j_m}}$ and $\psi(A)$ is contained in $\widetilde{A_{j_m}}$.

If $k \geq q + 1$, then $l - m + 1 \geq q + 1$, so the complementary arc $\psi(A)$ of F_{l-m+1} cannot contain in its interior a point of level q or less. Thus, the arc $\psi(A)$ cannot intersect the interior of the complementary arcs of F_q that separate the endpoints of $\widetilde{A_{j_m}}$ from the endpoints of A_{j_m} . In this case, we also have that $\psi(A) \subset \widetilde{A_{j_m}}$.

It follows as in Case 1 that $f \circ \psi = f^{\circ m}$ has bounded relative distortion on $\bigcup_{i=1}^p I_i$, which is contained in $\psi^{-1}(\widetilde{A_{j_m}})$. The same conclusion holds for g , using the same combinatorial analysis. Moreover, since $I_{i_0} \subset I$, we have

$$\text{diam } f^{\circ m}(I) \geq \text{diam } f^{\circ m}(I_{i_0}).$$

The point $a = f^{\circ m}(z_1)$ lies in F_1 , and the arc $f^{\circ m}(A) = \bigcup_{i=1}^p f^{\circ m}(I_i)$ is contained in $[a, z_0]$ or in $[z_0, a]$ for some $z_0 \neq a$. Since $i_0 \neq 1$, by Corollary 4.18 we conclude that

$$\text{diam } f^{\circ m}(I_{i_0}) \simeq \text{diam } f^{\circ m}(I_2).$$

This proves the desired inequality

$$\text{diam } f^{\circ m}(I) \gtrsim \text{diam } f^{\circ m}(I_2).$$

Finally, we prove the last statement of the lemma. In both cases, by the relative distortion bounds of $f^{\circ m}$ we have

$$\frac{\text{diam } f^{\circ m}(I)}{\text{diam } I} \simeq \frac{\text{diam } \widetilde{A_{j_m}}}{\text{diam } \widetilde{A}}.$$

Since $\widetilde{A} \subset U \subset U_{j_1}$, we have

$$\frac{\text{diam } f^{\circ m}(I)}{\text{diam } I} \gtrsim \frac{\text{diam } \widetilde{A_{j_m}}}{\text{diam } U_{j_1}}.$$

Let $M_0 = \max\{\text{diam } \widetilde{A_{k_1}} / \text{diam } U_{k_2} : k_1, k_2 \in \{0, \dots, r\}\}$. Then we have

$$\text{diam } f^{\circ m}(I) \gtrsim M_0 \text{diam } I,$$

as desired. $\square$

Lemma 4.21 (One-sided distortion). *Let $I, J \subset \mathbb{S}^1$ be adjacent closed arcs each of which has length $t \in (0, 1/2)$. Suppose that alternative (ii) of Lemma 4.20 occurs for the arc $I \cup J$ and consider points $a \in F_1$ and $b = h(a)$ as in Lemma 4.20.*

- If $a^\pm$ is hyperbolic and $b^\pm$ is hyperbolic, then

$$\frac{\text{diam } h(I)}{\text{diam } h(J)} \simeq 1.$$

- If $a^\pm$ is parabolic and $b^\pm$ is parabolic, then

$$\frac{\text{diam } h(I)}{\text{diam } h(J)} \simeq 1.$$

- If $a^\pm$ is hyperbolic and $b^\pm$ is parabolic, then

$$\max \left\{ \frac{\text{diam } h(J)}{\text{diam } h(I)}, \frac{\text{diam } h(I)}{\text{diam } h(J)} \right\} \lesssim \log(1/t).$$

Proof. Let m, n, p be as in Lemma 4.20 (ii) and set $I' = f^{\circ m}(I)$ and $J' = f^{\circ m}(J)$. Moreover, let $I_1, \dots, I_p$ be complementary arcs of F_{n+m} , $n \geq 1$, such that $I \cup J \subset \bigcup_{i=1}^p I_i$, $(I \cup J) \cap \bigcup_{i=2}^p I_i \neq \emptyset$, and set $I'_i = f^{\circ m}(I_i)$ for $i \in \{1, \dots, p\}$. Since $f^{\circ m}$ has bounded distortion on $I \cup J$, we have $\text{diam } I' \simeq \text{diam } J'$. Moreover, $\text{diam } I' \gtrsim \text{diam } I'_2$ and $\text{diam } J' \gtrsim \text{diam } I'_2$. We will work in the proof with a^+ and b^+ , so $\bigcup_{i=1}^p I_i \subset [a, z_0]$. We have $I' \cap \bigcup_{i=2}^p I'_i \neq \emptyset$ or $J' \cap \bigcup_{i=2}^p I'_i \neq \emptyset$. If $J' \cap \bigcup_{i=2}^p I'_i \neq \emptyset$ and I' is not contained in the arc between J' and the point a , then I' also has this intersection property. Hence, by reversing the roles of I' and J' if necessary, we assume that I' is closer to the point a than J' , i.e., I' is contained in the arc between J' and a , and that $J' \cap \bigcup_{i=2}^p I'_i \neq \emptyset$. We note that since $g^{\circ m}$ has bounded relative distortion on $h(I) \cup h(J)$ it suffices to derive the conclusions of the lemma for the arcs $h(I')$ and $h(J')$.

First, suppose that a^+ and b^+ are hyperbolic. Since $\text{diam } I' \simeq \text{diam } J' \gtrsim \text{diam } I'_2$, we can apply Lemma 4.19 to each of the arcs I', J' . We obtain

$$\text{diam } I' \simeq \text{diam } J' \simeq \lambda(a^+)^{-n/q},$$

where q is the orientation-preserving period of a . Since b^+ is hyperbolic, it follows from the same lemma that

$$\text{diam } h(I') \simeq \text{diam } h(J') \simeq \lambda(b^+)^{-n/q}.$$

Next, suppose that a^+ and b^+ are parabolic. We will apply again Lemma 4.19. We let k_1 be the smallest integer such that a complementary arc of F_{n+k_1} not having a as an endpoint intersects I' . Similarly, we define k_2 , corresponding to J' . Since $J' \cap \bigcup_{i=2}^p I'_i \neq \emptyset$, we conclude that $k_2 = 0$. Moreover, let l_1 be the smallest integer such that there exists a complementary arc of $F_{n+k_1+l_1}$ having a as an endpoint and not intersecting I' , and l_2 be the corresponding integer for J' . Since I' and J' are adjacent and I' is between J' and a , it follows that $k_1 = l_2$. By Lemma 4.19 we have, since $k_1 = l_2$ and $k_2 = 0$,

$$n^{-\alpha-1} \min\{k_1 + 1, n\} \simeq \text{diam } J' \simeq \text{diam } I' \simeq (n + k_1)^{-\alpha-1} (k_1 + \min\{l_1 + 1, n\}).$$

If $k_1 + 1 \geq n$, then we obtain

$$n^{-\alpha} \simeq (k_1 + 1)^{-\alpha-1} (k_1 + \min\{l_1 + 1, n\}) \lesssim (k_1 + 1)^{-\alpha-1} (k_1 + n) \lesssim (k_1 + 1)^{-\alpha}.$$

It follows that $n \gtrsim k_1 + 1$. So, in any case $0 \leq k_1 \lesssim n$. Therefore, from Lemma 4.19 we conclude that

$$\frac{\text{diam } h(I')}{\text{diam } I'} \simeq n^{\alpha-\beta} \simeq \frac{\text{diam } h(J')}{\text{diam } J'},$$

which implies that $\text{diam } h(I') \simeq \text{diam } h(J')$.

Finally, suppose that a^+ is hyperbolic, but b^+ is parabolic. First, note that $\text{diam } I' \simeq \text{diam } J' \simeq \lambda(a^+)^{-n}$ by Lemma 4.19. According to the very last inequality in Lemma 4.20 (ii), we have $\text{diam } I' \gtrsim \text{diam } I = t$. Hence, $\lambda(a^+)^{-n} \gtrsim t$. This implies that $\log(1/t) \gtrsim n$, since $t < 1/2$ by assumption. Since b^+ is parabolic, by Lemma 4.19 we have, as in the previous paragraph, that there exist $k_1, k_2, l_1, l_2 \in \mathbb{N} \cup \{0, \infty\}$ with $k_2 = 0$ and $k_1 = l_2$ such that

$$\begin{aligned} \text{diam } h(I') &\simeq (n + k_1)^{-\beta-1} (k_1 + \min\{l_1 + 1, n\}) \quad \text{and} \\ \text{diam } h(J') &\simeq n^{-\beta-1} \min\{k_1 + 1, n\}. \end{aligned}$$

Moreover, k_1 is uniformly bounded by Lemma 4.19, so

$$\text{diam } h(I') \simeq n^{-\beta-1} \min\{l_1 + 1, n\} \quad \text{and} \quad \text{diam } h(J') \simeq n^{-\beta-1} \min\{k_1 + 1, n\}.$$

It follows that

$$\max \left\{ \frac{\text{diam } h(J')}{\text{diam } h(I')}, \frac{\text{diam } h(I')}{\text{diam } h(J')} \right\} \lesssim n \lesssim \log(1/t).$$

The proof is complete. $\square$

Lemma 4.22 (Two-sided distortion). *Let $I, J \subset \mathbb{S}^1$ be adjacent arcs each of which has length $t \in (0, 1/2)$. Suppose that alternative (i) of Lemma 4.20 occurs for the arc $I \cup J$ and consider points $a \in F_1$ and $b = h(a)$ as in Lemma 4.20. Then we have*

$$\frac{\text{diam } h(I)}{\text{diam } h(J)} \simeq 1.$$

under condition $(\mathbf{H}/\mathbf{P} \rightarrow \mathbf{H}/\mathbf{P})$ for the points a, b , and

$$\max \left\{ \frac{\text{diam } h(J)}{\text{diam } h(I)}, \frac{\text{diam } h(I)}{\text{diam } h(J)} \right\} \lesssim \log(1/t).$$

under condition $(\mathbf{H} \rightarrow \mathbf{P})$ for the points a, b .

Proof. We let $I' = f^{\circ m}(I)$ and $J' = f^{\circ m}(J)$, where m is as in Lemma 4.20. We have $\text{diam } I' \simeq \text{diam } J'$ by the relative distortion bounds of $f^{\circ m}$. We note that since $g^{\circ m}$ has bounded relative distortion on $h(I) \cup h(J)$ it suffices to derive the conclusions of the lemma for the arcs $h(I')$ and $h(J')$.

Without loss of generality, suppose that $J' \subset [a, z_0]$ and $I' = K^+ \cup K^-$, where $K^+ \subset [a, z_0]$ and $K^- \subset [z_0, a]$. We may assume that I', J' are sufficiently small arcs, so that $K^+ \cup J'$ is contained in a complementary arc of F_1 and K^- is contained in a complementary arc of F_1 . Indeed, if I', J' have diameters comparable to 1, then by continuity, $h(I'), h(J')$ also have diameters comparable to 1 and, thus, to each other.

We let $n - 1 \in \mathbb{N}$ be the largest integer such that there exists a complementary arc J_1'' of F_{n-1} having a as an endpoint and containing J' . Then, there exists $p \leq r + 1$ and consecutive complementary arcs $J_1', J_2', \dots, J_p' \subset [a, z_0]$ of F_n , which are the children of J_1'' , such that $J' \subset \bigcup_{i=1}^p J_i'$ and $J' \cap \bigcup_{i=2}^p J_i' \neq \emptyset$. We note that $\text{diam } I' \cup J' \geq \text{diam } J_1'$, so $\text{diam } J' \gtrsim \text{diam } J_1'$. In view of Corollary 4.18, we also have $\text{diam } J' \gtrsim \text{diam } J_2'$. Similarly, we can also find consecutive complementary arcs $K_1^-, \dots, K_{p_1}^-$ of F_{n_1} such that $K^- \subset \bigcup_{i=1}^{p_1} K_i^-$, $K^- \cap \bigcup_{i=2}^{p_1} K_i^- \neq \emptyset$, and $\text{diam } K^- \gtrsim \text{diam } K_2^-$ and consecutive complementary arcs $K_1^+, \dots, K_{p_2}^+$ of F_{n_2}

such that $K^+ \subset \bigcup_{i=1}^{p_2} K_i^+$, $K^+ \cap \bigcup_{i=2}^{p_2} K_i^+ \neq \emptyset$. and $\text{diam } K^+ \gtrsim \text{diam } K_2^+$ for some $n_1, n_2 \in \mathbb{N}$ and $p_1, p_2 \leq r+1$. We are exactly in the setting of Lemma 4.19.

We first suppose that the alternative $(\mathbf{H}/\mathbf{P} \rightarrow \mathbf{H}/\mathbf{P})$ holds for the points $a^\pm, b^\pm$. That is, there exists $\mu > 0$ such that if $a^\pm$ is parabolic, then $b^\pm$ is necessarily parabolic with $\mu^{-1}N(a^\pm) = N(b^\pm)$ and if $a^\pm$ is hyperbolic, then $b^\pm$ is hyperbolic with $\lambda(a^\pm)^\mu = \lambda(b^\pm)$. Our goal is to prove that in all of these cases we have

$$\begin{aligned} \text{diam } h(J') &\simeq (\text{diam } J')^\mu \quad \text{and} \\ \text{diam } h(K^\pm) &\simeq (\text{diam } K^\pm)^\mu. \end{aligned}$$

These imply that

$$\begin{aligned} \text{diam } h(I') &\simeq \text{diam } h(K^+) + \text{diam } h(K^-) \simeq (\text{diam } K^+)^\mu + (\text{diam } K^-)^\mu \\ &\simeq (\text{diam } I')^\mu \simeq (\text{diam } J')^\mu \simeq \text{diam } h(J'), \end{aligned}$$

which is the desired conclusion.

Suppose that a^+ and b^+ are hyperbolic. Then by Lemma 4.19 we have

$$\begin{aligned} \text{diam } J' &\simeq \lambda(a^+)^{-n/q} \quad \text{and} \quad \text{diam } h(J') \simeq \lambda(b^+)^{-n/q} \simeq \lambda(a^+)^{-\mu n/q}, \quad \text{and} \\ \text{diam } K^+ &\simeq \lambda(a^+)^{-n_2/q} \quad \text{and} \quad \text{diam } h(K^+) \simeq \lambda(b^+)^{-n_2/q} \simeq \lambda(a^+)^{-\mu n_2/q}. \end{aligned}$$

It follows that

$$\begin{aligned} \text{diam } h(J') &\simeq (\text{diam } J')^\mu \quad \text{and} \\ \text{diam } h(K^+) &\simeq (\text{diam } K^+)^\mu. \end{aligned}$$

If a^- and b^- are hyperbolic, then with the same argument we have

$$\text{diam } h(K^-) \simeq (\text{diam } K^-)^\mu.$$

Next, assume that a^+ and b^+ are parabolic. Consider $k, l \in \mathbb{N} \cup \{0, \infty\}$ as in Lemma 4.19 corresponding to J' and K^+ . Since $J' \cap \bigcup_{i=2}^p J_i' \neq \emptyset$, we have $k = 0$. Similarly, $k_2 = 0$. Moreover, since a is an endpoint of K^+ , we have $l_2 = \infty$. Therefore, if we set $\alpha^+ = 1/N(a^+)$ and $\beta^+ = 1/N(b^+)$, we have

$$\begin{aligned} \text{diam } J' &\simeq n^{-\alpha^+-1} \min\{l+1, n\} \quad \text{and} \quad \text{diam } h(J') \simeq n^{-\beta^+-1} \min\{l+1, n\}, \quad \text{and} \\ \text{diam } K^+ &\simeq n_2^{-\alpha^+} \quad \text{and} \quad \text{diam } h(K^+) \simeq n_2^{-\beta^+}. \end{aligned}$$

We have $\text{diam } J' \simeq \text{diam}(I') \gtrsim \text{diam}(K^+)$. Therefore,

$$(4.13) \quad n_2^{-\alpha^+} \lesssim n^{-\alpha^+-1} \min\{l+1, n\}$$

Note that by the relative position of K^+ and J' , and the definition of l , there exists a complementary arc of F_{n+l} having a as an endpoint and being contained in K^+ . This, combined with (4.7) in Lemma 4.17, gives $(n+l)^{-\alpha^+} \lesssim n_2^{-\alpha^+}$. This inequality and (4.13) imply that $l+1 \gtrsim n$. Therefore,

$$\text{diam } J' \simeq n^{-\alpha^+} \quad \text{and} \quad \text{diam } h(J') \simeq n^{-\beta^+} \simeq n^{-\mu\alpha^+}.$$

It follows that

$$\begin{aligned} \text{diam } h(J') &\simeq (\text{diam } J')^\mu \quad \text{and} \\ \text{diam } h(K^+) &\simeq (\text{diam } K^+)^\mu. \end{aligned}$$

If a^- and b^- are parabolic, with the same argument we have

$$\text{diam } h(K^-) \simeq (\text{diam } K^-)^\mu.$$

Finally, we treat the alternative $(\mathbf{H} \rightarrow \mathbf{P})$. Suppose that a^+ and a^- are hyperbolic (with possibly different multipliers) and b is parabolic. We can apply Lemma 4.19 and obtain

$$\begin{aligned} \text{diam } J' &\simeq \lambda(a^+)^{-n/q} \quad \text{and} \quad \text{diam } h(J') \simeq n^{-\beta-1} \min\{l+1, n\}, \\ \text{diam } K^+ &\simeq \lambda(a^+)^{-n_2/q} \quad \text{and} \quad \text{diam } h(K^+) \simeq n_2^{-\beta}, \quad \text{and} \\ \text{diam } K^- &\simeq \lambda(a^-)^{-n_1/q} \quad \text{and} \quad \text{diam } h(K^-) \simeq n_1^{-\beta}, \end{aligned}$$

where $\beta = 1/N(b)$. Without loss of generality we suppose that $\text{diam } K^+ \leq \text{diam } K^-$, which implies that $\text{diam } K^- \simeq \text{diam } J'$. From these we deduce that $n_2 \gtrsim n_1$ and $n_1 \simeq n$. The first condition implies that

$$\text{diam } h(I') \simeq \text{diam } h(K^-).$$

and the latter condition implies that

$$\text{diam } h(K^-) \simeq n^{-\beta}$$

Altogether, we have

$$\frac{\text{diam } h(J')}{\text{diam } h(I')} \simeq n^{-1} \min\{l+1, n\}.$$

This implies that

$$\max \left\{ \frac{\text{diam } h(J')}{\text{diam } h(I')}, \frac{\text{diam } h(I')}{\text{diam } h(J')} \right\} \lesssim n.$$

Finally, the inequality $\text{diam } J' \gtrsim \text{diam } J = t$ from Lemma 4.20 implies that $n \lesssim \log(1/t)$. $\square$

Proof of Theorem 4.9. Let $I, J \subset \mathbb{S}^1$ be adjacent closed arcs, each of which has length $t \in (0, 1/2)$. Under the condition $(\mathbf{H}/\mathbf{P} \rightarrow \mathbf{H}/\mathbf{P})$, we obtain

$$\text{diam } h(I) \simeq \text{diam } h(J)$$

by Lemma 4.21 and Lemma 4.22. Therefore,

$$\rho_h(t) \simeq 1.$$

This implies that the map $h: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is quasimetric and has a quasiconformal extension in $\mathbb{D}$.

If condition $(\mathbf{H} \rightarrow \mathbf{P})$ is also allowed for some periodic points $a \in F_1$, then by the same distortion lemmas we obtain instead

$$\rho_h(t) \lesssim \log(1/t).$$

By Theorem 2.3 we have that h has a David extension in $\mathbb{D}$. $\square$

5. DAVID WELDING

A homeomorphism $h: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is a *welding homeomorphism* if there exists a Jordan curve J and conformal homeomorphisms H_1 from $\mathbb{D}$ onto the interior of J and H_2 from $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ onto the exterior of J so that $h = \widetilde{H}_2^{-1} \circ \widetilde{H}_1$, where $\widetilde{H}_1$ and $\widetilde{H}_2$ are the homeomorphic extensions of H_1 and H_2 to the closures of $\mathbb{D}$ and $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$, respectively. The Jordan curve J is called a *welding curve* that corresponds to h . We note that in general the curve J is not unique up to Möbius transformations. However, if there is a welding curve J corresponding to the welding homeomorphism h that is conformally removable, then J is unique up to Möbius transformations.

The goal of this section is to prove the existence of a new class of welding homeomorphisms.

Theorem 5.1. *Let $f, g: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be expansive covering maps with the same orientation and $\mathcal{P}(f; \{a_0, \dots, a_r\})$, $\mathcal{P}(g; \{b_0, \dots, b_r\})$ be Markov partitions satisfying conditions (4.1) and (4.2). Assume that each periodic point $a \in \{a_0, \dots, a_r\}$ of f and each periodic point $b \in \{b_0, \dots, b_r\}$ of g is either symmetrically hyperbolic or symmetrically parabolic. Then any conjugating homeomorphism $h: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ between f and g is a welding homeomorphism and the corresponding welding curve is unique up to Möbius transformations.*

Theorem 5.1 will be derived from the proof of the following mateability result for piecewise analytic circle coverings of $\mathbb{S}^1$. This mateability theorem is a direct implication of Theorem 4.9.

Theorem 5.2 (Mating piecewise analytic circle maps). *Let $f, g: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be expansive covering maps with the same orientation and $\mathcal{P}(f; \{a_0, \dots, a_r\})$, $\mathcal{P}(g; \{b_0, \dots, b_r\})$ be Markov partitions satisfying conditions (4.1) and (4.2). Suppose that the map $h: \{a_0, \dots, a_r\} \rightarrow \{b_0, \dots, b_r\}$ defined by $h(a_k) = b_k$, $k \in \{0, \dots, r\}$, conjugates f to g on the set $\{a_0, \dots, a_r\}$ and assume that each periodic point $a \in \{a_0, \dots, a_r\}$ of f and each periodic point $b \in \{b_0, \dots, b_r\}$ of g is either symmetrically hyperbolic or symmetrically parabolic. Then f and g are conformally mateable, so that for each $k \in \{0, \dots, r\}$ the point a_k is mated with the point $b_k = h(a_k)$.*

Recall that by condition (4.1), f and g have conformal extensions in open neighborhoods of the arcs $\overline{(a_k, a_{k+1})}$ and $\overline{(b_k, b_{k+1})}$, $k \in \{0, \dots, r\}$, respectively. The *conformal mateability* of f and g in this theorem means that there exist a Jordan curve J , a partition of J into open arcs J_k , and a map R that is analytic in an open neighborhood W_k of J_k , $k \in \{0, \dots, r\}$, such that R is conjugate to f on $W_k \cap \text{Int}(J)$ and conjugate to g on $W_k \cap \text{Ext}(J)$; here $\text{Int}(J)$ and $\text{Ext}(J)$ denote the interior and exterior open regions of the Jordan curve J , respectively. The map R need not be defined in the entire sphere, although it might extend analytically to open sets that are larger than W_k . However, the proof given below shows that if f and g are restrictions of Blaschke products, then the map R that realizes the mating is analytic everywhere and hence it is rational.

Proof. By Lemma 3.3 h extends to a homeomorphism of $\mathbb{S}^1$, conjugating f to g . Let $P(z)$ be the map $z \mapsto z^d$ or $z \mapsto \bar{z}^d$, depending on whether f and g are orientation-preserving or orientation-reversing, respectively. By Theorem 4.12 there exist orientation-preserving homeomorphisms $h_i: \mathbb{S}^1 \rightarrow \mathbb{S}^1$, $i \in \{1, 2\}$, that conjugate P to f and g , respectively, and have David extensions in $\mathbb{D}$. Note that

h_2 and $h \circ h_1$ conjugate P to g . By the uniqueness part in property (E3), we may precompose h_2 with a rotation so that it agrees with $h \circ h_1$. This will guarantee that the point a_k is mated with $b_k = h(a_k)$ for each $k \in \{0, \dots, r\}$, through the construction below.

We define a Beltrami coefficient μ in the sphere as follows. In $\mathbb{D}$ we let μ be the pullback of the standard complex structure under the David homeomorphism h_1 . In $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ we let μ be the pullback of the standard complex structure under map $\tilde{h}_2 = (1/\bar{z}) \circ h_2 \circ (1/\bar{z})$. By Proposition 2.5 (i) and (ii) (which also hold in the orientation-reversing setting), this map is a David homeomorphism. Therefore, μ is a David coefficient on $\widehat{\mathbb{C}}$.

By the David Integrability Theorem 2.1 there exists a David homeomorphism H of $\widehat{\mathbb{C}}$ with $\mu_H = \mu$. Consider the map

$$R = \begin{cases} H \circ h_1^{-1} \circ f \circ h_1 \circ H^{-1}, & \text{in } H(\mathbb{D}) \\ H \circ \tilde{h}_2 \circ g \circ \tilde{h}_2 \circ H^{-1}, & \text{in } H(\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}). \end{cases}$$

The two definitions agree on $H(\mathbb{S}^1)$. Since f and g are not globally defined in $\mathbb{D}$, we have to further restrict the domain of R to small neighborhoods of the open Jordan arcs $J_k := H((h_1^{-1}(a_k), h_1^{-1}(a_{k+1}))) = H((\tilde{h}_2^{-1}(b_k), \tilde{h}_2^{-1}(b_{k+1})))$, $k \in \{0, \dots, r\}$. We claim that J_k has an open neighborhood W_k in which R is analytic. Moreover, we claim that R is conformally conjugate to f in $W_k \cap H(\mathbb{D})$ and conformally conjugate to g in $W_k \cap H(\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}})$.

By Theorem 2.2, the map $h_1 \circ H^{-1}$ is conformal in $H(\mathbb{D})$ and the map $\tilde{h}_2 \circ H^{-1}$ is conformal in $H(\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}})$. This proves the claims regarding the conformal conjugacy.

By (4.1) f extends conformally to a neighborhood of the arc (a_k, a_{k+1}) and g extends conformally to a neighborhood of the arc (b_k, b_{k+1}) . This implies that R extends to a homeomorphism in a neighborhood W_k of J_k that is conformal in $W_k \setminus H(\mathbb{S}^1)$. Since $\mathbb{S}^1$ is removable for $W^{1,1}$ functions (e.g., it bounds a John domain), we conclude from Theorem 2.7 that $H(\mathbb{S}^1)$ is locally conformally removable. This implies that R is conformal in W_k . $\square$

Proof of Theorem 5.1. From the proof of Theorem 5.2, we see that the map $h_1 \circ H^{-1}$ is conformal in $H(\mathbb{D})$ and the map $\tilde{h}_2 \circ H^{-1}$ is conformal in $H(\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}})$. Therefore,

$$h = \tilde{h}_2 \circ h_1^{-1} = (\tilde{h}_2 \circ H^{-1}) \circ (h_1 \circ H^{-1})^{-1}.$$

This implies that the conjugating homeomorphism h between f and g is a welding homeomorphism. Moreover, the welding curve $H(\mathbb{S}^1)$ of h is conformally removable by Theorem 2.7, since it is the image of the unit circle under a David homeomorphism. $\square$

Example 5.3. For an illustration, consider the Blaschke product

$$B(z) = \frac{2z^3 + 1}{z^3 + 2},$$

which is expansive on $\mathbb{S}^1$ by Example 3.4. Moreover, there exists a Markov partition $\mathcal{P}(B; \{a_0, \dots, a_5\})$ satisfying (4.1) and (4.2) by Example 4.2, where $a_0, \dots, a_5$ are the 3-rd roots of 1 and -1 . The point 1 is a symmetrically parabolic fixed point of B and -1 is a symmetrically hyperbolic fixed point. Finally, we define a bijective map $h: \{a_0, \dots, a_5\} \rightarrow \{a_0, \dots, a_5\}$ that preserves the orientation of $\mathbb{S}^1$ and maps

1 to -1 and -1 to 1. It is easy to see that h conjugates B to itself on the set $\{a_0, \dots, a_5\}$. By Theorem 5.2, we can mate the Blaschke product B with itself so that the point 1 is mated with -1 and the point -1 is mated with the point 1. In this case, since B is analytic in the entire disk, the mating is realized by a rational R ; see the comments after the statement of Theorem 5.2. One can in fact compute this rational map, after doing some normalizations and obtain the formula

$$R(z) = \frac{4z^3 + 8 - 3(1 - \sqrt{3})}{(1 - \sqrt{3})z^3 + 8 + 4\sqrt{3}}.$$

The Julia set of R is shown in Figure 1. It is a Jordan curve with both inward and outward cusps. These cusps arise from the mating of the parabolic point 1 with the hyperbolic point -1 . Note that this is a conformally removable Jordan curve, since it is the image of the unit circle under a David homeomorphism.

6. REFLECTION GROUPS AND SCHWARZ REFLECTION MAPS

In this section, we will collect some preliminaries on Kleinian reflection groups and Schwarz reflection maps associated with quadrature domains.

6.1. Kleinian reflection groups. We denote by $\text{Aut}^\pm(\widehat{\mathbb{C}})$ be the group of all Möbius and anti-Möbius automorphisms of $\widehat{\mathbb{C}}$.

Definition 6.1. A discrete subgroup Γ of $\text{Aut}^\pm(\widehat{\mathbb{C}})$ is called a *Kleinian reflection group* if Γ is generated by reflections in finitely many Euclidean circles.

Remark 6.2. 1) For a Euclidean circle C , consider the upper hemisphere $S \subset \mathbb{H}^3 := \{(x, y, t) \in \mathbb{R}^3 : t > 0\}$ such that $\partial S \cap \partial \mathbb{H}^3 = C$; i.e., C bounds the upper hemisphere S . Note that the anti-Möbius reflection r with respect to C extends naturally to reflection in S , and defines an orientation-reversing isometry of $\mathbb{H}^3$. Hence, a Kleinian reflection Γ group can be thought of as a 3-dimensional hyperbolic reflection group.

2) Since Γ is discrete, by [VS93, Part II, Chapter 5, Proposition 1.4], we can and will always choose its generators to be reflections in Euclidean circles $C_1, \dots, C_{d+1}$ such that for each i , the closure of the bounded component of $\mathbb{C} \setminus C_i$ does not contain any other C_j .

Definition 6.3. Let Γ be a Kleinian reflection group. The *domain of discontinuity* of Γ , denoted $\Omega(\Gamma)$, is the maximal open subset of $\widehat{\mathbb{C}}$ on which Γ acts properly discontinuously (equivalently, the transformations in Γ form a normal family). The *limit set* of Γ , denoted by $\Lambda(\Gamma)$, is defined by $\Lambda(\Gamma) := \widehat{\mathbb{C}} \setminus \Omega(\Gamma)$.

Recall that for a Euclidean circle C , the bounded complementary component of C is denoted by $\text{Int } C$. A circle packing is a connected collection of oriented circles in $\mathbb{C}$ with disjoint interiors (where the interior is determined by the orientation). Up to a Möbius map, we can always assume that no circle of the circle packing contains ∞ in its interior; i.e., the interior of each circle C of the circle packing can be assumed to be the bounded complementary component $\text{Int } C$.

Definition 6.4. A *kissing reflection group* is a group generated by reflections in the circles of a finite circle packing (with at least three circles).

Combinatorially, a circle packing can be described by its *contact graph*, where we associate a vertex to each circle, and connect two vertices by an edge if and only if the two associated circles intersect. By the Circle Packing Theorem, every connected, simple, planar graph is the contact graph of some circle packing. According to [LLM20, Proposition 3.4], the limit set of a kissing reflection group is connected if and only if the contact graph of the underlying circle packing is 2-connected (i.e., the contact graph remains connected if any vertex is deleted).

Let Γ be a kissing reflection group generated by reflections in the circles $C_1, \dots, C_{d+1}$. Set

$$\mathcal{F}_\Gamma := \widehat{\mathbb{C}} \setminus \left(\bigcup_{i=1}^{d+1} \text{Int } C_i \bigcup_{j \neq k} (C_j \cap C_k) \right).$$

Proposition 6.5. *Let Γ be a kissing reflection group. Then $\mathcal{F}_\Gamma$ is a fundamental domain for the action of Γ on $\Omega(\Gamma)$.*

Proof. Let $\mathcal{P}_\Gamma$ be the convex hyperbolic polyhedron (in $\mathbb{H}^3$) whose relative boundary in $\mathbb{H}^3$ is the union of the hyperplanes S_i bounded by the circles C_i (see Remark 6.2). Then, by [VS93, Part II, Chapter 5, Theorem 1.2], $\mathcal{P}_\Gamma$ is a fundamental domain for the action of Γ on $\mathbb{H}^3$. It now follows that $\mathcal{F}_\Gamma = \overline{\mathcal{P}_\Gamma} \cap \Omega(\Gamma)$ (where the closure is taken in $\Omega(\Gamma) \cup \mathbb{H}^3$) is a fundamental domain for the action of Γ on $\Omega(\Gamma)$ [Mar16, §3.5]. $\square$

To a kissing reflection group Γ , we can associate a piecewise anti-Möbius reflection map ρ_Γ that will play an important role in the paper.

Definition 6.6. Let Γ be a kissing reflection group generated by reflections $(r_i)_{i=1}^{d+1}$ in circles $(C_i)_{i=1}^{d+1}$. We define the associated *Nielsen map* ρ_Γ by:

$$\begin{aligned} \rho_\Gamma : \bigcup_{i=1}^{d+1} \overline{\text{Int } C_i} &\rightarrow \widehat{\mathbb{C}} \\ z &\mapsto r_i(z) \text{ if } z \in \overline{\text{Int } C_i}. \end{aligned}$$

Definition 6.7. Let Γ be a kissing reflection group generated by reflections in the circles of a finite circle packing $C_1, \dots, C_{d+1}$. We say that Γ is a *necklace group* if

- (1) each circle C_i is tangent to C_{i+1} (with $i+1$ taken mod $(d+1)$), and
- (2) the boundary of the unbounded component of $\mathbb{C} \setminus \bigcup_i C_i$ intersects each C_i .

Remark 6.8. Necklace groups can be characterized as Kleinian reflection groups generated by reflections in the circles of a finite circle packing whose contact graph is 2-connected and outerplanar; i.e., the contact graph remains connected if any vertex is deleted, and has a face containing all the vertices on its boundary. Necklace groups can be compared with complex polynomials with connected Julia set. Indeed, the connected Julia set of a polynomial is the boundary of a simply connected, completely invariant Fatou component, namely, the basin of attraction of infinity. Similarly, as we will shortly see, the limit set of a necklace group is the boundary of a simply connected, invariant component of its domain of discontinuity (compare [LLM20, Theorem 1.2]).

We conclude this subsection with the definition of the regular ideal polygon reflection group, which will play a central role in the rest of the paper.

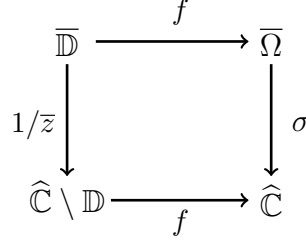


FIGURE 3. The rational map f semiconjugates the reflection map $1/\bar{z}$ of $\mathbb{D}$ to the Schwarz reflection map σ of Ω .

Definition 6.9. Consider the Euclidean circles $\mathbf{C}_1, \dots, \mathbf{C}_{d+1}$ where $\mathbf{C}_j$ intersects $\mathbb{S}^1$ at right angles at the roots of unity $\exp(\frac{2\pi i \cdot (j-1)}{d+1})$, $\exp(\frac{2\pi i \cdot j}{d+1})$. By [VS93, Part II, Chapter 5, Theorem 1.2], the group generated by reflections in these circles is discrete. Therefore, it is also a necklace group, and we denote it by Γ_{d+1} . Moreover, denoting the reflection map in the circle $\mathbf{C}_j$ by ρ_j , we have the following presentation

$$\Gamma_{d+1} = \langle \rho_1, \dots, \rho_{d+1} : \rho_1^2 = \dots = \rho_{d+1}^2 = 1 \rangle.$$

To ease notations, we will denote the Nielsen map of Γ_{d+1} by ρ_d . Note that ρ_d restricts to a degree d orientation-reversing covering of $\mathbb{S}^1$.

6.2. Quadrature domains and Schwarz reflections.

Definition 6.10 (Schwarz Function). Let $\Omega \subsetneq \widehat{\mathbb{C}}$ be a domain such that $\infty \notin \partial\Omega$ and $\text{int } \overline{\Omega} = \Omega$. A *Schwarz function* of Ω is a meromorphic extension of $\bar{z}|_{\partial\Omega}$ to all of Ω . More precisely, a continuous function $S : \overline{\Omega} \rightarrow \widehat{\mathbb{C}}$ of Ω is called a Schwarz function of Ω if it satisfies the following two properties:

- (1) S is meromorphic on Ω ,
- (2) $S(z) = \bar{z}$ on $\partial\Omega$.

It is easy to see from the definition that a Schwarz function of a domain (if it exists) is unique.

Definition 6.11 (Quadrature Domains). A domain $\Omega \subsetneq \widehat{\mathbb{C}}$ with $\infty \notin \partial\Omega$ and $\text{int } \overline{\Omega} = \Omega$ is called a *quadrature domain* if Ω admits a Schwarz function.

Therefore, for a quadrature domain Ω , the map $\sigma := \bar{S} : \overline{\Omega} \rightarrow \widehat{\mathbb{C}}$ is an anti-meromorphic extension of the Schwarz reflection map with respect to $\partial\Omega$ (the reflection map fixes $\partial\Omega$ pointwise). We will call σ the *Schwarz reflection map* of Ω .

Simply connected quadrature domains are of particular interest, and these admit a simple characterization (see [AS76, Theorem 1]).

Proposition 6.12 (Simply Connected Quadrature Domains). *A simply connected domain $\Omega \subsetneq \widehat{\mathbb{C}}$ with $\infty \notin \partial\Omega$ and $\text{int } \overline{\Omega} = \Omega$ is a quadrature domain if and only if the Riemann uniformization $f : \mathbb{D} \rightarrow \Omega$ extends to a rational map on $\widehat{\mathbb{C}}$.*

In this case, the Schwarz reflection map σ of Ω is given by $f \circ (1/\bar{z}) \circ (f|_{\mathbb{D}})^{-1}$. Moreover, if the degree of the rational map f is d , then $\sigma : \sigma^{-1}(\Omega) \rightarrow \Omega$ is a (branched) covering of degree $(d-1)$, and $\sigma : \sigma^{-1}(\text{int } \Omega^c) \rightarrow \text{int } \Omega^c$ is a (branched) covering of degree d .

Remark 6.13. If Ω is a simply connected quadrature domain with associated Schwarz reflection map σ , and M is a Möbius transformation, then $M(\Omega)$ is also a quadrature domain with Schwarz reflection map $M \circ \sigma \circ M^{-1}$.

7. A GENERAL DAVID SURGERY

This section is devoted to the proof of a technical lemma, that will serve as a motor for much of the rest of the paper. The result is sufficiently general to allow us to construct various kinds of conformal dynamical systems by replacing the attracting dynamics of an anti-rational map on suitable invariant Fatou components by Nielsen maps of ideal polygon groups.

We denote the Julia set of a(n) (anti-)rational map R by $\mathcal{J}(R)$. Recall that a rational map is called *subhyperbolic* if every critical orbit is either finite or converges to an attracting periodic orbit.

Lemma 7.1. *Let R be a subhyperbolic anti-rational map with connected Julia set, and for $i \in \{1, \dots, k\}$, let U_i be an invariant Fatou component of R such that $R|_{\partial U_i}$ has degree d_i . Then, there is a global David surgery that replaces the dynamics of R on each U_i by the dynamics of $\rho_{d_i}|_{\mathbb{D}}$, transferred to U_i via a Riemann map.*

More precisely, we will show that there exists a global David homeomorphism Ψ , and an anti-analytic map F , defined on a subset of $\widehat{\mathbb{C}}$, such that $F|_{\Psi(U_i)}$ is conformally conjugate to $\rho_{d_i}|_{\mathbb{D}}$, and F is conformally conjugate to R outside the grand orbit of $\bigcup_{i=1}^k \Psi(U_i)$.

Proof. For $i \in \{1, 2, \dots, k\}$, let $\phi_i : \mathbb{D} \rightarrow U_i$ be a Riemann map. Note that since R is subhyperbolic, each U_i is the immediate basin of attraction of an attracting fixed point. Hence, the Riemann map ϕ_i conjugates a degree d_i anti-Blaschke product B_i , with an attracting fixed point in $\mathbb{D}$, to R . In particular, B_i is expanding (with respect to some conformal metric) on its Julia set $\mathbb{S}^1$. Since connected, subhyperbolic Julia sets are locally connected [Mil06, Theorem 19.7], we have a continuous extension $\phi_i : \mathbb{S}^1 \rightarrow \partial U_i$ semiconjugating B_i to R .

By Theorem 4.13, there is a topological conjugacy H_i between $B_i|_{\mathbb{S}^1}$ and $\rho_{d_i}|_{\mathbb{S}^1}$ that continuously extends to $H_i : \mathbb{D} \rightarrow \mathbb{D}$ as a David homeomorphism.

We now define a new map $\tilde{R}$ by modifying the map R as follows:

$$\tilde{R} := \begin{cases} \phi_i \circ (H_i^{-1} \circ \rho_{d_i} \circ H_i) \circ \phi_i^{-1}, & \text{in } U_i \setminus \text{int } T_i, \quad i = 1, 2, \dots, k, \\ R, & \text{in } \widehat{\mathbb{C}} \setminus \bigcup_{i=1}^k U_i, \end{cases}$$

where $T_i = \phi_i(H_i^{-1}(T^0(\Gamma_{d_i+1})))$, and $T^0(\Gamma_{d_i+1})$ is a regular ideal (d_i+1) -gon in $\mathbb{D}$. The fact that $H_i^{-1} \circ \rho_{d_i} \circ H_i \equiv B_i$ on $\mathbb{S}^1$ combined with the semiconjugacy relation $\phi_i \circ B_i \equiv R \circ \phi_i$ on $\mathbb{S}^1$ imply that the piecewise definitions of the map $\tilde{R}$ match continuously. Thus, $\tilde{R}$ is a continuous orientation-reversing map of $\widehat{\mathbb{C}} \setminus \bigcup_{i=1}^k \text{int } T_i$ onto $\widehat{\mathbb{C}}$. If μ_i is the pullback to U_i , $i \in \{1, 2, \dots, k\}$, of the standard complex structure on $\mathbb{D}$ by the map $H_i \circ \phi_i^{-1}$, then we have $(\tilde{R}|_{U_i \setminus T_i})^*(\mu_i) = \mu_i$.

We now use the dynamics of R to spread the complex structure out to all the iterated preimage components of U_i , $i \in \{1, 2, \dots, k\}$, under all the iterates of R . Everywhere else we use the standard complex structure, i.e., the zero Beltrami coefficient. This way we obtain an $\tilde{R}$ -invariant measurable complex structure μ on $\widehat{\mathbb{C}}$.

We will now argue that μ is a David coefficient on $\widehat{\mathbb{C}}$; i.e., it satisfies condition (2.1). We say that an iterated preimage component U' of U_i ($i \in \{1, \dots, k\}$) has *rank* $r = r(U')$ if r is the smallest non-negative integer j such that $R^{\circ j}(U') = U_i$. By subhyperbolicity of R , the closures of at most finitely many iterated preimage components of U_i may intersect the critical orbits of R . Let m be the maximum of the ranks of the iterated preimage components of U_i ($i \in \{1, \dots, k\}$) whose closures intersect the critical orbits of R . We denote the iterated preimage components of U_i ($i \in \{1, \dots, k\}$) of rank at most $(m+1)$ by $V_1, \dots, V_l$.

Since R is subhyperbolic and $\mathcal{J}(R)$ is connected, each U_i is a John domain [CG93, §7, Theorem 3.1]. By part (iv) of Proposition 2.5, the map $H_i \circ \phi_i^{-1} : U_i \rightarrow \mathbb{D}$ is a David homeomorphism, and hence, μ is a David coefficient on U_i , for $i \in \{1, \dots, k\}$. Moreover, since μ is defined on a preimage component U' of U_i by pulling back $\mu|_{U_i}$ by $R^{\circ r(U')}$, it follows that

$$\mu|_{U'} = \left(H_i \circ \phi_i^{-1} \circ R^{\circ r(U')} \right)^* (\mu_0|_{\mathbb{D}}),$$

where μ_0 is the trivial Beltrami coefficient. By part (iii) of Proposition 2.5, μ satisfies condition (2.1) on U_i . It now follows that μ satisfies condition (2.1) on $\bigcup_{j=1}^l V_j$ with some constants $M, \alpha, \varepsilon_0 > 0$.

It remains to check the David condition on the union of the rest of the preimage components of U_i . If U' is an iterated preimage component of U_i ($i = 1, 2, \dots, k$) of rank larger than $(m+1)$, then it lands on a rank $(m+1)$ component in time $n(U') = (r(U') - m - 1)$ univalently. In other words, $n(U')$ is the *first landing time* of U' in $\bigcup_{j=1}^l V_j$. By our choice of m , there is a neighborhood of the closures of the rank $(m+1)$ components that is disjoint from the postcritical set of R . Hence, by the Koebe distortion theorem, $R^{\circ n(U')} \circ \lambda_{U'}$ is an L -bi-Lipschitz map between $\frac{1}{\text{diam } U'} U'$ and V_j , for some absolute constant $L \geq 1$ and a rank $(m+1)$ component V_j , where $\lambda_{U'}(z) = \text{diam } U' \cdot z$ is a scaling map. This implies that, given any $0 < \varepsilon \leq \varepsilon_0$,

$$\sigma(\{z \in U' : |\mu(z)| \geq 1 - \varepsilon\}) \leq L^2 (\text{diam } U')^2 \sigma(\{z \in V_j : |\mu(z)| \geq 1 - \varepsilon\}).$$

Moreover, since all the Fatou components U' are uniform John domains [Mih11, Proposition 10], it follows from [Nta18, p. 444] that there exists a constant $C > 0$ such that

$$(\text{diam } U')^2 \leq C \sigma(U'),$$

for all preimage components U' of V_j .

Therefore,

$$\begin{aligned} & \sigma(\{z \in \widehat{\mathbb{C}} : |\mu(z)| \geq 1 - \varepsilon\}) \\ &= \sum_{r(U') > m+1} \sigma(\{z \in U' : |\mu(z)| \geq 1 - \varepsilon\}) + \sum_{j=1}^l \sigma(\{z \in V_j : |\mu(z)| \geq 1 - \varepsilon\}) \\ &\leq \left(L^2 C \left(\sum_{U'} \sigma(U') \right) + 1 \right) \cdot \left(\sum_{j=1}^l \sigma(\{z \in V_j : |\mu(z)| \geq 1 - \varepsilon\}) \right) \\ &\leq \left(L^2 C l \sigma(\widehat{\mathbb{C}}) + l \right) M e^{-\alpha/\varepsilon}, \quad \varepsilon \leq \varepsilon_0. \end{aligned}$$

Theorem 2.1 then gives us an orientation preserving homeomorphism Ψ of $\widehat{\mathbb{C}}$ such that the pullback of the standard complex structure under Ψ is equal to μ .

We now proceed to show that the map $F := \Psi \circ \tilde{R} \circ \Psi^{-1}$ is anti-analytic. This is the desired map that replaces the dynamics of R on each U_i , $i = 1, 2, \dots, k$, with the dynamics of the Nielsen map ρ_{d_i} .

We will show that if V is an open set in $\widehat{\mathbb{C}} \setminus \bigcup_{i=1}^k \text{int } T_i$ on which $\tilde{R}$ restricts as a homeomorphism, then F is anti-analytic on $\Psi(V)$. This will imply that F is anti-analytic away from its critical points. Hence, by the Riemann removability theorem and the continuity of F , we can conclude that F is anti-analytic on the interior of its domain of definition.

To this end, let V be an open set in $\widehat{\mathbb{C}} \setminus \bigcup_{i=1}^k \text{int } T_i$ such that $\tilde{R}|_V$ is a homeomorphism. Since $F \circ \Psi = \Psi \circ \tilde{R}$, it is now enough to show that $\Psi \circ \tilde{R}$ is an orientation-reversing David homeomorphism on V . Indeed, if both Ψ and $\Psi \circ \tilde{R}$ are David homeomorphisms on V with opposite orientation, since both of them integrate μ , we can apply Theorem 2.2 to obtain that F is anti-analytic on $\Psi(V)$.

Since each U_i is a John domain, Theorem 2.10 tells us that $\bigcup_{i=1}^k \partial U_i$ is $W^{1,1}$ -removable. Therefore, by Lemma 2.8, it now suffices to show that $\Psi \circ \tilde{R}$ is an orientation-reversing David homeomorphism on $V \setminus \bigcup_{i=1}^k \partial U_i$.

Note that by construction, $\tilde{R}$ is anti-conformal on $V \setminus \bigcup_{i=1}^k \overline{U_i}$. Hence, by part (ii) of Proposition 2.5, $\Psi \circ \tilde{R}$ is an orientation-reversing David homeomorphism on $V \setminus \bigcup_{i=1}^k \overline{U_i}$.

Now let us look at $V \cap \left(\bigcup_{i=1}^k U_i \right)$. Let us set $V_i := V \cap U_i$, for $i \in \{1, \dots, k\}$. We can factorize $\Psi \circ \tilde{R}$ on V_i as follows:

$$(7.1) \quad \Psi \circ \tilde{R} = (\Psi \circ \phi_i \circ H_i^{-1}) \circ (\rho_{d_i} \circ H_i \circ \phi_i^{-1}).$$

Since ϕ_i^{-1} and ρ_{d_i} are conformal and anti-conformal (respectively), the composition $\rho_{d_i} \circ H_i \circ \phi_i^{-1}$ is an orientation-reversing David homeomorphism by parts (i) and (ii) of Proposition 2.5. By the same proposition, the map $H_i \circ \phi_i^{-1}$ is also a David homeomorphism. Moreover, both the maps Ψ and $H_i \circ \phi_i^{-1}$ integrate μ . Therefore, by Theorem 2.2, we conclude that $\Psi \circ \phi_i \circ H_i^{-1}$ is anti-conformal. By part (i) of Proposition 2.5 and Relation (7.1), we now have that $\Psi \circ \tilde{R}$ is an orientation-reversing David homeomorphism on each V_i , and hence on $V \cap \left(\bigcup_{i=1}^k U_i \right)$. This completes the proof of the fact that $\Psi \circ \tilde{R}$ is an orientation-reversing David homeomorphism on $V \setminus \bigcup_{i=1}^k \partial U_i$. $\square$

8. DAVID SURGERY FROM ANTI-RATIONAL MAPS TO KISSING REFLECTION GROUPS

An anti-rational map is called *critically fixed* if all of its critical points are fixed. Each fixed critical point c of a critically fixed anti-rational map R lies in an invariant Fatou component U_c . If the local degree of R at c is k , then $R|_{U_c}$ is conformally conjugate to $\bar{z}^k|_{\mathbb{D}}$. In particular, there are $k+1$ fixed internal rays in U_c . The *Tischler graph* of R is defined as the union of all fixed internal rays in the invariant Fatou components. Clearly, this graph contains the postcritical set of R . It can be seen as a natural generalization of Hubbard trees of postcritically finite polynomials

to the context of critically fixed maps. According to [LLM20, Lemma 4.9], the planar dual of the Tischler graph of a critically fixed anti-rational map is simple and 2-connected.

The goal of this section is to show that there is a global David surgery that turns a degree d critically fixed anti-rational map R into a kissing reflection group Γ of rank $d + 1$.

Theorem 8.1. *Suppose that the Tischler graph of a critically fixed anti-rational map R is dual to the contact graph of the circle packing defining a kissing reflection group Γ . Then there exists a David homeomorphism of $\widehat{\mathbb{C}}$ that conjugates $R_{\mathcal{J}(R)}$ to $\rho_\Gamma|_{\Lambda(\Gamma)}$.*

Proof. Let Γ be generated by reflections in the circles $C_1, \dots, C_{d+1}$. We denote the connected components of the fundamental domain $\mathcal{F}_\Gamma$ by $\mathcal{P}_1, \dots, \mathcal{P}_k$. Note that each $\mathcal{P}_i$ is a topological $(d_i + 1)$ -gon with vertices removed, for some $d_i \geq 2$, $i \in \{1, \dots, k\}$. Thus, the faces of the contact graph of the underlying circle packing are topological $(d_i + 1)$ -gons, $i \in \{1, \dots, k\}$. Since the Tischler graph of R is dual to this contact graph, it follows that R has k invariant Fatou components $U_1, \dots, U_k$ such that $R|_{\partial U_i}$ has degree d_i .

Applying Lemma 7.1 to the Fatou components $U_1, \dots, U_k$, we obtain a global David homeomorphism Ψ and an anti-meromorphic map σ on a subset of $\widehat{\mathbb{C}}$ such that σ is conformally conjugate to $\rho_{d_i}|_{\mathbb{D}}$ on $\Psi(U_i)$, $i \in \{1, \dots, k\}$, and topologically conjugate to $R|_{\mathcal{J}(R)}$ on $\Psi(\mathcal{J}(R))$ (the conjugacy Ψ maps the dynamical plane of R to that of σ). It follows from the proof of Lemma 7.1 that the domain of definition of σ is $\Psi\left(\widehat{\mathbb{C}} \setminus \bigcup_{i=1}^k \text{int } T_i\right)$, where each $T_i \subset U_i$ is a vertices-removed topological $(d_i + 1)$ -gon with vertices at the landing points of the $(d_i + 1)$ fixed internal rays of R in U_i . It follows from the construction that the domain of definition of σ is naturally homeomorphic to $\bigcup_{i=1}^{d+1} \overline{\text{Int } C_i}$. We denote these $(d + 1)$ Jordan domains by $\Omega_1, \dots, \Omega_{d+1}$ such that Ω_j corresponds to $\text{Int } C_j$ under the canonical homeomorphism between the domain of definition of σ and $\bigcup_{i=1}^{d+1} \overline{\text{Int } C_i}$. In particular, the domain of definition of σ is connected, and its interior is the union of $(d + 1)$ disjoint Jordan domains. Moreover, σ fixes the boundary of each of these domains pointwise. Thus, by definition, each of these Jordan domains is a quadrature domain, and σ restricts to each such domain as its Schwarz reflection map.

By [LLM20, Corollary 4.7], the map R carries each face of its Tischler graph onto the closure of its complement as an orientation reversing homeomorphism. Since Ψ conjugates $R|_{\mathcal{J}(R)}$ to $\sigma|_{\Psi(\mathcal{J}(R))}$, it follows that σ carries $\Psi(\mathcal{J}(R)) \cap \Omega_j$ univalently to $\Psi(\mathcal{J}(R)) \setminus \Omega_j$.

By Proposition 6.12, there exist rational maps ϕ_j , $j \in \{1, \dots, d + 1\}$, such that $\phi_j : \mathbb{D} \rightarrow \Omega_j$ is a conformal isomorphism. Furthermore, by the same proposition, the map $\sigma : \sigma^{-1}(\Omega_j) \rightarrow \Omega_j$ is a (branched) cover of degree $(\deg \phi_j - 1)$. Since $\Psi(\mathcal{J}(R))$ is completely invariant under σ , we conclude by the previous paragraph that $\deg \phi_j = 1$, $j \in \{1, \dots, d + 1\}$. Thus, each ϕ_j is a Möbius map. It follows that each Ω_j is a round disk, and $\sigma|_{\overline{\Omega_j}}$ is simply the reflection map in the boundary circle $C'_j := \partial \Omega_j$. Our labeling of the domains Ω_j also guarantees that the contact

graph of the circle packing $\{C'_1, \dots, C'_{d+1}\}$ is planar isomorphic to that of the circle packing $\{C_1, \dots, C_{d+1}\}$.

Let Γ' be the group generated by reflections in the circles C'_i . The above discussion implies that $\sigma \equiv \rho_{\Gamma'}$, and the David homeomorphism Ψ conjugates $R_{\mathcal{J}(R)}$ to $\rho_{\Gamma'}|_{\Lambda(\Gamma')}$.

We will now show that there exists a quasiconformal homeomorphism Φ of the sphere that conjugates Γ to Γ' by conjugating the reflection in C_j to the reflection in C'_j , $j \in \{1, \dots, d+1\}$. In particular, such a Φ would conjugate $\rho_{\Gamma}|_{\Lambda(\Gamma)}$ to $\rho_{\Gamma'}|_{\Lambda(\Gamma')}$. In light of Proposition 2.5 (part i), this would imply that $\Phi^{-1} \circ \Psi$ is the desired global David homeomorphism conjugating $R_{\mathcal{J}(R)}$ to $\rho_{\Gamma}|_{\Lambda(\Gamma)}$.

The existence of such a quasiconformal map Φ follows from general rigidity results for geometrically finite Kleinian groups [Tuk85, Theorem 4.2]; however, in our situation, we will give a more direct argument.

Lemma 8.2. *There exists a quasiconformal homeomorphism Φ of the sphere that conjugates $\rho_{\Gamma}|_{\Lambda(\Gamma)}$ to $\rho_{\Gamma'}|_{\Lambda(\Gamma')}$.*

Proof. First, we claim that there exists a K -quasiconformal homeomorphism φ of the sphere that takes the fundamental domain of Γ onto that of Γ' . Moreover, the map φ carries cusps to cusps preserving their labels. Note that cusps on the boundaries of the fundamental domains are naturally labeled as they are the points of intersections of the underlying labeled circle packings.

Indeed, we start by defining φ on the boundary of each connected component $\mathcal{P}$ of the fundamental domain $\mathcal{F}_{\Gamma}$ of Γ so that φ is a bi-Lipschitz map onto the boundary of the corresponding component $\mathcal{P}'$ of $\mathcal{F}_{\Gamma'}$. Such $\mathcal{P}$ and $\mathcal{P}'$ are topological polygons, say n -gons, with each side being an arc of a geometric circle. Moreover, the vertices of $\mathcal{P}$ and $\mathcal{P}'$ are cusps, i.e., they are common to two tangent circles.

Let v be a vertex of $\mathcal{P}$ and v' be the corresponding vertex of $\mathcal{P}'$. Let α_-, α_+ be the two sides of $\mathcal{P}$ that intersect at v , and α'_-, α'_+ be the corresponding two sides of $\mathcal{P}'$. Let β be the arc of a circle that is orthogonal to α_- and α_+ , and is contained in a small neighborhood of v intersected with $\mathcal{P}$. Let β' be an arc defined similarly for v' . If we define such arcs β and β' for each vertex v of $\mathcal{P}$ and v' of $\mathcal{P}'$, respectively, the components $\mathcal{P}$ and $\mathcal{P}'$ decompose into disjoint topological triangles T_v and $T'_{v'}$, where v and v' run over all vertices of $\mathcal{P}$ and $\mathcal{P}'$, respectively, as well as $2n$ -gons, which we denote by D and D' , respectively (see Figure 4). The triangles T_v and $T'_{v'}$ have each one cusp at v , respectively v' , and the other two angles being right angles. The polygons D and D' are right angled $2n$ -gons, in particular they are quasicircles.

For each v , we define $\varphi: T_v \rightarrow T'_{v'}$ to be the unique conformal map that takes vertices of T_v to the vertices of $T'_{v'}$ so that v goes to v' . Note that since T_v and $T'_{v'}$ have right angles at the vertices of β and β' , respectively, the map $\varphi: \beta \rightarrow \beta'$ is bi-Lipschitz. Also, by a simple direct computation or a special case of [War42], we conclude that φ is also bi-Lipschitz on each of the other two sides of T_v mapping it to the corresponding two sides of $T'_{v'}$.

For each side δ of the $2n$ -gon D , other than the n sides β where the map φ was already defined, we define φ to be the scaling map that takes δ to the corresponding side δ' of D' . Such maps $\varphi: \delta \rightarrow \delta'$ are bi-Lipschitz. We now have φ defined on the whole boundary of D and this map takes it to the boundary of D' . Since the polygons D and D' are right angled, and so are quasicircles, the map φ has a quasiconformal extension to a map between D and D' . Thus, we constructed a

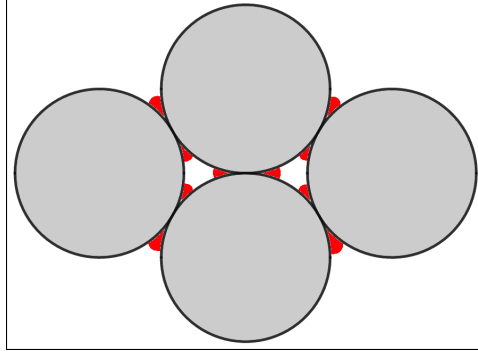


FIGURE 4. The decomposition of each component of the fundamental domain into topological triangles (red) and topological $2n$ -gons (white) is shown.

quasiconformal map for each component $\mathcal{P}$ of the fundamental domain $\mathcal{F}_\Gamma$, taking it to the corresponding component $\mathcal{P}'$ of the fundamental domain $\mathcal{F}_{\Gamma'}$.

To obtain a global quasiconformal map φ , we observe that it is already defined on each circle C_j , $j \in \{1, 2, \dots, d+1\}$, and φ takes it onto the corresponding circle C'_j . Moreover, from the definition of φ , we conclude that $\varphi: C_j \rightarrow C'_j$ is bi-Lipschitz for each $j \in \{1, 2, \dots, d+1\}$. Thus, φ has a quasiconformal extension into each disk D_j bounded by C_j , $j \in \{1, 2, \dots, d+1\}$, mapping it onto the disk D'_j bounded by C'_j . Since circles are removable for quasiconformal maps, the claim follows.

Now, we use the reflection groups Γ and Γ' to redefine the map φ so that for the new map Φ , we have

$$\Phi(\Lambda(\Gamma)) = \Lambda(\Gamma'), \text{ and } \Phi \circ \rho_\Gamma \circ \Phi^{-1} \equiv \rho_{\Gamma'} \text{ on } \Lambda(\Gamma').$$

We may and will assume that the circles $C_1, C_2, \dots, C_{d+1}$ are ordered so that the radii are non-increasing, and let γ_j be the reflection in C_j . Also, let γ'_j be the reflection in the corresponding circle C'_j . In D_1 , we redefine the map φ so that it is equal to

$$\gamma'_1 \circ \varphi \circ \gamma_1.$$

The new map, which we denote by φ_1 , is still K -quasiconformal and it agrees with φ outside of D_1 .

We next redefine the map φ_1 in the largest disk in the complement of the closure of the doubled fundamental domain $\mathcal{F}_\Gamma \cup \gamma_1(\mathcal{F}(\Gamma))$. We then argue inductively, and, provided a map φ_i , $i \in \mathbb{N}$, was already defined, we define φ_{i+1} by changing φ_i on the largest complementary disk of the closure of the union of the successive images of the fundamental domain similar to obtaining φ_1 from φ above. Note that on any copy of the fundamental domain $\mathcal{F}_\Gamma$ under a group element $\gamma \in \Gamma$, the map φ is only redefined finitely many times in this construction.

In this way we obtain a sequence of normalized K -quasiconformal maps $(\varphi_i)_{i \geq 1}$. By the standard compactness argument for uniformly quasiconformal maps, a subsequence converges to a K -quasiconformal map Φ . The map Φ agrees with the original map φ on the fundamental domain of Γ since each φ_i , $i \in \{1, 2, \dots\}$, does. By the construction, the map Φ takes the limit set $\Lambda(\Gamma)$ of the group Γ onto the limit set $\Lambda(\Gamma')$ of Γ' since the limit sets $\Lambda(\Gamma)$ and $\Lambda(\Gamma')$ are the complements of the

unions of reflected copies of the fundamental domains under the respective groups. Moreover, Φ conjugates the action of γ_j on the domain of discontinuity of Γ to the action of γ'_j on the domain of discontinuity of Γ' . By continuity, $\Phi \circ \gamma_j \circ \Phi^{-1} \equiv \gamma'_j$ on $\widehat{\mathbb{C}}$, for $j \in \{1, \dots, d+1\}$. Hence, Φ conjugates $\rho_\Gamma|_{\Lambda(\Gamma)}$ to $\rho_{\Gamma'}|_{\Lambda(\Gamma')}$. $\square$

As mentioned before the proof of the lemma, the map $\Phi^{-1} \circ \Psi$ is the desired global David homeomorphism conjugating $R_{\mathcal{J}(R)}$ to $\rho_\Gamma|_{\Lambda(\Gamma)}$. $\square$

Note that the Tischler graph of a critically fixed anti-polynomial has a vertex at ∞ , and this vertex lies on the boundary of each face of the Tischler graph. Hence, the planar dual of the Tischler graph of a critically fixed anti-polynomial has a face containing all the vertices on its boundary; i.e., the planar dual is outerplanar. By [LLM20, Theorem 1.2], the contact graph of the circle packing defining a kissing reflection group Γ is 2-connected and outerplanar if and only if Γ is a necklace group. Thus, if the Tischler graph of a critically fixed anti-polynomial P is dual to the contact graph of the circle packing defining a kissing reflection group Γ , then Γ is a necklace group. With these observations on mind, we now record a useful corollary of Theorem 8.1.

Corollary 8.3. *Suppose that the Tischler graph of a critically fixed anti-polynomial P is dual to the contact graph of the circle packing defining a necklace reflection group Γ . Then there exists a David homeomorphism of $\widehat{\mathbb{C}}$ that conjugates $R_{\mathcal{J}(R)}$ to $\rho_\Gamma|_{\Lambda(\Gamma)}$.*

8.1. Connections with known results. Let us now mention connections of Theorem 8.1 with some recent works on critically fixed anti-rational maps and kissing reflection groups.

In [LLMM19, §10], the existence of the David homeomorphism of Theorem 8.1 was proved in the special case where Γ is the classical Apollonian gasket reflection group and R is a cubic critically fixed anti-rational map (with four distinct critical points). (More generally, it was done for a class of Apollonian-like gaskets.)

Critically fixed anti-rational maps were classified in terms of their Tischler graphs in [Gey20] and [LLM20]. Furthermore, in [LLM20][Theorem 1.1], a dynamically natural bijection between the following classes of objects were established:

- {Critically fixed anti-rational maps of degree d up to Möbius conjugacy},
- {Kissing reflection groups of rank $d+1$ with connected limit set up to QC conjugacy}.

The bijection is such that the contact graph of the circle packing defining a kissing reflection group Γ is dual to the Tischler graph of the corresponding anti-rational map R . Moreover, if R and Γ correspond to each other under this bijection, then the dynamics of R on the Julia set $\mathcal{J}(R)$ is topologically conjugate to the action of the Nielsen map ρ_Γ on the limit set $\Lambda(\Gamma)$. Particular occasions of this bijection had been earlier exhibited in the aforementioned case of Apollonian-like gaskets (corresponding to triangulations) [LLMM19] and in the case of the correspondence between critically fixed anti-polynomials and necklace groups [LLM20, Theorem B] (also compare [LLM20, Theorem 1.2]).

Note that all periodic points of a hyperbolic anti-rational map on its Julia set are repelling, while the Nielsen map of a kissing reflection group has parabolic fixed

points on the limit set. Hence, the aforementioned conjugacies cannot be quasymmetric; i.e., they cannot admit quasiconformal extensions to the Riemann sphere. Theorem 8.1 shows that these conjugacies extend as David homeomorphisms of the sphere.

Combining Theorem 8.1 with the bijection results mentioned above, we have the following result.

Corollary 8.4. *Let Γ be a kissing reflection group of rank $d + 1$. Then, there exists a degree d critically fixed anti-rational map R such that $R_{\mathcal{J}(R)}$ is conjugate to $\rho_\Gamma|_{\Lambda(\Gamma)}$ by a David homeomorphism of $\widehat{\mathbb{C}}$.*

Moreover, if Γ is a necklace group, then R can be chosen as a critically fixed anti-polynomial.

9. CONFORMAL REMOVABILITY OF JULIA AND LIMIT SETS

9.1. Conformal removability of limit sets of necklace reflection groups.

An application of Theorem 2.11 and Theorem 8.1 is the following conformal removability result for limit sets of necklace reflection groups (see Figures 2 and 5 for examples of such limit sets).

Theorem 9.1. *Let Γ be a necklace group. Then, the limit set $\Lambda(\Gamma)$ is conformally removable.*

Proof. By Corollary 8.4, there exist a degree d critically fixed anti-polynomial P and a David homeomorphism Ψ of $\widehat{\mathbb{C}}$ such that $\Psi(\mathcal{J}(P)) = \Lambda(\Gamma)$. Since the basin of infinity of a hyperbolic polynomial is a John domain, it follows by Theorem 2.11 that $\Lambda(\Gamma)$ is conformally removable. $\square$

9.2. Conformal removability of geometrically finite polynomial Julia sets.

We recall that a rational map R is said to be *geometrically finite* if its postcritical set intersects the Julia set $\mathcal{J}(R)$ of R in a finite set, or equivalently, if every critical point of R is either preperiodic, or attracted to an attracting or parabolic cycle. The main result of this section is the following.

Theorem 9.2. *Let P be a geometrically finite polynomial of degree $d \geq 2$ with connected Julia set $\mathcal{J}(P)$. Then $\mathcal{J}(P)$ is conformally removable.*

We remark that special cases of Theorem 9.2 were already known in the 90s. Indeed, conformal removability of connected Julia sets of subhyperbolic polynomials was proved by Jones [Jon91]. In fact, it was shown there that in the subhyperbolic case, the basin of infinity is a John domain, which implies conformal removability of the Julia set. On the other hand, when a polynomial has parabolic cycles, the basin of infinity is no longer a John domain. However, for the polynomial $P(z) = z^2 + \frac{1}{4}$, the whole Julia set is the boundary of the immediate basin of attraction of a parabolic fixed point. In this case, it was shown in [CJY94] that the parabolic basin is John, and hence the corresponding Julia set is conformally removable.

The proof of Theorem 9.2 is indirect. We first realize the connected Julia set of a geometrically finite polynomial P as that of a subhyperbolic polynomial Q , whose Julia set is known to be $W^{1,1}$ -removable. We then use our David surgery technique

to construct a global David homeomorphism that carries the Julia set of Q onto the Julia set of P . The proof is completed by invoking Theorem 2.11.

For the first step, we need to recall some basic facts from polynomial dynamics. Let $\mathcal{P}_d$ be the space of all monic, centered, holomorphic polynomials of degree d ; i.e.,

$$\mathcal{P}_d := \{P(z) = z^d + a_{d-2}z^{d-2} + \cdots + a_1z + a_0 : a_0, \dots, a_{d-2} \in \mathbb{C}\}.$$

Two distinct members of $\mathcal{P}_d$ are affinely conjugate if and only if they are conjugate via rotation by a $(d-1)$ -st root of unity. The filled Julia set and the basin of infinity of a polynomial P are denoted by $\mathcal{K}(P)$ and $\mathcal{B}_\infty(P)$ respectively.

If P is a monic, centered, polynomial of degree d such that $\mathcal{J}(P)$ is connected, then there exists a conformal map $\phi_P : \mathbb{D}^* \rightarrow \mathcal{B}_\infty(P)$ that conjugates $z^d|_{\mathbb{D}^*}$ to $P|_{\mathcal{B}_\infty(P)}$, and satisfies $\phi'_P(\infty) = 1$. We will call ϕ_P the *Böttcher coordinate* for P . Furthermore, if $\partial\mathcal{K}(P) = \mathcal{J}(P)$ is locally connected, then ϕ_P extends to a semiconjugacy between $z^d|_{\mathbb{S}^1}$ and $P|_{\mathcal{J}(P)}$. The external dynamical ray of P at angle θ (the image of the radial line at angle θ under ϕ_P) is denoted by $R_P(\theta)$.

Our immediate goal is to associate a postcritically finite polynomial (in $\mathcal{P}_d$) to each geometrically finite member of $\mathcal{P}_d$ (with connected Julia set). To this end, we first recall the notion of *polynomial laminations*.

Definition 9.3. Let $p \in \mathcal{P}_d$ be such that $\mathcal{J}(P)$ is connected and locally connected. We define $\lambda(P)$ to be the smallest equivalence relation on $\mathbb{R}/\mathbb{Z}$ that identifies $s, t \in \mathbb{R}/\mathbb{Z}$ whenever the external dynamical rays $R_P(s)$ and $R_P(t)$ land at a common point on $\mathcal{J}(P)$. We call $\lambda(P)$ the *lamination* of P .

Proposition 9.4. Let $P \in \mathcal{P}_d$ be geometrically finite with connected $\mathcal{J}(P)$. Then, $\lambda(P)$ satisfies the following properties, where the map $m_d : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}$ is given by $\theta \mapsto d\theta$.

- (1) $\lambda(P)$ is closed in $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$.
- (2) Each equivalence class A of $\lambda(P)$ is a finite subset of $\mathbb{R}/\mathbb{Z}$.
- (3) If A is a $\lambda(P)$ -equivalence class, then $m_d(A)$ is also a $\lambda(P)$ -equivalence class.
- (4) If A is a $\lambda(P)$ -equivalence class, then $A \mapsto m_d(A)$ is *consecutive preserving*¹; i.e., for every connected component (s, t) of $\mathbb{R}/\mathbb{Z} \setminus A$, we have that (ds, dt) is a connected component of $\mathbb{R}/\mathbb{Z} \setminus m_d(A)$.
- (5) $\lambda(P)$ -equivalence classes are pairwise *unlinked*; i.e., if A and B are two distinct equivalence classes of $\lambda(P)$, then there exist disjoint intervals $I_A, I_B \subset \mathbb{R}/\mathbb{Z}$ such that $A \subset I_A$ and $B \subset I_B$.
- (6) $(\mathbb{R}/\mathbb{Z})/\lambda(P) \cong \mathcal{J}(P)$, and ϕ_P descends to a topological conjugacy between

$$m_d : (\mathbb{R}/\mathbb{Z})/\lambda(P) \rightarrow (\mathbb{R}/\mathbb{Z})/\lambda(P) \text{ and } P : \mathcal{J}(P) \rightarrow \mathcal{J}(P).$$

- (7) If $\gamma \subset (\mathbb{R}/\mathbb{Z})/\lambda(P)$ is a periodic simple closed curve, then the corresponding return map is not a homeomorphism.

¹This notion, which is slightly more general than that of being cyclic order preserving, is required to handle the case when A is formed by the arguments of the external rays landing at a critical point of P .

- (8) If c is a critical point of P on the Julia set $\mathcal{J}(P)$, then $\phi_P^{-1}(c) \subset \mathbb{Q}/\mathbb{Z}$, and $\phi_P^{-1}(c) \mapsto m_d(\phi_P^{-1}(c))$ has some degree $\delta > 1$. Every other equivalence class of $\lambda(P)$ maps injectively onto its image equivalence class under m_d .

Proof. First note that by [DH07, Exposé X, §1, Theorem 1] (also see [TY96, Theorem A]), the Julia set $\mathcal{J}(P)$ is locally connected, and hence $\lambda(P)$ can be defined following Definition 9.3. The first five properties now follow from [Kiw04, Theorem 1]. Moreover, local connectivity of $\mathcal{J}(P)$ implies that the Böttcher coordinate of P extends continuously to a surjection $\phi_P : \mathbb{R}/\mathbb{Z} \rightarrow \mathcal{J}(P)$ that semiconjugates m_d to P . By definition, the equivalence classes of $\lambda(P)$ are precisely the fibers of this map ϕ_P , from which property (6) follows. Property (7) follows from the fact that P has no Siegel disk. Finally, the last property is a consequence of geometric finiteness of P ; more precisely, of the fact that each critical point of P on $\mathcal{J}(P)$ is strictly pre-periodic. $\square$

Following [Kiw04], we say that an equivalence relation λ on $\mathbb{R}/\mathbb{Z}$ is a *Real lamination* (not to be confused with rational laminations; see [Kiw01, §1]) if it satisfies conditions (1)–(5). Moreover, if a Real lamination λ satisfies condition (7), it is called a *Real lamination with no rotation curves*. An equivalence class A of λ is called a *Julia critical element* if the degree of the map $m_d : A \rightarrow m_d(A)$ is greater than one. Finally, a Real lamination λ is called *postcritically finite* if every Julia critical element of λ is contained in $\mathbb{Q}/\mathbb{Z}$. Using this terminology, Proposition 9.4 can be restated as follows.

Corollary 9.5. *Let $P \in \mathcal{P}_d$ be geometrically finite with connected $\mathcal{J}(P)$. Then, $\lambda(P)$ is a postcritically finite Real lamination (with no rotation curves).*

The following folklore result can be easily derived from Kiwi’s theory of polynomial laminations, though it was not explicitly stated in [Kiw04].

Theorem 9.6. *For a postcritically finite Real lamination λ (with no rotation curves), there exists a monic, centered, postcritically finite polynomial Q with $\lambda(Q) = \lambda$. In particular, $\mathcal{J}(Q) \cong (\mathbb{R}/\mathbb{Z})/\lambda$.*

Proof. To prove this result, one needs to work with a more general definition of laminations (than the one given in Definition 9.3) involving prime end impressions. For some $Q \in \mathcal{P}_d$ with connected Julia set, the lamination $\lambda(Q)$ is defined as the smallest equivalence relation on $\mathbb{R}/\mathbb{Z}$ that identifies $s, t \in \mathbb{R}/\mathbb{Z}$ whenever $\text{Imp}(s) \cap \text{Imp}(t) \neq \emptyset$ (where $\text{Imp}(t)$ stands for the impression of an angle $t \in \mathbb{R}/\mathbb{Z}$ on the Julia set $\mathcal{J}(Q)$, see [Kiw04, Definition 2.3]). Clearly, if $\mathcal{J}(Q)$ is locally connected, then every impression is a singleton, and this definition agrees with the one given in Definition 9.3.

Since λ is a Real lamination with no rotation curves, [Kiw04, Lemma 6.34, Theorem 1] guarantees the existence of a monic, centered polynomial Q with connected Julia set such that each cycle of Q is either repelling or superattracting, each bounded Fatou component of Q contains a unique element in a critical grand orbit, and $\lambda(Q) = \lambda$. In particular, every critical point of Q in its Fatou set is eventually periodic. (At this point, it is unclear whether $\mathcal{J}(Q)$ is locally connected, which is why the more general definition of $\lambda(Q)$ is necessary here).

Now, it follows from postcritical finiteness of λ that each critical point of Q on its Julia set $\mathcal{J}(Q)$ lies in the impression of some rational (i.e., preperiodic under

m_d) angle. By [Kiw04, Corollary 1.2], the impression of a preperiodic angle is a singleton, and hence, every critical point of Q on $\mathcal{J}(Q)$ is strictly preperiodic. Therefore, Q is postcritically finite. In particular, $\mathcal{J}(Q)$ is locally connected, and the Böttcher coordinate of Q extends continuously to $\mathbb{R}/\mathbb{Z}$ to yield a topological conjugacy between

$$m_d : (\mathbb{R}/\mathbb{Z})/\lambda(Q) \rightarrow (\mathbb{R}/\mathbb{Z})/\lambda(Q) \text{ and } Q : \mathcal{J}(Q) \rightarrow \mathcal{J}(Q).$$

This completes the proof. $\square$

Finally, combining Property (6) of Proposition 9.4 with Corollary 9.5 and Theorem 9.6, we obtain the following result that allows us to associate a postcritically finite polynomial (in $\mathcal{P}_d$) to each geometrically finite member of $\mathcal{P}_d$ (with connected Julia set).

Corollary 9.7. *Let $P \in \mathcal{P}_d$ be geometrically finite with connected $\mathcal{J}(P)$. Then, there exists a postcritically finite polynomial $Q \in \mathcal{P}_d$ such that $\lambda(Q) = \lambda(P)$. In particular, $Q|_{\mathcal{J}(Q)}$ is topologically conjugate to $P|_{\mathcal{J}(P)}$.*

Remark 9.8. 1) Invariant laminations of complex polynomials were first defined in a slightly different language by Thurston (see [Thu09, §II.4]).

2) Theorem 9.6 is an instance of realization results in complex dynamics. Realization of postcritically finite maps with prescribed combinatorics was first proved by Thurston (see [DH93]), which was used by Hubbard, Bielefeld, and Fisher to classify critically pre-periodic polynomials using co-landing structure of suitable external dynamical rays [BFH92], and by Poirier to classify postcritically finite polynomials in terms of their Hubbard trees [Poi10]. An alternative way of constructing critically pre-periodic polynomials with prescribed laminations is to study the connectedness locus of $\mathcal{P}_d$ from ‘outside’; i.e., to approximate such polynomials from the *shift locus* (see [Kiw05]).

We are now prepared to prove the main theorem of this subsection.

Proof of Theorem 9.2. By conjugating P with an appropriate conformal linear map, we may assume that $P \in \mathcal{P}_d$. Furthermore, replacing P by a suitable iterate, we can assume that each periodic Fatou component of P is fixed.

The first step of the proof is to reduce the map P to its *rigid model*. Roughly speaking, the rigid model of P is a geometrically finite (but not necessarily postcritically finite) polynomial of the same degree, whose Julia dynamics is topologically conjugate to that of P , and whose Fatou critical points have the simplest possible dynamics in a suitable sense. To make this precise, let us first note that the Blaschke product

$$B_k(z) = \frac{(k+1)z^k + (k-1)}{(k-1)z^k + (k+1)}, \quad k \geq 2,$$

has a parabolic fixed point at 1, and $k-1$ distinct repelling fixed points on $\mathbb{S}^1$. Moreover, B_k has a unique critical point in $\mathbb{D}$; namely, at the origin.

By [McM, Proposition 6.9], one can perform quasiconformal surgeries on the Fatou set of P to produce a degree d geometrically finite polynomial $\hat{P}$, called the rigid model of P , such that the following properties hold true.

- (1) There exists a global quasiconformal map ψ conjugating $P|_{\mathcal{J}(P)}$ to $\hat{P}|_{\mathcal{J}(\hat{P})}$; in particular, each periodic Fatou component of $\hat{P}$ is fixed.

- (2) The restriction of $\hat{P}$ to each fixed Fatou component is conformally conjugate to $z^k|_{\mathbb{D}}$ (if the corresponding Fatou component of P is the immediate basin of attraction of an attracting fixed point) or $B_k|_{\mathbb{D}}$ (if the corresponding Fatou component of P is an immediate basin of attraction of a parabolic fixed point), where $k \geq 2$ is the degree of the restriction of P to the corresponding Fatou component.
- (3) If c is a critical point of $\hat{P}$ contained in a Fatou component, then there is a least $n \geq 0$ such that $\hat{P}^{\circ n}(c)$ lies in a fixed Fatou component U . Moreover, $\hat{P}^{\circ n}(c)$ is the unique critical point of $\hat{P}$ in U .

In particular, each Fatou component of $\hat{P}$, except for the immediate basins of the parabolic fixed points, contains a unique element in a critical grand orbit.

Since the property of being conformally removable is preserved under global quasiconformal maps, it now suffices to prove that $\mathcal{J}(\hat{P})$ is conformally removable.

By Corollary 9.7, there exists a postcritically finite polynomial $Q \in \mathcal{P}_d$ such that $Q|_{\mathcal{J}(Q)}$ is topologically conjugate to $\hat{P}|_{\mathcal{J}(\hat{P})}$. By construction, each periodic Fatou component of Q is fixed, and the restriction of Q to each fixed Fatou components is conformally conjugate to $z^k|_{\mathbb{D}}$, for some $k \geq 2$. Also note that the topological conjugacy between the Julia sets of Q and $\hat{P}$ induces a bijection between their fixed Fatou components.

Let $V_1, \dots, V_l$ be the fixed Fatou components of Q such that the corresponding Fatou components of $\hat{P}$ are parabolic. Recall from Example 4.2 that for any $k \geq 2$, the expansive covering map $B_k|_{\mathbb{S}^1}$ admits a Markov partition satisfying conditions (4.1) and (4.2). Hence, by Theorem 4.12, there exists a topological conjugacy between $z^k|_{\mathbb{S}^1}$ and $B_k|_{\mathbb{S}^1}$ that extends as a David homeomorphism of $\mathbb{D}$. One can now repeat the arguments of Lemma 7.1 to replace the power map dynamics on the Fatou component V_i by the dynamics of B_{k_i} , where k_i is the degree of Q restricted to the component V_i , for all $i \in \{1, \dots, l\}$. This yields a geometrically finite polynomial $\tilde{Q}$ and a global David homeomorphism H such that H carries $\mathcal{J}(Q)$ onto $\mathcal{J}(\tilde{Q})$, and conformally conjugates Q to $\tilde{Q}$ outside the grand orbit of $\cup_{i=1}^l V_i$. On the other hand, the restriction of $\tilde{Q}$ on $H(V_i)$ is conformally conjugate to $B_{k_i}|_{\mathbb{D}}$. By construction, the map $\tilde{Q}$ satisfies condition (2). Thanks to [McM, Proposition 6.9], possibly after performing quasiconformal surgeries on the Fatou set of $\tilde{Q}$, we can further assume that $\tilde{Q}$ satisfies condition (3). By Proposition 2.5 (i), we still have that $H(\mathcal{J}(Q)) = \mathcal{J}(\tilde{Q})$, where H is a global David homeomorphism.

It is now easy to see from the above construction of $\tilde{Q}$ that $\tilde{Q}|_{\mathcal{J}(\tilde{Q})}$ is topologically conjugate to $\hat{P}|_{\mathcal{J}(\hat{P})}$, and this conjugacy extends to a global topological conjugacy between $\tilde{Q}$ and $\hat{P}$ such that the conjugacy is conformal on the Fatou set of $\tilde{Q}$ (compare [McM, Proposition 6.10]). According to [Jon91], the basin of infinity $\mathcal{B}_{\infty}(Q)$ of the postcritically finite (hence, subhyperbolic) polynomial Q is a John domain. Hence, Theorem 2.11 combined with the fact that $\partial\mathcal{B}_{\infty}(Q) = \mathcal{J}(Q)$ yield that $H(\partial\mathcal{B}_{\infty}(Q)) = \mathcal{J}(\tilde{Q})$ is conformally removable. It follows that $\tilde{Q}$ is Möbius conjugate to $\hat{P}$, and $\mathcal{J}(\hat{P})$ is a Möbius image of $\mathcal{J}(\tilde{Q})$. Hence, $\mathcal{J}(\hat{P})$ is also conformally removable. $\square$

10. MATING REFLECTION GROUPS WITH ANTI-POLYNOMIALS: EXISTENCE
THEOREM

The goal of this section is to apply our result on David extensions of circle homeomorphisms to the theory of mating in conformal dynamics.

10.1. Necklace groups and Bers slice. For the purposes of mating necklace groups with anti-polynomials, it will be important to work with groups equipped with labeled generators (compare Remark 10.21). Moreover, the conformal conjugacy class of the action of a necklace group on the unbounded component of its domain of discontinuity will play no special role in the mating theory. Hence, we may freeze the ‘external class’ of the necklace groups under consideration. We now formalize this by defining a space of representations of the necklace group Γ_{d+1} (see Definition 6.6).

Definition 10.1. Let Γ be a discrete subgroup of $\text{Aut}^\pm(\widehat{\mathbb{C}})$. An isomorphism

$$\xi : \Gamma_{d+1} \rightarrow \Gamma$$

is said to be *weakly type-preserving*, or *w.t.p.*, if

- (1) $\xi(g)$ is orientation-preserving if and only if g is orientation-preserving, and
- (2) $\xi(g) \in \Gamma$ is a parabolic Möbius map for each parabolic Möbius map $g \in \Gamma_{d+1}$.

Definition 10.2. We define

$$\mathcal{D}(\Gamma_{d+1}) := \{\xi : \Gamma_{d+1} \rightarrow \Gamma \mid \Gamma \text{ is a discrete subgroup of } \text{Aut}^\pm(\widehat{\mathbb{C}}), \\ \text{and } \xi \text{ is a w.t.p. isomorphism}\}.$$

We endow $\mathcal{D}(\Gamma_{d+1})$ with the topology of *algebraic convergence*: we say that a sequence $(\xi_n)_{n=1}^\infty \subset \mathcal{D}(\Gamma_{d+1})$ converges to $\xi \in \mathcal{D}(\Gamma_{d+1})$ if $\xi_n(\rho_i) \rightarrow \xi(\rho_i)$ coefficient-wise (as $n \rightarrow \infty$) for $i \in \{1, \dots, d+1\}$.

Remark 10.3. Let $\xi \in \mathcal{D}(\Gamma_{d+1})$. Since for each $i \in \mathbb{Z}/(d+1)\mathbb{Z}$, the Möbius map $\rho_i \circ \rho_{i+1}$ is parabolic (this follows from the fact that each $\mathbf{C}_i$ is tangent to $\mathbf{C}_{i+1}$), the w.t.p. condition implies that $\xi(\rho_i) \circ \xi(\rho_{i+1})$ is also parabolic. As each $\xi(\rho_i)$ is an anti-conformal involution, it follows that $\xi(\rho_i)$ is Möbius conjugate to the circular reflection $z \mapsto 1/\bar{z}$ or the antipodal map $z \mapsto -1/\bar{z}$. A straightforward computation shows that the composition of $-1/\bar{z}$ with either the reflection or the antipodal map with respect to any circle has two distinct fixed points in $\widehat{\mathbb{C}}$, and hence not parabolic. Therefore, it follows that no $\xi(\rho_i)$ is Möbius conjugate to the antipodal map $-1/\bar{z}$. Hence, each $\xi(\rho_i)$ must be the reflection in some Euclidean circle C_i . Thus, $\Gamma = \xi(\Gamma_{d+1})$ is generated by reflections in the circles $C_1, \dots, C_{d+1}$. The fact that $\xi(\rho_i) \circ \xi(\rho_{i+1})$ is parabolic now translates to the condition that each C_i is tangent to C_{i+1} (for $i \in \mathbb{Z}/(d+1)\mathbb{Z}$). However, new tangencies among the circles C_i may arise. Moreover, that ξ is an isomorphism rules out non-tangential intersection between circles C_i, C_j (indeed, a non-tangential intersection between C_i and C_j would introduce a new relation between $\xi(\rho_i)$ and $\xi(\rho_j)$, compare [VS93, Part II, Chapter 5, §1.1]). Therefore, $\Gamma = \xi(\Gamma_{d+1})$ is a Kleinian reflection group satisfying properties (1) and (2) of necklace groups.

Recall that $\mathbb{D}^* = \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$.

Definition 10.4. Let $\text{Bel}_{\Gamma_{d+1}}$ denote those Beltrami coefficients μ invariant under Γ_{d+1} , satisfying $\mu = 0$ a.e. on $\mathbb{D}^*$. For a Beltrami coefficient μ , let $\tau_\mu : \mathbb{C} \rightarrow \mathbb{C}$ denote the quasiconformal integrating map of μ normalized so that $\tau_\mu(z) = z + O(1/z)$ as $z \rightarrow \infty$. The *Bers slice* of Γ is defined as

$$\beta(\Gamma_{d+1}) := \{\xi \in \mathcal{D}(\Gamma_{d+1}) : \exists \mu \in \text{Bel}_{\Gamma_{d+1}} \text{ such that } \xi(g) = \tau_\mu \circ g \circ \tau_\mu^{-1} \forall g \in \Gamma_{d+1}\}.$$

Remark 10.5. There is a natural free $\text{PSL}_2(\mathbb{C})$ -action on $\mathcal{D}(\Gamma_{d+1})$ given by conjugation, and so it is natural to consider the space $\text{AH}(\Gamma_{d+1}) := \mathcal{D}(\Gamma_{d+1})/\text{PSL}_2(\mathbb{C})$. The following definition of the Bers slice, where no normalization for τ_μ is specified, is more aligned with the classical Kleinian group literature:

$$(\star) \quad \{\xi \in \text{AH}(\Gamma_{d+1}) : \exists \mu \in \text{Bel}_{\Gamma_{d+1}} \text{ such that } \xi(g) = \tau_\mu \circ g \circ \tau_\mu^{-1} \forall g \in \Gamma_{d+1}\}.$$

Our Definition 10.4 of $\beta(\Gamma_{d+1})$ is simply a canonical choice of representative from each equivalence class of $(\star)$, and will be more appropriate for the present work.

Proposition 10.6. [LMM20, Proposition 2.23] *The Bers slice $\beta(\Gamma_{d+1})$ is precompact in $\mathcal{D}(\Gamma_{d+1})$, and for each $\xi \in \overline{\beta(\Gamma_{d+1})}$, the group $\xi(\Gamma_{d+1})$ is a necklace group.*

Definition 10.7. We refer to $\overline{\beta(\Gamma_{d+1})} \subset \mathcal{D}(\Gamma_{d+1})$ as the *Bers compactification* of the Bers slice $\beta(\Gamma_{d+1})$.

Remark 10.8. We will often identify $\xi \in \overline{\beta(\Gamma_{d+1})}$ with the group $\Gamma := \xi(\Gamma_{d+1})$, and simply write $\Gamma \in \overline{\beta(\Gamma_{d+1})}$, but always with the understanding of an associated representation $\xi : \Gamma_{d+1} \rightarrow \Gamma$. Since ξ is completely determined by its action on the generators $\rho_1, \dots, \rho_{d+1}$ of Γ_{d+1} , this is equivalent to remembering the ‘labeled’ circle packing $C_1, \dots, C_{d+1}$, where $\xi(\rho_i)$ is the reflection in the circle C_i , for $i \in \{1, \dots, d+1\}$.

For $\Gamma \in \overline{\beta(\Gamma_{d+1})}$, the unbounded component of the domain of discontinuity $\Omega(\Gamma)$ is denoted by $\Omega_\infty(\Gamma)$. We set $\mathcal{K}(\Gamma) := \mathbb{C} \setminus \Omega_\infty(\Gamma)$. We further denote the union of all bounded components of the fundamental domain $\mathcal{F}_\Gamma$ by $T^0(\Gamma)$, and the unique unbounded component of $\mathcal{F}_\Gamma$ by $\Pi^0(\Gamma)$. Finally, we set $\Pi(\Gamma) := \overline{\Pi^0(\Gamma)}$.

Note that a Kleinian group is said to be *geometrically finite* if its action on $\mathbb{H}^3$ admits a fundamental polyhedron with finitely many faces (see [Mar16, §3.6] for many equivalent definitions).

Proposition 10.9. *Let $\Gamma \in \overline{\beta(\Gamma_{d+1})}$. Then the following hold true.*

- (1) $\Omega_\infty(\Gamma)$ is simply connected, and Γ -invariant.
- (2) $\partial\Omega_\infty(\Gamma) = \Lambda(\Gamma)$; in particular, $\text{int } \mathcal{K}(\Gamma) = \Omega(\Gamma) \setminus \Omega_\infty(\Gamma)$.
- (3) $\Lambda(\Gamma)$ is connected.
- (4) All bounded components of $\Omega(\Gamma)$ are Jordan domains.

Proof. Except for the last one, all the statements follow from [LMM20, Proposition 2.28].

For the last statement, first note that by the proof of Proposition 6.5, the index two Kleinian subgroup of Γ is geometrically finite, and hence, its connected limit set $\Lambda(\Gamma)$ is locally connected [AM96]. Hence, each component of $\Omega(\Gamma) \setminus \Omega_\infty(\Gamma)$ is simply connected with a locally connected boundary. The fact that such a component $\mathcal{U}$ is Jordan follows from the fact that $\partial\mathcal{U} \subset \Lambda(\Gamma) = \partial\Omega_\infty(\Gamma)$. $\square$

Remark 10.10. We note that since the groups on the Bers boundary $\partial\beta(\Gamma_{d+1})$ are reflection groups, they are all geometrically finite. Thus, the boundary of the Bers

slice of an ideal polygon reflection group is considerably simpler than Bers slices of Fuchsian groups. Indeed, the boundary of the Bers slice of a Fuchsian group (except for the thrice punctured sphere group) necessarily contains degenerate groups; i.e., groups that are not geometrically finite (see [Ber70]).

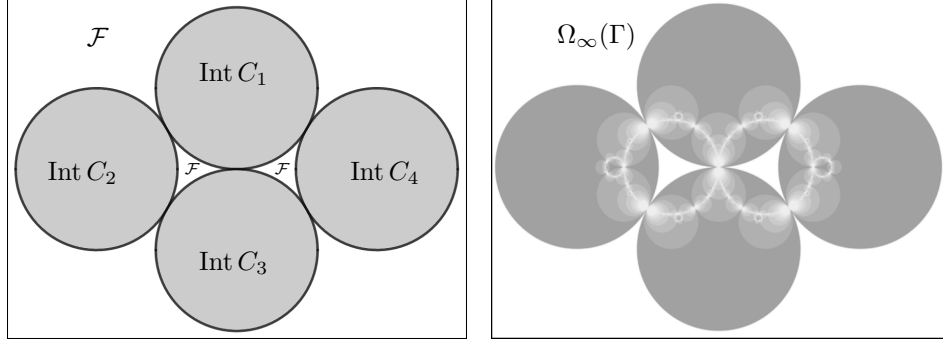


FIGURE 5. Left: The circles C_i generate a necklace group Γ . The Nielsen map ρ_Γ is defined piecewise on the union of the closed disks $\overline{\text{Int } C_i}$. The fundamental domain $\mathcal{F} = \mathcal{F}_\Gamma$ (for the action of Γ on $\Omega(\Gamma)$) is the complement of these open disks with the singular boundary points removed. The connected components of $\mathcal{F}$ are marked. Right: The unbounded component of the domain of discontinuity $\Omega(\Gamma)$ is $\Omega_\infty(\Gamma)$. The boundary of $\Omega_\infty(\Gamma)$ is the limit set $\Lambda(\Gamma)$. Every point in $\Omega(\Gamma)$ escapes to $\mathcal{F}$ under iterates of ρ_Γ . The dynamics of ρ_Γ on the limit set $\Lambda(\Gamma)$ is topologically conjugate to the Julia dynamics of the cubic anti-polynomial depicted in Figure 7.

Proposition 10.11. [LMM20, Proposition 2.31] *Let $\Gamma \in \overline{\beta(\mathbf{\Gamma}_{d+1})}$. The map ρ_Γ is orbit equivalent to Γ on $\widehat{\mathbb{C}}$; i.e., for any two points $z, w \in \widehat{\mathbb{C}}$, there exists $g \in \Gamma$ with $g(z) = w$ if and only if there exist non-negative integers n_1, n_2 such that $\rho_\Gamma^{n_1}(z) = \rho_\Gamma^{n_2}(w)$.*

Remark 10.12. Note that although ρ_Γ is not defined on $\text{int } \mathcal{F}_\Gamma$, the expression $\rho_\Gamma^{\circ n}(z)$ makes sense for $z \in \text{int } \mathcal{F}_\Gamma$ when $n = 0$ (with the convention that $\rho_\Gamma^{\circ 0} = \text{id}$).

Proposition 10.13. *Let $\Gamma \in \overline{\beta(\mathbf{\Gamma}_{d+1})}$. Then, the following hold true.*

- (1) $\overline{T^0(\Gamma)}$ is connected.
- (2) $\mathcal{K}(\Gamma) = \bigcup_{n \geq 0} \overline{\rho_\Gamma^{-n}(T^0(\Gamma))}$.
- (3) $\Lambda(\Gamma)$ is completely invariant under ρ_Γ .
- (4) If $\mathcal{U}$ is a connected component of $\text{int } \mathcal{K}(\Gamma)$ containing a component of $T^0(\Gamma)$, then $\rho_\Gamma|_{\partial \mathcal{U}}$ is topologically conjugate to $\rho_{d'}|_{\mathbb{S}^1}$, for some $d' \geq 2$.

Proof. 1) This follows from the fact that Γ is generated by reflections in the circles of a finite circle packing with 2-connected and outerplanar contact graph.

2) Recall that $\mathcal{F}_\Gamma = T^0(\Gamma) \sqcup \Pi^0(\Gamma)$. Since $\Omega_\infty(\Gamma) \supset \Pi^0(\Gamma)$ is an invariant component of $\Omega(\Gamma)$, it follows from Proposition 6.5 that

$$\Omega(\Gamma) \setminus \Omega_\infty(\Gamma) = \Gamma(T^0(\Gamma)).$$

Proposition 10.11 now implies that

$$\Omega(\Gamma) \setminus \Omega_\infty(\Gamma) = \bigcup_{n \geq 0} \rho_\Gamma^{-n}(T^0(\Gamma)).$$

As $\Omega(\Gamma) \setminus \Omega_\infty(\Gamma)$ is also Γ -invariant, we have that

$$\partial(\Omega(\Gamma) \setminus \Omega_\infty(\Gamma)) = \Lambda(\Gamma).$$

Hence,

$$\mathcal{K}(\Gamma) = \Lambda(\Gamma) \sqcup (\Omega(\Gamma) \setminus \Omega_\infty(\Gamma)) = \overline{\bigcup_{n \geq 0} \rho_\Gamma^{-n}(T^0(\Gamma))}.$$

3) Complete invariance of $\Lambda(\Gamma)$ under ρ_Γ follows from Γ -invariance of the limit set.

4) If $\mathcal{U}$ is a connected component of $\text{int } \mathcal{K}(\Gamma)$ containing a component of $T^0(\Gamma)$, then $\mathcal{U} \cap T^0(\Gamma)$ is a topological $(d' + 1)$ -gon (with vertices removed), for some $d' \geq 2$. Choose a quasiconformal homeomorphism $\kappa : T^0(\Gamma_{d'+1}) \rightarrow \mathcal{U} \cap T^0(\Gamma)$ preserving the vertices. By iterated Schwarz reflections (and quasiconformal removability of analytic arcs), we obtain a quasiconformal homeomorphism $\kappa : \mathbb{D} \rightarrow \mathcal{U}$ conjugating $\rho_{d'}$ to ρ_Γ . Since $\mathcal{U}$ is a Jordan domain (by Proposition 10.9), the quasiconformal homeomorphism κ extends continuously to a topological conjugacy between $\rho_{d'}|_{\mathbb{S}^1}$ and $\rho_\Gamma|_{\partial \mathcal{U}}$. $\square$

Proposition 10.14. [LMM20, Proposition 2.36] *Let $\Gamma \in \overline{\beta(\Gamma_{d+1})}$. There exists a conformal map $\phi_\Gamma : \mathbb{D}^* \rightarrow \Omega_\infty(\Gamma)$ such that*

$$(10.1) \quad \rho_d(z) = \phi_\Gamma^{-1} \circ \rho_\Gamma \circ \phi_\Gamma(z), \text{ for } z \in \mathbb{D}^* \setminus \text{int } \Pi(\Gamma_{d+1}).$$

The map ϕ_Γ extends continuously to a semiconjugacy $\phi_\Gamma : \mathbb{S}^1 \rightarrow \Lambda(\Gamma)$ between $\rho_d|_{\mathbb{S}^1}$ and $\rho_\Gamma|_{\Lambda(\Gamma)}$, and for each i , sends the cusp of $\partial \Pi(\Gamma_{d+1})$ at $\mathbf{C}_i \cap \mathbf{C}_{i+1}$ to the cusp of $\partial \Pi(\Gamma)$ at $C_i \cap C_{i+1}$.

The Nielsen map ρ_d of the group Γ_{d+1} , restricted to the limit set $\mathbb{S}^1$, admits the Markov partition

$$\mathcal{P}(\rho_d; \{1, e^{\frac{2\pi i}{d+1}} \dots, e^{\frac{2\pi i d}{d+1}}\}).$$

Note that the expanding map

$$z \mapsto \bar{z}^d : \mathbb{S}^1 \rightarrow \mathbb{S}^1,$$

or equivalently,

$$m_{-d} : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}, \theta \mapsto -d\theta$$

also admits the same Markov partition with the same transition matrix (identifying $\mathbb{S}^1$ with $\mathbb{R}/\mathbb{Z}$). By Lemma 3.3, there exists a homeomorphism

$$\mathcal{E}_d : \mathbb{S}^1 \rightarrow \mathbb{S}^1$$

conjugating ρ_d to $\bar{z}^d$ (or m_{-d}) with $\mathcal{E}_d(1) = 1$.

10.2. Conformal mating. We will now define the notion of conformal mating of the Nielsen map of a necklace group and an anti-polynomial. Our definitions follow the classical definition of conformal matings of two (anti-)polynomials, which we recall below (we refer the readers to [PM12] for a more extensive discussion on conformal mating of polynomials).

Let us first introduce some terminologies. We denote the Julia set of an anti-rational map R by $\mathcal{J}(R)$. The filled Julia set and the basin of infinity of an anti-polynomial P are denoted by $\mathcal{K}(P)$ and $\mathcal{B}_\infty(P)$ respectively.

Let P be a monic, centered, anti-polynomial of degree d such that $\mathcal{J}(P)$ is connected and locally connected. Denote by $\phi_P : \mathbb{D}^* \rightarrow \mathcal{B}_\infty(P)$ the Böttcher coordinate for P such that $\phi'_P(\infty) = 1$. We note that since $\partial\mathcal{K}(P) = \mathcal{J}(P)$ is locally connected by assumption, it follows that ϕ_P extends to a semiconjugacy between $z \mapsto \bar{z}^d|_{\mathbb{S}^1}$ and $P|_{\mathcal{J}(P)}$.

10.2.1. Conformal mating of anti-polynomials. An anti-rational map $R : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ of degree $d \geq 2$ is said to be the *conformal mating* of two degree d monic, centered, anti-polynomials P_1 and P_2 with connected and locally connected filled Julia sets if and only if there exist continuous maps

$$\tilde{\psi}_1 : \mathcal{K}(P_1) \rightarrow \widehat{\mathbb{C}} \text{ and } \tilde{\psi}_2 : \mathcal{K}(P_2) \rightarrow \widehat{\mathbb{C}},$$

conformal on $\text{int } \mathcal{K}(P_1)$, $\text{int } \mathcal{K}(P_2)$, respectively, such that

- (1) $\tilde{\psi}_1(\mathcal{K}(P_1)) \cup \tilde{\psi}_2(\mathcal{K}(P_2)) = \widehat{\mathbb{C}}$,
- (2) $\tilde{\psi}_i \circ P_i = R \circ \tilde{\psi}_i$, for $i \in \{1, 2\}$, and
- (3) $\tilde{\psi}_1(a) = \tilde{\psi}_2(b)$ if and only if $a \sim_1 b$, where the equivalence relation $\sim_1$ on $\mathcal{K}(P_1) \sqcup \mathcal{K}(P_2)$ is generated by $\phi_{P_1}(s) \sim_1 \phi_{P_2}(\bar{s})$ for all $s \in \mathbb{S}^1$.

10.2.2. Conformal mating of a necklace group and an anti-polynomial. Let $\Gamma \in \overline{\beta(\mathbf{\Gamma}_{d+1})}$ be a necklace group generated by reflections in circles $C_1, \dots, C_{d+1}$. By Proposition 10.14, there is a natural semiconjugacy $\phi_\Gamma : \mathbb{S}^1 \rightarrow \Lambda(\Gamma)$ between $\rho_d|_{\mathbb{S}^1}$ and $\rho_\Gamma|_{\Lambda(\Gamma)}$ such that $\phi_\Gamma(1)$ is the point of tangential intersection of C_1 and C_{d+1} . Recall also that $\mathcal{E}_d : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is a topological conjugacy between $\rho_d|_{\mathbb{S}^1}$ and $z \mapsto \bar{z}^d|_{\mathbb{S}^1}$.

Definition 10.15. We define the equivalence relation $\sim$ on $\mathcal{K}(\Gamma) \sqcup \mathcal{K}(P)$ generated by $\phi_\Gamma(t) \sim \phi_P(\mathcal{E}_d(t))$ for all $t \in \mathbb{S}^1$.

Definition 10.16. Let $\Gamma \in \overline{\beta(\mathbf{\Gamma}_{d+1})}$, and P be a monic, centered anti-polynomial such that $\mathcal{J}(P)$ is connected and locally connected. Further, let $\Omega \subsetneq \widehat{\mathbb{C}}$ be an open set, and $F : \bar{\Omega} \rightarrow \widehat{\mathbb{C}}$ be a continuous map that is anti-meromorphic on Ω . We say that F is a *conformal mating* of Γ with P if there exist continuous maps

$$\psi_P : \mathcal{K}(P) \rightarrow \widehat{\mathbb{C}} \text{ and } \psi_\Gamma : \mathcal{K}(\Gamma) \rightarrow \widehat{\mathbb{C}},$$

conformal on $\text{int } \mathcal{K}(P)$, $\text{int } \mathcal{K}(\Gamma)$, respectively, such that

- (1) $\psi_P(\mathcal{K}(P)) \cup \psi_\Gamma(\mathcal{K}(\Gamma)) = \widehat{\mathbb{C}}$,
- (2) $\Omega = \widehat{\mathbb{C}} \setminus \psi_\Gamma(\overline{T^0(\Gamma)})$,
- (3) $\psi_P \circ P = F \circ \psi_P$ on $\mathcal{K}(P)$,
- (4) $\psi_\Gamma \circ \rho_\Gamma = F \circ \psi_\Gamma$ on $\mathcal{K}(\Gamma) \setminus \text{int } T^0(\Gamma)$, and
- (5) $\psi_\Gamma(z) = \psi_P(w)$ if and only if $z \sim w$ where $\sim$ is as in Definition 10.15.

The following lemma connects the study of conformal matings of reflection groups and anti-polynomials to the theory of quadrature domains.

Lemma 10.17. *If $F : \overline{\Omega} \rightarrow \widehat{\mathbb{C}}$ is a conformal mating of Γ and P , then each component of Ω is a simply connected quadrature domain, and F is the piecewise defined Schwarz reflection map associated with these quadrature domains.*

Proof. As all but finitely many points of $\overline{T^0(\Gamma)}$ are contained in $\text{int } \mathcal{K}(\Gamma)$, injectivity of $\psi_\Gamma|_{\text{int } \mathcal{K}(\Gamma)}$ implies that ψ_Γ can introduce at most finitely many identifications on $\overline{T^0(\Gamma)}$. In fact, these identifications may happen only at the singular points of $\partial T^0(\Gamma)$. It follows that $\Omega = \widehat{\mathbb{C}} \setminus \psi_\Gamma(\overline{T^0(\Gamma)})$ has at most finitely many connected components, say $\Omega_1, \dots, \Omega_l$, and they satisfy the conditions

- (1) $\text{int } \overline{\Omega_i} = \Omega_i$, for $i \in \{1, \dots, l\}$, and
- (2) $\overline{\Omega} = \bigcup_{i=1}^l \overline{\Omega_i} = \widehat{\mathbb{C}} \setminus \psi_\Gamma(\text{int } T^0(\Gamma))$.

Since $\Gamma \in \overline{\beta(\mathbf{\Gamma}_{d+1})}$, the set $\overline{T^0(\Gamma)}$ is connected by Proposition 10.13. Hence, each Ω_i is simply connected. Possibly after conjugating F by a Möbius map, we can also assume that $\infty \notin \partial \Omega_i$, for $i \in \{1, \dots, l\}$. Finally, as ρ_Γ fixes $\partial T^0(\Gamma)$ pointwise, it follows that the anti-meromorphic map $F|_{\Omega_i}$ continuously extends to the identity map on $\partial \Omega_i$, for each i . We conclude that each component Ω_i is a simply connected quadrature domain, and $F|_{\overline{\Omega_i}}$ is the Schwarz reflection map associated with Ω_i . $\square$

10.3. A general mateability theorem. Let $\Gamma \in \overline{\beta(\mathbf{\Gamma}_{d+1})}$. By [LMM20, Theorem B, Remark 5.13], there exists a unique monic, centered, critically fixed anti-polynomial P_Γ of degree d such that

$$\rho_\Gamma : \Lambda(\Gamma) \rightarrow \Lambda(\Gamma) \quad \text{and} \quad P_\Gamma : \mathcal{J}(P_\Gamma) \rightarrow \mathcal{J}(P_\Gamma)$$

are topologically conjugate (see Figures 5 and 7); and

$$(10.2) \quad (\mathcal{E}_d \times \mathcal{E}_d)(\lambda(\Gamma)) = \lambda(P_\Gamma),$$

where $\lambda(\Gamma)$ (respectively, $\lambda(P_\Gamma)$) is the equivalence relation on $\mathbb{R}/\mathbb{Z}$ determined by the fibers of $\phi_\Gamma : \mathbb{R}/\mathbb{Z} \rightarrow \Lambda(\Gamma)$ (respectively, of $\phi_{P_\Gamma} : \mathbb{R}/\mathbb{Z} \rightarrow \mathcal{J}(P_\Gamma)$). Moreover, the topological conjugacy $\mathfrak{H} : \Lambda(\Gamma) \rightarrow \mathcal{J}(P_\Gamma)$ satisfies

$$(10.3) \quad \mathfrak{H}(\phi_\Gamma(t)) = \phi_{P_\Gamma}(\mathcal{E}_d(t)), \quad t \in \mathbb{S}^1.$$

Thanks to Lemma 7.1, the question of mateability of an anti-polynomial P and the group Γ can be reduced to the question of mateability of the pair of anti-polynomials P and P_Γ . The main idea is to pass from the conformal mating of two anti-polynomials to that of an anti-polynomial and a necklace group by gluing Nielsen maps of ideal polygon groups in suitable invariant Fatou components of anti-rational maps.

Proposition 10.18. *Let P be a monic, postcritically finite anti-polynomial of degree d , and $\Gamma \in \overline{\beta(\mathbf{\Gamma}_{d+1})}$; i.e., Γ is a necklace group. Then, Γ and P are conformally mateable if P_Γ and P are conformally mateable.*

Proof. We suppose that R is a conformal mating of P_Γ and P . Then, by definition (see Subsection 10.2.1), there exist continuous maps

$$\tilde{\psi}_{P_\Gamma} : \mathcal{K}(P_\Gamma) \rightarrow \widehat{\mathbb{C}} \text{ and } \tilde{\psi}_P : \mathcal{K}(P) \rightarrow \widehat{\mathbb{C}},$$

conformal on $\text{int } \mathcal{K}(P_\Gamma)$, $\text{int } \mathcal{K}(P)$, respectively, such that

- (1) $\tilde{\psi}_{P_\Gamma}(\mathcal{K}(P_\Gamma)) \cup \tilde{\psi}_P(\mathcal{K}(P)) = \widehat{\mathbb{C}}$,
- (2) $\tilde{\psi}_{P_\Gamma} \circ P_\Gamma = R \circ \tilde{\psi}_P$ on $\mathcal{K}(P_\Gamma)$,
- (3) $\tilde{\psi}_P \circ P = R \circ \tilde{\psi}_{P_\Gamma}$ on $\mathcal{K}(P)$, and
- (4) $\tilde{\psi}_{P_\Gamma}(a) = \tilde{\psi}_P(b)$ if and only if $a \sim_1 b$, where the equivalence relation $\sim_1$ on $\mathcal{K}(P_\Gamma) \sqcup \mathcal{K}(P)$ is generated by $\phi_{P_\Gamma}(s) \sim_1 \phi_P(\bar{s})$ for all $s \in \mathbb{S}^1$.

In particular, R is a postcritically finite (hence, subhyperbolic) anti-rational map.

Let $\mathcal{U}_i$, $i \in \{1, \dots, k\}$, be the components of $\text{int } \mathcal{K}(\Gamma)$ containing the components of $T^0(\Gamma)$. By Proposition 10.13(part 4), $\rho_\Gamma|_{\partial \mathcal{U}_i}$ is topologically conjugate to $\rho_{d_i}|_{\mathbb{S}^1}$ (for some $d_i \geq 2$), and hence is a degree d_i expansive orientation-reversing covering of $\partial \mathcal{U}_i$. Under $\mathfrak{H}$, the boundaries of these components $\mathcal{U}_i$ are mapped to the boundaries of invariant (bounded) Fatou components $\tilde{U}_1, \dots, \tilde{U}_k$ of P_Γ such that $P_\Gamma|_{\partial \tilde{U}_i}$ is topologically conjugate to $\bar{z}^{d_i}|_{\mathbb{S}^1}$. Thus,

$$U_i := \tilde{\psi}_{P_\Gamma}(\tilde{U}_i) \subset \tilde{\psi}_{P_\Gamma}(\mathcal{K}(P_\Gamma))$$

is an invariant Fatou component of the anti-rational map R . Moreover, the map $R|_{\partial U_i}$ is topologically semiconjugate to $\bar{z}^{d_i}|_{\mathbb{S}^1}$.

We will now glue the dynamics of $\rho_\Gamma|_{\mathcal{U}_i}$ in each U_i . This will be done in two steps. We first glue the Nielsen map of the regular ideal $(d_i + 1)$ -gon group in U_i , and then quasiconformally deform it to the Nielsen map $\rho_\Gamma|_{\mathcal{U}_i}$. To achieve the first goal, note that Lemma 7.1, applied to the Fatou components $U_1, \dots, U_k$ of R , provides us with a global David homeomorphism Ψ_1 and an anti-analytic map F_1 (defined on a subset of $\widehat{\mathbb{C}}$) such that F_1 is conformally conjugate to $\rho_{d_i}|_{\mathbb{D}}$ on $\Psi_1(U_i)$ ($i \in \{1, \dots, k\}$), and topologically conjugate to $R|_{\mathcal{J}(R)}$ on $\Psi_1(\mathcal{J}(R))$. Let us set

$$\Omega_1 := \widehat{\mathbb{C}} \setminus \bigcup_{i=1}^k \overline{\Psi_1(T_i)},$$

where T_i is as in the proof of Lemma 7.1. Note that since at most finitely many points on various $\partial \Psi_1(T_i)$ may touch, the domain of definition of F_1 is $\overline{\Omega_1}$. We now choose quasiconformal homeomorphisms from the topological $(d_i + 1)$ -gon $\Psi_1(U_i) \setminus \Omega_1$ onto the topological $(d_i + 1)$ -gon $\mathcal{U}_i \cap T^0(\Gamma)$ that preserve the vertices. We then pull back the standard complex structure on $\mathcal{U}_i \cap T^0(\Gamma)$ by this quasiconformal homeomorphism to $\Psi_1(U_i) \setminus \Omega_1$ ($i \in \{1, \dots, k\}$), spread it by the dynamics to all of $\text{int } \Psi_1(\tilde{\psi}_{P_\Gamma}(\mathcal{K}(P_\Gamma)))$, and put the standard complex structure on the rest of the sphere. The Measurable Riemann Mapping Theorem now gives us a global quasiconformal homeomorphism Ψ_2 that is conformal on $\text{int } \Psi_1(\tilde{\psi}_P(\mathcal{K}(P)))$, and conjugates F_1 to a continuous map $F : \overline{\Psi_2(\Omega_1)} \rightarrow \widehat{\mathbb{C}}$ that is anti-analytic on $\Psi_2(\Omega_1)$. By construction, setting $\Psi := \Psi_2 \circ \Psi_1$ and $\Omega := \Psi_2(\Omega_1)$, we have that $F : \overline{\Omega} \rightarrow \widehat{\mathbb{C}}$ is conformally conjugate to $\rho_\Gamma|_{\mathcal{U}_i}$ on $\Psi(U_i)$ ($i \in \{1, \dots, k\}$), and topologically conjugate to $R|_{\mathcal{J}(R)}$ on $\Psi(\mathcal{J}(R))$.

We will now argue that F is a conformal mating of Γ and P . To this end, let us set

$$\psi_P := \Psi \circ \tilde{\psi}_P : \mathcal{K}(P) \rightarrow \widehat{\mathbb{C}}.$$

Note that since $R \equiv \tilde{R}$ on $\tilde{\psi}_P(\mathcal{K}(P))$ and Ψ is conformal on $\tilde{\psi}_P(\text{int } \mathcal{K}(P))$, it follows that ψ_P is conformal on $\text{int } \mathcal{K}(P)$ and $\psi_P \circ P = F \circ \psi_P$ on $\mathcal{K}(P)$.

We now set

$$\psi_\Gamma := \Psi \circ \tilde{\psi}_{P_\Gamma} \circ \mathfrak{H} : \Lambda(\Gamma) \rightarrow \Psi(\mathcal{J}(R)).$$

Then, $\psi_\Gamma \circ \rho_\Gamma = F \circ \psi_\Gamma$ on $\Lambda(\Gamma)$. By our construction of F , we can extend $\psi_\Gamma|_{\Lambda(\Gamma)}$ to a conformal map

$$\psi_\Gamma : \text{int } \mathcal{K}(\Gamma) \rightarrow \Psi\left(\tilde{\psi}_{P_\Gamma}(\text{int } \mathcal{K}(P_\Gamma))\right)$$

such that $\psi_\Gamma(\overline{T^0(\Gamma)}) = \bigcup_{i=1}^k \overline{\Psi(T_i)}$, and $\psi_\Gamma \circ \rho_\Gamma = F \circ \psi_\Gamma$ on $\mathcal{K}(\Gamma) \setminus \text{int } T^0(\Gamma)$. It also follows that

$$\Omega = \Psi_2(\Omega_1) = \widehat{\mathbb{C}} \setminus \bigcup_{i=1}^k \overline{\Psi(T_i)} = \widehat{\mathbb{C}} \setminus \psi_\Gamma\left(\overline{T^0(\Gamma)}\right),$$

and

$$\psi_P(\mathcal{K}(P)) \bigcup \psi_\Gamma(\mathcal{K}(\Gamma)) = \Psi\left(\tilde{\psi}_P(\mathcal{K}(P)) \bigcup \tilde{\psi}_{P_\Gamma}(\mathcal{K}(P_\Gamma))\right) = \widehat{\mathbb{C}}.$$

Thus, F satisfies conditions (1)–(4) of Definition 10.16. It now remains to verify condition (5).

To this end, let us first choose $z = \phi_\Gamma(t) \in \Lambda(\Gamma)$ and $w = \phi_P(\overline{\mathcal{E}_d(t)}) \in \mathcal{J}(P)$, for some $t \in \mathbb{S}^1$. Then,

$$\psi_\Gamma(z) = (\Psi \circ \tilde{\psi}_{P_\Gamma} \circ \mathfrak{H})(\phi_\Gamma(t)) = \Psi(\tilde{\psi}_{P_\Gamma}(\phi_{P_\Gamma}(\mathcal{E}_d(t)))) = \Psi(\tilde{\psi}_P(\phi_P(\overline{\mathcal{E}_d(t)}))) = \psi_P(w).$$

Thus, $\psi_\Gamma(z) = \psi_P(w)$ whenever $z \sim w$.

Conversely, let

$$\psi_\Gamma(z) = \psi_P(w)$$

for some $z \in \Lambda(\Gamma)$ and $w \in \mathcal{J}(P)$. By definition of ψ_Γ and ψ_P , and the fact that Ψ is a homeomorphism, the above implies that $\tilde{\psi}_{P_\Gamma}(\mathfrak{H}(z)) = \tilde{\psi}_P(w)$. Hence, there exists $s \in \mathbb{S}^1$ such that

$$\mathfrak{H}(z) = \phi_{P_\Gamma}(s), \quad w = \phi_P(\overline{s}).$$

Now set $t = \mathcal{E}_d^{-1}(s)$. Then,

$$\mathfrak{H}(z) = \phi_{P_\Gamma}(\mathcal{E}_d(t)) = \mathfrak{H}(\phi_\Gamma(t)) \implies z = \phi_\Gamma(t).$$

Therefore, $z = \phi_\Gamma(t)$ and $w = \phi_P(\overline{\mathcal{E}_d(t)})$; and hence, $z \sim w$. The proof is now complete. $\square$

In order to apply the above proposition, we quote a mateability result for anti-polynomials from [LLM20].

Proposition 10.19. [LLM20, Proposition 4.21] *Let P and Q be marked anti-polynomials of equal degree $d \geq 2$, where P is postcritically finite, hyperbolic, and Q is critically fixed. There is an anti-rational map that is the conformal mating of P and Q if and only if $\mathcal{K}(P) \sqcup \mathcal{K}(Q) / \sim_1$ is homeomorphic to $\mathbb{S}^2$, where the equivalence relation $\sim_1$ on $\mathcal{K}(P) \sqcup \mathcal{K}(Q)$ is generated by $\phi_P(s) \sim_1 \phi_Q(\overline{s})$ for all $s \in \mathbb{S}^1$.*

We are now ready to prove a precise version of Theorem D. The equivalence relation $\sim$ appearing in the statement below is the one introduced in Definition 10.15.

Theorem 10.20 (Criterion for Mateability). *Let P be a monic, postcritically finite, hyperbolic anti-polynomial of degree d , and $\Gamma \in \overline{\beta(\Gamma_{d+1})}$; i.e., Γ is a necklace group. Then, P and Γ are conformally mateable if and only if $\mathcal{K}(P) \sqcup \mathcal{K}(\Gamma)/\sim$ is homeomorphic to $\mathbb{S}^2$.*

Proof. It is obvious from Definition 10.16 that if P and Γ are mateable, then $\mathcal{K}(P) \sqcup \mathcal{K}(\Gamma)/\sim$ must be homeomorphic to $\mathbb{S}^2$.

For the converse, let us assume that $\mathcal{K}(P) \sqcup \mathcal{K}(\Gamma)/\sim$ is homeomorphic to $\mathbb{S}^2$. It is easy to check using the definitions of $\sim$ and $\sim_1$, and Relation (10.3) that the topological spaces $\mathcal{K}(P) \sqcup \mathcal{K}(\Gamma)/\sim$ and $\mathcal{K}(P) \sqcup \mathcal{K}(P_\Gamma)/\sim_1$ are homeomorphic. In particular, $\mathcal{K}(P) \sqcup \mathcal{K}(P_\Gamma)/\sim_1$ is homeomorphic to $\mathbb{S}^2$. Hence, Proposition 10.19 implies that the anti-polynomials P and P_Γ are conformally mateable. The desired conclusion that P and Γ are conformally mateable now follows from Proposition 10.18. $\square$

Remark 10.21. In the statements of Proposition 10.18 and Theorem 10.20, being mateable depends on the representation $\xi : \Gamma_{d+1} \rightarrow \Gamma$ (or equivalently, the labeling of the circles $C_1, \dots, C_{d+1}$, where $\xi(\rho_i)$ is the reflection in the circle C_i , for $i \in \{1, \dots, d+1\}$). Indeed, the lamination $\lambda(P_\Gamma)$ of the anti-polynomial P_Γ is determined by the lamination $\lambda(\Gamma)$ of the group Γ (via relation (10.2)), and the lamination $\lambda(\Gamma)$ depends on the choice of the conformal map ϕ_Γ , which, in turn, depends on the labeling of the circles $C_1, \dots, C_{d+1}$ (see Proposition 10.14). Roughly speaking, different representations give rise to different ways of gluing the limit set of Γ with the Julia set of an anti-polynomial, and the choice of gluing determines whether a conformal mating exists.

For example, in light of Proposition 10.18 and [LLM20, Corollary 4.22], it is easy to see that the anti-polynomial $P_1(z) = \bar{z}^3 - \frac{3i}{\sqrt{2}}\bar{z}$ (see Figure 6) is conformally mateable with the group Γ from Figure 5 with the labeling of the underlying circle packing shown in Figure 5(left). However, if we consider the same group Γ with a different labeling of the underlying circle packing such that the circles C_2 and C_4 touch at a point (note that this amounts to looking at a different element of $\overline{\beta(\Gamma_{d+1})}$), then it is no longer conformally mateable with the anti-polynomial P_1 since with this new marking of the generators of Γ , the quotient space $\mathcal{K}(P_1) \sqcup \mathcal{K}(\Gamma)/\sim$ is not homeomorphic to $\mathbb{S}^2$.

11. MATING REFLECTION GROUPS WITH ANTI-POLYNOMIALS: EXAMPLES

The goal of this section is to illustrate Proposition 10.18 and Theorem 10.20 by producing various examples of matings of anti-polynomials and necklace reflection groups. While these results guarantee the existence of conformal matings of suitable anti-polynomials and necklace reflection groups, in general, it may be hard to find explicit Schwarz reflection maps realizing the conformal matings.

However, there are two ways to achieve this in low degrees. The first one, implemented in [LLMM18a, LLMM18b, LMM20], is to study the dynamics and parameter spaces of specific families of Schwarz reflection maps, and recognize such maps as matings of anti-polynomials and reflection groups. In Subsection 11.1, we will indicate how the examples studied in these papers fit in our general mating framework.

In the opposite direction, to explicitly characterize the conformal mating of a given necklace group and an anti-polynomial, let us first recall that by Lemma 10.17, the conformal mating is a piecewise defined Schwarz reflection map associated with a finite collection of simply connected quadrature domains. This allows one to uniformize the domains by rational maps of suitable degrees (compare Proposition 6.12), and use the desired dynamical properties to explicitly find these rational maps. We note that this approach requires care when both the anti-polynomial and the necklace group have non-trivial laminations; indeed, one needs to read off the contact pattern of the finite collection of simply connected quadrature domains (whose Schwarz reflection maps define the conformal mating) and the degrees of the corresponding uniformizing rational maps from the laminations of the anti-polynomial and the necklace group (equipped with a labeling of the underlying circle packing). This will be illustrated with a couple of worked out examples in Subsections 11.2 and 11.3. The final Subsection 11.4 underscores the additional analytic steps required to characterize conformal matings of parabolic anti-polynomials and necklace groups.

11.1. Some known examples. 1. For $\Gamma := \mathbf{\Gamma}_{d+1}$, the associated anti-polynomial P_Γ is given by $P_\Gamma(z) = \bar{z}^d$. Since $\bar{z}^d$ is conformally mateable with every anti-polynomial P of degree d , we conclude from Proposition 10.18 that $\mathbf{\Gamma}_{d+1}$ is mateable with every postcritically finite anti-polynomial P of degree d . In the particular case of $d = 2$, the mating of $\mathbf{\Gamma}_3$ and $\bar{z}^2$ is realized as the Schwarz reflection map with respect to a deltoid [LLMM18a, §5], and the matings of $\mathbf{\Gamma}_3$ and all other postcritically finite quadratic anti-polynomials are realized as the Schwarz reflection maps associated with a fixed cardioid and a family of circumscribed circles [LLMM18b].

2. Since $\bar{z}^d$ can be conformally mated with every critically fixed degree d anti-polynomial, it follows once again from Proposition 10.18 and the fact that the limit set of each group in the Bers slice closure is homeomorphic to the Julia set of some critically fixed anti-polynomial (see the comments before Proposition 10.18) that all necklace reflection groups can be mated with the anti-polynomial $\bar{z}^d$. By [LMM20], these conformal matings are realized as Schwarz reflection maps associated with the quadrature domains $f(\mathbb{D}^*)$, where

$$f \in \Sigma_d^* := \left\{ g(z) = z + \frac{a_1}{z} + \cdots + \frac{a_d}{z^d} : a_d = -\frac{1}{d} \text{ and } g|_{\mathbb{D}^*} \text{ is conformal} \right\}.$$

11.2. Schwarz reflections in an ellipse and two inscribed disks. Consider the anti-polynomial $P_1(z) = \bar{z}^3 - \frac{3i}{\sqrt{2}}\bar{z}$. Each finite critical point of P_1 forms a 2-cycle (see Figure 6).

Note that the $1/4$ and $1/2$ rays (respectively, the 0 and $3/4$ rays) of P_1 land at a common fixed point on $\mathcal{J}(P_1)$. They cut $\mathcal{K}(P_1)$ into three components. We will denote the component containing the two critical points by $\mathcal{K}_1(P_1)$, the component containing the critical value in the left half-plane (respectively, the critical value in the right half-plane) by $\mathcal{K}_2(P_1)$ (respectively, $\mathcal{K}_3(P_1)$). We have the following mapping properties of the action of P_1 on its filled Julia set.

- (1) $P_1 : \mathcal{K}_i(P_1) \rightarrow \mathcal{K}(P_1) \setminus \overline{\mathcal{K}_i(P_1)}$ has degree 1, for $i \in \{2, 3\}$, and
- (2) $P_1 : P_1^{-1}(\mathcal{K}_1(P_1)) \cap \mathcal{K}_1(P_1) \rightarrow \mathcal{K}_1(P_1)$ has degree 1.

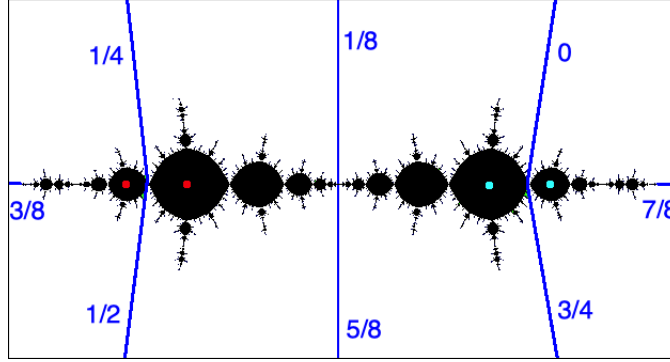


FIGURE 6. The dynamical plane of $P_1(z) = \bar{z}^3 - \frac{3i}{\sqrt{2}}\bar{z}$; each critical point of P_1 forms a 2-cycle (the figure displayed is a $\frac{\pi}{4}$ -rotate of the actual dynamical plane). The external dynamical rays of period 1 and 2 are marked.

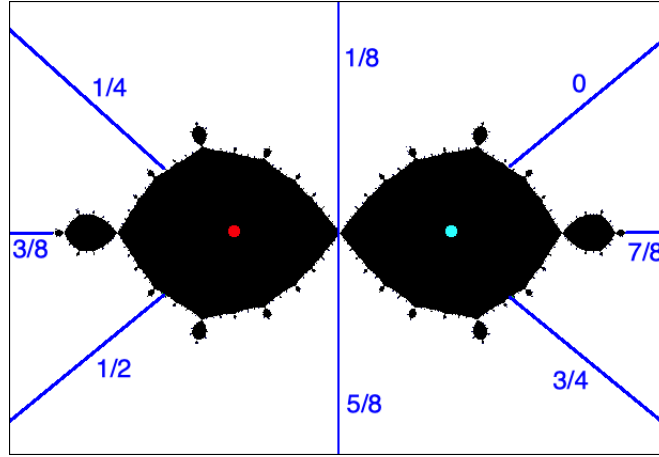


FIGURE 7. The dynamical plane of the critically fixed cubic anti-polynomial $P_\Gamma(z) = \bar{z}^3 - \frac{3i}{2}\bar{z}$ (the figure displayed is a $\frac{\pi}{4}$ -rotate of the actual dynamical plane). The external dynamical rays of period 1 and 2 are marked.

Also consider the cusp reflection group Γ shown in Figure 5, and the associated reflection map ρ_Γ . The monic, centered, critically fixed anti-polynomial P_Γ associated to Γ in Subsection 10.3 is given by $P_\Gamma(z) = \bar{z}^3 - \frac{3i}{2}\bar{z}$ (see Figure 7).

We will now argue that P_Γ and P_1 are conformally mateable. Note that the only rays landing at the separating repelling fixed point (i.e., the repelling fixed point that is a cut-point of the filled Julia set) of the critically fixed anti-polynomial P_Γ have angles $1/8$ and $5/8$, while for the other anti-polynomial P_1 , the rays at angles $-1/8 = 7/8$ and $-5/8 = 3/8$ land at non-cut points of $\mathcal{J}(P_1)$. Therefore, the principal ray equivalence class for P_Γ and P_1 contains no cycle, and hence by [LLM20, Corollary 4.22, Figure 4.2], the maps P_Γ and P_1 are conformally mateable.

By Proposition 10.18, there exists a conformal mating $F : \overline{\Omega} \rightarrow \widehat{\mathbb{C}}$ of P_1 and Γ . We set $T^0(F) := \psi_\Gamma(T^0(\Gamma))$. Since ψ_Γ is conformal on $\text{int } \mathcal{K}(\Gamma)$, each of the two components of $T^0(F)$ is a topological triangle with its vertices removed. Moreover, by Lemma 10.17, each component of Ω is a simply connected quadrature domain, and F is the piecewise defined Schwarz reflection map of these quadrature domains.

According to [LMM20, Lemma 4.22 (part i)], the four cusp points of $\partial T^0(\Gamma)$ (that are not cut-points of $\partial T^0(\Gamma)$) have external angles $0, 1/4, 1/2$, and $3/4$ (as points on $\Lambda(\Gamma)$). Since the $1/4$ and $1/2$ rays (respectively, the 0 and $3/4$ rays) of P_1 land at a common fixed point, the landing points of the $1/2$ and $3/4$ rays (respectively, the landing points of the 0 and $1/4$ rays) of P_Γ are identified in the Julia set of the conformal mating of P_Γ and P_1 . There is no other identification involving the landing points of the fixed rays of P_Γ . It follows from Relation (10.3) that each vertex of a component of $T^0(F)$ is identified with a vertex of the other component of $T^0(F)$. Hence, Ω has three connected components Ω_1, Ω_2 , and Ω_3 such that

- (1) each Ω_i is a Jordan quadrature domain,
- (2) $\psi_{P_1}(\mathcal{K}_i(P_1)) \subset \overline{\Omega_i}$, $i = 1, 2, 3$,
- (3) $\partial\Omega_i \cap \partial\Omega_j$ is a singleton, for each $i \neq j \in \{1, 2, 3\}$.

(See Figure 8.)

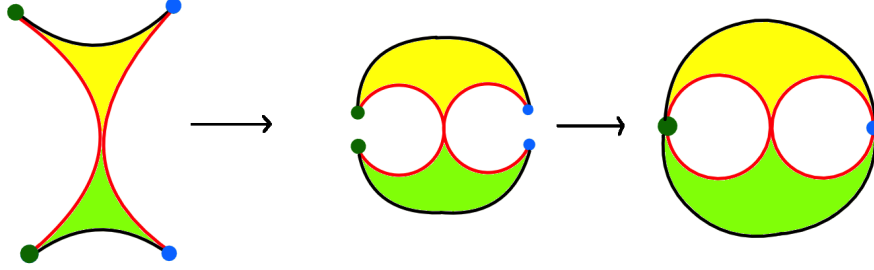


FIGURE 8. The figure depicts the identifications on the boundary of the fundamental domain $T^0(\Gamma)$ that give rise to three pairwise touching Jordan domains.

Denoting the Schwarz reflection maps of Ω_i by σ_i , we have that

$$F(w) = \sigma_i(w), \text{ if } w \in \overline{\Omega_i}.$$

It follows from the mapping degrees of P_1 on $\mathcal{K}_1(P_1)$, $\mathcal{K}_2(P_1)$, and $\mathcal{K}_3(P_1)$ that $\sigma_1 : \sigma_1^{-1}(\text{int } \Omega_1) \rightarrow \text{int } \Omega_1$ is a branched cover of degree 2, and $\sigma_i : \sigma_i^{-1}(\text{int } \Omega_i^c) \rightarrow \text{int } \Omega_i^c$ has degree 1, for $i \in \{2, 3\}$. By Proposition 6.12, there exist rational maps f_1, f_2 , and f_3 such that

- (1) $f_i : \mathbb{D}^* \rightarrow \Omega_i$ are univalent, for $i \in \{1, 2, 3\}$,
- (2) $F|_{\Omega_i} \equiv f_i \circ (1/\bar{\zeta}) \circ (f_i|_{\mathbb{D}^*})^{-1}$, and
- (3) $\deg f_1 = 2$, $\deg f_i = 1$, $i \in \{2, 3\}$.

Note that 0 is a repelling fixed point of P_1 . Conjugating F by a Möbius map, we can assume that $\psi_{P_1}(0) = \infty \in \Omega_1$, and hence, $F(\infty) = \infty$. We can choose f_1

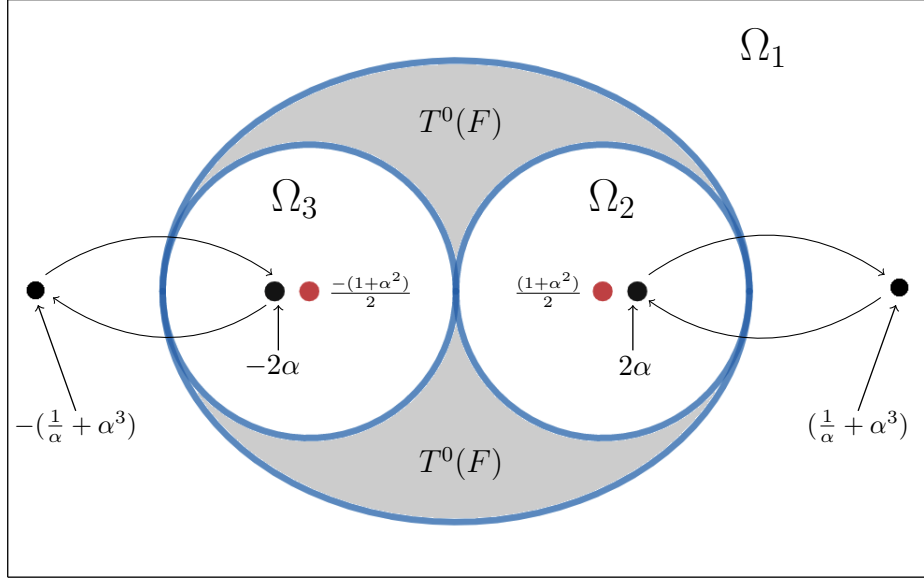


FIGURE 9. The conformal mating of $P_1(z) = \bar{z}^3 - \frac{3i}{\sqrt{2}}\bar{z}$ and the necklace group from Figure 5 is given by the piecewise Schwarz reflection map F associated with the quadrature domains Ω_i , $i \in \{1, 2, 3\}$; where Ω_1 is the exterior of the ellipse $\frac{x^2}{(1+\alpha^2)^2} + \frac{y^2}{(1-\alpha^2)^2} = 1$, and Ω_2, Ω_3 are the round disks $|z \pm \frac{1+\alpha^2}{2}| = \frac{1+\alpha^2}{2}$ ($\alpha = \frac{1}{2} \left((1 + \sqrt{5}) - \sqrt{2 + 2\sqrt{5}} \right)$). The two components of $T^0(F)$ are topological triangles with vertices removed. Each vertex of a component of $T^0(F)$ is identified with a vertex of the other component of $T^0(F)$. Each of the two critical points of F forms a 2-cycle.

so that $f_1(\infty) = \infty$ and $f'_1(\infty) > 0$; i.e.,

$$f_1(z) = \frac{az^2 + bz + c}{z + d},$$

where $a > 0$, $b, c, d \in \mathbb{C}$. Since ∞ is a fixed point of F , it follows from the formula

$$F|_{\Omega_1} \equiv f_1 \circ (1/\bar{\zeta}) \circ (f_1|_{\mathbb{D}^*})^{-1}$$

that $f_1(0) = \infty$; i.e., $d = 0$. Thus, f_1 reduces to the form

$$f_1(z) = az + b + c/z.$$

We are now only allowed to conjugate F by affine maps as conjugating by non-affine maps will, in general, destroy the normalization $f_1(0) = \infty$. Conjugating F by a translation, we can now assume that $f_1(z) = az + c/z$. Finally, conjugating F by a dilation and rotation, we can choose f_1 to be

$$f_1(z) = z + \alpha^2/z,$$

for some $\alpha > 0$.

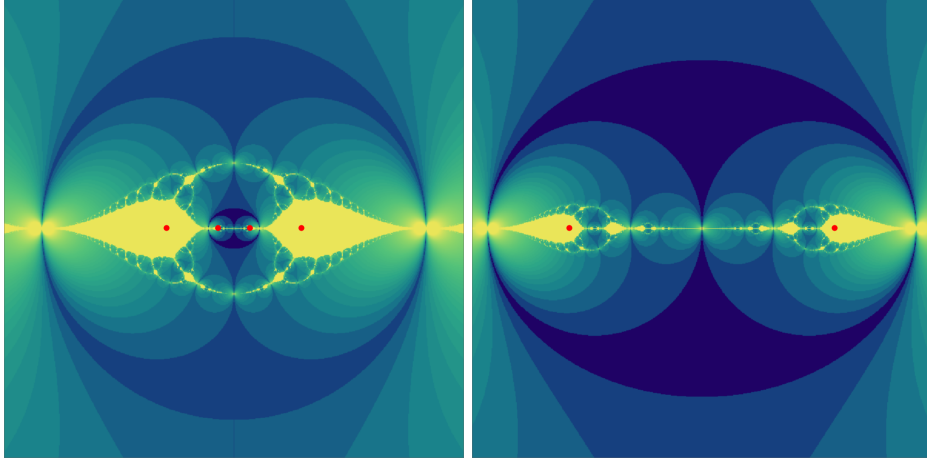


FIGURE 10. Left: The dynamical plane of F is shown. The blue/green region is the tiling set and the yellow region is the non-escaping set. The large yellow components contain the two 2-periodic critical points (in red) of F . The corresponding critical values lie inside the ellipse. Right: A blow-up of the interior of the ellipse is displayed with the two critical values marked in red. The two dark blue topological triangles are the components of $T^0(F)$.

A direct computation now shows that Ω_1 is the exterior of the ellipse

$$\frac{x^2}{(1+\alpha^2)^2} + \frac{y^2}{(1-\alpha^2)^2} = 1.$$

The major and minor axes of the ellipse are along the real and imaginary axes. Moreover, the quadrature domains Ω_2 and Ω_3 are interiors of round circles (contained in the interior of the ellipse) with $\partial\Omega_i \cap \partial\Omega_j$ a singleton, for each $i \neq j \in \{1, 2, 3\}$.

We note that since the critical points of f_1 are $\pm\alpha$, it is easy to see by direct computation that the critical points of σ_1 are $\pm(1/\alpha + \alpha^3)$. The corresponding critical values of σ_1 are $\pm 2\alpha$. Since the critical points of P_1 are 2-periodic, the same is true for the conformal mating F of Γ and P_1 . Therefore, the circle reflection σ_2 (respectively, σ_3) maps $2\alpha \in \mathbb{R}$ (respectively, $-2\alpha \in \mathbb{R}$) to $(1/\alpha + \alpha^3) \in \mathbb{R}$ (respectively, $-(1/\alpha + \alpha^3) \in \mathbb{R}$). Hence, the centers of the circles $\partial\Omega_i$ ($i = 2, 3$) must lie on the real axis. Finally, by symmetry, the radii of the circles $\partial\Omega_i$ ($i = 2, 3$) must be equal; i.e., the centers of the circles are at $\pm \frac{1+\alpha^2}{2}$, and the common radius is $\frac{1+\alpha^2}{2}$. A direct computation using the fact $\sigma_1(2\alpha) = 1/\alpha + \alpha^3$ now yields that

$$\alpha = \frac{1}{2} \left((1 + \sqrt{5}) - \sqrt{2 + 2\sqrt{5}} \right).$$

Proposition 11.1. *Let $P_1(z) = \bar{z}^3 - \frac{3i}{\sqrt{2}}\bar{z}$, Γ be the cusp reflection group from Figure 5, and $\alpha = \frac{1}{2} \left((1 + \sqrt{5}) - \sqrt{2 + 2\sqrt{5}} \right)$. Then, the piecewise defined Schwarz*

reflection map F in the exterior of the ellipse

$$\frac{x^2}{(1+\alpha^2)^2} + \frac{y^2}{(1-\alpha^2)^2} = 1,$$

and in the interiors of the circles

$$\left| z \pm \frac{1+\alpha^2}{2} \right| = \frac{1+\alpha^2}{2}$$

is a conformal mating of P_1 and Γ .

11.3. Schwarz reflections in an extremal quadrature domain and a circumscribed disk. Consider the group Γ as in Subsection 11.2 and the unicritical cubic anti-polynomial $P_2(z) = \bar{z}^3 + \frac{(1+i)}{\sqrt{2}}$, which has a superattracting 2-cycle (see Figure 11).

Note that the 0 and $1/4$ rays of P_2 land at a common fixed point ζ , and cut $\mathcal{K}(P_2)$ into two components. We will denote the component containing the critical value (respectively, the critical point) by $\mathcal{K}_1(P_2)$ (respectively, $\mathcal{K}_2(P_2)$). Note that the set $\mathcal{K}_1(P_2) \cup \{\zeta\}$ is mapped injectively onto $\mathcal{K}_2(P_2) \cup \{\zeta\}$ under P_2 . On the other hand, under P_2 , the set $\mathcal{K}_2(P_2) \cup \{\zeta\}$ covers itself twice, and $\mathcal{K}_1(P_2) \cup \{\zeta\}$ thrice.

As in the Subsection 11.2, one can show that P_Γ (defined in Subsection 11.2) and P_2 are conformally mateable. Indeed, the only rays landing at the separating repelling fixed point (i.e., the repelling fixed point that is a cut-point of the filled Julia set) of the critically fixed anti-polynomial P_Γ have angles $1/8$ and $5/8$, while for the other anti-polynomial P_2 , the rays at angles $-1/8 = 7/8$ and $-5/8 = 3/8$ land at non-cut points of $\mathcal{J}(P_2)$. Therefore, the principal ray equivalence class for P_Γ and P_2 contains no cycle, and hence by [LLM20, Corollary 4.22], the maps P_Γ and P_2 are conformally mateable.

By Proposition 10.18, there exists a conformal mating $F : \bar{\Omega} \rightarrow \hat{\mathbb{C}}$ of P_1 and Γ . We set $T^0(F) := \psi_\Gamma(T^0(\Gamma))$. Since ψ_Γ is conformal on $\text{int } \mathcal{K}(\Gamma)$, each of the two components of $T^0(F)$ is a topological triangle with its vertices removed. Moreover, by Lemma 10.17, each component of Ω is a simply connected quadrature domain, and F is the piecewise defined Schwarz reflection map of these quadrature domains.

Note that the four cusp points of $\partial T^0(\Gamma)$ (that are not cut-points of $\partial T^0(\Gamma)$) have external angles $0, 1/4, 1/2$, and $3/4$ (as points on $\Lambda(\Gamma)$). Since the 0 and $1/4$ rays of P_2 land at a common fixed point, the landing points of the 0 and $3/4$ rays of P_Γ are identified in the Julia set of the conformal mating of P_Γ and P_2 . There is no other identification involving the landing points of the fixed rays of P_Γ . It follows that two vertices of a component of $T^0(F)$ are identified. Hence, Ω has two connected components Ω_1 and Ω_2 such that

- (1) Ω_1 and Ω_2 are simply connected quadrature domains,
- (2) $\psi_{P_2}(\mathcal{K}_1(P_2)) \subset \bar{\Omega}_1$, and $\psi_{P_2}(\mathcal{K}_2(P_2)) \subset \bar{\Omega}_2$,
- (3) Ω_1 is a Jordan domain, and $\partial\Omega_2$ has a unique cut-point,
- (4) $\partial\Omega_1 \cap \partial\Omega_2$ is a singleton.

Denoting the Schwarz reflection maps of Ω_i by σ_i , we have that

$$F(w) = \begin{cases} \sigma_1(w) & \text{if } w \in \bar{\Omega}_1, \\ \sigma_2(w) & \text{if } w \in \bar{\Omega}_2. \end{cases}$$

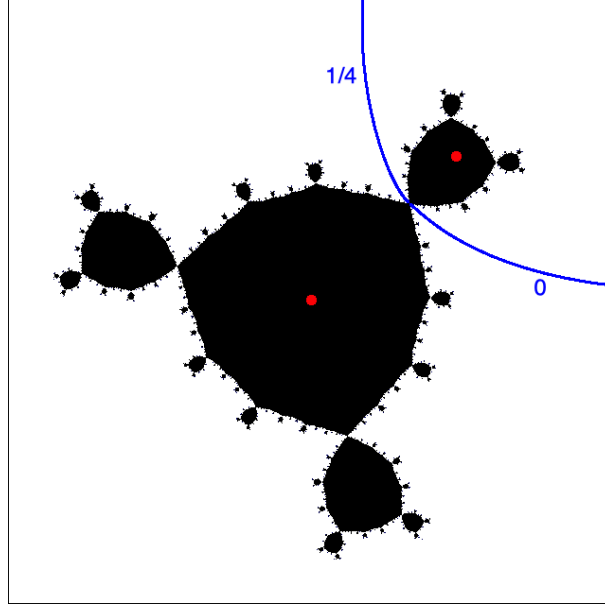


FIGURE 11. The dynamical plane of the unicritical cubic anti-polynomial $P_2(z) = \bar{z}^3 + \frac{(1+i)}{\sqrt{2}}$. The critical point of P_2 is 2-periodic.

Conjugating F by a Möbius map, we can assume that unique critical point of F is $0 \in \Omega_2$, the unique critical value is $\infty \in \Omega_1$, and the conformal radius of Ω_2 (with conformal center at 0) is 1.

By mapping properties of P_2 , we have that $\sigma_1 : \sigma_1^{-1}(\text{int } \Omega_1^c) \rightarrow \text{int } \Omega_1^c$ has degree 1, and $\sigma_2 : \sigma_2^{-1}(\text{int } \Omega_2^c) \rightarrow \text{int } \Omega_2^c$ is a branched cover of degree 3. By Proposition 6.12, there exist rational maps f_1 and f_2 such that

- (1) $f_i : \mathbb{D} \rightarrow \Omega_i$ is univalent,
- (2) $F|_{\Omega_i} \equiv f_i \circ (1/\zeta) \circ (f_i|_{\mathbb{D}})^{-1}$, and
- (3) $\deg f_1 = 1$, $\deg f_2 = 3$.

The assumption that the conformal radius of Ω_2 (with conformal center at 0) is 1 allows us to normalize f_2 so that $f_2(0) = 0$ and $f_2'(0) = 1$. By the dynamics of F , it now follows that f_2 has a triple pole at ∞ ; so f_2 is a cubic polynomial.

Let $f_2(z) = z + az^2 + bz^3$. Since F (in particular, σ_2) has a unique critical point, the two finite critical points of f_2 must lie on $\mathbb{S}^1$. A simple calculation now implies that $|b| = 1/3$. Conjugating f_2 by a rotation, we can assume that

$$f_2(z) = z + az^2 + z^3/3,$$

for some $a \in \mathbb{C}$. We will now use the condition that $\partial\Omega_2$ has a unique cut-point (equivalently, a double point) to determine a . In fact, a simple numerical computation shows that the only map f_2 of the above form with a double point on $f_2(\mathbb{S}^1)$ is $f_2(z) = z + \frac{2\sqrt{2}}{3}z^2 + \frac{z^3}{3}$ (up to conjugation by $z \mapsto -z$). Below we give a rigorous proof of this fact.

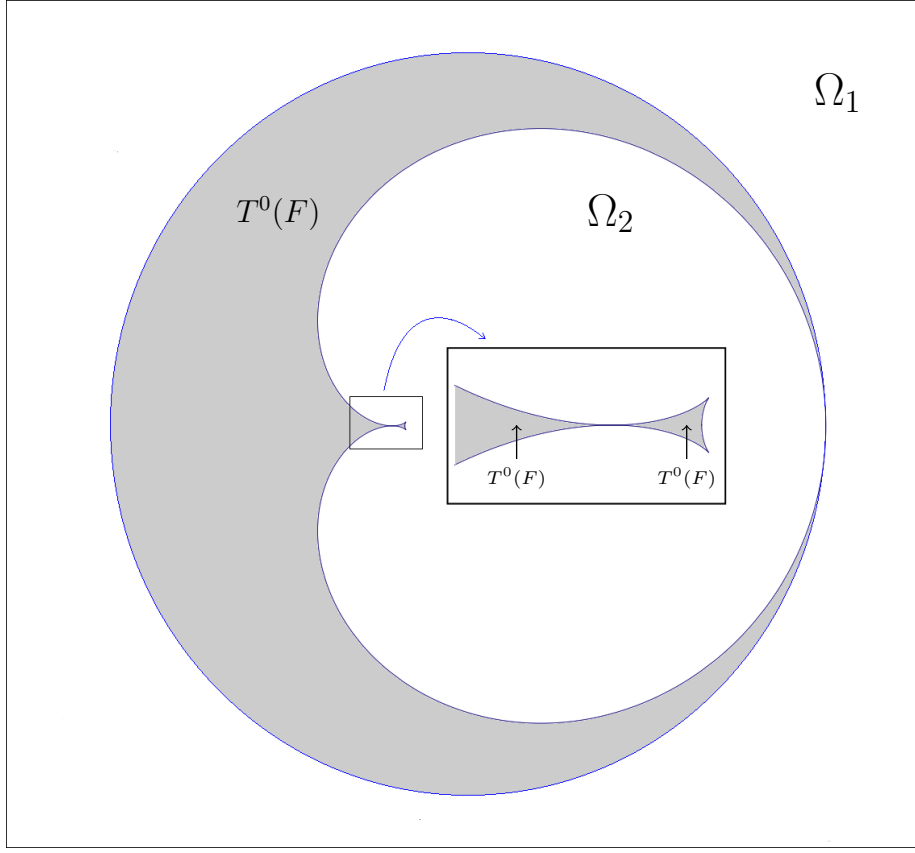


FIGURE 12. The conformal mating of $P_2(z) = \bar{z}^3 + \frac{1+i}{\sqrt{2}}$ and the necklace group from Figure 5 is given by the piecewise Schwarz reflection map F associated with the quadrature domains Ω_i , $i \in \{1, 2\}$; where $\Omega_1 = \{|z| > \frac{4}{3} + \frac{2\sqrt{2}}{3}\}$, and $\Omega_2 = f(\mathbb{D})$, $f(z) = z + \frac{2\sqrt{2}}{3}z^2 + \frac{z^3}{3}$. The two components of $T^0(F)$ are topological triangles with vertices removed. The closure of one of the components of $T^0(F)$ is a topological triangle, and two vertices of the other component of $T^0(F)$ are identified. The unique critical point 0 of F forms a 2-cycle $0 \leftrightarrow \infty$.

Consider the space of cubic polynomials

$$S_3^* := \{f(z) = z + az^2 + z^3/3 : a \in \mathbb{C}, f|_{\mathbb{D}} \text{ is univalent}\}.$$

By [Bra67, Theorem 2], the maps $z \pm \frac{2\sqrt{2}}{3}z^2 + \frac{z^3}{3}$ maximize the absolute value of the coefficient of z^2 in the space S_3^* . Moreover, for any $f \in S_3^*$, we have $a \in \mathbb{R}$ (the fact that the coefficient of z^3 is $1/3$ implies that both critical points of f lie on $\mathbb{S}^1$). Hence, the maps $z \pm \frac{2\sqrt{2}}{3}z^2 + \frac{z^3}{3}$ are extremal points of S_3^* ; i.e., they cannot be written as proper convex combinations of two distinct maps in S_3^* . It now follows from [LM14, Theorem 3.1] that the image of $\mathbb{S}^1$ under each of the maps $z \pm \frac{2\sqrt{2}}{3}z^2 + \frac{z^3}{3}$ has a double point. Finally, by [LMM19, Theorem B], up to

conjugation by $z \mapsto -z$, there is a unique member f in S_3^* with the property that $f(\mathbb{S}^1)$ has a double point. Therefore, we can choose $f_2(z) = z + \frac{2\sqrt{2}}{3}z^2 + \frac{z^3}{3}$.

Since $F(\infty) = 0$, it follows that Ω_1 is the exterior of a round circle centered at the origin. It is easy to see that $\text{dist}(0, f_2(\mathbb{S}^1)) = d(0, f_2(1)) = \frac{4}{3} + \frac{2\sqrt{2}}{3}$. As $\partial\Omega_1$ touches $\partial\Omega_2 = f_2(\mathbb{S}^1)$, it follows that Ω_1 is the exterior of the circle $\{|z| = \frac{4}{3} + \frac{2\sqrt{2}}{3}\}$.

Proposition 11.2. *Let $P_2(z) = \bar{z}^3 + \frac{(1+i)}{\sqrt{2}}\bar{z}$, Γ be the cusp reflection group from Figure 5, and $f(z) = z + \frac{2\sqrt{2}}{3}z^2 + \frac{z^3}{3}$. Then, the piecewise defined Schwarz reflection map F in the quadrature domains $f(\mathbb{D})$ and $\{|z| > \frac{4}{3} + \frac{2\sqrt{2}}{3}\}$ is a conformal mating of P_2 and Γ .*

11.4. Mating the cauliflower anti-polynomial with the ideal triangle group.

In this subsection, we will prove the existence of the conformal mating of the ‘cauliflower’ anti-polynomial $P(z) = \bar{z}^2 + 1/4$ and the ideal triangle reflection group, and give an explicit description of this conformal mating.

Note that the parabolic basin of attraction of P is equal to $\text{int } \mathcal{K}(P)$, and it is a Jordan domain. Mapping $\mathcal{K}(P)$ to $\mathbb{D}^c = \widehat{\mathbb{C}} \setminus \mathbb{D}$ by a Riemann map ϕ that sends the critical point 0 (of P) to ∞ and the parabolic fixed point $1/2$ to 1, we see that $\phi \circ P \circ \phi^{-1} : \mathbb{D}^c \rightarrow \mathbb{D}^c$ is equal to the anti-Blaschke product

$$B(z) = \frac{3\bar{z}^2 + 1}{3 + \bar{z}^2}$$

(compare [DH07, Exposé IX, §II, Corollary 1]). In other words, $P|_{\mathcal{K}(P)}$ is conformally conjugate to $B|_{\mathbb{D}^c}$.

Lemma 11.3. *There is a homeomorphism $H : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ with $H(1) = 1$ that conjugates B to ρ_2 , and extends as a David homeomorphism of $\mathbb{D}$.*

Proof. Recall from Examples 3.4 and 3.5 that both $B|_{\mathbb{S}^1}$ and $\rho_2|_{\mathbb{S}^1}$ are expansive.

Note that B has three fixed points on $\mathbb{S}^1$; namely, at 1 and $\left(\frac{-1 \pm 2\sqrt{2}i}{3}\right)$. Moreover, 1 is a parabolic fixed point of B , while the other two are repelling. On the other hand, all three fixed points $1, \omega, \omega^2$ of the Nielsen map ρ_2 of the ideal triangle reflection group Γ_2 are parabolic. We consider the Markov partitions $\mathcal{P}\left(B, \left\{1, -\frac{1}{3} + \frac{2\sqrt{2}i}{3}, -\frac{1}{3} - \frac{2\sqrt{2}i}{3}\right\}\right)$ and $\mathcal{P}\left(\rho_2, \{1, \omega, \omega^2\}\right)$.

It was shown in Example 4.3 that ρ_2 admits piecewise conformal extensions satisfying conditions (4.1) and (4.2) (with respect to $\mathcal{P}(\rho_2, \{1, \omega, \omega^2\})$). Thanks to Theorem 4.9, it now suffices to prove that the map $B|_{\mathbb{S}^1}$ also admits piecewise conformal extensions satisfying conditions (4.1) and (4.2) (with respect to the partition $\mathcal{P}\left(B, \left\{1, -\frac{1}{3} + \frac{2\sqrt{2}i}{3}, -\frac{1}{3} - \frac{2\sqrt{2}i}{3}\right\}\right)$). In fact, the desired extension of $B|_{\mathbb{S}^1}$ near the fixed points satisfying condition (4.2) is given by $z \mapsto 1/\overline{B(z)}$. We now proceed to find open neighborhoods $U_i, V_i, i \in \{1, 2, 3\}$, such that U_i s (respectively, V_i s) contain the interiors of the above Markov partition pieces (respectively, the B -images of the Markov partition pieces), the map $z \mapsto 1/\overline{B(z)}$ carries U_i onto V_i conformally, and the sets satisfy condition (4.1).

To this end, we set

$$\tilde{B}(z) := \frac{1}{\overline{B(z)}} = \frac{3 + z^2}{3z^2 + 1}.$$

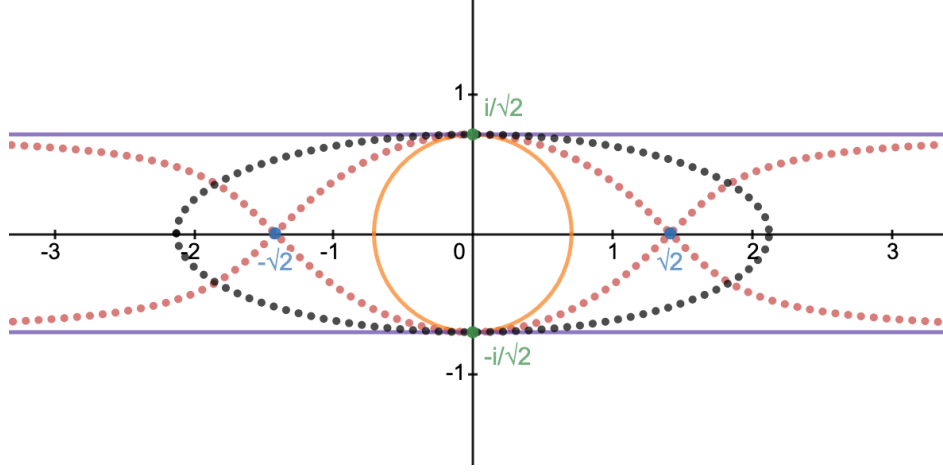


FIGURE 13. The set $\check{U}_1$ (respectively, $\check{U}_2$) is the region above (respectively, below) the purple horizontal line $y = \frac{1}{\sqrt{2}}$ (respectively, $y = -\frac{1}{\sqrt{2}}$), and the set $\check{U}_3$ is the disk bounded by the orange circle $\{x^2 + y^2 = \frac{1}{2}\}$. The $\check{B}$ -image of $\check{U}_1$ (respectively, of $\check{U}_2$) is the region below (respectively, above) the dotted red curve passing through $\frac{i}{\sqrt{2}}$ and $\pm\sqrt{2}$ (respectively, through $-\frac{i}{\sqrt{2}}$ and $\pm\sqrt{2}$). Finally, the $\check{B}$ -image of $\check{U}_3$ is the exterior of the dotted black ellipse $\frac{2x^2}{9} + 2y^2 = 1$.

It will be convenient to conjugate $\check{B}$ by a Möbius map

$$M(z) = \frac{z+1}{z-1},$$

that preserves the real line, sends 1 to ∞ , the other two fixed points to $\frac{-i}{\sqrt{2}}$, and the unit circle to the imaginary axis. The conjugated map is given by

$$\check{B}(w) := M \circ \check{B} \circ M^{-1}(w) = -w - \frac{1}{w}.$$

By construction, M carries the interiors of the Markov partition pieces (in the B -plane) to $(-i\infty, \frac{-i}{\sqrt{2}})$, $(\frac{-i}{\sqrt{2}}, \frac{i}{\sqrt{2}})$, and $(\frac{i}{\sqrt{2}}, i\infty)$ (in the $\check{B}$ -plane). We now set

$$\check{U}_1 = \left\{ \operatorname{Im}(w) > \frac{1}{\sqrt{2}} \right\}, \quad \text{and} \quad \check{U}_2 = \left\{ \operatorname{Im}(w) < \frac{-1}{\sqrt{2}} \right\}.$$

The image $\check{V}_1$ of $\check{U}_1$ under $\check{B}$ is the region below the curve

$$\left\{ \left(1 + \frac{2x^2}{(2 + \sqrt{2}y)^2} \right) (1 + \sqrt{2}y) = 2 \right\},$$

that passes through $\frac{i}{\sqrt{2}}$, intersects the real line at $\pm\sqrt{2}$, and is contained in the horizontal strip $\left\{ \frac{-1}{\sqrt{2}} \leq \operatorname{Im}(w) \leq \frac{1}{\sqrt{2}} \right\}$. Likewise, the image $\check{V}_2$ of $\check{U}_2$ under $\check{B}$ is the

region above the curve

$$\left\{ \left(1 + \frac{2x^2}{(2 - \sqrt{2}y)^2} \right) (1 - \sqrt{2}y) = 2 \right\},$$

that passes through $\frac{-i}{\sqrt{2}}$, intersects the real line at $\pm\sqrt{2}$, and is contained in the horizontal strip $\left\{ \frac{-1}{\sqrt{2}} \leq \text{Im}(w) \leq \frac{1}{\sqrt{2}} \right\}$. Moreover, $\check{B}$ maps $\check{U}_1, \check{U}_2$ conformally onto $\check{V}_1, \check{V}_2$. Finally, we define

$$\check{U}_3 := \{|z| < 1/\sqrt{2}\} \subset \check{B}(\check{U}_1) \cap \check{B}(\check{U}_2).$$

Then, $\check{U}_3$ does not contain the critical points ± 1 of β . Furthermore,

$$\check{B} : \check{U}_3 \rightarrow \check{V}_3$$

is a conformal isomorphism, where $\check{V}_3$ is the exterior of an ellipse with $\check{V}_3 \supset \check{U}_1 \cup \check{U}_2$ (see Figure 13). Transporting the sets $\check{U}_i, \check{V}_i$, $i \in \{1, 2, 3\}$, back to the B -plane via the change of coordinate M , we obtain our desired open sets U_i, V_i , $i \in \{1, 2, 3\}$ satisfying condition (4.1). $\square$

We define an orientation-reversing continuous map on a subset of $\mathbb{S}^2$:

$$\tilde{B} = \begin{cases} H^{-1} \circ \rho_2 \circ H, & \text{in } \mathbb{D} \setminus \text{int } H^{-1}(\Pi), \\ B, & \text{in } \mathbb{D}^*, \end{cases}$$

where Π is a regular ideal triangle in $\mathbb{D}$. We define $\mu|_{\mathbb{D}}$ to be the pullback of the standard complex structure on $\mathbb{D}$ by the map H , and set μ equal to zero everywhere else. Since H is a David homeomorphism of $\mathbb{D}$, it follows that μ is a David coefficient on $\hat{\mathbb{C}}$; i.e., it satisfies condition (2.1).

Theorem 2.1 then gives us an orientation-preserving homeomorphism Ψ of $\hat{\mathbb{C}}$ such that the pullback of the standard complex structure under Ψ is equal to μ . Setting $\Omega := \hat{\mathbb{C}} \setminus \overline{\Psi(H^{-1}(\Pi))}$, we see as in the proof of Lemma 7.1 that the map

$$\sigma := \Psi \circ \tilde{B} \circ \Psi^{-1} : \bar{\Omega} \rightarrow \hat{\mathbb{C}}$$

is continuous and anti-analytic on Ω . Moreover, $B|_{\mathbb{D}^c}$ is conformally conjugate to $\sigma|_{\Psi(\mathbb{D}^c)}$ via Ψ , and $\rho_2|_{\mathbb{D}}$ is conformally conjugate to $\sigma|_{\Psi(\mathbb{D})}$ via $\Psi \circ H^{-1}$ (conformality of $\Psi \circ H^{-1}$ follows from Theorem 2.2).

Since $P|_{\mathcal{K}(P)}$ is conformally conjugate to $B|_{\mathbb{D}^c}$, it follows from the previous paragraph and Definition 10.16 that σ is a conformal mating of P and the ideal triangle group Γ_2 . Moreover, by Theorem 2.7, $\Psi(\mathbb{S}^1)$ is conformally removable. We claim that σ is the unique (up to Möbius conjugacy) conformal mating of P and Γ_2 . Indeed, if $\tilde{\sigma}$ is another conformal mating of P and Γ_2 , then there exists a homeomorphism of $\hat{\mathbb{C}}$, conformal on $\hat{\mathbb{C}} \setminus \Psi(\mathbb{S}^1)$, conjugating σ to $\tilde{\sigma}$. But the conformal removability of $\Psi(\mathbb{S}^1)$ implies that this homeomorphism is a Möbius map; i.e., σ and $\tilde{\sigma}$ are Möbius conjugate.

Note that P and ρ_2 commute with $\iota : z \mapsto \bar{z}$. It follows that $\iota \circ \sigma \circ \iota$ is another conformal mating of P and ρ_2 . If we normalize Ψ so that the unique critical point, critical value, and ‘parabolic’ fixed point of σ are real, then by uniqueness of the conformal mating of P and ρ_2 , the maps σ and $\iota \circ \sigma \circ \iota$ must be conjugate via a Möbius map M fixing these three dynamically marked points. Hence, $M = \text{id}$, and $\sigma = \iota \circ \sigma \circ \iota$. Here is an alternative way of seeing the real-symmetry of σ . By

Remark 2.4, the David homeomorphism H is real-symmetric. It follows from the construction that the map $\tilde{B}$ and the David coefficient μ are also real-symmetric. By the uniqueness part of Theorem 2.1, we conclude that the David homeomorphism Ψ is real-symmetric, from which real-symmetry of σ follows.

We now proceed to an explicit characterization of σ . By construction, Ω is a Jordan domain. Also note that by Lemma 10.17, Ω is a quadrature domain, and σ is its Schwarz reflection map. Since σ commutes with the complex conjugation map, the domain Ω is real-symmetric. Since $\sigma : \sigma^{-1}(\text{int } \Omega^c) \rightarrow \text{int } \Omega^c$ has degree 3, Proposition 6.12 now provides us with a rational map R of degree 3 such that $R : \mathbb{D}^c \rightarrow \bar{\Omega}$ is univalent. Since Ω is real-symmetric, we can assume that R has real coefficients.

For each critical point ξ of R in $\mathbb{D}$, the point $R(1/\bar{\xi}) \in \Omega$ is a critical point of σ . Since σ has exactly one (simple) critical point, it follows that R has exactly one (simple) critical point in $\mathbb{D}$. By the real-symmetry of R , this critical point must be real. By the univalence of $R|_{\mathbb{D}^c}$, the other three critical points of R lie on $\mathbb{S}^1$ and are simple. Once again, the real-symmetry of R implies that one of these two critical points is real, and the other two are complex conjugates of each other.

Note that postcomposing R with Möbius transformations and precomposing R with Möbius maps that preserve the unit disk do not change the Möbius conjugacy class of σ . Thus, pre and postcomposing R with real-symmetric Möbius maps, we can assume the following:

- (1) $R|_{\mathbb{D}^c}$ is univalent,
- (2) $\overline{R(z)} = R(\bar{z})$,
- (3) $\text{Crit}(R) = \{0, 1, \alpha, \bar{\alpha}\}$ for some $\alpha \in \mathbb{S}^1 \setminus \{\pm 1\}$,
- (4) $\sigma^{\circ n} \rightarrow R(1)$ locally uniformly on $\Psi(\mathbb{D}^c)$,
- (5) $R(\infty) = \infty$, $R'(\infty) = 1$, and
- (6) $R(0) = 2$.

The conditions that $R(\infty) = \infty$, $R'(\infty) = 1$, and $R(0) = 2$ immediately imply that

$$R(z) = \frac{z^3 + az^2 + bz + 2c}{z^2 + dz + c},$$

for some $a, b, c, d \in \mathbb{R}$ with $c \neq 0$. The fact that 0 is a critical point of R now shows that $b = 2d$; and hence,

$$\implies R'(z) = \frac{z(z^3 + 2dz^2 + (ad + 3c - 2d)z + (2ac - 4c))}{(z^2 + dz + c)^2}.$$

The fact that the cubic map R is univalent on $\mathbb{D}^c$ and fixes ∞ implies that none of the critical points $1, \alpha, \bar{\alpha}$ is a pole of R . Hence, in light of Vieta's formulas, the expression of R' obtained above yields that

$$(11.1) \quad 1 + \alpha + \bar{\alpha} = -2d = ad + 3c - 2d, \quad 4c - 2ac = 1.$$

Hence, $a = \frac{4c-1}{2c}$, and $d = \frac{6c^2}{1-4c}$ ($c \neq \frac{1}{4}$). Therefore, we have the following form of R :

$$(11.2) \quad R(z) = \frac{z^3 + \frac{4c-1}{2c}z^2 + \frac{12c^2}{1-4c}z + 2c}{z^2 + \frac{6c^2}{1-4c}z + c},$$

for some $c \in \mathbb{R} \setminus \{0, 1/4\}$.

Claim. $c < \frac{1}{6}$.

Proof of Claim. By way of contradiction, let us assume that $c \geq \frac{1}{6}$.

Case i. $c > \frac{1}{4}$. Then, Equation (11.1) combined with the fact that $\alpha \in \mathbb{S}^1 \setminus \{\pm 1\}$ imply

$$-2 < \alpha + \bar{\alpha} < 2 \implies -3 < 2d \implies -3 < \frac{12c^2}{1-4c} \implies (2c-1)^2 < 0,$$

a contradiction.

Case ii. $\frac{1}{6} < c < \frac{1}{4}$. In this case, we have

$$-2 < \alpha + \bar{\alpha} < 2 \implies 2d < 1 \implies \frac{12c^2}{1-4c} < 1 \implies (2c+1)(6c-1) < 0,$$

a contradiction.

Case iii. $c = \frac{1}{6}$. In this case, ± 1 are critical points of R , which contradicts the fact that R has exactly one real critical point on $\mathbb{S}^1$. This completes the proof of the claim. $\square$

Let us note that for $c < \frac{1}{6}$, we have that

$$(11.3) \quad R(1) - R(-1) = \frac{2(1-4c)}{(1-c)(1-5c)} > 0 \implies R(1) > R(-1).$$

Claim. $c \notin (\frac{1}{10}, \frac{1}{6})$.

Proof of Claim. We will look at the local geometry of $R(\mathbb{S}^1)$ near the cusp $R(1)$. To do so, we parametrize $\mathbb{S}^1$ as $\{e^{it} : 0 \leq t \leq 2\pi\}$, and compute that for $t \approx 0$,

$$R(e^{it}) = \left(\frac{4c^2 - 8c + 1}{2c(c-1)} + \frac{3(4c-1)}{2(c-1)^2} t^2 + O(t^3) \right) + i \left(\frac{(4c-1)(1-10c)}{2(c-1)^2(2c-1)} t^3 + O(t^4) \right).$$

Now, for $c \in (\frac{1}{10}, \frac{1}{6})$, the coefficient of t^3 in the imaginary part of $R(e^{it})$ is negative. It follows that for $t \approx 0$, we have that $\text{Im } R(e^{it}) > 0$ for $t < 0$, and $\text{Im } R(e^{it}) < 0$ for $t > 0$. Since $R(\mathbb{S}^1)$ is a real-symmetric Jordan curve with $R(1) > R(-1)$, the above observation forces R to be orientation-reversing on $\mathbb{S}^1$, which is a contradiction. Therefore, $c \notin (\frac{1}{10}, \frac{1}{6})$. $\square$

Thanks to the above claims, our search of c is reduced to the (infinite) parameter interval $c \leq \frac{1}{10}$. To find the exact value of c , let us first note that since $R(1) > R(-1)$, and $\lim_{x \rightarrow \pm\infty} R(x) = \pm\infty$, it follows from univalence of $R|_{\mathbb{D}^c}$ that $(R(1), +\infty) \subset \Omega$. Now, the real-symmetry of R and the local uniform convergence of σ^{on} to $R(1)$ on $\Psi(\mathbb{D}^c)$ imply that $(R(1), +\infty)$ is an attracting direction for the ‘parabolic’ fixed point $R(1)$ of σ . This implies, in particular, that

$$(11.4) \quad R\left(\frac{1}{1+\varepsilon}\right) = \sigma(R(1+\varepsilon)) < R(1+\varepsilon)$$

for $\varepsilon > 0$ sufficiently small. In terms of the Taylor series of $R(x)$ and $R(\frac{1}{x})$ at $x = 1$, Inequality (11.4) can be rewritten as

$$\frac{(4c-1)(10c-1)}{(c-1)^2(2c-1)} \varepsilon^3 + O(\varepsilon^4) > 0,$$

for $\varepsilon > 0$ sufficiently small. If $c \neq \frac{1}{10}$; i.e., if $c < \frac{1}{10}$, then the above inequality implies that

$$\frac{(4c-1)(10c-1)}{(c-1)^2(2c-1)} > 0,$$

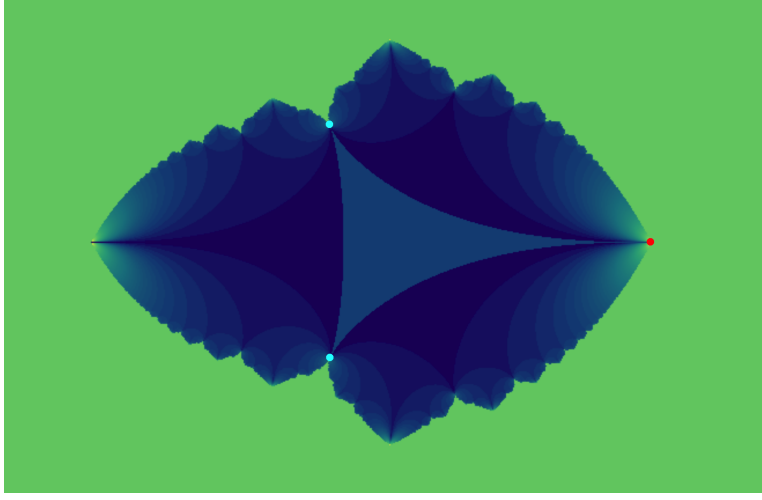


FIGURE 14. The dynamical plane of the Schwarz reflection map σ , which is the conformal mating of $\bar{z}^2 + \frac{1}{4}$ and the ideal triangle reflection group Γ_2 , is shown. The forward orbit of every point of the non-escaping set (shaded in green) converges to the unique real cusp point $R(1)$ on $R(\mathbb{S}^1)$ (marked in red). The non-escaping set contains an attracting petal subtending an angle $4\pi/3$ at this cusp. On the other hand, the two real-symmetric cusps on $R(\mathbb{S}^1)$ (marked in light blue) repel nearby points in the non-escaping set, and the non-escaping set subtends zero angles at these two cusps. (Picture courtesy: Seung-Yeop Lee.)

which is impossible. Therefore, we must have $c = \frac{1}{10}$. Plugging $c = \frac{1}{10}$ in Formula (11.2), we finally have

$$(11.5) \quad R(z) = \frac{10z^3 - 30z^2 + 2z + 2}{10z^2 + z + 1}.$$

(See Figure 14 for the dynamical plane of the associated Schwarz reflection map σ .)

To conclude, let us convince the readers that the map R of Formula (11.5) satisfies Inequality (11.4). Indeed, in terms of the Taylor series of $R(x)$ and $R(\frac{1}{x})$ at $x = 1$, we have that

$$(11.6) \quad R(1 + \varepsilon) - \sigma(R(1 + \varepsilon)) = R(1 + \varepsilon) - R\left(\frac{1}{1 + \varepsilon}\right) = \frac{5}{36}\varepsilon^5 + O(\varepsilon^6) > 0,$$

for $\varepsilon > 0$ sufficiently small. In fact, Formula (11.6) can be used to compute the asymptotics of σ at the cusp point $R(1) = -4/3$:

$$(11.7) \quad \sigma\left(-\frac{4}{3} + \delta\right) = -\frac{4}{3} + \delta - c \cdot \delta^{5/2} + O(\delta^3),$$

for some $c > 0$, and all $\delta > 0$ sufficiently small (where $\delta^{1/2}$ stands for the positive square root of δ). It follows from Formula (11.7) that the non-escaping set of σ contains an attracting petal subtending an angle $4\pi/3$ at the ‘parabolic’ point $R(1)$.

Proposition 11.4. *Let $P(z) = \bar{z}^2 + \frac{1}{4}$, and*

$$R(z) := \frac{10z^3 - 30z^2 + 2z + 2}{10z^2 + z + 1}.$$

Then, R is univalent on $\mathbb{D}^c$, and the Schwarz reflection map σ of the quadrature domain $R(\mathbb{D}^)$ is the unique conformal mating of P and the ideal triangle reflection group Γ_2 .*

Remark 11.5. It is tempting to construct the conformal mating of $P(z) = \bar{z}^2 + \frac{1}{4}$ and ρ_2 by starting with the dynamical plane of P , and replacing the dynamics of P on its basin of infinity by Nielsen map ρ_2 . However, our David surgery methods do not permit us to carry out this construction since the basin of infinity of P is not a John domain, and hence, we cannot apply Proposition 2.5 to obtain an invariant David coefficient for the modified map (compare the proof of Lemma 7.1). It would be interesting to know if one can directly pass from the dynamical plane of P to the conformal mating of P and ρ_2 using a more general surgery technique.

12. EXTREMAL POINTS IN SPACES OF SCHLICHT FUNCTIONS

The well-known De Brange's theorem (earlier known as the Bieberbach conjecture) asserts that each member f of the class of *schlicht* functions

$$\mathcal{S} := \{f(z) = z + a_2 z^2 + \cdots + a_n z^n + \cdots : f|_{\mathbb{D}} \text{ is univalent}\}$$

satisfies $|a_n| \leq n$, for all $n \in \mathbb{N}$. Moreover, the bound is sharp; the Koebe function $\sum_{n \geq 1} n z^n$ simultaneously maximizes all the coefficients.

The analogous coefficient problem for the class of *external univalent* maps

$$\Sigma := \left\{ f(z) = z + \frac{a_1}{z} + \cdots + \frac{a_d}{z^d} + \cdots : f|_{\mathbb{D}^*} \text{ is univalent} \right\},$$

where $\mathbb{D}^* = \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$, is still open. Note that the area theorem implies the upper bound

$$|a_n| \leq n^{-1/2},$$

(see [Dur83, Theorem 2.1]). But this bound is far from sharp; we refer the readers to [Dur83, §4.7] for a survey of known results.

The related question of establishing coefficient bounds for the truncated families

$$\Sigma_d^* := \left\{ f(z) = z + \frac{a_1}{z} + \cdots + \frac{a_{d-1}}{z^{d-1}} - \frac{1}{d \cdot z^d} : f|_{\mathbb{D}^*} \text{ is conformal} \right\}$$

is motivated by the work of Suffridge on coefficient bounds for polynomials in class $\mathcal{S}$ [Suf69, Suf72]. In fact, one can adapt the proof of [Suf72, Theorem 10] for the space Σ to show that

$$\Sigma = \overline{\bigcup_{d \geq 1} \Sigma_d^*}.$$

It is worth mentioning that the choice of $-1/d$ as the last coefficient stems from the fact that if $f(z) = z + \frac{a_1}{z} + \cdots + \frac{a_d}{z^d} \in \Sigma$, then the absolute value of the product of the non-zero critical points of f is $d|a_d|$, and the univalence of $f|_{\mathbb{D}^*}$ implies that $d|a_d| \leq 1$; i.e., $|a_d| \leq \frac{1}{d}$.

Viewing Σ_d^* as a compact subset of a finite dimensional Euclidean space (given by the coefficients of the members of Σ_d^*), the problem of maximizing the coefficients

of the members of Σ_d^* boils down to finding extremal points of Σ_d^* . Here, a map $f \in \Sigma_d^*$ is said to be *extremal* if it has no representation of the form

$$f = tf_1 + (1-t)f_2, \quad 0 < t < 1,$$

as a proper convex combination of two distinct maps $f_1, f_2 \in \Sigma_d^*$.

Before we proceed to study the extremal points of Σ_d^* , we need to recall some general facts on singularities on the boundary of $f(\mathbb{D}^*)$, for $f \in \Sigma_d^*$, $d \geq 2$. A boundary point $p \in \partial f(\mathbb{D}^*)$ is called *regular* if there is a disc $B = B(p, \varepsilon)$ such that $f(\mathbb{D}^*) \cap B$ is a Jordan domain, and $\partial f(\mathbb{D}^*) \cap B$ is a simple non-singular real-analytic arc; otherwise p is called a *singular* point. Singular points on $\partial f(\mathbb{D}^*)$ come in two varieties. A *cusp* singularity $\zeta_0 \in \partial f(\mathbb{D}^*)$ is a critical value of f ; it has the property that for sufficiently small $\varepsilon > 0$, the intersection $B(\zeta_0, \varepsilon) \cap f(\mathbb{D}^*)$ is a Jordan domain. Moreover, by conformality of $f|_{\mathbb{D}^*}$, each cusp on $\partial f(\mathbb{D}^*)$ points in the inward direction towards $f(\mathbb{D}^*)$. On the other hand, a singular point $\zeta_0 \in \partial f(\mathbb{D}^*)$ is said to be a *double point* if for all sufficiently small $\varepsilon > 0$, the intersection $B(\zeta_0, \varepsilon) \cap f(\mathbb{D}^*)$ is a union of two Jordan domains, and ζ_0 is a non-singular boundary point of each of them. In particular, two distinct non-singular (real-analytic) local branches of $\partial f(\mathbb{D}^*)$ intersect tangentially at a double point ζ_0 .

In the rest of this section, we will assume that $d \geq 2$.

Lemma 12.1. *For each $f \in \Sigma_d^*$, the boundary of $f(\mathbb{D}^*)$ is a piecewise analytic curve with precisely $(d+1)$ cusps.*

Proof. Clearly, each $f \in \Sigma_d^*$ has a critical point of multiplicity $(d-1)$ at the origin and $(d+1)$ non-zero critical points (counting multiplicity). If $f \in \Sigma_d^*$, then the absolute value of the product of all the non-zero critical points of f is 1. As all these critical points must lie in $\mathbb{D}$, it follows that each non-zero critical point of f must have absolute value 1 and hence lies on $\mathbb{S}^1$. Moreover, conformality of $f|_{\mathbb{D}^*}$ implies that each of these critical points of f must be simple. Thus, f has $(d+1)$ distinct simple critical points on $\mathbb{S}^1$. The result follows. $\square$

By Lemma 12.1 and [LM14, Lemma 2.4], for each $f \in \Sigma_d^*$, the boundary of $f(\mathbb{D}^*)$ has exactly $(d+1)$ cusps and at most $(d-2)$ double points.

Definition 12.2. $f \in \Sigma_d^*$ is called a *Suffridge polynomial* if $\partial f(\mathbb{D}^*)$ has $(d+1)$ cusps and $(d-2)$ double points. The curve $f(\mathbb{S}^1)$ is called a *Suffridge curve*.

The following result yields a connection between extremal points of Σ_d^* and Suffridge polynomials.

Theorem 12.3. [LM14, Theorem 2.5] *Extremal points of Σ_d^* are Suffridge polynomials.*

We now describe a procedure to assign an angled tree to each Suffridge polynomial.

Definition 12.4. Let $f \in \Sigma_d^*$ be a Suffridge polynomial, and $\Omega := f(\mathbb{D}^*)$, $T := \mathbb{C} \setminus \Omega$. We set $T^0 := T \setminus \{\text{Cusps and double points on } \partial T\}$. The connected components $T_1, \dots, T_{d-1}$ of T^0 are called *fundamental tiles* of f .

Remark 12.5. Note that the cusp points on ∂T are not cut-points of T , while the double points are. Since there are exactly $(d-2)$ double points on ∂T for a Suffridge polynomial, they disconnect T into $(d-1)$ components.

Definition 12.6.

- (1) A bi-angled tree $\mathcal{T}$ is a topological tree each of whose vertices are of valence at most 3, and that is equipped with an angle function $\angle$, defined on pairs of edges incident at a common vertex, and taking values in $\{0, 2\pi/3, 4\pi/3\}$, satisfying the following conditions:
 - (a) for each pair of distinct edges e and e' incident at a vertex v , we have $\angle_v(e, e') \in \{2\pi/3, 4\pi/3\}$, and $\angle_v(e, e) = 0$,
 - (b) $\angle_v(e, e') = -\angle_v(e', e) \pmod{2\pi}$, and
 - (c) $\angle_v(e, e') + \angle_v(e', e'') = \angle_v(e, e'') \pmod{2\pi}$, where e, e', e'' are edges incident at a vertex v .
- (2) Two bi-angled trees $\mathcal{T}_1, \mathcal{T}_2$ are said to be *isomorphic* if there exists a tree isomorphism $f : \mathcal{T}_1 \rightarrow \mathcal{T}_2$ that satisfies $\angle_{f(v)}(f(e), f(e')) = \angle_v(e, e')$, for each pair of edges e, e' incident at a vertex v of $\mathcal{T}_1$.

We remark that the angle data of a bi-angled tree is purely combinatorial; in other words, for a planar embedding of a bi-angled tree, we neither require the edges to be straight line segments, nor require the Euclidean angle between two edges to be $\pm 2\pi/3$. However, since the function $\angle$ induces a cyclic order on the edges incident at each vertex, there is a preferred (isotopy class of) embedding of a bi-angled tree into the complex plane.

Definition 12.7. For a Suffridge polynomial $f \in \Sigma_d^*$, we define its bi-angled tree $\mathcal{T}(f) = (V_\Omega, E_\Omega)$ in the following manner. Denote by $T_1, \dots, T_{d-1}$ the fundamental tiles of f . Associate a vertex v_i to the component T_i , and connect the vertices v_i and v_j by an edge if and only if $\bar{T}_i$ and $\bar{T}_j$ intersect. We now equip the tree with an *angle function* $\angle$. If the valence of a vertex v_i is 3, then for each pair of consecutive edges e, e' incident at v_i (in the counter-clockwise circular order around v_i), we set $\angle_{v_i}(e, e') = 2\pi/3$. On the other hand, if the valence of v_i is 2, and e, e' are the edges incident at v_i , then $\angle_{v_i}(e, e') = 2\pi/3$ (respectively, $4\pi/3$) if the double points of ∂T corresponding to the edges e and e' are consecutive (respectively, are not consecutive) singular points of ∂T_i in the counter-clockwise orientation.

Using David surgery techniques, we will give a new proof of the following theorem which recently appeared in [LMM19]. The proof given below is essentially different from the existence proof given in [LMM19, Theorem 4.1], which used a pinching deformation technique for Schwarz reflection maps combined with compactness of Σ_d^* .

Theorem 12.8. *Given a bi-angled tree $\mathcal{T}$ with $(d-1)$ vertices, there exists a Suffridge polynomial $f \in \Sigma_d^*$ such that $\mathcal{T}(f)$ is isomorphic to $\mathcal{T}$. Moreover, f is unique up to conjugation by multiplication by a $(d+1)$ -st root of unity.*

Proof of Existence. By [LMM20, Proposition 5.4], there exists a critically fixed anti-polynomial P of degree d whose angled Hubbard tree is isomorphic to $\mathcal{T}$ equipped with the *local degree function* $\deg : V(\mathcal{T}) \rightarrow \mathbb{N}$, $\deg(v) = 2$ (see [Poi13] for more general discussions on angled Hubbard trees). In particular, P has $(d-1)$ distinct, simple, fixed critical points $c_1, \dots, c_{d-1}$. These critical points correspond to the vertices $v_1, \dots, v_{d-1}$ of $\mathcal{T}$, respectively. Let us denote the corresponding immediate basins of attraction by U_i , $i \in \{1, \dots, d-1\}$. By [LMM20, Proposition 5.5], we

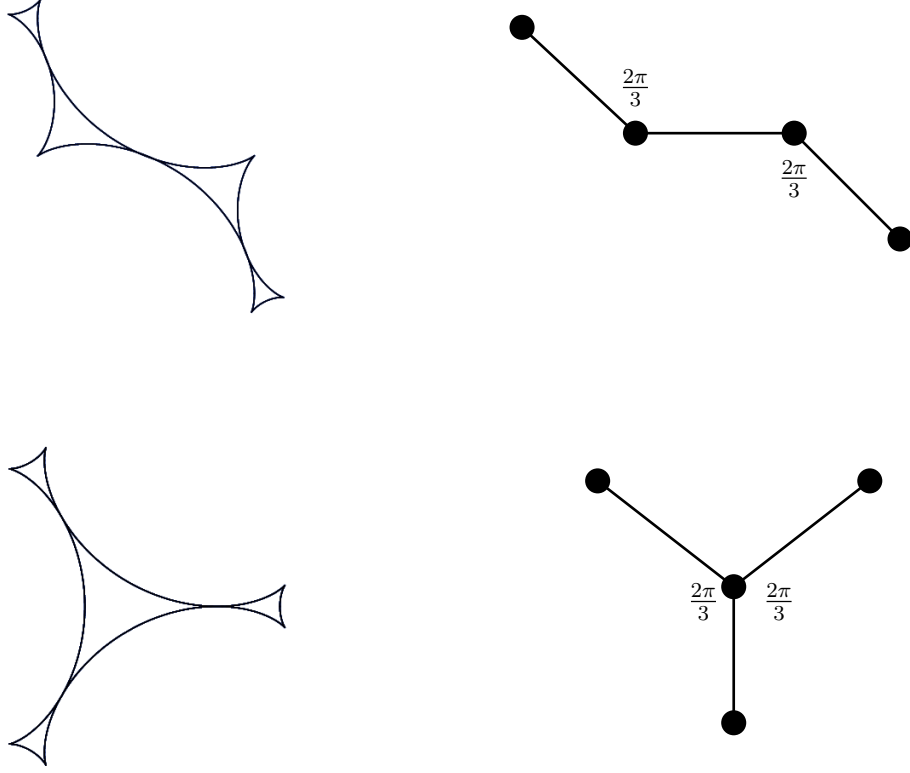


FIGURE 15. The images of the unit circle under the Suffridge polynomials $f(z) \approx z - \frac{0.71i}{z} + \frac{0.71i}{3z^3} - \frac{1}{5z^5}$ and $g(z) = z + \frac{2\sqrt{2}}{5z^2} - \frac{1}{5z^5}$, along with their associated bi-angled trees, are shown.

have that

$$\hat{U} := \bigcup_{i=1}^{d-1} \overline{U_i}$$

is connected, and $\overline{U_i}$ intersects $\overline{U_j}$ if and only if there is an edge in $\mathcal{T}$ connecting v_i and v_j .

Applying Lemma 7.1 on the fixed Fatou components $U_1, \dots, U_{d-1}$ of P , we obtain a global David homeomorphism Ψ and an anti-meromorphic map σ_1 defined on a closed, connected subset of $\hat{\mathbb{C}}$ that is conformally conjugate to $\bar{z}^d|_{\mathbb{D}^*}$ on $\Psi(\mathcal{B}_\infty(P))$, and to the Nielsen map $\rho_2|_{\mathbb{D}}$ on each $\Psi(U_i)$. Moreover, if Ω_1 is the interior of the domain of definition of σ_1 , then σ_1 fixes $\partial\Omega_1$ pointwise. So Ω_1 is a quadrature domain, and σ_1 is its Schwarz reflection map.

It also follows from the construction and the fullness of $\hat{U}$ that Ω_1 is simply connected. Thus, there exists a rational map f_1 , univalent on $\mathbb{D}^*$, such that

$$f_1(\mathbb{D}^*) = \Omega, \text{ and } \sigma \equiv f_1 \circ (1/\bar{z}) \circ (f_1|_{\mathbb{D}^*})^{-1} \text{ on } \Omega_1.$$

After possibly replacing Ω_1 by a Möbius image of it, we can assume that Ω_1 has conformal radius 1 with conformal center at ∞ . Hence, we can normalize f_1 such

that $f_1(\infty) = \infty$, $f_1'(\infty) = 1$. Since ∞ is a critical point of multiplicity $(d-1)$ for σ_1 , and $\sigma_1^{-1}(\infty) = \{\infty\}$, it follows that $\sigma_1 : \sigma_1^{-1}(\Omega_1) \rightarrow \Omega_1$ is a branched covering of degree d . Hence, by Lemma 6.12, we have that $\deg f_1 = d+1$, and f_1 has a pole of order d at the origin. Therefore, we have

$$f_1(z) = z + a_0 + \frac{a_1}{z} + \cdots + \frac{a_d}{z^d}.$$

As σ_1 has no critical point other than ∞ , it follows that f_1 has all of its $(d+1)$ non-zero critical points on $\mathbb{S}^1$. Moreover, the conformality of $f_1|_{\mathbb{D}^*}$ implies that these $(d+1)$ critical points of f_1 are simple and distinct.

Setting $\Omega_2 := A_1(\Omega_1)$ and $f_2 := A_1 \circ f_1$, where $A_1(z) = z + b$ (for some $b \in \mathbb{C}$), we can assume that

$$f_2(z) = z + \frac{a_1}{z} + \cdots + \frac{a_d}{z^d}.$$

A simple computation (using the fact that all critical points of f_2 are on $\mathbb{S}^1$ and at the origin) now shows that $|a_d| = 1/d$. Finally, setting $\Omega := A_2(\Omega_2)$ and $f := A_2 \circ f_2 \circ A_2^{-1}$, where $A_2(z) = \alpha z$ (for some $\alpha \in \mathbb{S}^1$), we can further assume that f is of the above form with $a_d = -1/d$. Clearly, $f \in \Sigma_d^*$. Moreover, $\partial f(\mathbb{D}^*)$ has $(d+1)$ cusps and $(d-2)$ double points; i.e. f is a Suffridge polynomial in Σ_d^* . It is now easy to see from the construction of f that its bi-angled tree $\mathcal{T}(f)$ is isomorphic to $\mathcal{T}$. $\square$

Proof of Uniqueness. Assume that $g \in \Sigma_d^*$ is another Suffridge polynomial realizing the bi-angled tree $\mathcal{T}$. We set

$$T^\infty(\sigma_f) = \bigcup_{n=0}^{\infty} \sigma_f^{-n}(T^0(f)), \quad T^\infty(\sigma_g) = \bigcup_{n=0}^{\infty} \sigma_g^{-n}(T^0(g)).$$

Also note that both σ_f and σ_g have a superattracting fixed point at ∞ . We denote the corresponding basins of attraction by $\mathcal{B}_\infty(\sigma_f)$ and $\mathcal{B}_\infty(\sigma_g)$. Since σ_f, σ_g do not have any other critical points in $\mathcal{B}_\infty(\sigma_f)$, $\mathcal{B}_\infty(\sigma_g)$ (respectively), it follows from the proof of [Mil06, Theorem 9.3] that these basins are simply connected, and the Schwarz reflection maps restricted to these basins are conformally conjugate to $\bar{z}^d|_{\mathbb{D}}$.

Further, we denote the singular points (i.e., the cusps and double points) on $\partial T(f)$ (respectively, on $\partial T(g)$) by $S^0(f)$ (respectively, $S^0(g)$), and define

$$S^\infty(f) := \bigcup_{n=0}^{\infty} \sigma_f^{-n}(S^0(f)), \quad S^\infty(g) := \bigcup_{n=0}^{\infty} \sigma_g^{-n}(S^0(g)).$$

By [LMM20, Corollary 4.11], we have that

$$\widehat{\mathbb{C}} = \mathcal{B}_\infty(\sigma_g) \sqcup \Lambda(\sigma_g) \sqcup T^\infty(\sigma_g),$$

where $\Lambda(\sigma_g)$ is the common boundary of $\mathcal{B}_\infty(\sigma_g)$ and $T^\infty(\sigma_g)$. That the same holds for f , follows both from [LMM20, Corollary 4.11] and the construction of f .

By construction, $S^\infty(g) \subset \Lambda(\sigma_g)$. As $\Lambda(\sigma_g)$ is locally connected by [LMM20, Proposition 4.2], the conformal conjugacy between $\bar{z}^d|_{\mathbb{D}}$ and $\sigma_g|_{\mathcal{B}_\infty(\sigma_g)}$ extends as a continuous semiconjugacy between $\bar{z}^d|_{\mathbb{S}^1}$ and $\sigma_g|_{\Lambda(\sigma_g)}$. It follows that $S^\infty(g)$ is dense on $\Lambda(\sigma_g)$. The same also holds for f .

Since f and g have isomorphic bi-angled trees, there exists an orientation-preserving homeomorphism $\Phi : T(f) \rightarrow T(g)$ that carries cusps to cusps and double

points to double points, and is conformal on $\text{int } T(f)$. By iterated Schwarz reflection (equivalently, by iterated lifting under the Schwarz reflection maps σ_f and σ_g), the map Φ can be extended to a homeomorphism

$$\Phi : T^\infty(\sigma_f) \bigcup S^\infty(f) \longrightarrow T^\infty(\sigma_g) \bigcup S^\infty(g),$$

such that Φ is a topological conjugacy between σ_f to σ_g , and is conformal on $T^\infty(\sigma_f)$.

Claim. Φ extends to a homeomorphism $\Phi : \overline{T^\infty(\sigma_f)} \longrightarrow \overline{T^\infty(\sigma_g)}$ conjugating σ_f to σ_g .

Proof of claim. First note that the points of $S^0(f)$ (respectively, $S^0(g)$) determine a Markov partition $\mathcal{P}(f) \equiv \mathcal{P}(\sigma_f, S^0(f))$ (respectively, $\mathcal{P}(g) \equiv \mathcal{P}(\sigma_g, S^0(g))$) for $\sigma_f|_{\Lambda(\sigma_f)}$ (respectively, for $\sigma_g|_{\Lambda(\sigma_g)}$). Since $\Lambda(\sigma_f)$ (respectively, $\Lambda(\sigma_g)$) contains no critical point of σ_f (respectively, of σ_g), the arguments of [DU91, Theorem 4] apply to the current setting to show that $\sigma_f|_{\Lambda(\sigma_f)}$ and $\sigma_g|_{\Lambda(\sigma_g)}$ are expansive. In particular, the diameters of the iterated preimages of the above Markov partition pieces shrink to zero uniformly (alternatively, shrinking of diameters of the iterated preimages of the pieces of $\mathcal{P}(f)$ and $\mathcal{P}(g)$ can be proved using [LMM20, Lemma 4.1] and the parabolic dynamics of σ_f, σ_g at points of $S^0(f), S^0(g)$, respectively).

We will first show that $\Phi : S^\infty(f) \rightarrow S^\infty(g)$ admits a continuous extension $\Phi : \Lambda(\sigma_f) \rightarrow \Lambda(\sigma_g)$. Since $S^\infty(f)$ (respectively, $S^\infty(g)$) is dense on $\Lambda(\sigma_f)$ (respectively, on $\Lambda(\sigma_g)$), to prove the existence of a continuous extension $\Phi : \Lambda(\sigma_f) \rightarrow \Lambda(\sigma_g)$, it suffices to show that $\Phi : S^\infty(f) \rightarrow S^\infty(g)$ is uniformly continuous. To this end, let us fix $\varepsilon > 0$. Now choose N so that the diameters of the $\sigma_g^{\circ N}$ -preimages of the pieces of $\mathcal{P}(g)$ are less than ε . Next, choose $\delta > 0$ so that any two non-adjacent $\sigma_f^{\circ N}$ -preimages of the pieces of $\mathcal{P}(f)$ are at least δ distance away. If $x, y \in S^\infty(f)$ are at most δ distance apart, then they lie in two adjacent $\sigma_f^{\circ N}$ -preimages of the pieces of $\mathcal{P}(f)$. It follows from the construction of Φ that $\Phi(x), \Phi(y)$ lie in two adjacent $\sigma_g^{\circ N}$ -preimages of the pieces of $\mathcal{P}(g)$, and hence, $d(\Phi(x), \Phi(y)) < 2\varepsilon$. This proves uniform continuity of $\Phi : S^\infty(f) \rightarrow S^\infty(g)$.

Applying the same argument on Φ^{-1} , we get a continuous inverse $\Phi^{-1} : \Lambda(\sigma_g) \rightarrow \Lambda(\sigma_f)$. Thus, $\Phi : \Lambda(\sigma_f) \rightarrow \Lambda(\sigma_g)$ is a homeomorphism extending $\Phi : S^\infty(f) \rightarrow S^\infty(g)$.

We have now defined a bijective map $\Phi : \overline{T^\infty(\sigma_f)} \rightarrow \overline{T^\infty(\sigma_g)}$, that is continuous restricted to $T^\infty(\sigma_f)$ and $\Lambda(\sigma_f)$ separately, and conjugates σ_f to σ_g . To complete the proof of the claim, we only need to justify that $\Phi|_{\Lambda(\sigma_f)}$ continuously extends $\Phi|_{T^\infty(\sigma_f)}$.

To this end, first observe that each component of $T^\infty(\sigma_f), T^\infty(\sigma_g)$ is a Jordan domain. Hence, $\Phi|_{T^\infty(\sigma_f)}$ extends homeomorphically to the boundary of each component of $T^\infty(\sigma_f)$. Moreover, this extension agrees with $\Phi|_{\Lambda(\sigma_f)}$ at points of $S^\infty(f)$. Since $\overline{S^\infty(f)} = \Lambda(\sigma_f)$, we conclude that the homeomorphic extension of $\Phi|_{T^\infty(\sigma_f)}$ to the boundary of each component of $T^\infty(\sigma_f)$ agrees with $\Phi|_{\Lambda(\sigma_f)}$. Finally, local connectedness of $\Lambda(\sigma_f), \Lambda(\sigma_g)$ imply that the diameters of the components of $T^\infty(\sigma_f), T^\infty(\sigma_g)$ go to zero, from which continuity of $\Phi : \overline{T^\infty(\sigma_f)} \rightarrow \overline{T^\infty(\sigma_g)}$ follows. $\square$

We now note that both σ_f and σ_g are conformally conjugate to $\bar{z}^d|_{\mathbb{D}}$ on their basins of infinity via their Böttcher coordinates $\phi_{\sigma_f} : \mathbb{D} \rightarrow \mathcal{B}_\infty(\sigma_f)$ and $\phi_{\sigma_g} : \mathbb{D} \rightarrow$

$\mathcal{B}_\infty(\sigma_g)$. Since $\Lambda(\sigma_f)$ and $\Lambda(\sigma_g)$ are locally connected, ϕ_{σ_f} and ϕ_{σ_g} extend continuously to $\mathbb{S}^1$. Moreover, by [LMM20, Lemma 4.14], the continuous extension of ϕ_{σ_f} (respectively, of ϕ_{σ_g}) sends the $(d+1)$ -st roots of unity to the cusp points on $\partial T(f)$ (respectively, on $\partial T(g)$). After possibly precomposing ϕ_{σ_g} with multiplication by a $(d+1)$ -st root of unity, we can assume that the conformal map

$$\widehat{\Phi} := \phi_{\sigma_g} \circ \phi_{\sigma_f}^{-1} : \mathcal{B}_\infty(\sigma_f) \rightarrow \mathcal{B}_\infty(\sigma_g)$$

continuously extends to the cusps of $\partial T(f)$, and agrees with Φ (constructed above) at these points. Hence, $\widehat{\Phi}$ must also continuously extend to the iterated σ_f -preimages of the cusps on $\partial T(f)$, and agree with Φ at these points. As the iterated preimages of the cusps on $\partial T(f)$ are dense on $\Lambda(\sigma_f)$, it is now easy to see that $\Phi|_{\Lambda(\sigma_f)}$ continuously extends $\widehat{\Phi}|_{\mathcal{B}_\infty(\sigma_f)}$.

Thus, we have constructed a topological conjugacy between σ_f and σ_g that is conformal away from $\Lambda(\sigma_f)$. On the other hand, since the basin of infinity of the hyperbolic anti-polynomial P (used to construct f in the existence part) is a John domain, and Ψ is a global David homeomorphism, Theorem 2.11 tells us that $\Lambda(\sigma_f)$ is conformally removable. This implies that σ_f and σ_g are Möbius conjugate. As this conjugacy must send the superattracting fixed point of σ_f at ∞ to the superattracting fixed point of σ_g at ∞ , it follows that the conjugacy is affine. In particular, there exists an affine map A carrying $f(\mathbb{D}^*)$ to $g(\mathbb{D}^*)$. Therefore,

$$M := (g|_{\mathbb{D}^*})^{-1} \circ A \circ f : \mathbb{D}^* \rightarrow \mathbb{D}^*$$

is a conformal map. Since A is affine, it follows that M fixes ∞ , and thus is a rotation. Note that since $(g|_{\mathbb{D}^*})^{-1}(w)$ is of the form $w + O(\frac{1}{w})$ near ∞ , it follows that $A(w) = \alpha w$, for some $\alpha \in \mathbb{C}^*$. The fact that both f and g have derivative 1 at ∞ implies that $A \equiv M$. Finally, as the coefficient of $1/z^d$ is $-\frac{1}{d}$ for both f and g , it follows that $\alpha^{d+1} = 1$. We conclude that $g = M \circ f \circ M^{-1}$, where M is rotation by a $(d+1)$ -st root of unity. $\square$

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