Selecta Mathematica (2022) 28:90
https://doi.org/10.1007/s00029-022-00803-0

Selecta Mathematica New Series

On the P = W conjecture for SL_n

Mark Andrea de Cataldo[1] · Davesh Maulik[2] · Junliang Shen[3]

Accepted: 7 August 2022 / Published online: 13 October 2022

Abstract

Let p be a prime number. We prove that the $P = W$ conjecture for SL_p is equivalent to the $P = W$ conjecture for GL_p. As a consequence, we verify the $P = W$ conjecture for genus 2 and SL_p. For the proof, we compute the perverse filtration and the weight filtration for the variant cohomology associated with the SL_p-Hitchin moduli space and the SL_p-twisted character variety, relying on Gröchenig–Wyss–Ziegler's recent proof of the topological mirror conjecture by Hausel–Thaddeus. Finally we discuss obstructions of studying the cohomology of the SL_n-Hitchin moduli space via compact hyper-Kähler manifolds.

Mathematics Subject Classification 14H60, 14F45, 14D20

Contents

0. Introduction . . . 2
1 Hitchin moduli spaces and character varieties . . . 3
  1.1 Moduli spaces . . . 4
  1.2 Perverse filtrations . . . 4
  1.3 Γ-actions . . . 5
  1.4 The variant cohomology . . . 6
  1.5 Main results . . . 7
2 Characterizations for k-sequences . . . 8
  2.1 k-sequences . . . 8
  2.2 The first criterion . . . 8

✉ Junliang Shen
junliang.shen@yale.edu

Mark Andrea de Cataldo
mark.decataldo@stonybrook.edu

Davesh Maulik
maulik@mit.edu

1 Stony Brook University, Stony Brook, NY, USA

2 Massachusetts Institute of Technology, Cambridge, MA, USA

3 Yale University, New Haven, CT, USA

2.3 The second criterion 9
3 Perverse filtrations and weight filtrations 10
3.1 The topological mirror symmetry conjecture 11
3.2 Proof of Theorem 1.5 (a). 12
3.3 A symmetry 13
3.4 E-polynomials of character varieties 13
3.5 Vanishing and Hodge–Tate 15
3.6 Proof of Theorem 1.5 (b). 16
4 Hitchin moduli spaces and compact hyper-Kähler manifolds 17
4.1 Overview 17
4.2 An obstruction for SL_2 18
4.3 Compact hyper-Kähler manifolds 19
References 20

0. Introduction

Throughout the paper, we work over the complex numbers $\mathbb{C}$.

Let C be a nonsingular projective curve of genus $g \geq 2$, and let G be a reductive group. The P=W conjecture of de Cataldo, Hausel, and Migliorini [3] predicts a surprising connection between the topology of G-Hitchin systems and the Hodge theory G-character varieties via the non-abelian Hodge correspondence. More precisely, it suggests that the perverse filtration for the Hitchin system associated with the G-Dolbeault moduli space $\mathcal{M}_{\mathrm{Dol}}$ coincides with the weight filtration associated with the corresponding G-Betti moduli space $\mathcal{M}_B$,

$$\text{“}P = W\text{"}: \quad P_k H^d(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q}) = W_{2k} H^d(\mathcal{M}_B, \mathbb{Q}) = W_{2k+1} H^d(\mathcal{M}_B, \mathbb{Q}), \quad \forall k, d \geq 0; \tag{1}$$

see Sect. 1 for a brief review.

When $G = \mathrm{GL}_n$, the $P = W$ conjecture was established for any genus g and rank $n = 2$ in [3], and very recently, for genus $g = 2$ and arbitrary rank n in [4]. Furthermore, for aribitrary genus and rank, [4] shows $P = W$ for the tautological generators of the cohomology, and reduces the full $P = W$ conjecture to the multiplicativity of the perverse filtration.

The $G = \mathrm{PGL}_n$ case is equivalent to the GL_n case for a fixed curve C; see [4] the paragraph following Theorem 0.2. It is natural to explore non-trivial examples of the $P = W$ phenomenon for a reductive group G other than GL_n and PGL_n.

The purpose of this paper is to study $P = W$ for $G = \mathrm{SL}_n$. The case of SL_2 was already established in [3]. We provide in the following theorem an affirmative answer to the $P = W$ conjecture when the curve has genus $g = 2$ and the rank n is any prime number.

Theorem 0.1 *The $P = W$ conjecture (1) holds when C has genus $g = 2$ and $G = \mathrm{SL}_n$ with n a prime number.*

We refer to Sect. 1.5 for more precise statements. Here we briefly explain the main difference between the $G = \mathrm{GL}_n$ case and the $G = \mathrm{SL}_n$ case.

Let $\mathcal{M}_{\mathrm{Dol}}$ be the SL_n-Dolbeault moduli space assocated with a curve C of genus $g \geq 2$ and a line bundle L with $\gcd(c_1(L), n) = 1$ (see Sect. 1). There is a natural action of the finite group $\Gamma = \mathrm{Pic}^0(C)[n]$ on $\mathcal{M}_{\mathrm{Dol}}$ via tensor product. This group action yields a decomposition with respect to the irreducible characters of Γ,

$$H^*(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q}) = H^*(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q})^{\Gamma} \bigoplus H^*_{\mathrm{var}}(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q}). \tag{2}$$

Here the Γ-invariant part $H^*(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q})^{\Gamma}$ corresponds to the trivial character, and the variant cohomology $H^*_{\mathrm{var}}(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q})$ corresponds to all the non-trivial characters. Note that we have the same decomposition (2) for the Betti moduli space $\mathcal{M}_B$. The Γ-invariant part

$$H^*(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q})^{\Gamma} \subset H^*(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q}) \tag{3}$$

is canonically identified with the cohomology of the corresponding moduli of stable PGL_n-Higgs bundles (see (8)). In particular, it is the sub-vector space of $H^*(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q})$ generated by the tautological classes with respect to a universal family.

As a consequence, $P = W$ for GL_n is equivalent to $P = W$ for the invariant part (3). The following theorem proves $P = W$ for the variant cohomology for any genus when n is prime.

Theorem 0.2 *We have $P = W$ for the variant cohomology $H^*_{\mathrm{var}}(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q})$ for any genus $g \geq 2$ with n a prime number,*

$$P_k H^d_{\mathrm{var}}(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q}) = W_{2k} H^d_{\mathrm{var}}(\mathcal{M}_B, \mathbb{Q}), \quad \forall k, d \geq 0.$$

Theorem 0.2 shows that, for a curve C of genus $g \geq 2$, the $P = W$ conjecture for the groups GL_n, SL_n, and PGL_n are equivalent when n is prime. The proof of Theorem 0.2 relies on the recent proof [8] of the topological mirror conjecture [14], and the calculations of E-polynomials for character varieties [13, 20].

For general rank n. we refer to [19, Sect. 5] for a discussion on the connection between the Hausel–Thaddeus topological mirror conjecture and the $P = W$ conjecture for SL_n. We expect that the $P = W$ conjecture for SL_n is reduced to the $P = W$ conjecture for GL_d where d runs through all divisors of n. Such a reduction can be achieved by proving the compatibility between the endoscopic correspondence for the Hitchin moduli spaces with the weight filtrations for the character varieties; see [19, Question 5.5] and the paragraph follwing it for more details.

In Sect. 4, we discuss obstructions of studying the cohomology of $\mathcal{M}_{\mathrm{Dol}}$ via compact hyper–Kähler manifolds; see Propositions 4.2 and 4.3 . In particular, we provide obstructions to extend the method of [4] for proving the $P = W$ conjecture for genus 2 and GL_n to the genus 2 and SL_n case.

1 Hitchin moduli spaces and character varieties

Througout the section, we let C be a nonsingular projective curve of genus $g \geq 2$. We also fix 2 integers n, d satisfying $n \geq 2$ and $\gcd(n, d) = 1$, and a line bundle $L \in \mathrm{Pic}^d(C)$.

1.1 Moduli spaces

We review the two moduli spaces $\mathcal{M}_{\mathrm{Dol}}$ and $\mathcal{M}_B$ associated with the curve C, the group SL_n, and the line bundle $L \in \mathrm{Pic}^d(C)$. We refer to [3, 11, 13, 15, 16] for more details.

The Dolbeault moduli space $\mathcal{M}_{\mathrm{Dol}}$ parametrizes stable Higgs bundles

$$(\mathcal{E}, \theta), \quad \theta : \mathcal{E} \to \mathcal{E} \otimes \Omega_C$$

satisfying the conditions

$$\mathrm{trace}(\theta) = 0, \quad \det(\mathcal{E}) = L.$$

The Hitchin system associated with $\mathcal{M}_{\mathrm{Dol}}$ is a proper surjective morphism $\pi : \mathcal{M}_{\mathrm{Dol}} \to \Lambda$ sending $(\mathcal{E}, \theta)$ to the characteristic polynomial

$$\mathrm{char}(\theta) \in \Lambda := \oplus_{i=2}^n H^0(C, \Omega_C^{\otimes i}).$$

It is Lagrangian with respect to the canonical hyper-Kähler metric on $\mathcal{M}_{\mathrm{Dol}}$. The Betti moduli space $\mathcal{M}_B$ is the SL_n-twisted character variety,

$$\mathcal{M}_B := \left\{ a_k, b_k \in \mathrm{SL}_n,\ k = 1, 2, \ldots, g : \prod_{j=1}^{g} [a_j, b_j] = e^{\frac{2\pi\sqrt{-1}d}{n}} \mathrm{Id}_n \right\} /\!\!/ \mathrm{SL}_n, \tag{4}$$

which is obtained as an affine GIT quotient with respect to the action by conjugation.

Both $\mathcal{M}_{\mathrm{Dol}}$ and $\mathcal{M}_B$ are nonsingular quasi-projective varieties satisfying

$$\dim(\mathcal{M}_{\mathrm{Dol}}) = 2\dim(\Lambda) = \dim(\mathcal{M}_B) = (n^2 - 1)(2g - 2).$$

The non-abelian Hodge theory [25, 26] provides a diffeomorphism between $\mathcal{M}_{\mathrm{Dol}}$ and $\mathcal{M}_B$, which identifies the cohomology

$$H^*(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q}) = H^*(\mathcal{M}_B, \mathbb{Q}). \tag{5}$$

1.2 Perverse filtrations

The $P = W$ conjecture (1) predicts the match of two completely different structures under the identification (5), namely the perverse filtration associated with $\pi : \mathcal{M}_{\mathrm{Dol}} \to \Lambda$ and the weight filtration with respect to the mixed Hodge structure on $\mathcal{M}_B$.

The perverse filtration

$$\begin{aligned} &P_0 H^*(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q}) \subset P_1 H^*(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q}) \subset \cdots \subset \\ &P_k H^*(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q}) \subset \cdots \subset H^*(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q}) \end{aligned} \tag{6}$$

is an increasing filtration defined via the perverse truncation functor [3, Sect. 1.4.1]. It is governed by the topology of the Hitchin system $\pi : \mathcal{M}_{\mathrm{Dol}} \to \Lambda$. We recall the following useful characterization of the perverse filtration (6) by de Cataldo–Migliorini [5].

Theorem 1.1 (de Cataldo–Migliorini [5]) *Let $\Lambda^s \subset \Lambda$ denote an s-dimensional general linear sub-space. Then we have*

$$P_i H^{i+k}(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q}) = \mathrm{Ker}\left(H^{i+k}(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q}) \to H^{i+k}(\pi^{-1}(\Lambda^{k-1}), \mathbb{Q})\right).$$

1.3 Γ-actions

Let $\mathcal{L} \in \mathrm{Pic}^0(C)[n]$ be a n-torsion line bundle. Then for $(\mathcal{E}, \theta) \in \mathcal{M}_{\mathrm{Dol}}$, we have $(\mathcal{L} \otimes \mathcal{E}, \theta) \in \mathcal{M}_{\mathrm{Dol}}$. Hence the finite abelian group

$$\Gamma = \mathrm{Pic}^0(C)[n] \simeq (\mathbb{Z}/n\mathbb{Z})^{2g}$$

acts on $\mathcal{M}_{\mathrm{Dol}}$, with the quotient

$$\hat{\mathcal{M}}_{\mathrm{Dol}} = \mathcal{M}_{\mathrm{Dol}}/\,\Gamma$$

a Deligne–Mumford stack parametrizing stable PGL_n-Higgs bundles. The Hitchin map $\pi : \mathcal{M}_{\mathrm{Dol}} \to \Lambda$ is Γ-equivariant with the trivial action on the Hitchin base Λ. The PGL_n-Hitchin map $\hat{\pi} : \hat{\mathcal{M}}_{\mathrm{Dol}} \to \Lambda$ fits into the commutative diagram

$$\begin{tikzcd} \mathcal{M}_{\mathrm{Dol}} \arrow[rr] \arrow[dr, "\pi"'] & & \hat{\mathcal{M}}_{\mathrm{Dol}} \arrow[dl, "\hat{\pi}"] \\ & \Lambda & \end{tikzcd} \tag{7}$$

where the horizontal arrow is the quotient map. We obtain from (7) the canonical isomorphism

$$H^*(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q})^{\Gamma} = H^*(\hat{\mathcal{M}}_{\mathrm{Dol}}, \mathbb{Q}) \tag{8}$$

compatible with the perverse filtrations,

$$P_k H^*(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q})^{\Gamma} = P_k H^*(\hat{\mathcal{M}}_{\mathrm{Dol}}, \mathbb{Q}),$$

Here the perverse filtration for $\hat{\mathcal{M}}_{\mathrm{Dol}}$ is associated with $\hat{\pi} : \hat{\mathcal{M}}_{\mathrm{Dol}} \to \Lambda$.

We also have the corresponding Γ-action on the Betti moduli space $\mathcal{M}_B$. More precisely, we view Γ as a sub-group of $(\mathbb{C}^*)^{\times 2g}$, which acts on the matrices $a_i, b_i \in \mathrm{SL}_n$ of (4) by multiplication. The Γ-action on $\mathcal{M}_B$ is induced by the action of the rank 1 character variety $(\mathbb{C}^*)^{\times 2g}$ on the GL_n-twisted character variety, which, via the non-abelian Hodge correspondence, coincides with the action of the rank 1 Hitchin moduli space $T^*\mathrm{Pic}^0(C)$ on the GL_n-Hitchin moduli space. Hence the Γ-decomposition

$$H^*(\mathcal{M}_B, \mathbb{Q}) = H^*(\mathcal{M}_B, \mathbb{Q})^\Gamma \bigoplus H^*_{\mathrm{var}}(\mathcal{M}_B, \mathbb{Q})$$

matches the Γ-decomposition (2) for $\mathcal{M}_{\mathrm{Dol}}$ via the non-abelian Hodge correspondence (5). Analagous to (8), we have a canonical isomorphism of mixed Hodge structures

$$H^*(\mathcal{M}_B, \mathbb{Q})^\Gamma = H^*(\hat{\mathcal{M}}_B, \mathbb{Q}) \tag{9}$$

with $\hat{\mathcal{M}}_B$ the PGL_n-character variety diffeomorphic to $\hat{\mathcal{M}}_{\mathrm{Dol}}$ via the non-abelian Hodge correspondence for PGL_n.

In conclusion, we have the following proposition concerning the $P = W$ for the Γ-invariant cohomology.

Proposition 1.2 *Assume that the $P = W$ conjecture (1) holds for the curve C, the group $G = \mathrm{GL}_n$, and the degree d. Then we have*

$$P_k H^*(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q})^\Gamma = W_{2k} H^*(\mathcal{M}_B, \mathbb{Q})^\Gamma, \quad \forall k \geq 0.$$

The following is a consequence of Proposition 1.2 and [4, Theorem 0.2].

Corollary 1.3 *When the curve C has genus $g = 2$, we have*

$$P_k H^*(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q})^\Gamma = W_{2k} H^*(\mathcal{M}_B, \mathbb{Q})^\Gamma, \quad \forall k \geq 0.$$

1.4 The variant cohomology

In view of Proposition 1.2 and Corollary 1.3, our main purpose of this paper is to understand the perverse filtration and the weight filtration on the variant cohomology

$$H^*_{\mathrm{var}}(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q}) = H^*_{\mathrm{var}}(\mathcal{M}_B, \mathbb{Q}).$$

Proposition 1.4 *Let p be the smallest prime divisor of n. We have*

$$P_{k-n(n-n/p)(g-1)} H^k_{\mathrm{var}}(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q}) = H^k_{\mathrm{var}}(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q}). \tag{10}$$

Proof The argument here is a generalization of the first part of the proof of [3, Theorem 4.4.6] which treated the case $n = 2$. Here we apply results of Hausel–Pauly [12] and Theorem 1.1.

Let $\Lambda' \subset \Lambda$ be a general linear subspace of dimension

$$\dim(\Lambda') = n(n - n/p)(g - 1) - 1. \tag{11}$$

Assume $\mathcal{M}_{\Lambda'} = \pi^{-1}(\Lambda') \subset \mathcal{M}_{\mathrm{Dol}}$. In order to prove (10), by Theorem 1.1 it suffices to show

$$r\left(H^k_{\mathrm{var}}(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q})\right) = 0 \tag{12}$$

where r is the restriction morphism

$$r : H^k(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q}) \to H^k(\mathcal{M}_{\Lambda'}, \mathbb{Q}). \tag{13}$$

We consider the endoscopic loci $\Lambda_{\mathrm{endo}} \subset \Lambda$ defined in [12, Corollary 1.3], which is formed by $a \in \Lambda$ such that the Prym variety $\mathrm{Prym}(C_a/C)$ associated with the corresponding spectral curve C_a is not connected. By [12, Lemma 7.1], we have

$$\mathrm{codim}_\Lambda(\Lambda_{\mathrm{endo}}) = n(n - n/p)(g-1). \tag{14}$$

Since Λ' is general, it is completely contained in $\Lambda \setminus \Lambda_{\mathrm{endo}}$ by (11) and (14). An identical argument as in the first paragraph of [12, Proof of Theorem 1.4] implies that Γ acts trivially on $H^k(\mathcal{M}_{\Lambda'}, \mathbb{Q})$, *i.e.*,

$$H^k_{\mathrm{var}}(\mathcal{M}_{\Lambda'}, \mathbb{Q})) = 0.$$

On the other hand, the Γ-action is fiberwise with respect to the Hitchin map $\pi : \mathcal{M}_{\mathrm{Dol}} \to \Lambda$, and the restriction morphism (13) is Γ-equivariant. In particular, we see that

$$r\left(H^k_{\mathrm{var}}(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q})\right) \subset H^k_{\mathrm{var}}(\mathcal{M}_{\Lambda'}, \mathbb{Q})) = 0.$$

This completes the proof of (12). □

1.5 Main results

The following theorem is our main result, which generalizes [3, Theorems 4.4.6 and 4.4.7] for $n = 2$. It computes the perverse filtration and the weight filtration explicitly on the variant cohomology for SL_n with n a prime number.

Theorem 1.5 *Assume n is a prime number, and assume*

$$c_n := n(n-1)(g-1).$$

(a) We have

$$0 = P_{k-c_n-1}H^k_{\mathrm{var}}(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q}) \subset P_{k-c_n}H^k_{\mathrm{var}}(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q}) = H^k_{\mathrm{var}}(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q}).$$

(b) We have

$$0 = W_{2(k-c_n)-1}H^k_{\mathrm{var}}(\mathcal{M}_B, \mathbb{Q}) \subset W_{2(k-c_n)}H^k_{\mathrm{var}}(\mathcal{M}_B, \mathbb{Q}) = H^k_{\mathrm{var}}(\mathcal{M}_B, \mathbb{Q}).$$

We prove Theorem 1.5 in Sect. 3. It is clear that Theorem 1.5 implies Theorem 0.2. Hence we complete the proof of Theorem 0.1 by combining Corollary 1.3.

Remark 1.6 Proposition 1.2 and Theorem 1.5 combined shows that, when n is a prime number, the $P = W$ conjecture for SL_n is equivalent to the $P = W$ conjecture for GL_n.

For general n, the perverse filtration on the variant cohomology $H^k_{\mathrm{var}}(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q})$ for the SL_n-Hitchin moduli space $\mathcal{M}_{\mathrm{Dol}}$ is expected to be more complicated. In view of [14], the variant cohomology is governed by the Hitchin moduli spaces of endoscopic groups attached to irreducible non-trivial characters of $\Gamma = \mathrm{Pic}^0(C)[n]$. These endoscopic moduli spaces are further related to the $\mathrm{GL}_{n/d}$-Hitchin moduli space associated with a curve $\widetilde{C}$ given by a degree d Galois cover of C, where d runs through *all* divisors of n. We will discuss this in a future paper.

In particular, when n is prime, the relevant endoscopic Higgs bundles are of rank 1, with the corresponding moduli space the total cotangent bundle of a Prym variety. Therefore the associated perverse filtrations are trivial. This is the heuristic reason that the perverse filtrations on the variant cohomology are of the form Theorem 1.5 (a).

2 Characterizations for k-sequences

2.1 *k*-sequences

We consider double indexed sequences

$$\{v_{i,j} \in \mathbb{N}\}_{i,j} \tag{15}$$

satisfying $v^{i,j} = 0$ when $i < 0$ or $j < 0$. For convenience, we assume that all indices are non-negative integers.

We say that (15) is a *k-sequence* if $v^{i,j} = 0$ when $j \neq k$. The purpose of Sect. 2 is to give two criteria for k-sequences.

2.2 The first criterion

Proposition 2.1 *For fixed $m, k \in \mathbb{N}_{>0}$, we assume that (15) satisfies the following conditions:*

(i) $v^{i,j} = 0$ if $j < k$;
(ii) $v^{m-i,j} = v^{m+i,j}$ for any i, j;
(iii) The following identify holds for any $l \geq 0$,

$$\sum_{i+j=m+k-l} v^{i,j} = \sum_{i+j=m+k+l} v^{i,j}.$$

Then (15) is a k-sequence.

Proof By (i), it suffices to show that

$$v^{i,j} = 0, \quad \text{if } \; k < j. \tag{16}$$

We prove this by induction on the value $i+j$. The induction base is the case $i+j=k$ where (16) is clearly true.

We now assume that (16) holds if $i+j<d_0$. To complete the induction, we need to show that $v^{d_0-j,j}=0$ for $k<j$. The condition (ii) implies that $v^{d_0-j,j}=v^{2m-d_0+j,j}$. On the other hand, by (iii), we have

$$v^{2m-d_0+j,j}+v^{2m-d_0+2j-k,k} \\ \leq \sum_{i+j=2m-d_0+2j} v^{i,j} = \sum_{i+j=d_0-2j+2k} v^{i,j} = v^{d_0-2j+k,k} \tag{17}$$

where we apply the induction assumption in the last equation (since $d_0-2j+2k<d_0$). We deduce from (17) and (ii) that

$$v^{2m-d_0+j,j} \leq v^{d_0-2j+k,k} - v^{2m-d_0+2j-k,k} = 0.$$

Hence we have $v^{d_0-j,j}=v^{2m-d_0+j,j}=0$ which completes the induction. □

2.3 The second criterion

Proposition 2.2 *For fixed $m,k\in\mathbb{N}_{>0}$, we assume that (15) satisfies the following conditions:*

(i) $v^{i,j}=v^{2m+2k-i-2j,j}$ *for any* i,j.
(ii) The following identity holds for any $l\geq 0$,

$$\sum_{i+j=k+l} v^{i,j} = \sum_j v^{l,j}.$$

(iii) The following identify holds for any $i\geq 0$,

$$\sum_j v^{m+i,j} = \sum_j v^{m-i,j}.$$

Then (15) is a k-sequence.

Proof We prove that

$$v^{i,j}=0, \quad \text{if} \quad k\neq j \tag{18}$$

by induction on the value $i+j$.

If $i+j\leq k$ and $j<k$, we have $v^{i,j}=v^{2m+2k-i-2j,j}$ by (i). Then (iii) implies that

$$v^{2m+2k-i-2j,j} \leq \sum_l v^{2m+2k-i-2j,l} = \sum_l v^{i+2j-2k,l} = 0,$$

since $i+2j-2k<0$. Hence $v^{i,j}=0$ if $i+j<k$, and $v^{i,k-i}=0$ if $i>0$. This provides the induction base.

Now assume that (18) holds if $i+j<d_0$. We first show that $v^{d_0-j,j}=0$ if $j>k$. In fact, by (ii) we have

$$v^{d_0-j,j}+v^{d_0-j,k}\leq\sum_{j'}v^{d_0-j,j'}=\sum_{i_1+i_2=k+(d_0-j)}v^{i_1,i_2}. \tag{19}$$

Then, since $k+(d_0-j)<d_0$, the induction assumption further implies

$$\sum_{i_1+i_2=k+(d_0-j)}v^{i_1,i_2}=v^{d_0-j,k}. \tag{20}$$

Combining (19) and (20), we have $v^{d_0-j,j}=0$ if $j>k$.

It remains to show that $v^{d_0-j,j}=0$ if $j<k$. In this case, we have

$$v^{d_0-j,j}=v^{2m+2k-d_0-j,j}$$

by (i). The condition (ii) further implies that

$$\begin{aligned}&v^{2m+2k-d_0-j,j}+v^{2m+2k-d_0-j,k}\\&\leq\sum_{j'}v^{2m+2k-d_0-j,j'}=\sum_{i_1+i_2=2m+3k-d_0-j}v^{i_1,i_2}.\end{aligned} \tag{21}$$

For $i_1+i_2=2m+3k-d_0-j$, we have by (i) that $v^{i_1,i_2}=v^{j_1,j_2}$ with

$$j_1+j_2=2m+2k-(2m+3k-d_0-j)=d_0+j-k<d_0.$$

Hence (i) and the induction assumption yield

$$\sum_{i_1+i_2=2m+3k-d_0-j}v^{i_1,i_2}=v^{2m+2k-d_0-j,k}. \tag{22}$$

Combining (21) and (22), we obtain

$$v^{d_0-j,j}=v^{2m+2k-d_0-j,j}=0$$

which completes the induction. □

3 Perverse filtrations and weight filtrations

Throughout the section, we assume that n is a prime number, and complete the proof of Theorem 1.5. For the proof, we apply the numerical criteria of Sect. 2 combined with the following ingredients:

(a) Hausel–Thaddeus' topological mirror symmetry conjecture for Hitchin systems [14], and its recent proof by Gröcheneg–Wyss–Ziegler [8].
(b) The E-polynomials of character varieties calculated by Hausel– Rodriguez–Villegas [13] and Mereb [20] via point counting over finite fields.

3.1 The topological mirror symmetry conjecture

Recall that the virtual Hodge polynomial $H(X; u, v)$ of an algebraic variety X is

$$H(X; t, u, v) = \sum_{i,j,k} h^{j,k}\left(\mathrm{Gr}^W_{j+k} H^i_c(X, \mathbb{C})\right) t^i u^j v^k$$

where Gr^W_* is the graded piece with respect to the weight filtration. The E-polynomial of X is the specialization

$$E(X; u, v) = H(X; -1, u, v).$$

The topological mirror symmetry conjecture proposed by Hausel–Thaddeus [14] relates the E-polynomial of the SL_n-Hitchin moduli space $\mathcal{M}_{\mathrm{Dol}}$ to the *stringy* E-polynomial of the PGL_n-Hitchin moduli space $\hat{\mathcal{M}}_{\mathrm{Dol}}$. A generalized version of the Hausel–Thaddeus conjecture was proven by Gröchenig–Wyss–Ziegler [8] via the method of p-adic integrations; see also [9].

When n is a prime number, we obtain the following closed formula for the E-polynomial of the variant cohomology of $\mathcal{M}_{\mathrm{Dol}}$ from a direct calculation of the stringy E-polynomial of the PGL_n-Hitchin moduli space $\hat{\mathcal{M}}_{\mathrm{Dol}}$; see [14, Proposition 8.2].

Proposition 3.1 *(Topological mirror symmetry [8, 14]) Let n be a prime number. Then we have*

$$\begin{aligned} &E(\mathcal{M}_{\mathrm{Dol}}; u, v) - E(\hat{\mathcal{M}}_{\mathrm{Dol}}; u, v) \\ &\quad = \frac{n^{2g}-1}{n}(uv)^{(n^2-1)(g-1)}\left(((u-1)(v-1))^{(n-1)(g-1)}\right. \\ &\quad \left. - \left((1+u+\cdots+u^{n-1})(1+v+\cdots+v^{n-1})\right)^{(g-1)}\right). \end{aligned} \tag{23}$$

We denote $E(q)$ to be the polynomial by setting $u = v = q$ on the righthand side of (23),

$$E(q) := \frac{n^{2g}-1}{n} q^{\dim(\mathcal{M}_{\mathrm{Dol}})}\left((q-1)^{(n-1)(2g-2)} - (1+q+\cdots+q^{n-1})^{2g-2}\right), \tag{24}$$

which is palindromic satisfying

$$E(q) = q^{(2g-2)(2n^2+n-3)} E\left(\frac{1}{q}\right). \tag{25}$$

We denote $[E(q)]_{q^i}$ to be the coefficient of q^i in the polynomial expansion of $E(q)$.

Corollary 3.2 *We have*

$$\dim\left(H^d_{\mathrm{var}}(\mathcal{M}_{\mathrm{Dol}},\mathbb{Q})\right) = (-1)^d[E(q)]_{q^{2\dim(\mathcal{M}_{\mathrm{Dol}})-d}}.$$

Proof Since the cohomology groups H^k of the moduli spaces $\mathcal{M}_{\mathrm{Dol}}$ and $\hat{\mathcal{M}}_{\mathrm{Dol}}$ are pure of weights k, their E-polynomials recover the virtual Hodge polynomials. Corollary 3.2 follows from the Poincaré duality and (8). □

3.2 Proof of Theorem 1.5 (a).

We define

$$v_P^{i,j} := \dim\left(\mathrm{Gr}_i^P H^{i+j}_{\mathrm{var}}(\mathcal{M}_{\mathrm{Dol}},\mathbb{Q})\right) \tag{26}$$

with Gr_*^P the graded piece of the perverse filtration. Recall c_n from Theorem 1.5. It suffices to show that (26) forms a c_n-sequence.

We check that (26) satisfies (i,ii,iii) of Proposition 2.1 for

$$k = c_n, \quad m = \frac{1}{2}\dim(\mathcal{M}_{\mathrm{Dol}}) = (n^2-1)(g-1). \tag{27}$$

The condition (i) follows directly from Proposition 1.4. The condition (ii),

$$\dim\left(\mathrm{Gr}^P_{m-i} H^{m-i+j}_{\mathrm{var}}(\mathcal{M}_{\mathrm{Dol}},\mathbb{Q})\right) = \dim\left(\mathrm{Gr}^P_{m+i} H^{m+i+j}_{\mathrm{var}}(\mathcal{M}_{\mathrm{Dol}},\mathbb{Q})\right),$$

follows from the Relative Hard Lefschetz [2] with respect to the Hitchin map $\pi : \mathcal{M}_{\mathrm{Dol}} \to \Lambda$, and its compatibility with the Γ-decomposition.

Since

$$\dim\left(H^d_{\mathrm{var}}(\mathcal{M}_{\mathrm{Dol}},\mathbb{Q})\right) = \sum_{i+j=d} v_P^{i,j},$$

the condition (iii) is equivalent to

$$\dim\left(H^{m+c_n-i}_{\mathrm{var}}(\mathcal{M}_{\mathrm{Dol}},\mathbb{Q})\right) = \dim\left(H^{m+c_n+i}_{\mathrm{var}}(\mathcal{M}_{\mathrm{Dol}},\mathbb{Q})\right),$$

which follows from Corollary 3.2 and the symmety (25),

$$[E(q)]_{q^i} = [E(q)]_{q^j}, \quad \text{if} \quad i+j = 6m-2c_n = (2n^2+n-3)(2g-2).$$

This completes the proof. □

3.3 A symmetry

We see from Corollary 3.2 that $H^2(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q}) = H^2(\hat{\mathcal{M}}_{\mathrm{Dol}}, \mathbb{Q})$. So there is only one class η spanning $H^2(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q})$ (see [18]), and it is relatively ample with respect to the Hitchin map. As a consequence of Theorem 1.5 (a), we obtain the following symmetry on the cohomology of $\mathcal{M}_{\mathrm{Dol}}$.

Corollary 3.3 *Cupping with a power of the class η induces an isomorphism*

$$\eta^i : H^{m+c_n-i}_{\mathrm{var}}(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q}) \xrightarrow{\simeq} H^{m+c_n+i}_{\mathrm{var}}(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q}), \quad \forall i, m. \tag{28}$$

Proof The Relative Hard Lefschetz Theorem implies that

$$\eta^i : \mathrm{Gr}^P_{m-i} H^{m+j-i}_{\mathrm{var}}(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q}) \xrightarrow{\simeq} \mathrm{Gr}^P_{m+i} H^{m+j+i}_{\mathrm{var}}(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q}).$$

Since (26) is a c_n-decomposition by Theorem 1.5 (a), the only non-trivial isomorphisms (3.3) are those with $j = c_n$, and Corollary 3.3 follows. □

Remark 3.4 In general, if n is not prime, (28) does not hold. In particular, Corollary 3.3 relies heavily on the fact that (26) is a c_n-sequence, which, by the proof of Theorem 1.5 (a), further relies on the symmetries of the coefficients of the polynomial $E(q)$.

3.4 *E*-polynomials of character varieties

Recall the polynomial $E(q)$ introduced in (24). In view of (9), We define the variant E-polynomial

$$E_{\mathrm{var}}(\mathcal{M}_B; u, v) := E(\mathcal{M}_B; u, v) - E(\hat{\mathcal{M}}_B; u, v).$$

The following proposition calculates the variant E-polynomial for $\mathcal{M}_B$. We note that the two sides of the equation (29) are of completely different flavors. The left-hand side is governed by point counting over finite fields via the character tables of $\mathrm{GL}_n(\mathbb{F}_q)$ and $\mathrm{SL}_n(\mathbb{F}_q)$, while the right-hand side calculates suitable cohomology groups of the moduli of certain endoscopic Higgs bundles.

Proposition 3.5 *We have*

$$E_{\mathrm{var}}(\mathcal{M}_B; u, v) \cdot (uv)^{(n^2+n-2)(g-1)} = E(uv). \tag{29}$$

Proof The result of Katz [13, Appendix] and the calculations of [13, 20] imply that the E-polynomials $E(\mathcal{M}_B; u, v)$ and $E(\hat{\mathcal{M}}_B; u, v)$ are polynomials in the variable $q = uv$. Hence it suffices to show that

$$E_{\mathrm{var}}(\mathcal{M}_B; \sqrt{q}, \sqrt{q}) = \frac{n^{2g} - 1}{n} q^{(n^2-n)(g-1)}$$

$$\left((q-1)^{(n-1)(2g-2)}-(1+q+\cdots+q^{n-1})^{(2g-2)}\right). \tag{30}$$

By [13, Equation (3.2.4)] and [20, Theorem 3.4], we have

$$E(\hat{\mathcal{M}}_B;\sqrt{q},\sqrt{q})=\sum_{\tau}\left(q^{\frac{n^2}{2}}\frac{\mathcal{H}_{\tau'}(q)}{q-1}\right)^{2g-2}C^0_{\tau}; \tag{31}$$

$$E(\mathcal{M}_B;\sqrt{q},\sqrt{q})=\sum_{\tau,t}\left(q^{\frac{n^2}{2}}\frac{\mathcal{H}_{\tau'}(q)}{q-1}\right)^{2g-2}t^{2g-1}C^t_{\tau}. \tag{32}$$

Here we follow the notation of [13, 20]: the summation in (31) is taken over types τ of size n multi-partitions and the summation in (32) is taken over τ of size n multi-partitions and divisors t of n (see [20, Sect. 2.5]); the polynomial $\mathcal{H}_{\tau'}(q)$ is the *normalized hook polynomial* associated with the conjugate τ' of the partition τ [20, Sect. 3.6]; the constant C^0_{τ} is given by [20, Equation (7)], and [20, Equation (33)] expresses every C^n_{τ} in terms of C^0_{τ}.[1]

Now we calculate the difference of (31) and (32).

For our purpose, we focus on 2 types of multi-partitions τ_1 and τ_2 as follows. Recall the type $\tau=(m_{\lambda,d})_{\lambda,d\geq 1}$ of a multi-partition from [20, Definition 2.1]. Let τ_1 be the type of the multi-partition with the only non-trivial multiplicity $m_{(1^1),1}=n$, and we calculate directly that

$$\mathcal{H}_{\tau_1'}(q)=\left(q^{-\frac{1}{2}}(1-q)\right)^n=q^{-\frac{n}{2}}(1-q)^n. \tag{33}$$

Let τ_2 be the type of the multi-partition with the only non-trivial multiplicity $m_{(1^1),n}=1$, and we have

$$\mathcal{H}_{\tau_2'}(q)=q^{-\frac{n}{2}}(1-q^n). \tag{34}$$

Furthermore, by a direct calculation using the concrete formula [20, Equation (33)] for the constants C^t_{τ}, we obtain that

(a) $C^1_{\tau}=C^0_{\tau},\quad C^n_{\tau}=0$, for $\tau\neq\tau_1,\tau_2$;
(b) $C^1_{\tau_1}=C^0_{\tau_1}-\frac{1}{n}$, $C^n_{\tau_1}=1$;
(c) $C^1_{\tau_2}=C^0_{\tau_2}+\frac{1}{n}$, $C^n_{\tau_2}=-1$.

Since n is a prime number and t divides n, the integer t is either 1 or n on the right-hand side of (32),

$$E(\mathcal{M}_B;\sqrt{q},\sqrt{q})=\sum_{\tau}\left(q^{\frac{n^2}{2}}\frac{\mathcal{H}_{\tau'}(q)}{q-1}\right)^{2g-2}C^1_{\tau}+\sum_{\tau}\left(q^{\frac{n^2}{2}}\frac{\mathcal{H}_{\tau'}(q)}{q-1}\right)^{2g-2}n^{2g-1}C^n_{\tau}. \tag{35}$$

[1] See [20, Equation (34)] for the connection between C^0_{τ} and the coefficients C_{τ} used in [13].

By (a,b,c), (33), and (31), we have

$$\sum_{\tau}\left(q^{\frac{n^2}{2}}\frac{\mathcal{H}_{\tau'}(q)}{q-1}\right)^{2g-2}C_{\tau}^{1} = E(\hat{\mathcal{M}}_B;\sqrt{q},\sqrt{q})$$
$$+\left(q^{\frac{n^2}{2}}\frac{q^{-\frac{n}{2}}(1-q)^n}{q-1}\right)^{2g-2}\cdot\left(-\frac{1}{n}\right)+\left(q^{\frac{n^2}{2}}\frac{q^{-\frac{n}{2}}(1-q^n)}{q-1}\right)^{2g-2}\cdot\left(\frac{1}{n}\right). \tag{36}$$

Similarly, (a,b,c) and (34) yield

$$\sum_{\tau}\left(q^{\frac{n^2}{2}}\frac{\mathcal{H}_{\tau'}(q)}{q-1}\right)^{2g-2}n^{2g-1}C_{\tau}^{n}$$
$$= +\left(q^{\frac{n^2}{2}}\frac{q^{-\frac{n}{2}}(1-q)^n}{q-1}\right)^{2g-2}n^{2g-1}+\left(q^{\frac{n^2}{2}}\frac{q^{-\frac{n}{2}}(1-q^n)}{q-1}\right)^{2g-2}n^{2g-1}\cdot(-1). \tag{37}$$

We complete the proof of (30) by combining (35), (36), and (37). ◻

3.5 Vanishing and Hodge–Tate

We prove some properties of the variant cohomology of $\mathcal{M}_B$ which play a crucial role in the proof of Theorem 1.5 (b). We denote

$$w^{i,j} := \dim\left(\mathrm{Gr}_i^W H^j_{\mathrm{var},c}(\mathcal{M}_B,\mathbb{Q})\right)$$

where $H^*_{\mathrm{var},c}$ is the variant part of the compactly support cohomology.

Lemma 3.6 *If i is odd, or $i = 2i'$ with $i'+j$ odd, we have $w^{i,j} = 0$.*

Proof By Proposition 3.5, we have

$$E_{\mathrm{var}}(\mathcal{M}_B;q,q) = \sum_{i,j}(-1)^j w^{i,j}\cdot q^i = q^{-(n^2+n-2)(2g-2)}E(q^2).$$

In particular $w^{i,j} = 0$ if i is odd. Together with Corollary 3.2, we have the expressions

$$E_{\mathrm{var}}(\mathcal{M}_B;q,q) = \sum_{i',j}(-1)^j w^{2i',j}q^{2i'},\quad E(q^2) = \sum_{i',j}(-1)^j w^{2i',j}q^{2(2i'+j)}. \tag{38}$$

Proposition 3.5 further implies that

$$\sum_{i',j}(-1)^{i'+j}w^{2i',j} = \sum_{i',j}w^{2i',j}$$

by setting $q^2 = -1$ in the equations (38). Thus $w^{2i',j} = 0$ if $i' + j$ is odd. $\square$

The vanishing of Lemma 3.6 implies that there is no cancellation of Hodge numbers in calculating each term of the E-polynomial $E_{\mathrm{var}}(\mathcal{M}_B; u, v)$. In particular, we deduce the following lemma from Proposition 3.5 that the mixed Hodge structures on the variant cohomology groups $H^d_{\mathrm{var},c}(\mathcal{M}_B, \mathbb{Q})$ are of Hodge–Tate types.

Lemma 3.7 *The mixed Hodge structure on $H^d_{\mathrm{var},c}(\mathcal{M}_B, \mathbb{Q})$ is of Hodge–Tate type,* i.e.,

$$h^{i,j}(\mathrm{Gr}^W_{i+j} H^d_{\mathrm{var},c}(\mathcal{M}_B, \mathbb{Q})) = 0, \quad \text{if } i \neq j.$$

As a corollary of Lemma 3.7 and the Poincaré duality, we obtain that $H^d_{\mathrm{var}}(\mathcal{M}_B, \mathbb{Q})$ is also of Hodge–Tate type.

Corollary 3.8 *The mixed Hodge structure on $H^d_{\mathrm{var}}(\mathcal{M}_B, \mathbb{Q})$ is of Hodge–Tate type.*

3.6 Proof of Theorem 1.5 (b).

We use $F^\bullet H^*(X, \mathbb{C})$ to denote the Hodge filtration on the cohomology of an algebraic variety X. The Hodge filtration on $H^*(\mathcal{M}_B, \mathbb{C})$ induces a Hodge filtration $F^\bullet H^*_{\mathrm{var}}(\mathcal{M}_B, \mathbb{C})$ on the variant cohomology.

We define the sub-vector spaces

$${}^k\mathrm{Hdg}^d_{\mathrm{var}}(\mathcal{M}_B) := F^k H^d_{\mathrm{var}}(\mathcal{M}_B, \mathbb{C}) \cap W_{2k} H^d_{\mathrm{var}}(\mathcal{M}_B, \mathbb{Q}) \subset H^d_{\mathrm{var}}(\mathcal{M}_B, \mathbb{Q}).$$

We obtain from Corollary 3.8 that

$$\dim\left({}^k\mathrm{Hdg}^d_{\mathrm{var}}(\mathcal{M}_B)\right) = \dim\left(\mathrm{Gr}^W_{2k} H^d(\mathcal{M}_B, \mathbb{Q})\right). \tag{39}$$

Recall the class $\eta \in H^2(\mathcal{M}_B, \mathbb{Q})$ introduced in Sect. 3.3, which lies in ${}^2\mathrm{Hdg}^2_{\mathrm{var}}(\mathcal{M}_B)$ by [24]. Hence, Corollary 3.3 implies that cupping with η^i induces an isomorphism

$$\eta^i : {}^r\mathrm{Hdg}^{r+c_n-i}_{\mathrm{var}}(\mathcal{M}_B) \xrightarrow{\simeq} {}^{r+2i}\mathrm{Hdg}^{r+c_n+i}_{\mathrm{var}}(\mathcal{M}_B), \quad \forall r \in \mathbb{N}. \tag{40}$$

Now we consider

$$v_W^{i,j} := \dim\left({}^i\mathrm{Hdg}^{i+j}_{\mathrm{var}}(\mathcal{M}_B)\right).$$

In view of (39), it suffices to check that $\{v_W^{i,j}\}_{i,j}$ satisfies (i,ii,iii) of Proposition 2.2 with k and m given by (27).

The condition (i) follows from (40). Next, we verify the condition (iii). By Lemma 3.6 and Proposition 3.5, each summation $\sum_j v_W^{i,j}$ is given by a coefficient of the polynomial $E(q)$, and the condition (iii) follows from the symmetry (25).

Finally, we obtain from Proposition 3.5 and the equation (25) that

$$E\left(\frac{1}{q}\right)q^{2\dim(\mathcal{M}_B)} = E_{\mathrm{var}}(\mathcal{M}_B; \sqrt{q}, \sqrt{q})q^{c_n}. \tag{41}$$

By Corollary 3.2, the left-hand side of (41) computes

$$\dim\left(H^d_{\mathrm{var}}(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q})\right) = \sum_{i+j=d} v_W^{i,j},$$

while the right-hand side computes $\sum_j v_W^{d-c_n,j}$ by the definition of E-polynomials and the vanishing of Lemma 3.6. Hence the condition (ii) holds. This completes the proof. □

4 Hitchin moduli spaces and compact hyper-Kähler manifolds

4.1 Overview

A crucial step in the proof of the $P = W$ conjecture for genus 2 and GL_n in [4] is to use degenerations connecting certain compact hyper–Kähler manifolds and Hitchin moduli spaces. More precisely, we embed a genus 2 curve C into an abelian surface A,

$$j : C \hookrightarrow A. \tag{42}$$

The degeneration to the normal cone associated with (42) yields a flat family

$$\mathcal{M} \to \mathbb{A}^1. \tag{43}$$

Its general fiber is a compact (non-simply connected) hyper–Kähler manifold $\mathcal{M}_{n[C],A}$ which is the moduli of certain stable 1-dimensional sheaves supported on the curve class

$$n[C] \in H_2(A, \mathbb{Z}),$$

and its central fiber is the GL_n-Hitchin moduli space $\mathcal{M}_{\mathrm{Dol}}^{\mathrm{GL}_n}$. See [10] and [4, Sect. 4.2] for more details about this degeneration.

We construct in [4, Sect. 4.3] a *surjective* specialization morphism

$$\mathrm{sp}^! : H^*(\mathcal{M}_{n[C],A}, \mathbb{Q}) \to H^*(\mathcal{M}_{\mathrm{Dol}}^{\mathrm{GL}_n}, \mathbb{Q}) \tag{44}$$

which is a morphism of $\mathbb{Q}$-algebras preserving the perverse filtrations and tautological classes constructed from universal families. Hence the morphism (44) governs the tautological generators in $H^*(\mathcal{M}_{\mathrm{Dol}}^{\mathrm{GL}_n}, \mathbb{Q})$.

A degeneration similar to $\mathcal{M} \to \mathbb{A}^1$ can also be constructed for the SL_n-Hitchin moduli space $\mathcal{M}_{\mathrm{Dol}}$. More precisely, under the degeneration (43), the albenese map (see [29])

$$\mathcal{M}_{n[C],A} \to \mathrm{Pic}^d(A) \times A$$

degenerates to the morphism

$$\det \times \mathrm{trace} : \mathcal{M}_{\mathrm{Dol}}^{\mathrm{GL}_n} \to \mathrm{Pic}^d(C) \times \mathbb{A}^2.$$

By taking fibers, we obtain a flat family $\mathcal{M}^{\mathrm{SL}} \to \mathbb{A}^1$ with general fiber $\mathcal{K}_{n[C],A}$ an irreducible hyper–Kähler manifold of Kummer type [2] and central fiber the SL_n-Hitchin moduli space $\mathcal{M}_{\mathrm{Dol}}$. Moreover, the variety $\mathcal{K}_{n[C],A}$ admits a Lagrangian fibration

$$\mathcal{M}_{\mathrm{Dol}} \to \mathbb{P}^N = |nC|$$

degenerating to the Hitchin map $\pi : \mathcal{M}_{\mathrm{Dol}} \to \Lambda$.[3] By the construction in [4, Sect. 4.3], this yields a specialization morphism

$$\mathrm{sp}^! : H^*(\mathcal{K}_{n[C],A}, \mathbb{Q}) \to H^*(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q}) \tag{45}$$

preserving the perverse filtrations. It is natural to ask whether (45) is surjective. More general, we are interested in exploring whether the cohomology of $H^*(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q})$ can be governed by the cohomology of a compact irreducible hyper-Kähler manifold, so that we can extend the method of [4] to studying the perverse filtration for the SL_n-Hitchin system $\pi : \mathcal{M}_{\mathrm{Dol}} \to \Lambda$.

Question 4.1 *Does there exist a grading preserved surjective morphism*

$$f : H^*(M, \mathbb{Q}) \to H^*(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q}) \tag{46}$$

of graded $\mathbb{Q}$-algebras such that M is a compact irreducibel hyper-Kähler manifold?

In this Section, we discuss obstructions to the existence of (46).

4.2 An obstruction for SL_2

From now on, let $\mathcal{M}_{\mathrm{Dol}}$ be the moduli space of stable Higgs bundles attached to a genus 2 curve C, the group SL_2, and a degree 1 line bundle $L \in \mathrm{Pic}^1(C)$; see Sect. 1. The variety $\mathcal{M}_{\mathrm{Dol}}$ is nonsingular of dimension 6.

The following proposition provides a necessary condition for the cohomology of $\mathcal{M}_{\mathrm{Dol}}$ to be governed by the cohomology of another manifold M.

---

[2] We call M an *irreducible hyper-Kähler manifold* if M is simply connected satisfying that $H^0(M, \Omega_M)$ is generated by a non-where degenerate holomorphic 2-form. We say that a hyper-Kähler manifold is of Kummer type if it deforms to a generalized Kummer variety.

[3] Since this construction is not essentially used in the present paper, we omit further details.

Proposition 4.2 *Assume M is a manifold with a grading preserved surjective morphism*

$$f: H^*(M, \mathbb{Q}) \to H^*(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q}) \tag{47}$$

of graded $\mathbb{Q}$-algebras. Then we have

$$\dim \left(H^5(M, \mathbb{Q}) / \left[(H^2(M, \mathbb{Q}) \cup H^3(M, \mathbb{Q}) \right] \right) \geq 30. \tag{48}$$

Proof Assume that (47) is surjective. Recall the decomposition (2). By [15], we have

$$H^*_{\mathrm{var}}(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q}) = H^5_{\mathrm{var}}(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q}), \quad \dim \left(H^5_{\mathrm{var}}(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q}) \right) = 30. \tag{49}$$

Since $H^2(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q})$ and $H^3(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q})$ lie in the invariant part $H^*(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q})^\Gamma$ and f is grading preserved, we have

$$f \left[(H^2(M, \mathbb{Q}) \cup H^3(M, \mathbb{Q}) \right] \subset H^5(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q})^\Gamma.$$

Hence we obtain a surjective morphism

$$H^5(M, \mathbb{Q}) / \left[(H^2(M, \mathbb{Q}) \cup H^3(M, \mathbb{Q}) \right] \to H^5_{\mathrm{var}}(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q})$$

which implies (48). □

4.3 Compact hyper-Kähler manifolds

Recall that all known examples of compact irreducible hyper-Kähler manifolds belong to the following families:

(a) The $K3$ type and the Kummer type [1];
(b) O'Grady's 6-dimensional family (OG6 type) [23];
(c) O'Grady's 10-dimensional family (OG10 type) [22].

Combining with structural results of the cohomology of hyper–Kähler manifolds [17, 27, 28], Proposition 4.2 implies that M cannot be one of the known examples listed above of irreducible hyper–Kähler 6-folds for a surjective morphism (47) to exist.

Proposition 4.3 *Assume M is a hyper-Kähler 6-fold of $K3$, Kummer, or OG6 type, then any grading preserved morphism*

$$f: H^*(M, \mathbb{Q}) \to H^*(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q})$$

is not surjective. In particular, the specialization morphism (45) is not a surjection.

Proof By (49), the variety $\mathcal{M}_{\mathrm{Dol}}$ has non-trivial odd cohomology. Therefore the calculations of [6, 21] imply that M is not of $K3$ or OG6 type whose odd cohomology vanishes.

The cohomology of a manifold of Kummer type admits an action of the Looijenga–Lunts–Verbitsky (LLV) Lie algebra $\mathfrak{so}(4, 5)$; see [17, 27, 28]. If M is 6-dimensional, the precise form of the LLV decomposition of $H^*(M, \mathbb{R})$ with respect to $\mathfrak{so}(4, 5)$-representations was calculated in [7, Corollary 3.6]. In particular, the odd cohomology $H^{\mathrm{odd}}(M, \mathbb{R})$ is an irreducible $\mathfrak{so}(4, 5)$-module whose highest weight vector lying in $H^3(M, \mathbb{R})$. Therefore we obtain that

$$H^2(M, \mathbb{R}) \cup H^3(M, \mathbb{R}) = H^5(M, \mathbb{R}).$$

This contradicts Proposition 4.2. □

Acknowledgements We are grateful to Chen Wan and Zhiwei Yun for helpful discussions. We also thank the anonymous referee for careful reading of the paper. The first-named author is partially supported by NSF DMS Grant 1901975. The third-named author is partially supported by NSF DMS Grant 2134315.

References

1. Beauville, A.: Variétés kählériennes dont la première classe de Chern est nulle. J. Differ. Geom. **18**(4), 755–782 (1984)
2. Beĭlinson, A.A., Bernstein, J., Deligne, P.: Faisceaux pervers, Analysis and topology on singular spaces, I (Luminy, 1981), 5–171, Astérisque, 100. Soc. Math. France, Paris (1982)
3. de Cataldo, M.A., Hausel, T., Migliorini, L.: Topology of Hitchin systems and Hodge theory of character varieties: the case A_1. Ann. Math. (2) **175**(3), 1329–1407 (2012)
4. de Cataldo, M.A., Maulik, D., Shen, J.: Hitchin fibrations, abelian surfaces, and the $P = W$ conjecture. J. Am. Math. Soc. **35**(3), 911–953 (2022)
5. de Cataldo, M.A., Migliorini, L.: The perverse filtration and the Lefschetz hyperplane section theorem. Ann. Math. **171**(3), 2089–2113 (2010)
6. Göttsche, L.: The Betti numbers of the Hilbert scheme of points on a smooth projective surface. Math. Ann. **286**, 193–207 (1990)
7. Green, M., Kim, Y.-J., Laza, R., Robles, C.: The LLV decomposition of hyper–Kähler cohomology (the known cases and the general conjectural behavior). arXiv:1906.03432v2. Math. Ann., to appear
8. Groechenig, M., Wyss, D., Ziegler, P.: Mirror symmetry for moduli spaces of Higgs bundles via p-adic integration. Invent. Math. **221**(2), 505–596 (2022)
9. Groechenig, M., Wyss, D., Ziegler, P.: Geometric stabilisation via p-adic integration. J. Amer. Math. Soc. **33**, 807–873 (2020)
10. Donagi, R., Ein, L., Lazarsfeld, R.: Nilpotent cones and sheaves on K3 surfaces, Birational algebraic geometry (Baltimore, MD, 1996), pp. 51–61. Contemp. Math., 207, Amer. Math. Soc., Providence (1997)
11. Hausel, T., Letellier, E., Rodriguez-Villegas, F.: Arithmetic harmonic analysis on character and quiver varieties. Duke Math. J. **160**(2), 323–400 (2011)
12. Hausel, T., Pauly, C.: Prym varieties of spectral covers. Geom. Topol. **16**(3), 1609–1638 (2012)
13. Hausel, T., Rodriguez-Villegas, F.: Mixed Hodge polynomials of character varieties, with an appendix by Nicholas M. Katz. Invent. Math. **174**(3), 555–624 (2008)
14. Hausel, T., Thaddeus, M.: Mirror symmetry, Langlands duality, and the Hitchin system. Invent. Math. **153**(1), 197–229 (2003)
15. Hitchin, N.J.: The self-duality equations on a Riemann surface. Proc. Lond. Math. Soc. (3) **55**(1), 59–126 (1987)
16. Hitchin, N.J.: Stable bundles and integrable systems. Duke Math. J. **54**, 91–114 (1987)

17. Looijenga, E., Lunts, V.A.: A Lie algebra attached to a projective variety. Invent. Math. **129**(2), 361–412 (1997)
18. Markman, E.: Generators of the cohomology ring of moduli spaces of sheaves on symplectic surfaces. J. Reine Angew. Math. **544**, 61–82 (2002)
19. Maulik, D., Shen, J.: Endoscopic decompositions and the Hausel–Thaddeus conjecture. Forum Math. Pi, 9, No. e8 (2021)
20. Mereb, M.: On the E -polynomials of a family of Sl_n-character varieties. Math. Ann. **363**(3–4), 857–892 (2015)
21. Mongardi, G., Rapagnetta, A., Saccá, G.: The Hodge diamond of O'Grady's six-dimensional example. Compos. Math. **154**(5), 984–1013 (2018)
22. O'Grady, K.G.: Desingularized moduli spaces of sheaves on a K3. J. Reine Angew. Math. **512**, 49–117 (1999)
23. O'Grady, K.G.: A new six-dimensional irreducible symplectic variety. J. Algebraic Geom. **12**(3), 435–505 (2003)
24. Shende, V.: The weights of the tautological classes of character varieties. Int. Math. Res. Not. **22**, 6832–6840 (2017)
25. Simpson, C.T.: Higgs bundles and local systems. Inst. Hautes Études Sci. Publ. Math. No. **75**, 5–95 (1992)
26. Simpson, C.T.: Moduli of representations of the fundamental group of smooth projective varieties II," Inst. Hautes. Etudes Sci. Publ. Math. No. **80**(1994), 5–79 (1995)
27. Verbitsky, M.: Cohomology of compact hyperkaehler manifolds. Thesis (Ph.D.)–Harvard University (1995)
28. Verbitsky, M.: Cohomology of compact hyper-Kähler manifolds and its applications. Geom. Funct. Anal. **6**(4), 601–611 (1996)
29. Yoshioka, K.: Moduli spaces of stable sheaves on abelian surfaces. Math. Ann. **321**(4), 817–884 (2001)