UNIFORM *A PRIORI* BOUNDS FOR NEUTRAL RENORMALIZATION.

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Abstract. We prove *uniform a priori bounds* for Siegel disks of bounded type that give a uniform control of oscillations of their boundaries in all scales. As a consequence, we construct the *Mother Hedgehog* for any quadratic polynomial with a neutral periodic point.

Contents

1. Introduction

Local dynamics near a neutral fixed point, and a closely related dynamical theory of circle homeomorphisms, is a classical story going back to Poincaré, Fatou, and Julia. It followed up in the next two decades with breakthroughs by Denjoy (1932) and Siegel (1942) on the linearization of circle diffeomorphisms and local maps, respectively. The local theory received an essentially complete treatment in the second half of the last century in the work by Arnold (in the KAM framework), Herman, Yoccoz, and Perez-Marco.

About at the same time (1980–90s) a global and semi-local theory for neutral quadratic polynomials $f_\theta : z \mapsto e^{2\pi i\theta} z + z^2$ with rotation numbers θ of *bounded type* was designed on the basis of the *Douady-Ghys surgery.* And in the 2000s, in the framework of the *parabolic implosion phenomenon*, Inou and Shishikura established *uniform a priori bounds* for quadratic polynomials f_θ with rotation numbers of *high type.* This theory found numerous applications, from constructing examples of Julia sets of positive area Buff-Cheritat (2000s, [BC]), Avila-Lyubich (2010s, [AL2])) to a complete description, for high type rotation numbers, of the topological structure of the *Mother Hedgehogs* that capture the semi-local dynamics of neutral quadratic polynomials (Shishikura-Yang, Cheraghi (2010s)). (See §1.3 below for a more detailed historical account.)

In this paper, we will prove *uniform a priori bounds* for neutral maps f_θ with arbitrary rotation numbers. It gives an opening for removing the high-type assumption in the results just mentioned and alluded to. As a first illustration, we prove that the *Mother Hedgehog exists for an arbitrary rotation number.*

Our proof is based upon analysis of *degenerating* Siegel disks of bounded type. The *degeneration principles*, in the quadratic-like renormalization context, were originally designed by Jeremy Kahn [K], with a key analytic tool, the *Covering Lemma*, appeared in [KL1]. They serve as an entry point for our paper. One of the major subtleties of our situation is that Siegel disks of bounded type *do not* have uniformly bounded geometry since they may develop long fjords in all scales. (Otherwise, Cremer points would not have existed.) To deal with this problem, we design a regularization machinery of filling-in the fjords to gradually turn Siegel disks into uniform quasidisks.

Let us note in conclusion that our inductive argument goes in the opposite direction compared with the quadratic-like renormalization theory [K, KL2]. Indeed, we show that high degeneration on a certain level implies even higher degeneration on a deeper rather than a shallower level.

1.1. **Results.** Due to the Douady-Ghys surgery, for a Siegel map $f = f_\theta$ of bounded type, the dynamics on the Siegel disk Z, all the way up to its boundary ∂Z, is qc conjugate to the rigid rotation by θ, which provides us with the *rotation combinatorial model* for $f\,|\,\partial Z$.

Let $\mathfrak{p}_n/\mathfrak{q}_n$ be the continued fraction approximants for θ, so for any $x \in \partial Z$, $f^{\mathfrak{q}_n} x$ are the closest combinatorial returns of $\operatorname{orb} x$ back to x. A *combinatorial interval* $I \equiv I^n_f(x) \subset \partial Z$ of level n is the combinatorially shortest interval bounded by x and $f^{\mathfrak{q}_n} x$. For a combinatorial interval $I \subset \partial Z$, we let $\widetilde{I} \supset I$ be the enlargement of I by two attached combinatorial intervals.

Given a combinatorial interval $I \subset \partial Z$, let us consider the family $\mathcal{F}^+_3(I)$ of curves $\gamma \subset \widehat{\mathbb{C}} \setminus Z$ connecting I to points of $\partial Z \setminus \widetilde{I}$. The *external modulus* $\mathcal{W}^+_3(I)$ is the *extremal width* (i.e., the inverse of the extremal length) of the family $\mathcal{F}^+_\lambda(I)$.

Uniform Bounds Theorem 1.1. *There exists an absolute constant* $\mathbf{K}$ *such that* $\mathcal{W}^+_3(I) \le \mathbf{K}$ *for all Siegel quadratic polynomials* $f = f_\theta$ *of bounded type and all combinatorial intervals* $I = I^n_f(x)$.

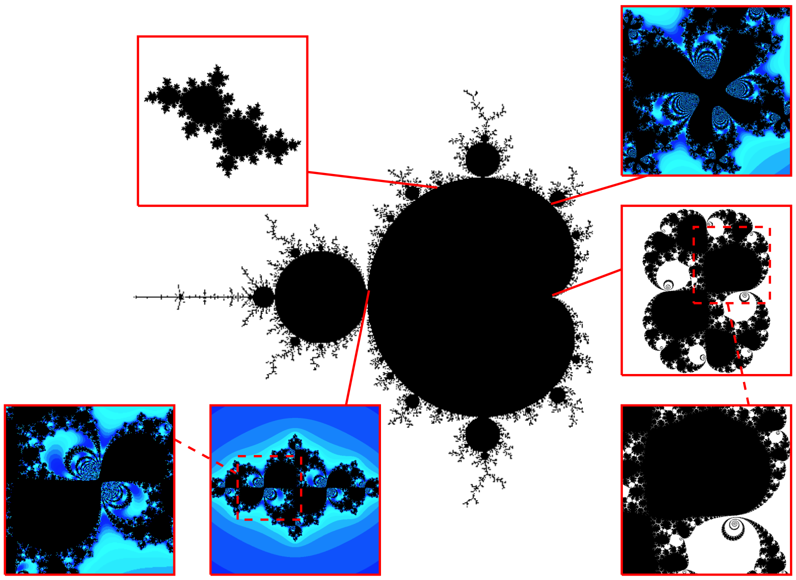

FIGURE 1. Different types of Siegel disks: golden-ratio (top-left), near-Basilica (bottom-left), near-cauliflower (bottom-right), near-1/4-Rabbit (top-right).

A *hull* $Q \subset \mathbb{C}$ is a compact connected full set. The *Mother Hedgehog* [Chi] for a neutral polynomial f_θ is an invariant hull containing both the fixed point 0 and the critical point $c_0(f) \coloneqq -e^{2\pi i\theta}/2$.

Mother Hedgehog Theorem 1.2. *Any neutral quadratic polynomial $f = f_\theta$, $\theta \notin \mathbb{Q}$, has a Mother Hedgehog $H_f \ni c_0(f)$ such that $f\colon H_f \to H_f$ is a homeomorphism.*

The last theorem is a consequence of the following result:

Quasidisk Approximation Theorem 1.3. *There exists an absolute constant $\mathbf{K}$ such that for any Siegel quadratic polynomial f of bounded type there exists a $\mathbf{K}$-quasidisk $\widehat{Z}_f \supset \overline{Z}_f$ such that $f|\,\widehat{Z}_f$ is injective.*

In [IS], Inou and Shishikura constructed a compact renormalization operator for high type rotation numbers. Theorems 1.1 and 1.3 imply that a compact renormalization operator of a similar nature exists for all rotation numbers.

1.2. **Quick outline of the proof.** Let us first give an informal description of Siegel disk degenerations. Let us denote by

$$\theta = [0; a_1, a_2, \dots, a_n, a_{n+1}, \dots], \qquad a_n \le M_\theta.$$

the rotation number of f. Its Siegel disk Z_f is a K_θ-quasidisk by the Douady-Ghys surgery. If we start increasing a_n with the remaining a_i fixed, then Z_f will be developing parabolic fjords towards the α-fixed point on the renormalization scale n, see Figure 2. These fjords can approach α arbitrary close. Cremer points are obtained by developing fjords in many scales so that in the limit the α-fixed point is not an interior point of the filled Julia set. In the paper, we will justify that this "star-like" degeneration is the only possible degeneration of bounded type Siegel

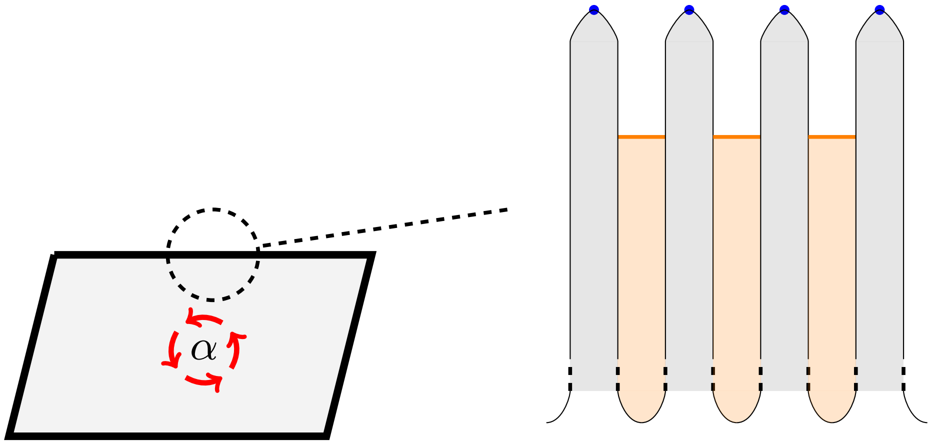

FIGURE 2. For a rotation number $[0; 1, \dots, 1, a_{n+1}, 1, \dots]$, the Siegel disk develops parabolic fjords on scale n towards the α-fixed point as $a_{n+1} \to \infty$. The critical points of $f^{\mathfrak{q}_{n+1}}$ (blue) are beacons (on the top of) parabolic peninsulas. After adding appropriately truncated parabolic fjords (orange) to the Siegel disk, the resulting pseudo-Siegel disk is almost quasi-invariant up to $\mathfrak{q}_{n+1}$ iterates.

Disks. We will work in the near-degenerate regime where wide rectangles impose non-crossing constraints on the geometry. (One may call it "1.5-dimensional real dynamics".)

In Section 4 we will justify (in the near-degenerate regime) that parabolic fjords have translational geometry – reminiscent of the Fatou coordinates for near-parabolic maps. It will follow from Calibration Lemma 9.1 that the critical points of $f^{\mathfrak{q}_{n+1}}$ are "beacons" (on the top of) the level-n parabolic peninsulas.

A pseudo-Siegel disk $\widehat{Z}^m$ is constructed by adding to $\overline{Z}$ all truncated parabolic fjords on scales $\geq m$, see Figure 2. We will show that $\widehat{Z}^m$ is qausi-invariant (in particular, injective) for all f^i with $i \leq \mathfrak{q}_{m+1}$. Moreover, the pseudo-Siegel disk $\widehat{Z}_f = \widehat{Z}^{-1}$ is uniformly qc (Theorem 1.3).

Theorem 3.8 establishes certain beau bounds to control the inner geometry of $\widehat{Z}^m$. The bounds imply, in particular, that the errors do not accumulate under the regularization

$$\overline{Z} \rightsquigarrow \dots \rightsquigarrow \widehat{Z}^{m+2} \rightsquigarrow \widehat{Z}^{m+1} \rightsquigarrow \widehat{Z}^m \rightsquigarrow \dots \rightsquigarrow \widehat{Z}^{-1} = \widehat{Z}_f$$

Furthermore, the outer geometry of the Siegel disk is almost unaffected under $\overline{Z} \rightsquigarrow \widehat{Z}^m$ – most of the outer harmonic measure of $\overline{Z}$ sits on tops of peninsulas, see §5.2. In other words, a random walk in $\widehat{\mathbb{C}} \setminus \overline{Z}$ starting at ∞ is unlikely to enter any truncated parabolic fjord of any level if the truncation is chosen sufficiently deep.

To control the outer geometry and its interaction with the inner geometry of $\widehat{Z}^m$, we will introduce the following degeneration parameters. For an interval $I \subset \partial \widehat{Z}^m$, we denote by $\lambda I \subset \partial \widehat{Z}^m$ the λ-rescaling of I in the linearized coordinates of ∂Z. Then $\mathcal{W}_\lambda(I)$ is the width of the family of curves connecting I to $\partial \widehat{Z}^m \setminus (\lambda I)$. Similarly, $\mathcal{W}^+_\lambda(I)$ is the width of the outer family of curves (i,e., in $\widehat{\mathbb{C}} \setminus Z$) connecting I to $\partial \widehat{Z}^m \setminus (\lambda I)$. If $\mathcal{W}_\lambda(I) = K \gg 1$, then iterating Snake Lemma 6.1, we can eventually find J with $\mathcal{W}^+_\lambda(J) \succeq K$ and $|J| \le |I|$, where "$|\ |$" denotes the length in the linearized coordinates of ∂Z.

If for a combinatorial interval $I \subset \partial \widehat{Z}^m$ on the renormalization level m we have $\mathcal{W}^+_\lambda(I) = K \ge \mathbf{K}$ for an absolute threshold $\mathbf{K} \gg 1$, then the Covering Lemma allows us to spread the associated degeneration around $\partial \widehat{Z}^m$. Then Snake-Lair Lemma 8.6 finds a bigger degeneration: there will exist a combinatorial interval $J_2 \subset \partial \widehat{Z}^n$ for some $n > m$ such that $\mathcal{W}^+_\lambda(J_2) \ge 2K$ and $|J_2| \le |J|$. Proceeding by induction, we obtain a sequence of shrinking intervals J_n such that $\mathcal{W}^+(J_n) \ge 2^n K$ contradicting eventually that $\overline{Z}$ is a K_θ-quasidisk. This establishes Theorem 1.1.

Theorem 1.3 is obtained by justifying a universal combinatorial bound for the truncation depth. Theorem 1.3 allows us to control Hausdorff limits of $\overline{Z}_{f_\theta}$ as θ approaches any irrational number; the resulting limits are Mother Hedgehogs (Theorem 1.2).

The paper is organized as follows. In Part 1 we will show that near-rotation domains and parabolic fjords are coarse qs-equivalent to rotations of the unit disk – see Proposition 3.3 and Theorems 3.8, 4.1. In Part 2, we will introduce pseudo-Siegel disks, show that they quasi-behave as uniformly bounded Siegel disks, establish Snake Lemma 6.1. Corollary 7.3 states that either the regularization $\widehat{Z}^m \rightsquigarrow \widehat{Z}^{m-1}$ is possible or there is a much bigger degeneration on some scale $\ge m$. In Part 3 we will prove Theorem 8.1 (application of the Covering Lemma followed by Snake-Lair Lemma 8.6) and Calibration Lemma 9.1; they say that if the outer geometry of $\widehat{Z}^m$ is sufficiently degenerate on scale m, then the outer geometry of $\widehat{Z}^n$ is even more degenerate on some scale deeper scale $n > m$. The main theorems are proven in Part 4.

For the reader's convenience, in the beginning of each section, we provide its detailed outline.

1.3. Historical retrospective and further perspective.

As we have already mentioned, the local theory for neutral holomorphic germs and circle homeomorphisms was completed by Arnold, Herman, Yoccoz and Perez-Marco in the second half of the last century. In particular, Yoccoz showed the *Bruno's linearization condition* is sharp for germs [Yo], while Perez-Marco introduced a topological object, a *hedgehog* that greatly clarified the local structure of non-linearizable Cremer maps [PM].

Another line of thought was related to the *quasiconformal surgery* machinery introduced to the field by Sullivan, Douady and Hubbard in the early 1980s. By means of the *Douady-Ghys surgery* (see [D1]), it led to a precise topological model for the Julia set of a neutral quadratic polynomial $f_\theta : z \mapsto e^{2\pi i\theta}z + z^2$ with rotation number θ of bounded type. In particular, it allowed Petersen to justify the local connectivity of the corresponding Julia set [Pe].

The Douady-Ghys surgery is based upon *real a priori bounds* for critical circle maps proved by Swiatek [Sw] and Herman [H]. (*A priori bounds* mean a uniform geometric control of a system(s) under consideration in all dynamical scales.) The Swiatek-Herman bounds were promoted to *complex a priori bounds* by de Faria for bounded combinatorics [dF] and by Yampolsky in general [Ya2].

Yet another direction was the theory of *parabolic implosion* designed by Doaudy and Lavaurs in the 1980s. A remarkable breakthrough in this theory appeared in the work by Inou and Shishikura in the mid-2000s, providing us with *uniform a* *priori bounds* for rotation numbers θ of *high type* [IS] (see also Cheritat [Che]). Besides applications mentioned above (to produce Julia sets of positive area [BC, AL2] and to the description of Mother Hedgehogs [ShY, Ch2]), the Inou-Shishikura bounds were instrumental in the proof of the Marmi-Moussa-Yoccoz Conjecture for rotation numbers of high type [ChC] and in the description of the measurable dynamics on the Julia sets of positive measure in the Inou-Shishikura class [Ch1, ACh]. It also provided an opening to a partial description of the global topological structure of Cremer Julia sets, which have been viewed as most mysterious objects in holomorphic dynamics [BBCO].

Another potential implication of our *a priori bounds* is a construction of a *hyperbolic full renormalization horseshoe* for all rotation numbers simultaneously, unifying the pacman renormalization periodic points [McM, Ya1, DLS] with the Inou-Shishikura horseshoe of high type [IS]. Such a structure would imply various scaling features of the Mandelbrot set near the main cardioid. (Compare with the scaling impact of the pacman renormalization periodic points [DLS, DL].)

And last but not least, uniform bounds for the neutral renormalization give control of the satellite quadratic-like renormalization that is relevant to the MLC Conjecture and the area problem for Julia sets (see [CS, DL]).

1.4. **Main notations and conventions.** We state here our main notations and conventions; see §2 for more details. We denote by

- $\Theta_{\text{bnd}} = \{\theta = [0; a_1, a_2, \dots] \mid a_i \leq M_\theta\}$ the set of bounded (irrational) rotation numbers;
- $\mathbf{e}(\theta) \coloneqq e^{2\pi i\theta}$;
- Z the Siegel disk of $f = f_\theta$ for $\theta \in \Theta_{\text{bnd}},\quad 0 < \theta < 1$;
- $c_0, c_1 \in \partial Z$ the critical point and critical value of f;
- more generally: $c_n \coloneqq (f \mid \partial Z)^n(c_0)$;
- $h\colon (Z, \alpha) \to (\mathbb{D}, 0)$ the conformal conjugacy between f and $z \mapsto e(\theta)z$;
- $|I| \coloneqq |h(I)|_{\mathbb{R}/\mathbb{Z}}$ the (combinatorial) Euclidean length of an interval $I \subset \partial Z$;
- $\theta_n \in (-1/2, 1/2)$ the rotation number of $f^{\mathfrak{q}_n} \mid Z$ and $\ell_n \coloneqq |\theta_n|$ the length of a *level n combinatorial* interval $[x, f^{\mathfrak{q}_n}(x)] \subset \partial Z$, where $\mathfrak{p}_n/\mathfrak{q}_n \approx \theta$ are best approximations;
- $x \boxplus \nu \coloneqq h^{-1}(h(x) + \nu)$, where $x \in \partial Z$, $\nu \in \mathbb{R}$, and $h(x) + \nu \in \mathbb{R}/\mathbb{Z} \simeq \partial\mathbb{D}$;
- $\mathcal{W}_\lambda(I) \coloneqq \mathcal{W}(\mathcal{F}_\lambda(I))$ and $\mathcal{W}^+_\lambda(I) \coloneqq \mathcal{W}(\mathcal{F}^+_\lambda(I))$ the width of the full and outer families measuring degeneration of $\overline{Z}$ at an interval $I \subset \partial Z$, see §2.3.1;
- "$<$" denotes a clockwise orientation on ∂Z;
- for an interval $I \subset \partial Z$ and $x, y \in I$ we write $x < y$ rel I if x is on the left of y in I, i.e. $\partial Z \setminus I, x, y$ are clockwise oriented;
- for a pair of disjoint intervals $I, J \subset \partial Z$ we define $\lfloor I, J \rfloor \coloneqq I \cup L \cup J$, where L is the complementary interval between I, J so that $I < L < J$; in most cases L will be the shortest interval between I and J;
- $x \oplus y = (x^{-1} + y^{-1})^{-1},\quad x, y > 0$ the harmonic sum – see the Grötzsch inequality (A.3);
- $\gamma \# \beta$ the concatenation of curves γ and β.

By default, curves are considered up reparametrization and are usually parameterized by the unit interval $[0, 1]$. We say a curve $\gamma\colon (0, 1) \to \widehat{\mathbb{C}} \setminus (A \cup B)$ *connects* A and B if

$$\lim_{\tau \to 0} \gamma(\tau) = \gamma(0) \in A \qquad \text{and} \qquad \lim_{\tau \to 0} \gamma(1 - \tau) = \gamma(1) \in B.$$

A *Jordan* disk is a closed or open topological disk bounded by a Jordan curve.

We write

$$f(x) \preceq g(x) \quad \text{if } \ f(x) \le Cg(x), \qquad f(x), g(x) > 0$$

for an absolute constant $C > 0$. Similarly:

$$f(x) \preceq_\kappa g(x) \quad \text{if } \ f(x) \le C_\kappa g(x), \qquad f(x), g(x) > 0$$

for a constant $C_\kappa > 0$ depending on κ. The big $O(\ \ )$ notation describes at most liner dependence on the argument: $O(f(x)) \preceq f(x)$. Similarly, $O_\kappa(f(x)) \preceq_\kappa f(x)$ is at most liner dependence on $f(x)$ with a constant depending on κ.

We will write "$A \gg B$" to assume that A is sufficiently bigger than B. Similarly, "$A \gg_\kappa B$" means that A is sufficiently big than B depending on a parameter κ.

We will often need to truncate laminations $\mathcal{F}, \mathcal{G}, \mathcal{H}$ by removing buffers of certain sizes. We will use upper indices "new, New, NEW" to denote new truncated families with the convention

$$\mathcal{F} \supset \mathcal{F}^{\text{new}} \supset \mathcal{F}^{\text{New}} \supset \mathcal{F}^{\text{NEW}}.$$

Slightly abusing notations, we will often identify a lamination with its support.

A *vertical* curve of a rectangle $\mathcal{R}$ is a curve that becomes vertical after conformal identification $\mathcal{R}$ with a standard Euclidean rectangle.

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The results of this paper were announced at the Fields Institute Symposium celebrating Artur Avila's Fields medal (November 2019), and during the SCGP Renormalization program in December 2020 (see the mini-course on http://scgp.stonybrook.edu/video/results.php?event_id=317).

Part 1. **Rotation geometry**

2. Preparation

We fix a neutral quadratic polynomial $f : z \mapsto \mathbf{e}(\theta)z + z^2$ with bounded type rotation number $\theta \in (0, 1)$; i.e. the α-fixed point of f has multiplier

$$\mathbf{e}(\theta) = e^{2\pi i\theta}, \qquad \theta \in \Theta_{\text{bnd}}.$$

2.1. **Siegel Disk.**

Let us denote by Z the Siegel disk of f. Recall that $\overline{Z}$ is a qc disk because $\theta \in \Theta_{\text{bnd}}$. Consider a Riemann map

$$h\colon \overline{Z} \to \overline{\mathbb{D}}, \qquad h(\alpha) = 0$$

conjugating $f \mid \overline{Z}$ to $z \mapsto \mathbf{e}(\theta)z$.

The *(combinatorial) length* of an interval $I \subset \partial Z \simeq \mathbb{R}/\mathbb{Z}$ is defined by

$$|I| \coloneqq |h(I)|_{\mathbb{R}/\mathbb{Z}} \in (0, 1).$$

Similarly, the *(combinatorial) distance*

$$\operatorname{dist}(x, y) \coloneqq \operatorname{dist}_{\mathbb{R}/\mathbb{Z}}(h(x), h(y)) \in [0, 1/2], \qquad x, y \in \partial Z$$

is defined. It also induces the distance between subsets of ∂Z.

Given $t \in \mathbb{R}/\mathbb{Z}$ and $x \in \partial Z$, we set

$$x \boxplus t = h^{-1}\big(\mathbf{e}(t)h(x)\big),$$

i.e. $x \boxplus t$ is x rotated by angle t. We have

$$f(x) = x \boxplus \theta, \qquad \text{for } x \in \partial Z.$$

2.1.1. *Closest returns of $f|\partial Z$.* Let $\theta = [0; a_1, a_2, \dots]$ be the continued fraction expansion and consider the sequence of the best approximations of θ

$$\mathfrak{p}_n/\mathfrak{q}_n := \begin{cases} [0; a_1, a_2, a_3 \dots, a_n] & \text{if } a_1 > 1 \\ [0; 1, a_2, a_3, \dots, a_{n+1}] & \text{if } a_1 = 1 \end{cases},$$

and set $\mathfrak{q}_0 := 1$. Then $f^{\mathfrak{q}_0} = f, f^{\mathfrak{q}_1}, f^{\mathfrak{q}_2}, \dots$ is the sequence of the closest returns of $f \mid \partial Z$; i.e.

$$\operatorname{dist}\big(f^i(x), x\big) > \operatorname{dist}\big(f^{\mathfrak{q}_n}(x), x\big) =: \mathfrak{l}_n, \quad x \in \partial Z \qquad \text{for all } i < \mathfrak{q}_n.$$

For $n \geq 0$, we specify $\theta_n \in (-1/2, 1/2)$ so that

$$f^{\mathfrak{q}_n}(x) = x \boxplus \theta_n, \quad x \in \partial Z.$$

In particular, $\theta_0 = \theta$ if $\theta < 1/2$ and $\theta_0 = \theta - 1$ otherwise. By construction, $\mathfrak{l}_n = |\theta_n|$. The sequence θ_n is alternating: $\theta_n \theta_{n+1} < 0$ – reflecting the fact that θ is between $\mathfrak{p}_n/\mathfrak{q}_n$ and $\mathfrak{p}_{n+1}/\mathfrak{q}_{n+1}$. Since $\mathfrak{l}_n \geq \mathfrak{l}_{n+1} + \mathfrak{l}_{n+2}$ and $\mathfrak{l}_{n+2} < \mathfrak{l}_{n+1}$, we have

$$\mathfrak{l}_{n+2} < \mathfrak{l}_n/2. \tag{2.1}$$

2.1.2. *Intervals.* Consider two points $x, y \in \partial Z$. Unless otherwise is stated, we denote by $[x, y]$ the shortest closed interval of ∂Z between x and y. The corresponding open interval is denoted by (x, y). Most of the intervals will be closed.

For an interval $I \subset \partial Z$, we denote by $I^c = \overline{\partial Z \setminus I}$ its complement.

We denote by "$<$" the clockwise order on $\partial \mathbb{D}$ and on ∂Z. Given two non-equal points a, b in an interval I with $|I| < 1/2$, we say that a is on the *left* of b, and write $a < b$, if I^c, a, b have the clockwise order. This convention is consistent with drawing intervals on the upper side of ∂Z, see Figures 3, 4. (Note that $x \mapsto x \boxplus \varepsilon$ is a counterclockwise rotation for a small $\varepsilon > 0$.)

Given intervals $I, J \subset \partial Z$, we define $\lfloor I, J \rfloor \subset \partial Z$ to be the interval $I \cup L \cup J$, where L is the complementary interval of I, J (i.e., a component of $\partial Z \setminus (I \cup J)$) specified so that $I < L < J$ with respect to the clockwise order. In other words, $\lfloor I, J \rfloor$ is the shortest interval containing $I \cup J$ such that $I < J$ in $\lfloor I, J \rfloor$. In most cases, $\lfloor I, J \rfloor$ will be the shortest interval containing I, J.

Given an interval $I \subset \partial Z$ and $\lambda \geq 1$, we define

$$\lambda I := \{x \in \partial Z : \quad \operatorname{dist}(x, I) \leq (\lambda - 1)|I|/2\} \tag{2.2}$$

to be the λ-rescaling of I with respect to its center.

2.1.3. *Combinatorial intervals.* For $n \geq 0$, a *combinatorial level n interval* is an interval $I \subset \partial Z$ with length $\mathfrak{l}_n$. It has the form

$$I = [x, f^{\mathfrak{q}_n}(x)] = [x, x \boxplus \theta_n] \qquad \text{where } x \in \partial Z.$$

Since the sequence θ_n is alternating, we have:

Lemma 2.1. *The return time of points in a level $n \geq 0$ combinatorial interval $I = [x, x \boxplus \theta_n]$ is at least $\mathfrak{q}_{n+1}$:*

$$f^i(y) \notin I \qquad \text{for} \quad y \in (x, x \boxplus \theta_n), \quad i < \mathfrak{q}_{n+1}. \tag{2.3}$$

□

As a consequence, the commuting pair

$$\big(f^{\mathfrak{q}_{n+1}} \mid [f^{\mathfrak{q}_n}(x), x], \quad f^{\mathfrak{q}_n} \mid [x, f^{\mathfrak{q}_{n+1}}(x)]\big) \tag{2.4}$$

realizes the first return map to $[f^{\mathfrak{q}_n}(x), f^{\mathfrak{q}_{n+1}}(x)]$. The renormalization theory of circle maps is often set up using commuting pairs.

2.1.4. *Renormalization tilings.* Given $x \in \partial Z$ and $n \geq 0$, we denote by $\mathfrak{J}_n(x)$ the associated *renormalization tiling of level* n:

$$\bigcup_{i=0}^{\mathfrak{q}_{n+1}-1} f^i[f^{\mathfrak{q}_n}(x), x] \quad \cup \quad \bigcup_{i=0}^{\mathfrak{q}_n-1} f^i[x, f^{\mathfrak{q}_{n+1}}(x)]. \tag{2.5}$$

We also set $\mathfrak{J}_n \coloneqq \mathfrak{J}_n(c_0)$, where c_0 is the free critical point in ∂Z.

Note that most of the intervals in (2.5) are in the orbit of $[f^{\mathfrak{q}_n}(x), x]$ (explaining the subindex n in $\mathfrak{J}_n(x)$). Moreover, level n intervals in $\mathfrak{J}_n$ form an almost tiling, with gaps being intervals of level $n+1$:

Lemma 2.2. *Level* $n+1$ *combinatorial intervals in* (2.5) *are disjoint.*

Proof. It is a well-known statement that easily follows by induction. In $\mathfrak{J}_0$, there is a single level 1 interval and $\mathfrak{q}_1 \geq 2$ level 0 intervals. The tiling $\mathfrak{J}_{n+1}$ is obtained from $\mathfrak{J}_n$ by decomposing every level n interval into level $n+1$ intervals and a single level $n+2$ interval either on the level or on the right depending on the parity of n; i.e. level $n+2$ intervals are disjoint in $\mathfrak{J}_{n+1}$. □

For $n > m$, we say that level n combinatorial intervals are on *deeper scale* than level m combinatorial intervals, while level m combinatorial intervals are on *shallower scale* than level n combinatorial intervals.

2.1.5. *Spreading around a combinatorial interval.* Consider a combinatorial level n interval I. We say that the intervals

$$\{f^i(I) \mid i \in \{0, 1, \dots, \mathfrak{q}_n - 1\}\}$$

are obtained by *spreading around* I. We enumerate these intervals counterclockwise starting with $I = I_0$

$$I_0 = I, \quad I_1 = f^{i_1}(I), \dots, I_{\mathfrak{q}_n - 1} = f^{i_{\mathfrak{q}_n-1}-1}(I), \qquad i_j \in \{1, 2, \dots, \mathfrak{q}_n - 1\}. \tag{2.6}$$

It follows from Lemma 2.2 that either I_i is attached to I_{i+1} or there is a level $n+1$ combinatorial complementary interval between I_i and I_{i+1}.

2.1.6. *Diffeo-tilings.* For $n \geq -1$, we denote by $\mathrm{CP}_n = \mathrm{CP}_n(f) = \mathrm{CP}(f^{\mathfrak{q}_{n+1}})$ the set of critical points of $f^{\mathfrak{q}_{n+1}}$. The *diffeo-tiling* $\mathfrak{D}_n$ *of level* n is the partition of ∂Z induced by CP_n: every interval in $\mathfrak{D}_n$ is the closure of a component of $\partial Z \setminus \mathrm{CP}_n$. For $n = -1$, the tiling consists of a single "interval" viewed as $[c_0, c_0 \boxplus 1]$.

For $n \geq 0$, every point in CP_n is an endpoint of an interval in $\mathfrak{J}_n(c_{-\mathfrak{q}_n+1})$; we see that $\mathfrak{D}_n$ is an enlargement of $\mathfrak{J}_n(c_{-\mathfrak{q}_{n+1}+1})$. By Lemma 2.2, every interval in $\mathfrak{D}_n$ has length either $\mathfrak{l}_n$ or $\mathfrak{l}_n + \mathfrak{l}_{n+1}$.

Enumerating counterclockwise intervals in $\mathfrak{D}_n$ as $P_0, P_1, \dots, P_{\mathfrak{q}_{n+1}}$, we have $f^{i\mathfrak{q}_{n+1}}(P_k) \approx P_{k+i\mathfrak{p}_{n+1}}$, where the "rotational error" with respect to the combinatorial length is small if $\mathfrak{l}_{n+1} \ll \mathfrak{l}_n$.

Let us set

$$\mathcal{K}_n \coloneqq f^{-\mathfrak{q}_{n+1}}(\overline{Z}), \tag{2.7}$$

Every interval of $\mathfrak{D}_n$ is between two components, called *limbs*, of $\mathcal{K}_n \setminus \overline{Z}$.

For $T \in \mathfrak{D}_n$, we write $T' := T \cap f^{\mathfrak{q}_{n+1}}(T) \subset T$ so that $f^{\mathfrak{q}_{n+1}} \colon T' \boxminus \theta_n \to T'$ is a homeomorphism. If $n = -1$, then T' is the longest interval connecting c_1 and c_0.

The *nest of diffeo-tilings* is

$$\mathfrak{D} := [\mathfrak{D}_n]_{n \geq -1} \tag{2.8}$$

It follows from (2.1) that

Lemma 2.3. *Every interval of* $\mathfrak{D}_n$ *consists of at least* 2 *intervals of* $\mathfrak{D}_{n+2}$. $\square$.

2.1.7. *Fjords.* Consider an interval $T = [a, b]$ in $\mathfrak{D}_n$, $n \geq -1$ and let $\ell \subset \widehat{\mathbb{C}} \setminus \mathcal{K}_n$ be the hyperbolic geodesic connecting a and b. Then the connected component $\mathfrak{T} = \mathfrak{T}(T)$ of $\widehat{\mathbb{C}} \setminus (\overline{Z} \cup \ell)$ attached to T is called the *fjord associated with* T.

More generally, a level n *fjord* is subdomain of $\mathfrak{T}(T)$ bounded by a simple arc connecting points in T, where $T \in \mathfrak{D}_n$.

Lemma 2.4. *If* $\mathfrak{T}$ *is a fjord attached to* T*, then* $f^{\mathfrak{q}_{n+1}} \mid \mathfrak{T}$ *is injective.*

Proof. The lemma follows from the observation that

$$f^{\mathfrak{q}_{n+1}} \colon \widehat{\mathbb{C}} \setminus \mathcal{K}_m \to \widehat{\mathbb{C}} \setminus \overline{Z}$$

is a covering map of degree $2^{\mathfrak{q}_{n+1}}$ and the harmonic measure of T in $(\widehat{\mathbb{C}} \setminus \mathcal{K}_m, \infty)$ is $\leq 2^{-\mathfrak{q}_{n+1}} \leq \dfrac{1}{2}$. $\square$

2.2. **Inner geometry of $\overline{Z}$.** For disjoint intervals $I, J \subset \partial Z$, the *inner family* $\mathcal{F}^-(I, J) = \mathcal{F}_{\overline{Z}}(I, J)$ is the family of all curves in $\overline{Z}$ connecting I, J; see also §A.1.6. Its width $\mathcal{W}^-(I, J)$ can be explicitly computed:

Lemma 2.5 (Log-Rule). *Consider intervals* $I, J \subset \partial Z$. *If* $\operatorname{dist}(I, J) \leq \min\{|I|, |J|\}$, *then*

$$\mathcal{W}^-(I, J) \asymp \log \frac{\min\{|I|, |J|\}}{\operatorname{dist}(I, J)} + 1; \tag{2.9}$$

otherwise

$$\mathcal{W}^-(I, J) \asymp \left(\log \frac{\operatorname{dist}(I, J)}{\min\{|I|, |J|\}} + 1\right)^{-1}. \tag{2.10}$$

We will later generalize these estimates to near-rotation domains (Proposition 3.3 and Theorem 3.8) and to parabolic fjords (Theorem 4.1).

Proof. Since $\overline{Z}$ and $\overline{\mathbb{D}}$ are conformally identified, it is sufficient to prove the lemma for $\overline{\mathbb{D}}$. Observe first that for $A, B \subset \partial\mathbb{D}$

$$\mathcal{W}^-(A, B) \asymp \mathcal{W}(\mathcal{R}(A, B)) \asymp 1 \qquad \text{if } \operatorname{dist}(A, B) \asymp \min\{|A|, |B|\}, \tag{2.11}$$

where $\mathcal{R}(A, B)$ is the geodesic rectangle, see §A.1.12. Indeed, the condition $\operatorname{dist}(A, B) \asymp \min\{|A|, |B|\}$ implies that the cross-ratio of 4 endpoints of A, B is comparable to 1. Applying a Möbius transformation, we can assume that all 4 intervals (i.e., A, B and two complementary intervals between A, B) have comparable lengths, and the claim follows by compactness.

Suppose $\operatorname{dist}(I, J) \leq \min\{|I|, |J|\}$. We also assume that $|I| \leq |J|$, and we present I and J as concatenations

$$I = I_1 \# I_2 \# \ldots \# I_n \qquad \text{and} \qquad J_1 \# J_2 \# \ldots \# J_n,$$

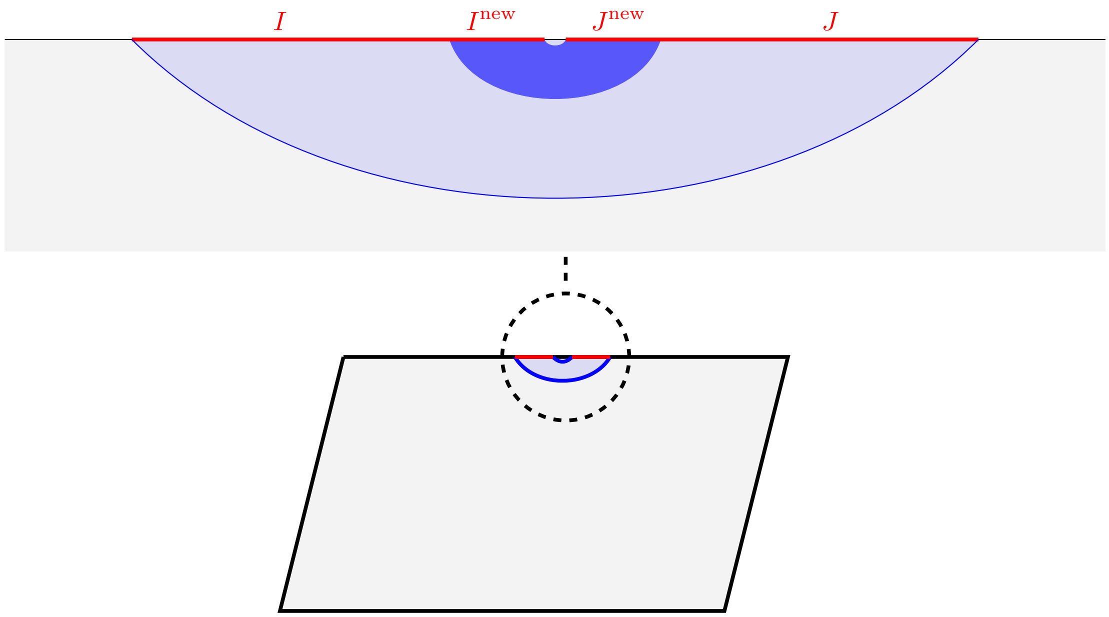


FIGURE 3. Illustration to the Localization Property: the width $\mathcal{F}^-(I,J)$ is within $\mathcal{F}^-(I^{\text{new}}, J^{\text{new}})$ up to $O(\log\lambda)$, where $I^{\text{new}}, J^{\text{new}}$ is an innermost subpair.

where $n \asymp \log \dfrac{\min\{|I|,|J|\}}{\operatorname{dist}(I,J)} + 1$, such that

$$\operatorname{dist}(I_k, J_k) \asymp |I_k| \asymp \operatorname{dist}(I_k, J).$$

By the Parallel Law, we obtain:

$$\mathcal{W}^-(I,J) \le \sum_{k=1}^{n} \mathcal{W}(I_k, J) \preceq n,$$

$$\mathcal{W}^-(I,J) \ge \sum_{k=1}^{n} \mathcal{W}(\mathcal{R}(I_k, J_k)) \succeq n.$$

This proves (2.9). If $\operatorname{dist}(I,J) \ge \min\{|I|,|J|\}$, then $\mathcal{W}^-(I,J) = 1/\mathcal{W}^-(A,B)$, where A, B are complementary intervals between I, J; i.e., (2.10) follows from (2.9).
□

Remark 2.6 (Splitting Argument)**.** *Note that in the proof of Lemma 2.5 we established* (2.9) *and* (2.10) *from* (2.11) *by splitting I and J into an appropriate number of intervals. We will call it the* Splitting Argument *– this argument will be used several times later.*

2.2.1. *Localization Property.* Consider a pair $I, J \subset \partial D$, where $D \subset \mathbb{C}$ is a closed Jordan disk. A subpair $I^{\text{new}} \subset I, \;\; J^{\text{new}} \subset J$ is called *innermost* if

$$I \setminus I^{\text{new}} < I^{\text{new}} < J^{\text{new}} < J \setminus J^{\text{new}} \qquad \text{in } \lfloor I, J \rfloor.$$

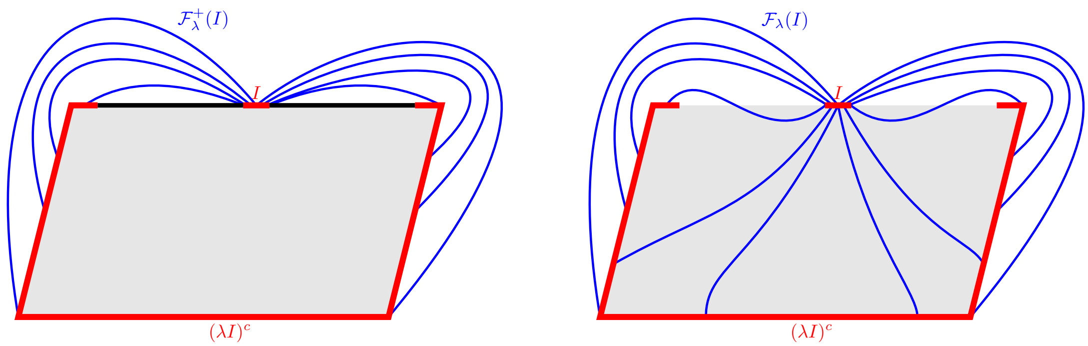


FIGURE 4. Parameters measuring degeneration of the Siegel disk: $\mathcal{W}_\lambda(I)$ is the width of the full family of curves connecting I and $[\lambda I]^c$ (right), while $\mathcal{W}^+_\lambda(I)$ is the width of outer family (left).

Lemma 2.5 implies the following localization property; see Figure 3. For a pair of intervals I, J with $|\lfloor I, J\rfloor| \leq 1 - \frac{1}{\lambda}\min\{|I|, |J|\}$, define $I^{\text{new}} \subset I$ and $J^{\text{new}} \subset J$ to be the closest innermost subpair such that

$$|I^{\text{new}}| = |J^{\text{new}}| = \frac{1}{\lambda}\min\{|I|, |J|\}.$$

Then most of the width of $\mathcal{F}^-(I, J)$ is in $\mathcal{F}^-(I^{\text{new}}, J^{\text{new}})$:

$$\mathcal{W}^-(I^{\text{new}}, J) + \mathcal{W}^-(I, J^{\text{new}}) = O(\log \lambda).$$

2.2.2. *Squeezing Property.* A counterpart to the localization property is the following squeezing property which also follows from Lemma 2.5.

There is a constant $C > 0$ such that the following holds. Suppose $I, J \subset \partial Z$ is a pair of intervals such that

$$\mathcal{W}^-(I, J) \geq C \log \lambda, \qquad \lambda > 2.$$

Then

$$\text{dist}(I, J) \leq \frac{1}{\lambda}\min\{|I|, |J|\}.$$

We will later generalize Localization and Squeezing Properties to pseudo-Siegel disks, see §5.5.

2.3. **Outer geometry $\overline{Z}$.** In this section we will define $\mathcal{W}_\lambda(I) = \mathcal{W}(\mathcal{F}_\lambda(I))$, $\mathcal{W}^+_\lambda(I) = \mathcal{W}(\mathcal{F}^+_\lambda(I))$ and other quantities to measure degeneration of $\overline{Z}$, see Figure 4.

2.3.1. *Full and outer families.* Recall (2.2) that λI denotes the rescaling of $I \subset \partial Z$ by λ with respect to the center of I. Recall also that $[\lambda I]^c$ denotes the complement of λI in ∂Z.

Given disjoint intervals $I, J \subset \partial Z$ and $\lambda \geq 2$, we denote by

- $\mathcal{F}(I, J)$ the family of curves in $\widehat{\mathbb{C}} \setminus (I \cup J)$ connecting I and J;
- $\mathcal{W}(I, J) = \mathcal{W}(\mathcal{F}(I, J))$ the extremal width of $\mathcal{F}(I, J)$;
- $\mathcal{F}_\lambda(I) \coloneqq \mathcal{F}(I, [\lambda I]^c)$;

- $\mathcal{W}_\lambda(I) = \mathcal{W}(\mathcal{F}_\lambda(I))$;
- $\mathcal{F}^+(I,J)$ the family of curves in $\mathbb{C}\setminus Z$ connecting I and J;
- $\mathcal{W}^+(I,J) = \mathcal{W}(\mathcal{F}^+(I,J))$ the extremal width of $\mathcal{F}^+(I,J)$;
- $\mathcal{F}^+_\lambda(I) := \mathcal{F}^+\left(I, [\lambda I]^c\right)$;
- $\mathcal{W}^+_\lambda(I) = \mathcal{W}(\mathcal{F}^+_\lambda(I))$.

We call $\mathcal{F}$ and $\mathcal{F}^+$ the *full* and *outer* families respectively.

We say that an interval I is

- $[K,\lambda]$-*wide* if $\mathcal{W}_\lambda(I) \ge K$, and
- $[K,\lambda]^+$-*wide* if $\mathcal{W}^+_\lambda(I) \ge K$.

Clearly, for every $K > 1$ and $\lambda \ge 2$, there is $K_\lambda > 1$ such that if $\overline{Z}$ is a K-quasidisk, then $\mathcal{W}(\mathcal{F}_\lambda(I)) \le K_\lambda$ for every $I \subset \partial Z$. See §11.1 for a converse statement.

For a closed Jordan disk $D \subset \mathbb{C}$ and disjoint intervals $I, J \subset \partial D$, the objects

$$\mathcal{F}^-_D(I,J),\ \ \mathcal{W}^-_D(I,J),\ \ \mathcal{F}^+_D(I,J),\ \ \mathcal{W}^+_D(I,J),\ \ \mathcal{F}_D(I,J),\ \ \mathcal{W}_D(I,J)$$

are defined in the same way as in the Siegel case $D = \overline{Z}$ (see also §A.1.6). We say that a rectangle $\mathcal{R}$ is *based on an interval* $I \subset \partial D$ if

$$\mathcal{R} \subset \widehat{\mathbb{C}} \setminus \operatorname{int} D \quad \text{and} \quad \partial^h \mathcal{R} \subset I. \tag{2.12}$$

2.3.2. *External and diving families.* Consider an interval $I \subset \partial Z$. Recall from (2.7) that $\mathcal{K}_m := f^{-\mathfrak{q}_{m+1}}(\overline{Z})$. A curve γ in $\mathcal{F}^+_\lambda(I)$ is called

- *external* rel $\mathcal{K}_m$ if γ minus its endpoints is in $\widehat{\mathbb{C}} \setminus \mathcal{K}_m$, and
- *diving* (rel $\mathcal{K}_m$) otherwise;

i.e., diving curves submerge into limbs of $\mathcal{K}_m$.

We denote by $\mathcal{F}^+_{\text{ext},m}(I,J)$ and $\mathcal{F}^+_{\text{div},m}(I,J)$ the subfamilies of $\mathcal{F}^+_\lambda(I,J)$ consisting of external and diving curves respectively. As usual, we write

$$\mathcal{W}^+_{\text{ext},m}(I,J) = \mathcal{W}\left(\mathcal{F}^+_{\text{ext},m}(I,J)\right) \quad \text{and} \quad \mathcal{W}^+_{\text{div},m}(I,J) = \mathcal{W}\left(\mathcal{F}^+_{\text{div},m}(I,J)\right).$$

The families $\mathcal{F}^+_{\lambda,\text{ext},m}(I), \mathcal{F}^+_{\lambda,\text{div},m}(I)$ are defined accordingly.

Lemma 2.7. *Consider an interval $T = [a,b] \in \mathfrak{D}_m$ in the diffeo-tiling §2.1.6 and let L_a, L_b be the limbs of $\mathcal{K}_m$ attached to a, b. Consider intervals $I \subset T$ and $J \subset \partial Z$. We have:*

$$\mathcal{W}^+(I,J) = \mathcal{W}^+_{\text{ext},m}(I,J) + \mathcal{W}^+_{\text{div},m}(I,J) - O(1). \tag{2.13}$$

Moreover, there are laminations $\mathcal{G}_{\text{ext}} \subset \mathcal{W}^+_{\text{ext},m}(I,J)$, $\mathcal{G}_{\text{div}} \subset \mathcal{F}^+_{\text{div},m}(I,J)$ consisting of at most two rectangles each such that

$$\mathcal{W}(\mathcal{G}_{\text{ext}}) = \mathcal{W}^+_{\text{ext},m}(I,J) - O(1), \qquad \mathcal{W}(\mathcal{G}_{\text{div}}) = \mathcal{W}^+_{\text{div},m}(I,J) - O(1). \tag{2.14}$$

Moreover, we can assume that $\mathcal{G}_{\text{ext}}$ consists of rectangles $\mathcal{R}_a, \mathcal{R}_b \subset \mathbb{C}\setminus Z$ with $\partial^{h,0}\mathcal{R}_a, \partial^{h,0}\mathcal{R}_b \subset I$ such that every curve in $\mathcal{F}(\mathcal{R}_a)$ intersects L_a before intersecting $\mathcal{K}_m \setminus L_a$ while every curve in $\mathcal{F}(\mathcal{R}_b)$ intersects L_b before intersecting $\mathcal{K}_m \setminus L_b$.

If $J \subset T^c$, then $\mathcal{W}^+_{\text{ext},m}(I,J) = O(1)$ and $\mathcal{W}^+(I,J) = \mathcal{W}^+_{\text{div},m}(I,J) + O(1)$.

Proof. Present J as a concatenation of intervals $J_x \cup J_y \cup J_z$ such that $J_x, J_z \subset T$ while $J_y \subset T^c$; we allow some of the intervals to be empty. Consider the canonical rectangle $\mathcal{R}$ of $\mathcal{F}^+(I,J)$, see §A.1.6. Then $\mathcal{R}$ splits into a union of genuine subrectangles $\mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3 \cup \mathcal{R}_4 \cup \mathcal{R}_5$, where some of them can be empty, such that

- $\mathcal{R}_1 \subset \mathcal{F}^+_{\text{ext},m}(I, J_x)$ and $\mathcal{R}_5 \subset \mathcal{F}^+_{\text{ext},m}(I, J_z)$;
- every $\gamma \in \mathcal{F}(\mathcal{R}_2)$ intersects L_a before intersecting $\mathcal{K}_m \setminus L_a$;
- similarly, every $\gamma \in \mathcal{F}(\mathcal{R}_4)$ intersects L_b before intersecting $\mathcal{K}_m \setminus L_b$;
- every $\gamma \in \mathcal{F}(\mathcal{R}_3) \subset \mathcal{F}^+(I, J_y)$ is either disjoint from $L_a \cup L_b$ or it intersects $\mathcal{K}_m \setminus (L_a \cup L_b)$ before intersecting $L_a \cup L_b$.

In particular, $\mathcal{R}_2, \mathcal{R}_4 \subset \mathcal{F}^+_{\text{div},m}(I, J)$.

Since the harmonic measure in $(\widehat{\mathbb{C}} \setminus \mathcal{K}_m, \infty)$ of $\partial L_a, \partial L_b$ is bigger than the harmonic measure of T, we have $\mathcal{W}(\mathcal{R}_3) = O(1)$. And by removing $O(1)$ buffers, we can assume that $\mathcal{R}_2, \mathcal{R}_4 \subset \mathbb{C} \setminus Z$. By (A.7), we have:

$$\mathcal{W}(\mathcal{R}) = \mathcal{W}(\mathcal{R}_1) + \mathcal{W}(\mathcal{R}_2) + \mathcal{W}(\mathcal{R}_4) + \mathcal{W}(\mathcal{R}_5) - O(1).$$

□

2.3.3. *Univalent push-forward.* Consider an interval $T = [a, b] \subset \mathfrak{D}_m$ in the diffeo-tiling and let L_a, L_b be two limbs of $\mathcal{K}_m$ attached to a, b. Consider a rectangle $\mathcal{R} \subset \mathbb{C} \setminus Z$ with $\partial^{h,0}\mathcal{R} \subset T$ and $\partial^{h,1}\mathcal{R} \subset \partial Z$ such that every $\gamma \in \mathcal{F}(\mathcal{R})$ is either external rel $\mathcal{K}_m$ or intersects L_a before intersecting $\mathcal{K}_m \setminus L_a$. (Similar, the case of L_b is considered.)

For every vertical curves $\gamma\colon [0, 1] \to \mathbb{C}$ in $\mathcal{R}$, let $t_\gamma \le 1$ the first intersection of γ with L_a. We denote by $\mathcal{G} = \{\gamma \mid [0, t_\gamma] \mid \gamma \in \mathcal{R}\}$ be the corresponding restriction, and let $\mathcal{R}_0$ be the rectangle in $\mathbb{C} \setminus \operatorname{int} \mathcal{K}_m$ with $\partial^{h,0}\mathcal{R}_0 = \partial^{h,0}\mathcal{R}$ and $\partial^{h,1}\mathcal{R}' \subset \partial\mathcal{K}_m$ be the rectangle bounded by the leftmost and rightmost curves of $\mathcal{G}$, see §A.1.8. We have

$$\mathcal{W}(\mathcal{R}') \ge \mathcal{W}(\mathcal{G}) \ge \mathcal{W}(\mathcal{R}).$$

If $\mathcal{R}$ is external rel $\mathcal{K}_m$, then $\mathcal{R}' = \mathcal{R}$. Let $\mathcal{R}^{\text{new}}$ be the rectangle obtained from $\mathcal{R}'$ by removing two 1-buffers. By Lemma A.10, for every $i \le \mathfrak{q}_{m+1}$, the map

$$f^i\colon \mathcal{R}^{\text{new}} \xrightarrow{1:1} f^i(\mathcal{R}^{\text{new}}) \tag{2.15}$$

is injective. We will refer to (2.15) as the *univalent push-forward* of $\mathcal{R}$.

2.3.4. $\mathcal{F}(I^+, J^+)$*-families.* For a closed Jordan disk $D \subset \mathbb{C}$, consider two disjoint intervals $I, J \subset \partial D$. Let us view $\widehat{\mathbb{C}} \setminus (I \cup J)$ as a Riemann surface; with respect to this Riemann surface both I, J have two sides: the *outer sides* I^+, J^+ and the *inner sides* I^-, J^-. Ignoring the endpoints of I, J, a curve $\gamma\colon (0, 1) \to \widehat{\mathbb{C}} \setminus (I \cup J)$ *lands* at $\gamma(1) \in I^+$ if

$$\gamma[1 - \varepsilon, 1) \subset \widehat{\mathbb{C}} \setminus Z \;\; \forall \varepsilon > 0 \qquad \text{and} \qquad \lim_{\tau \to 1} \gamma(\tau) = \gamma(1).$$

Similarly, the landing at I^- is defined. Let

- $\mathcal{F}(I^+, J^+)$ be the family of curves in $\widehat{\mathbb{C}} \setminus (I \cup J)$ connecting I^+ and J^+;
- $\mathcal{W}(I^+, J^+) = \mathcal{W}(\mathcal{F}(I^+, J^+))$ be the extremal width of $\mathcal{F}(I^+, J^+)$.

The *central arc* in $\mathcal{F}^-_D(I, J)$ is the curve $\ell \in \mathcal{F}^-_D(I, J)$ that splits D, viewed as a rectangle with horizontal sides I, J, into two genuine subrectangles of equal width.

Lemma 2.8 (Trading $\mathcal{F}(I, J)$ into $\mathcal{F}(I^+, J^+)$)**.** *Consider a closed Jordan disk $D \subset \mathbb{C}$ and a family $\mathcal{F}^-(I, J) = \mathcal{F}^-_D(I, J)$ for $I, J \subset \partial D$. Let $A, B \subset \partial D$ be two complementary intervals between I and J and let $\widetilde{I} \supset I$ and $\widetilde{J} \supset J$ be thickenings of I, J so that $\widetilde{I}, \widetilde{J}$ are disjoint intervals of ∂D. Set*

$$C := \mathcal{W}^-(A, B) + \mathcal{W}^-(I, \widetilde{I}^c) + \mathcal{W}^-(J, \widetilde{J}^c).$$

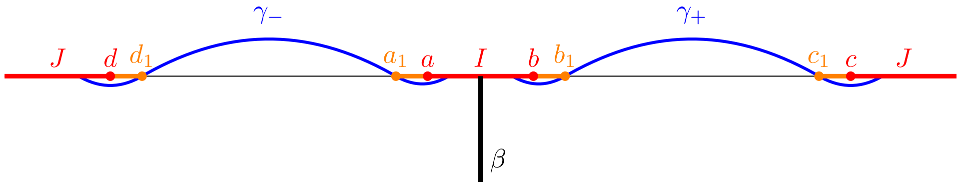


FIGURE 5. The curves γ_- and γ_+ specify the intervals $\widehat{I} = [a_1, b_1]$ and $\widehat{J} = [c_1, d_1]$. Here $I = [a, b]$ and $J = [c, d]$.

Let $\mathcal{R} \subset \mathcal{F}(I, J)$ be a lamination.

Then there are intervals $\widehat{I}, \widehat{J} \subset \partial Z$ with $I \subset \widehat{I} \subset \widetilde{I}$ and $J \subset \widehat{J} \subset \widetilde{J}$ such that there is a restriction $\mathcal{G}$ of a sublamination of $\mathcal{R}$ with

- *$\mathcal{G} \subset \mathcal{F}\big(\widehat{I}^+, \widehat{J}^+\big)$;*
- *$\mathcal{W}(\mathcal{R}|\mathcal{G}) = \mathcal{W}(\mathcal{R}) - O(C)$, see (A.6);*
- *$\mathcal{G}$ is disjoint from the central arc in $\mathcal{F}^-(I, J)$.*

In particular, by taking $\mathcal{R}$ to be the vertical family of $\mathcal{F}(I, L)$, see §A.1.6, we obtain $\mathcal{W}(\widehat{I}^+, \widehat{J}^+) \geq \mathcal{W}(I, J) - O(C)$.

Proof. Let $\mathcal{G}^{\text{new}}$ be the lamination obtained from $\mathcal{G}$ by removing all leaves ℓ satisfying one of the following properties:

- ℓ intersects the central arc β of $\mathcal{F}^-(I, J)$;
- ℓ contains a subarc in $\mathcal{F}^-(I, \widetilde{I}^c)$;
- ℓ contains a subarc in $\mathcal{F}^-(J, \widetilde{J}^c)$.

By assumption, $\mathcal{W}(\mathcal{G}^{\text{new}}) = \mathcal{W}(\mathcal{G}) - O(C)$.

Since curves in $\mathcal{G}^{\text{new}}$ do not intersect β, they possess a left-right order. Denote by γ_-, γ_+ the leftmost and rightmost arc in $\mathcal{G}$. We assume that $\gamma_- < \gamma_+ < \beta$ with respect to the clockwise order around I, see Figure 5. Write $I = [a, b], J = [c, d]$, where $a < b < c < d$. Intersecting γ_-, γ_+ with ∂D^+, we obtain the intervals $\widehat{I} = [a_1, b_1], \widehat{J} = [c_1, d_1]$ as follows (Figure 5):

- If γ_- starts in I^+, then $a_1 \coloneqq a$; otherwise a_1 is the first intersection of γ_- with $\partial D \setminus I$.
- If γ_+ starts in I^+, then $b_1 \coloneqq b$; otherwise b_1 is the first intersection of γ_+ with $\partial D \setminus I$.
- If γ_- ends at J^+, then $d_1 \coloneqq d$; otherwise d_1 is the last intersection of γ_- with $\partial D \setminus J$.
- If γ_+ ends at J^+, then $c_1 \coloneqq c$; otherwise c_1 is the last intersection of γ_+ with $\partial D \setminus J$.

By construction, $\widehat{I} \subset I$ and $\widehat{J} \subset J$. Restricting $\mathcal{G}^{\text{new}}$ to the family $\mathcal{F}(\widehat{I}, \widehat{J})$, we obtain a required lamination $\mathcal{G}^{\text{New}} \subset \mathcal{F}(\widehat{I}^+, \widehat{J}^+)$. □

2.3.5. *$\mathcal{F}^\circ_L(I, J)$-families.* Consider a pair $I, J \subset \partial D$ of disjoint intervals, and let $L \subset \partial D$ be one of the complementary intervals between I and J. We define

- $\mathcal{F}^\circ_L(I, J)$ to be the set of curves $\gamma \in \mathcal{F}(I^+, J^+)$ such that γ is disjoint from $\partial D \setminus (I \cup L \cup J)$;
- $\mathcal{W}^\circ_L(I, J) \coloneqq \mathcal{W}\left(\mathcal{F}^\circ_L(I, J)\right)$.

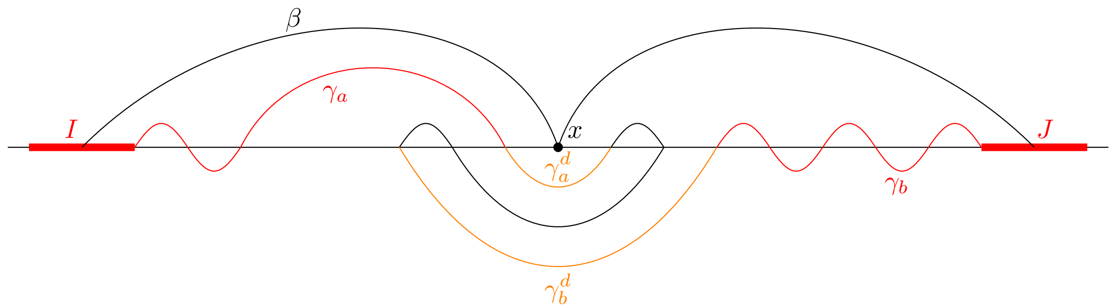


FIGURE 6. The subcurves of γ: γ_a (red), γ_a^d (orange), γ_b^d (orange), γ_b (red). Note that γ_a and γ_b are disjoint.

If $I < L < J$, then we write:

$$\mathcal{F}^\circ(I,J) \coloneqq \mathcal{F}^\circ_L(I,J) \qquad \text{and} \qquad \mathcal{W}^\circ(I,J) \coloneqq \mathcal{W}^\circ_L(I,J).$$

Lemma 2.9 (Snakes in $\mathcal{F}^\circ \setminus \mathcal{F}^+$)**.** *Consider a closed topological disk D, intervals $I, J \subset \partial D$, and $\mathcal{F}^\circ_L(I,J)$ as above, if*

$$K = \mathcal{W}^\circ(I,J) - \mathcal{W}^+_L(I,J) \gg 1,$$

then $\mathcal{F}^\circ_L(I,J)$ contains a rectangle $\mathcal{R}$, called a snake*, with $\mathcal{W}(\mathcal{R}) = K - O(1)$ such that every vertical curve of $\mathcal{R}$ intersects L.*

Proof. Let $\mathcal{R}$ be the canonical rectangle of $\mathcal{F}^\circ_L(I,J)$, §A.1.6; i.e. the semi-closed rectangle realizing the width between I^+, J^+ in the open topological disk $\widehat{\mathbb{C}} \setminus L^c$. Let $\gamma \in \mathcal{F}(\mathcal{R})$ be the unique vertical curve intersecting L such that $\gamma \subset \widehat{\mathbb{C}} \setminus \operatorname{int} D$. Then γ splits $\mathcal{R}$ into two rectangles $\mathcal{R}^{\text{out}} \subset \mathcal{F}^+(I,J)$ and $\mathcal{R}^{\text{inn}} \subset \mathcal{F}^\circ_L(I,J)$, where the latter rectangle submerges into D. By Lemma A.2, $\mathcal{W}(\mathcal{R}^{\text{out}}) = \mathcal{F}^+(I,J) - O(1)$; this will imply that $\mathcal{W}(\mathcal{R}^{\text{inn}}) = K - O(1)$. □

2.4. **Series Decompositions.** In this subsection, we will discuss how to take restrictions (compare with §A.1.5) of families submerging into topological disks. For a closed topological disk D consider a lamination $\mathcal{R}$ in $\mathcal{F}^\circ_L(I,J) \setminus \mathcal{F}^+(I,J)$; i.e. every curve in $\mathcal{R}$ intersects L. We assume that $I < L < J$ is the order of intervals in ∂D. Let us introduce a topological decomposition for $\mathcal{R}$.

We will use the inner/outer order on curves in $\mathcal{R}$, see §A.1.7: the innermost curve in $\mathcal{R}$ is the closest curve to $(L^c)^-$ in $\widehat{\mathbb{C}} \setminus L^c$, where $L^c = \partial D \setminus L$.

Let $\beta\colon [0,1] \to \widehat{\mathbb{C}}$, $\beta(0) \in I$, $\beta(1) \in J$ be the outermost curve of $\mathcal{R}$; i.e. all other curves of $\mathcal{R}$ are between β and $(L^c)^-$. Let $x = \beta(t) \in \beta \cap L$ be the first (for the smallest t) intersection between L and β.

Consider a vertical curve $\gamma\colon [0,1] \to \mathbb{C}$ in $\mathcal{R}$ with $\gamma(0) \in I$ and $\gamma(1) \in J$. Since x is the first entry of β into $\lfloor x, J \rfloor$, we obtain that the first entry of γ into $\lfloor x, J \rfloor$ is from $\operatorname{int} D$. And since γ starts and ends at I^+ and J^+ respectively, we can define (see Figure 6)

- $\gamma(a_2) \in \partial D$ to be the first intersection of γ with $\lfloor x, J \rfloor$;
- $\gamma(a) \in \partial D$ to be the last before a_2 intersection of γ with $\lfloor I, x \rfloor$;
- γ_a to be the subcurve of γ between I and $\gamma(a)$;

- γ_a^d to be the subcurve of γ between $\gamma(a)$ and $\gamma(a_2)$;
- $\gamma(b_2) \in \partial D$ to be the last intersection of γ with $\lfloor I, x\rfloor$;
- $\gamma(b) \in \partial D$ to be the first after b_2 intersection of γ with $\lfloor x, J\rfloor$,
- γ_b^d to be the subcurve of γ between $\gamma(b_2)$ and $\gamma(b)$; and
- γ_b to be the subcurve of γ between $\gamma(b)$ and J.

We say that

- $\gamma_a^d \subset D$ is the *first passage* of γ under x;
- γ_a is the subcurve of γ *before* γ_a^d;
- $\gamma_b^d \subset D$ is the *last passage* of γ under x;
- γ_b is the subcurve of γ *after* γ_b^d.

Clearly, γ_a and γ_b are disjoint because a_2, b_2 are between a and b. Also $\gamma_a \cup \gamma_b$ is disjoint from $\gamma_a^d \cup \gamma_b^d$. The curves γ_a^d, γ_b^d may or may not coincide.

Remark 2.10. *Since x is the first intersection of β with L, the curve β is outside of D before it reaches x. After x, the curve β may have a complicated intersection pattern with ∂D. For example, β may pass under x to intersect the left interval of $L \setminus \{x\}$; but then β must go back under x and intersect the right interval of $L \setminus \{x\}$.*

Let us specify the following laminations

$$\widetilde{\mathcal{F}}_a \coloneqq \{\gamma_a \mid \gamma \in \mathcal{R}\}, \qquad \widetilde{\mathcal{F}}_b \coloneqq \{\gamma_b \mid \gamma \in \mathcal{R}\},$$
$$\Gamma_a \coloneqq \{\gamma_a^d \mid \gamma \in \mathcal{R}\}, \qquad \Gamma_b \coloneqq \{\gamma_b^d \mid \gamma \in \mathcal{R}\},$$

$$\Gamma \coloneqq \Gamma_a \cup \Gamma_b = \{\gamma_a^d \mid \gamma \in \mathcal{R}\} \cup \{\gamma_b^d \mid \gamma \in \mathcal{R}\}. \tag{2.16}$$

Then $\mathcal{R}$ consequently overflows $\widetilde{\mathcal{F}}_a, \Gamma, \widetilde{\mathcal{F}}_b$.

Let ℓ be the lowest curve in Γ with respect to x; i.e. ℓ separates Γ from L^c in D. We denote by $J_a \subset L$ the interval between the left endpoint of ℓ and x and we denote by $I_b \subset L$ the interval between x and the right endpoint of ℓ. Define

$$\mathcal{F}_a = \{\gamma' \mid \gamma' \text{ is the first shortest subcurve of } \gamma \in \widetilde{\mathcal{F}}_a \text{ connecting } I^+, J_a^+\} \tag{2.17}$$

to be the restriction of $\widetilde{\mathcal{F}}_a$ to $\mathcal{F}(I, J_a)$ – compare to §A.1.5; and

$$\mathcal{F}_b = \{\gamma' \mid \gamma' \text{ is the first shortest subcurve of } \gamma \in \widetilde{\mathcal{F}}_b \text{ connecting } I_b^+, J^+\} \tag{2.18}$$

to be the restriction of $\widetilde{\mathcal{F}}_b$ to $\mathcal{F}(I_b, J)$. Since curves in $\widetilde{\mathcal{F}}_a \sqcup \widetilde{\mathcal{F}}_b$ are disjoint from ℓ, we obtain

$$\mathcal{F}_a \subset \mathcal{F}^\circ(I, J_a), \qquad \mathcal{F}_b \subset \mathcal{F}^\circ(I_b, J);$$

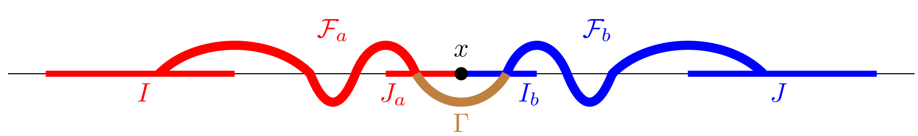
in particular, curves in $\mathcal{F}_a, \mathcal{F}_b$ land at J_a^+, I_b^+ respectively.

We summarize:

Lemma 2.11. *A lamination $\mathcal{R} \subset \mathcal{F}_L^\circ(I, J) \setminus \mathcal{F}^+(I, J)$ as above consequently overflows the pairwise disjoint laminations*

$$\mathcal{F}_a \subset \mathcal{F}^\circ(I, J_a), \qquad \Gamma \subset \mathcal{F}^-(J_a, I_b), \qquad \mathcal{F}_b \subset \mathcal{F}^\circ(I_b, J).$$

defined in (2.17), (2.16), (2.18) *respectively.*



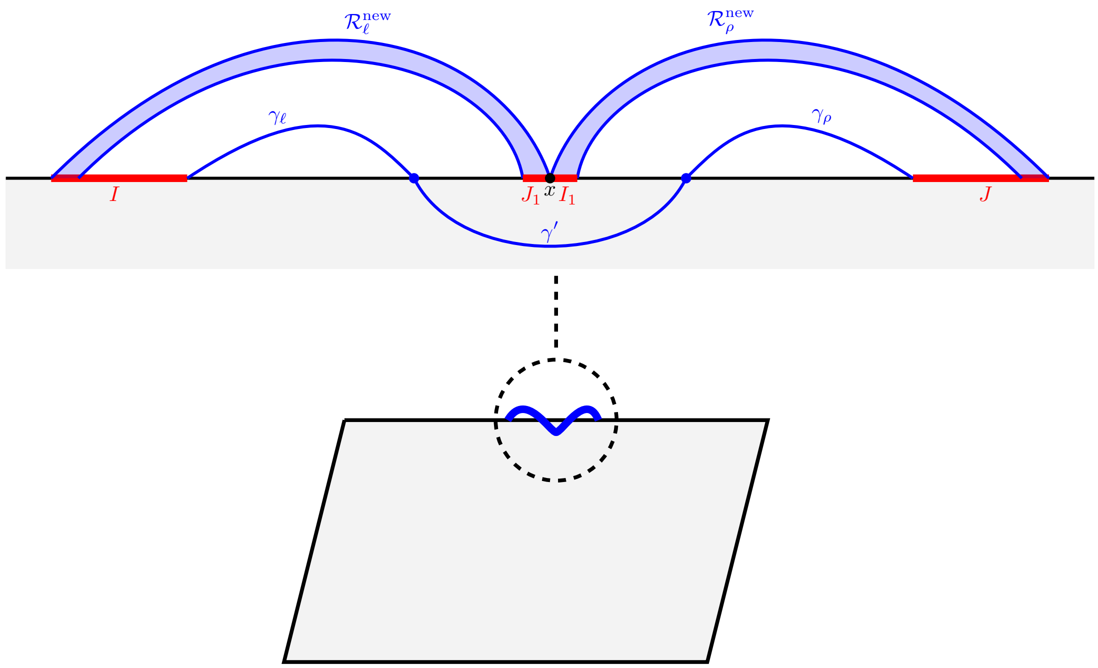


FIGURE 7. Illustration to Snake Lemma 2.12: if a "snake" with width $K \gg 1$ submerges, then either $\mathcal{R}_\ell^{\text{new}}$ or $\mathcal{R}_\rho^{\text{new}}$ has width $2K - O(\log \lambda)$.

□

2.5. **Snake Lemma.** The following lemma allows us to control submergence of $\mathcal{F}^\circ(I, J)$ into Z (see §2.3.5). The Snake Lemma for pseudo-Siegel disks will be proven as Lemma 6.1.

Snake Lemma 2.12 (See Figure 7)**.** *Let $I, J \subset \partial Z$ be a pair of intervals with $\lfloor I, J \rfloor < 1/2$ and let $L \coloneqq \lfloor I, J \rfloor \setminus (I \cup J)$ be the complementary interval between I, J with $I < L < J$. Set*

$$K \coloneqq \mathcal{W}_L^\circ(I, J) - \mathcal{W}^+(I, J).$$

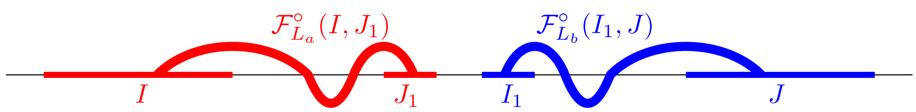
If $K \gg \log \lambda$ with $\lambda > 2$, then there are intervals

$$J_1, I_1 \subset L, \qquad |J_1| < \frac{\operatorname{dist}(I, J_1)}{\lambda}, \quad |I_1| < \frac{\operatorname{dist}(I_1, J)}{\lambda}, \qquad I < J_1 < I_1 < J$$

such that

$$\mathcal{W}_{L_a}^\circ(I, J_1) \oplus \mathcal{W}_{L_b}^\circ(I_1, J) \geq K - O(\log \lambda), \tag{2.19}$$

where $L_a, L_b \subset L$ are the complementary intervals between I, J_1 and I_1, J respectively:



In particular, either $\mathcal{W}^\circ_{L_a}(I, J_1)$ or $\mathcal{W}^\circ_{L_b}(I_1, J)$ has width $\geq 2K - O(\log\lambda)$.

The Snake Lemma is a consequence of the Localization Property §2.2.1 applied to Series Decomposition §2.4.

Proof. Let $\mathcal{R} \subset \mathcal{F}^\circ_L(I, J) \setminus \mathcal{F}^+(I, J)$ with $\mathcal{W}(\mathcal{R}) = K - O(1)$ be a rectangle (a snake) from Lemma 2.9 realizing K. Apply Series Decomposition §2.4 to $\mathcal{R}$, we obtain that $\mathcal{F}(\mathcal{R})$ consequently overflows the laminations

$$\mathcal{F}_a \subset \mathcal{F}^\circ(I, J_a), \qquad \Gamma \subset \mathcal{F}^-(J_a, I_b), \qquad \mathcal{F}_b \subset \mathcal{F}^\circ(I_b, J).$$

By the Localization Property §2.2.1, J_a, I_b contain an innermost subpair J_1, I_1 such that

$$|I_1|,\ |J_1| \leq \frac{1}{2\lambda}\{|I_a|,\ |J_b|\}$$

and up to $O(\log\lambda)$-width the family $\mathcal{F}^-(J_a, I_b)$ is in $\mathcal{F}^-(J_1, I_1)$:

$$\mathcal{W}^-(J_a \setminus J_1,\ \ I_b) + \mathcal{W}^-(J_a,\ \ I_b \setminus I_1) = O(\log\lambda)$$

Let $\mathcal{R}^{\text{new}}$ be the lamination obtained from $\mathcal{R}$ by removing all $\gamma \in \mathcal{F}(\mathcal{R})$ with $\gamma^d_a \notin \mathcal{F}^-(J_1, I_1)$ or $\gamma^d_b \notin \mathcal{F}^-(J_1, I_1)$. Then $\mathcal{W}(\mathcal{R}^{\text{new}}) = K - O(\log\lambda)$.

Applying Series Decomposition §2.4 to $\mathcal{R}^{\text{new}}$, we obtain that $\Gamma^{\text{new}} \subset \mathcal{F}^-(J_1, I_1)$; i.e. $J^{\text{new}}_a \subset J_1$ and $I^{\text{new}}_b \subset I_1$.

□

2.6. Trading $\mathcal{F}$ into $\mathcal{F}^+$.

Corollary 2.13 (Trading $\mathcal{W}^\circ$ into $\mathcal{W}^+$)**.** *Under the assumptions of Lemma 6.1, there is an interval $I^{\text{new}} \subset L$ such that $\mathcal{W}^+_\lambda(I^{\text{new}}) \succeq K$.*

Proof. It follows from (6.2) and $K \gg \log\lambda$ that either $\mathcal{W}^\circ_{L_a}(I, J_1)$ or $\mathcal{W}^\circ_{L_b}(I_1, J)$ has width $\geq 2K - O(\log\lambda) \geq \frac{7}{4}K$. Assume that $\mathcal{W}^\circ_{L_a}(I, J_1) \geq \frac{7}{4}K$. Since $\mathcal{F}^+_\lambda(J_1) \supset \mathcal{F}^+_{L_a}(I, J_1)$, either $\mathcal{W}^+_\lambda(J_1) \geq \frac{1}{5}K$ or $\mathcal{W}^\circ(I, J_1) - \mathcal{W}^+(I, J_1) \geq \frac{3}{2}K$; in the latter case, we can again apply the Snake Lemma and construct intervals I_2, J_2 such that

$$\mathcal{W}^\circ(I, J_2) \oplus \mathcal{W}^\circ(I_2, J_1) \geq \frac{3}{2}K - O(\log\lambda), \qquad \text{where}$$

$$\mathcal{F}^+(I, J_2) \subset \mathcal{F}^+_\lambda(J_2) \qquad \text{and} \qquad \mathcal{F}^+(I_2, J_1) \subset \mathcal{F}^+_\lambda(I_2).$$

Applying induction, we either find an interval I^{new} with $\mathcal{W}^+_\lambda(I^{\text{new}}) \geq \frac{3^n}{2^n 5}K$, or construct an infinite sequence of shrinking intervals $I^{\text{new}}_n, J^{\text{new}}_n, L^{\text{new}}_n$ with

$$\mathcal{F}^\circ_{L^{\text{new}}_n}(I^{\text{new}}_n, J^{\text{new}}_n) \geq \frac{3^n}{2^n}K \qquad |L^{\text{new}}_n| \geq \min\{|I^{\text{new}}_n|,\ \ |J^{\text{new}}_n|\}.$$

Such an infinite sequence does not exist because $\overline{Z}$ is a (non-uniform) qc disk. □

3. Near-Rotation Systems

For $r > 0$, we denote by $|x-y|_r$ the Euclidean distance between x, y on the circle $\mathbb{R}/r = \mathbb{R}/(r\mathbb{Z})$. We also write $|x-y| = |x-y|_1$, which is consistent with the combinatorial distance introduce for $\partial\mathbb{D}, \partial Z$.

Fix $\mu > 0$. A *μ-near-rotation* system with rotation number $\mathfrak{p}/\mathfrak{q} \in \mathbb{Q}$ is $\mathfrak{F}_\mathfrak{q} = \left(f^t \colon U \to U_t\right)_{0 \le t \le \mathfrak{q}}$ such that (see Figure 8)

(A) $\overline{U}$ and $\overline{U}_t$ are closed Jordan disks;
(B) $f^t \colon U \to U_t$ is conformal for $t \le \mathfrak{q}$;
(C) ∂U is a cyclic (clockwise or counterclockwise) concatenation of simple arcs:

$$\partial U = L_0 \# L_1 \# L_2 \# \dots \# L_{\mathfrak{q}-1}; \tag{3.1}$$

(D) for every k there is an annuls A_k with $\operatorname{mod}(A_k) \ge \mu$ such that
  (D1) the bounded component B_k of $\mathbb{C} \setminus A_k$ compactly contains L_k as well as all $f^t(L_{k-\mathfrak{p}t})$ for $t \le \mathfrak{q}$, and
  (D2) A_k is disjoint from A_i for $|i-k|_\mathfrak{q} > 1$.

In other words, f^t maps L_k approximately onto $L_{k+t\mathfrak{p}}$ so that $L_k, L_{k+t\mathfrak{p}} \subset B_k$; this "error" is controlled (surrounded) by A_k. Let us write $\widetilde{A}_k = A_k \cup B_k$ – the filling-in of A_k; and

$$\widetilde{U} \coloneqq U \cup \bigcup_{k=0}^{\mathfrak{q}-1} B_k.$$

We call the L_i *unit intervals* of ∂U and we call $\overline{U}$ a *μ-near-rotation domain.*

3.0.1. *Motivation and outline.* Recall that $\theta \approx \mathfrak{p}_n/\mathfrak{q}_n$ (see §2.1.1) and f rotates the diffeo-tiling $\mathfrak{D}_n$ by approximately $\mathfrak{p}_{n+1}/\mathfrak{q}_{n+1}$, see §2.1.6. If $\mathfrak{l}_{n+1} \ll \mathfrak{l}_n$, then the "rotational error" is small with respect to the combinatorial metric of ∂Z. However, with respect to the conformal geometry of $\mathbb{C}$, the rotational error will be big due to parabolic fjords; see Figure 2. To deal with this issue, we will approximate $\overline{Z}$ by a $\boldsymbol{\delta}$-near-rotation domain Z^n with a universal $\boldsymbol{\delta} > 0$ and add all such Z^m to $\overline{Z}$, see §7.

For $\mathbb{S}^1 = \partial\mathbb{D}$, the Euclidean metric on $\mathbb{R}/\mathbb{Z} \simeq \mathbb{S}^1$ is a unique invariant metric under all rotations $z \mapsto \mathbf{e}(\phi)z$. For near-rotation domains we have almost-rotations $f^i \mid U, i \le \mathfrak{q}$, where the error is controlled by annuli A_i with $\operatorname{mod} A_i \ge \mu > 0$. It is natural to expect that as μ is fixed and $\mathfrak{q} \to \infty$, "almost invariant metrics" on ∂U converge to the Euclidean metric on $\mathbb{R}/\mathbb{Z} \simeq \partial\mathbb{D}$ after a conformal uniformization. In this section, we will prove a slightly weaker statement: almost invariant metrics will eventually be universally close to the Euclidean metric, see Theorem 3.8. These beau bounds will imply that the error does not increase during iterative construction of pseudo-Siegel disks $\dots \rightsquigarrow \widehat{Z}^{m+1} \rightsquigarrow \widehat{Z}^m \rightsquigarrow \widehat{Z}^{m-1} \rightsquigarrow \dots$, see Remark 7.2.

Theorem 3.8 is proven using the Shift Argument (Figure 9): if there is an unexpected wide rectangle, then its appropriate shift will have a substantial cross-intersection with itself contradicting Non-Crossing Principle §A.2.1. From this, the estimates in Theorem 3.8 are established in the same way as in Lemma 2.5. The main subtlety is that shifted curves can sneak through the B_k. We will first establish estimates on scale $\ge 40/\mathfrak{q}$ with an error depending on μ (Proposition 3.3), then we will upgrade them to universal estimates on scale $\gg_\mu 1/\mathfrak{q}$.

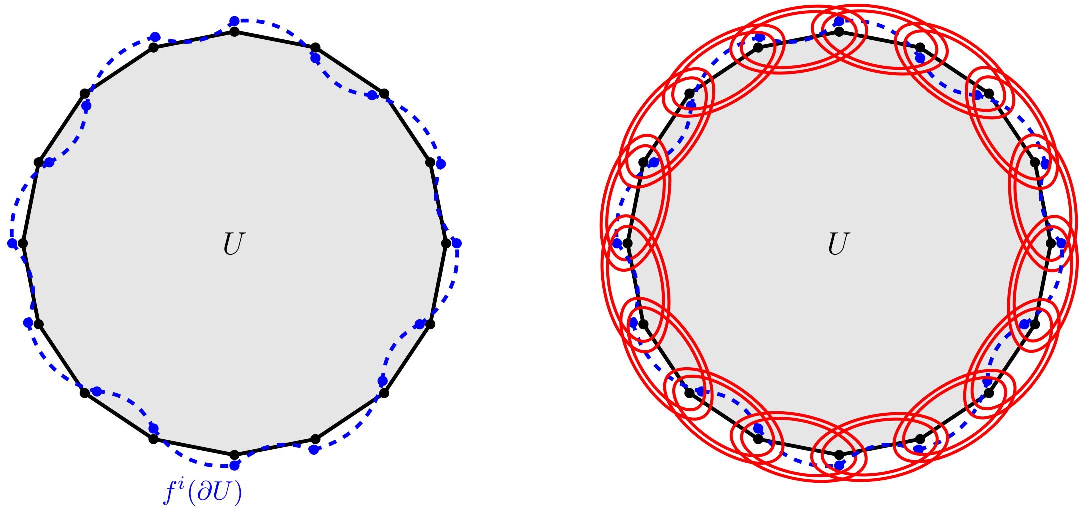


FIGURE 8. Illustration to near-rotation systems: ∂U is a finite concatenation of arcs, and $f^i(\partial U)$ (blue) is approximately ∂U rotated by $i\mathfrak{p}/\mathfrak{q}$, where the error is controlled by a system of annuli (right side).

3.1. **Standard intervals of ∂U.** A *discrete interval*

$$S \subset \{0, 1, 2 \dots, \mathfrak{q} - 1\} \simeq \mathbb{Z}/\mathfrak{q}$$

with *length* b is a finite subset $\{a, a+1, a+2, \dots, a+b-1\}$ of $\mathbb{Z}/\mathfrak{q}$ consisting of consecutive numbers. Set

$$L_S \coloneqq \bigcup_{s\in S} L_s, \qquad B_S \coloneqq \bigcup_{s\in S} B_s.$$

By construction, $f^t(L_S) \subset B_{S+\mathfrak{p}t}$ for $t \le \mathfrak{q}$, where $S + j = \{s + j \mid s \in S\}$.

For $r > 1$, we define the rescaling of S with respect to its center as

$$rS \coloneqq \left\{ n \in \mathbb{Z}/\mathfrak{q} : |n - a - (b-1)/2|_{\mathfrak{q}} \le r|S|_{\mathfrak{q}}/2 \right\}.$$

Let $(rS)^c \coloneqq \{0, 1, \dots, \mathfrak{q} - 1\} \setminus (rS)^c$ be the complement of rS. Similar to §2.2, we define:

- $\mathcal{F}^-(L_V, L_W)$ to be the family of curves in U connecting L_V and L_W;
- $\mathcal{W}^-(L_V, L_W) = \mathcal{W}(\mathcal{F}^-(L_V, L_W))$;
- $\mathcal{F}^-_r(L_S) = \mathcal{F}^-(L_S, L_{(rS)^c})$;
- $\mathcal{W}^-_r(L_S) = \mathcal{W}^-(L_S, L_{(rS)^c})$.

We say that an interval L_S is $[K, r]^-$*-wide* if $\mathcal{W}_r(L_S) \ge K$.

We call an interval $L_S \subset \partial U$ *standard*. Any interval $I \subset \partial U$ can be approximated from above or below by a standard interval with an error within $L_a \cup L_b$ for some $a, b \in \{0, 1, \dots, \mathfrak{q} - 1\}$.

3.2. **Inner geometry of U.** The following lemma is a corollary of Lemma A.8.

Lemma 3.1. *Consider a rectangle*

$$\mathcal{R} \subset f^i(\overline{U}) \subset \widetilde{U}, \quad \partial^h\mathcal{R} \subset f^i(\partial U), \qquad i \le \mathfrak{q}$$

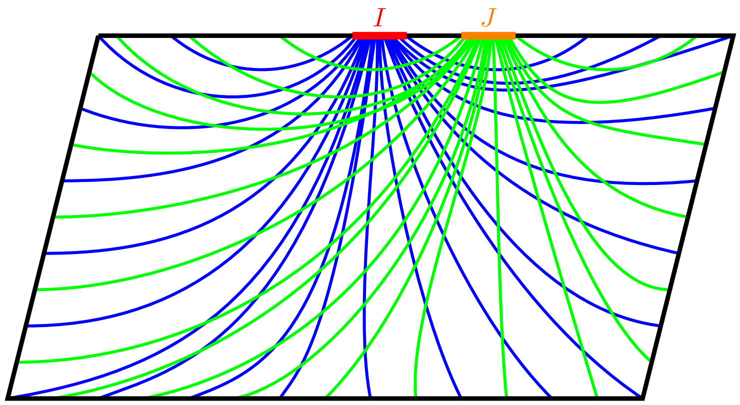


Figure 9. Illustration to Lemma 3.2: since $\mathcal{F}_{20}(I)$ (blue) crosses its shift $\mathcal{F}_{20}(J)$ (green), $\mathcal{F}_{20}(I)$ is not wide.

$$\textit{with} \quad \partial^{h,0}\mathcal{R} \subset B_V, \quad \partial^{h,1}\mathcal{R} \subset B_W,$$

where V and W are discrete intervals. After removing two $C'_\mu := 1/\mu$-wide buffers from $\mathcal{R}$, the new rectangle $\mathcal{R}^{\text{new}}$ is disjoint from B_s for every $s \in \mathbb{Z}/\mathfrak{q}$ at distance at least 3 from $V \cup W$. □

Lemma 3.2. *Set $C_\mu := 30 + 2C'_\mu = 30 + 2/\mu$. There are no $[C_\mu, 20]^-$-wide intervals L_S, where $1 \le |S|_\mathfrak{q} \le \mathfrak{q}/40$.*

Proof. Suppose $I = L_S$ is such a $[C_\mu, 20]^-$-wide interval. Let $\mathcal{R}$, $\partial^{h,0}\mathcal{R} = I$ be the canonical rectangle of $\mathcal{F}^-_{20}(L_S)$, see §A.1.6. We will construct below a shift $\mathcal{R}_J$ of $\mathcal{R}$ so that $\mathcal{R}, \mathcal{R}_J$ have substantial cross-intersection, see Figure 9.

Fix $k \in \mathbb{N}$ such that $S + k$ has $\mathbb{Z}/\mathfrak{q}$-distance at least 3 from $S \cup [20S]^c$. Let $j < \mathfrak{q}$ be so that $\mathfrak{p}j = k$ in $\mathbb{Z}/\mathfrak{q}$. Define

$$J := f^j(I) \subset B_{S+k} \qquad \text{and} \qquad \mathcal{R}_J := f^j(\mathcal{R}).$$

Let $\mathcal{R}_J^{\text{new}}$ be the rectangle obtained from $\mathcal{R}_J$ by removing 5-buffers. By Lemma A.6 (with $n = 1$), we can remove from $\mathcal{R}$ and $\mathcal{R}_J^{\text{new}}$ buffers with width less than 5 so that the new rectangles $\mathcal{R}^{\text{NEW}}$ and $\mathcal{R}_J^{\text{NEW}}$ have disjoint vertical boundaries. Let V be the minimal discrete interval such that $\partial^{h,1}\mathcal{R}^{\text{NEW}} \subset L_V$. By construction:

$$\mathcal{W}\left(\mathcal{R}_J^{\text{NEW}}\right) \ge 10 + 2/\mu, \quad \partial^{h,0}\mathcal{R}_J^{\text{NEW}} \subset B_{S+k}, \quad \partial^{h,1}\mathcal{R}_J^{\text{NEW}} \subset B_{V+k}, \tag{3.2}$$
$$f^j\left(\mathcal{R}^{\text{NEW}}\right) \supset \mathcal{R}_J^{\text{new}} \supset \mathcal{R}_J^{\text{NEW}}.$$

Since $\mathcal{R}^{\text{NEW}} \subset \overline{U}$ with $\partial^h\mathcal{R}^{\text{NEW}} \subset \partial U$, we can choose a vertical boundary component $\beta \in \{\partial^{v,\ell}\mathcal{R}^{\text{NEW}}, \partial^{v,\rho}\mathcal{R}^{\text{NEW}}\}$ that separate L_{S+k} from $\mathcal{R}^{\text{NEW}} \setminus \beta$; i.e., L_{S+k}, $\mathcal{R}^{\text{NEW}} \setminus \beta$ are in different components of $\overline{U} \setminus \beta$. Suppose β starts in L_a and ends at L_b.

By construction, $\{a, b\}$ has distance at least 3 from $[S+k] \cup [V+k]$. Since the horizontal boundary of $\mathcal{R}_J^{\text{NEW}}$ is within B_{S+k} and B_{V+k} (see (3.2)), by Lemma 3.1, the rectangle $\mathcal{R}_J^{\text{NEW}}$ has a vertical curve γ disjoint from $B_a \cup B_b$, i.e. γ is disjoint from $\beta \cup B_a \cup B_b$. This is a contradiction as the endpoints of γ are separated by $\beta \cup B_a \cup B_b$ in $\widetilde{U}$. □

3.3. **Coarse bounds for Near-Rotation domains.** We now extend the estimates from Lemma 2.5 to near-rotation domains. Let us rescale the distance on ∂U by $1/\mathfrak{q}$:

$$|I| \coloneqq \frac{1}{\mathfrak{q}}|I|_{\mathfrak{q}}, \qquad \operatorname{dist}(I,J) \coloneqq \frac{1}{\mathfrak{q}} \operatorname{dist}_{\mathfrak{q}}(I,J), \qquad \text{for } I = L_V,\ J = L_W \subset \partial U,$$

and we choose any continuous extension of the distance function dist(,) to all point in ∂U. The objects

$$\mathcal{F}^-(I,J) = \mathcal{F}^-_{\overline{U}}(I,J), \qquad \mathcal{W}^-(I,J) = \mathcal{W}^-_{\overline{U}}(I,J)$$

for intervals $I, J \subset \partial U$ are defined in §2.3.1.

Proposition 3.3 (Coarse bounds)**.** *Consider intervals*

$$I, J \subset \partial U \qquad \textit{such that} \quad |I|,\ |J|,\ \operatorname{dist}(I,J) \geq 40/\mathfrak{q}.$$

If $\operatorname{dist}(I,J) \leq \min\{|I|,|J|\}$, *then*

$$\mathcal{W}^-(I,J) \asymp_\mu \log \frac{\min\{|I|,|J|\}}{\operatorname{dist}(I,J)} + 1; \tag{3.3}$$

otherwise

$$\mathcal{W}^-(I,J) \asymp_\mu \left(\log \frac{\operatorname{dist}(I,J)}{\min\{|I|,|J|\}} + 1\right)^{-1}. \tag{3.4}$$

Corollary 3.4 (∂U is a coarse quasi-line)**.** *Choose a homeomorphism*

$$h\colon \partial U \to \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}, \qquad h(L_i) = [i/\mathfrak{q}, (i+1)/\mathfrak{q}].$$

Let $I, J \subset \partial U$ *be two intervals with* $\min\{|I|,|J|,\operatorname{dist}(I,J)\} \geq 40/\mathfrak{q}$. *Then*

$$\mathcal{W}^-_U(I,J) \asymp_\mu \mathcal{W}^-_{\mathbb{D}}(h(I),h(J)).$$

□

Proof of Proposition 3.3.

Claim 1. *Suppose* I, J *are intervals with*

$$\min\{|I|,|J|\} \asymp \operatorname{dist}(I,J) \qquad \textit{and} \quad |I|,|J|,\operatorname{dist}(I,J) \geq 40/\mathfrak{q}. \tag{3.5}$$

Then $\mathcal{W}^-(I,J) \asymp_\mu 1$.

Proof. We can approximate I from above by a concatenation of standard intervals

$$\widetilde{I} = I_1 \# I_2 \# \dots \# I_n \quad \text{ such that } \quad \mathcal{F}^-(I,J) \subset \bigcup_{k=1}^{n} \mathcal{F}^-_{20}(I_k), \qquad \widetilde{I} \setminus I \subset L_a \cup L_b,$$

where n depends on the constant representing "$\asymp$" in (3.5). Using Lemma 3.2 and Parallel Law (A.4), we obtain

$$\mathcal{W}^-(I,J) \leq \mathcal{W}^-_{20}(I_1) + \mathcal{W}^-_{20}(I_2) + \dots + \mathcal{W}^-_{20}(I_n) \preceq_\mu 1. \tag{3.6}$$

Let X, Y be the connected components of $\partial U \setminus (I \cup J)$. We have:

$$\min\{|X|,|Y|\} \asymp \operatorname{dist}(X,Y) \qquad \text{and} \quad |X|,|Y|,\operatorname{dist}(X,Y) \geq 40/\mathfrak{q}.$$

Repeating the above argument for X, Y, we obtain:

$$\big(\mathcal{W}^-(I,J)\big)^{-1} = \mathcal{W}^-(X,Y) \preceq_\mu 1, \quad \text{ i.e. } \ \mathcal{W}^-(I,J) \succeq_\mu 1.$$

Therefore, $\mathcal{W}^-(I,J) \asymp_\mu 1$. □

The proposition follows from Claim 1 by applying the Splitting Argument, see Remark 2.6.

□

Remark 3.5. *We note that the comparison "$\asymp_\mu$" in* (3.3) *and* (3.4) *depends only on the constant C_μ from Lemma 3.2 – this lemma was used only in* (3.6)*. The constant C_μ depends only on the constant $C'_\mu = 1/\mu$ from Lemma 3.1. In §3.4, we will improve Lemma 3.1 and obtain beau coarse-bounds on scale $\gg_\mu 1/\mathfrak{q}$.*

3.4. **Beau coarse-bounds for near-rotation domains.** Let us start by improving Lemma 3.1 on scale $\gg_\mu 1/\mathfrak{q}$:

Lemma 3.6. *There is a constant $T_\mu > 1$ such that the following holds. Consider a rectangle*

$$\mathcal{R} \subset f^i(\overline{U}) \subset \widetilde{U}, \qquad \partial^h \mathcal{R} \subset f^i(\partial U), \qquad i \le \mathfrak{q}$$
$$\text{with} \quad \partial^{h,0}\mathcal{R} \subset B_V, \quad \partial^{h,1}\mathcal{R} \subset B_W,$$

where V and W are discrete intervals. After removing two 1-buffers from $\mathcal{R}$, the new rectangle $\mathcal{R}^{\text{new}}$ is disjoint from B_s for every $s \in \mathbb{Z}/\mathfrak{q}$ with $\operatorname{dist}_\mathfrak{q}(s, V \cup W) \ge T_\mu$.

Proof. Follows essentially from Proposition 3.3 because B_s is protected by a wide family $\mathcal{F}^-(L_G, L_H)$, where G, H are discrete intervals separating s from $V \cup W$. A slight complication is that $\mathcal{R}$ is a rectangle in $\widetilde{U}$ and not in U.

Suppose T_μ is sufficiently big. There is a sequence of pairs of discrete intervals $G_i, H_i \subset \mathbb{Z}/\mathfrak{q}$ such that all G_i, H_i have pairwise distances at least three, $|G_i| = |H_i| = \operatorname{dist}(G_i, H_i)$, every pair G_i, H_i separates s from $V \cup W$, and $n = n(T_\mu)$ is big.

Using Proposition 3.3, we can choose a subfamily $\mathcal{F}_i$ in $\mathcal{F}^-(G_i, H_i)$ such that $\mathcal{W}(\mathcal{F}_i) \asymp_\mu 1$ and such that the $\mathcal{F}_i$ are pairwise disjoint.

For every $g \in G_i$, the set of vertical curves in $\mathcal{R}$ that intersect B_g forms a buffer of $\mathcal{R}$ by Lemma A.9; let us choose $g_i \in G_i$ such that the buffer is maximal. Similarly, we choose $h_i \in H_i$ such that the buffer of curves in $\mathcal{R}$ intersecting B_{h_i} is maximal. Then for every i and every curve $\gamma \in \mathcal{F}$ intersecting B_s either

(1) γ intersects B_{g_i}; or
(2) γ intersects B_{h_i}; or
(3) γ intersects every curve in $\mathcal{F}_i$.

The modulus of curves in $\mathcal{F}$ satisfying (1), (2), and (3) is $\preceq_\mu 1$ because B_{g_i}, B_{h_i} are separated from $V \cup W \cup \{s\}$ by A_{g_i}, A_{h_i}. Therefore, the modulus of vertical curves in $\mathcal{R}$ intersecting B_s is $\preceq_\mu 1/n$. Since n is big, the lemma follows. □

Lemma 3.7. *There is a universal constant $C > 0$ and a constant $T_\mu > 0$ depending on μ such that there are no $[C, 5]^-$-wide intervals L_S with $|S| \ge T_\mu$.*

Proof. Follows from Lemma 3.6 in the same way as Lemma 3.2 follows from Lemma 3.1.

□

Theorem 3.8 (Beau coarse-bounds)**.** *Let $\mathfrak{F}_\mathfrak{q} = \left(f^t \colon U \to U_t\right)_{0 \le t \le \mathfrak{q}}$ be a μ-near rotation domain. There is a constant $T_\mu > 1$ depending on μ such that the following holds. Consider intervals*

$$I, J \subset \partial Z \qquad \text{such that} \quad |I|,\ |J|,\ \operatorname{dist}(I, J) \ge T_\mu/\mathfrak{q}.$$

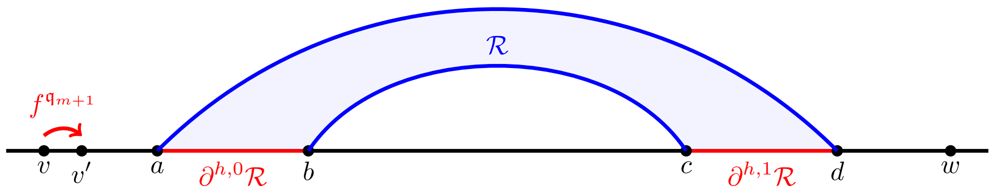


FIGURE 10. A parabolic rectangle $\mathcal{R}$ on $T' = [v', w]$, following Notations (4.3) and (4.2).

If $\operatorname{dist}(I, J) \leq \min\{|I|, |J|\}$, *then*

$$\mathcal{W}^-(I, J) \asymp \log \frac{\min\{|I|, |J|\}}{\operatorname{dist}(I, J)} + 1; \tag{3.7}$$

otherwise

$$\mathcal{W}^-(I, J) \asymp \left(\log \frac{\operatorname{dist}(I, J)}{\min\{|I|, |J|\}} + 1\right)^{-1}. \tag{3.8}$$

Proof. Follows from Lemma 3.7 in the same way as Proposition 3.3 follows from Lemma 3.2 – see Remark 3.5. □

In Theorem 5.12, we will extend beau coarse-bounds for pseudo-Siegel disks.

4. Parabolic fjords

In this section we fix an interval $T \in \mathfrak{D}_m$, $m \geq -1$ in the diffeo-tiling, see §2.1.6. We recall that $|T| \in \{\ell_m, \ell_m + \ell_{m+1}\}$ and $T' \coloneqq T \cap f^{\mathfrak{q}_{m+1}}(T)$. If $m = -1$, then $T = [c_0, c_0 \boxplus 1] \simeq \partial Z$ and T' is the longest interval between c_1 and c_0.

Recall from (2.12) that a rectangle $\mathcal{R}$ is based on T' if $\mathcal{R} \subset \widehat{\mathbb{C}} \setminus Z$ and $\partial^h \mathcal{R} \subset T'$. We assume that $\partial^{h,0}\mathcal{R} < \partial^{h,1}\mathcal{R}$ in T' so that $\lfloor \partial \mathcal{R} \rfloor \subset T'$.

If $m > -1$, then we set $\operatorname{dist}_T(x, y) \coloneqq \operatorname{dist}(x, y)$. For $m = -1$ and $x, y \neq c_0$, we define $\operatorname{dist}_T(x, y)$ to be the length of the interval (x, y) that does not contain c_0. In other words, we view T as $(c_0, c_0 \boxplus 1)$ with the induced Euclidean metric.

A rectangle based on T' or T is called *parabolic* if

$$\operatorname{dist}_T(\partial^{h,0}\mathcal{R}, \partial^{h,1}\mathcal{R}) \geq 6 \min\{|\partial^{h,0}\mathcal{R}|, \quad |\partial^{h,1}\mathcal{R}|\} + 3\mathfrak{l}_{m+1} \tag{4.1}$$

i.e. the gap between $\partial^{h,0}\mathcal{R}$ and $\partial^{h,1}\mathcal{R}$ is bigger than the minimal horizontal side of $\mathcal{R}$. We say that a parabolic rectangle $\mathcal{R}$ is *balanced* if $|\partial^{h,0}\mathcal{R}| = |\partial^{h,1}\mathcal{R}|$.

Let us assume that

$$T = [v, w], \quad v < w, \qquad \theta_{m+1} < 0, \qquad T' = [v', w], \qquad \text{where } v' = v \boxplus \theta_{m+1}, \tag{4.2}$$

i.e. $f^{\mathfrak{q}_{m+1}} \mid T$ moves points clockwise towards w, see Figure 10. The case $\theta_{m+1} > 0$ is equivalent. For a parabolic rectangle $\mathcal{R}$ based on T' we will often write

$$\partial^{h,0}\mathcal{R} = [a, b], \quad \partial^{h,1}\mathcal{R} = [c, d], \qquad \text{where} \quad a < b < c < d. \tag{4.3}$$

Following §2.3.2, we say that a parabolic rectangle $\mathcal{R}$ is *external* if $\operatorname{int} \mathcal{R} \subset \widehat{\mathbb{C}} \setminus \mathcal{K}_m$. The following result describes wide external families based on T:

Theorem 4.1. *If there is a sufficiently wide external parabolic rectangle based on T (see (4.2)), then T contains a subinterval*

$$T_{\text{par}} = [x, y] \subset T', \qquad v' < x < y < w, \qquad \operatorname{dist}(x, v) \le \operatorname{dist}(y, w)$$

with the following properties.

(I) *If $\mathcal{R}$ is an external parabolic rectangle based on T with $\mathcal{W}(\mathcal{R}) \gg 1$, then $\mathcal{R}$ contains a balanced parabolic subrectangle*

$$\mathcal{R}^{\text{new}} \subset \mathcal{R}, \qquad \partial^h \mathcal{R}^{\text{new}} \subset T_{\text{par}}, \qquad \mathcal{W}(\mathcal{R}^{\text{new}}) = \mathcal{W}(\mathcal{R}) - O(1)$$

such that $x < \partial^{h,0}\mathcal{R}^{\text{new}} < \partial^{h,1}\mathcal{R}^{\text{new}} < y, \quad \operatorname{dist}(x, \partial^{h,0}\mathcal{R}^{\text{new}}) = \operatorname{dist}(\partial^{h,1}\mathcal{R}^{\text{new}}, y),$

$$\mathcal{W}(\mathcal{R}^{\text{new}}) \asymp \log \frac{|\partial^{h,0}\mathcal{R}^{\text{new}}|}{\operatorname{dist}(\partial^{h,0}\mathcal{R}^{\text{new}}, v)} = \log \frac{|\partial^{h,1}\mathcal{R}^{\text{new}}|}{\operatorname{dist}(\partial^{h,0}\mathcal{R}^{\text{new}}, v)}. \tag{4.4}$$

(II) *If $I, J \subset T_{\text{par}}, \quad x < I < J < y$ are two intervals with*

$$\operatorname{dist}(x, I) \asymp \operatorname{dist}(J, y) \preceq |I| \asymp |J| \preceq \operatorname{dist}_T(I, J)$$

and $|I|, |J|, \operatorname{dist}_T(I, J) \ge \mathfrak{l}_{m+1}$, then

$$\mathcal{W}^+(I, J) - O(1) = \mathcal{W}^+_{\text{ext},m}(I, J) \asymp \log^+ \frac{\min\{|I|, |J|\}}{\operatorname{dist}(v, I)} + 1. \tag{4.5}$$

(III) *If $I, J \subset T_{\text{par}}, \quad x < I < J < y$ are two intervals with*

$$|\lfloor I, J \rfloor| < \frac{1}{2}|T_{\text{par}}| \quad \text{and} \quad \min\{|I|, |J|\} \succeq \operatorname{dist}(I, J) > 3\mathfrak{l}_{m+1}$$

and $|I|, |J| \ge \mathfrak{l}_{m+1}$, then

$$\mathcal{W}^+(I, J) - O(1) = \mathcal{W}^+_{\text{ext}}(I, J) \asymp \log^+ \frac{\min\{|I|, |J|\}}{\operatorname{dist}(I, J)} + 1. \tag{4.6}$$

Theorem 4.3 will be proven in §4.3. We remark that narrow families based on T can be estimated by evaluating their dual families using Theorem 4.1.

4.0.1. *Outline and Motivation.* Theorem 4.1 says that Siegel disks develop fjords in a controllable way. Roughly, as Figures 2 and 11 illustrate, fjords are vertical strips towards the α fixed point and wide parabolic rectangles are horizontal. After conformal uniformization, $f^{q_{m+1}} \mid$ fjord becomes a quasi-rotation of the unit disk; Theorem 4.1 describes the geometry of this quasi-rotation. During the conformal uniformalization "fjord $\to \mathbb{D}$", the hyperbolic geodesic $\ell_{x,y} \subset \partial(\text{fjord})$ connecting x, y will get the length $\asymp \operatorname{dist}(v, x)$ – this explains v in the estimates of Theorem 4.1; taking this into account, the estimates in Theorem 4.1 are similar to the estimates in Lemma 2.5.

The central theme of this section is designing "shifts" for rectangles based on T; after that, the proof of Theorem 4.1 is similar to Lemma 2.5. Shifts towards v (pullbacks) are relatively easy: the external condition "$\mathcal{R} \subset \widehat{\mathbb{C}} \setminus \operatorname{int} \mathcal{K}_m$" is almost equivalent to "non-winding around the Siegel disk", hence $\mathcal{R}$ can be efficiently moved towards v using $f^{-\mathfrak{q}_{m+1}}$, see §4.1. Shifts towards w (push-forwards) are more delicate because curves may hit $\mathcal{K}_m$. Since pullbacks are well-defined, we can choose the closest to v' outermost external parabolic rectangle $\mathcal{R}_{\text{out}}$ with a certain fixed width; then we set $T_{\text{par}} \coloneqq [x, y]$ to be the complementary interval between $\partial^{h,0}\mathcal{R}_{\text{out}}$ and $\partial^{h,1}\mathcal{R}_{\text{out}}$. Thanks to the "protection" by $\mathcal{R}_{\text{out}}$, rectangles based on $[x, y]$ can be efficiently shifted towards y using $f^{\mathfrak{q}_{m+1}}$, see §4.2. We note that our

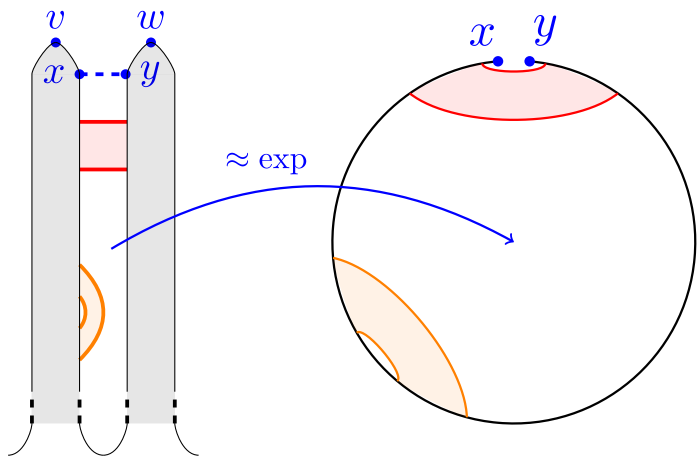


FIGURE 11. Two types of wide rectangles in a parabolic fjord; compare with Figures 1 and 2. The red (parabolic) rectangle has a small combinatorial distance towards x and y relative its horizontal sides, while the orange rectangle has a small distance between its horizontal sides.

arguments are not local: the global branched structure of $f^{\mathfrak{q}_{m+1}}$ is essential in the proofs.

4.1. **Pullbacks in fjords.** We say that a parabolic rectangle $\mathcal{R}$ based on T is *non-winding* if every vertical curve in $\mathcal{R}$ is homotopic in $\mathbb{C} \setminus Z$ to a curve in T; i.e. vertical curves in a non-winding parabolic rectangle do not go around ∞. For a non-winding parabolic rectangle $\mathcal{R}$, we will write

$$\overset{\bullet}{\mathcal{R}} := \mathcal{R} \cup \overline{O},$$

where O is the bounded component of $\mathbb{C} \setminus \left(\overline{Z} \cup \partial^{v,\text{inn}}\mathcal{R}\right)$; i.e. O is the component between $\mathcal{R}$ and T.

Lemma 4.2. *Let $\mathcal{R}$ be an external parabolic rectangle based on T with $\mathcal{W}(\mathcal{R}) > 1$. Let $\mathcal{R}^{\text{new}}$ be the rectangle obtained from $\mathcal{R}$ by removing the outer 1-buffer. Then $\mathcal{R}^{\text{new}}$, $f^{\mathfrak{q}_{m+1}}(\mathcal{R}^{\text{new}})$ are non-winding and $f^{\mathfrak{q}_{m+1}} \mid \overset{\bullet}{\mathcal{R}}{}^{\text{new}}$ is injective.*

Proof. Let $\mathfrak{T}$ be the fjord attached to T, see §2.1.7. By Lemma A.5, $\operatorname{int}\mathcal{R}^{\text{new}} \subset \mathfrak{T} \not\ni \infty$; hence $\mathcal{R}^{\text{new}}$ is non-winding. By Lemma 2.4, $f^{\mathfrak{q}_{m+1}} \mid \overset{\bullet}{\mathcal{R}}{}^{\text{new}}$ is injective. The image $f^{\mathfrak{q}_{m+1}}(\mathcal{R}^{\text{new}})$ is non-winding. □

Lemma 4.3 (Pullbacks)**.** *Let $\mathcal{R}$ be a parabolic non-winding rectangle on T'. Then the pullback of $\mathcal{R}$ along $f^{\mathfrak{q}_{m+1}} \colon T' \boxminus \theta_{m+1} \to T'$ is a parabolic non-winding rectangle on T.*

Proof. For every vertical curve $\ell \in \mathcal{R}$ there is a homotopy τ in $\mathbb{C} \setminus \left(Z \cup T'^c\right)$ between ℓ and a curve $\bar{\ell} \subset T'$, where $T'^c = \partial Z \setminus T'$. This homotopy τ lifts under $f^{\mathfrak{q}_{m+1}}$ into a homotopy between $\bar{\ell} \boxminus \theta_m \subset T$ and a curve ℓ_1; all such curves ℓ_1 form a parabolic non-winding rectangle $\mathcal{R}_1$ which is the pullback of $\mathcal{R}$ along $f^{\mathfrak{q}_{m+1}} \colon T' \boxminus \theta_{m+1} \to T'$. □

Lemma 4.4. *Let $\mathcal{R}$ be a parabolic non-winding rectangle based on T'. Then $\mathcal{R}$ contains a parabolic non-winding subrectangle $\mathcal{R}^{\text{new}}$ with*

$$\partial^{h,0}\mathcal{R}^{\text{new}}| \le |\partial^{h,1}\mathcal{R}^{\text{new}}| \qquad \text{and} \qquad \mathcal{W}(\mathcal{R}^{\text{new}}) \ge \mathcal{W}(\mathcal{R}) - 2, \tag{4.7}$$

where we assume Notations (4.2) *and* (4.3).

Proof. Assume that $|\partial^{h,0}\mathcal{R}^{\text{new}}| > |\partial^{h,1}\mathcal{R}^{\text{new}}|$. Present

$$\partial^{h,0}\mathcal{R} = I_1 \# I_2, \quad I_1 \le I_2, \qquad |I_1| = |\partial^{h,1}\mathcal{R}|,$$

and let $\mathcal{R}_2$ be the subrectangle of $\mathcal{R}$ consisting of vertical curves emerging from I_2. We claim that $\mathcal{W}(\mathcal{R}_2) \le 2$, this implies the lemma.

The claim follows from the Shift Argument §A.3. Let k be the smallest integer such that $k \ge |I_1|/\mathfrak{l}_{m+1}$. Pulling back $\mathcal{R}_2$ under $f^{\mathfrak{q}_{m+1}k}$, we obtain a rectangle $\mathcal{R}'_2$ linked with $\mathcal{R}_2$, see (A.9): $\partial^{h,0}\mathcal{R}_2 < \partial^{h,1}\mathcal{R}'_2 < \partial^{h,1}\mathcal{R}_2$. Lemma A.11 completes the proof. □

Lemma 4.5. *Let $\mathcal{R}$ be a parabolic non-winding rectangle based on T' with $\mathcal{W}(\mathcal{R}) > 2$. Assume Notations* (4.2) *and* (4.3)*. Then $|\partial^{h,0}\mathcal{R}| > 1$ and, moreover,*

$$\log \frac{|\partial^{h,0}\mathcal{R}|}{\operatorname{dist}(v, \partial^{h,0}\mathcal{R})} + 1 \succeq \mathcal{W}(\mathcal{R}), \tag{4.8}$$

i.e., $\operatorname{dist}(v, \partial^{h,0}\mathcal{R})$ *is small compared to* $|\partial^{h,0}\mathcal{R}|$ *if* $\mathcal{W}(\mathcal{R})$ *is big.*

Proof. Follows from the Shift Argument §A.3 and Lemma 4.3. If $|\partial^{h,0}\mathcal{R}| < 1$, then the pullback $\mathcal{R}_1$ of $\mathcal{R}$ under $f^{\mathfrak{q}_{m+1}}$ would be linked to $\mathcal{R}$ – impossible because $\mathcal{W}(\mathcal{R}) > 2$.

By Lemma 4.4, there is a parabolic subrectangle $\mathcal{R}^{\text{new}} \subset \mathcal{R}$ satisfying (4.7); thus $|\partial^{h,0}\mathcal{R}^{\text{new}}| < \operatorname{dist}_{T'}(\partial^{h,0}\mathcal{R}^{\text{new}}, \partial^{h,1}\mathcal{R}^{\text{new}}) + \mathfrak{l}_{m+1}$. Let $k \ge 1$ be the integer part of $\operatorname{dist}(v, \partial^{h,0}\mathcal{R}^{\text{new}})/\mathfrak{l}_{m+1}$. Decompose $\partial^{h,0}\mathcal{R}^{\text{new}}$ into the concatenation of closed intervals

$$I_1 \# I_2 \# \dots \# I_n, \quad v' \le I_1 \le I_2 \le \dots \le I_n < w$$

such that

$$|I_1| = k\mathfrak{l}_{m+1}, \quad |I_2| = 2k\mathfrak{l}_{m+1}, \dots, |I_{n-1}| = 2^{n-1}k\mathfrak{l}_{m-1}, \quad |I_n| \le 2^n k\mathfrak{l}_{m-1}.$$

We claim that $n \ge \mathcal{W}(\mathcal{R}^{\text{new}})/2$ – this will imply the lemma.

Let $\mathcal{R}_t$ be the subrectangle of $\mathcal{R}^{\text{new}}$ consisting of vertical curves connecting I_t and $\partial^{h,1}\mathcal{R}^{\text{new}}$. Then $\mathcal{R}_t$ is linked to its pullback $\mathcal{R}'_t$ under $f^{\mathfrak{q}_{m+1}2^{t-1}k}$, see (A.9): $\partial^{h,0}\mathcal{R}'_t < \partial^{h,0}\mathcal{R}_t < \partial^{h,1}\mathcal{R}'_t$. By Lemma A.11, $\mathcal{W}(\mathcal{R}_t) \le 2$ for every t. By the Parallel Law §A.1.4, $n \ge \mathcal{W}(\mathcal{R}^{\text{new}})/2 \ge \mathcal{W}(\mathcal{R})/2 + 1$. □

The following lemma is a counterpart to Lemma 4.2.

Lemma 4.6. *Let $\mathcal{R}$ be a parabolic non-winding rectangle based on T'. If $\mathcal{W}(\mathcal{R}) > 2$, then after removing the outermost 2-buffer from $\mathcal{R}$, we obtain an external parabolic rectangle $\mathcal{R}^{\text{new}}$ with $f^{\mathfrak{q}_{m+1}}\left(\dot{\mathcal{R}}^{\text{new}}\right) \subset \dot{\mathcal{R}}$. In particular, $f^{\mathfrak{q}_{m+1}} \mid \dot{\mathcal{R}}^{\text{new}}$ is injective.*

Proof. As before, we assume Notations (4.2) and (4.3). Consider the pullback $\mathcal{R}_1$ of $\mathcal{R}$ under $f^{\mathfrak{q}_{m+1}}$ (see Lemma 4.3); clearly, $\operatorname{int}(\mathcal{R}_1) \subset \mathbb{C} \setminus \operatorname{int}\mathcal{K}_m$. We claim that after removing the outermost 2-buffer from $\mathcal{R}$, the new rectangle $\mathcal{R}^{\text{new}}$ is within $\dot{\mathcal{R}}_1$. This would imply the lemma because $f^{\mathfrak{q}_{m+1}} \mid \dot{\mathcal{R}}_1$ is injective.

Denote by $\mathcal{X}$ the outermost 1-buffer of $\mathcal{R}$, and denote by $\mathcal{Y}$ the outermost 1-buffer of $\mathcal{R} \setminus \mathcal{X}$. We have $\mathcal{R} = \mathcal{X} \cup \mathcal{Y} \cup \mathcal{R}^{\text{new}}$. Let $\mathcal{X}_1 \subset \mathcal{R}_1$ be the pullback of $\mathcal{X}$ under $f^{\mathfrak{q}_{m+1}}$. Since the distance between $\partial^{h,0}\mathcal{R}$ and $\partial^{h,1}\mathcal{X}$ is bigger than $\mathfrak{l}_{m+1}$ (see (4.1)), we have

$$\partial^{h,0}\mathcal{X}_1 < \partial^{h,0}\mathcal{Y} \cup \partial^{h,0}\mathcal{R}^{\text{new}} < \partial^{h,1}\mathcal{X}_1;$$

thus at most 1-wide part of $\mathcal{Y} \cup \mathcal{R}^{\text{new}}$ can cross $\mathcal{X}_1$, see §A.2.1. Hence $\mathcal{R}^{\text{new}} \subset \mathcal{R}_1$. □

Combined with Lemma 4.4, we obtain:

Corollary 4.7. *If $\mathcal{R}$ is a parabolic non-winding rectangle on T' with $\mathcal{W}(\mathcal{R}) \geq 5$, then $|\partial^{h,0}\mathcal{R}|, |\partial^{h,1}\mathcal{R}| \geq 1$.* □

4.2. **Push-forwards in fjords.** As before, we assume Notations (4.2), (4.3).

Let us select a simple arc $\delta \subset \widehat{\mathbb{C}} \setminus Z$ connecting a point in $\delta(0) \in \partial Z \setminus (v, w)$ to $\delta(1) = \infty$ such that δ is disjoint from $\partial Z \setminus \delta(0)$. Then $\Delta \coloneqq \widehat{\mathbb{C}} \setminus (\overline{Z} \cup \delta)$ is an open topological disk.

For a rectangle $\mathcal{R}$ based on T, we will define below the push-forward $\mathcal{R}_k$ of $\mathcal{R}$ under $f^{\mathfrak{q}_{m+1}k}$ assuming that $\operatorname{dist}(\partial^{h,1}\mathcal{R}, w) > k\mathfrak{l}_{m+1}$. The result $\mathcal{R}_k$ will be a lamination in Δ.

Let us orient all vertical curves in $\mathcal{R}$ from $\partial^{h,1}\mathcal{R}$ to $\partial^{h,0}\mathcal{R}$:

$$\gamma(0) \in \partial^{h,1}\mathcal{R}, \quad \gamma(1) \in \partial^{h,0}\mathcal{R} \qquad \text{for} \qquad [\gamma\colon [0,1] \to \widehat{\mathbb{C}} \setminus Z] \in \mathcal{R}. \tag{4.9}$$

Let Δ_{-k} be the component of $f^{-\mathfrak{q}_{m+1}k}(\Delta)$ attached to $[v, w \boxminus k\theta_{m+1}] \subset T$. For a vertical curve $\ell\colon [0,1] \to \widehat{\mathbb{C}}$ in $\mathcal{R}$, let $t^\ell_k > 0$ be the first moment such that $\ell\big(t^\ell_k\big) \in \partial\Delta_{-k}$. We define

$$\mathcal{R}'_k \coloneqq \{\ell \mid [0, t^\ell_k] \quad \text{for} \quad \ell \in \mathcal{R}\}, \qquad \mathcal{R}_k \coloneqq f^{\mathfrak{q}_{m+1}k}(\mathcal{R}'_k).$$

In other words, $\mathcal{R}'_k$ is the restriction (see §A.1.5) of $\mathcal{R}$ to Δ_{-k} and $\mathcal{R}_k$ is the appropriate conformal image of $\mathcal{R}'_k$. We say that the curve $\ell \mid [0, t^\ell_k]$ in $\mathcal{R}'_k$ and its image $f^{\mathfrak{q}_{m+1}k}\big(\ell \mid [0, t^\ell_k]\big)$ in $\mathcal{R}_k$ is of

- Type I if $t^\ell_k = 1$,
- Type II if $t^\ell_k < 1$ but $f^{\mathfrak{q}_{m+1}k} \circ \ell\big(t^\ell_k\big) \in T$;
- Type III otherwise.

We denote by $\mathcal{R}^I_k$, $\mathcal{R}^{II}_k$, $\mathcal{R}^{III}_k$ the sublaminations of $\mathcal{R}_k$ consisting of Type I, II, III curves respectively. Similarly are defined the sublaminations $\mathcal{R}'^I_k$, $\mathcal{R}'^{II}_k$, $\mathcal{R}'^{III}_k$ of $\mathcal{R}'^k$. Since $\mathcal{R}$ overflows $\mathcal{R}'_k$, we have

$$\mathcal{W}(\mathcal{R}) \geq \mathcal{W}(\mathcal{R}'_k) = \mathcal{W}(\mathcal{R}_k). \tag{4.10}$$

Lemma 4.8. *In $\widehat{\mathbb{C}} \setminus Z$, the lamination $\mathcal{R}^{II}_k$ separates $\mathcal{R}^I_k$ from $\mathcal{R}^{III}_k$ and $\{v, w\}$; i.e., $\mathcal{R}^{III}_k \cup \{v, w\}$ and $\mathcal{R}^I_k$ are in different components of $\widehat{\mathbb{C}} \setminus (Z \cup \gamma)$ for every $\gamma \in \mathcal{R}^{II}_k$, see Figure 12.*

Proof. Consider the preimage $T_{-k} \coloneqq f^{-\mathfrak{q}_{m+1}k}(T) \cap \partial\Delta_{-k}$ of T under $f^{\mathfrak{q}_{m+1}k}\colon \overline{\Delta}_{-k} \to \overline{\Delta}$. Observe that T_{-k} contains v but not w. The point v splits T_{-k} into two intervals T^b_{-k} and T^a_{-k}, we assume that $T^a_{-k} \subset T$ while T^b_{-k} is disjoint from ∂Z. Then

- $\mathcal{R}'^I_k$ is the sublamination of $\mathcal{R}'_k$ landing at T^a_{-k},
- $\mathcal{R}'^{II}_k$ is the sublamination of $\mathcal{R}'_k$ landing at T^b_{-k},

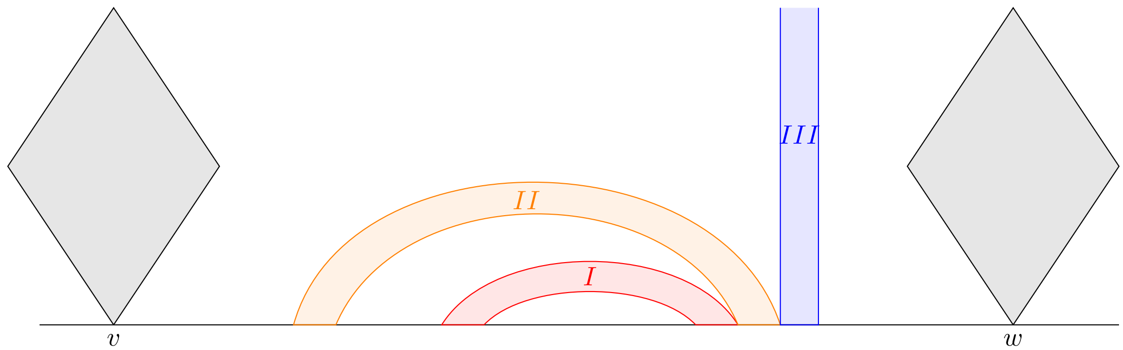


FIGURE 12. Types I (red), II (orange), and III (blue) curves in $\mathcal{R}_k$.

- $\mathcal{R}_k'^{III}$ is the sublamination of $\mathcal{R}_k'$ landing at $\partial\Delta_{-k}\setminus T_{-k}$.

The lemma now follows from the observation that in Δ_{-k}, the lamination $\mathcal{R}_k'^{II}$ separates $\mathcal{R}_k'^{I}$ from $\mathcal{R}_k'^{III}$ and $w \boxminus \theta_{m+1} k$. □

Let $\mathcal{X}$ be an external parabolic rectangle based on T' with $\mathcal{W}(\mathcal{X}) \geq 10$. Let $P \subset T'$ be the complementary interval between $\partial^{h,0}\mathcal{X}$ and $\partial^{h,1}\mathcal{X}$. We say that P is *protected* by $\mathcal{X}$.

Lemma 4.9 (Push-forwards)**.** *Let P be an interval protected by an external parabolic rectangle $\mathcal{X}$ as above. If $\mathcal{R}$ is a parabolic rectangle based on P such that*

$$\partial^h \mathcal{R} \boxplus i\mathfrak{l}_{m+1} \subset P \qquad \textit{for all} \qquad i \in \{0,1,2,\dots,k\},$$

then after removing the 1-outermost buffer, the rectangle $\mathcal{R}^{\text{new}}$ has univalent push-forwards:

$$f^{\mathfrak{q}_{m+1} i}(\mathcal{R}^{\text{new}}) \subset \dot{\mathcal{X}} \qquad \textit{for all} \qquad i \in \{0,1,2,\dots,k\}. \tag{4.11}$$

Proof. Let us choose δ to be disjoint from $\dot{\mathcal{X}}$ and let $\mathcal{X}^{\text{new}}$ be the rectangle obtained by removing the outermost 5-buffer from $\mathcal{X}$. By Lemmas 4.2 and 4.6, $f^{\mathfrak{q}_{m+1}}$ is injective on $\dot{\mathcal{X}}^{\text{new}}$ and:

$$f^{\mathfrak{q}_{m+1}}\left(\dot{\mathcal{X}}^{\text{new}}\right) \subset \dot{\mathcal{X}} \qquad \text{hence} \qquad \operatorname{int} \dot{\mathcal{X}}^{\text{new}} \subset \Delta_{-1}. \tag{4.12}$$

Note that $\mathcal{W}(\mathcal{X}^{\text{new}}) \geq 5$. Let us prove by induction that

$$\mathcal{W}\left(\mathcal{R}_i^I\right) \geq \mathcal{W}(\mathcal{R}) - 4/5. \tag{4.13}$$

for all $i \leq k$. This will imply (4.11) because at most $\frac{1}{5}$-wide family of $\mathcal{R}_i^I$ can cross the protection $\mathcal{X}^{\text{new}}$.

It follows from (4.12) that

$$\mathcal{R}_{i+1}^I \supseteq \left\{ f^{\mathfrak{q}_{m+1}}(\gamma) \quad | \quad \gamma \in \mathcal{R}_i^I \quad \text{and} \quad \gamma \text{ is disjoint from } \partial\Delta_{-1}\setminus T \right\}. \tag{4.14}$$

If $\mathcal{W}(\mathcal{R}_i^I) > \mathcal{W}(\mathcal{R}) - 3/5$, then at most $1/5$ curves in $\mathcal{R}_i^I$ can cross the protection $\mathcal{X}^{\text{new}}$ and hit $\partial\Delta_{-1}\setminus T$. We obtain that $\mathcal{W}(\mathcal{R}_{i+1}^I) > \mathcal{W}(\mathcal{R}) - 4/5$.

Assume now that

$$\mathcal{W}(\mathcal{R}_i^I) \le \mathcal{W}(\mathcal{R}) - 3/5 \quad \text{hence} \quad \mathcal{W}(\mathcal{R}_i^{II}) + \mathcal{W}(\mathcal{R}_i^{III}) \ge 3/5,$$

by (4.10). Since $\mathcal{R}_i^{III}$ crosses $\mathcal{X}^{\text{new}}$, we obtain $\mathcal{W}(\mathcal{R}_i^{III}) \le 1/5$ and $\mathcal{W}(\mathcal{R}_i^{II}) \ge 2/5$. At most 1/5 curves in $\mathcal{R}^I \cup \mathcal{R}^{II}$ can cross $\mathcal{X}^{\text{new}}$ and hit $\partial\Delta_{-1} \setminus T$; and all such curves must be in $\mathcal{R}^{II}$ – they are outermost by Lemma 4.8. We obtain that all curves in $\mathcal{R}_i^I$ are inside $\overset{\bullet}{\mathcal{X}}{}^{\text{new}}$ and $\mathcal{W}(\mathcal{R}_{i+1}^I) \ge \mathcal{W}(\mathcal{R}_i^I)$. □

4.3. **Proof of Theorem 4.1.**

Lemma 4.10. *Let $\mathcal{R}$ be a parabolic non-winding rectangle based on T' with $\mathcal{W}(\mathcal{R}) > 50$. Then $\mathcal{R}$ contains a parabolic non-winding balanced geodesic rectangle $\mathcal{R}^{\text{new}}$ with $\mathcal{W}(\mathcal{R}^{\text{new}}) \ge \mathcal{W}(\mathcal{R}) - 25$.*

Proof. We assume Notations (4.2) and (4.3). Let $\mathcal{R}^{\text{new}}$ be the rectangle obtained from $\mathcal{R}$ by removing the outermost 18-buffer $\mathcal{X}$. Then Lemma 4.9 (push-forwards) is applicable in $\overset{\bullet}{\mathcal{R}}{}^{\text{new}}$.

Choose the maximal intervals $I \subset \partial^{h,0}\mathcal{R}^{\text{new}}$ and $J \subset \partial^{h,1}\mathcal{R}^{\text{new}}$ so that the geodesic rectangle $\mathcal{R}(I,J)$ is in $\mathcal{R}^{\text{new}}$. By Lemma A.5, $\mathcal{R}(I,J)$ contains most of the width of $\mathcal{R}^{\text{new}}$: we have $\mathcal{W}(\mathcal{R}(I,J)) \ge \mathcal{W}(\mathcal{R}) - 20$.

Assume that $|I| > |J|$. As in the proof of Lemma 4.4, we present

$$I = I_1 \# I_2, \quad I_1 \le I_2, \qquad |I_1| = |J|.$$

Let k be the smallest integer such that $k \ge |I_1|/\mathfrak{l}_{m+1}$. Since the geodesic rectangle $\mathcal{R}(I_2, J)$ is linked to its pullback under $f^{\mathfrak{q}_{m+1}k}$, we have $\mathcal{W}(\mathcal{R}(I_2,J)) \le 2$; hence $\mathcal{W}(\mathcal{R}(I_1,J)) \ge \mathcal{W}(\mathcal{R}) - 25$.

Assume that $|I| < |J|$. We present

$$J = J_2 \# J_1, \quad J_2 \le J_1, \qquad |J_1| = |I|.$$

Let k be the smallest integer such that $k \ge |J_1|/\mathfrak{l}_{m+1}$. After removing the outermost 1-buffer from the geodesic rectangle $\mathcal{R}(I, J_2)$, we obtain a rectangle linked to its push-forward under $f^{\mathfrak{q}_{m+1}k}$ (Lemma 4.9). We have $\mathcal{W}(\mathcal{R}(I,J_2)) \le 3$; hence $\mathcal{W}(\mathcal{R}(I,J_1)) \ge \mathcal{W}(\mathcal{R}) - 25$. □

Proof of Theorem 4.1. For $x \in T'$ with $\operatorname{dist}_T(v', x) < |T'|/10$, define $m_{10} \in T'$ so that $\operatorname{dist}_T(v', m_{10}) = 10 \operatorname{dist}_T(v', m)$.

Let $\mathcal{Z}$ be a sufficiently wide external parabolic rectangle based on T' with $\partial^{h,0}\mathcal{Z} < \partial^{h,1}\mathcal{Z}$ in T'. By removing $O(1)$ buffers from $\mathcal{Z}$ we can assume that $|\partial^{h,0}\mathcal{Z}| \le |\partial^{h,1}\mathcal{Z}|$ (Lemma 4.4) and that $|\lfloor v', \partial^{h,0}\mathcal{Z} \rfloor|$ is small compare to $\operatorname{dist}_{T'}(\partial^{h,0}\mathcal{Z}, \partial^{h,1}\mathcal{Z})$ (Lemma 4.8). Therefore, we can define the shortest interval

$$S \coloneqq [v', m] \subset T', \quad |S| \ge \mathfrak{l}_{m+1} \quad \text{such that} \quad \mathcal{W}^+_{\text{ext},m}(S, [m_{10}, w]) \ge 500. \tag{4.15}$$

By Lemmas 4.2 and 4.10, we can select two disjoint parabolic balanced non-winding geodesic rectangles based on T' satisfying

$$\mathcal{X}, \mathcal{Y} \subset \mathcal{F}^+_{\text{ext},m}(S, [m_{10}, w]), \quad \mathcal{X} \subset \mathcal{Y}^\bullet \setminus \mathcal{Y} \quad \text{with} \quad \mathcal{W}(\mathcal{X}) \ge 400, \;\; \mathcal{W}(\mathcal{Y}) \ge 10.$$

We set, see Figure 13:

$$\widetilde{T}_{\text{par}} = [\tilde{x}, \tilde{y}] \coloneqq \lfloor \partial^h \mathcal{X} \rfloor \subset T', \quad \text{and} \quad T_{\text{par}} = [x, y] \coloneqq \widetilde{T}_{\text{par}} \setminus \partial^h \mathcal{X},$$

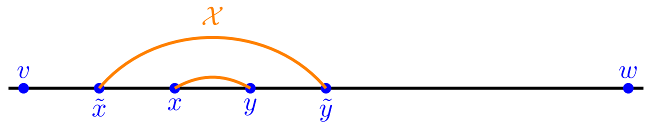


FIGURE 13. Intervals $T_{\text{par}} = [x, y]$ and $\widetilde{T} = [\tilde{x}, \tilde{y}]$.

where $\tilde{x} < x < y < \tilde{y}$ in T'. Since $\mathcal{Y}$ protects $\widetilde{T}_{\text{par}}$, wide rectangles based on $\widetilde{T}_{\text{par}}$ can be push-forward. By Lemma 4.5, $|\partial^{h,0}\mathcal{X}| > \operatorname{dist}(v', \partial^{h,0}\mathcal{X})$ hence

$$\operatorname{dist}_{T'}(v', z) \asymp \operatorname{dist}_{T'}(\tilde{x}, z) \quad \text{for all} \quad z \in T_{\text{par}}. \tag{4.16}$$

Claim 1. *For an interval $I \subset T_{\text{par}}$ with*

$$\operatorname{dist}(I, \{\tilde{x}, \tilde{y}\}) \succeq |I| \qquad \textit{and} \qquad \operatorname{dist}(I, \{\tilde{x}, \tilde{y}\}), |I| \geq \mathfrak{l}_{m+1}$$

we have $\mathcal{W}_3^+(I) \preceq 1$.

Proof. By splitting I into finitely many intervals (depending on the constant representing "$\succeq$"), it is sufficient to assume that $\operatorname{dist}(I, \{\tilde{x}, \tilde{y}\}) \geq |I| + \mathfrak{l}_{m+1}$ or $|I| = \mathfrak{l}_{m+1}$. Write $I = [a, b]$ with $\tilde{x} < a < b < \tilde{y}$. Let us present $\mathcal{F}_3^+(I)$ as $\mathcal{F}_- \sqcup \mathcal{F}_+ \sqcup \mathcal{F}'$ where

- $\mathcal{F}'$ consists of curves crossing $\mathcal{X}$;
- $\mathcal{F}_-$ consists of curves in $\dot{\mathcal{X}}$ connecting I and $[\tilde{x}, a] \cap (3I)^c$;
- $\mathcal{F}_+$ consists of curves in $\dot{\mathcal{X}}$ connecting I and $[b, \tilde{y}] \cap (3I)^c$.

Clearly, $\mathcal{W}(\mathcal{F}') \leq 1/10$. We will estimate the width of $\mathcal{F}_-, \mathcal{F}_+$ using the Shift Argument. Let $\mathcal{R}_- \subset \mathcal{F}_-$ and $\mathcal{R}_+ \subset \mathcal{F}_+$ be the canonical rectangles; i.e. $\mathcal{W}(\mathcal{R}_-) = \mathcal{W}(\mathcal{F}_-)$ and $\mathcal{W}(\mathcal{R}_+) = \mathcal{W}(\mathcal{F}_+)$. Let k be the smallest integer such that $k \geq |I|/\mathfrak{l}_{m+1}$. Then $\mathcal{R}_+$ is linked to its pullback under $f^{-k\mathfrak{q}_{m+1}}$ implying that $\mathcal{W}(\mathcal{R}_+) \leq 2$. Since $\widetilde{T}_{\text{par}}$ is protected by $\mathcal{Y}$, the rectangle $\mathcal{R}_-^{\text{new}}$ obtained by removing the outermost 1-buffer from $\mathcal{R}_-$ is linked to its push-forward under $f^{k\mathfrak{q}_{m+1}}$ (Lemma 4.9); this implies $\mathcal{W}(\mathcal{R}-) \leq 3$. □

Claim 2. *If $I, J \subset T_{\text{par}}, \quad I < J$ are two intervals with*

$$\frac{1}{2}|\lfloor I, J \rfloor| < |T_{\text{par}}|, \qquad \min\{|I|, |J|\} \asymp \operatorname{dist}(I, J), \qquad |I|, |J|, \operatorname{dist}(I, J) \geq \mathfrak{l}_{m+1},$$

then $\mathcal{W}^+(I, J) \asymp 1$.

Proof. Assume $|I| \leq |J|$. Let $L \subset T_{\text{par}}$ be the complementary interval between I, J. Applying Claim 1, and subdividing if necessary I and L into finitely many intervals we obtain

$$\mathcal{W}^+(I, J) \preceq 1 \qquad \text{and} \qquad \big(\mathcal{W}^+(I, J)\big)^{-1} = \mathcal{W}^+(L, \lfloor I, J \rfloor^c) \preceq 1.$$

Therefore, $\mathcal{W}^+(I, J) \asymp 1$. □

Statement **(III)** of Theorem 4.1 follows from Claims 1 and 2 using the Splitting Argument, see Remark 2.6.

Claim 3. *If $I, J \subset T_{\text{par}}, I < J$ are two intervals with*

$$\operatorname{dist}_{T'}(\tilde{x}, I) \asymp \operatorname{dist}_{T'}(J, \tilde{y}) \asymp |I| \asymp |J| \preceq \operatorname{dist}_{T'}(I, J) \tag{4.17}$$

and $|I|, |J|, \operatorname{dist}_{T'}(I, J) \geq \mathfrak{l}_{m+1}$. Then $\mathcal{W}^+(I, J) \asymp 1$.

Proof. The property $\mathcal{W}^-(I,J) \preceq 1$ follows from Claim 1 by splitting, if necessary, I into finitely many intervals.

Denote by $L \subset T_{\text{par}}$ the complementary interval between I and J. Let us show that that the dual family $\mathcal{G} = \mathcal{F}^+(L, \lfloor I,J \rfloor^c)$ satisfies $\mathcal{W}(\mathcal{G}) \preceq 1$; this will imply the claim. Denote by $N \subset [\tilde{x}, \tilde{y}]$ the interval between $\tilde{x}$ and $\lfloor I,J \rfloor$ and by $M \subset [\tilde{x}, \tilde{y}]$ the interval between $\lfloor I,J \rfloor$ and $\tilde{y}$. As in the proof of Claim 1, we decompose $\mathcal{G}$ as $\mathcal{G}' \sqcup \mathcal{G}_- \sqcup \mathcal{G}_+$ where

- $\mathcal{G}'$ consists of curves crossing $\mathcal{X}$;
- $\mathcal{G}_-$ consists of curves in $\overset{\bullet}{\mathcal{X}}$ connecting N and L;
- $\mathcal{G}_+$ consists of curves in $\overset{\bullet}{\mathcal{X}}$ connecting L and M.

Let $\mathcal{R}_- \subset \mathcal{G}_-$ and $\mathcal{R}_+ \subset \mathcal{G}_+$ be the canonical rectangles; i.e. $\mathcal{W}(\mathcal{R}_-) = \mathcal{W}(\mathcal{G}_-)$ and $\mathcal{W}(\mathcal{R}_+) = \mathcal{W}(\mathcal{G}_+)$. Set

$$\tau \coloneqq \min\{\text{dist}_{T'}(\tilde{x}, I), |I|, |J|, \text{dist}_{T'}(J, \tilde{y})\} - \mathfrak{l}_{m+1};$$

if $\tau < \mathfrak{l}_{m+1}$, then replace $\tau \coloneqq \mathfrak{l}_{m+1}$. We decompose N and M into finitely many intervals $\cup_i N_i$ and $\cup_i M_i$ so that $|N_i|, |M_i| \leq \tau$ for all i. The number of intervals depends on the constants representing "$\asymp$" and "$\preceq$" in (4.17).

Denote by $\mathcal{R}_{-,i} \subset \mathcal{R}_-$ the subrectangle consisting of vertical curves landing at N_i. Similarly, $\mathcal{R}_{+,i} \subset \mathcal{R}_+$ is the subrectangle consisting of vertical curves landing at M_i. Define k to be smallest integer such that $k\mathfrak{l}_{m+1} \geq \tau$. Then $\mathcal{R}_{-,i}$ is linked to its push-forward under $f^{k\mathfrak{q}_{m+1}}$ (Lemma 4.9); i.e. $\mathcal{W}(\mathcal{R}_{-,i}) \leq 3$. And $\mathcal{R}_{+,i}$ is linked to its pullback under $f^{k\mathfrak{q}_{m+1}}$; i.e. $\mathcal{W}(\mathcal{R}_{+,i}) \leq 2$. □

Statement **(II)** of Theorem 4.1 follows from Claims 1 and 3 using the Splitting Argument, see Remark 2.6.

Claim 4. *Consider $s \in \mathbb{N}$ such that $\text{dist}_{T'}(y, \tilde{y}) < s\mathfrak{l}_{m+1} < \text{dist}(\tilde{y}, w) - \mathfrak{l}_{m+1}$. Define $Z_s \coloneqq f^s[y, \tilde{y}]$ and note that Z_s is between $Z_0 \coloneqq [y, \tilde{y}]$ and w. Then*

$$\mathcal{W}^+_{\text{div}}(Z_s, (T')^c \cup [v', x]) \geq 100. \tag{4.18}$$

Proof. See Figure 14 for illustration. Let $\mathcal{X}_s$ be the push-forward of $\mathcal{X}$ under $f^{\mathfrak{q}_{m+1}s}$ as in §4.2. Then $\mathcal{W}(\mathcal{X}_s) \geq \mathcal{W}(\mathcal{X}) \geq 400$. Since most of the curves in $\mathcal{X}_s$ do not cross $\mathcal{X}$, we obtain $\mathcal{W}^+(Z_s, (T')^c \cup [v', x]) \geq 399$. The width of external curves in $\mathcal{F}^+(Z_s, (T')^c \cup [v', x])$ landing at S is at most 100 because, otherwise, $[v', x] \subset S$ would not be the shortest interval satisfying (4.15). This implies

$$\mathcal{W}^+(Z_s, (T')^c) + \mathcal{W}^+_{\text{div}}(Z_s, [v', x]) \geq 299.$$

By Lemma 2.7, $\mathcal{W}^+_{\text{ext}}(Z_s, T^c) \leq 5$. We claim that $\mathcal{W}^+_{\text{ext}}([v, v'], Z_s) \leq 150$; this will imply (4.18).

If $\mathcal{W}^+_{\text{ext}}([v, v'], Z_s) \geq 150$, then $\mathcal{F}^+_{\text{ext}}([v, v'], Z_s)$ contains an external parabolic rectangle $\mathcal{R}$ with $\mathcal{W}(\mathcal{R}) \geq 150$; applying Lemma 4.2 we obtain a non-winding parabolic rectangle $\mathcal{R}_2 \coloneqq f^{\mathfrak{q}_{m+1}}(\mathcal{R}^{\text{new}})$ in $\mathcal{F}^+_{\text{ext}}([v, v'] \boxplus \theta_{m+1}, Z_{s+1})$ with $\mathcal{W}(\mathcal{R}_2) \geq 149$. Since most curves in $\mathcal{R}_2$ are external (Lemma 4.6), we obtain a contradiction with the property that S is the shortest interval satisfying (4.15).

□

We will later need the following fact:

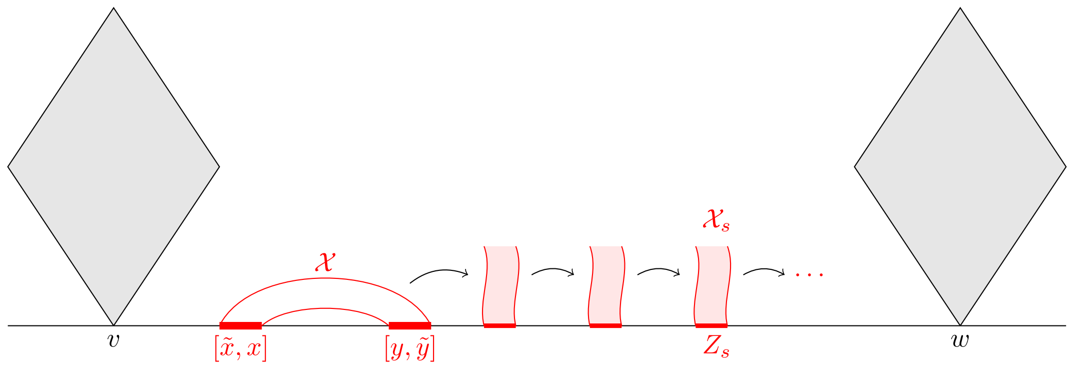


FIGURE 14. Laminations $\mathcal{X}_s$ are push-forwards of the rectangle $\mathcal{X}$.

Corollary 4.11. *For every $\lambda \gg 1$, the following holds. If $\text{dist}_{T'}(y,w) \gg_\lambda \text{dist}_{T'}(v,x)$, then there is an interval $J \subset [\widetilde{y}, w]$ such that*

$$\mathcal{W}^+_{\lambda,\text{div},m}(J) \succeq_\lambda \frac{\text{dist}_{T'}(y,w)}{\text{dist}_T(v,x)}.$$

Proof. We will use notations of Claim 4. Set k to be the minimal integer bigger than $\text{dist}_{T'}(y,\tilde{y})/\mathfrak{l}_{m+1}$. Since $\text{dist}_{T'}(y,w) \gg_\lambda \text{dist}_T(v,x)$, we can find

$$J := \lfloor Z_s, Z_{s+kj} \rfloor \supset Z_s \sqcup S_{s+k} \sqcup \cdots \sqcup S_{s+jk} \qquad j \asymp_\lambda \frac{\text{dist}_{T'}(y,w)}{\text{dist}_T(v,x)}$$

so that $(\lambda J)^c \supset (T')^c \cup [v', x]$. By (4.18) and the Parallel Law, we have $\mathcal{W}^+_{\lambda,\text{div},m}(J) \geq 90j$. □

Let us prove Statement **(I)**. By Lemmas 4.5 and 4.10, $\mathcal{R}$ contains a balanced non-winding geodesic subrectangle $\mathcal{R}_1$ with

$$\mathcal{W}(\mathcal{R}_1) = \mathcal{W}(\mathcal{R}) - O(1) \qquad \text{and} \qquad \partial^{h,0}\mathcal{X} < \partial^{h,0}\mathcal{R}_1.$$

Consider $J := \lfloor \partial \mathcal{R}_1 \rfloor \setminus \partial^h \mathcal{R}_1$. Using Claim 4 and its notations, J contains neither Z_0 nor Z_s for s satisfying Claim 4. We deduce that $J \subset \lfloor \partial^h \mathcal{X} \rfloor$. By removing a 2-buffer, we obtain that the new rectangle $\mathcal{R}_1^{\text{new}}$ is disjoint from $\mathcal{X}$.

It follows from Claim 1 that $\mathcal{R}_1^{\text{new}}$ contains a balanced geodesic subrectangle $\mathcal{R}_2$ such that $\mathcal{W}(\mathcal{R}_2) \geq \mathcal{W}(\mathcal{R}_1^{\text{new}})$ and

$$\text{dist}(x, \partial^{h,0}\mathcal{R}_2) \asymp \text{dist}(\partial^{h,1}\mathcal{R}_2, y) \preceq |\partial^{h,0}\mathcal{R}_2| \asymp |\partial^{h,1}\mathcal{R}_2|$$

Statement **(II)** is now applicable for $\mathcal{R}_2$. □

4.4. **Submergence Rule.** Let us underline the following fact, see Figure 15. Suppose that we have a wide parabolic rectangle $\mathcal{N}$, $\mathcal{W}(\mathcal{N}) \asymp K \gg 1$ based on $T_{\text{par}} = [x, y]$. We note that $\mathcal{N}$ is non-winding after removing $O(1)$-buffer. Assume that a parabolic rectangle $\mathcal{R}$ with $\mathcal{W}(\mathcal{R}) \succeq 1$ is protected by $\mathcal{N}$; i.e. $\mathcal{R}$ is based on the interval between $\partial^{h,0}\mathcal{N}$ and $\partial^{h,1}\mathcal{N}$. Then by Theorem 4.1

$$\log \frac{|\partial^{h,0}\mathcal{R}|}{\text{dist}(v,x)}, \quad \log \frac{|\partial^{h,1}\mathcal{R}|}{\text{dist}(v,x)} \succeq K. \tag{4.19}$$

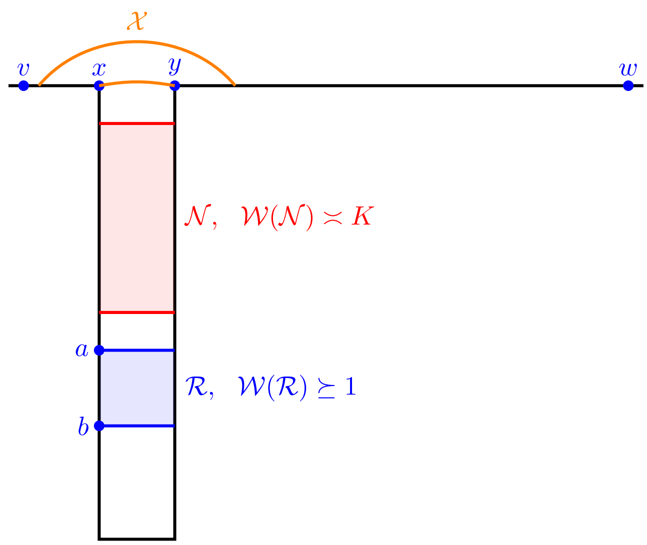


FIGURE 15. Submergence Rule. Suppose a buffer $\mathcal{N}$ has width $\asymp K \gg 1$. If a parabolic rectangle $\mathcal{R}$ has width $\succeq 1$, then $\log \dfrac{\operatorname{dist}(a,b)}{\operatorname{dist}(v,x)} \succeq K$.

4.5. **Central rectangles.** We say that a parabolic rectangle $\mathcal{R}$ based on T' is *central* if

$$0.9 < \frac{\operatorname{dist}_T(v, \lfloor \partial^h \mathcal{R}' \rfloor)}{\operatorname{dist}_T(\lfloor \partial^h \mathcal{R}' \rfloor, w)} < 1.1;$$

i.e. if the distances from $\partial^h \mathcal{R}$ to v and w are essentially the same.

Lemma 4.12 (Central subrectangles)**.** *Consider a parabolic non-winding rectangle $\mathcal{R}$ based on T' with $\mathcal{W}(\mathcal{R}) \gg_\lambda 1$. Then*

- *either $\mathcal{R}$ contains a parabolic non-winding central balanced geodesic subrectangle $\mathcal{R}^{\text{new}}$ with $\mathcal{W}(\mathcal{R}^{\text{new}}) \geq \mathcal{W}(\mathcal{R})/2$;*
- *or there is an interval*

$$I \subset T', \qquad |I| > \mathfrak{l}_{m+1} \qquad \textit{such that} \qquad \log \mathcal{W}^+_{\lambda,\text{div},m}(I) \succeq \mathcal{W}(\mathcal{R}).$$

Proof. Write $K \coloneqq \mathcal{W}(\mathcal{R}) \gg_\lambda 1$. Let $\mathcal{R}^{\text{new}}$ be the rectangle obtained from $\mathcal{R}$ by removing the outermost $K/3$ buffer $\mathcal{N}$. By Theorem 4.1, I, we have

$$\log \frac{|\partial^{h,0}\mathcal{N}|}{\operatorname{dist}_{T'}(v, \partial^{h,0}\mathcal{N})}, \quad \log \frac{|\partial^{h,1}\mathcal{N}|}{\operatorname{dist}_{T'}(v, \partial^{h,0}\mathcal{N})} \succeq K. \tag{4.20}$$

Since $\mathcal{W}(\mathcal{R}^{\text{new}}) = 2K/3$, using Part II, we can select intervals $I \subset \partial^{h,0}\mathcal{R}^{\text{new}}$ and $J \subset \partial^{h,1}\mathcal{R}^{\text{new}}$ such that the geodesic rectangle $\mathcal{R}^{\text{New}} \coloneqq \mathcal{R}(I,J) \subset \widehat{\mathbb{C}} \setminus Z$ between I, J is in $\mathcal{R}^{\text{new}}$ and satisfies:

$$|I| = |J|, \qquad \operatorname{dist}_{T'}(x, I) = \operatorname{dist}_{T'}(J, y), \qquad \text{and} \qquad \mathcal{W}(\mathcal{R}^{\text{New}}) \geq K/2.$$

Assume that $\mathcal{R}^{\text{New}}$ is not central. Then

$$\operatorname{dist}_{T'}(y, w) = \operatorname{dist}_{T'}(J, w) - \operatorname{dist}_{T'}(x, I) > 0.1 \operatorname{dist}_{T'}(J, y) \succeq |\partial^{h,0}\mathcal{N}|.$$

Using (4.20), we have:

$$\log \frac{\operatorname{dist}_{T'}(y,w)}{\operatorname{dist}(v,x)} \succeq \log \frac{|\partial^{h,0}\mathcal{N}|}{\operatorname{dist}(v,x)} \succeq K.$$

Corollary 4.11 now implies the existence of a required interval I with $\log \mathcal{W}^+_{\lambda,\mathrm{div},m}(I) \succeq \mathcal{W}(\mathcal{R})$. □

Part 2. Pseudo-Siegel disks and Snakes

5. Pseudo-Siegel disks

A pseudo-Siegel disk $\widehat{Z}^m$ is obtained from $\overline{Z}$ by filling-in deep parts of parabolic fjords of levels $\geq m$. We will show in Theorem 11.1 that $\widehat{Z}^{-1}$ can be constructed to be a uniform quasidisk. Consider a sufficiently small $\boldsymbol{\delta} > 0$.

Definition 5.1. A $\boldsymbol{\delta}$-pseudo-Siegel disk $\widehat{Z}^m$ of level m is a disk inductively constructed as follows:

- $\widehat{Z}^n = \overline{Z}$ for $n \gg 0$,
- either $\widehat{Z}^m := \widehat{Z}^{m+1}$,
- or $\widehat{Z}^m := \widehat{Z}^{m+1} \cup Z^m$, where is Z^m is a $\boldsymbol{\delta}/2$-near rotation domain (see §3), called the *core* of $\widehat{Z}^m$, satisfying the compatibility conditions with $\widehat{Z}^{m+1}$ stated in §5.1.

If $\widehat{Z}^m \neq \widehat{Z}^{m+1}$, then we call $\widehat{Z}^m := \widehat{Z}^{m+1} \cup Z^m$ a *regularization* of $\widehat{Z}^{m+1}$ at *level* m. Given $\widehat{Z}^m$, all its levels of regularization m_i are enumerated as

$$\cdots > m_{i+1} > m_i > m_{i-1} > \cdots \geq m.$$

We say that m_{i+1} is the level *before* m_i while m_{i-1} is the level *after* m_i.

5.0.1. *Outline and Motivation.* Pseudo-Siegel disks are inductively constructed as

$$\widehat{Z}^m = \widehat{Z}^{m+1} \cup Z^m = \overline{Z} \cup \cdots \cup Z^{m_i} \cup Z^{m_{i-1}} \cup \cdots \cup Z^{m_k} \cup Z^m,$$

where Z^m is obtained from $\widehat{Z}^{m+1} = \widehat{Z}^{m_k}$ by smoothing its boundary on level m. The boundary ∂Z^m is a cyclic concatenation $\alpha_0\#\beta_0\#\alpha_1\#\beta_1\#\dots$, where α_i are "channels" through peninsulas of $\widehat{Z}^{m+1}$ and β_i are "dams" in parabolic fjords of $\widehat{Z}^{m+1}$, see Figure 16. We require that there is a system of annuli around α_i, β_i making Z^m a near-rotation domain §3.

We will require in Assumption 5 that dams are sufficiently deep in fjords so that the outer geometries of Z and $\widehat{Z}^m$ are close: if the endpoints of intervals $I, J \subset \partial Z$ are in upper parts peninsulas, then $\mathcal{W}^+_Z(I,J) = (1\pm\varepsilon)\mathcal{W}^+_{\widehat{Z}^m}(I^m, J^m)$, where I^m, J^m are the "projections" of I, J onto $\widehat{Z}^m$, see details in §5.2. Here ε is uniformly small independently of the number of regularizations.

We define the combinatorial distance on $\partial\widehat{Z}^m$ to be induced from ∂Z:

$$\operatorname{dist}_{\partial\widehat{Z}^m}(x,y) := \operatorname{dist}_{\partial Z}(x,y) \qquad \text{for} \qquad x,y \in \partial\widehat{Z}^m \cap \partial Z. \tag{5.1}$$

With respect to this metric, the inner geometry of $\widehat{Z}^m$ has a description similar to Z – see estimates in Theorem 5.12. The estimates depend on $\boldsymbol{\delta}$; however on scale $\gg_{\boldsymbol{\delta}} \mathfrak{l}_m$, the estimates are uniform. Lemma 5.15 relates the inner geometry of peninsulas of $\widehat{Z}^m$ with the inner geometry of $\widehat{Z}^{m+1}$. As a consequence, Localization and Squeezing Lemmas 5.16, 5.17 hold for $\widehat{Z}^m$ – compare with §2.2.1, §2.2.2. The constant $\boldsymbol{\delta} > 0$ will be fixed in Section 7 so that the regularization $\widehat{Z}^{m+1} \rightsquigarrow \widehat{Z}^m$ can be iterated if Z has deep parabolic fjords of level m.

5.0.2. *Regular intervals.* A *regular* point of $\partial \widehat{Z}^m$ is a point in $\partial \widehat{Z}^m \cap \partial Z$. A *regular* interval $I \subset \partial \widehat{Z}^m$ is an interval with regular endpoints. An interval $I \subset \partial Z$ is regular rel $\widehat{Z}^m$ if the endpoints of I are in $\partial \widehat{Z}^m \cap \partial Z$.

The *projection* of a regular interval $I \subset \partial \widehat{Z}^m$ onto ∂Z is the interval $I^\bullet \subset \partial Z$ with the same endpoints and the same orientation as I. All regular points of I are in $I^\bullet$. We define the combinatorial length of I by $|I| \coloneqq |I^\bullet|$. Similarly is defined the *projection* I^k of a regular interval $I \subset \partial \widehat{Z}^m$ onto $\partial \widehat{Z}^k$ for $k > m$.

For an interval $I \subset \partial Z$, the *projection* I^m onto ∂Z^m is the shortest regular interval whose projection onto ∂Z contains I. Similarly is defined the projection of an interval $I \subset \partial \widehat{Z}^m$ onto $\partial \widehat{Z}^n$ for $n < m$. An interval $I \subset \partial Z$ is regular rel $\widehat{Z}^m$ if and only if $I = (I^m)^\bullet$

As for ∂Z, given $I, J \subset \partial \widehat{Z}^m$, we set $\lfloor I, J \rfloor \coloneqq I \cup L \cup J$, where $L \subset \partial \widehat{Z}^m$ is the complementary interval between I and J so that I, L, J are clockwise oriented.

5.1. **Compatibility between $\widehat{Z}^{m+1}$ and Z^m.** In this subsection, we inductively define $\widehat{Z}^m = \widehat{Z}^{m+1} \cup Z^m$ completing Definition 5.1.

We say γ is an *external arc* of a closed topological disk D if γ is a simple arc in $\widehat{\mathbb{C}} \setminus \operatorname{int} D$ such that $\gamma \cap \partial D$ consists of two endpoints of γ. Similarly, an *internal arc* of D is a simple arc $\ell \subset D$ such that $\ell \cap \partial D$ consists of two endpoints of ℓ.

5.1.1. *Channels and Dams.* Recall from §2.1.6 that $\mathfrak{D}_m$ denotes the diffeo-tiling of level $m \geq -1$. Let us enumerate intervals in $\mathfrak{D}_m$ clockwise as $T_i = [a_i, a_{i+1}]$; i.e.

$$\partial Z = T_0 \# T_1 \# \dots \# T_{\mathfrak{q}_{m+1}-1}$$

is the level m tessellation of ∂Z into diffeo-intervals. Then $f^{\mathfrak{q}_{m+1}}$ maps the T_i almost into $T_{i-\mathfrak{p}_{m+1}}$. We also recall that $T_i' = T_i \cap f^{\mathfrak{q}_{m+1}}(T_i)$ (with a slight adjustment for $m = -1$). Let us denote by $T_i'^{m+1}, T_i^{m+1}$ the projections of T_i', T_i onto $\widehat{Z}^{m+1}$. By Assumptions 1, T_i, T_i' are regular rel $\widehat{Z}^n, \ \ n \geq m$:

Assumption 1 (Channels and dams)**.** *The clockwise tessellation* (3.1) *of* ∂Z^m *into unit intervals*

$$\partial Z^m = L_0 \# L_1 \# \dots \# L_{\mathfrak{q}_m - 1}$$

satisfies $L_i = \alpha_i \# \beta_i$*, where (see Figure 16):*

- $\alpha_i = \alpha_i^m$ *is an internal arc of* $\widehat{Z}^{m+1}$ *connecting*
$$y_{i-1} \in T_{i-1}'^{m+1} \qquad \textit{and} \qquad x_i \in T_i'^{m+1},$$
- $\beta_i = \beta_i^m$ *is an external arc of* $\widehat{Z}^{m+1}$ *connecting* $x_i, y_i \in T_i'^{m+1}$,
- x_i *is on the left of* y_i *in* $T_i'^{m+1}$.

Moreover, $x_i, y_i \in \mathrm{CP}_{m+1} \setminus \mathrm{CP}_m$.

We say that α_i *is a* level m channel *and* β_i *is a* level m dam. △

We will require in Assumption 7 that the α_i^m, β_j^m are pairwise disjoint except possibly at endpoints and that the α_i^m are disjoint from the α_j^n for all $n > m$. Components of $\widehat{Z}^m \setminus \widehat{Z}^{m+1} = Z^m \setminus \widehat{Z}^{m+1}$ and of $\widehat{Z}^m \setminus Z^m = \widehat{Z}^{m+1} \setminus Z^m$ will be called fjords and peninsulas, see §5.1.5.

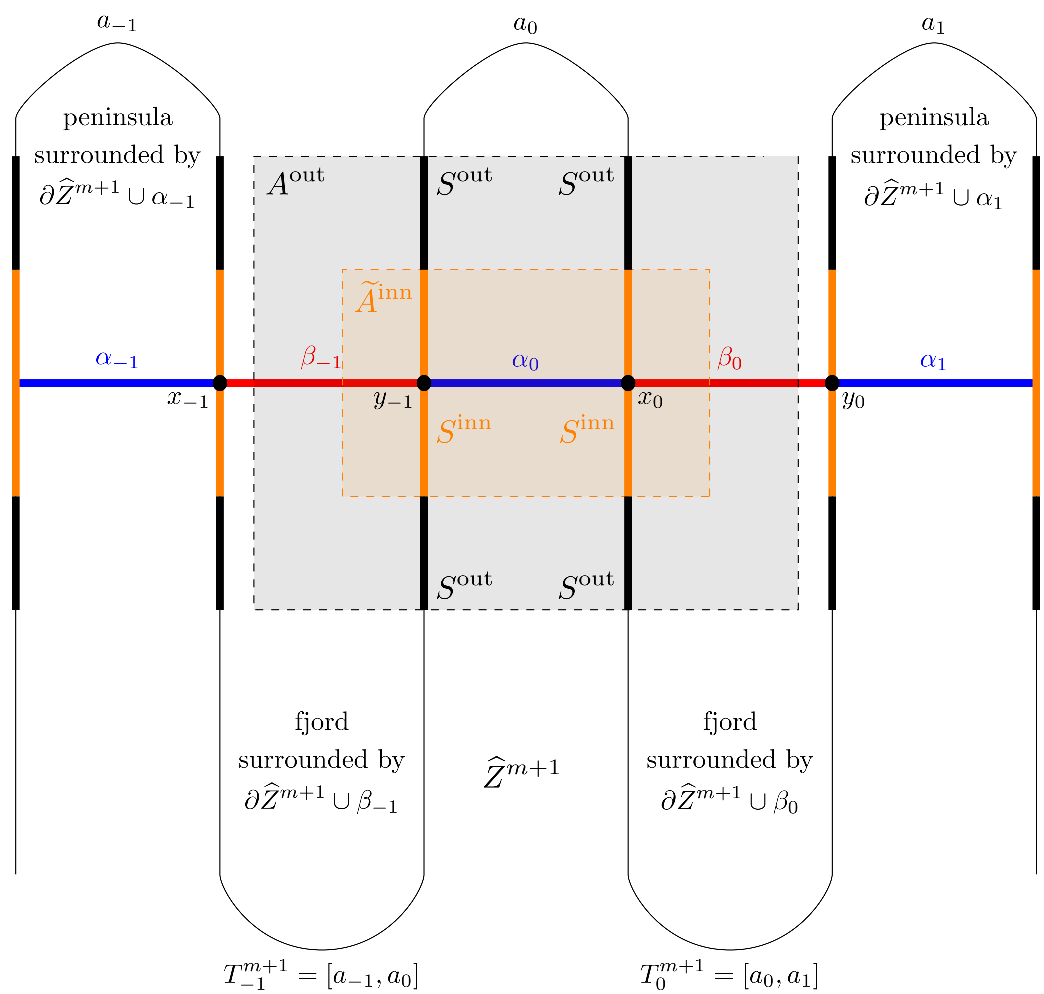


FIGURE 16. The channels α_i, the dams β_i, the collars $A^{\text{inn}}(\alpha_0), A^{\text{out}}(\alpha_0)$, and buffers $S^{\text{inn}}, S^{\text{out}}$ of a pseudo-Siegel disk $\widehat{Z}^{m+1}$. (The inner boundary of A^{inn} is omitted.)

5.1.2. *Collars and S-buffers.*

Assumption 2 (Collars)**.** *There are closed collars around α_i and β_i*

$$A(\alpha_i)=A^{\text{inn}}(\alpha_i)\cup A^{\text{out}}(\alpha_i) \quad \textit{and} \quad A(\beta_i)=A^{\text{inn}}(\beta_i)\cup A^{\text{out}}(\beta_i),$$
$$\partial^{\text{inn}}A^{\text{out}}(\alpha_i)=\partial^{\text{out}}A^{\text{inn}}(\alpha_i), \qquad \partial^{\text{inn}}A^{\text{out}}(\beta_i)=\partial^{\text{out}}A^{\text{inn}}(\beta_i)$$

with

$$\operatorname{mod}A^{\text{inn}}(\alpha_i),\quad \operatorname{mod}A^{\text{out}}(\alpha_i),\quad \operatorname{mod}A^{\text{inn}}(\beta_i),\quad \operatorname{mod}A^{\text{out}}(\beta_i)\quad \geq \boldsymbol{\delta}$$

such that for all i and all $k\in\{0,+1,\dots,\mathfrak{q}_{m+1}\}$ we have

- $A(\alpha_{i-k\mathfrak{p}_{m+1}})$ *encloses* $f^k(\alpha_i)$*;*
- $A(\alpha_{i+k\mathfrak{p}_{m+1}})$ *encloses the unique f^k-lift of α_i starting and ending at ∂Z;*

- $A(\beta_{i-k\mathfrak{p}_{m+1}})$ *encloses* $f^k(\beta_i)$*;*
- $A(\beta_{i+k\mathfrak{p}_{m+1}})$ *encloses the unique* f^k*-lift of* β_i *starting and ending at* ∂Z*.*

In other words, $A(\alpha_i)$ and $A(\beta_i)$ control the difference between Z^m and its image under f^k for $|k| \le \mathfrak{q}_{m+1}$. The inner colors A^{inn} will be used later in this section to describe the inner geometry of $\widehat{Z}^m$. The outer colors A^{out} will be used on several occasions; for instance, to "tame snakes," see Lemmas 6.3, 6.4.

Assumption 3 (Intersection Pattern)**.** *For all* α_i^m, β_i^m*, the simple closed curves*

$$\partial^{\text{out}} A^{\text{out}}(\alpha_i^m), \quad \partial^{\text{inn}} A^{\text{out}}(\alpha_i^m) = \partial^{out} A^{\text{inn}}(\alpha_i^m), \quad \partial^{\text{inn}} A^{\text{inn}}(\alpha_i^m),$$
$$\partial^{\text{out}} A^{\text{out}}(\beta_i^m), \quad \partial^{\text{inn}} A^{\text{out}}(\beta_i^m) = \partial^{out} A^{\text{inn}}(\beta_i^m), \quad \partial^{\text{inn}} A^{\text{inn}}(\beta_i^m)$$

intersect $\partial \widehat{Z}^m$ *at exactly* 4 *points and these intersection points are in* $\text{CP}_{m+1} \setminus \text{CP}_m$*.*

Moreover, all 12 *intersection points in*

$$P(\beta_i^m) \coloneqq \partial \widehat{Z}^m \cap \left(\partial^{\text{out}} A^{\text{out}}(\beta_i^m) \cup \partial^{\text{inn}} A^{\text{out}}(\beta_i^m) \cup \partial^{\text{inn}} A^{\text{inn}}(\beta_i^m)\right)$$

are within T_i*. The* 6 *most left points of* $P(\beta_i^m) \cap T_i$ *are within*

$$P(\alpha_i^m) \coloneqq \partial \widehat{Z}^m \cap \left(\partial^{\text{out}} A^{\text{out}}(\alpha_i^m) \cup \partial^{\text{inn}} A^{\text{out}}(\alpha_i^m) \cup \partial^{\text{inn}} A^{\text{inn}}(\alpha_i^m)\right),$$

the 6 *most right points of* $P(\beta_i^m) \cap T_i$ *are within* $P(\alpha_{i+1}^m)$*.*

Let us denote by $\widetilde{A}(\alpha_i^m), \widetilde{A}^{\text{inn}}(\alpha_i^m), \widetilde{A}(\beta_i^m), \widetilde{A}^{\text{inn}}(\beta_i^m)$ the disks obtained by filling-in $A(\alpha_i^m)$, $A^{\text{inn}}(\alpha_i^m)$, $A(\beta_i^m)$, $A^{\text{inn}}(\beta_i^m)$. It follows from the Assumption 3 that $\widetilde{A}, \widetilde{A}^{\text{inn}}$ have the following intersection properties with $\partial \widehat{Z}^{m+1}$:

$$\widetilde{A}(\alpha_i) \cap \partial \widehat{Z}^{m+1} = S_{y_{i-1}} \cup S_{x_i}, \qquad \widetilde{A}(\beta_i) \cap \partial \widehat{Z}^{m+1} = S_{x_i} \cup S_{y_i},$$
$$\widetilde{A}^{\text{inn}}(\alpha_i) \cap \partial \widehat{Z}^{m+1} = S^{\text{inn}}_{y_{i-1}} \cup S^{\text{inn}}_{x_i}, \qquad \widetilde{A}^{\text{inn}}(\beta_i) \cap \partial \widehat{Z}^{m+1} = S^{\text{inn}}_{x_i} \cup S^{\text{inn}}_{y_i},$$

where

- $S^{\text{inn}}_{x_i} \subset S_{x_i}$ are sub-intervals of $T_i'^{m+1}$ containing x_i,
- $S^{\text{inn}}_{y_i} \subset S_{y_i}$ are sub-intervals of $T_i'^{m+1}$ containing y_i,
- all S_{x_i}, S_{y_i} are pairwise disjoint. △

We say that S_{x_i}, S_{y_i} are *S-buffers of level m* and we say that $S^{\text{inn}}_{x_i}$, $S^{\text{inn}}_{y_i}$ are S^{inn}-*buffers of level m*. Note that $\partial \widehat{Z}^{m+1}$ may also contain many S- and S^{inn}-buffers of deeper levels We also write:

$$S^{\text{inn}}(\beta_i^m) \coloneqq \widetilde{A}^{\text{inn}}(\beta_i^m) \cap \partial \widehat{Z}^m = \left(S^{\text{inn}}_{x_i} \cup \beta_i^m \cup S^{\text{inn}}_{y_i}\right) \setminus \text{int}(\widehat{Z}^m), \tag{5.2}$$

$$S^{\text{inn}}(\alpha_i^m) \coloneqq \widetilde{A}^{\text{inn}}(\alpha_i^m) \cap \partial \widehat{Z}^m = \left(S^{\text{inn}}_{y_{i-1}} \cup S^{\text{inn}}_{x_i}\right) \setminus \text{int}(\widehat{Z}^m), \tag{5.3}$$

and similar with $S(\beta_i^m), S(\alpha_i^m)$.

Lemma 5.2. *The disk* Z^m *(see Assumption 1) is a* $\boldsymbol{\delta}/2$*-near rotation domain (see Section 3) with respect to*

$$A_i = A(L_i) \coloneqq A^{\text{inn}}(\alpha_i) \square A^{\text{inn}}(\beta_i);$$

see §A.1.11 for the definition of "□".

Proof. By Assumption 2, the annulus A_i controls the difference between $f^k(L_i)$ and $L_{i-k\mathfrak{p}_{m+1}}$. It follows from Assumptions 3 and 7 that A_i intersects only A_{i-1} and A_{i+1}. Since $\text{mod}(A^{\text{inn}}(\alpha_i)), \text{mod}(A^{\text{inn}}(\beta_i)) \ge \boldsymbol{\delta}$, we have $\text{mod}(A_i) \ge \boldsymbol{\delta}/2$ by Lemma A.3. □

Assumption 4 (Combinatorial space)**.** *Each of the* 15 *intervals in* $T_i \setminus \big(P(\beta_i^n) \cup \{x_i, y_i\}\big)$ *has length at least* $200\mathfrak{l}_{m+1}$*.*

Moreover, the subinterval $[x_i, y_i] \subset T_i^{m+1}$ *has length at least* $\dfrac{4}{5}|T_i^{m+1}|$*.* △

In particular, most of T_i^{m+1} is "reclaimed" during the regularization.

5.1.3. *Extra geometry constrains.*

Assumption 5 (Conformal separation)**.** *For all* i *we have*

$$\mathcal{W}(S_{x_i}, S_{y_i}), \quad \mathcal{W}(S_{y_i}, S_{x_{i+1}}) \le 1/\boldsymbol{\delta}.$$

△

Recall from (2.12) that a rectangle $\mathcal{R}$ is based on $T_i'^{m+1}$ if $\mathcal{R} \subset \widehat{\mathbb{C}} \setminus \operatorname{int} \widehat{Z}^{m+1}$ and $\partial^h \mathcal{R} \subset T_i'^{m+1}$. A rectangle $\mathcal{R}$ based on $T_i'^{m+1}$ is

- *parabolic* if $\operatorname{dist}_{T_i'^{m+1}}(\partial^{h,0}\mathcal{R}, \partial^{h,1}\mathcal{R}) \ge 6 \min\{|\partial^{h,0}\mathcal{R}|, \quad |\partial^{h,1}\mathcal{R}|\} + 3\mathfrak{l}_{m+1}$;
- balanced if $|\partial^{h,0}\mathcal{R}| = |\partial^{h,1}\mathcal{R}|$;
- *non-winding* if, in addition, every vertical curve in $\mathcal{R}$ is homotopic in $\mathbb{C} \setminus \operatorname{int} \widehat{Z}^m$ to a subcurve of $T_i'^{m+1}$

Consider a sufficiently big $\mathbb{\Delta} \gg 1$ – it will be fixed in §5.2, see Remark 5.11.

Assumption 6 (Extra outer protection)**.** *For every dam* β_i^m *there is a balanced parabolic non-winding rectangle* $\mathcal{X}_i^m = \mathcal{X}(\beta_i^m)$ *based on* $T_i'^{m+1}$ *such that* $\mathcal{W}(\mathcal{X}_i^m) \ge \mathbb{\Delta}$ *and* $\widetilde{A}(\beta_i^m) \setminus \widehat{Z}^{m+1}$ *is in the bounded component of* $\mathbb{C} \setminus (\widehat{Z}^{m+1} \cup \mathcal{X}_i^m)$*.* △

In particular, β_i is deep in the fjord associated with T_i, see 2.1.7. We assume that $\partial^{h,0}\mathcal{X}_i^m < \partial^{h,1}\mathcal{X}_i^n$ in T_i; i.e. $\lfloor \mathcal{X}_i^n \rfloor \subset T_i^{m+1}$. We denote by $\lfloor \mathcal{X}_i^n \rfloor^m$ the projection of $\lfloor \mathcal{X}_i^n \rfloor$ onto $\partial \widehat{Z}^m$.

5.1.4. *Minimal position of the collars.* A collection $\Gamma = (\gamma_k)$ of external arcs of a topological disk D is in *minimal position rel* ∂D if every two arcs γ_k, γ_t have the minimal intersection number up to homotopy in $(\widehat{\mathbb{C}} \setminus D, \partial D)$. This means that $|\gamma_k \cap \gamma_t| \le 1$ and $|\gamma_k \cap \gamma_t| = 1$ if and only if the endpoints of γ_k are linked rel ∂D to the endpoints of γ_t. Similarly, the minimal position for internal arcs is defined.

By Assumption 3, every simple closed curve

$$\partial^{\text{out}} A^{\text{out}}(\alpha_i^m), \quad \partial^{\text{inn}} A^{\text{out}}(\alpha_i^m), \quad \partial^{\text{inn}} A^{\text{inn}}(\alpha_i^m),$$
$$\partial^{\text{out}} A^{\text{out}}(\beta_i^m), \quad \partial^{\text{inn}} A^{\text{out}}(\beta_i^m), \quad \partial^{\text{inn}} A^{\text{inn}}(\beta_i^m)$$

is a cyclic concatenation of 2 external and 2 internal arcs of $\widehat{Z}^m$. We denote by Γ^m_{ext} and Γ^m_{inn} the set of external and internal arcs of the above boundaries of collars.

Assumption 7. *The set* $\Gamma^m_{\text{inn}} \cup \{\alpha_i^m\}_i$ *is in minimal position rel* $\partial \widehat{Z}^m$*. The set*

$$\bigcup_{n \ge m} \Gamma^n_{\text{ext}} \cup \{\beta_i^n\}_i \cup \{\partial^{v,0}\mathcal{X}_i^n, \partial^{v,1}\mathcal{X}_i^n\}_i$$

is in minimal position rel ∂Z*.*

Moreover, all landing points of curves in $\Gamma^m_{\text{ext}} \cup \Gamma^m_{\text{inn}} \cup \{\alpha_i^m, \beta_i^m\}_i \cup \{\partial^{v,0}\mathcal{X}_i^m, \partial^{v,1}\mathcal{X}_i^m\}_i$ *are within* $\mathrm{CP}_{m+1} \setminus \mathrm{CP}_m$*.* △

In particular, the $\mathcal{X}_i^n$ geometrically separate dams of all levels.

5.1.5. *Fjords and Peninsulas.* A connected component of $\widehat{Z}^m \setminus \widehat{Z}^{m+1} = Z^m \setminus \widehat{Z}^{m+1}$ is called a *level m fjord* while a connected component of $\widehat{Z}^m \setminus Z^m = \widehat{Z}^{m+1} \setminus Z^m$ is called a *level m peninsula*. Every peninsula $\mathfrak{P}$ has a unique channel α_i^m on its boundary; we will often write $\mathfrak{P} = \mathfrak{P}(\alpha_i^m)$. The *coast* of $\mathfrak{P}$ is

$$\partial^c \mathfrak{P} := \overline{\partial \mathfrak{P} \setminus \alpha_i^m}.$$

The boundary $\partial \widehat{Z}^m$ is a concatenation of dams β_i^m and coasts of peninsulas.

Consider a peninsula $\mathfrak{P}(\alpha_j^n)$ of $\widehat{Z}^n$, where $n > m$. By construction (Assumption 1), $\mathfrak{P}$ contains a unique point $z \in \mathrm{CP}_n$. There are three possibilities:

- $\partial^c \mathfrak{P}$ (as well as $\mathfrak{P}$) is in the interior of $\widehat{Z}^m$;
- $\partial^c \mathfrak{P} \subset \partial \widehat{Z}^m$;
- one of the components of $\partial^c \mathfrak{P} \setminus \{z\}$ is in $\operatorname{int} \widehat{Z}^m$ while the other component is in $\partial \widehat{Z}^m$.

In the last case, z is an endpoint of a dam $\beta_s^k \subset \partial \widehat{Z}^m$ of generation $k < n$ (by Assumption 7).

5.1.6. $S^{\mathrm{inn}}(\widehat{Z}^m) \subset S(\widehat{Z}^m) \subset S^{\mathrm{well}}(\widehat{Z}^m)$. Let us write

$$S^{\mathrm{inn}}(\widehat{Z}^m) := \bigcup_{n,j} S^{\mathrm{inn}}(\beta_j^n), \quad S(\widehat{Z}^m) := \bigcup_{n,j} S(\beta_j^n), \quad S^{\mathrm{well}}(\widehat{Z}^m) := \bigcup_{n,j} \lfloor \mathcal{X}_j^n \rfloor,$$

where the unions are taken over all $n \geq m$ and j.

By construction: $\mathrm{CP}_m \subset \partial \widehat{Z}^m \setminus S^{\mathrm{well}}(\widehat{Z}^m)$ and, moreover:

Lemma 5.3. *Every connected component of $S^{\mathrm{well}}(\widehat{Z}^m)$ is $\lfloor \mathcal{X}_j^n \rfloor$ for some $n \geq m, j$.*

Similarly, every connected component of $S(\widehat{Z}^m)$ is $S(\beta_i^n)$ for some β_i^n.

Similarly, every connected component of $S^{\mathrm{inn}}(\widehat{Z}^m)$ is $S^{\mathrm{inn}}(\beta_i^n)$ for some β_i^n. □

We say that an interval $I \subset \partial \widehat{Z}^m$ is *well-grounded* if its endpoints are in $\partial \widehat{Z}^m \setminus S^{\mathrm{well}}(\widehat{Z}^m)$. An interval $I \subset \partial Z$ is *well-grounded* rel $\widehat{Z}^m$ if I is regular and its projection $I^m \subset \partial \widehat{Z}^m$ is well-grounded.

5.1.7. $\mathcal{U}(\widehat{Z}^m) \subset \mathcal{X}(\widehat{Z}^m)$. Let us denote by $U(\alpha_i^n) := \widetilde{A}(\alpha_i^n) \setminus A(\alpha_i^n)$ and $U(\beta_i^n) := \widetilde{A}(\beta_i^n) \setminus A(\beta_i^n)$ the topological disks surrounded by $A(\alpha_i^n)$ and $A(\beta_i^n)$. Let us write

$$\mathcal{U}(\widehat{Z}^m) := \widehat{Z}^m \cup \bigcup_{n,i} \big(U(\alpha_i^n) \cup U(\beta_i^n)\big) = \widehat{Z}^m \cup \bigcup_{n,i} U(\beta_i^n),$$

where the equality follows from Assumption 3. Since the $A(\alpha_i^n), A(\beta_i^n)$ control the difference between $\widehat{Z}^m$ and the induced image under f^k, $|k| \leq \mathfrak{q}_{m+1}$ (Assumption 2), the map $(f \mid \overline{Z})^k \colon \overline{Z} \circlearrowleft$ extends uniquely to

$$f^k \colon \ \widehat{Z}^m \xrightarrow{1:1} (f^k)_*(\widehat{Z}^m) \subset \mathcal{U}(\widehat{Z}^m), \qquad \text{where} \quad |k| \leq \mathfrak{q}_{m+1}. \tag{5.4}$$

Let us also set

$$\mathcal{X}(\widehat{Z}^m) := \text{Filling-in of } \left(\bigcup_{n,j} \mathcal{X}_j^n \cup \widehat{Z}^m \right). \tag{5.5}$$

By construction, $\mathcal{X}(\widehat{Z}^m)$ contains all $A(\alpha_i^n)$ and $A(\beta_i^n)$.

5.1.8. *Pullbacks of* $\widehat{Z}^m$. Let $\operatorname{stab}(\widehat{Z}^m) \in \mathbb{N}_{\geq 0}$ be the smallest number such that the distance between $\partial^h \mathcal{X}_j^{m_s}$ and the endpoints of $T_j'^{m_s}$ is at least $(\operatorname{stab}(\widehat{Z}^m)+1)\mathfrak{l}_{m_s+1}$ for every level of regularization $m_s \geq m$ and every j. Then pullbacks of $\widehat{Z}^m$ are well defined up to $\operatorname{stab}(\widehat{Z}^m)\mathfrak{q}_{m+1}$ iterates:

Lemma 5.4. *For every $t \leq \operatorname{stab}(\widehat{Z}^m)\mathfrak{q}_{m+1}$ all the $\alpha_i^n, \beta_i^n, A(\alpha_i^n), A(\beta_i^n), \mathcal{X}(\beta_i^n)$ have univalent lifts along $f^t\colon \overline{Z} \to \overline{Z}$; the resulting lifts $\alpha_{i,-t}^n$, $\beta_{i,-t}^n$, $A(\alpha_{i,-t}^n)$, $A(\beta_{i,-t}^n), \mathcal{X}(\beta_{i,-t}^n)$ form a $(\boldsymbol{\delta}, \mathbb{A})$ pseudo-Siegel disk $\widehat{Z}_{-t}^m$ such that $f^t\colon \widehat{Z}_{-t}^m \to \widehat{Z}^m$ is conformal.*

Proof. Since the $\mathcal{X}_i^n$ are non-winding, they have univalent lifts for all $t \leq \operatorname{stab}(\widehat{Z}^m)\mathfrak{q}_{n+1}$ – compare with Lemma 4.3. Since the $\mathcal{X}_i^n$ separate $A(\beta_i^n) \setminus \widehat{Z}^{n+1}$ from CP_n (Assumption 6) and since $A(\alpha_i^n) \setminus \widehat{Z}^{n+1} \subset A(\beta_{i-1}^n) \cup A(\beta_i^n)$, the filled-in collars $\widetilde{A}(\alpha_i^n), \widetilde{A}(\beta_i^n)$ also have univalent lifts and the statement follows. □

5.1.9. *Geodesic pseudo-Siegel disks.* We say $\widehat{Z}^m$ is a *geodesic pseudo-Siegel disk* if

- $\bigcup_{n\geq m} \Gamma_{\text{ext}}^n \cup \{\beta_i^n\}_i \cup \{\partial^{v,0}\mathcal{X}_i^n, \partial^{v,1}\mathcal{X}_i^n\}_i$ consists of hyperbolic geodesics of $\widehat{\mathbb{C}} \setminus \overline{Z}$;
- $\Gamma_{\text{inn}}^n \cup \{\alpha_i^m\}_i$ consists of hyperbolic geodesics of $\operatorname{int} \widehat{Z}^{n+1}$ for every regularization level $n \geq m$; and
- $\operatorname{stab}(\widehat{Z}^m) \geq 10$.

Consider a parabolic non-winding rectangle $\mathcal{R}$ based on $T_i' \subset \partial Z$. By Lemma 4.3, $\mathcal{R}$ has a univalent pullback along $f^{\mathfrak{q}_{m+1}}: T' \boxminus \theta_{m+1} \to T$ for $j \in \{0, 1, \dots, \mathfrak{q}_{m+1}-1\}$. Let $\mathcal{R}_{-j}$ be the lift of $\mathcal{R}$ along $f^j\colon \overline{Z} \circlearrowleft$. We set

$$\operatorname{orb}_{-\mathfrak{q}_{m+1}+1} \mathcal{R} \coloneqq \bigcup_{j=0}^{\mathfrak{q}_{m+1}-1} \mathcal{R}_{-j},$$

i.e., $\operatorname{orb}_{-\mathfrak{q}_{m+1}+1} \mathcal{R}$ is the set of rectangles obtained by spreading around $\mathcal{R}$ using pullbacks.

We say that a regularization $\widehat{Z}^m = Z^{m+1} \cup \widehat{Z}^{m+1}$ is *within* $\operatorname{orb}_{-\mathfrak{q}_{m+1}+1} \mathcal{R}$ if

$$\Gamma_{\text{ext}}^m \cup \{\beta_i^m\}_i \cup \{\partial^{v,0}\mathcal{X}_i^m, \partial^{v,1}\mathcal{X}_i^m\}_i \subset \operatorname{orb}_{-\mathfrak{q}_{m+1}+1} \mathcal{R}.$$

Remark 5.5. *If $\widehat{Z}^m$ is geodesic, then $\operatorname{stab}(\widehat{Z}^m) \geq 10$ implies that*

$$\left(\bigcup_{|i|\leq 2\mathfrak{q}_{m+1}} [f \mid \partial Z]^i(\mathrm{CP}_m)\right) \cap S^{\text{well}}(\widehat{Z}^m) = \emptyset. \tag{5.6}$$

Therefore, if the endpoints of an interval $J \subset \partial Z$ are in CP_m, then $f^i(J)$ is well-grounded rel $\widehat{Z}^m$ for all $i \leq 2\mathfrak{q}_{m+1}$.

For instance, let $I \subset \partial Z$ be a combinatorial level-m interval such that one of the endpoints of I is in CP_m. Let I_s, $s < \mathfrak{q}_{m+1}$ be the intervals obtained by spreading around $I = I_0$, see §2.1.5. Then all I_s are well-grounded rel $\widehat{Z}^m$.

5.2. **Outer geometry of $\widehat{Z}^m$.** In this subsection, we will show that $\overline{Z}$ and $\widehat{Z}^m$ have comparable outer geometries with respect to grounded intervals.

5.2.1. *Removing small components from a rectangle.* Fix a big $\Delta \gg 1$ and some $\tau > 1$. Consider a rectangle $\mathcal{R} \subset \widehat{\mathbb{C}}$. Let $D_i, i \in I$ be a finite set of open Jordan disks in $\mathcal{R}$ with pairwise disjoint closures satisfying the following properties:

(A) every D_i is a attached to a unique side T_i of $\mathcal{R}$: the intersection $\partial D_i \cap \partial \mathcal{R}$ is a closed arc within T_i;

(B) there is a rectangle

$$\mathcal{Y}_i \subset \mathcal{R}, \quad \partial^h \mathcal{Y}_i = \partial \mathcal{Y}_i \cap \partial \mathcal{R} \subset T_i, \quad \mathcal{W}(\mathcal{Y}_i) \geq \Delta$$

protecting D_i in the following sense: $\partial \mathcal{R} \backslash T_i$ and D_i are in different components of $\mathcal{R} \setminus \mathcal{Y}_i$.

Let us denote by $T_i^{\text{opp}} \subset \partial \mathcal{R}$ the opposite side to T_i; i.e. $T_i \cup T_i^{\text{opp}}$ is either $\partial^v \mathcal{R}$ or $\partial^h \mathcal{R}$. We will also consider the following relaxations of (B):

(B_v) if T_i is a vertical side of $\mathcal{R}$, then there is a rectangle

$$\mathcal{Y}_i \subset \mathcal{R}, \quad \partial^h \mathcal{Y}_i = \partial \mathcal{Y}_i \cap \partial \mathcal{R} \subset T_i, \quad \mathcal{W}(\mathcal{Y}_i) \geq \tau$$

protecting D_i in the following sense: T_i^{opp} and D_i are in different components of $\mathcal{R} \setminus \mathcal{Y}_i$.

(B_h) if T_i is a horizontal side of $\mathcal{R}$, then there is a rectangle

$$\mathcal{Y}_i \subset \mathcal{R}, \quad \partial^h \mathcal{Y}_i = \partial \mathcal{Y}_i \cap \partial \mathcal{R} \subset \partial \mathcal{R} \setminus T_i^{\text{opp}}, \quad \mathcal{W}(\mathcal{Y}_i) \geq \Delta$$

protecting D_i: T_i^{opp} and D_i are in different components of $\mathcal{R} \setminus \mathcal{Y}_i$.

(B'_h) if T_i is a horizontal side of $\mathcal{R}$, then there is a rectangle

$$\mathcal{Y}_i \subset \mathcal{R}, \quad \partial^h \mathcal{Y}_i = \partial \mathcal{Y}_i \cap \partial \mathcal{R} \subset \partial \mathcal{R} \setminus T_i^{\text{opp}}, \quad \mathcal{W}(\mathcal{Y}_i) \geq \Delta_i$$

protecting D_i: T_i^{opp} and D_i are in different components of $\mathcal{R} \setminus \mathcal{Y}_i$, where:

$$\Delta_i \geq \tau \quad \text{for} \quad i \in S \qquad \text{and} \qquad \Delta_i \geq \Delta \quad \text{for} \quad i \notin S.$$

(Here S is an index set.)

Set

$$\mathcal{R}' \coloneqq \overline{\operatorname{int} \mathcal{R} \setminus \bigcup_{i \in I} D_i}$$

and view $\mathcal{R}'_i$ as a rectangle with the same vertices as $\mathcal{R}$ and with the same labeling of sides; i.e. $\partial^h \mathcal{R}$ and $\partial^h \mathcal{R}'$ have an infinite intersection. In other words, $\mathcal{R}'$ is obtained from $\mathcal{R}$ by slightly moving its boundaries towards the interior; the motion is geometrically controlled by rectangles $\mathcal{Y}_i$. If we view $\mathcal{R}$ as an outer rectangle in $\widehat{\mathbb{C}}$ (i.e., $\infty \in \operatorname{int} \mathcal{R}$), then $\mathcal{R}'$ is obtained from $\mathcal{R}$ by filling in fjords, see Figure 17.

Lemma 5.6. *For every $\varepsilon > 0$ and $\Delta \gg_\varepsilon 1$ the following holds. For $\mathcal{R}$ and $\mathcal{R}'$ satisfying the above (A) and (B) we have:*

$$1 - \varepsilon < \frac{\mathcal{W}(\mathcal{R}')}{\mathcal{W}(\mathcal{R})} < 1 + \varepsilon. \tag{5.7}$$

Proof. Let us conformally replace $\mathcal{R}$ with a Euclidean rectangle, see §A.1.1; i.e. we assume that $\mathcal{R} = [0, x] \times [0, 1]$. Since every D_i is separated from three sides of $\mathcal{R}$ by a wide rectangle $\mathcal{Y}_i$, we obtain that D_i has a Euclidean diameter less than $\delta \min\{x, 1\}$, where $\delta \to 0$ as $\Delta \to \infty$. Therefore, the width of $\mathcal{R}'$ is estimated from below and above by the width of $[\delta, x - \delta] \times [0, 1]$ and of $[0, x] \times [\delta, 1 - \delta]$. This implies (5.7). $\square$

Lemma 5.7. *For every $\varepsilon > 0$ and $\Delta \gg_\varepsilon 1$ the following holds.*

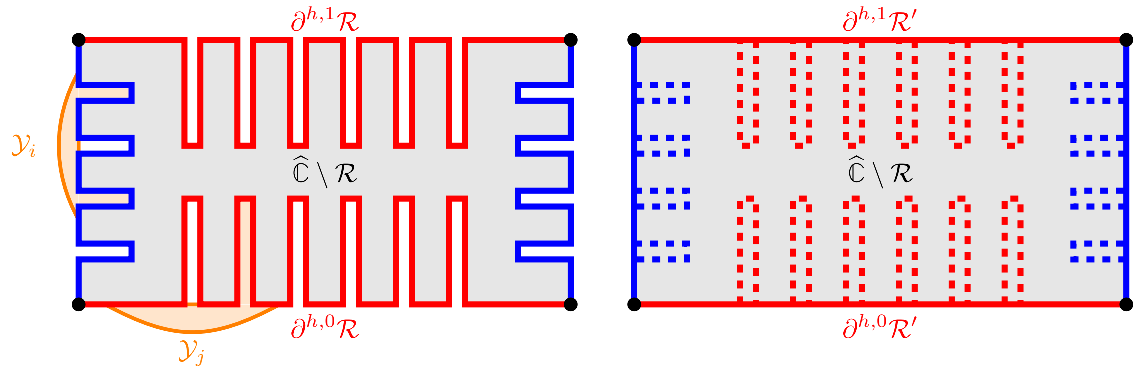


FIGURE 17. The (outer) rectangle $\mathcal{R}'$ is obtained from $\mathcal{R}$ by filling fjords. If all the fjords are protected by wide rectangles $\mathcal{Y}_j$ (orange), i.e., fjords have narrow entrances, then $\mathcal{W}(\mathcal{R})$ is close to $\mathcal{W}(\mathcal{R}')$.

- *if $\mathcal{R}$ and $\mathcal{R}'$ satisfy (A) and (B_v), then:*

$$\mathcal{W}(\mathcal{R}) - O_\tau(1) < \mathcal{W}(\mathcal{R}') \tag{5.8}$$

- *if $\mathcal{R}$ and $\mathcal{R}'$ satisfy (A) and (B_h), then:*

$$\mathcal{W}(\mathcal{R}') < (1+\varepsilon)\mathcal{W}(\mathcal{R}) \tag{5.9}$$

- *if $\mathcal{R}$ and $\mathcal{R}'$ satisfy (A) and (B_h'), then:*

$$\mathcal{W}(\mathcal{R}') < (1+\varepsilon)\mathcal{W}(\mathcal{R}) + O_\tau(|S|). \tag{5.10}$$

Proof. As in the proof of Lemma 5.6, we assume that $\mathcal{R} = [0,x]\times[0,1]$ is a Euclidean rectangle. We also assume that $x = \mathcal{W}(\mathcal{R}) > 1$ – the only relevant case.

Assume $\mathcal{R}, \mathcal{R}'$ satisfy (A) and (B_v). If T_i is vertical, then $\operatorname{diam} D_i < C_\tau$ for some $C_\tau > 0$. Therefore, $\mathcal{W}(\mathcal{R}')$ is estimated from below by the width of $[C_\tau, x - C_\tau]\times[0,1]$ and (5.8) follows.

Assume $\mathcal{R}, \mathcal{R}'$ satisfy (A) and (B_h). If T_i is horizontal, then $\operatorname{diam} D_i < c_\Delta$, where $c_\Delta \to 0$ as $\Delta \to \infty$. Therefore, $\mathcal{W}(\mathcal{R}')$ is estimated from above by the width of

$$\mathcal{R}^{\text{new}} := [0,x]\times[c_\Delta, 1-c_\Delta]$$

and (5.9) follows.

Assume $\mathcal{R}, \mathcal{R}'$ satisfy (A) and (B_h'). Consider the vertical foliation $\mathcal{F}'$ of $\mathcal{R}'$. The width of curves in $\mathcal{F}'$ landing at $\bigcup_{i\in S} D_i$ is $O_\tau(|S|)$. The width of the remaining curves in $\mathcal{F}'$ is bounded by $\mathcal{W}(\mathcal{R}^{\text{new}})$ because every remaining curve contains a subcurve in $\mathcal{R}^{\text{new}}$ connecting $\partial^h\mathcal{R}^{\text{new}}$. This implies (5.10). ☐

5.2.2. *Well grounded intervals.* Recall from §5.1.6 that a regular interval $I^m \subset \partial\widehat{Z}^m$ is well-grounded if its endpoints are not in $S^{\text{well}}(\widehat{Z}^m)$. Recall also that $\mathbb{\Delta}$ is a parameter from Assumption 6.

Lemma 5.8. *For every $\varepsilon > 0$ and $\mathbb{A} \gg_\varepsilon 1$ the following holds. For every pseudo-Siegel disk $\widehat{Z}^m$ and every pair of well-grounded intervals $I^m, J^m \subset \partial \widehat{Z}^m$, we have:*

$$1 - \varepsilon \le \frac{\mathcal{W}^+_{\widehat{Z}^m}(I^m, J^m)}{\mathcal{W}^+_Z(I, J)} \le 1 + \varepsilon,$$

where I, J are the projections of I^m, J^m onto ∂Z. Similarly, if $\mu^+_{\overline{Z}}, \mu^+_{\widehat{Z}^m}$ are the outer harmonic measures of $\overline{Z}, \widehat{Z}^m$ rel ∞, then

$$1 - \varepsilon \le \frac{\mu^+_{\widehat{Z}^m}(I^m)}{\mu^+_Z(I)} \le 1 + \varepsilon.$$

Proof. Let us enumerate all components of $\operatorname{int} \widehat{Z}^m \setminus \overline{Z}$ as D_i, $i \in I^m$. Every component D_i is bounded by $\overline{Z}$ and a level $n_i \ge m$ dam $\beta_i^{n_i}$. By Assumption 6, there is a wide rectangle

$$\mathcal{X}_i^{n_i} \subset \mathbb{C} \setminus \operatorname{int}(\widehat{Z}^{n_i+1}), \qquad \partial^h \mathcal{X}_i^{n_i} \subset \partial \widehat{Z}^{n_i+1} \qquad \text{with} \qquad \mathcal{W}(\mathcal{X}_i) \ge \mathbb{A}$$

separating the endpoints of $\beta_i^{n_i}$ from the endpoints of I, J.

We denote by $\overset{\star}{\mathcal{X}}{}_i^{n_i}$ the rectangle obtained from $\mathcal{X}_i^{n_i}$ by adding all bounded components of $\mathbb{C} \setminus (\overline{Z} \cup \partial^h \mathcal{X}_i^{n_i})$; i.e. we adjust the horizontal boundary of $\mathcal{X}_i^{n_i}$ by adding fjords so that $\partial^h \overset{\star}{\mathcal{X}}{}_i^{n_i} \subset \partial Z$.

Lemma 5.9. *If $\mathbb{A} \gg_{\boldsymbol{\delta}} 1$, then $\mathcal{W}(\overset{\star}{\mathcal{X}}{}_i^{n_i}) \ge \mathbb{A}/2$ (independently of the number of regularizations).*

Proof. We proceed by induction on n: we assume that the statement is verified for all $\overset{\star}{\mathcal{X}}{}_i^{n_i}$ with $n_i \ge n+1$, and we will prove it for all $\overset{\star}{\mathcal{X}}{}_i^{n_i}$ with $n_i = n$.

Consider a rectangle $\mathcal{X}_i^n$ and recall that $\overset{\star}{\mathcal{X}}{}_i^n$ is obtained from $\mathcal{X}_i^n$ by adding fjords of level $> n$. Every such fjord is separated from $\operatorname{int} \mathcal{X}_i^n$ by a dam $\beta_j^{n_j}$ with $n_j > n$. Consider the protection $\mathcal{X}_j^{n_j}$ from Assumption 6. By construction, $\mathcal{X}_j^{n_j} \subset \overset{\star}{\mathcal{X}}{}_i^n$ and hence $\overset{\star}{\mathcal{X}}{}_j^{n_j} \subset \overset{\star}{\mathcal{X}}{}_i^n$. By the induction assumption, $\mathcal{W}(\overset{\star}{\mathcal{X}}{}_j^{n_j}) \ge \mathbb{A}/2 \gg_\varepsilon 1$. Applying Lemma 5.6 for the rectangles $\mathcal{R} = \overset{\star}{\mathcal{X}}{}_i^n$, $\mathcal{R}' = \mathcal{X}_i^n$ and protections $\{\overset{\star}{\mathcal{X}}{}_j^{n_j}\}$, we obtain that their widths are close; in particular, $\mathcal{W}(\overset{\star}{\mathcal{X}}{}_i^n) > \mathcal{W}(\mathcal{X}_i^n)/2 \ge \mathbb{A}/2$. □

Lemma 5.8 now follows from Lemmas 5.6 and 5.9 by viewing $\widehat{\mathbb{C}} \setminus Z$ as a rectangle with horizontal sides I, J. □

5.2.3. *Grounded intervals.* A regular interval $I^m \subset \partial \widehat{Z}^m$ is *grounded* if its endpoints are not in $S^{\text{inn}}(\widehat{Z}^m)$. An interval $I \subset \partial Z$ is *grounded rel* $\widehat{Z}^m$ if its projection $I^m \subset \partial \widehat{Z}^m$ is grounded.

Lemma 5.10. *For every $\varepsilon > 0$ and $\mathbb{A} \gg_{\boldsymbol{\delta}, \varepsilon} 1$ the following holds. If*

$$I, J \subset \partial Z \qquad \textit{with} \qquad \operatorname{dist}(I, J) \ge 3 \min\{|I|, |J|\}$$

is a pair of grounded rel $\widehat{Z}^m$ intervals, then

$$\mathcal{W}^+_Z(I, J) - O_\delta(1) < \mathcal{W}^+_{\widehat{Z}^m}(I^m, J^m) < (1+\varepsilon)\mathcal{W}^+_Z(I, J) + O_\delta(1), \tag{5.11}$$

where $I^m, J^m \subset \partial \widehat{Z}^m$ are the projections of I, J onto $\widehat{Z}^m$.

Proof. We view $\widehat{\mathbb{C}}\setminus Z$ as a rectangle $\mathcal{R}$ with horizontal sides I, J and we view $\widehat{\mathbb{C}}\setminus Z^m$ as a rectangle $\mathcal{R}'$ with horizontal sides I', J'.

Consider a dam β_i^n. By Assumption 3, $A^{\text{inn}}(\beta_i^n)\setminus \widehat{Z}^{n+1}$ consists of 2 rectangles; let the rectangle

$$\mathcal{A}_i^n \subset \widehat{\mathbb{C}}\setminus \operatorname{int}\widehat{Z}^{n+1}, \qquad \partial^h \mathcal{A}_i^n = \mathcal{A}_i^n \cap \partial \widehat{Z}^{n+1}$$

be the closure of the connected component of $A^{\text{inn}}(\beta_i^n)\setminus \widehat{Z}^{n+1}$ separating (i.e., protecting) β_i^n from CP_n. Since $\text{mod}(A(\beta_i^n)) \geq \boldsymbol{\delta}$ we have $\mathcal{W}(\mathcal{A}_i^n)\geq \boldsymbol{\delta}$.

As in the proof of Lemma 5.8, we define $\overset{\star}{\mathcal{A}}{}_i^n$ to be the rectangle obtained from $\mathcal{A}_i^n$ by adding all fjords between $\overline{Z}$ and $\partial^h\mathcal{A}_i^n$. Since $\partial^h\mathcal{A}_i^n$ consists of a pair of well-grounded intervals of $\partial\widehat{Z}^n$, the argument in Lemma 5.8 is applicable to $\mathcal{A}_i^n$ and shows that $\mathcal{W}\left(\overset{\star}{\mathcal{A}}{}_i^n\right) \geq \mathcal{W}(\mathcal{A}_i^n)/(1+\epsilon) > \boldsymbol{\delta}/2 =: \tau$ because $\mathbb{\Delta} \gg_{\boldsymbol{\delta}} 1$.

Since $\overset{\star}{\mathcal{A}}{}_i^n$ satisfy (B_v) for $\mathcal{R}, \mathcal{R}'$, the first inequality in (5.11) follows from (5.8). Below we will remove $O(1)$ buffers from $\mathcal{R}, \mathcal{R}'$ so that the new rectangles $\mathcal{R}^{\text{new}}, \mathcal{R}'^{\text{new}}$ satisfy (5.10) with $|S|\leq 3$.

Assume that $|I|\leq |J|$. Let k be the level of I: the unique number satisfying $\frac{4}{5}\mathfrak{l}_k > |I| \geq \frac{4}{5}\mathfrak{l}_{k+1}$, where $\frac{4}{5}$ is from Assumption 4. Consider dams $\beta_i^n\subset I^n\cup J^n$. We distinguish the following three cases.

Assume $n>k$. Since

$$\operatorname{dist}(I^n, J^n) \geq 3\min\{|I^n|, |J^n|\} = 3|I^n| > 2\mathfrak{l}_{k+1} > |\lfloor \partial^h \mathcal{X}_i^n\rfloor|,$$

$\lfloor \partial^h \mathcal{X}_i^n\rfloor$ is disjoint from either I^n or J^n; i.e. $\overset{\star}{\mathcal{X}}{}_i^n$ satisfy (B_h) for $\mathcal{R}, \mathcal{R}'$.

Assume $n=k$. Then $\beta_i^n \subset J^m$ because $|\beta_i^n| > |I|$ by Assumption 4. There are at most two level $n=k$ dams β_i^n such that the associated rectangles $\overset{\star}{\mathcal{X}}{}_i^n$ intersect I^n. Such exceptional dams are protected $\overset{\star}{\mathcal{A}}{}_i^n$, and we add these dams into S. The remaining dams β_i^n are protected by $\overset{\star}{\mathcal{X}}{}_i^n$.

Assume $n<k$. Again $\beta_i^n\subset J^n$. For every β_i^n, we recognize disjoint genuine subrectangles $\mathcal{X}_i^{n,\text{inn}}, \mathcal{X}_i^{n,\text{out}} \subset \mathcal{X}_i^n$ such that $\operatorname{dist}(\partial^h\mathcal{X}_i^{n,\text{out}},\ \ \partial^h\mathcal{X}_i^{n,\text{inn}}) > |I|$ and

$$\mathcal{W}(\mathcal{X}_i^{n,\text{inn}}), \mathcal{W}(\mathcal{X}_i^{n,\text{out}}) \geq \mathbb{\Delta}/3, \qquad \lfloor\partial^h\mathcal{X}_i^{n,\text{inn}}\rfloor \subset \lfloor\partial^h\mathcal{X}_i^{n,\text{out}}\rfloor.$$

If I^n is disjoint from all $\partial^h\mathcal{X}_i^{n,\text{inn}}$ for $n<k$, then $\overset{\star}{\mathcal{X}}{}_i^{n,\text{inn}}$ satisfy (B_h) and the lemma follows from (5.10). Suppose that n is minimal so that for a dam β_i^n the interval I^n intersects $\partial^h\mathcal{X}_i^{n,\text{inn}}$; note that the dam β_i^n is unique. Then

$$\partial^{h,0}\mathcal{X}_i^{n,\text{out}} < I^n < \mathcal{X}_i^{n,\text{out}}$$

and for every $t\geq n,\ \ \beta_j^t\neq\beta_i^n$, the rectangle $\mathcal{X}_i^{n,\text{out}}$ separates I^n from β_j^t. By removing a $O_\delta(1)$-buffer from $\mathcal{R}, \mathcal{R}'$ we can assume that the new rectangles $\mathcal{R}^{\text{new}}, \mathcal{R}'^{\text{new}}$ do not cross $\mathcal{X}_i^{n,\text{out}}$; the new rectangles are disjoint from all $\beta_j^t\neq \beta_i^n$ with $t\geq k$. We add β_i^n to S and apply (5.10). □

5.2.4. *Restrictions of rectangles.* Consider a rectangles $\mathcal{R}\subset\widehat{\mathbb{C}}\setminus Z$ and assume that there is a connected graph $G\subset\mathcal{R}$ containing all vertices of $\mathcal{R}$ such that $G\subset\widehat{\mathbb{C}}\setminus\operatorname{int}\mathcal{X}(\widehat{Z}^m)$, see (5.5). Then the *restriction* $\mathcal{R}^m$ of $\mathcal{R}$ to the complement of $\widehat{Z}^m$ is the connected component of $\mathcal{R}\setminus\operatorname{int}\widehat{Z}^m$ containing G. We view $\mathcal{R}^m$ as a rectangle with the same vertex set and the same orientation of sides as $\mathcal{R}$. Observe that every connected component of $\mathcal{R} \setminus \mathcal{R}^m$ is separated from G by a restriction of some $\mathcal{F}(\overset{\star}{\mathcal{X}}{}_i^{n_i})$. Therefore, Lemmas 5.9 and 5.6 imply that

$$1-\varepsilon < \frac{\mathcal{W}(\mathcal{R})}{\mathcal{W}(\mathcal{R}^m)} < 1+\varepsilon, \tag{5.12}$$

for a small $\varepsilon > 0$, where $\mathbb{A} \gg_\varepsilon 1$.

Remark 5.11 (Fixing $\mathbb{A}$)**.** *From now on we assume that $\mathbb{A} \gg 1$ is fixed so that ε in Lemmas 5.8 and 5.10 and in* (5.12) *is small.*

5.3. **The geometry on shallow scales.** On scale $\geq \mathfrak{l}_m$, the geometry of $\widehat{Z}^m$ is controlled by its core Z^m:

Theorem 5.12. *Consider two regular intervals*

$$I, J \subset \partial \widehat{Z}^m \qquad \text{such that} \qquad |I|,\ |J|,\ \operatorname{dist}(I,J) \geq 1/\mathfrak{q}_{m+1}.$$

If $\operatorname{dist}(I,J) \leq \min\{|I|,|J|\}$*, then*

$$\mathcal{W}^-(I,J) \asymp_{\boldsymbol{\delta}} \log \frac{\min\{|I|,|J|\}}{\operatorname{dist}(I,J)} + 1; \tag{5.13}$$

otherwise

$$\mathcal{W}^-(I,J) \asymp_{\boldsymbol{\delta}} \left(\log \frac{\operatorname{dist}(I,J)}{\min\{|I|,|J|\}} + 1\right)^{-1}. \tag{5.14}$$

Moreover, there is a constant $T_{\boldsymbol{\delta}} > 1$ such that if $|I|,\ |J|,\ \operatorname{dist}(I,J) \geq T_{\boldsymbol{\delta}}/\mathfrak{q}_{m+1}$, then (5.13) *and* (5.14) *are independent of $\boldsymbol{\delta}$: if* $\operatorname{dist}(I,J) \leq \min\{|I|,|J|\}$*, then*

$$\mathcal{W}^-(I,J) \asymp \log \frac{\min\{|I|,|J|\}}{\operatorname{dist}(I,J)} + 1; \tag{5.15}$$

otherwise

$$\mathcal{W}^-(I,J) \asymp \left(\log \frac{\operatorname{dist}(I,J)}{\min\{|I|,|J|\}} + 1\right)^{-1}. \tag{5.16}$$

We refer to (5.15) and (5.16) as *beau coarse-bounds*, compare with Theorem 3.8.

Proof. For a channel $\alpha_i \subset \partial Z^m$, let us denote by $\mathcal{W}^-_{Z^m,3}(\alpha_i)$ the width of curves starting at α_i and ending at $\partial Z^m \setminus (\beta_{i-1} \# \alpha_i \# \beta_i)$. Similarly, $\mathcal{W}^-_{Z^m,3}(\beta_i)$ is the width of curves starting at β_i and ending at $\partial Z^m \setminus (\alpha_i \# \beta_i \# \alpha_{i+1})$.

Claim 1. *For all channels and dams of ∂Z^m, we have:*

$$\mathcal{W}^-_{Z^m,3}(\alpha_i) \asymp_{\boldsymbol{\delta}} 1, \qquad \mathcal{W}^-_{Z^m,3}(\beta_i) \asymp_{\boldsymbol{\delta}} 1.$$

Proof. We have $\mathcal{W}^-_{Z^m,3}(\alpha_i) \preceq_{\boldsymbol{\delta}} 1$ because $A^{\text{inn}}(\alpha_i)$ separates α_i from $\partial Z^m \setminus (\beta_{i-1} \# \alpha_i \# \beta_i)$. Similarly, $\mathcal{W}^-_{Z^m,3}(\beta_i) \preceq_{\boldsymbol{\delta}} 1$. Since

$$\left(\mathcal{W}^-_{Z^m,3}(\alpha_i)\right)^{-1} = \mathcal{W}^-_{Z^m}(\beta_{i-1},\beta_i) \leq \mathcal{W}^-_{Z^m,3}(\beta_{i-1}) \preceq_{\boldsymbol{\delta}} 1,$$

where $\mathcal{W}^-_{Z^m}(\beta_{i-1},\beta_i)$ is the width of curves connecting β_{i-1} and β_i, we obtain $\mathcal{W}^-_{Z^m,3}(\alpha_i) \asymp_{\boldsymbol{\delta}} 1$. Similarly, $\mathcal{W}^-_{Z^m,3}(\beta_i) \asymp_{\boldsymbol{\delta}} 1$. □

Claim 2. *If* $\min\{|I|,|J|\} \asymp \operatorname{dist}(I,J)$*, then* $\mathcal{W}^-_{\widehat{Z}^m}(I,J) \asymp_{\boldsymbol{\delta}} 1$.

Proof. Let us define the *projection* I° of I onto ∂Z^m to be the minimal concatenation

$$\alpha_i \# \beta_{i+1} \# \beta_{i+1} \# \dots \# \beta_k \# \alpha_{k+1}$$

(starting and ending with a channel) such that every component of $I \setminus I^\circ$ is in the peninsula whose channel is in I°. Similarly, J° is defined. Since $\operatorname{dist}(I,J) \geq 1/\mathfrak{q}_{m+1}$, the intervals I° and J° are separated by at least one dam (follows from Assumption 4) and we still have $\min\{|I^\circ|, |J^\circ|\} \asymp \operatorname{dist}(I^\circ, J^\circ)$ with respect to the distance of ∂Z^m.

The case $\min\{|I|,\ |J|\},\ \operatorname{dist}(I,J) \asymp 1/\mathfrak{q}_{m+1}$ follows from Claim 1 by spiting intervals into finitely many channels and dams. Let us assume that $|I|,\ |J|,\ \operatorname{dist}(I,J) \geq 50/\mathfrak{q}_{m+1}$.

By Lemma A.9 that the set of curves in $\mathcal{F} := \mathcal{F}^-_{\widehat{Z}^m}(I,J)$ entering a peninsula with a channel in $\partial Z^m \setminus (I^\circ \cup J^\circ)$ is a buffer of $\mathcal{F}^-_{\widehat{Z}^m}(I,J)$. By removing two $O_{\boldsymbol\delta}(1)$-buffers from $\mathcal{F}^-_{\widehat{Z}^m}(I,J)$, we obtain that curves in the new family $\mathcal{F}^{\text{new}}$ do not enter any peninsula whose channel is in $\partial Z^m \setminus (I^\circ \cup J^\circ)$ – such channels are separated from $I^\circ \cup J^\circ$ by a definite annulus. Let $\mathcal{F}^{\text{New}}$ be the restriction of $\mathcal{F}^{\text{new}}$ to the family $\mathcal{F}^-_{Z^m}(I^\circ, J^\circ)$ (see §A.1.5); i.e., $\mathcal{F}^{\text{New}}$ consists of the first shortest subcurves γ' of $\gamma \in \mathcal{F}^{\text{new}}$ such that γ' connects I° and J°. We have:

$$\mathcal{W}^-_{\widehat{Z}^m}(I,J) - O_{\boldsymbol\delta}(1) \leq \mathcal{W}(\mathcal{F}^{\text{new}}) \leq \mathcal{W}(\mathcal{F}^{\text{New}}) \leq \mathcal{W}^-_{Z^m}(I^\circ, J^\circ) \asymp 1$$

by Proposition 3.3; i.e. $\mathcal{W}^-_{\widehat{Z}^m}(I,J) \preceq_{\boldsymbol\delta} 1$. If A, B are the components of $\partial\widehat{Z}^m \setminus (I,J)$, then the above argument shows $\mathcal{W}^-_{\widehat{Z}^m}(A,B) \preceq_{\boldsymbol\delta} 1$. Therefore, $\mathcal{W}^-_{\widehat{Z}^m}(I,J) \asymp_{\boldsymbol\delta} 1$. ☐

The case $\operatorname{dist}(I,J) \leq \min\{|I|,|J|\}$ follows from Claim 2 by applying the Splitting Argument (Remark 2.6). By the same reason, the second part of the theorem follows from the following claim:

Claim 3. *There is a constant $T_{\boldsymbol\delta} > 50$ such that if*

$$|I|,\ |J|,\ \operatorname{dist}(I,J) \geq T_{\boldsymbol\delta}/\mathfrak{q}_{m+1} \qquad \text{and} \qquad \min\{|I|,|J|\} \asymp_{\boldsymbol\delta} \operatorname{dist}(I,J),$$

then $\mathcal{W}^-(I,J) \asymp 1$.

Proof. Assume $T_{\boldsymbol\delta} \gg 50$ is sufficiently big and let us consider I° and J° as in the proof of Claim 2. Let us enlarge I°, J° by adding $\sqrt{T_{\boldsymbol\delta}}$ unit intervals on both sides of I°, J°; the new intervals $\widetilde{I}^\circ, \widetilde{J}^\circ$ still satisfy $\min\{|\widetilde{I}^\circ|, |\widetilde{J}^\circ|\} \asymp \operatorname{dist}(\widetilde{I}^\circ, \widetilde{J}^\circ)$.

We claim that by removing two 1-buffers from $\mathcal{F}^-_{\widehat{Z}^m}(I,J)$ we obtain the family $\mathcal{F}^{\text{new}}$ so that curves in $\mathcal{F}^{\text{new}}$ do not enter any peninsula with a channel in $\partial Z^m \setminus (\widetilde{I}^\circ \cup \widetilde{J}^\circ)$. Indeed, consider a channel $\alpha_k \in \partial Z^m \setminus (\widetilde{I}^\circ \cup \widetilde{J}^\circ)$, and let $A, B \subset \partial\widehat{Z}^m$ be intervals of length $\sqrt{T_{\boldsymbol\delta}}$ attached to α_k. Since we already established (5.13), $\mathcal{W}^-_{\widehat{Z}^k}(A,B) \asymp_{\boldsymbol\delta} \log\sqrt{T_{\boldsymbol\delta}} \gg 1$; removing a $O_{\boldsymbol\delta}(1)$-buffer from $\mathcal{F}^-_{\widehat{Z}^k}(A,B)$, we obtain a family of curves $\mathcal{R}_k$ such that $\mathcal{W}(\mathcal{R}_k) > 1$ and $\mathcal{R}_k$ separates α_k from $I^\circ \cup J^\circ$. The set of vertical curves in $\mathcal{F}^-_{\widehat{Z}^m}(I,J)$ that intersect α form a buffer of $\mathcal{R}$ by Lemma A.9; this buffer has width less than 1 by $\mathcal{W}(\mathcal{R}_k) > 1$.

As in the proof of Claim 2, we now define $\mathcal{F}^{\text{New}}$ to be the restriction of $\mathcal{F}^{\text{new}}$ to the family $\mathcal{F}^-_{Z^m}(\widetilde{I}^\circ, \widetilde{J}^\circ)$ (see §A.1.5); i.e., $\mathcal{F}^{\text{New}}$ consists of the first shortest subcurves γ' of $\gamma \in \mathcal{F}^{\text{new}}$ such that γ' connects $\widetilde{I}^\circ$ and $\widetilde{J}^\circ$. We have:

$$\mathcal{W}^-_{\widehat{Z}^m}(I,J) - O(1) \leq \mathcal{W}(\mathcal{F}^{\text{new}}) \leq \mathcal{W}(\mathcal{F}^{\text{New}}) \leq \mathcal{W}^-_{Z^m}(I^\circ, J^\circ) \asymp 1$$

by Theorem 3.8; i.e. $\mathcal{W}^-_{\widehat{Z}^m}(I,J) \preceq 1$. And similarly, $\Big(\mathcal{W}^-_{\widehat{Z}^m}(I,J)\Big)^{-1} = \mathcal{W}^-_{\widehat{Z}^m}(A,B) \preceq 1$, where A,B are the components of $\partial \widehat{Z}^m \setminus (I,J)$. □

□

5.3.1. *Hyperbolic geodesics in* $\widehat{Z}^m$*.* Let us extend continuously the distance function $\operatorname{dist}_{\partial \widehat{Z}^m}(\ ,\ )$ specified in (5.1) to all points of $\partial \widehat{Z}^m$. Given two sets X,Y intersecting $\partial \widehat{Z}^m$, we define the $\partial \widehat{Z}^m$*-distance between* X,Y to be $\operatorname{dist}_{\partial \widehat{Z}^m}\Big(X\cap \partial \widehat{Z}^m,\ Y\cap \partial \widehat{Z}^m\Big)$.

Lemma 5.13. *There is an* $M = M(\boldsymbol{\delta}) \ge 1$ *such that the following properties hold for every hyperbolic geodesic* $\gamma \subset \widehat{Z}^m$ *connecting* $x,y\in \partial \widehat{Z}^m$*.*

(I) *If the* $\partial \widehat{Z}^m$*-distance from* $\{x,y\}$ *to* $\widetilde{A}^{\text{inn}}(\beta^n_i)$ *is at least* $M\mathfrak{l}_m$*, then* γ *is disjoint* $\widetilde{A}^{\text{inn}}(\beta^n_i)$*.*

(II) *Similarly, if the* $\partial \widehat{Z}^m$*-distance from* $\{x,y\}$ *to* $\widetilde{A}^{\text{inn}}(\alpha^n_i)$ *is at least* $M\mathfrak{l}_m$*, then* γ *is disjoint* $\widetilde{A}^{\text{inn}}(\alpha^n_i)$*.*

Let $\gamma_k = (f^k)_*(\gamma) \subset \mathcal{U}\big(\widehat{Z}^m\big)$ *be the image of* γ *under* (5.4)*, and assume that the endpoints of* γ_k *are in* $U_{x,k}, U_{y,k}$ *– the components of*

$$\{\widetilde{A}(\alpha^n_i)\setminus A(\alpha^n_i),\ \widetilde{A}(\beta^n_i)\setminus A(\beta^n_i) \mid n,i\}, \qquad \textit{see §5.1.7.} \tag{5.17}$$

(III) *If a component* U' *of* (5.17) *has* $\partial \widehat{Z}^m$*-distance at least* $M\mathfrak{l}_m$ *to* $U_{x,k}\cup U_{y,k}$*, then* γ_k *is disjoint from* U'*.*

Proof. Choose a sufficiently big $M = M(\boldsymbol{\delta})$ and assume that $\widetilde{A}^{\text{inn}}(\beta^n_i)$ has distance at least $M\mathfrak{l}_{m+1}$ to $\{x,y\}$. By Theorem 5.12, there is a sufficiently wide geodesic rectangle $\mathcal{R}\subset \widehat{Z}^m$ with $\partial^h\mathcal{R}\subset \partial \widehat{Z}^m$ such that $\partial^h \mathcal{R}$ separates $\widetilde{A}(\beta^n_i)\cap \partial \widehat{Z}^m$ from $\{x,y\}$ in $\partial \widehat{Z}^m$. By removing $1/\boldsymbol{\delta}$ buffers from $\mathcal{R}$, we obtain a rectangle $\mathcal{R}^{\text{new}}$ that separates $\widetilde{A}^{\text{inn}}(\beta^n_i)$ from γ. This proves (I) and (II) is similar.

Property (III) follow from (I) and (II) by applying f^{-k}. □

5.4. **The inner geometry of peninsulas.** For a level $n\ge m$ peninsula $\mathfrak{P}$ of $\widehat{Z}^n$ with $\partial^c\mathfrak{P}\subset \partial \widehat{Z}^m$ and intervals $I,J\subset \partial^c\mathfrak{P}$, we denote by $\mathcal{F}^-_{\mathfrak{P}}(I,J)$ the family of curves in $\mathfrak{P}$ connecting I and J. We write $\mathcal{W}^-_{\mathfrak{P}}(I,J) \coloneqq \mathcal{W}(\mathcal{F}^-_{\mathfrak{P}}(I,J))$.

By a *grounded pair* of intervals $I,J\subset \partial \widehat{Z}^m$, we mean a pair of disjoint grounded intervals §5.2.3.

Lemma 5.14. *Let* $I,J\subset \partial \widehat{Z}^m$ *be a grounded pair of intervals, and assume that* I *is within a level* $k\ge m$ *peninsula* $\mathfrak{P}$ *of* $\widehat{Z}^m$*. Set* $J^{\text{new}} \coloneqq J\cap \partial\mathfrak{P}$*. Then* $\partial^c\mathfrak{P}\subset\partial \widehat{Z}^m$ *and*

$$\mathcal{W}^-_{\mathfrak{P}}(I,J^{\text{new}}) = \mathcal{W}^-_{\widehat{Z}^{k+1}}(I,J^{\text{new}}) - O_{\boldsymbol{\delta}}(1) = \mathcal{W}^-_{\widehat{Z}^m}(I,J) - O_{\boldsymbol{\delta}}(1). \tag{5.18}$$

If $J^{\text{new}}\neq\emptyset$*, then* I,J^{new} *is a grounded pair of* $\widehat{Z}^{k+1}$*.*

Proof. Since the endpoints of I are grounded, $\partial^c\mathfrak{P}$ is non in any S^{inn} buffer of level $< k$; this implies that $\partial^c\mathfrak{P}\subset \partial\widehat{Z}^m$. Let α_k be the channel of $\mathfrak{P}$. If $J^{\text{new}}=\emptyset$, then every curve in $\mathcal{F}^-_{\widehat{Z}^m}(I,J)$ crosses $A^{\text{inn}}(\alpha_k)$ before intersecting α_k; i.e. all parts of (5.18) are equal to $O_{\boldsymbol{\delta}}(1)$ and the lemma follows.

If $J^{\text{new}}\neq\emptyset$, then J^{new} is a grounded interval of $\widehat{Z}^{k+1}$ because the endpoints of α are not in any S^{inn}-buffers of level $\ge k+1$. Since $I\subset \partial^c\mathfrak{P}\setminus S^{\text{inn}}(\alpha)$, the width of curves in $\mathcal{F}^-_{\widehat{Z}^m}(I,J)$ that are not in $\mathfrak{P}$ is $O_{\boldsymbol\delta}(1)$ because every such curve crosses $A^{\text{inn}}(\alpha_k)$. This implies the lemma. $\square$

The following lemma combined with Theorem 5.12 allows us to inductively calculate the width between grounded intervals.

Lemma 5.15. *Let $I,J\subset\partial\widehat{Z}^m$ be a grounded pair with $|\lfloor I,J\rfloor\leq 1/2$. Assume that both I,J intersect a level m peninsula $\mathfrak{P}=\mathfrak{P}(\alpha_i^m)$. Set*

$$I^{\text{new}}:=I\cap\partial^c\mathfrak{P},\qquad I':=I\setminus(\beta^m_{i-1}\cup I^{\text{new}})\qquad J^{\text{new}}:=J\cap\partial^c\mathfrak{P},\qquad J':=J\setminus(J^{\text{new}}\cup\beta^m_i).$$

Then:

$$\begin{aligned}\mathcal{W}^-_{\widehat{Z}^m}(I,J)\qquad &=\mathcal{W}^-_{\widehat{Z}^m}(I',J')+\mathcal{W}^-_{\mathfrak{P}}(I^{\text{new}},J^{\text{new}})+O_{\boldsymbol\delta}(1)\\ &=\mathcal{W}^-_{\widehat{Z}^m}(I',J')+\mathcal{W}^-_{\widehat{Z}^{m+1}}(I^{\text{new}},J^{\text{new}})+O_{\boldsymbol\delta}(1)\end{aligned}\ .$$

Proof. The intervals I^{new} and J^{new} are grounded intervals of $\widehat{Z}^{m+1}$ because the endpoints y_{i-1},x_i of α_i^m are not in any S^{inn}-buffer of level $\geq m+1$. We need to show that the width of the set $\mathcal{F}'$ of curves in $\mathcal{F}^-_{\widehat{Z}^m}(I,J)$ intersecting α_i^m is $O_{\boldsymbol\delta}(1)$. Let $\mathcal{A}$ be the set of curves in $\mathcal{F}'$ that are in $\widetilde{A}^{\text{inn}}(\alpha_i^m)$. Note that $\mathcal{W}(\mathcal{F}'\setminus\mathcal{A})=O_{\boldsymbol\delta}(1)$ because every curve in $\mathcal{F}'\setminus\mathcal{A}$ crosses $A^{\text{inn}}(\alpha_i^m)$.

Assume $\mathcal{A}\neq\emptyset$. This is only possible if

$$\beta^m_{i-1}\cup(S^{\text{inn}}(y_{i-1})\cap\partial\widehat{Z}^m)\subset I\qquad\text{and}\qquad(S^{\text{inn}}(x_i)\cap\partial\widehat{Z}^m)\cup\beta^m_i\subset J$$

because I,J are grounded. Therefore, every curve in $\mathcal{A}$ has a subcurve connecting $S^{\text{inn}}_{y_{i-1}}$ and $S^{\text{inn}}_{x_i}$. By Assumption 5, we obtain $\mathcal{W}(\mathcal{A})=O_{\boldsymbol\delta}(1)$. $\square$

5.5. **Localization and Squeezing lemmas.** We are now ready to establish Localization and Squeezing Properties for $\partial\widehat{Z}^m$, compare with §2.2.1 and §2.2.2. See §2.2.1 for the notion of an innermost subpair.

Localization Lemma 5.16. *For every $\lambda>1$ the following holds. If $I,J\subset\partial\widehat{Z}^m$ is a grounded pair with $|\lfloor I,J\rfloor|\leq 1-\dfrac{1}{\lambda}\min\{|I|,|J|\}$, then there is an innermost subpair*

$$I^{\text{new}},J^{\text{new}}\qquad\textit{with}\qquad I^{\text{new}}\subset I,\quad J^{\text{new}}\subset J$$

satisfying

$$|I^{\text{new}}|,|J^{\text{new}}|\leq\frac{1}{\lambda}\min\{|I|,|J|\}$$

such that, up to $O_{\boldsymbol\delta}(\log\lambda)$, the width of $\mathcal{F}^-(I,J)$ is contained in $\mathcal{F}^-(I^{\text{new}},J^{\text{new}})$:

$$\mathcal{W}^-(I\setminus I^{\text{new}},J)+\mathcal{W}^-(I,J\setminus J^{\text{new}})=O_{\boldsymbol\delta}(\log\lambda).\tag{5.19}$$

Moreover, we can assume that $\max\{|I^{\text{new}}|,|J^{\text{new}}|\}<2\min\{|I^{\text{new}}|,|J^{\text{new}}|\}$. *The subpair $I^{\text{new}},J^{\text{new}}$ is grounded rel $\widehat{Z}^n$, where $n\geq m$ is the deepest level such that $I^{\text{new}},J^{\text{new}}\subset\partial\widehat{Z}^n$.*

Squeezing Lemma 5.17. *There is a constant $C_{\boldsymbol\delta}$ such that the following holds. If $I,J\subset\partial\widehat{Z}^m$ is a grounded pair of intervals with $|\lfloor I,J\rfloor|\leq 1/2$ such that*

$$\mathcal{W}^-(I,J)\geq C_{\boldsymbol\delta}\log\lambda,\qquad\lambda>2,$$

then $\operatorname{dist}(I,J)\leq\dfrac{1}{\lambda}\min\{|I|,|J|\}$.

Proof of Lemmas 5.16 and 5.17. By splitting I, J into shorter grounded intervals, we can assume that $|\lfloor I, J\rfloor| \le 1/2$.

Let $L \subset \lfloor I, J\rfloor$ be the shortest complementary interval between I and J. The case $\operatorname{dist}(I,J) = |L| \ge \mathfrak{l}_m$ follows from Theorem 5.12: we can find points $a, b \in \mathrm{CP}_m$ so that the right interval I^{new} of $I \setminus \{a\}$ and the left interval J^{new} of $J\setminus\{b\}$ satisfy the conclusion of Lemma 5.16. (The intervals $I^{\mathrm{new}}, J^{\mathrm{new}}$ are grounded because the set CP_m is away from S^{inn}-buffers.)

Let $m_j \ge m$ be the smallest regularization level such that $|I|, |J| \ge \mathfrak{l}_{m_j}$. If $m < m_j$, then by Lemma 5.14, up to $O_{\boldsymbol\delta}(1)$ the width of $\mathcal{F}^-_{\widehat Z^m}(I,J)$ is in $\mathcal{F}^-_{\widehat Z^{m_j}}(I^{\mathrm{new}}, J^{\mathrm{new}})$; set $I_1 = I^{\mathrm{new}}, J_1 = J^{\mathrm{new}} \subset \partial \widehat Z^m \cap \partial \widehat Z^{m_j}$. If $m = m_j$, set $I_1 \coloneqq I,\ \ J_1 \coloneqq J$. Note that L is still the shortest complementary interval between I_1, J_1.

The case $\operatorname{dist}(I_1, J_1) = |L| \ge \mathfrak{l}_{m_j}$ follows again from Theorem 5.12. Otherwise, as in Lemma 5.15, we recognize subintervals $I_1', I_1^{\mathrm{new}} \subset I_1$ and $J_1^{\mathrm{new}}, J_1' \subset J_1$. Up to $O_{\boldsymbol\delta}(1)$, the family $\mathcal{F}^-_{\widehat Z^{m_j}}(I_1,J_1)$ is $\mathcal{F}^-_{\widehat Z^{m_j}}(I_1',J_1') \sqcup \mathcal{F}^-_{\widehat Z^{m_j+1}}(I_1^{\mathrm{new}}, J_1^{\mathrm{new}})$. Applying Theorem 5.12 to $\mathcal{F}^-_{\widehat Z^{m_j}}(I_1', J_1')$, we either conclude the lemma if $\min\{|I_1'|, |J_1'|\} \ge \lambda |L|$ or we reduce the problem to the pair $I_2 \coloneqq I_1^{\mathrm{new}}, J_2 \coloneqq J_1^{\mathrm{new}}$ with a new $\lambda_2 < \lambda/2$ because $|I_2|, |J_2| \le 2\mathfrak{l}_{m_k}/5$ by Assumption 4. Proceeding by induction, we conclude the lemma with at most $\log_2 \lambda$ steps. □

5.6. **Grounded subintervals.** As before, we extend continuously the distance function $\operatorname{dist}_{\widehat Z^m}$ specified in (5.1) to all points of $\partial \widehat Z^m$. Then $\mathcal{F}^-_\lambda(I)$ and $\mathcal{W}^-_\lambda(I)$ are well defined for all intervals $I \subset \partial \widehat Z^m$.

Given an interval $J \subset \partial Z$, we define:

- $J^{\mathrm{GRND}} \subset J$ be the biggest grounded interval in J; and
- $J^{\mathrm{grnd}} \supset J$ be the smallest grounded interval containing J.

We allow $J^{\mathrm{grnd}} = \emptyset$ or $J^{\mathrm{GRND}} = \partial \widehat Z^m$.

Lemma 5.18 (X vs X^{grnd})**.** *Consider a family $\mathcal{F}(I,J)$, where $I, J \subset \partial\widehat Z^m$. Let $A, B \subset \partial \widehat Z^m$ be two complementary intervals to I, J.*

- (i) *If $\operatorname{dist}(I,J) > \mathfrak{l}_m$, then for every interval $X \subset \partial \widehat Z^m$, there are at most $O_{\boldsymbol\delta}(1)$ curves in $\mathcal{F}(I,J)$ intersecting $X \setminus X^{\mathrm{grnd}}$.*
- (ii) *If $|I|, |J|, |A| \ge \mathfrak{l}_m$, then for every interval $X \subset A$, there are at most $O_{\boldsymbol\delta}(1)$ curves in $\mathcal{F}(I,J)$ intersecting $X \setminus X^{\mathrm{grnd}}$.*

Proof. Consider Case (i). Every component (out of at most 2) of $X \setminus X^{\mathrm{grnd}}$ is within $S^{\mathrm{inn}}(\beta)$ for a dam $\beta \subset L$. Since $|A|, |B| \ge \mathfrak{l}_m$, the color $A^{\mathrm{out}}(\beta)$ is disjoint (and hence separate $S^{\mathrm{inn}}(\beta)$) from either I or J. The lemma now follows from $\operatorname{mod} A^{\mathrm{out}}(\beta) \ge \boldsymbol\delta$.

Case (ii) follows from a similar argument. Every component of $X \setminus X^{\mathrm{grnd}}$ is within $S^{\mathrm{inn}}(\beta)$. Since $|I|, |J|, |A| \ge \mathfrak{l}_m$, the annulus $A^{\mathrm{out}}(\beta)$ separates $S^{\mathrm{inn}}(\beta)$ from either I or J. □

5.7. **Pseudo-bubbles.** A *bubble* Z_ℓ is the closure of a connected component of $f^{-k}(Z)\setminus Z$. The *generation* of Z_ℓ is the minimal k such that $f^k(Z_\ell) = \overline{Z}$; i.e. $f^k\colon Z_\ell \to \overline Z$ is the first landing. Given a pseudo-Siegel disk $\widehat Z^m$, the *pseudo-bubble* $\widehat Z_\ell$ is the closure of the connected component of $f^{-k}\big(\operatorname{int} \widehat Z^m\big)$ containing $\operatorname{int} Z_\ell$. In other words, $\widehat Z_\ell$ is obtained from Z_ℓ by adding the lifts of all reclaimed fjords (components of $\widehat Z^m \setminus \overline Z$) along $f^k\colon Z_\ell \to \overline Z$.

Channels, dams, collars $A^{\text{inn}}, A^{\text{out}}$, extra protections $\mathcal{X}_i^n$ are defined for $\widehat{Z}_\ell$ as pullbacks of the corresponding objects along $f^k\colon \widehat{Z}_\ell \to \widehat{Z}^m$. For instance, $\mathcal{X}(\widehat{Z}_\ell)$ is the pullback of $\mathcal{X}(\widehat{Z}^m)$, see (5.5), under f^k. The length of an interval $I \subset \partial \widehat{Z}_\ell$ is the length of its image $f^k(I) \subset \partial \widehat{Z}^m$. All results of this section are valid for pseudo-bubbles. In particular, the results concerning the inner geometry of $\widehat{Z}^m$ (such as Theorem 5.12, Lemmas 5.16 and 5.17) are obtained by identifying $\widehat{Z}_\ell$ with $\widehat{Z}^m$ via f^k. The results concerning the outer geometry of $\widehat{Z}_\ell$ (see §5.2) are obtained by repeating the arguments.

6. Snakes

Consider the family $\mathcal{F}^\circ_L(I,J) = \mathcal{F}^\circ_{L,\widehat{Z}^m}(I,J)$ as in §2.5. Our principal result of the section is the following generalization of Lemma 2.12:

Snake Lemma 6.1. *Let $I, J \subset \partial\widehat{Z}^m$ be a pair of grounded intervals, and let L be a complementary interval between I, J. Normalize $I < L < J$ and set*

$$K \coloneqq \mathcal{W}^\circ_L(I,J) - \mathcal{W}^+(I,J).$$

Assume that $|N| \geq \mathfrak{l}_m$, where $N = \partial\widehat{Z}^m \setminus (I \cup L \cup J)$ is the second complementary interval between I and J.

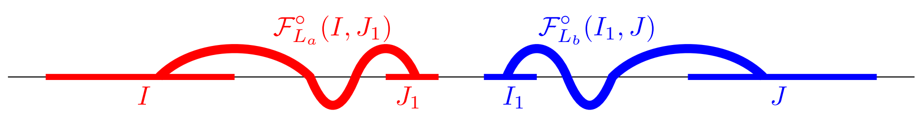
If $K \gg_{\boldsymbol{\delta}} \log\lambda$ with $\lambda > 2$, then there are grounded intervals

$$\text{(6.1)} \qquad J_1, I_1 \subset L, \qquad |J_1| < \frac{\operatorname{dist}(I,J_1)}{\lambda}, \quad |I_1| < \frac{\operatorname{dist}(I_1,J)}{\lambda}, \qquad I < J_1 < I_1 < J$$

such that

$$\text{(6.2)} \qquad \mathcal{W}^\circ_{L_a}(I,J_1) \oplus \mathcal{W}^\circ_{L_b}(I_1,J) \geq K - O_{\boldsymbol{\delta}}(\log\lambda),$$

where $L_a, L_b \subset L$ are the intervals between I, J_1 and I_1, J respectively:



Note that (6.2) implies

$$\text{(6.3)} \qquad \max\{\mathcal{W}^\circ_{L_a}(I,J_1), \mathcal{W}^\circ_{L_b}(I_1,J)\} \geq 2K - O_{\boldsymbol{\delta}}(\log\lambda).$$

Corollary 6.2. *Under the assumption of Lemma 6.1, there is an interval $I^{\text{new}} \subset L^\bullet \subset \partial Z$ grounded rel $\widehat{Z}^m$ such that $\mathcal{W}^+_\lambda(I^{\text{new}}) \succeq K$, where $L^\bullet$ is the projection of L onto ∂Z.*

6.0.1. *Outline and motivation of the section.* The proof of Lemma 6.1 repeats the argument of Lemma 2.12 (the Snake Lemma for Z) with an additional input from Lemma 6.4. More precisely, the Series Decomposition §2.4 yields families $\mathcal{F}^\circ_{L_a}(I,J_1)$ and $\mathcal{F}^\circ_{L_b}(I_1,J)$ shown on the figure in Lemma 6.1. By Localization Lemma 5.16, we can assume that $|J_1|, |I_1|$ are small compared with $\operatorname{dist}(I,J_1), \operatorname{dist}(I_1,J)$ respectively. And by Lemma 6.4, we can assume that J_1, I_1 are grounded rel $\widehat{Z}^m$; i.e., Snake Lemma 6.1 can be iterated.

This allows to trade families entering int $\widehat{Z}^m$ into outer families (Corollary 6.2). Indeed, assuming that $\mathcal{W}^\circ_{L_a}(I,J_1) \geq 2K - O_{\boldsymbol{\delta}}(\log\lambda)$, either $\mathcal{W}^+_{L_a}(I,J_1) \geq K/3$ or repeating the Snake Lemma we find I_2, J_2 with $\mathcal{W}^\circ_{L_2}(I_2,J_2) \geq \frac{3}{2}K$ enlarging the family. Since $\widehat{Z}^m$ is a non-uniformly qc disk, the process eventually stops.

For applications, we will need several variations of the Snake Lemma. In §6.4 we state the Snake Lemma "with toll barriers": if $\mathcal{F}^\circ_L(I,J)$ contains a lamination $\mathcal{R}$ submerging into L at least n times, then $2K$ in (6.3) can be replaced by nK. In §6.13 we state the Sneaking Lemma when $\mathcal{F}^\circ(I,J)$ "sneaks" through a wide outer rectangle $\mathcal{R}$, see Figure 19 for illustration. Both versions will be used in Snake-Lair Lemma 8.6 to amplify the width of degeneration.

6.1. **Proof of Snake Lemma 6.1.** We will prove the Snake Lemma in §6.1.2 after introducing an auxiliary subfamily $\mathcal{F}^\star_L(I,J) \subset \mathcal{F}^\circ_L(I,J)$. The family $\mathcal{F}^\star_L(I,J)$ consists of curves omitting dams (and some space around them) that have an endpoint in L^c. We will show that $\mathcal{W}^\circ_L(I,J) - \mathcal{W}^\star_L(I,J) = O_{\boldsymbol{\delta}}(1)$ and that at most $O_{\boldsymbol{\delta}}(1)$-curves in $\mathcal{F}^\star_L(I,J)$ intersects $X \setminus X^{\mathrm{grnd}}$ for every interval $X \subset L$.

6.1.1. $\mathcal{F}^\star$*-family.* Let us fix an interval $L \subset \partial \widehat{Z}^m$. For a dam $\alpha = \alpha^n_i \subset \widehat{Z}^m$ with $|\alpha \cap L^c| \geq 1$ (i.e., at least one of the endpoints of α is in L^c), define

- $\alpha^{\star L^c} \coloneqq \alpha$ if $|\alpha \cap L^c| = 2$;
- $\alpha^{\star L^c}$ to be the connected component of $A^{\mathrm{inn}}(\alpha) \setminus L^c$ intersecting $\operatorname{int} \widehat{Z}^m$ if $|\alpha \cap L^c| = 1$ (i.e., this connected component is attached to $(L^c)^-$).

We define

$$\left(L^c\right)^\star \coloneqq \text{ filling in of } \left[L^c \cup \bigcup_{|\alpha \cap L^c| \geq 1} \alpha^{\star L^c} \right].$$

We remark that if $\mathfrak{l}_{m_{i+1}} \leq |L| < \mathfrak{l}_{m_i}$, where the m_i are level of regularizations, then $\widehat{Z}^m \setminus \left(L^c\right)^\star$ is within the level m_i peninsula containing L on its boundary.

Finally, we define $\mathcal{F}^\star_L(I,J)$ to be the the set of curves in $\mathcal{F}^\circ_L(I,J)$ that are in $\widehat{\mathbb{C}} \setminus \operatorname{int}(L^c)^\star$.

Lemma 6.3 (Trading $\mathcal{F}^\circ$ into $\mathcal{F}^\star$)**.** *Under the assumption of Lemma 6.1, the family* $\mathcal{F}^\star_L(I,J) \setminus \mathcal{F}^+(I,J)$ *contains a rectangle* $\mathcal{R}$ *such that*

$$\mathcal{W}(\mathcal{R}) = \mathcal{W}^\circ_L(I,J) - \mathcal{W}^+(I,J) - O_{\boldsymbol{\delta}}(1).$$

Proof. Write $K \coloneqq \mathcal{W}^\circ_L(I,J) - \mathcal{W}^+(I,J)$. By Lemma 2.9, $\mathcal{F}^\circ_L(I,J)$ contains a rectangle $\mathcal{R}$ submerging into $\widehat{Z}^m$ with $\mathcal{W}(\mathcal{R}) = K - O(1)$. Let $\mathcal{R}^{\mathrm{new}}$ be the rectangle obtained by removing the $1/\boldsymbol{\delta}$ innermost buffer (attached to $(L^c)^-$) from $\mathcal{R}$. We claim that $\mathcal{R} \subset \widehat{\mathbb{C}} \setminus \left(L^c\right)^\star$.

Assume first that $|\alpha \cap L^c| = 2$. Since $\alpha = \alpha^{\star L^c}$ is attached to $(L^c)^-$, vertical curves in $\mathcal{R}$ intersecting α form a buffer of $\mathcal{R}$, see Lemma A.9. Since curves in this buffer cross A^{inn} after entering $\operatorname{int} \widehat{Z}^m$ through L, the width of the buffer is $\leq 1/\delta$.

Assume now that $|\alpha \cap L^c| = 1$. We **claim** that $A^{\mathrm{out}}(\alpha)$ is disjoint from either I or J. The claim will imply that the width of the buffer formed by curves intersecting $\alpha^{\star L^c}$ (these curves form a buffer of $\mathcal{R}$ by Lemma A.9) is $\leq 1/\boldsymbol{\delta}$ because $\alpha^{\star L^c}$ is separated from either I or J by $A^{\mathrm{out}}(\alpha)$.

Proof of the claim. Write $\alpha = \alpha^n_i = [y_{n-1}, x_n]$ as in Figure 16 and assume that $y_{n-1} \in L^c$; the opposite case is analogous.

Suppose first that $x_n \in \operatorname{int} \widehat{Z}^n$. Then the unique point z in $\partial^c \mathfrak{P}(\alpha_i^n) \cap \mathrm{CP}_n$ is an endpoint of a dam $\beta_s^k \subset \partial \widehat{Z}^m$ for $k < n$, see §5.1.5. Therefore, $A^{\mathrm{out}}(\alpha_i^n) \cap \partial \widehat{Z}^m \subset S_z^{\mathrm{inn}}$. Since I, J are grounded, S_z^{inn} (and hence $A^{\mathrm{out}}(\alpha_i^n)$) is disjoint from either I or J.

Suppose now that $x_n \in L$. Since L is grounded, it also contains β_i^n – the dam after α_i^n. We obtain that β_i^n separates S_{x_n} from J while N separates $S_{y_{n-1}}$ from J because $|N| \geq \mathfrak{l}_m$. Therefore, $A^{\mathrm{out}}(\alpha_i^n)$ separates $\widetilde{A}^{\mathrm{inn}}(\alpha_i^n)$ from J. □

□

Lemma 6.4 (X vs X^{grnd})**.** *Consider a family $\mathcal{F}_L^\star(I,J)$ from Lemma 6.3. Then for every interval $X \subset L$, there are at most $O_{\boldsymbol{\delta}}(1)$ curves in $\mathcal{F}_L^\star(I,J)$ intersecting $X \setminus X^{\mathrm{grnd}}$.*

Combined with Lemma 6.3, there are at most $O_{\boldsymbol{\delta}}(1)$ curves in $\mathcal{F}^\circ(I,J)$ intersecting $X \setminus X^{\mathrm{grnd}}$.

Proof. Let us start the proof with the following two properties.

Claim 1. *Let α be a channel such that either both endpoints of α are in L or one of the endpoints of α is in L and the second endpoint is in $\operatorname{int} \widehat{Z}^m$. Then at most $O_{\boldsymbol{\delta}}(1)$ curves in $\mathcal{F}_L^\star(I,J)$ intersect $\widetilde{A}^{\mathrm{inn}}(\alpha)$.*

Proof of the claim. Assume first that both endpoints of α are in L. Then we have $\partial^c \mathfrak{P}(\alpha) \subset L$. Since L is grounded, it also contains two dams attached to α. Therefore, $\widetilde{A}^{\mathrm{out}}(\alpha)$ is disjoint from $I \cup J$. The claim now follows from $\operatorname{mod} A^{\mathrm{out}}(\alpha) \geq \boldsymbol{\delta}$.

Assume that one of the endpoints of α is in $\operatorname{int} Z$. Then the unique point z in $\partial^c \mathfrak{P}(\alpha_i^n) \cap \mathrm{CP}_n$ is an endpoint of a dam $\beta_s^k \subset \partial \widehat{Z}^m$ for $k < n$, see §5.1.5. We have $A^{\mathrm{out}}(\alpha_i^n) \cap \partial \widehat{Z}^m \subset S_z^{\mathrm{inn}}$ and $A^{\mathrm{out}}(\alpha_i^n)$ separates $\widetilde{A}^{\mathrm{inn}}(\alpha)$ from I, J. □

Claim 2. *Let $\beta \subset L$ be a dam. Then at most $O_{\boldsymbol{\delta}}(1)$ curves in $\mathcal{F}_L^\star(I,J)$ intersect $S^{\mathrm{inn}}(\beta)$.*

Proof of the claim. Since L is grounded, $A^{\mathrm{inn}}(\beta)$ separates β from $I \cup J$; thus at most $O_{\boldsymbol{\delta}}(1)$ curves in $\mathcal{F}_L^\star(I,J)$ intersect β. Write $\beta = \beta_i^n$; then $S^{\mathrm{inn}}(\beta) \subset S^{\mathrm{inn}}(\alpha_i^n) \cup S^{\mathrm{inn}}(\alpha_{i+1}^n)$. By Lemma 6.3 and Claim 2, at most $O_{\boldsymbol{\delta}}(1)$ curves in $\mathcal{F}_L^\star(I,J)$ intersect $S^{\mathrm{inn}}(\alpha_i^n) \cup S^{\mathrm{inn}}(\alpha_{i+1}^n)$. □

The lemma now follows from Claim 2 because every component (out of at most 2) of $X \setminus X^{\mathrm{grnd}}$ is within $S^{\mathrm{inn}}(\beta)$ for a dam $\beta \subset L$. □

6.1.2. *Proof of Snake Lemma 6.1.* Let $\mathcal{R} \subset \mathcal{F}_L^\star(I,J) \setminus \mathcal{F}^\circ = (I,J)$ with $\mathcal{W}(\mathcal{R}) = K - O_{\boldsymbol{\delta}}(1)$ be a rectangle (a snake) from Lemma 6.3 realizing K. Applying Series Decomposition §2.4 to $\mathcal{R}$, we obtain that $\mathcal{F}(\mathcal{R})$ consequently overflows the laminations

$$\mathcal{F}_a \subset \mathcal{F}^\circ(I, J_a), \quad \Gamma \subset \mathcal{F}^-(J_a, I_b), \quad \mathcal{F}_b \subset \mathcal{F}^\circ(I_b, J), \tag{6.4}$$

where $J_a, I_b \subset L$. Let $J_a^{\mathrm{grnd}}, I_b^{\mathrm{grnd}}$ be the biggest grounded intervals in J_a, I_b, see §5.6. By Lemma 6.4, the width of vertical curves in $\mathcal{R}$ intersecting $\big(J_a \setminus J_a^{\mathrm{grnd}}\big) \cup \big(I_b \setminus I_b^{\mathrm{grnd}}\big)$ is $O_{\boldsymbol{\delta}}(1)$; removing these curves from $\mathcal{R}$ and their restrictions from the laminations in (6.4), we obtain that the new rectangle $\mathcal{R}^{\rm new}, \mathcal{W}(\mathcal{R}^{\rm new}) \geq K - O_{\boldsymbol{\delta}}(1)$ such that $\mathcal{F}(\mathcal{R}^{\rm new})$ consequently overflows

$$\text{(6.5)}\quad \mathcal{F}^{\rm new}_a \subset \mathcal{F}^{\circ}(I, J^{\rm grnd}_a), \qquad \Gamma^{\rm new} \subset \mathcal{F}^{-}(J^{\rm grnd}_a, I^{\rm grnd}_b), \qquad \mathcal{F}^{\rm new}_b \subset \mathcal{F}^{\circ}(I^{\rm grnd}_b, J).$$

By Localization and Squeezing Lemmas 5.16, 5.17, $J^{\rm grnd}_a, I^{\rm grnd}_b$ contains an innermost pair J_1, I_1 such that

$$|\lfloor J_1, I_1 \rfloor| \leq \frac{1}{5\lambda}\{|I^{\rm grnd}_a|,\ |J^{\rm grnd}_b|\}$$

and up to $O_{\boldsymbol{\delta}}(\log\lambda)$-width the family $\mathcal{F}^{-}(J_a, I_b)$ is in $\mathcal{F}^{-}(J_1, I_1)$:

$$\text{(6.6)}\qquad \mathcal{W}^{-}(J_a \setminus J_1,\ \ I_b) + \mathcal{W}^{-}(J_a,\ \ I_b \setminus I_1) = O_{\boldsymbol{\delta}}(\log\lambda).$$

Let $\mathcal{R}^{\rm New}$ be the lamination obtained from $\mathcal{R}^{\rm new}$ by removing all $\gamma \in \mathcal{F}(\mathcal{R})$ with $\gamma^d_a \notin \mathcal{F}^{-}(J_1, I_1)$ or $\gamma^d_b \notin \mathcal{F}^{-}(J_1, I_1)$. Then $\mathcal{W}(\mathcal{R}^{\rm New}) = K - O_{\boldsymbol{\delta}}(\log\lambda)$.

Applying the Series Decomposition §2.4 to $\mathcal{R}^{\rm New}$, we obtain that $\mathcal{R}^{\rm New}$ consequently overflows

$$\text{(6.7)}\quad \mathcal{F}^{\rm New}_a \subset \mathcal{F}^{\circ}(I, J^{\rm New}_a), \qquad \Gamma^{\rm New} \subset \mathcal{F}^{-}(J^{\rm New}_a, I^{\rm New}_b), \qquad \mathcal{F}^{\rm New}_b \subset \mathcal{F}^{\circ}(I^{\rm New}_b, J),$$

where $J^{\rm New}_a \subset J_1$, $I^{\rm New}_b \subset I_1$ and $\lfloor J^{\rm New}_a, I^{\rm New}_b \rfloor \subset \lfloor J_1, I_1 \rfloor$. Set $J_2 \coloneqq \left(J^{\rm New}_a\right)^{\rm grnd}$ and $I_2 \coloneqq \left(I^{\rm New}_b\right)^{\rm grnd}$. By Lemma 6.4, the width of curves in $\mathcal{R}^{\rm New}$ intersecting $J^{\rm New}_a \setminus J_2$ or $I^{\rm New}_b \setminus I_2$ is at most $O_{\boldsymbol{\delta}}(1)$. Removing these curves from $\mathcal{R}^{\rm New}$ and their restrictions from the laminations in (6.7), we obtain that the new $\mathcal{R}^{\rm NEW}$ consequently overflows

$$\text{(6.8)}\qquad \mathcal{F}^{\rm NEW}_a \subset \mathcal{F}^{\circ}(I, J_2), \qquad \Gamma^{\rm NEW} \subset \mathcal{F}^{-}(J_2, I_2), \qquad \mathcal{F}^{\rm NEW}_b \subset \mathcal{F}^{\circ}(I_2, J),$$

where $\lfloor J_2, I_2 \rfloor \subset \lfloor J_1, I_1 \rfloor$. Therefore, (6.6) and (6.8) imply the lemma.

Remark 6.5. *Let us summarize the steps in the proof of Lemma 6.1:*

(a) *first we apply Series Decomposition §2.4 to $\mathcal{F}(\mathcal{R})$, see* (6.4)*;*
(b) *then we apply Lemma 6.4 to obtain grounded intervals $J^{\rm grnd}_a, I^{\rm grnd}_b$, see* (6.5)*;*
(c) *then we apply Localization Lemma 5.16 to $J^{\rm grnd}_a, I^{\rm grnd}_b$;*
(d) *then we reapply the Series Decomposition §2.4, see* (6.7)*;*
(e) *finally, we reapply Lemma 6.4 to obtain required* (6.8)*.*

In Steps (b), (c), (e) we remove at most $O_{\boldsymbol{\delta}}(\log\lambda)$ curves from $\mathcal{R}$.

6.1.3. *Submerging laminations.* We can refine the Snake Lemma as follows:

Lemma 6.6. *Under the assumptions of Lemma 6.1, consider a lamination $\mathcal{R} \subset \mathcal{F}^{\circ}_L(I, J) \setminus \mathcal{F}^{+}(I, J)$ with $\mathcal{W}(\mathcal{R}) \gg_{\boldsymbol{\delta}} \log\lambda$. Then there are intervals J_1, I_1 satisfying* (6.1) *and there are laminations $\mathcal{R}_a \subset \mathcal{F}_{L_a}(I, J_1)$, $\mathcal{R}_b \subset \mathcal{F}^{\circ}_{L_b}(I_1, J)$ such that*

$$\mathcal{W}(\mathcal{R}_a) \oplus \mathcal{W}(\mathcal{R}_b) \geq \mathcal{W}(\mathcal{R}) - O_{\boldsymbol{\delta}}(\log\lambda),$$

and such that $\mathcal{R}_a, \mathcal{R}_b$ are restrictions of sublaminations of $\mathcal{R}$.

Proof. By Lemma 6.3, we can remove $O_{\boldsymbol{\delta}}(1)$ buffers from $\mathcal{R}$ so that the new lamination $\mathcal{R}^{\rm new}$ is in $\mathcal{F}^{\star}_L(I, J)$. We apply the argument of §6.1.2 to $\mathcal{R}^{\rm new}$ until (6.8), and then we set $\mathcal{R}_a \coloneqq \mathcal{F}^{\rm NEW}_a$ and $\mathcal{R}_b \coloneqq \mathcal{F}^{\rm NEW}_b$. □

Remark 6.7. *The condition $|N| \geq \mathfrak{l}_m$ in Snake Lemmas 6.1 6.6 can be omitted if the lamination $\mathcal{R}$ in Lemma 6.6 has the following property:*

(X) *for every interval* $X \subset \partial \widehat{Z}_\ell$*, there are at most* $O_{\boldsymbol{\delta}}(1)$ *vertical curves in* $\mathcal{R}$ *intersecting* $X \setminus X^{\mathrm{grnd}}$.

In the proof, Property (X) substitutes Lemma 6.4.

6.2. **Trading $\mathcal{W}$ into $\mathcal{W}^+$.** In thus subsection we will prove Corollary 6.2 as well as several of its variations.

Proof of Corollary 6.2. Snake Lemma 6.1 implies (6.3). Assume that $\mathcal{W}^\circ_{L_a}(I, J_1) \geq 2K - O_{\boldsymbol{\delta}}(\log \lambda)$; the second case is analogous.

If $\mathcal{W}^+(I, J_1) \geq \frac{1}{3}K$, then we set $I^{\mathrm{new}} = J_1^\bullet$ to be the projections of J_1 onto ∂Z. By Lemma 5.10 we have

$$\mathcal{W}^+_{\lambda,Z}(I^{\mathrm{new}}) \geq \mathcal{W}^+_Z(I^\bullet, J_1^\bullet) \geq \mathcal{W}^+_{\widehat{Z}^m}(I, J_1) - O_{\boldsymbol{\delta}}(1) \succeq K.$$

If $\mathcal{W}^+_{L_a}(I, J_1) < \frac{5}{3}K$, then applying Snake Lemma 6.1 again, we find $I_2, J_2 \subset L$ with

$$\mathcal{W}^\circ_{L_2}(I_2, J_2) \geq \frac{3}{2}K \qquad \text{and} \qquad \min\{|I_2|, |J_2|\} < \frac{1}{\lambda}\operatorname{dist}(I_2, J_2).$$

The case $\mathcal{W}^+(I_2, J_2) \geq \frac{1}{3}K$ is treated as above. If $\mathcal{W}^+(I_2, J_2) < \frac{1}{3}K$, then applying Snake Lemma 6.1 again, we find $I_3, J_3 \subset L$ with

$$\mathcal{W}^\circ_{L_3}(I_3, J_3) \geq \left(\frac{3}{2}\right)^2 K \qquad \text{and} \qquad \min\{|I_3|, |J_3|\} < \frac{1}{\lambda}\operatorname{dist}(I_3, J_3),$$

i.e., $\mathcal{W}^\circ_{L_n}(I_n, J_n) \geq \left(3/2\right)^n K$ growth exponentially fast. Since $\widehat{Z}^m$ is a non-uniformly qc disk, the process eventually stops: we obtain $\mathcal{W}^+(I_n, J_n) \geq \frac{1}{3}K$ for some n and grounded intervals I_n, J_n with $\min\{|I_n|, |J_n|\} < \frac{1}{\lambda}\operatorname{dist}(I_n, J_n)$. Lemma 5.10 allows to replace I_n, J_n with their projections onto ∂Z. □

6.2.1. *Scale* $\geq \mathfrak{l}_m$. Trading $\mathcal{W}$ into $\mathcal{W}^+$ is more straightforward if intervals have length $\geq \mathfrak{l}_m$ thanks to Lemma 5.18.

Lemma 6.8. *Let* $I, L \subset \partial \widehat{Z}$ *be disjoint intervals such that*

$$|I| \geq \mathfrak{l}_m, \qquad \operatorname{dist}(I, L) \geq \max\{\mathfrak{l}_{m+1}, |I|/\lambda\}, \qquad |L| \geq \frac{1}{2}, \qquad |L^c| \leq \lambda \mathfrak{l}_{m+1},$$
$$K \coloneqq \mathcal{W}(I, L) - \mathcal{W}^+(I, L) \gg_{\boldsymbol{\delta},\lambda} 1, \qquad \text{where} \qquad \lambda \geq 3.$$

Then there is an interval $I^{\mathrm{new}} \subset L^c \setminus I \subset \partial Z$ *grounded rel* $\widehat{Z}^m$ *such that*

$$|I^{\mathrm{new}}| < \frac{1}{\lambda}|I| \qquad \text{and} \qquad \mathcal{W}^+_\lambda(I^{\mathrm{new}}) \succeq K.$$

Proof. Let $\mathcal{R}$ be the vertical family of $\mathcal{F}(I, L)$, see §A.1.6. By Lemma 5.18, Property (X) in Remark 6.7 holds for $\mathcal{R}$.

Let $\widetilde{I}, \widetilde{L}$ be slight enlargements of $I^{\mathrm{grnd}}, L^{\mathrm{grnd}}$ such that every interval in $\widetilde{I} \setminus I^{\mathrm{grnd}}, \widetilde{L} \setminus L^{\mathrm{grnd}}$ has length $\frac{1}{5\lambda^2}|I|$. Applying Lemma 2.8, we construct intervals $\widehat{I}, \widehat{L}$ with $I \subset \widehat{I} \subset \widetilde{I}$ and $L \subset \widehat{L} \subset \widetilde{L}$ such that there is a restriction $\mathcal{G} \subset \mathcal{F}(\widehat{I}^+, \widehat{L}^+)$ of a sublamination of $\mathcal{R}$ satisfying

- $\mathcal{W}(\mathcal{R}|\mathcal{G}) = \mathcal{W}(I, L) - O(C) = K - O_{\boldsymbol{\delta}}(\log \lambda)$;
- $\mathcal{G}$ is disjoint from the central arc in $\mathcal{F}^-(I, L)$.

Here $C = O_{\boldsymbol\delta}(\log\lambda)$ by Lemmas 5.16 and 5.18.

Let A, B be the complementary intervals to $\widehat I, \widehat L$, and let I_a, I_b be two intervals (possibly empty) in $\widehat I \setminus I$. Similarly, let L_a, L_b be two intervals (possibly empty) in $\widehat L \setminus L$. Since $\mathcal W(I,L) - \mathcal W^+(I,L) = K$, there are two possibilities:

(I) either $\mathcal G$ contains a lamination $\mathcal H$ in $\mathcal F^+\big(I_a, \widehat L\big) \cup \mathcal F^+\big(I_b, \widehat L\big) \cup \mathcal F^+\big(\widehat I, L_a\big) \cup \mathcal F^+\big(\widehat I, L_b\big)$ with $\mathcal W(\mathcal R|\mathcal H) \ge K/3$, see §A.1.9;

(II) or $\mathcal G$ contains a lamination $\mathcal H$ intersecting $A \cup B$ with $\mathcal W(\mathcal R|\mathcal H) \ge K/3$, where A, B are two intervals of $\partial \widehat Z^m \setminus \big(\widehat I \cup \widehat L\big)$.

Case (I) follows from Lemma 5.10 and Property (X) by defining I^{new} to be the projection of either I_a^{grnd}, or I_b^{grnd}, or L_a^{grnd}, or L_b^{grnd} onto ∂Z.

Consider Case (II). Let $\mathcal H_A$ be the lamination of curves in $\mathcal H$ intersecting A before intersecting B. Similar, $\mathcal H_B \subset \mathcal H$ consists of curves intersecting B before A. We have $\mathcal H = \mathcal H_A \sqcup \mathcal H_B$. Below we assume that $\mathcal W(\mathcal H_A) \ge K/6$; the case $\mathcal W(\mathcal H_B) \ge K/6$ is analogous.

Since $\mathcal H_A$ is disjoint from the central arc in $\mathcal F^-(I,J)$, we can restrict $\mathcal H_A$ to the lamination $\mathcal P$ in $\mathcal F^\circ_A\big(\widehat I, \widehat L \cup B\big)$; i.e., $\mathcal P$ consists of the first shortest subcurve γ' of $\gamma \in \mathcal H_A$ such that γ' connects $\widehat I^+$ to $\big(\widehat J \cup B\big)^+$.

Because of Property (X), Snake Lemma 6.1 is applicable to $\mathcal P$, see Remark 6.7. Therefore, there are grounded intervals $J_1 \subset A, L_a$ such that $\mathcal W^\circ_{L_a}(\widehat I^{\text{grnd}}, J_1) \succeq K$ for and $|J_1| < \lambda \operatorname{dist}(J_1, \widehat I^{\text{grnd}})$. Corollary 6.2 applied to $\mathcal F^\circ_{L_a}(I, J_1) \succeq K$ finishes the proof. □

6.3. **Rectangles crossing pseudo-bubbles.** Consider a rectangle $\mathcal R$ and a closed topological disk D such that $\partial^h \mathcal R \subset \widehat{\mathbb C} \setminus D$. Assume that all vertical curves in $\mathcal R$ intersect D. We denote by $x, y \subset \partial D$ the first intersections of $\partial^{v,\ell}\mathcal R, \partial^{v,\rho}\mathcal R$ with D; and let $I = [x,y] \subset \partial D$ be an interval with endpoints x, y. We say that $\mathcal R$ *crosses* D *through* I if for every $\gamma \in \mathcal F(\mathcal R)$

- the first intersection of γ with D is in I; and
- the last intersection of γ with D is in $I^c = \partial D \setminus I$.

Lemma 6.9. *Assume that a rectangle $\mathcal R$, $\mathcal W(\mathcal R) = K$ crosses a pseudo-bubble $\widehat Z_\ell$ (see §5.7) through $I \subset \partial \widehat Z_\ell$. Assume also that either $\partial^{h,0}\mathcal R$ or $\partial^{h,1}\mathcal R$ is disjoint from $\mathcal X(\widehat Z_\ell)$, see* (5.5). *Then one of the following holds for every $\lambda > 2$.*

(I) *There is a grounded interval $B \subset \big[(1+\lambda^{-2})I\big] \setminus I \subset \partial \widehat Z_\ell$ and there is a sublamination $\widetilde{\mathcal Q} \subset \mathcal F(\mathcal R)$ with $\mathcal W(\widetilde{\mathcal Q}) \succeq K - O_{\boldsymbol\delta}(\log\lambda)$ such that the restriction (see §A.1.5) $\mathcal Q$ of $\widetilde{\mathcal Q}$ to the family from ∂D to $\partial^{h,1}\mathcal R$ starts in B.*

(II) *There is a grounded interval $B \subset \big[(1+\lambda^{-2})I\big] \setminus I \subset \partial \widehat Z_\ell$ and there is a lamination $\mathcal Q \subset \mathcal F^+\Big(B, \big[(\lambda B)^c\big]^{\text{grnd}}\Big)$ such that $\mathcal W(\mathcal Q) \succeq K - O_{\boldsymbol\delta}(\log\lambda)$ and such that $\mathcal Q$ is a restriction (see §A.1.5) of a sublaminatrion of $\mathcal R$.*

Proof. Since either $\partial^{h,0}\mathcal R$ or $\partial^{h,1}\mathcal R$ is disjoint from $\mathcal X(\widehat Z_\ell)$, we have

(X) for every interval $X \subset \partial \widehat Z_\ell$, there are at most $O_{\boldsymbol\delta}(1)$ vertical curves in $\mathcal R$ intersecting $X \setminus X^{\text{grnd}}$.

(This is Property (X) of Remark 6.7.) Write

$$\widetilde I := \big[(1+\lambda^{-5})I\big]^{\text{grnd}} \quad \text{and} \quad N := \Big[\partial \widehat Z_\ell \setminus (1+\lambda^{-5})I\Big]^{\text{grnd}}.$$

Let $\mathcal{F}^{\text{new}}$ be the lamination obtained from $\mathcal{F}(\mathcal{R})$ by removing all $\gamma \in \mathcal{F}(\mathcal{R})$ that have a subarc in $\widehat{Z}_\ell$ connecting I and $\widetilde{I}^c$. By Property (X), the width of removed curves is bounded by $\mathcal{F}^-(I^{\text{grnd}}, N)$, and by Squeezing Lemma 5.17, $\mathcal{F}^-(I^{\text{grnd}}, N) = O_{\boldsymbol{\delta}}(\log\lambda)$. Therefore, $\mathcal{W}(\mathcal{F}^{\text{new}}) \geq K - O_{\boldsymbol{\delta}}(\log\lambda)$.

Following notations of §2.4, for every $\gamma \in \mathcal{F}^{\text{new}}$, let

- $\gamma_b^d \subset \widehat{Z}_\ell$ be the last subarc of γ connecting I to $\widetilde{I} \setminus I$; and
- γ_b be the subarc of γ after γ_b^d.

We set

$$\Gamma_b := \{\gamma_b^d \mid \gamma \in \mathcal{F}^{\text{new}}\} \qquad \text{and} \qquad \widetilde{\mathcal{F}}_b := \{\gamma_b \mid \gamma \in \mathcal{F}^{\text{new}}\},$$

where $\mathcal{W}\big(\mathcal{R}|\widetilde{\mathcal{F}}_b\big) \succeq K - O_{\boldsymbol{\delta}}(\log\lambda)$ following conventions of §A.1.9. Let $\widetilde{I}^{\text{new}} \subset \widetilde{I}$ be the subinterval bounded by the leftmost and rightmost endpoints of curves in Γ_b. And let $\mathcal{F}_b$ be the restriction (see §A.1.5) of $\widetilde{\mathcal{F}}_b$ to the family of curves from $\widetilde{I}^{\text{new}}$ to $\partial^{h,1}\mathcal{R}$. Since Γ_b is a lamination, every curve in $\mathcal{F}_b$ starts in $\Big[\widetilde{I}^{\text{new}} \setminus I\Big]^+$.

If a sufficient part of $\mathcal{F}_b$ is outside of $\widehat{Z}_\ell$, then we obtain Case (I) of the lemma (we apply Property (X) to construct a grounded interval $B \subset \widetilde{I}^{\text{new}} \setminus I$).

Assume converse. Then using Property (X) we find a grounded interval $J \subset \widetilde{I}^{\text{new}} \setminus I$ and a sublamination $\mathcal{H}$ of $\mathcal{F}_b$ with $\mathcal{W}(\mathcal{R}|\mathcal{H}) \succeq K - O_{\boldsymbol{\delta}}(\log\lambda)$ such that curves in $\mathcal{H}$ start in J^+ and every $\gamma \in \mathcal{H}$ intersects $\partial\widehat{Z}_\ell \setminus J$.

Let $\mathcal{H}_1$ be the sublamination of $\mathcal{H}$ consisting of $\gamma \in \mathcal{H}$ such that the first intersection of γ with $\partial\widehat{Z}_\ell \setminus J$ is in $[(\lambda J)^c]^{\text{grnd}}$. And we set $\mathcal{H}_2 := \mathcal{H} \setminus \mathcal{H}_1$.

If $\mathcal{W}(\mathcal{R}|\mathcal{H}_1) \geq \mathcal{W}(\mathcal{R}|\mathcal{H}_2)$, then the Case (II) of the lemma is obtained by restricting $\mathcal{H}_1$ to the family $\mathcal{F}^+\Big(J, \big[(\lambda J)^c\big]^{\text{grnd}}\Big)$.

Assume that $\mathcal{W}(\mathcal{R}|\mathcal{H}_1) \leq \mathcal{W}(\mathcal{R}|\mathcal{H}_2)$. Let J_a, J_b be the connected components of $(\lambda J)^{\text{grnd}} \setminus J^{\text{grnd}}$ with $J_a < J < J_b$ in λJ. Let

- $\mathcal{H}_a$ be the set of curves in $\mathcal{H}_2$ intersecting J_a before intersecting $(J_a)^c$; and
- $\mathcal{H}_b$ be the set of curves in $\mathcal{H}_2$ intersecting J_b before intersecting $(J_b)^c$.

By Property (X), at most $O_{\boldsymbol{\delta}}(1)$ curves in $\mathcal{R}$ intersect $(\lambda J) \setminus (\lambda J)^{\text{grnd}}$ before intersecting $(\lambda J)^{\text{grnd}} \cup (\lambda J)^c$. Therefore, either $\mathcal{W}(\mathcal{R}|\mathcal{H}_a) \succeq K - O_{\boldsymbol{\delta}}(\log\lambda)$ or $\mathcal{W}(\mathcal{R}|\mathcal{H}_b) \succeq K - O_{\boldsymbol{\delta}}(\log\lambda)$. Below we will assume that $\mathcal{W}(\mathcal{R}|\mathcal{H}_b) \succeq K - O_{\boldsymbol{\delta}}(\log\lambda)$; the case of $\mathcal{H}_a$ is analogous.

Let $\rho \in \mathcal{H}_b$ be the curve with the rightmost starting point in J, let z be the first intersection of ρ with J_b, and let $J' \subset J_b$ be the subinterval of J_b between J and z. Following notations of §2.4, for every $\gamma \in \mathcal{H}_b$, let

- $\gamma_a^d \subset \widehat{Z}_\ell$ be the first subarc of γ connecting J' to $(J \cup J')^c$; and
- γ_a be the subarc of γ before γ_a^d.

We set

$$\Gamma_a := \{\gamma_a^d \mid \gamma \in \mathcal{H}_b\} \qquad \text{and} \qquad \widetilde{\mathcal{P}} := \{\gamma_a \mid \gamma \in \mathcal{H}_b\},$$

where $\mathcal{W}\big(\mathcal{R}|\widetilde{\mathcal{P}}\big) \succeq K - O_{\boldsymbol{\delta}}(\log\lambda)$. Let $N \subset J'$ be the subinterval bounded by the leftmost starting point of curves in Γ_a and z. We now apply Steps (b), (c), (d), (e) (see Remark 6.5) of §6.1.2, where Property (X) substitutes Lemma 6.4, to localize N and to contract a lamination

$$\mathcal{Q} \subset \mathcal{F}^\circ(J, N), \qquad |N| < \frac{1}{2\lambda}\operatorname{dist}(J, N), \qquad \mathcal{W}(\mathcal{R}|\mathcal{Q}) \succeq K - O_{\boldsymbol{\delta}}(\log\lambda).$$

Applying iteratively Lemma 6.6 and repeating the argument of Corollary 6.2, we can replace $\mathcal{Q}$ with a required $\mathcal{Q}^{\text{new}}$ in the complement of int $\widehat{Z}_\ell$. □

6.4. **Snakes with barriers.** Consider a grounded pair $I, J \subset \widehat{Z}^m$ with $|J| > 1/2$. Let A, B be two complementary intervals between I and J. We assume that the intervals are clockwise oriented as $A < I < B < J$.

Let $\ell_1, \ell_2, \dots, \ell_n$ be pairwise disjoint simple arcs in $\mathbb{C} \setminus \widehat{Z}^m$ such that every ℓ_i connects $a_i \in A$ and $b_i \in B$ with the orientation $a_i < a_{i-1} < b_{i-1} < b_i$ for every $i > 1$. We say that $\ell_1, \dots, \ell_n$ are *barriers* for a lamination $\mathcal{R} \subset \mathcal{F}(I, J)$ if no curves in $\mathcal{R}$ intersect $\bigcup_{i=1}^{n} \ell_i$.

We say that a curve $\gamma \in \mathcal{R}$

- *skips under* $[a_i, a_{i-1}]$ if $\gamma \cap \widehat{Z}^m$ contains a subcurve connecting two different components of $A \setminus [a_i, a_{i-1}]$;
- *skips under* $[b_{i-1}, b_i]$ if $\gamma \cap \widehat{Z}^m$ contains a subcurve connecting two different components of $B \setminus [b_{i-1}, b_i]$;
- *skips under* $[a_i, a_{i-1}] \cup [b_{i-1}, b_i]$ if γ skips under $[a_i, a_{i-1}]$ or under $[b_{i-1}, b_i]$.

Definition 6.10 (Toll barriers)**.** Let $\mathcal{R} \subset \mathcal{F}(I, J)$ be a lamination with barriers $\ell_1, \dots, \ell_n$ as above, where $I, J \subset \partial\widehat{Z}^m$ is a grounded pair with $|J| > 1/2$. Then $\ell_1, \dots, \ell_n$ are *toll barriers for* $\mathcal{R}$ if for all i, no curves in $\mathcal{R}$ skip under $[a_i, a_{i-1}] \cup [b_{i-1}, b_i]$.

Lemma 6.11. *For every $\lambda, \chi > 1$ the following holds. Assume that $\ell_1, \dots, \ell_n$ are barriers for a lamination $\mathcal{R} \subset \mathcal{F}(I, J)$, where $I, J \subset \partial\widehat{Z}$ is a grounded pair with $|J| > 1/2$. Assume moreover, that*

$$1/\chi < \frac{|A|}{|I|}, \frac{|B|}{|I|} < \chi \qquad \textit{and} \qquad |A|, |B| \geq 2\mathfrak{l}_m,$$

where A, B are the complementary intervals between I and J as above.

Then either

- $\operatorname{dist}_{\partial\widehat{Z}^m}(a_i, a_{i-1}) < |I|/\lambda$ *or* $\operatorname{dist}_{\partial\widehat{Z}^m}(b_i, b_{i-1}) < |I|/\lambda$ *for some i with respect to any extension of* $\operatorname{dist}_{\partial\widehat{Z}^m}(\ ,\ )$ *from regular to all points;*
- *or after removing $O_{\chi,n,\boldsymbol{\delta}}(\log\lambda)$-curves from $\mathcal{R}$, we obtain a lamination $\mathcal{R}^{\text{new}}$ for which $\ell_1, \dots, \ell_n$ are toll barriers.*

Let us say that a curve $\gamma \in \mathcal{R}$ *skip under* $[a_{i+1}, a_i] \cup [b_i, b_{i+1}]$ if γ intersects the interval $\{z \mid b_{i+1} < z < a_{i+1}\} \supset J$ before intersecting $[a_{i+1}, a_i] \cup [b_i, b_{i+1}]$.

Proof. If the width of curves skipping under $[a_{i+1}, a_i]$ or under $[b_i, b_{i+1}]$ is at least $C \gg_{\boldsymbol{\delta},\chi} \log\lambda$, then Squeezing Lemma 5.17 applied to $A \setminus [a_{i+1}, a_i]^{\text{GRND}}$ or to $B \setminus [b_i, b_{i+1}]^{\text{GRND}}$ implies that either $\operatorname{dist}_{\partial\widehat{Z}^m}(a_i, a_{i-1}) < |I|/\lambda$ or $\operatorname{dist}_{\partial\widehat{Z}^m}(b_i, b_{i-1}) < |I|/\lambda$ holds. □

Snake Lemma 6.12 (with toll barriers)**.** *Suppose that a lamination $\mathcal{R} \subset \mathcal{F}(I, J)$ has $n \geq 3$ toll barriers, where $I, J \subset \widehat{Z}^m$ is a grounded pair with $|J| > 1/2$. Assume also that $|A|, |B| \geq 2\mathfrak{l}_m$, where A, B are the complementary intervals between I, J.*

Then for every $\lambda > 1$ there is an interval $T \subset \partial Z$ grounded rel $\widehat{Z}^m$ such that

$$\mathcal{W}^+_\lambda(T) \succeq n\mathcal{W}(\mathcal{R}) - O_{\boldsymbol{\delta},n}(\log\lambda).$$

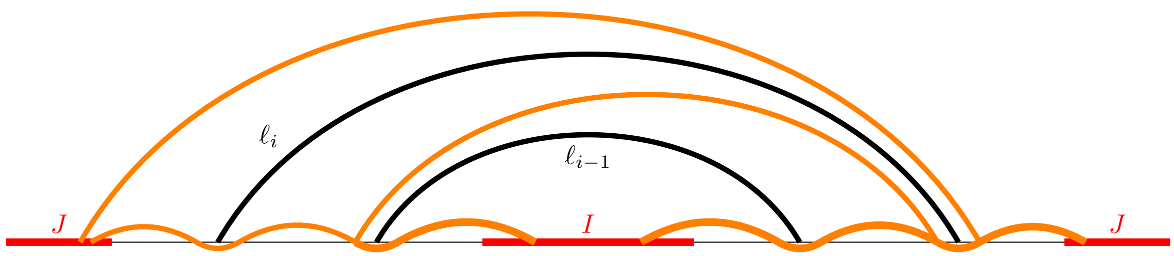


Figure 18. An example of a lamination (orange) with two toll barriers (black).

We will give a proof of Lemma 6.12 after introducing the Series Decomposition for $\mathcal{R}$.

6.4.1. *Series Decomposition for laminations with tall barriers.* The construction below is an adaptation of Series Decomposition §2.4. We assume that $I = [a_0, b_0]$ and $J = [b_{n+1}, a_{n+1}]$ so that $a_{n+1} < a_n < \cdots < a_0 < b_0 < \cdots < b_{n+1}$. Consider a curve $\gamma\colon [0,1] \to \mathbb{C}$ in $\mathcal{R}$. For $i \in \{2, 3 \dots, n\}$, we will define below:

- γ_i^{df} the first passage of γ under $\{a_i, b_i\}$;
- γ_{i-1}^{dl} the last passage of γ under $\{a_{i-1}, b_{i-1}\}$ before γ_i^{df};
- γ_i the subcurve of γ between γ_{i-1}^{dl} and γ_i^{df}.

Then we will specify the laminations $\mathcal{F}_i$ and Γ_i.
Definition of $\gamma_i^{df}, \gamma_{i-1}^{dl}, \gamma_i$. Set:

- $\gamma(\tilde{t}_i) \in \partial \widehat{Z}^m$ to be the first intersection of γ with $[a_{i+1}, a_i] \cup [b_i, b_{i+1}]$;
- $\gamma(t_i) \in \partial \widehat{Z}^m$ to be the last before $\tilde{t}_i$ intersection of γ with $[a_i, a_{i-1}] \cup [b_{i-1}, b_i]$;
- γ_i^{df} to be the subcurve $\gamma \mid [t_i, \tilde{t}_i]$;
- $\gamma(\tau_{i-1}) \in \partial \widehat{Z}^m$ to be the last intersection of γ with $[a_{i-1}, a_{i-2}] \cup [b_{i-2}, b_{i-1}]$ before t_i;
- $\gamma(\tilde{\tau}_{i-1}) \in \partial \widehat{Z}^m$ to be the first after τ_{i-1} intersection of γ with $[a_i, a_{i-1}] \cup [b_{i-1}, b_i]$;
- γ_{i-1}^{dl} to be the subcurve $\gamma \mid [\tau_{i-1}, \tilde{\tau}_{i-1}]$;
- γ_i is the subcurve of γ between γ_{i-1}^{dl} and γ_i^{df}.

By construction, γ_i

- is disjoint from $\ell_{-1} \cup \ell_i$;
- can only submerge into int $\widehat{Z}^m$ thorough $[a_i, a_{i-1}] \cup [b_{i-1}, b_i]$; and
- can not travel between $[a_i, a_{i-1}]$ and $[b_{i-1}, b_i]$ within $\widehat{Z}^m$.

Laminations $\Gamma_i = \Gamma_{a_i} \cup \Gamma_{b_i}$ and $\mathcal{F}_i = \mathcal{F}_{a_i} \cup \mathcal{F}_{b_i}$. Set

$$\widetilde{\mathcal{F}}_i \coloneqq \{\gamma_i \mid \gamma \in \mathcal{R}\} \qquad \text{and} \qquad \Gamma_i \coloneqq \left\{\gamma_i^{df}, \gamma_i^{dl} \mid \gamma\mathcal{R}\right\}.$$

Every curve $\beta \in \Gamma_i$ is either under a_i or under b_i depending on whether β connects $[a_i, a_{i-1}]$ to $[a_{i+1}, a_i]$ or $[b_{i-1}, b_i]$ to $[b_i, b_{i+1}]$. Let $\Gamma_{a_i}, \Gamma_{b_i}$ be the sublaminations of Γ_i consisting of curves that are below a_i and b_i respectively. One of the $\Gamma_{a_i}, \Gamma_{b_i}$ can be empty.

Similarly, we decompose $\widetilde{\mathcal{F}}_i = \widetilde{\mathcal{F}}_{a_i} \cup \widetilde{\mathcal{F}}_{a_i}$ as follows:

- $\widetilde{\mathcal{F}}_{a_i}$ consists of curves $\gamma_i \in \widetilde{\mathcal{F}}_i$ such that $\gamma_i^{df} \in \Gamma_{a_i}$; and
- $\widetilde{\mathcal{F}}_{b_i}$ consists of curves $\gamma_i \in \widetilde{\mathcal{F}}_i$ such that $\gamma_i^{df} \in \Gamma_{b_i}$.

For every a_i and b_i set β_{a_i} and β_{b_i} to be the lowest curves in Γ_{a_i} and Γ_{b_i} and specify the following intervals:

- J_{a_i} to be between the right endpoint of β_{a_i} and a_i,
- I_{a_i} to be between the left endpoint of β_{a_i} and a_i,
- J_{b_i} to be between the left endpoint of β_{b_i} and b_i,
- I_{b_i} to be between the right endpoint of β_{b_i} and b_i.

If Γ_{a_i} (resp Γ_{b_i}) is empty, then J_{a_i}, I_{a_i} (resp J_{b_i}, I_{b_i}) are trivial.

By construction every curve in $\widetilde{\mathcal{F}}_i = \widetilde{\mathcal{F}}_{a_i} \cup \widetilde{\mathcal{F}}_{b_i}$ connects $I_{a_{i-1}} \cup I_{b_{i-1}}$ to $J_{a_i} \cup J_{b_i}$ and is disjoint from barriers ℓ_{i-1}, ℓ_i and arcs $\beta_{a_{i-1}}, \beta_{b_{i-1}}, \beta_{a_i}, \beta_{b_i}$.

We define

$$\mathcal{F}_{a_i} \subset \mathcal{F}^\circ(J_{a_i},\ \lfloor I_{a_{i-1}}, J_{b_i} \rfloor) \qquad \text{and} \qquad \mathcal{F}_{b_i} \subset \mathcal{F}^\circ(\lfloor J_{a_i}, I_{b_{i-1}} \rfloor, J_{b_i}) \tag{6.9}$$

to be the restrictions of $\widetilde{\mathcal{F}}_{a_i}, \widetilde{\mathcal{F}}_{b_i}$ to the associated families; i.e.:

- $\mathcal{F}_{a_i}$ consists of the first shortest subcurves in $\widetilde{\mathcal{F}}_{a_i}$ connecting $\lfloor I_{a_{i-1}}, J_{b_i} \rfloor$ and J_{a_i};
- $\mathcal{F}_{b_i}$ consists of the first shortest subcurves in $\widetilde{\mathcal{F}}_{b_i}$ connecting $\lfloor J_{a_i}, I_{b_{i-1}} \rfloor$ and J_{b_i}.

Series Decomposition. We obtain that $\mathcal{R}$ consistently overflows

$$\Gamma_1 = \Gamma_{a_1} \cup \Gamma_{b_1},\ \ \mathcal{F}_2 = \mathcal{F}_{a_2} \cup \mathcal{F}_{b_2},\ \dots,\ \mathcal{F}_n = \mathcal{F}_{a_n} \cup \mathcal{F}_{b_n},\ \ \Gamma_n = \Gamma_{a_n} \cup \Gamma_{b_n} \tag{6.10}$$

6.4.2. *Proof of Lemma 6.12.* Follows by repeating the steps in the proof of Lemma 6.1, see §6.1.2 and Remark 6.5.

Consider Series Decomposition 6.10. We recall that every $\mathcal{F}_i$ satisfies (6.9).

By Lemma 5.18, at most $O_{\boldsymbol{\delta}}(n)$ curves in $\mathcal{R}$ intersect

$$\bigcup_{i=2}^{n} \Big(\big(I_{a_i} \setminus I_{a_i}^{\text{grnd}}\big) \cup \Big(I_{b_i} \setminus I_{b_i}^{\text{grnd}}\Big) \cup \big(J_{a_i} \setminus J_{a_i}^{\text{grnd}}\big) \cup \Big(J_{b_i} \setminus J_{b_i}^{\text{grnd}}\Big) \Big);$$

removing all such curves from $\mathcal{R}$, we can assume that $I_{a_i}, I_{b_i}, J_{a_i}, J_{b_i}$ are grounded (by replacing them with $I_{a_i}^{\text{grnd}}, I_{b_i}^{\text{grnd}}, J_{a_i}^{\text{grnd}}, J_{b_i}^{\text{grnd}}$).

By Localization and Squeezing Lemmas 5.16, 5.17 applied to

$$\mathcal{F}^-(I_{a_i}, J_{a_i}) \supset \Gamma_{a_i} \qquad \text{and} \qquad \mathcal{F}^-(I_{b_i}, J_{b_i}) \supset \Gamma_{b_i},$$

I_{a_i}, J_{a_i} and I_{b_i}, J_{b_i} have innermost subpairs $I_{a_i}^{\text{new}}, J_{a_i}^{\text{new}}$ and $I_{b_i}^{\text{new}}, J_{b_i}^{\text{new}}$ such that, up to $O_{\boldsymbol{\delta}}(\log\lambda)$, the width of $\mathcal{F}^-(I_{a_i}, J_{a_i}), \mathcal{F}^-(I_{b_i}, J_{b_i})$ is contained in $\mathcal{F}^-(I_{a_i}^{\text{new}}, J_{a_i}^{\text{new}})$, $\mathcal{F}^-(I_{b_i}^{\text{new}}, J_{b_i}^{\text{new}})$ and such that $\lfloor I_{a_i}^{\text{new}}, J_{a_i}^{\text{new}} \rfloor, \lfloor J_{b_i}^{\text{new}}, I_{b_i}^{\text{new}} \rfloor$ are at least 5λ times smaller than $I_{a_i}, J_{a_i}, I_{b_i}, J_{b_i}$ respectively. Removing all curves in $\mathcal{R}$ intersecting

$$\big(I_{a_i} \setminus I_{a_i}^{\text{new}}\big) \cup \big(J_{a_i} \setminus J_{a_i}^{\text{new}}\big) \cup \big(I_{b_i} \setminus I_{b_i}^{\text{new}}\big) \cup \big(J_{b_i} \setminus J_{b_i}^{\text{new}}\big),$$

and then reapplying Series Decomposition 6.10, we obtain that the new J_{a_i} and J_{b_i} have small length compare to their distances to I_{a_i} and I_{b_i} respectively.

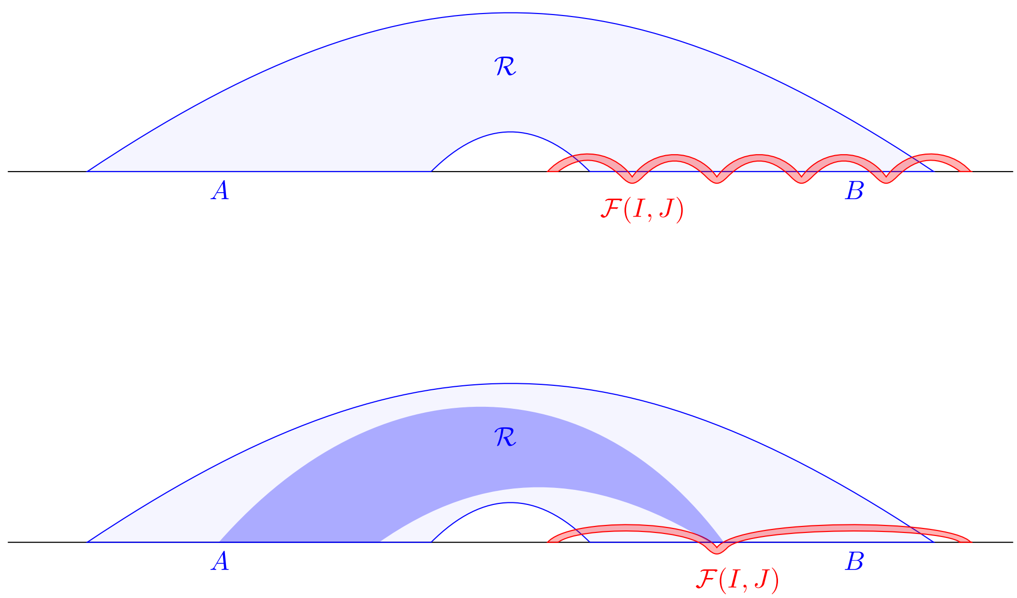


FIGURE 19. Two patterns for $\mathcal{F}(I,J)$ to sneak through $\mathcal{R}$.

Since $\mathcal{R}$ consequently overflows the $\mathcal{F}_i$, there is an i such that

$$\mathcal{W}(\mathcal{F}_i) = \mathcal{W}(\mathcal{F}_{a_i}) + \mathcal{W}(\mathcal{F}_{b_i}) \succeq n\mathcal{W}(\mathcal{R}) - O_{\boldsymbol{\delta},n}(\log \lambda).$$

The lemma now follows by applying Lemma 6.8 to either $\mathcal{F}_{a_i}$ or $\mathcal{F}_{b_i}$ – they satisfy (6.9) and the λ-separation. □

6.5. **Sneaking Lemma.**

Sneaking Lemma 6.13. *Let $\widehat{Z}^m$ be a $\boldsymbol{\delta}$ pseudo-Siegel disk. For all $t, \chi, \lambda > 2$ there is a $\kappa(t) > 2$ such that the following holds.*

Suppose

$$I, J \subset \partial \widehat{Z}^m, \quad |I| \geq \mathfrak{l}_m, \quad |A|, |B| \geq 2\mathfrak{l}_m, \quad |J| > 1/2$$

is a grounded pair and denote by A, B two complementary intervals between I and J; i.e., $A \cup B = \overline{\partial \widehat{Z}^m \setminus (I \cup J)}$. Suppose also that

$$1/\chi \leq \frac{|A|}{|I|}, \frac{|B|}{|I|} \leq \chi.$$

If

- $\mathcal{W}(I,J) =: K \gg_{\chi,\lambda,\boldsymbol{\delta}} 1$*; and*
- $\mathcal{W}^+(A,B) \geq \kappa(t)K$,

then there is a $[tK, \lambda]^+$-wide interval $T \subset \partial Z$ with $|T| < |I|$ such that T is grounded rel $\widehat{Z}^m$.

Proof. Let us denote by $\mathcal{R}$ the canonical rectangle of $\mathcal{F}^+(A,B)$. The idea of the proof is illustrated on Figure 19: either the family $\mathcal{F}_(I,J)$ submerges many times

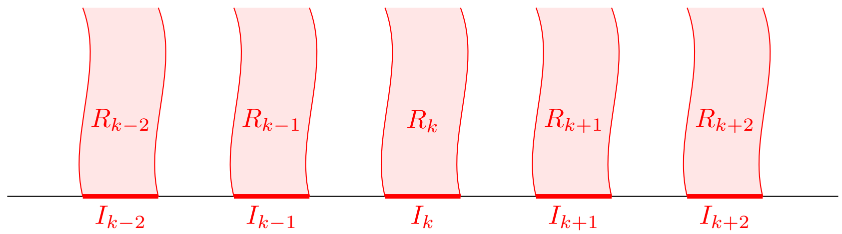


FIGURE 20. Illustration to Lemma 6.14: rectangles R_k do not exist because, otherwise, they would block each other.

in A or B, or a substantial part of $\mathcal{F}^+(A,B)$ is focused; i.e. it starts or terminates in a sufficiently small interval.

Let us select in $\mathcal{R}$ a disjoint union of rectangles

$$\mathcal{R}_1 \sqcup \mathcal{R}_2 \sqcup \cdots \sqcup \mathcal{R}_m, \qquad \text{with} \quad \mathcal{W}(\mathcal{R}_i) = (t+1)K, \qquad \text{and} \quad m \approx \kappa(t)/(t+1).$$

We assume that A_i and B_i is the base and the roof of $\mathcal{R}_i$ respectively, and that they have the following orientation:

$$A_m < A_{m-1} < \cdots < A_1 < B_1 < \cdots < B_m;$$

in particular, $\mathcal{R}_{i+1}$ is above $\mathcal{R}_i$.

By Lemma A.6, we can forget $O_m(1)$- curves in $\mathcal{F}(I,J)$ and we can choose a vertical curve β_i in the inner $O_m(1)$-buffer of every $\mathcal{R}_i$ such that the remaining part $\mathcal{L}$ of $\mathcal{F}(I,J)$ is disjoint from every β_i. We assume that β_i connects $a_i \in A_i$ with $b_i \in B_i$. We denote by $\mathcal{R}_i^{\text{new}}$ the rectangle obtained from $\mathcal{R}_i$ by removing an inner $O(1)$-buffer so that the horizontal sides of $\mathcal{R}_i^{\text{new}}$ are within $[a_{i+1}, a_i] \sqcup [b_i, b_{i+1}]$.

We obtain that β_i are barriers for $\mathcal{L}$ as in §6.4 and that $\mathcal{R}_i^{\text{new}}$ is between β_{i+1} and β_i. By Lemma 6.11, we have two possibilities (depending whether a sufficiently wide part of $\mathcal{L}$ skips under $[a_{i+1}, a_i] \cup [b_i, b_{i+1}]$):

Case I: $[a_{i+1}, a_i]$ or $[b_i, b_{i+1}]$ is smaller than $|I|/\lambda$. Then either $\mathcal{F}_\lambda[a_{i+1}, a_i]$ or $\mathcal{F}_\lambda[b_{i+1}, b_i]$ contains $\mathcal{R}_i^{\text{new}}$. The statement follows from Lemmas 5.10 and 5.18 by setting T to be the projection of either $[a_{i+1}, a_i]^{\text{grnd}}$ or $[b_i, b_{i+1}]^{\text{grnd}}$ on ∂Z.

Case II: we can remove $O_{\lambda,\boldsymbol{\delta},\chi}(1)$-part from $\mathcal{L}$ so that $\beta_1, \dots, \beta_m$ are toll barriers for the remaining $\mathcal{L}^{\text{new}}$. The statement now follows from Snake Lemma 6.12 (with toll barriers). □

6.6. **Families that block each other.** We will need the following simple fact.

Lemma 6.14. *Let $\widehat{Z}^m$ be a pseudo-Siegel disk. There is no sequence of pairwise disjoint intervals*

$$I_n = I_0, I_1, \dots, I_{n-1} \subset \partial \widehat{Z}^m, \qquad \mathcal{W}^+(I_k, L_k^c) \ge 3$$

enumerated either counterclockwise or clockwise such that $I_{k-1} \cup I_k \cup I_{k+1} \subset L_k$.

Proof. Suppose converse. Let $\mathcal{F}_k$ be the canonical rectangle of $\mathcal{F}^+(I_k, L_k^c)$ in $\widehat{\mathbb{C}} \setminus \text{int}(\widehat{Z}^m)$. By removing 1-buffers on each side of $\mathcal{F}_k$, we obtain closed rectangles $R_k \subset \mathcal{F}^+(I_k, L_k^c)$ such that the R_k are disjoint and have width at least 1. This is impossible because R_k block each other, see Figure 20. Indeed, let us choose

$$\left[\gamma\colon [0,1] \to \widehat{\mathbb{C}} \setminus \operatorname{int}(\widehat{Z}^m)\right] \in \bigsqcup_{i=1}^{n} R_k$$

so that $\operatorname{dist}_{\partial \widehat{Z}^m}(\gamma(1), \gamma(0))$ is the minimal possible. Assuming $\gamma \in R_k$ and using $I_{k-1} \sqcup I_{k+1} \subset L_k$, we find an $\ell \in R_{k-1} \sqcup R_k$ with smaller $\operatorname{dist}_{\partial \widehat{Z}^m}(\ell(1), \ell(0))$. □

7. Welding of $\widehat{Z}^{m+1}$ and parabolic fjords

Let $\widehat{Z}^{m+1}$ be a $\boldsymbol{\delta}$-pseudo-Siegel disk. For an interval $J^{m+1} = [x, y] \subset \partial \widehat{Z}^{m+1}$, let $[x, y]_{\widehat{Z}^{m+1}}$ and $[x, y]_{\widehat{\mathbb{C}} \setminus \widehat{Z}^{m+1}}$ be the closed hyperbolic geodesics of $\operatorname{int} \widehat{Z}^{m+1}$ and of $\widehat{\mathbb{C}} \setminus \widehat{Z}^{m+1}$ connecting x and y. Define $O_{J^{m+1}} \supset J^{m+1}$ to be the closed topological disk bounded by $[x, y]_{\widehat{Z}^{m+1}} \cup [x, y]_{\widehat{\mathbb{C}} \setminus \widehat{Z}^{m+1}} =: \partial O_{J^{m+1}}$, see Figure 21.

Consider $T \in \mathfrak{D}_m$, $m \geq -1$ and let T' be as in §2.1.6; i.e. $T' \coloneqq T \cap f^{\mathfrak{q}_{m+1}}(T)$ for $m \geq 0$ with an appropriate adjustment for $m = -1$. Assume that there is a sufficiently wide non-winding parabolic rectangle based on T'. By Theorem 4.1, all such wide rectangles are essentially based on $T_{\text{par}} \subset T'$. Theorem 4.1 also describes the outer geometry of $\overline{Z}$ above T_{par} on scale $\geq \mathfrak{l}_{m+1}$.

Welding Lemma 7.1. *Consider a concatenation of intervals $J = N \# I \# M \subset T_{\text{par}}$ with $|J| < \dfrac{1}{2}|T_{\text{par}}|$, where T_{par} is from Theorem 4.1, such that the endpoints of N, I, M are within CP_{m+1}. Assume that*

$$|N| \asymp |I| \asymp |M|. \tag{7.1}$$

Then there is a constant $\boldsymbol{\varepsilon} > 0$ depending only on "$\asymp$" in (7.1) such that the following holds for all $\lambda > 2$. If $\widehat{Z}^{m+1}$ is a $\boldsymbol{\delta}$-pseudo-Siegel disk and if $\nu \coloneqq |I|/\mathfrak{l}_{m+1} \gg_{\boldsymbol{\delta}, \lambda} 1$, then either

$$\operatorname{mod}\left(O_{J^{m+1}} \setminus I^{m+1}\right) \geq \boldsymbol{\varepsilon} \tag{7.2}$$

holds, where I^{m+1}, J^{m+1} are the projections of I, J onto $\partial \widehat{Z}^{m+1}$, or there is an interval $S \subset \partial Z$ with $|S| < \mathfrak{l}_{m+1}$ such that S is grounded rel $\widehat{Z}^{m+1}$ and

$$\log \mathcal{W}^+_\lambda(S) \succeq \log \nu. \tag{7.3}$$

Remark 7.2. *We emphasize that "$\boldsymbol{\varepsilon}$ and $\succeq$" in (7.2) and (7.3) are independent of $\boldsymbol{\delta}$. Only the scale on which the Welding Lemma works (i.e., how big is ν) depends on $\boldsymbol{\delta}$. This independence of $\boldsymbol{\delta}$ follows from beau coarse-bounds for $\widehat{Z}^{m+1}$ (Theorem 5.12) that are based on beau coarse-bounds for near rotation domains (Theorem 3.8). The independence of $\boldsymbol{\delta}$ implies that the error does not increase during the regularization $\dots \widehat{Z}^{m+1} \rightsquigarrow \widehat{Z}^m \rightsquigarrow \widehat{Z}^{m-1} \rightsquigarrow \dots$ – see Corollary 7.3.*

Recall from §5.1.9 that a regularization $\widehat{Z}^m = Z^{m+1} \cup \widehat{Z}^{m+1}$ is within $\operatorname{orb}_{-\mathfrak{q}_{m+1}+1} \mathcal{R}$ if all relevant objects are within the backward orbit of a rectangle $\mathcal{R}$. The following result is our primary tool of constructing pseudo-Siegel disks.

Corollary 7.3. *There is a sufficiently small $\boldsymbol{\delta} > 0$ with the following properties. Consider $T \in \mathfrak{D}_m$ and let T' be as above. Let $\mathcal{R}$ be a non-winding parabolic rectangle based on T' with $\mathcal{W}(\mathcal{R}) \gg_{\boldsymbol{\delta}} 1$. Let $\widehat{Z}^{m+1}$ be a geodesic $\boldsymbol{\delta}$-pseudo-Siegel disk, see §5.1.9. Then:*

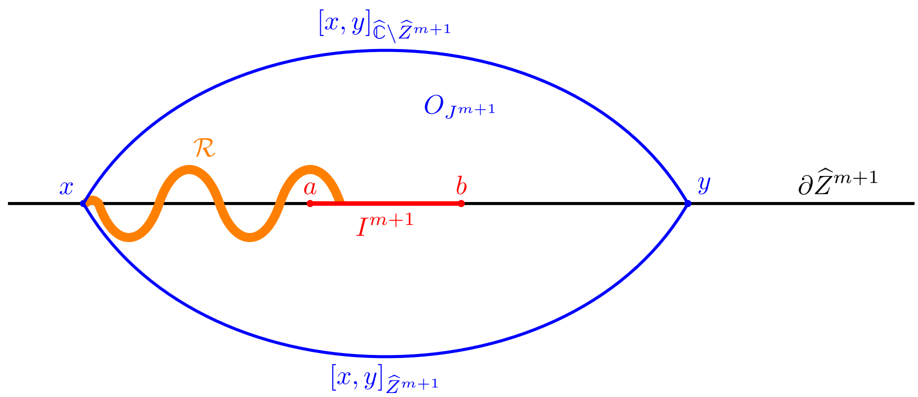


FIGURE 21. The open disk $O_{J^{m+1}}$ is bounded by hyperbolic geodesics in the interior and exterior of $\widehat{Z}^{m+1}$. If $\text{mod}(O_{J^{m+1}} \setminus I^{m+1})$ is small, then there is a wide lamination $\mathcal{R}$ submerging many times into $\widehat{Z}^{m+1}$.

(1) *either there is a geodesic $\boldsymbol{\delta}$-pseudo-Siegel disk $\widehat{Z}^m = Z^m \cup \widehat{Z}^{m+1}$ with its level-m regularization within* $\text{orb}_{-\mathfrak{q}_{m+1}+1}\mathcal{R}$*;*

(2) *or there is an interval*
$$I \subset T', \quad |I| > \mathfrak{l}_{m+1} \quad \textit{such that} \quad \log \mathcal{W}^+_{\lambda,\text{div},m}(I) \succeq \mathcal{W}(\mathcal{R});$$

(3) *or there is a grounded rel $\widehat{Z}^{m+1}$ interval*
$$I \subset \partial Z \quad \textit{with} \quad |I| \le \mathfrak{l}_{m+1} \quad \textit{such that} \quad \log \mathcal{W}^+_\lambda(I) \succeq \mathcal{W}(\mathcal{R}).$$

We refer to Cases (2) and (3) as *exponential boosts.*

Remark 7.4. *Calibration Lemma 9.1 will reduce Case* (2) *to Case* (3)*.*

Remark 7.5. *Starting with (the next) Section 3, we fix a sufficiently small $\boldsymbol{\delta} > 0$ so that Corollary 7.3 is applicable.*

7.0.1. *Outline and Motivation.* Note that we already have a control of

- the outer geometry of $\overline{Z}$ on scale $\ge \mathfrak{l}_{m+1}$ above T_{par} – Theorem 4.1;
- the outer geometry of $\widehat{Z}^{m+1}$ on scale $\ge \mathfrak{l}_{m+1}$ above T^{m+1}_{par} – because the outer geometries of $\overline{Z}$ and $\widehat{Z}^{m+1}$ are close (Lemma 5.8);
- the inner geometry of $\widehat{Z}^{m+1}$ with the estimates depending on $\boldsymbol{\delta}$ – see (5.13) and (5.14) in Theorem 5.12;
- the inner geometry of $\widehat{Z}^{m+1}$ with the estimates independent of $\boldsymbol{\delta}$ on scale $\gg_{\boldsymbol{\delta}} \mathfrak{l}_{m+1}$, see (5.15) and (5.16) in Theorem 5.12.

Therefore, the families $\mathcal{F}^+_{\widehat{Z}^{m+1}}(N^{m+1}, M^{m+1})$ and $\mathcal{F}^-_{\widehat{Z}^{m+1}}(N^{m+1}, M^{m+1})$ have width $\asymp 1$. Since these families (after a slight adjustment) separate I^{m+1} from $\partial O_{J^{m+1}}$, most of the curves in the vertical family $\mathcal{G} := \mathcal{F}(O_{J^{m+1}} \setminus I^{m+1})$ intersect $M^{m+1} \cup N^{m+1}$

(assuming $\mathcal{W}(\mathcal{G}) \gg 1$). Moreover, we should expect that a typical curve in $\mathcal{G}$ submerges $\asymp \nu$ times into $N \cup M$ because we have control of the inner and outer geometries on scale $\geq \mathfrak{l}_{m+1}$. Combined with Lemma 6.8, this would have implied the existence of an interval S with $\mathcal{W}^+_\lambda(S) \succeq \nu$. Our proof gives a somewhat weaker estimate $\log \mathcal{W}^+_\lambda(S) \succeq \log \nu$ (i.e. (7.3)) that is sufficient for our purposes.

The main step in Corollary 7.3 is construction of annuli $A(\alpha_i^m), A(\beta_i^m)$ around channel and dams. The annuli are of the form illustrated on Figure 29: $A = O_\ell \cup \mathcal{Y} \cup O_\rho \setminus \mathcal{X}$. Rectangles $\mathcal{Y}^\pm, \mathcal{X}$ are constructed using Theorems 4.1 and 5.12. To assemble all such rectangles in a chain around $\widehat{Z}^{m+1}$ we need the property that $\mathcal{R}$ contains central subrectangles. If this is not the case, then Lemma 4.12 implies Case (2) of the corollary. If $\mathcal{R}$ contains central subrectangles, then applying the Welding Lemma to construct O_ℓ, O_ρ, we obtain either Case (1) or Case (3) of the corollary.

7.1. **Proof of the Welding Lemma.** Since the endpoints of M, N are within CP_{m+1}, these intervals are well-grounded rel $\widehat{Z}^{m+1}$. By Lemma 5.8, we have $\mathcal{W}^+_{\widehat{Z}^{m+1}}(N^{m+1}, M^{m+1}) \asymp 1$. Since $|N|, |I|, |M| \gg_{\boldsymbol{\delta}} \mathfrak{l}_{m+1}$, we have $\mathcal{W}^-_{\widehat{Z}^{m+1}}(N^{m+1}, M^{m+1}) \asymp 1$ by Theorem 5.12, Equations (5.15) and (5.16).

Let $\mathcal{R}$, $\mathcal{W}(\mathcal{R}) = K := 1/\varepsilon$ be the vertical lamination of the annulus $O_{J^{m+1}} \setminus I^{m+1}$. Let us assume that $K \gg 1$. Let us write

$$F \approx_\ell G \quad \text{if both} \quad F \asymp G \quad \text{and} \quad F = G + O(\ell) \quad \text{hold.}$$

To simplify notations, we will omit below the upper index "$m+1$" for intervals. All intervals will be in $\partial \widehat{Z}^{m+1}$. Let us write

$$N = [x, a], \quad I = [a, b], \quad M = [b, y] \quad \in \partial \widehat{Z}^{m+1}, \qquad N \leq I \leq M \text{ in } T.$$

Let N_1 and M_1 be middle 1/3 well-grounded subintervals of N and M:

$$\mathrm{dist}(x, N_1) \approx_{\mathfrak{l}_{m+1}} |N_1| \approx_{\mathfrak{l}_{m+1}} \mathrm{dist}(N_1, a) \asymp \nu \mathfrak{l}_{m+1}/3,$$
$$\mathrm{dist}(b, M_1) \approx_{\mathfrak{l}_{m+1}} |M_1| \approx_{\mathfrak{l}_{m+1}} \mathrm{dist}(M_1, y) \asymp \nu \mathfrak{l}_{m+1}/3.$$

Claim 1. *At least* $0.99K$ *curves in* $\mathcal{R}$ *intersect* $N_1 \cup M_1$ *before intersecting* $\lfloor N_1, M_1 \rfloor^c \cup \partial^{\mathrm{out}} O_J$*, see Figure 22.*

Proof. Consider the outer and inner geodesic rectangles (see §A.1.12)

$$\mathcal{F}_+ \subset \widehat{\mathbb{C}} \setminus \widehat{Z}^m, \quad \mathcal{F}_- \subset \widehat{Z}^m, \quad \partial^{h,0}\mathcal{F}_+ = \partial^{h,0}\mathcal{F}_- = N_1, \quad \partial^{h,1}\mathcal{F}_+ = \partial^{h,1}\mathcal{F}_- = M_1$$

between N_1 and M_1. Since $|M_1| \asymp |N_1| \asymp \mathrm{dist}(M_1, N_1) \gg_{\boldsymbol{\delta}} \nu \mathfrak{l}_{m+1}$, Theorems 4.1 and 5.12 imply that $\mathcal{W}(\mathcal{F}_-) \asymp \mathcal{W}(\mathcal{F}_+) \asymp 1$. Since $\mathcal{F}_- \cup \mathcal{F}_+$ separate I from $\partial O_{[x,y]}$, most of the curves in $\mathcal{R}$ must intersect $N_1 \cup M_1$ before intersecting $\lfloor N_1 \cup M_1 \rfloor^c \cup \partial^{\mathrm{out}} O_J$. □

Let $X, Y \subset \partial \widehat{Z}^{m+1}$ be a pair of intervals and let A be one of the complementary intervals to X, Y. We denote by $\mathcal{F}_A(X, Y)$ the subfamily of $\mathcal{F}(X, Y)$ consisting of curves that are disjoint from $\partial \widehat{Z}^{m+1} \setminus \big(X \cup A \cup Y\big)$.

Let L_a be the shortest complementary interval between N_1 and I, and let L_b be the shortest complementary interval between I and M_1. Claim 1 implies that either

$$\mathcal{W}_{L_a}(N_1, I) \geq 0.49K \qquad \text{or} \qquad \mathcal{W}_{L_b}(I, M_1) \geq 0.49K.$$

Setting $I_0 := I$ and I_1, L_1 to be either N_1, L_a or M_1, L_b, we obtain $\mathcal{F}_{L_1}(I_1, I_0) \geq 0.49K$. Note that $|L_1| \approx_{\mathfrak{l}_{m+1}} |I_1|$. We can now proceed by induction:

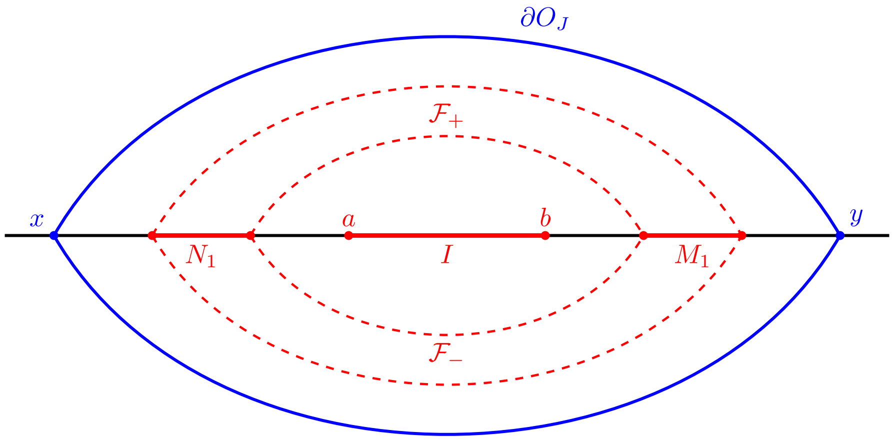


FIGURE 22. Since the families $\mathcal{F}_+$ and $\mathcal{F}_-$ separate I from ∂O_J (compare with Figure 21), most of the curves in the family $\mathcal{R}$ intersect $N_1 \cup M_1$.

Claim 2. *There is a sequence of grounded intervals*

$$I_1, I_2, \dots, I_k, \qquad |I_t| \asymp \nu \mathfrak{l}_{m+1}/3^k, \qquad k \asymp \log \nu, \qquad |I_k| \asymp \mathfrak{l}_{m+1}$$

such that for every I_t there is a $j(t) \in \{0, 1, \dots, t-1\}$ with

$$\mathcal{W}_{L_t}(I_t, I_{j(t)}) \geq 0.49 \cdot 1.9^t K, \qquad |L_t| \approx_{\mathfrak{l}_{m+1}} |I_t| \asymp \nu \mathfrak{l}_{m+1}/3^k,$$

where L_t is the shortest complementary interval between I_t and $I_{j(t)}$. Moreover,

$$\mathcal{W}_{L_t}(I_t, I_{j(t)}) - \mathcal{W}^+_{L_t}(I_t, I_{j(t)}) \geq 0.48 \cdot 1.9^t K \qquad \textit{for } t < k \tag{7.4}$$

and $\operatorname{dist}(I_{k-1}, I_k) \geq 0.6|I_k| \geq 4\mathfrak{l}_{m+1}$.

Proof. Assume that I_t is constructed for $t \geq 1$. Set

$$I_{t+1} \subset L_t \qquad \text{with} \qquad |I_{t+1}| \approx_{\mathfrak{l}_{m+1}} \operatorname{dist}(I_{t+1}, I_t) \approx_{\mathfrak{l}_{m+1}} \operatorname{dist}(I_{t+1}, I_{j(t)})$$

to be a middle 1/3-subinterval of L_t, see Figure 23. Let us show that either

$$\mathcal{W}_{L_a}(I_{t+1}, I_t) \geq 0.49 \cdot 1.9^{t+1} K \qquad \text{or} \qquad \mathcal{W}_{L_b}(I_{t+1}, I_{j(t)}) \geq 0.49 \cdot 1.9^{t+1} K, \tag{7.5}$$

where L_a and L_b are the shortest complementary intervals between I_{t+1}, I_t and $I_{t+1}, I_{j(t)}$. This would finish the construction of I_{t+1} with $j(t+1) \in \{t, j(t)\}$.

Consider a grounded interval $X \subset \partial \widehat{Z}^{m+1}$ attached to I_t so that $|I_t| \approx_{\mathfrak{l}_{m+1}} |X|$ and I_t is between X and I_{t+1} in T. As in Claim 1, consider the outer and inner geodesic rectangles

$$\mathcal{F}_+ \subset \widehat{\mathbb{C}} \setminus \widehat{Z}^m, \quad \mathcal{F}_- \subset \widehat{Z}^m, \quad \partial^{h,0}\mathcal{F}_+ = \partial^{h,0}\mathcal{F}_- = X, \quad \partial^{h,1}\mathcal{F}_+ = \partial^{h,1}\mathcal{F}_- = I_{t+1}$$

between X and I_{t+1}. By Theorems 4.1 and 5.12:

- if $|I_{t+1}| \gg_{\boldsymbol{\delta}} 1$, then $\mathcal{W}(\mathcal{F}_-) \asymp \mathcal{W}(\mathcal{F}_+) \asymp 1$;
- otherwise $\mathcal{W}(\mathcal{F}_-) \asymp_{\boldsymbol{\delta}} 1 \asymp_{\boldsymbol{\delta}} \mathcal{W}(\mathcal{F}_+)$.

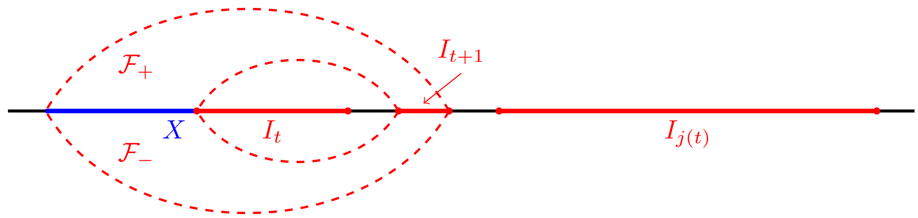


FIGURE 23. The interval I_{t+1} is a middle 1/3 interval between I^t and $I^t_{j(t)}$. Since $\mathcal{F}_-$ and $\mathcal{F}_+$ separate I^t and $I_{j(t)}$, most of the curves in $\mathcal{F}(I_t, I_{j(t)})$ must intersect I_{t+1}.

In the first case, after removing $O(1)$ curves, the family $\mathcal{F}_{L_t}(I_t, I_{j(t)})$ overflows consequently $\mathcal{F}_{L_a}(I_{t+1}, I_t)$ and then $\mathcal{F}_{L_b}(I_{t+1}, I_{j(t)})$. The Equation (7.5) follows.

In the second case, we have $t \gg_{\boldsymbol{\delta}} 1$ because $\nu \gg_{\boldsymbol{\delta}} 1$. Therefore, we can still remove $O_{\boldsymbol{\delta}}(1) \ll \mathcal{W}_{L_t}(I_t, I_{j(t)})$ curves from $\mathcal{F}_{L_t}(I_t, I_{j(t)})$; the remaining family overflows consequently $\mathcal{F}_{L_a}(I_{t+1}, I_t)$ and then $\mathcal{F}_{L_b}(I_{t+1}, I_{j(t)})$. The Equation (7.5) follows.

Note that we also established (7.4) for t. The induction can be proceed until $|I_t| > 20\mathfrak{l}_{m+1}$.

□

The Welding Lemma now follows from Lemma 6.8 applied to (7.4) with $t = k-1$. □

7.2. **Proof of Corollary 7.3.** Write $K \coloneqq \mathcal{W}(\mathcal{R})$ and let $\mathcal{R}^{\text{new}}$ be the rectangle obtained from $\mathcal{R}$ by removing the outermost $K/2$ buffer. If $\mathcal{R}^{\text{new}}$ is not central, then Lemma 4.12 implies Case (2) of the corollary.

Assume that $\mathcal{R}^{\text{new}}$ is central and write $T = [a_0, a_1]$ with $a_0 < \partial^{h,0}\mathcal{R}^{\text{new}} < \partial^{h,1}\mathcal{R}^{\text{new}} < a_1$. By Theorem 4.1,

$$\log \frac{\operatorname{dist}(a_0, \partial^{h,0}\mathcal{R}^{\text{new}})}{\mathfrak{l}_{m+1}}, \quad \log \frac{\operatorname{dist}(\partial^{h,1}\mathcal{R}^{\text{new}}, a_1)}{\mathfrak{l}_{m+1}} \succeq K \tag{7.6}$$

because the removed outermost buffer from $\mathcal{R}$ has width $K/2$. As in §5.1.9, let $\mathcal{R}^{\text{new}}_{-j}$ for $j < \mathfrak{q}_{m+1}$ be the pullback of $\mathcal{R}^{\text{new}}$ along $f^j \colon \overline{Z} ⟲$. Then every $\mathcal{R}^{\text{new}}_{-j}$ is based on a certain $T_i = [a_i, a_{i+1}] \in \mathfrak{D}_m$, $i = t(j)$, where T_i are enumerated from left-to-right. We write $\mathcal{R}^{\text{new}}_{i(j)} = \mathcal{R}^{\text{new}}_{-j}$.

Similarly, by spreading around $\mathcal{R} \setminus \mathcal{R}^{\text{new}}$, we construct a wide rectangle $\mathcal{B}_i$, $\mathcal{W}(\mathcal{B}_i) \asymp K$ based on T_i such that $\mathcal{B}_i$ separates $\mathcal{R}^{\text{new}}_i$ from $\mathcal{K}_m \setminus \overline{Z}$.

Fix a big $S \gg 1$ independent of (and much smaller than) K. We can select intervals $X_i \subset \partial^{h,0}\mathcal{R}^{\text{new}}_i$ and $Y_i \subset \partial^{h,1}\mathcal{R}^{\text{new}}_i$ such that

(A) the endpoints of X_i, Y_i are in CP_{m+1};
(B) all X_i are obtained by spreading around X_0 and, similar, all Y_i are obtained by spreading around Y_0;
(C) $0.99 < \dfrac{|X_i|}{|Y_i|} = \dfrac{|X_0|}{|Y_0|} < 1.01$;
(D) $S \operatorname{dist}(X_i, a_i) < |X_i| < (S+1) \operatorname{dist}(X_i, a_i)$;

(E) $S\operatorname{dist}(Y_i, a_{i+1}) < |Y_i| < (S+1)\operatorname{dist}(Y_i, a_{i+1})$;
(F) the geodesic rectangle $\mathcal{G}_i \subset \widehat{\mathbb{C}}\setminus Z$ between X_i, Y_i splits $\mathcal{R}_i^{\text{new}}$ into two rectangles with width $\asymp K$.

Here, to achieve (D) and (E) we use the property that $\mathcal{R}^{\text{new}}$ is central. Property (F) is achieved by combining Theorem 4.1 with (C), (D) (E). By construction (i.e., by (7.6) and Theorem 4.1), we also have

(F) $\log|X_i|, \log|Y_i| \succeq K$.

By (C), (D), (E) and Theorem 4.1, we have $\mathcal{W}(\mathcal{G}_i) \asymp \log S \gg 1$.

Let X_i^{m+1}, Y_i^{m+1} be the projections of X_i, Y_i onto $\partial \widehat{Z}^{m+1}$. Similar, let $\mathcal{G}_i^{m+1}$ be the restriction of $\mathcal{G}_i$ onto $\widehat{\mathbb{C}} \setminus \operatorname{int} \widehat{Z}^{m+1}$. By (5.12), $\mathcal{W}(\mathcal{G}_i^{m+1}) \asymp \log S \gg 1$.

Let $\mathcal{H}_i \subset \widehat{Z}^{m+1}$ be the geodesic rectangles between Y_{i-1}^{m+1} and X_i^{m+1}. By (C), (D), (E), (F), and Theorem 5.12, we have $\mathcal{W}(\mathcal{H}_i) \asymp \log S \gg 1$. Below we will construct relevant objects for $\widehat{Z}^m$ to satisfy §5.1.9.

7.2.1. *Channels α_i and dams β_i, see Figure 16.* Let us select points $x_i \in X_i \cap \mathrm{CP}_{m+1}$, $y_i \in Y_i \cap \mathrm{CP}_{m+1}$ such that the hyperbolic geodesics $\alpha_i = [y_{i-1}, x_i]_{\widehat{Z}^{m+1}}$ and $\beta_i = [x_i, y_i]_{\widehat{\mathbb{C}}\setminus Z}$ split every $\mathcal{H}_i$ and $\mathcal{G}_i^{m+1}$ into two subrectangles of width $\asymp \log S$ respectively. Moreover, we can choose genuine subrectangles $\mathcal{G}_{i,a}^{m+1}, \mathcal{G}_{i,b}^{m+1}, \mathcal{G}_{i,c}^{m+1}$ in $\mathcal{G}_i^{m+1}$ and genuine subrectangles $\mathcal{H}_{i,a}, \mathcal{H}_{i,b}, \mathcal{H}_{i,c}$ in $\mathcal{H}_i$ such that

- all subrectangles have width $\asymp \log S$;
- $\mathcal{G}_{i,b}^{m+1}$ is between $\mathcal{G}_{i,a}^{m+1}$ and $\mathcal{G}_{i,c}^{m+1}$ and contains β_i in the middle, i.e. β_i splits $\mathcal{G}_{i,b}^{m+1}$ into two subrectangles of width $\asymp \log S$;
- $\operatorname{dist}(\partial^h \mathcal{G}_{i,b}^{m+1}, \partial^h \mathcal{G}_{i,a}^{m+1}), \operatorname{dist}(\partial^h \mathcal{G}_{i,b}^{m+1}, \partial^h \mathcal{G}_{i,c}^{m+1}) \gg \mathfrak{l}_{m+1}$;
- $\mathcal{H}_{i,b}$ is between $\mathcal{H}_{i,a}$ and $\mathcal{H}_{i,c}$ and contains α_i in the middle, i.e. α_i splits $\mathcal{H}_{i,b}$ into two subrectangles of width $\asymp \log S$;
- $\operatorname{dist}(\partial^h \mathcal{H}_{i,b}, \partial^h \mathcal{H}_{i,a}), \operatorname{dist}(\partial^h \mathcal{H}_{i,b}, \partial^h \mathcal{H}_{i,c}) \gg \mathfrak{l}_{m+1}$.

Following (5.4), we denote by $f_*^k(\alpha_i), f_*^k(\beta_i)$ either the f^k-images of α_i, β_i if $k > 0$ or the lifts under f^{-k} starting and ending at ∂Z if $k < 0$, where $|k| \le \mathfrak{q}_{m+1}$. Similarly, $f_*^k(\mathcal{G}_i), f_*^k(\mathcal{H}_i)$ are defined.

Consider $f_*^k(\beta_i)$ for $|k| \le \mathfrak{q}_{m+1}$. Then $f_*^k(\beta_i)$ is in the appropriate $\mathcal{G}_j$ because at most $O(1)$ curves in $f_*^k(\mathcal{G}_{i,b})$ can cross $\mathcal{G}_{j,a}, \mathcal{G}_{j,c}$. Since the $\mathcal{B}_t$ with $\mathcal{W}(\mathcal{B}_t) \asymp K$ separate the $\mathcal{R}_t^{\text{new}}$ from $\mathcal{K}_m \setminus \overline{Z}$, we obtain that $f_*^k(\beta_i)$ is in the $\varepsilon = \varepsilon(K)$ hyperbolic neighborhood of the geodesic of $\widehat{\mathbb{C}} \setminus \overline{Z}$ connecting the endpoints of $f_*^k(\beta_i)$, where $\varepsilon(K) \to 0$ as $K \to \infty$. Therefore, $f_*^k(\beta_i)$ is in $\mathcal{G}_j^{m+1}$ because components of $\mathcal{G}_j \setminus \mathcal{G}_j^{m+1}$ are separated by the $\mathcal{X}_r^n$, see Assumption 6.

Similarly, $f_*^k(\alpha_i) \cap \widehat{Z}^{m+1}$ is in the appropriate $\mathcal{H}_j$ because $f_*^k(\mathcal{H}_{i,b})$ is disjoint from $\partial^h \mathcal{H}_{j,a} \cup \partial^h \mathcal{H}_{j,c}$ by Lemma 5.13, (III), and hence at most $O(1)$ curves in $f_*^k(\mathcal{H}_{i,b})$ can cross $\mathcal{H}_{j,a}, \mathcal{H}_{j,c}$. By Lemma 5.13, (III), $f_*^k(\alpha_i)$ can intersect only components of $f_*^k\left(\widehat{Z}^{m+1}\right) \setminus \widehat{Z}^{m+1}$ that are close to the endpoints of $f_*^k(\alpha_i)$. We conclude that the orbits $f_*^k(\alpha_i), f_*^k(\beta_i)$ with $|k| \le \mathfrak{q}_{m+1}$ are within $\bigcup_i \big(\mathcal{G}_i^{m+1} \cup \mathcal{H}_i^{m+1}\big)$ and Assumption 1 follows.

7.2.2. *Collars* $A(\alpha_i), A(\beta_i)$. By (F), we can choose well-grounded enlargements of intervals

$$X_i = X_{i,0} \subset X_{i,1} \subset X_{i,2} \subset X_{i,3} \subset X_{i,4} \subset \partial^{h,0}\mathcal{R}_i^{\text{new}},$$

$$Y_i = Y_{i,0} \subset Y_{i,1} \subset Y_{i,2} \subset Y_{i,3} \subset Y_{i,4} \subset \partial^{h,1}\mathcal{R}_i^{\text{new}}$$

such that for every $t \in \{1,2,3,4\}$ and every i

- $X_{i,t} \setminus X_{i,t-1}$ consists of a pair of intervals $X_{i,t}^{\pm}$ with $X_{i,t}^{+} < X_{i,t-1} < X_{i,t}^{-}$ such that $|X_{i,t}^{+}| \asymp_S |X_{i,t}^{-}| \asymp_S \text{dist}(X_{i,t}^{+}, a_i) \asymp_S \text{dist}(X_{i,t}^{-}, a_i) \asymp_S |X_i|$;
- $Y_{i,t} \setminus Y_{i,t-1}$ consists of a pair of intervals $Y_{i,t}^{\pm}$ with $Y_{i,t}^{-} < Y_{i,t-1} < Y_{i,t}^{+}$ such that $|Y_{i,t}^{-}| \asymp_S |Y_{i,t}^{+}| \asymp_S \text{dist}(Y_{i,t}^{-}, a_{i+1}) \asymp_S \text{dist}(Y_{i,t}^{+}, a_{i+1}) \asymp_S |Y_i|$.

Taking the projection of the new intervals onto $\partial \widehat{Z}^{m+1}$ and using Theorems 4.1 and 5.12, we obtain that

- the geodesic rectangles $\mathcal{G}_{i,t}^{+}$ and $\mathcal{G}_{i,t}^{-}$ of $\widehat{\mathbb{C}}\setminus\widehat{Z}^{m+1}$ between $(X_{i,t}^{+})^{m+1}$, $(Y_{i,t}^{+})^{m+1}$ and between $(X_{i,t}^{-})^{m+1}$, $(Y_{i,t}^{-})^{m+1}$ have width $\asymp_S 1$;
- the geodesic rectangles $\mathcal{H}_{i,t}^{+}$ and $\mathcal{H}_{i,t}^{-}$ of $\widehat{Z}^{m+1}$ between $(Y_{i-1,t}^{+})^{m+1}$, $(X_{i,t}^{+})^{m+1}$ and between $(Y_{i-1,t}^{-})^{m+1}$, $(X_{i,t}^{-})^{m+1}$ have width $\asymp_S 1$.

Applying Welding Lemma 7.1, we obtain either Case (3) of the corollary, or:

$$\text{mod}(O_{X_{i,t}^{m+1}} \setminus X_{i,t-1}^{m+1}), \quad \text{mod}(O_{Y_{i,t}^{m+1}} \setminus Y_{i,t-1}^{m+1}) \geq \varepsilon = \varepsilon(S). \tag{7.7}$$

We now construct collars

$$A^{\text{inn}}(\alpha_i), \quad A^{\text{out}}(\alpha_i), \quad A^{\text{inn}}(\beta_i), \quad A^{\text{out}}(\beta_i)$$

as annuli bounded by hyperbolic geodesics of $\widehat{\mathbb{C}} \setminus \overline{Z}$ and of $\widehat{Z}^{m+1}$ such that their outer boundaries pass through the endpoints of

$$Y_{i-1,2}^{m+1} \cup X_{i,2}^{m+1}, \quad Y_{i-1,4}^{m+1} \cup X_{i,4}^{m+1}, \quad X_{i,2}^{m+1} \cup Y_{i,2}^{m+1}, \quad X_{i,4}^{m+1} \cup Y_{i,4}^{m+1}$$

while their inner boundaries pass through the endpoints of

$$Y_{i-1,0}^{m+1} \cup X_{i,0}^{m+1}, \quad Y_{i-1,2}^{m+1} \cup X_{i,2}^{m+1}, \quad X_{i,0}^{m+1} \cup Y_{i,0}^{m+1}, \quad X_{i,2}^{m+1} \cup Y_{i,2}^{m+1}$$

respectively. The moduli of the collars are bounded by Lemma A.4 (see also Figure 29) by $\boldsymbol{\delta} = \boldsymbol{\delta}(S)$ because we have bounds on the width of $\mathcal{G}_{i,t}^{\pm}, \mathcal{H}_{i,t}^{\pm}$ and the moduli bounds (7.7).

This verifies Assumptions 2 and 3. Assumption 4 follows from (7.6). Assumption 5 follows from Theorems 4.1 and 5.12.

Extra protections $\mathcal{X}_i^m$ for Assumption 6 can be selected as subrectangles of $\mathcal{R}_i^{\text{new}}$. Assumption 7 and conditions in §5.1.9 hold by construction.

□

Part 3. Covering and Calibration lemmas

8. Covering and Lair Lemmas

In the section, we will prove the following theorem that can be characterized by the principle "if the life is bad now, then it will be worse tomorrow"[1]:

[1]Compare with Kahn's principle: "If the life is bad now, then it was even worse yesterday."

Amplification Theorem 8.1. *There are increasing functions*

$$\boldsymbol{\lambda}_{\mathbf{t}}, \quad \mathbf{K}_{\mathbf{t}} \qquad \textit{for } \mathfrak{t} > 1$$

such that the following holds. Suppose that there is a combinatorial interval

$$I \subset \partial Z \qquad \textit{such that} \quad \mathcal{W}^{+}_{\boldsymbol{\lambda}_{\mathbf{t}}}(I) =: K \geq \mathbf{K}_{\mathbf{t}} \quad \textit{and} \quad |I| \leq |\theta_0|/(2\boldsymbol{\lambda}_{\mathbf{t}}).$$

Consider a geodesic pseudo-Siegel disk $\widehat{Z}^m$*, where* m *is the level of* I*. Then there is a grounded rel* $\widehat{Z}^m$ *interval*

$$J \subset \partial Z \qquad \textit{such that} \quad \mathcal{W}^{+}_{\boldsymbol{\lambda}_{\mathbf{t}}}(J) \geq \mathbf{t}K \quad \textit{and} \quad |J| \leq |I|.$$

8.0.1. *Motivation and outline.* Recall from §2.1.4 that a forward orbit of a combinatorial interval up to the first return almost tiles ∂Z. If a combinatorial interval $I \subset \partial Z$ witnesses a big degeneration, say that I is $[K, \lambda]^+$-wide with $K \gg_\lambda 1$, then, using the Covering Lemma, we spread this degeneration around ∂Z and obtain an almost tiling I_k of ∂Z so that, *roughly* I_k is $[CK, \lambda]$-wide for an absolute $C > 0$. (Covering Lemma 8.5 has two possibilities; we are omitting the "local" Case (1) in this outline.) The constant C is independent of λ; the λ influences only the degeneration threshold $\mathbf{K}_{\mathbf{t}} \gg_\lambda 1$. In short, Snake-Lair Lemma 8.6 states that if $\lambda \gg_{C,\mathbf{t}} 1$, then λ "beats" C and produces a $[\mathbf{t}K, \lambda]^+$-wide interval J on a deeper scale. More precisely, since wide families $\mathcal{F}_\lambda(I_k)$ combinatorially block each other, they must submerge under each other resulting in long snakes. Then Snake Lemmas 6.12 and 6.13 are applicable.

A key technical issue is that the new wide interval J may be far from being combinatorial. Namely, the resulting wide family $\mathcal{F}^+_\lambda(J)$ can be within a wide non-winding parabolic rectangle – such rectangles exist and are described by Theorem 4.1. To deal with this issue, we apply the Covering and Snake-Lair Lemmas to the pseudo-Siegel disk $\widehat{Z}^m$ instead of $\overline{Z}$. Pseudo-Siegel disks are almost invariant up to $\sim \mathfrak{q}_{m+1}$ iterates §5.1.8 – this is sufficient to spread the degeneration around using the Covering Lemma. Lemma 5.10 allows us to trade $\mathcal{W}^+$ wide families between $\overline{Z}$ and $\widehat{Z}^m$.

In Section 10 (see §10.0.1, (a)), we will inductively construct (from the deep to shallow scales) $\widehat{Z}^m$ so that it absorbs "most" of the non-winding parabolic rectangles. Then the Calibration Lemma will replace J with a combinatorial interval on a deeper scale.

8.1. **Applying the Covering Lemma.** As in §5.0.2, we will denote by I^m the projection of a regular interval $I \subset \partial Z$ onto $\partial \widehat{Z}^m$.

Lemma 8.2. *For every* $\boldsymbol{\kappa} > 1$ *and* $\lambda > 10$*, there is* $\mathbf{K}_{\lambda,\boldsymbol{\kappa}} > 1$ *and* C_κ *(independent of* $\boldsymbol{\lambda}$*) such that the following holds. Suppose that there is a combinatorial interval*

$$I \subset \partial Z \qquad \textit{such that} \quad \mathcal{W}^{+}_{\lambda+2}(I) = K \geq \mathbf{K}_{\lambda,\boldsymbol{\kappa}}, \qquad |I| \leq \theta/(2\lambda+4), \qquad m = \mathrm{Level}(I)$$

and such that one of the endpoints of I *is in* CP_m*. Let* $\widehat{Z}^m$ *be a geodesic pseudo-Siegel disk (see §5.1.9), and*

$$I_s \subset \partial Z, \qquad I_s = f^{i_s}(I), \qquad s \in \{0, 1, \dots, \mathfrak{q}_{m+1} - 1\}$$

be the intervals obtained by spreading around $I = I_0$ *(as in §2.1.5). Then every interval* I_s *is well-grounded rel* $\widehat{Z}^{m+1}$ *and its projection* $I^m_s \subset \partial \widehat{Z}^m$ *is*

(1) either $[\boldsymbol{\kappa}K, 10]$*-wide;*
(2) or $[C_{\boldsymbol{\kappa}}K, \lambda]$*-wide.*

Proof. Since one of the endpoints of I is in CP_m, all intervals I_s, $s < \mathfrak{q}_{m+1}$ are well grounded, see the Remark 5.5.

We will start the proof by introducing appropriate branched covering restrictions of the f^{i_s} with uniformly bounded degrees. Then we will apply the Covering Lemma. The condition "$|I| \le |\theta_0|/(2\lambda+4)$" will be used in removing slits.

8.1.1. *Projections onto* $\widehat{Z}^m$. Let us first approximate $10I$ and λI with well-grounded intervals. Choose intervals L and T whose endpoints are in CP_m such that

$$10I \subset T \subset 12I \qquad \text{and} \qquad \lambda I \subset L \subset (\lambda+2)I.$$

Applying f^{i_s} to T and L we obtain the intervals T_s and L_s respectively satisfying

$$10I_s \subset T_s \subset 12I_s \qquad \text{and} \qquad \lambda I_s \subset L_s \subset (\lambda+2)I_s.$$

Then T_s, L_s are well-grounded rel $\widehat{Z}^m$, see (5.6).

8.1.2. *Covering structure around* $f^{i_s} \mid I$. Observe first that $I \subset \partial Z$ contains at most one critical point of f^{i_s} because the map $f^{i_s}\colon I \to I_s$ realizes the first landing of points in I onto I_s, see Lemma 2.1.

Since $|I| \le |\theta_0|/(2\lambda+4) < 1/2$, the interval $(L_s)^c = \partial Z \setminus L_s$ has length greater than $1/2$. Consider a simple arc $\gamma_s \subset \mathbb{C} \setminus \overline{Z}$ connecting $(L_s)^c$ to ∞; we will specify γ_s in §8.1.5. Then

$$V \coloneqq \mathbb{C} \setminus (\gamma_s \cup (L_s)^c) \tag{8.1}$$

is an open topological disk. Define U_{-s} to be the pullback of V along $f^{i_s} \mid I$. We obtain a branched covering

$$f^{i_s}\colon U_{-s} \to V. \tag{8.2}$$

Lemma 8.3. *The degree of* (8.2) *is at most* $4^{\lambda+2}$.

Proof. Let us present (8.2) as the composition of branched coverings

$$U_{-s} = X_0 \xrightarrow{f} X_1 \xrightarrow{f} X_2 \xrightarrow{f} \dots \xrightarrow{f} X_n = V.$$

Observe that $X_j \cap \partial Z$ is the interior of the interval $f^j(L)$. The map $f\colon X_j \to X_{j+1}$ has degree 2 if and only if $f^j(L)$ contains c_0 in its interior. Since $f^{i_s}\colon I \to I_s$ is the first landing, there are at most $2(\lambda+2)$ moments $t \in \{0,1,\dots,i_s\}$ such that $(\lambda+2)f^t(I) \supset f^t(L) \ni c_0$. The lemma follows. □

□

8.1.3. *Covering structure around* $f^{i_s} \mid I^m$. Consider the projection $L_s^m \subset \partial\widehat{Z}^m$ of L_s. Similar to §8.1.2, we choose a simple arc $\gamma_s^m \subset \widehat{\mathbb{C}} \setminus \widehat{Z}^m$ connecting

$$(\widehat{L}_s^m)^c = \partial\widehat{Z}^m \setminus \widehat{L}_s^m \qquad \text{and} \quad \infty.$$

By Lemma 5.4, $\widehat{Z}^m$ has the conformal pullback $f^{i_s}\colon \widehat{Z}^m_{-s} \to \widehat{Z}^m$ such that $\widehat{Z}^m_{-s}$ is also a pseudo-Siegel disk. We denote by $I^{m,-s}$ the projection of I onto $\widehat{Z}^m_{-s}$. Then $I_s^m = f^{i_s}(I^{m,-s})$ is the projection of I onto $\widehat{Z}^m$.

Similar to (8.2), we define the branched covering

$$f^{i_s}\colon U^m_{-s} \to V^m \coloneqq \mathbb{C} \setminus \left(\gamma_s^m \cup \left(\widehat{L}_s^m\right)^c\right), \tag{8.3}$$

where U^m_{-s} is the pullback of V^m along $f^{i_s}\colon I^{m,-s} \to I_s^m$.

Lemma 8.4. *The degree of* (8.2) *is at most* $4^{\lambda+2}$. □

Proof. All critical values of f^{i_s} are in $\partial Z \cap \partial \widehat{Z}^m$ and we can repeat the argument of Lemma 8.3. □

8.1.4. *Covering Lemma.* The Covering Lemma was proven in [KL1]; for our convenience we will state it in terms of the width $\mathcal{W}(A)$ instead of $\operatorname{mod}(A) = 1/\mathcal{W}(A)$ for an annulus A. We will also state the Collar Assumption (1) as one of the alternatives.

Lemma 8.5 (Covering Lemma)**.** *Fix some* $\kappa > 1$. *Let* $U \supset \Lambda' \supset \Lambda$ *and* $V \supset B' \supset B$ *be two nests of Jordan disks. Let*

$$f : (U, \Lambda', \Lambda) \to (V, B', B)$$

be a branched covering between the respective disks, and let $D = \deg(U \to V)$, $d = \deg(\Lambda' \to B')$. *Then there is a* $K_1 > 0$ *(depending on* κ *and* D*) such that the following holds. If*

$$\mathcal{W}(U \setminus \Lambda) > K_1,$$

then either

(1) $\mathcal{W}(B' \setminus B) > \kappa \mathcal{W}(U \setminus \Lambda)$, *or*

(2) $\mathcal{W}(V \setminus B) > \left(2\kappa d^2\right)^{-1} \mathcal{W}(U \setminus \Lambda)$.

Consider (8.3) and recall that $I^{m,-s}$ to be the projection of I onto $\widehat{Z}^m_{-s}$. We denote by $T^m_s \subset \partial \widehat{Z}^m$ the projection of T_s. Set

- $B \coloneqq I^m_s$;
- Λ to be the connected component of $f^{-i_s}(I^m_s)$ containing $I^{m,-s}$;
- $B' \coloneqq \mathbb{C} \setminus \left(\gamma^m_s \cup \left(T^m_s\right)^c\right)$;
- Λ' to be the connected component of $f^{-i_s}(B')$ containing $I^{m,-s}$.

By Lemma 8.4 applied to the case $\lambda = 12$, the degree of $f\colon \Lambda' \to B'$ is at most $d \coloneqq 4^{12}$. Clearly,

$$\mathcal{W}(U^m_{-s} \setminus \Lambda) \geq \mathcal{W}^+_{\widehat{Z}^m_{-s}}(I^{m,-s}) \geq \mathcal{W}^+_Z(I) - O(1) = K - O(1).$$

Applying the Covering Lemma to

$$f^{i_s}\colon (U^m_{-s}, \Lambda', \Lambda) \to (V^m, B', B)$$

with $\kappa = 3\boldsymbol{\kappa}$, we obtain that either

- $\mathcal{W}(B' \setminus B) \geq 3\boldsymbol{\kappa} K$; or
- $\mathcal{W}(V \setminus B) \geq C_{\boldsymbol{\kappa}} K$ otherwise.

8.1.5. *Removing* γ_s. It remains to remove γ_s from

$$V^m = \mathbb{C} \setminus \left(\gamma^m_s \cup \left(L^m_s\right)^c\right) \qquad \text{and} \qquad B' = \mathbb{C} \setminus \left(\gamma^m_s \cup \left(T^m_s\right)^c\right)$$

without decreasing much $\mathcal{W}(B' \setminus B)$ and $\mathcal{W}(V \setminus B)$.

Consider the outer harmonic measure of $\partial \widehat{Z}^m$ – it is the harmonic measure of $\widehat{\mathbb{C}} \setminus \partial \widehat{Z}^m$ relative ∞. If the outer harmonic measure of L^m_s is less than $2/3$, then we can choose γ^m_s so that the width of curves in V^m connecting B and γ_s is $O(1)$. We obtain

$$\mathcal{W}_{10}(I^m_s) \geq \mathcal{W}(B' \setminus B) - O(1), \quad \mathcal{W}_\lambda(I^m_s) \geq \mathcal{W}(V \setminus B) - O(1)$$

and the lemma follows.

Consider the remaining case when the outer harmonic measure of L^m_s is bigger than 2/3. Let $\widehat{Z}^m_\bullet$ be the pullback of $\widehat{Z}^m$ under f; i.e. $\widehat{Z}^m_\bullet$ is the pseudo-Siegel disk so that $f\colon \widehat{Z}^m_\bullet \to \widehat{Z}^m$ is conformal. Let $I'^m, L'^m, \subset \partial \widehat{Z}^m_\bullet$ be the preimages of I^m_s, L^m_s under $f\colon \widehat{Z}^m_\bullet \to \widehat{Z}^m$. By Lemma 5.8, the outer harmonic measures of $L'^m \subset \partial \widehat{Z}^m_\bullet$ and $L^m_s \subset \partial \widehat{Z}^m$ are very close to the outer harmonic measures of $L', L_s \subset \partial Z$, where L' is the projection of L'^m onto $\overline{Z}$. Since L', L_s are disjoint, the outer harmonic measure of $L' \subset \partial \widehat{Z}^m_\bullet$ is less than 2/3. Repeating the above argument for I'^m, we obtain that either

- I'^m is $[2\boldsymbol{\kappa} K, 10]$-wide; or
- I'^m is $[C_{\boldsymbol{\kappa}} K, \lambda]$-wide

relative $\widehat{Z}^m_\bullet$. Applying f which has the global degree 2, we obtain (see (A.8)) that either

- I^m_s is $[\kappa K, 10]$-wide; or
- I^m_s is $[C_{\mathbf{t}} K/2, \boldsymbol{\lambda}_{\mathbf{t}}]$-wide.

8.2. **Lair of snakes.** For our convenience, we enumerate intervals clockwise in the following lemma.

Snake-Lair Lemma 8.6. *For every $\mathbf{t} > 2$ there are $\boldsymbol{\kappa}, \boldsymbol{\lambda}, \mathbf{K} \gg_{\boldsymbol{\delta}} 1$ such that the following holds. Suppose that $\widehat{Z}^m$ is a pseudo-Siegel disk with $\mathfrak{l}_m < \boldsymbol{\lambda}/4$. Let*

$$I_{n+1} = I_0, I_1, \dots, I_n \subset \partial \widehat{Z}^m, \qquad |I_k| = \mathfrak{l}_m, \quad \operatorname{dist}(I_k, I_{k+1}) \le \mathfrak{l}_m$$

be a sequence of well-grounded intervals enumerated clockwise such that every I_s is one of the following two **types**

(1) either I_s is $[\boldsymbol{\kappa} K, 10]$-wide,
(2) or I_s is $[C_{\boldsymbol{\kappa}} K, \boldsymbol{\lambda}]$-wide,

where $K \ge \mathbf{K}$ and C_κ is a constant (from Lemma 8.2) independent of λ. Then there is a $[\mathbf{t}K, 3\boldsymbol{\lambda}]^+$-wide interval $J \subset \partial Z$ grounded rel $\widehat{Z}^m$ with $|J| < |I|$.

Proof. The first three claims below show that families of Type (2) appear with certain frequency. The last three claims amplify their width (by the snake lemmas). We assume that $\mathbf{K} \gg \boldsymbol{\lambda} \gg \boldsymbol{\kappa} \gg \mathbf{t}$. The first claim follows immediately from Lemma 6.8.

Claim 1. *Lemma 8.6 holds if there is a Type* (1) *interval I_j such that*

$$\mathcal{W}_{10}(I_j) - \mathcal{W}^+_{10}(I_j) \ge \boldsymbol{\kappa} K/2.$$

□

We assume from now on that for every Type (1) interval I_j we have

$$\mathcal{W}^+_{10}(I_j) \ge \boldsymbol{\kappa} K/2. \tag{8.4}$$

Let us enlarge every I_i into a well-grounded interval $\widehat{I}_i \subset \partial \widehat{Z}^m$ by adding to I_i the interval between I_i and I_{i+1} if I_i and I_{i+1} are disjoint. Since the distances between the I_i and I_{i+1} are $\le \mathfrak{l}_{m+1}$ (see §2.1.4), we have $|\widehat{I}_i| \le 2\mathfrak{l}_m$.

Claim 2. *There is a sub-sequence*

$$\widehat{I}_i, \widehat{I}_{i+1}, \dots, \widehat{I}_{i+\boldsymbol{\lambda}/20} \subset \partial \widehat{Z}^m, \qquad |\widehat{I}| \le 2\mathfrak{l}_m \tag{8.5}$$

such that every interval $\widehat{I}_j$ in (8.5) *is not $[3, \boldsymbol{\lambda}/4]^+$ wide.*

Proof. Suppose converse. Then $(\widehat{I}_k)$ has a sub-sequence (L_k) with $L_k = \widehat{I}_{\ell(k)}$ such that $\ell(k) < \ell(k+1) \leq \ell(k) + \boldsymbol{\lambda}/20$ and such that every L_k is $[3, \boldsymbol{\lambda}/4]^+$-wide. This is impossible by Lemma 6.14 (such families would block each other). □

Claim 3. *There is* $\mathbf{k}$ *depending on* $\mathbf{t}$ *and* $\boldsymbol{\kappa}$ *but not on* $\boldsymbol{\lambda}$ *such that Lemma 8.6 holds if* (8.5) *has* $\mathbf{k}$ *consecutive Type* (1) *intervals in* (8.5).

Proof. Suppose that (8.5) has a consecutive sequence of Type (1) intervals $I_a, I_{a+1}, \dots, I_b$ with $b - a = \mathbf{k} - 1$. Consider the intervals

$$I_{a,b} \coloneqq \lfloor I_a, I_b \rfloor \qquad \text{and} \qquad L \coloneqq \bigcap_{j=a}^{b} \left(\frac{\boldsymbol{\lambda}}{4} I_j \right)^c$$

and observe that $\mathcal{W}^+(I, L) = O(b-a)$ by Claim 2. Since the $\mathcal{F}^+_{10}(I_j), \mathcal{F}^+_{10}(I_{j+1})$ have small overlaps (they block each other), there is a rectangle

$$\mathcal{R} \subset \mathcal{F} \coloneqq \mathcal{F}^+_{10}(I_a) \cup \mathcal{F}^+_{10}(I_{a+1}) \cup \dots \cup \mathcal{F}^+_{10}(I_b) \qquad \text{with} \quad \mathcal{W}(\mathcal{R}) \succeq (b-a)\kappa K.$$

Let J_a, J_b be two intervals forming $L \setminus I_{a,b}$. We assume that $J_a < I_{a,b} < J_b < L$. Since at most $O(b-a)$ curves in $\mathcal{R}$ land at L, we can select a subrectangle $\mathcal{R}_2$ in $\mathcal{R}$ with $\mathcal{W}(\mathcal{R}) \succeq (b-a)\kappa K = \mathbf{k}\kappa K$ such that, without loss of generality, $\mathcal{R}_2$ is lands at J_a.

By removing $O(1)$-buffer, we can assume that $\mathcal{R}_2$ skips over $I_{a-1} \subset (10 I_a)$. Since Type (1) intervals block each other, $\mathcal{R}_2$ goes above a Type (2) interval $I_x \subset J_a$. The claim now follows by applying Sneaking Lemma 6.13 to $\mathcal{R}_2$ and $\mathcal{F}(I_x)$. □

We may now assume that among $\mathbf{k}$ consecutive intervals in Sequence (8.5) there is at least one Type (2) interval. Let us enumerate Type (2) intervals in Sequence (8.5) as

$$I_{i_0}, I_{i_2}, \dots, I_{i_s}, \qquad i_j < i_{j+1} < i_j + \mathbf{k},$$

where $s \geq \boldsymbol{\lambda}/(22\mathbf{k})$.

Let us enlarge I_{i_t} to well grounded intervals $\widetilde{I}_{i_t} \supset I_{i_t}$ such that

- $\widetilde{I}_{i_t}$ ends where $I_{i_{t+1}}$ starts; and
- $\widetilde{I}_{i_0}$ and $\widetilde{I}_{i_s}$ have length between $\boldsymbol{\lambda}/4 + 1$ and $\boldsymbol{\lambda}/4 + 3$.

It follows from Claim 2 that most of the curves in $\mathcal{F}^+_{\boldsymbol{\lambda}}(\widetilde{I}_{i_t})$ do not bypass $\widetilde{I}_{i_0} \cup \widetilde{I}_{i_s}$:

Claim 4. *Write* $L \coloneqq \lfloor \widetilde{I}_{i_0}, \widetilde{I}_{i_s} \rfloor$. *Then for every* $i \in \{1, \dots, s-1\}$, *we have*

$$\mathcal{W}^+(\widetilde{I}_{i_t}, L) = O(\mathbf{k}) \qquad \textit{for } 1 \leq t \leq s-1.$$

□

Choose a big $T \gg 1$ (but still much smaller than $\mathbf{K}$; the T depends on $\mathfrak{r}, \mathbf{g}$ from Claims 5, 6). We consider the following fundamental arc diagram $\mathfrak{G}$:

- the vertices of $\mathfrak{G}$ are the intervals $\widetilde{I}_{i_j}$ for $j \in \{1, 2, \dots, s-1\}$.
- there is an edge between $\widetilde{I}_{i_a}$ and $\widetilde{I}_{i_b}$ if and only if

$$|a - b| \geq 2 \qquad \text{and} \qquad \mathcal{W}^+(\widetilde{I}_{i_a}, \widetilde{I}_{i_b}) \geq T.$$

Claim 5. *There is a constant* $\mathfrak{r} = \mathfrak{r}(\mathbf{t})$ *depending on* $\mathbf{t}$ *and* $\boldsymbol{\kappa}$ *but not on* $\boldsymbol{\lambda}$ *such that Lemma 8.6 holds if* $\mathfrak{G}$ *has a vertex with degree* $\mathfrak{r}$.

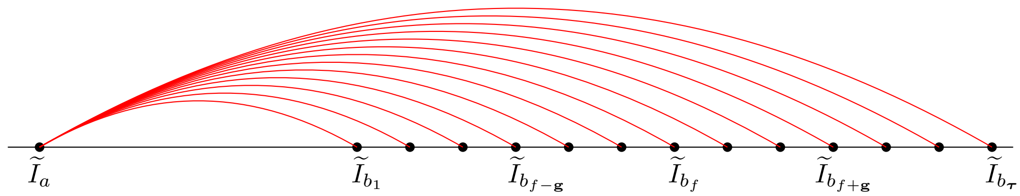


Figure 24. Illustration to the argument in Claim 5.

Proof. For our convenience, we replace $\mathfrak{r}$ with $2\mathfrak{r}$ and we also introduce a big constant $\mathbf{g}$ with $\mathfrak{r} \gg \mathbf{g} \gg 1$.

Assume that $\widetilde{I}_a$ is a vertex of $\mathfrak{G}$ with degree at least $2\mathfrak{r}$. Without loss of generality, we assume that $\mathfrak{r}$ neighbors of I_a in $\mathfrak{G}$ are on the right of I_a; we enumerate these neighbors as $\widetilde{I}_{b_1}, \widetilde{I}_{b_2}, \dots, \widetilde{I}_{b_\mathfrak{r}}$, see Figure 24. We will show below that either there is an $\widetilde{I}_{b_f}$ such that $\mathcal{F}_{\boldsymbol\lambda}(I_{b_k})$ submerges many times in the I_{b_k}, or the family $\mathcal{F}^+\left(\widetilde{I}_a, \lfloor \widetilde{I}_{b_2}, \widetilde{I}_{b_\mathfrak{r}} \rfloor\right)$ has width $\gg_\mathbf{t} K$. In the former case, we will apply the Snake Lemma with toll barriers. In the latter case, we will use the Sneaking Lemma.

Consider $\widetilde{I}_{b_f}$, where $f \in \{\mathbf{g}, \mathbf{g}+1, \dots, \mathfrak{r}-\mathbf{g}\}$, see Figure 24. Observe that the width of curves in $\mathcal{F}_{\boldsymbol\lambda}(I_{b_f})$ omitting $\widetilde{I}_{b_{f-\mathbf{g}}} \cup \widetilde{I}_{b_{f+\mathbf{g}}} \cup \widetilde{I}_a$ is at most $O_{\boldsymbol\lambda}(1)$ because

$$\mathcal{F}^+(\widetilde{I}_a, \widetilde{I}_{b_{f-\mathbf{g}}}), \qquad \mathcal{F}^+(\widetilde{I}_a, \widetilde{I}_{b_{f+\mathbf{g}}}), \qquad \mathcal{F}^-(\widetilde{I}_{b_{f-\mathbf{g}}}, \widetilde{I}_{b_{f+\mathbf{g}}})$$

have $\succeq_{\boldsymbol\lambda} 1$ width. We orient curves in $\mathcal{F}_{\boldsymbol\lambda}(\widetilde{I}_{b_f})$ from $\widetilde{I}_{b_f}$ towards $(\boldsymbol\lambda \widetilde{I}_{b_f})^c$.

Case (A). Suppose there is an $f \in \{\mathbf{g}, \mathbf{g}+1, \dots, \mathfrak{r}-\mathbf{g}\}$ such that a $\dfrac{1}{2}C_{\boldsymbol\kappa} K$ part of $\mathcal{F}_{\boldsymbol\lambda}(I_{b_f})$, call it $\mathcal{F}$, intersects $\widetilde{I}_{b_{f-\mathbf{g}}} \cup \widetilde{I}_{B_{f+\mathbf{g}}}$ before intersecting $\widetilde{I}_a$. By the Small Overlapping Principle §A.2.2, there are pairwise disjoint simple closed arcs

$$\ell_j \in \mathcal{F}^+(\widetilde{I}_a, \widetilde{I}_{f+j}) \qquad \text{for} \quad 1 \le |j-f| \le \mathbf{g}$$

such that at most $O_{\boldsymbol\lambda}(1)$ curves in $\mathcal{F}$ intersect $\displaystyle\bigcup_{1\le|j-f|\le\mathbf{g}} \ell_j$. Removing $O_{\boldsymbol\lambda}(1)$ curves in $\mathcal{F}$ we obtain a lamination $\mathcal{F}^{\text{new}} \subset \mathcal{F}$ whose curves are disjoint from any ℓ_j.

Suppose ℓ_j lands at $x_j \in \widetilde{I}_{b_j}$. Since $\operatorname{dist}(x_j, x_{j+2}) \ge \mathfrak{l}_m$, we can remove $O_{\boldsymbol\lambda}(1)$-curves from $\mathcal{F}^{\text{new}}$ so that every curve in the new family $\mathcal{F}^{\text{New}}$ intersects

$$\lfloor x_{b_{f-2j-2}}, x_{b_{f-2j}} \rfloor \cup \lfloor x_{b_{f+2j}}, x_{b_{f+2j+2}} \rfloor$$

before intersecting $\lfloor x_{b_{f-2j-2}}, x_{b_{f+2j+2}} \rfloor^c$. For $|j| \le \mathbf{g}/2$, define the arc $\beta'_j \subset \ell_{f-2j} \cup \widetilde{I}_a \cup \ell_{f+2j}$ to be the concatenation of ℓ_{f-2j}, followed by the subarc of $\widetilde{I}_a$, and followed by ℓ_{f+2j}. Note that β'_j is a simple arc connecting $\widetilde{I}_{f-2j}$ and $\widetilde{I}_{f+2j}$. Moreover, β'_j is disjoint from β'_k away from $\widetilde{I}_a$. Let us slightly move the arcs β'_j away from $\widetilde{I}_a$ so that the new arcs β_j are pairwise disjoint and so that at most $O(1)$ curves in $\mathcal{F}^{\text{New}}$ intersect any of β_j. We denote by $\mathcal{F}^{\text{NEW}}$ the family obtained from $\mathcal{F}^{\text{New}}$ by removing curves intersecting at least one β_j. Case (A) now follows from Snake

Lemma 6.12 applied to $\mathcal{F}^{\rm NEW}$, $\mathcal{W}(\mathcal{F}^{\rm NEW}) \geq \dfrac{1}{2}K - O_{\boldsymbol\lambda}(1)$ with toll barriers β_j, $j \leq \mathbf{g}/2$.

Case (B). If Case (A) never occurs, then $C_\kappa K/2$-wide part of $\mathcal{F}^+(\widetilde I_a, \widetilde I_{b_f})$ is disjoint from $\mathcal{F}^+(\widetilde I_a, \widetilde I_{b_{f\pm\mathbf{g}}})$. Applying the Parallel Law A.1.4, we obtain that

$$\mathcal{F} \coloneqq \mathcal{F}^+\left(\widetilde I_a, \lfloor \widetilde I_{b_2}, \widetilde I_{b_{\mathbf{k}}}\rfloor\right) \qquad \text{has width} \quad \succeq \frac{\mathfrak{r}}{\mathbf{g}} C_{\boldsymbol\kappa} K \gg_{\mathbf{t}} K.$$

We now apply Sneaking Lemma 6.13 to $\mathcal{F}$ and $\mathcal{F}_\lambda(I_{b_1})$. □

Claim 6. *Suppose that the degree of every vertex in $\mathfrak{G}$ is bounded by $\mathfrak{r}$ from Claim 5. For every $\mathbf{g} \gg 1$, if $\boldsymbol\lambda \gg_{\mathbf{g}} 1$, then the following holds. There is an interval $\widetilde I_s$ such that $\mathcal{F}_{\boldsymbol\lambda}(\widetilde I_s)$ contains a lamination $\mathcal{F}$, $\mathcal{W}(\mathcal{F}) \geq C_{\boldsymbol\kappa} K - O_{\boldsymbol\lambda}(1)$ that has toll barriers $\ell_1, \ell_2, \dots, \ell_{\mathbf{g}}$.*

Proof. Consider the dual graph $\mathfrak{G}^\vee$:

- vertices of $\mathfrak{G}^\vee$ are faces of $\mathfrak{G}$,
- edges of $\mathfrak{G}^\vee$ are orthogonal to edges of $\mathfrak{G}$.

We denote by X the outermost vertex of $\mathfrak{G}^\vee$ corresponding to the unbounded face. Since $\boldsymbol\lambda \gg_{\mathbf{g}} 1$, it is easy to check that either:

(1) there is a simple path in $\mathfrak{G}^\vee$ of length $3\mathfrak{r}\mathbf{g}$ starting at X, or
(2) there is a face Y of $\mathfrak{G}$ (a vertex of $\mathfrak{G}^\vee$) containing at least $3\mathbf{g}$ vertices of $\mathfrak{G}$.

In the first case, we can choose edges $\tilde\ell_i, i \leq \mathbf{g}$ of $\mathfrak{G}$ such that each $\tilde\ell_i$, connects $\widetilde I_{a_i}$ and $\widetilde I_{b_i}$ with

$$a_{i+1} + 1 < a_i, \quad a_i + 1 < s < b_i - 1, \quad b_i < b_{i+1} - 1. \tag{8.6}$$

Since $\mathcal{F}^+(\widetilde I_{a_i}, \widetilde I_{b_i}) \geq T$, Small Overlapping Principle §A.2.2 implies that $\mathcal{F}_{\boldsymbol\lambda}(I_s)$ contains a lamination $\mathcal{F}$ with $\mathcal{W}(\mathcal{F}) \geq C_{\boldsymbol\kappa} K - O_{\boldsymbol\lambda}(1)$ that is disjoint from pairwise disjoint curves $\ell_i \in \mathcal{F}^+(\widetilde I_a, \widetilde I_b)$. Since at most $O_{\boldsymbol\lambda}(1)$ curves in $\mathcal{F}$ can pass under $\widetilde I_{a_i+1}$ or $\widetilde I_{b_i+1}$ (Squeezing Lemma 5.17), we can remove $O_{\boldsymbol\lambda}(1)$ curves from $\mathcal{F}$ such that ℓ_i are toll barriers for the new lamination $\mathcal{F}^{\rm new}$ with $\mathcal{W}(\mathcal{F}^{\rm new}) \geq C_{\boldsymbol\kappa} K - O_{\boldsymbol\lambda}(1)$.

Consider the second case. We can choose intervals $\widetilde I_s, \widetilde I_{a_i}, \widetilde I_{b_i}$ $i \leq \mathbf{g}$ on the boundary of the face Y such that (8.6) holds. Since Y is a face of the arc diagram $\mathfrak{G}$, most curves (up to $O_{\boldsymbol\lambda}(1)$-width) in $\mathcal{F}_{\boldsymbol\lambda}(\widetilde I_s)$ intersect $I_{a_i} \cup I_{b_i}$ before intersecting $I_{a_{i+1}} \cup I_{b_{i+1}}$. Therefore, we can select pairwise disjoint arcs $\ell_i \in \mathcal{F}^+(\widetilde I_{a_i}, \widetilde I_{b_i})$ such that at most $O_{\boldsymbol\lambda}(1)$ curves in $\mathcal{F}_{\boldsymbol\lambda}(\widetilde I_s)$ intersect $\bigcup_i \ell_i$. Since at most $O_{\boldsymbol\lambda}(1)$ curves in $\mathcal{F}_{\boldsymbol\lambda}(\widetilde I_s)$ can pass under $\widetilde I_{a_i+1}$ or $\widetilde I_{b_i+1}$, the family $\mathcal{F}_{\boldsymbol\lambda}(\widetilde I_s)$ contains a lamination $\mathcal{F}$ with $\mathcal{W}(\mathcal{F}) \geq C_{\boldsymbol\kappa} K - O_{\boldsymbol\lambda}(1)$ such that ℓ_i are toll barriers for $\mathcal{F}$. □

The lemma now follows from Snake Lemma 6.12 applied to $\mathcal{F}$ and toll barriers $\ell_1, \ell_2, \dots, \ell_{\mathbf{g}}$. □

Proof of Theorem 8.1. Consider a $[\mathbf{K}, \boldsymbol\lambda_{\mathbf{t}}]^+$ combinatorial interval $I \subset \partial Z$ of level m as in the statement of Theorem 8.1. There are two level m combinatorial intervals I_a, I_b such that $I \subset I_a \cup I_b$ and at least one of the endpoints of I_a and of I_b is in CP_m. Then either I_a or I_b is $[\mathbf{K}/2, \boldsymbol\lambda_{\mathbf{t}}/2]^+$-wide. The theorem now follows from Lemmas 8.2 and 8.6. □

8.3. **Bounded type regime.** Recall that we are considering eventually golden-mean rotation numbers $\theta = [0; a_1, a_2, \dots]$ with $a_n = 1$ for $n \ge \mathbf{n}_\theta$.

Corollary 8.7. *There are absolute constant* $\mathbf{K}, \mathbf{n} > 2$ *such that for every* θ *we have*

$$\mathcal{W}^+_3(I) \le \mathbf{K} \qquad \text{for every interval } I \subset \partial Z \text{ with } |I| \le \mathfrak{l}_{\max\{\mathbf{n}_\theta, \mathbf{n}\}}.$$

Proof. It is sufficient to prove $\mathcal{W}^+_{\boldsymbol{\lambda}}(I) \le \mathbf{K}$ for some $\boldsymbol{\lambda} \ge 3$ and $\mathbf{K}$; the case $\boldsymbol{\lambda} = 3$ follows by spiting I and increasing $\mathbf{K}$. For a sufficiently big $\mathbf{t} \gg 1$, let $\boldsymbol{\lambda} = \boldsymbol{\lambda}_\mathbf{t}$ and $\mathbf{K} = \mathbf{K}_\mathbf{t}$ be the constants from Theorem 8.1. Set $\mathbf{n}$ to be the integer part of $2\log_2(2\boldsymbol{\lambda}) + 2$; then $\mathfrak{l}_\mathbf{n} \le \dfrac{|\theta_0|}{2\boldsymbol{\lambda}_\mathbf{t}}$ by (2.1).

Assume converse: $\mathcal{W}^+_{\boldsymbol{\lambda}}(I) = K \ge \mathbf{K}$ for some I with $|I| \le \mathfrak{l}_{\max\{\mathbf{n}_\theta, \mathbf{n}\}}$. Then I contains a combinatorial subinterval I' with $|I'| \asymp |I|$ such that $\mathcal{W}^+_{\boldsymbol{\lambda}}(I') \succeq K$. Applying Theorem 8.1 with $\mathbf{Z}^m = \overline{Z}$, we find an interval I_2 with $\mathcal{W}(I_2) \succeq \mathbf{t}K \gg K$ and $|I_2| \le |I|$. Continuing the process, we obtain a sequence of interval I_k with $|I_k| \le |I_{k-1}|$ such that $\mathcal{W}^+_{\boldsymbol{\lambda}}(I_k) \to +\infty$. This is impossible. □

9. The Calibration Lemma

Recall from §2.3.2 that the diving family $\mathcal{F}^+_{\lambda,\text{div},m}(I) \subset \mathcal{F}^+_\lambda(I)$ consists of curves intersecting (or diving into) $\mathcal{K}_m \setminus \overline{Z}$.

Calibration Lemma 9.1. *There is an absolute constant* $\boldsymbol{\chi} > 1$ *such that the following holds for every* $\lambda \ge 10$. *Let* $\widehat{Z}^{m+1}$ *be a geodesic pseudo-Siegel disk and consider an interval* $T \subset \partial Z$ *in the diffeo-tiling* $\mathfrak{D}_m$. *If there are intervals*

$$I \subset T, \quad L \subset \partial Z \quad \text{such that} \quad \mathfrak{l}_{m+1} \le |I| \le \mathfrak{l}_m, \quad \operatorname{dist}(I, L^c) \ge \lambda \mathfrak{l}_{m+1}$$
$$\text{and} \qquad \mathcal{W}^+_{\text{div},m}(I, L^c) = \boldsymbol{\chi} K \gg_\lambda 1,$$

then

- (I) *either there is a* $[K, \lambda]^+$*-wide level-*$(m+1)$ *combinatorial interval,*
- (II) *or there is a* $[\boldsymbol{\chi}^{1.5}K, \lambda]^+$*-wide interval* $I' \subset \partial Z$, $|I'| < \mathfrak{l}_{m+1}$ *grounded rel* $\widehat{Z}^{m+1}$.

In applications, we will often take $L = \lambda I$.

9.0.1. *Outline and Motivation.* The Calibration Lemma allows us to trade $\mathcal{W}^+_{\lambda,\text{div}}$-wide intervals into $\mathcal{W}^+_\lambda$-wide combinatorial intervals. In Section 10 (see §10.0.1, (a)), we will construct a pseudo-Siegel disk $\widehat{Z}^{m+1}$ to absorb most of the external families. Therefore, the Calibration Lemma can be inductively applied if Case (II) happens.

The main idea of the proof is illustrated on Figure 26. If Case (I) does not happen, then we should expect a wide rectangle $\mathcal{P}$ that essentially overflows its two conformal pullbacks $\mathcal{P}_-, \mathcal{P}_+$; this leads to the "$1 \le 1 \oplus 1 = 0.5$" contradiction. To construct such a wide rectangle $\mathcal{P}$, we will first spread the family $\mathcal{F}^+_{\lambda,\text{div},m}(I)$ around $\mathfrak{D}_m$ (using univalent push-forwards §2.3.3). Next we will find a wide rectangle $\mathcal{P}$ between two neighboring intervals of $\mathfrak{D}_m$. Finally either the argument illustrated on Figure 26 is applicable, or the roof of $\mathcal{P}$ is shorter than $\mathfrak{l}_{m+1}$ – this leads to Case (II).

9.1. **Proof of Calibration Lemma 9.1.** We split the proof into several subsections. We assume that $\boldsymbol{\chi} \gg 1$ is sufficiently big.

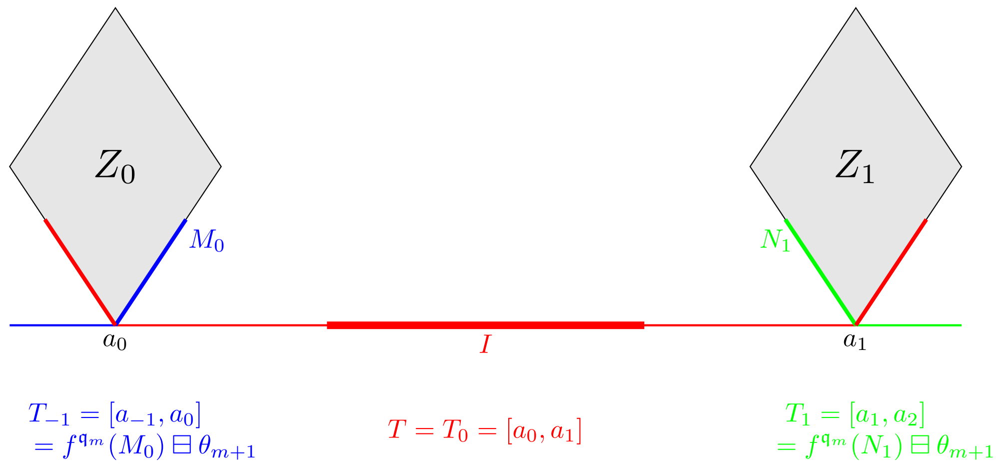


FIGURE 25. Various intervals on ∂Z and ∂Z_i.

9.1.1. *Bubbles Z_i.* We enumerate intervals in $\mathfrak{D}_m$ as T_i, $T_0 = T$ from left-to-right. Denote by a_i the common endpoint of T_{i-1} and T_i. We recall from §2.1.6 that a_i is a critical point of generation $t_i \le \mathfrak{q}_{m+1}$. Thus there is a bubble Z_i attached to a_i, see Figure 25, with the first landing $f^{t_i} \colon Z_i \to Z$.

Let us pullback the diffeo-tiling $\mathfrak{D}_m$ to a partition of ∂Z_i under

$$f^{-\mathfrak{q}_m} \circ f^{\mathfrak{q}_m} \colon \partial Z \to \partial Z_i \qquad \text{(equivalently, under } f^{-t_i} \circ f^{t_i} \colon \partial Z \to \partial Z_i).$$

We specify the following elements of the new partition:

- N_i is the preimage of $T_i \boxplus \theta_{m+1} = f^{\mathfrak{q}_m}(T_{i+1})$ under $f^{\mathfrak{q}_m} \mid \partial Z_i$;
- M_i is the preimage of $T_{i-1} \boxplus \theta_{m+1} = f^{\mathfrak{q}_m}(T_i)$ under $f^{\mathfrak{q}_m} \mid \partial Z_i$.

The point a_i is the common endpoint of M_i, N_i.

By the following properties N_i, M_i are intervals of $\mathcal{K}_m = f^{-\mathfrak{q}_{m+1}}(\overline{Z})$.

Claim 1. *The interiors of M_i, N_{i+1} contain no branched points of $f^{-\mathfrak{q}_{m+1}}(\partial Z)$.*

Proof. By definition, the interiors of the T_i contain no critical points of $f^{\mathfrak{q}_{m+1}}$. Therefore, the interiors of the $f^{\mathfrak{q}_{m+1}}(T_i)$ contain no critical values of $f^{\mathfrak{q}_{m+1}}$. Applying $f^{-\mathfrak{q}_{m+1}}$, we obtain a required claim for the M_i, N_i. □

9.1.2. *Rectangle $\mathcal{R}_i$.* By Lemma 2.7, most of the width of $\mathcal{F}^+_{\text{div},m}(I, L^c)$ is within two rectangles; thus we can select a rectangle $\mathcal{R} \subset \mathcal{F}^+_{\text{div},m}(I, L^c)$ satisfying

$$\mathcal{W}(\mathcal{G}) = \boldsymbol{\chi} K/2 - O(1).$$

We orient the vertical curves in $\mathcal{R}$ from I to L^c. By shrinking I, we can assume that $\partial^{h,0}\mathcal{R} = I$. For an interval $J \subset I$, we denote by $\mathcal{R}_J \subset \mathcal{R}$ the genuine subrectangle consisting of vertical curves of $\mathcal{R}$ starting at J.

Assuming that (I) does not hold, we obtain

(A) For every combinatorial subinterval $J \subset I$ with $|J| = \mathfrak{l}_{m+1}$, we have

$$\mathcal{W}(\mathcal{R}_J) \le K.$$

In particular, $|I| > \boldsymbol{\chi}\mathfrak{l}_{m+1}/3 \gg 1$. Let us present I as a concatenation

$$I = I_a \cup I_0 \cup I_b \qquad \text{with} \qquad I_a < I_0 < I_b$$

such that $\mathcal{W}(\mathcal{R}_{I_a}) = \mathcal{W}(\mathcal{R}_{I_b}) = 6K$. In particular, $|I_a|, |I_b| \geq 6\mathfrak{l}_{m+1}$ by (A).

We set

$$\mathcal{R}_0 \coloneqq \mathcal{R}_{I_0}, \qquad \text{where} \qquad \mathcal{W}(\mathcal{G}_0) = \boldsymbol{\chi}K/2 - O(K).$$

Since $\mathcal{R}_0$ is obtained from $\mathcal{R}$ by removing sufficiently wide buffers, we can spread around $\mathcal{R}_0$ using the univalent push-forward (2.15). We denote by $\mathcal{R}_i$ the resulting image of $\mathcal{R}_0$ in T_i. As for $\mathcal{R}$, we denote by $\mathcal{R}_{i,J}$ the subrectangle of $\mathcal{R}_i$ consisting of curves starting at $J \subset \partial^{h,0}\mathcal{R}_i$.

Claim 2 ((A) holds for all $\mathcal{R}_i$)**.** *For every level $m+1$ combinatorial interval $J \subset \partial^{h,0}\mathcal{R}_i$, we have $\mathcal{W}(\mathcal{R}_{i,J}) \leq K + O(1)$.*

Proof. Assume that $\mathcal{W}(\mathcal{R}_{i,J}) \geq K + C$ for $C \gg 1$. Pushing $\mathcal{R}_{i,J}$ forward under (2.15) towards T_0 we obtain the violation of (A) for $\mathcal{R}$. □

9.1.3. *Almost invariance of $\mathcal{R}_i$.* For every $\mathcal{R}_{i,J}$, let us denote by $\mathcal{R}^*_{i,J}$ the lift of $\mathcal{R}_{i,J}$ under $f^{\mathfrak{q}_{m+1}}$ starting at $J \boxminus \theta_{m+1}$.

Claim 3. *Every $J \subset \partial^{h,0}\mathcal{R}_i$ contains a subinterval $J^{\text{new}} \subset J$ such that*

$$\mathcal{W}(\mathcal{R}_{i,J^{\text{new}}}) = \mathcal{W}(\mathcal{R}_{i,J}) - O(K)$$

and such that $\mathcal{R}_{i,J^{\text{new}}}$ vertically overflows $\mathcal{R}^*_{i,J}$.

Moreover, at most $O(K)$ curves of $\mathcal{R}_i$ land at $T_i \boxplus \theta_{m+1}$.

Proof. Let $\mathcal{R}_{i,J^{\text{new}}}$ be the rectangle obtained from $\mathcal{R}_{i,J}$ by removing two $3K$-buffers. Then the length of each of the intervals in $J \setminus J^{\text{new}}$ is at least $2\mathfrak{l}_{m+1}$ by (A), and at most $O(1)$ curves in $\mathcal{R}_{i,J^{\text{new}}}$ can cross the buffers of $\mathcal{R}^*_{i,J}$ starting at $\big(J \boxminus \theta_{m+1}\big) \setminus J^{\text{new}}$.

Similarly, up to $O(K)$, curves of $\mathcal{R}^*_{i,J}$ are in $\mathcal{R}_{i,J^{\text{new}}}$. Since $\mathcal{R}_{i,J^{\text{new}}}$ is in $\mathcal{F}^+_{\text{div},m}$, we obtain that, up to $O(K)$, curves of $\mathcal{R}^*_{i,J}$ intersect $\partial Z_i \cup \partial Z_{i+1}$. Taking $J = \partial^{h,0}\mathcal{R}_i$ and applying $f^{\mathfrak{q}_{m+1}}$, we obtain the second claim. □

Claim 4 (A rectangle between T_s and $T_{s\pm1}$)**.** *There is an $\mathcal{R}_s$ such that, up to removing $O(K)$ buffers, the roof of $\mathcal{R}_s$ is in $\big(T_{s-1} \cup T_{s+1}\big) \boxplus \theta_{m+1}$.*

Moreover, up to removing $O(K)$ buffers from $\mathcal{R}_s$, we can assume that

$$\operatorname{dist}(\partial^{h,1}\mathcal{R}_s, \{a_{s-1}, a_{s+2}\}) \geq 5\mathfrak{l}_{m+1}.$$

Proof. Let us show that, up to removing $O(K)$ buffers, there is a rectangle $\mathcal{R}_s$ such that $\partial^{h,1}\mathcal{R}_s \subset \lfloor \partial^{h,0}\mathcal{R}_{s-1}, \partial^{h,0}\mathcal{R}_{s+1} \rfloor \setminus (T_s \boxplus \theta_{m+1})$. This will imply the claim because $\partial^{h,0}\mathcal{R}_j$ has distance $> 5\mathfrak{l}_{m+1}$ to $\{a_j, a_{j+1}\}$, see §9.1.2.

Assuming converse and using Claim 3 (the second part), we can choose in every $\mathcal{R}_i$ a wide rectangle $\mathcal{S}_i$ whose roof is outside of $\lfloor \partial^{h,0}\mathcal{R}_{i-1}, \partial^{h,0}\mathcal{R}_{i+1} \rfloor$. This is a contraction by Lemma 6.14. □

9.1.4. *Proof of Calibration Lemma.* We now fix a rectangle $\mathcal{R}_s$ from Claim 4. We assume that most of the curves (up to $O(K)$) in $\mathcal{R}_s$ land at T_{s+1}; the case of T_{s-1} is similar. By removing $O(K)$ buffers from $\mathcal{R}_s$, we obtain the new $\mathcal{R}^{\text{new}}_s$ with

$$\partial^{h,0}\mathcal{R}^{\text{new}}_s \subset T_s, \quad \partial^{h,1}\mathcal{R}^{\text{new}}_s \subset T_{s+1} \quad \operatorname{dist}(\partial^{h,1}\mathcal{R}^{\text{new}}_s, a_{s+2}) \geq 5\mathfrak{l}_{m+1}. \tag{9.1}$$

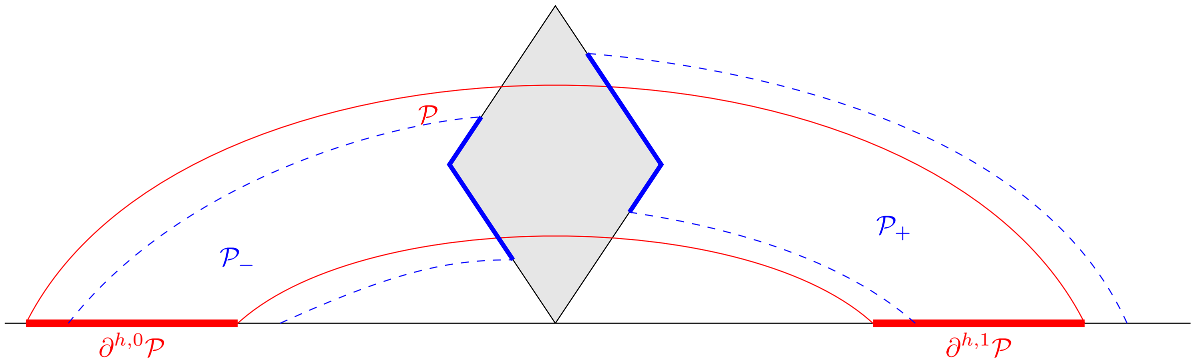


FIGURE 26. Illustration to Claim 5. If $\mathcal{P}$ essentially overflows its univalent pullbacks $\mathcal{P}_\pm$, then $\mathcal{W}(\mathcal{P}_-) \oplus \mathcal{W}(\mathcal{P}_+) \geq \mathcal{W}(\mathcal{P}) - O(\boldsymbol{\chi}^{0.9}K)$.

Let $\mathcal{P}, \mathcal{P}_1$ be the restrictions (see §5.2.4) of $\mathcal{R}_s^{\text{new}}$ onto $\widehat{Z}^{m+1}$ and $f^{\mathfrak{q}_{m+1}}(\widehat{Z}^{m+1})$ respectively. Subrectangles $\mathcal{P}_J$ of $\mathcal{P}$ are defined in the same way as subrectangles $\mathcal{R}_{i,J}$ for $\mathcal{R}_i$.

Claim 5 (See Figure 26)**.** *There is an interval $J \subset \partial^{h,0}\mathcal{P}$ well-grounded rel $\widehat{Z}^{m+1}$ with $\mathcal{W}(\mathcal{P}_J) \geq \boldsymbol{\chi}^{0.9}K$ such that $|\partial^{h,1}\mathcal{P}_J| \leq \mathfrak{l}_{m+1}/5$.*

Proof. Let $\mathcal{P}_-$ and $\mathcal{P}_+$ be the lifts of $\mathcal{P}_1$ under $f^{\mathfrak{q}_{m+1}}$ such that $\mathcal{P}_-$ starts in $T_s^{m+1} \boxplus \theta_{m+1}$ while $\mathcal{P}_+$ lands $T_{s+1}^{m+1} \boxplus \theta_{m+1}$, where T_s^{m+1}, T_{s+1}^{m+1} are the projections of T_s, T_{s+1} onto $\partial\widehat{Z}^{m+1}$. Assume a required interval J does not exist. Combining this assumption with Claim 2, we can remove $O(\boldsymbol{\chi}^{0.9}K)$ buffers from $\mathcal{P}$ such that $\mathcal{P}^{\text{new}}$ consequently overflows $\mathcal{P}_-$ and $\mathcal{P}_+$. This contradicts the Grötzsch inequality:

$$\boldsymbol{\chi}K - O(\boldsymbol{\chi}^{0.9}K) = \mathcal{W}(\mathcal{P}) - O(\boldsymbol{\chi}^{0.9}K) \leq \mathcal{W}(\mathcal{P}_-) \oplus \mathcal{W}(\mathcal{P}_+) = 0.5\boldsymbol{\chi}K.$$

By removing $O(K)$-buffers from $\mathcal{P}_J$, we can assume that J is well-grounded. $\square$

Let $\widehat{Z}_{s+1}$ be the pseudo-bubble (see §5.7) around Z_{s+1} such that $f^{\mathfrak{q}_{m+1}}$ maps $\widehat{Z}_{s+1}$ onto $f^{\mathfrak{q}_{m+1}}\big(\widehat{Z}^{m+1}\big)$. We denote by N_{s+1}^{m+1} the projection of N_{s+1} onto $\partial\widehat{Z}_{s+1}$.

Consider an interval J from Claim 5. Write $X := \partial^{h,1}\mathcal{P}_J^1$, where $|X| \leq \mathfrak{l}_{m+1}/5$, and let $X^* \subset N_{s+1}^{m+1}$ be the lift of X under $f^{\mathfrak{q}_{m+1}}$. We denote by $\mathcal{P}_J^*$, $\partial^{h,1}\mathcal{P}_J^* = X^*$ the full lift of $\mathcal{P}_{1,J}$ under $f^{\mathfrak{q}_{m+1}}$.

By Claim 3, we can remove $O(K)$ buffers from $\mathcal{P}_J$ so that the new rectangle $\mathcal{P}_J^{\text{new}}$ overflows $\mathcal{P}_J^*$. Let us denote by $V \subset X^*$ the subinterval of X^* between the first intersections of $\partial^{v,\ell}\mathcal{P}_J^{\text{new}}, \partial^{v,\rho}\mathcal{P}_J^{\text{new}}$ with X^*. Set

- $\mathcal{F}_1$ to be the restriction (see §A.1.5) of $\mathcal{F}(\mathcal{P}_J^{\text{new}})$ to $\mathcal{F}(\partial^{h,0}\mathcal{P}_J^{\text{new}}, V)$;
- $\mathcal{F}_2$ to be the restriction of $\mathcal{F}(\mathcal{P}_J^{\text{new}})$ to $\mathcal{F}(V, \partial^{h,0}\mathcal{P}_J^{\text{new}})$.

In other words:

$$\mathcal{F}_1 = \{\gamma_1 \mid \gamma \in \mathcal{F}(\mathcal{P}_J^{\text{new}})\} \qquad \text{and} \qquad \mathcal{F}_1 = \{\gamma_2 \mid \gamma \in \mathcal{F}(\mathcal{P}_J^{\text{new}})\},$$

where γ_1 and γ_2 are the shortest subarcs of γ connecting $\partial^{h,0}\mathcal{P}_J^{\text{new}}, V$ and $V, \partial^{h,0}\mathcal{P}_J^{\text{new}}$ respectively. We remark that γ_1 lands at V^+ while γ_2 starts in V^-.

Claim 6. *We have $\mathcal{W}(\mathcal{F}_2) \geq t^{1.7}K$.*

Proof. By the Grötzsch inequality $\mathcal{W}(\mathcal{P}^*_J) \oplus \mathcal{W}(\mathcal{F}_2) \geq \mathcal{W}(\mathcal{F}_1) \oplus \mathcal{W}(\mathcal{F}_2) \geq \mathcal{W}(\mathcal{P}^{\text{new}}_J)$, we have:

$$\mathcal{W}(\mathcal{P}^*_J) \oplus \mathcal{W}(\mathcal{F}_2) = \boldsymbol{\chi}^{0.9} K \oplus \mathcal{W}(\mathcal{F}_2) \geq \mathcal{W}(\mathcal{P}^{\text{new}}_J) = \boldsymbol{\chi}^{0.9} K - O(K),$$

$$\frac{1}{\boldsymbol{\chi}^{0.9} K} + \frac{1}{\mathcal{W}(\mathcal{F}_2)} \leq \frac{1}{\boldsymbol{\chi}^{0.9} K - O(K)},$$

$$\mathcal{W}(\mathcal{F}_2) \geq \frac{\boldsymbol{\chi}^{1.8} K^2}{O(K)} \geq \boldsymbol{\chi}^{1.7} K$$

because $\boldsymbol{\chi} \gg 1$. □

Consider the rectangle $\mathcal{G}$ bounded by the leftmost and rightmost curves of $\mathcal{F}_2$:

$$\mathcal{G}, \quad \partial^{h,0}\mathcal{G} \subset V^-, \quad \partial^{h,1}\mathcal{G} \subset \partial^{h,1}\mathcal{P}^{\text{new}}_J, \qquad \mathcal{W}(\mathcal{G}) \geq \mathcal{W}(\mathcal{F}_2) \geq \boldsymbol{\chi}^{1.7} K.$$

Applying Lemma 6.9 to $\mathcal{G}$, we obtain an interval $B \subset [(1+\lambda^{-2})V] \setminus V \subset \partial \widehat{Z}_{s+1}$ together with a lamination

$$\mathcal{Q} \subset \widehat{\mathbb{C}} \setminus \operatorname{int} \widehat{Z}_{s+1}, \qquad \mathcal{W}(\mathcal{Q}) \succeq \boldsymbol{\chi}^{1.7} K$$

from B to either $\left[(\lambda B)^c\right]^{\text{grnd}} \subset \partial \widehat{Z}_{s+1}$ (Case (II)) or to $\partial^{h,1}\mathcal{P}^{\text{new}}_J$ (Case (I)). In both cases, $\mathcal{Q}$ is disjoint from $\operatorname{int} \widehat{Z}^m$ as a restriction of a sublamination of $\mathcal{P}^{\text{new}}_J$. Write $N^{m+1}_{s+1} = [a_{s+1}, b_{s+1}]$, where $a_{s+1} \in \partial Z$, see Figure 25. Observe that

$$\text{(9.2)} \qquad \operatorname{dist}_{\partial \widehat{Z}_{s+1}}(B, b_{s+1}) \geq 4\mathfrak{l}_{m+1}, \qquad \operatorname{dist}_{\partial \widehat{Z}_{s+1} \cup \partial \widehat{Z}^{m+1}}(B, \partial^{h,1}\mathcal{P}^{\text{new}}_J) \geq \mathfrak{l}_{m+1}/5.$$

Indeed, the first inequality follows from (9.1) because $f^{\mathfrak{q}_{m+1}}(b_{s+1}) = a_{s+2}$. The second inequality follows from the observation that if $X^*, |X^*| \leq \mathfrak{l}_{m+1}/5$ is close to a_{s+1}, then X will be close to $a_{s+1} \boxplus \theta_{m+1}$.

For every $\gamma \in \mathcal{Q}$, let γ' be the first subarc of γ connecting B to $\partial \mathcal{K}_m$. By §2.3.3 and (9.2), $f^{\mathfrak{q}_{m+1}}$ injectively maps most of the $\{\gamma' \mid \gamma \in \mathcal{Q}\}$ into a sublamination of $\mathcal{F}\left(B_2, \left[(\lambda B_2)^c\right]^{\text{grnd}}\right)$, where $B_2 = f^{\mathfrak{q}_{m+1}}(B) \subset \partial f^{\mathfrak{q}_{m+1}}(\widehat{Z}^{m+1})$ is a grounded interval. By Lemma 5.10, the projection $B_2^\bullet$ of B_2 onto ∂Z satisfies

$$\mathcal{W}_\lambda(B_2^\bullet) \succeq \boldsymbol{\chi}^{1.7} K \geq \boldsymbol{\chi}^{1.5} K;$$

which is Case (II) of the Calibration Lemma.

Part 4. Conclusions

10. Proof of the main result

Recall that we are considering eventually golden-mean rotation numbers:

$$\text{(10.1)} \qquad \theta = [0; a_1, a_2, \dots] \quad \text{with} \quad a_n = 1 \quad \text{for all} \quad n \geq \mathbf{n}_\theta.$$

In this section we will establish the following results:

Theorem 10.1. *There is an absolute constant $\mathbf{K} \gg 1$ such that $\mathcal{W}^+_3(I) \leq \mathbf{K}$ for every combinatorial interval $I \subset \partial Z$ and every θ satisfying* (10.1).

Theorem 10.2. *There are absolute constants $\mathbf{N} \gg 1$ and $\mathbf{K} \gg 1$ such that for every θ satisfying* (10.1)*, there is a sequence of geodesic pseudo-Siegel disks (§5.1.9)*

$$\widehat{Z}^{\mathbf{n}_\theta} = \overline{Z}, \quad \widehat{Z}^{\mathbf{n}_\theta - 1}, \quad \widehat{Z}^{\mathbf{n}_\theta - 2}, \quad \dots, \quad \widehat{Z}^{-1} = \widehat{Z}$$

satisfying the following properties. If $I \subset \partial Z$ is a regular interval rel $\widehat{Z}^m$ with $\mathfrak{l}_{m+1} \le |I| \le \mathfrak{l}_m$, then

$$\text{(10.2)} \qquad \textit{either} \quad |I| \ge \mathfrak{l}_m/2 \qquad \textit{or} \qquad |I| \le \mathbf{N}\mathfrak{l}_{m+1}$$

and $\mathcal{W}_3^+(I) \le \mathbf{K}$.

10.0.1. *Outline and motivation.* Let us note that Theorem 10.1 implies Theorem 1.1. Indeed, for every $\theta = [0; a_1, a_2, a_3, \dots]$, $|a_i| \le M_\theta$ of bounded type, define the approximating sequence

$$\theta_n := [0; a_1, a_2, \dots, a_n, 1, 1, 1, \dots] \to \theta.$$

Since $Z_{\theta_n} \to Z_\theta$ as qc disks (with dilatation depending on M_θ), the estimate $\mathcal{W}_3^+(I) \le \mathbf{K}$ in Theorem 10.1 also holds for Z_θ, where $\mathbf{K} > 1$ is independent of M_θ.

We will prove Theorem 10.1 by induction from deep to shallow levels as follows. We will show that there are $\boldsymbol{\lambda} \gg 1$ and $\mathbf{K} \gg_{\boldsymbol{\lambda}} 1$ such that the following properties hold for every level m:

(a) Existence of a geodesic pseudo-Siegel disk $\widehat{Z}^m$ so that

$$\mathcal{W}^+_{\boldsymbol{\lambda},\text{ext},m}(I) = O(\sqrt{\mathbf{K}})$$

for every interval I grounded rel $\widehat{Z}^m$ with $\mathfrak{l}_{m+1} \le |I| \le \mathfrak{l}_m$.

(In other words, we construct $\widehat{Z}^m$ in (a) so that it *absorbs* all but $O(\sqrt{\mathbf{K}})$-external rel m families: if $\mathcal{W}^+_{\boldsymbol{\lambda},\text{ext},m}(I) \gg \sqrt{\mathbf{K}}$ for an interval $I \subset \partial Z$, then most of the I together with most of the family $\mathcal{F}^+_{\boldsymbol{\lambda},\text{ext},m}(I)$ submerges into $\widehat{Z}^m$.)

(b) $\mathcal{W}^+_{\boldsymbol{\lambda}}(I) \le (2\boldsymbol{\chi})\mathbf{K}$ for every grounded rel $\widehat{Z}^m$ interval m with $|I| \le \mathfrak{l}_m$, where $\boldsymbol{\chi}$ is the constant from Calibration Lemma 9.1.

(c) $\mathcal{W}^+_{\boldsymbol{\lambda}}(I) \le \mathbf{K}$ for every combinatorial interval I of level $\ge m$.

The proof of the induction step is illustrated in Figure 27:

- If a pseudo-Siegel disk $\widehat{Z}$ can not be constructed to satisfy Statement (a), i.e. to absorb all but $O(\sqrt{\mathbf{K}})$-external rel m families, then by the exponential boost in Corollary 7.3 there will be a degeneration of order $a^{\sqrt{\mathbf{K}}} \gg (2\boldsymbol{\chi})\mathbf{K}$ on levels $\ge m+1$, where $a > 1$ is fixed.
- If Statement (b) is violated, then it follows from $\mathcal{F}^+_{\boldsymbol{\lambda}}(I) = \mathcal{F}^+_{\boldsymbol{\lambda},\text{div},m}(I) \sqcup \mathcal{F}^+_{\boldsymbol{\lambda},\text{ext},m}(I)$ that either $\widehat{Z}^m$ was not properly constructed, i.e. the violation of Statement (a) with $\mathfrak{l}_m \ge |I| \ge \mathfrak{l}_{m+1}$, or there is a diving degeneration of order $> (1.5\boldsymbol{\chi})\mathbf{K}$.
- If there is a diving degeneration of scale $> (1.5\boldsymbol{\chi})\mathbf{K}$ with $\mathfrak{l}_m \ge |I| \ge \mathfrak{l}_{m+1}$, then by Calibration Lemma 9.1, either Statement (c) or Statement (b) is violated on levels $\ge m+1$.
- If Statement (c) is violated, then by Amplification Theorem 8.1, Statement (b) is violated with $|I| < \mathfrak{l}_m$.

After the induction, there might still be finitely many renormalization levels where Amplification Theorem 8.1 is not applicable because of the condition $|I| \le |\theta_0|/(2\boldsymbol{\lambda}_{\mathbf{t}})$. The number of such levels is bounded in terms of $\boldsymbol{\lambda}$; the estimates for these levels are established by increasing $\mathbf{K}$.

Let us stress that regularizations on different levels do not interact much. Corollary 7.3, Theorem 8.1, and Calibration Lemma 9.1 are stated in terms of the outer

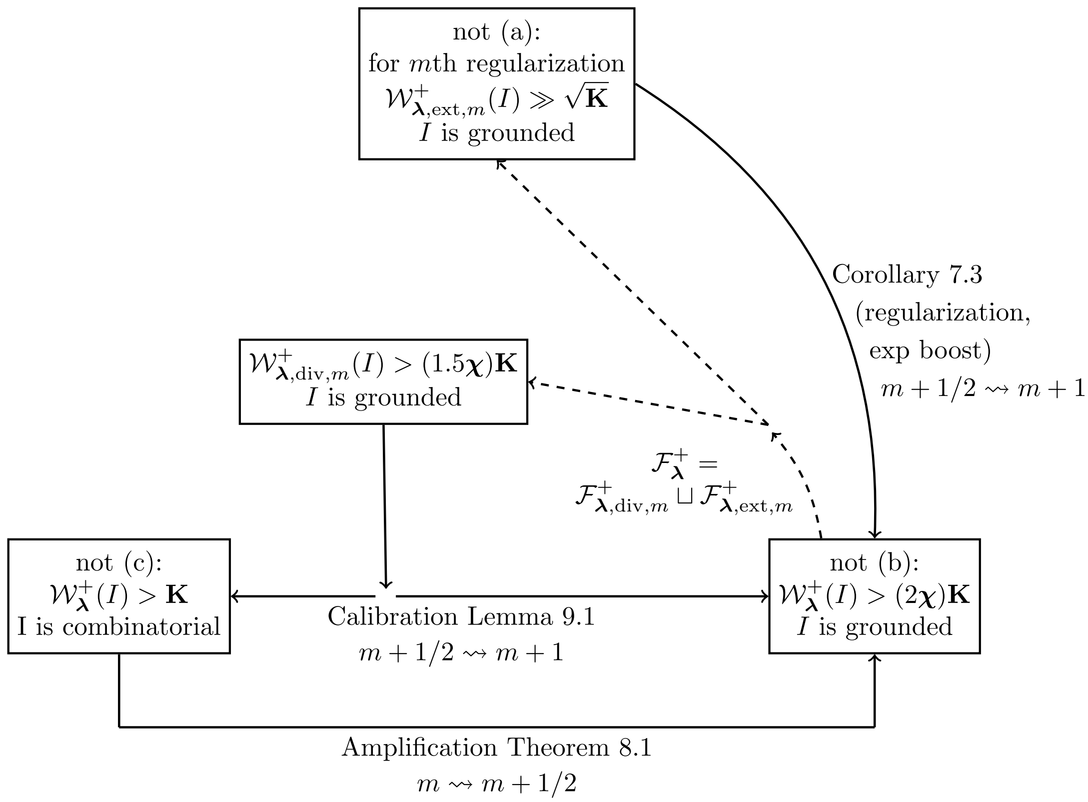


FIGURE 27. Statements (a), (b), and (c) are proved by contradiction: if one of the statements is violated on levels m or $m+1/2$, then there will be even bigger violation on deeper scales. Here "m" indicates level m combinatorial intervals, "$m+1/2$" indicates intervals with $\mathfrak{l}_m \geq |I| \geq \mathfrak{l}_{m+1}$, and "$m+1$" indicates intervals with $|I| \leq \mathfrak{l}_{m+1}$. The dashed arrows illustrate the decomposition $\mathcal{F}^+_{\boldsymbol{\lambda}}(I) = \mathcal{F}^+_{\boldsymbol{\lambda},\text{div},m}(I) \sqcup \mathcal{F}^+_{\boldsymbol{\lambda},\text{ext},m}(I)$.

geometry of the Siegel disk Z with only indirect references to $\widehat{Z}^m$. Lemma 5.10 implies that the outer geometries of $\widehat{Z}^m$ and $\overline{Z}$ are close rel grounded intervals independently of the number of regularizations. Our estimates for the inner geometry of $\widehat{Z}^m$ are also independent of the number of regularizations – see the discussion in §3.0.1 and in §5.0.1.

To prove Theorem 10.2, we will show that external families can not be unexpectedly narrow (Lemma 10.5); otherwise, the dual family will be very wide – contradicting the estimates established for Theorem 10.1. Combined with the Parallel Law, this will imply the existence of the combinatorial threshold $\mathbf{N}$. We will refer to $\mathbf{N}$ in Theorem 10.2 as the *high-type* condition. This a near-degenerate analogy of the high-type condition in the Inou-Shishikura theory [IS].

10.1. **Proof of Theorem 10.1.** We choose constants $\boldsymbol{\lambda}, \mathbf{K}$ and a parameter $\mathbf{t}$ from Theorem 8.1 such that

$$\boldsymbol{\chi} \ll \mathbf{t}, \qquad \boldsymbol{\lambda} \coloneqq \boldsymbol{\lambda}_{\mathbf{t}}, \qquad \mathbf{K}_{\mathbf{t}} \ll \mathbf{K},$$

where $\boldsymbol{\chi} > 1$ is the constant in Calibration Lemma 9.1 and $\boldsymbol{\lambda}_{\mathbf{t}}, \mathbf{K}_{\mathbf{t}}$ are from Theorem 8.1. Theorem 10.1 on levels $\geq \mathbf{n}_\theta$ follows from Corollary 8.7. Let $\mathbf{s} = \mathbf{s}_\theta$ be the smallest so that $\mathfrak{l}_{\mathbf{s}} \leq |\theta_0|/(2\boldsymbol{\lambda}+4)$.

Lemma 10.3 (Induction)**.** *There is a sequence of geodesic pseudo-Siegel disks*

$$\widehat{Z}^{\mathbf{n}_\theta} = \overline{Z}, \quad \widehat{Z}^{\mathbf{n}_\theta - 1}, \quad \widehat{Z}^{\mathbf{n}_\theta - 2}, \quad \dots, \quad \widehat{Z}^{\mathbf{s}_\theta}$$

with the following properties:

(A) *If Z does not have an external level-m parabolic rectangle of width $\sqrt{\mathbf{K}}$, then $\widehat{Z}^m = \widehat{Z}^{m+1}$; otherwise $\widehat{Z}^m$ is the regularization of $\widehat{Z}^{m+1}$ constructed by Corollary 7.3. Moreover, $\mathcal{W}^+_{\boldsymbol{\lambda},\text{ext},m}(J) \leq 2\sqrt{\mathbf{K}} + O(1)$ for every interval $J \subset \partial Z$ grounded rel $\widehat{Z}^m$ with $\mathfrak{l}_{m+1} \leq |J| \leq \mathfrak{l}_m$.*

(B) *For every interval $J \subset \partial Z$ grounded rel $\widehat{Z}^m$ with $\mathfrak{l}_{m+1} \leq |J| \leq \mathfrak{l}_m$, we have $\mathcal{W}^+_{\boldsymbol{\lambda}}(J) \leq 2\boldsymbol{\chi}\mathbf{K}$.*

(C) *For every level-m combinatorial interval $I \subset \partial Z$, we have $\mathcal{W}^+_{\boldsymbol{\lambda}}(I) \leq \mathbf{K}$.*

Proof. We proceed by induction from deep to shallow scales. The base case $m = \mathbf{n}_\theta$ follows from Corollary 8.7. Let us assume that the lemma is true for levels $> m$. In particular, $\widehat{Z}^{m+1}$ is constructed. Let us verify the lemma for m.

Suppose that Z has a level-m external parabolic rectangle $\mathcal{R}$ with $\mathcal{W}(\mathcal{R}) \geq \sqrt{\mathbf{K}}$. Assume that $\mathcal{R}$ is based on $T \in \mathfrak{D}_m$ and let T' be as in §2.1.6. We replace $\mathcal{R}$ with an outermost external rel $\widehat{\mathbb{C}} \setminus \mathcal{K}_m$ geodesic rectangle $\mathcal{R}_{\text{new}}$ based on T' with $\mathcal{W}(\mathcal{R}_{\text{new}}) \geq \sqrt{K} - O(1)$. In particular, $\mathcal{R}_{\text{new}}$ is non-winding. Let us apply Corollary 7.3 to $\mathcal{R}_{\text{new}}$. We **claim** that Case (1) of Corollary 7.3 occurs.

Proof of the Claim. Assume Case (3) of Corollary 7.3 occurs. We obtain an interval $I \subset \partial Z$ grounded rel $\widehat{Z}^{m+1}$ with $|I| \leq \mathfrak{l}_{m+1}$ such that $\log \mathcal{W}^+_{\boldsymbol{\lambda}}(I) \succeq \sqrt{\mathbf{K}}$. Since $\mathbf{K} \gg \boldsymbol{\chi} > 1$, we have

$$\mathcal{W}^+_{\boldsymbol{\lambda}}(I) \geq a^{\sqrt{\mathbf{K}}} \gg 2\boldsymbol{\chi}\mathbf{K}, \qquad \text{where} \quad a > 1 \ \text{ represents “}\succeq\text{”}$$

contradicting the induction assumption that Statement (B) holds on levels $\geq m+1$.

Calibration Lemma 9.1 reduces Case (2) of Corollary 8.7 to Case (3). □

By construction, $\sqrt{\mathbf{K}} + O(1)$ bounds the width of level-m external parabolic rectangles $\mathcal{R}$ such that $\partial^h \mathcal{R}$ is a pair of grounded rel $\widehat{Z}^m$ intervals – wider rectangles are absorbed by $\widehat{Z}^m$. By splitting J into at most 2 intervals, we obtain the estimate $\mathcal{W}^+_{\boldsymbol{\lambda},\text{ext},m}(J) \leq 2\sqrt{\mathbf{K}} + O(1)$. This proves Statement (A).

Let us verify Statement (B). Assuming otherwise and using $\mathcal{W}^+_{\boldsymbol{\lambda},\text{ext},m}(J) \leq 2\sqrt{\mathbf{K}} + O(1)$ (Statement (A)), we obtain $\mathcal{W}^+_{\boldsymbol{\lambda},\text{div},m}(J) > 1.5\boldsymbol{\chi}\mathbf{K}$. Applying Calibration Lemma 9.1, there would exist

- either a combinatorial $[1.5\mathbf{K}, \boldsymbol{\lambda}]^+$-wide level-$(m+1)$ combinatorial interval – contradicting Statement (C) on level $m+1$,
- or a $[1.5\boldsymbol{\chi}^{1.5}\mathbf{K}, \boldsymbol{\lambda}]^+$-wide interval $I' \subset \partial Z$ grounded rel $\widehat{Z}^{m+1}$ with $|I'| < \mathfrak{l}_{m+1}$ – contradicting Statement (B) on levels $\geq m+1$.

It remain to verify Statement (C). Let us assume converse: $\mathcal{W}^+_{\boldsymbol\lambda}(I) > \mathbf{K}$ for a combinatorial level m interval $I \subset \partial Z$. Applying Theorem 8.1, we obtain a $[\mathbf{tK}, \boldsymbol\lambda]^+$ wide interval J grounded rel $\widehat{Z}^m$ with length $\le \mathfrak{l}_m$. This contradicts Statement (B). □

Since $\mathfrak{l}_{n+2} < \mathfrak{l}_n/2$, see (2.1), we have:

$$\mathfrak{l}_{2n} < \mathfrak{l}_0/2^n = |\theta_0|/2^n \le |\theta_0|/(2\boldsymbol\lambda + 4) \qquad \text{if} \quad n > \log_2(2\lambda+4).$$

We obtain that $\mathbf{s}_\theta \le \mathbf{s} \coloneqq 2\log_2(2\lambda+4)$. Set $\mathbf{K}_i \coloneqq (2\boldsymbol\chi)^i\mathbf{K}$.

Lemma 10.4 (A few shallow levels)**.** *The sequence of geodesic pseudo-Siegel disks in Lemma 10.3 can be continued with a sequence of geodesic pseudo-Siegel disks*

$$\widehat{Z}^{\mathbf{s}_\theta-1},\quad \widehat{Z}^{\mathbf{s}_\theta-2},\quad \widehat{Z}^{\mathbf{n}_\theta-3},\quad \dots,\quad \widehat{Z}^{-1} \coloneqq \widehat{Z}_{f_\theta}, \qquad \mathbf{s}_\theta \le \mathbf{s}$$

with the following properties for $m < \mathbf{s}_\theta$:

(A) *If Z does not have an external level-m parabolic rectangle of width $\sqrt{(2\boldsymbol\chi)^{\mathbf{s}_\theta-m-1}\mathbf{K}}$, then $\widehat{Z}^m = \widehat{Z}^{m+1}$; otherwise $\widehat{Z}^m$ is the regularization of $\widehat{Z}^{m+1}$ constructed by Corollary 7.3. Moreover, $\mathcal{W}^+_{\boldsymbol\lambda,\mathrm{ext},m}(J) \le 2\sqrt{(2\boldsymbol\chi)^{\mathbf{s}_\theta-m-1}\mathbf{K}} + O(1)$ for every interval $J \subset \partial Z$ grounded rel $\widehat{Z}^m$ with $\mathfrak{l}_{m+1} \le |J| \le \mathfrak{l}_m$.*

(B) *For every interval $J \subset \partial Z$ grounded rel $\widehat{Z}^m$ with $\mathfrak{l}_{m+1} \le |J| \le \mathfrak{l}_m$, we have $\mathcal{W}^+_{\boldsymbol\lambda}(J) \le (2\boldsymbol\chi)^{\mathbf{s}_\theta-m}\mathbf{K}$.*

Proof. Statements (A) and (B) are proven in the same way as the corresponding statements in Lemma 10.3 where Statement (C) of Lemma 10.3 replaced with a weaker Statement (B) of Lemma 10.4. □

10.1.1. *Proof of Theorem 10.1.* We have shown in Lemmas 10.3 and 10.4 that there are absolute $\boldsymbol\lambda \gg 1$, $\mathbf{K} \gg_{\boldsymbol\lambda} 1$ such that $\mathcal{W}^+_{\boldsymbol\lambda}(I) \le \mathbf{K}$ for every combinatorial interval I. We need to show that $\mathcal{W}^+_3(I) \le \mathbf{K}_2$ for some $\mathbf{K}_2$.

Assume I is a level m combinatorial interval. For simplicity, let us round up $\boldsymbol\lambda$ to the smallest integer number. Choose the minimal $n > m$ such that $\mathfrak{l}_m/\mathfrak{l}_n > 2\boldsymbol\lambda+1$. We can decompose I as a concatenation

$$I = I_{-\boldsymbol\lambda} \cup I_{-\boldsymbol\lambda+1} \cup \dots I_{-1} \cup I_0 \cup I_1 \cup \dots \cup I_{\boldsymbol\lambda}$$

so that for $k \ne 0$ the interval I_k is level-n combinatorial while I_0 is a grounded rel $\widehat{Z}^n$ interval. By construction, Then

$$\mathcal{F}^+_3(I) \subset \bigcup_j \mathcal{F}^+(I_j, (3I)^c) \qquad \text{and} \qquad \mathcal{F}^+(I_k, (3I)^c) \subset \mathcal{F}^+_{\boldsymbol\lambda}(I_k) \quad \text{for } k \ne 0.$$

For $k \ne 0$, we have $\mathcal{W}^+_{\boldsymbol\lambda}(I_k) \le \mathbf{K}$. If $\mathcal{W}^+(I_0, (3I)^c) \gg_{\mathbf{K}} 1$, then applying Calibration Lemma 9.1 to $\mathcal{F}^+(I_0, (3I)^c)$, we obtain an interval J grounded rel $\widehat{Z}^n$ with $|J| \le \mathfrak{l}_n$ such that $\mathcal{W}^+_{\boldsymbol\lambda}(J) \gg_{\mathbf{K}} 1$ – contradicting the estimates in Lemmas 10.3 and 10.4. Therefore, $\mathcal{W}_3(I)$ is bounded in terms of $\boldsymbol\lambda$ and $\mathbf{K}$. □

10.2. **Proof of Theorem 10.2.** Consider a renormalization level $m \ge -1$ with $\mathfrak{l}_m/\mathfrak{l}_{m+1} \gg 1$, and let $T = [v, w]$ be an interval in the diffeo-tiling $\mathfrak{D}_m$. As in §4, we assume that $v < w$ in T and that $T' = [v', w]$ is $T \cap f^{\mathfrak{q}_{m+1}}(T)$ (with necessary adjustments for $m = -1$).

For $k < \log_2[\mathfrak{l}_m/(20\mathfrak{l}_{m+1})]$ we define $v_k, w_k \in T'$ to be the points at distance $10(2^k-1)\mathfrak{l}_{m+1}$ from v' and w respectively with $v_k < w_k$ in T'. We set

$$T^k \coloneqq [v_k, w_k] \subset T', \qquad X^{k+1} \coloneqq [v_k, v_{k+1}], \qquad Y^{k+1} \coloneqq [w_{k+1}, w_k]$$

i.e. $T^0 = T'$ and

$$T^k = X^{k+1} \cup T^{k+1} \cup Y^{k+1}, \qquad |X^{k+1}| = |Y^{k+1}| = 2^k 10\mathfrak{l}_{m+1}.$$

Lemma 10.5. *For a constant* $\mathbf{K}$ *in Theorem 10.1 and every above well-defined pair* X^k, Y^k *with* $k \geq 1$*, we have*

$$\mathcal{W}^+_{\text{ext},m}(X^k, Y^k) \succeq_{\mathbf{K}} 1.$$

Proof. Assume converse; then we have the following estimate of the dual family:

$$\mathcal{W}^+_{\mathcal{K}_m}(T^k, \partial\mathcal{K}_m \setminus T^{k-1}) = K \gg_{\mathbf{K}} 1.$$

Up to $O(1)$, the family $\mathcal{F}^+_{\mathcal{K}_m}(T^k, \partial\mathcal{K}_m \setminus T^{k-1})$ is within two rectangles $\mathcal{R}_x, \mathcal{R}_y$ in $\mathbb{C} \setminus \operatorname{int} \mathcal{K}_m$. Applying Lemma A.10, we can push-forward $\mathcal{F}^+_{\mathcal{K}_m}(T^k, \partial\mathcal{K}_m \setminus T^{k-1})$ almost univalently under $f^{\mathfrak{q}_{m+1}} \colon \widehat{\mathbb{C}} \setminus \mathcal{K}_{m+1} \to \widehat{\mathbb{C}} \setminus \overline{Z}$; we obtain that

$$\mathcal{W}^+_{\overline{Z}}(T_k \boxplus \theta_{m+1}, \ \ (T_{k-1} \boxplus \theta_{m+1})^c) \geq K - O(1).$$

Below we recognize three types of the curves in

$$\mathcal{F} \coloneqq \mathcal{F}^+_{Z}(T_k \boxplus \theta_{m+1}, \ \ (T_{k-1} \boxplus \theta_{m+1})^c)$$

and prove that the width of each type curve family can be bounded in terms of $\mathbf{K}$.

Curves diving into $\mathcal{K}_m \setminus \overline{Z}$*.* If the width of such curves is sufficiently big, then applying Calibration Lemma 9.1 to such curves, we obtain a sufficiently wide interval on deeper scale contradicting the estimates in Lemmas 10.3 and 10.4.

Curves landing at $[v, v']$*.* Since $[v, v']$ is combinatorial, the width of such curves is bounded by $\mathbf{K}$.

External curves landing at $T' \cap (T_{k-1} \boxplus \theta_{m+1})^c$*.* Note that $T' \cap (T_{k-1} \boxplus \theta_{m+1})^c$ consists of two intervals of length $\asymp 2^k \mathfrak{l}_{m+1}$. Since the distance between

$$T' \cap (T_{k-1} \boxplus \theta_{m+1})^c \qquad \text{and} \qquad T_k \boxplus \theta_{m+1}$$

is $\asymp 2^k \mathfrak{l}_{m+1}$, the width of curves of this last type is $O(1)$ by Theorem 4.1. □

Let us now choose a sufficiently big $\mathbf{N} \gg_{\mathbf{K}} 1$. Write $\mathbf{M} \coloneqq \log_2 \mathbf{N}/10^3$.. If $\mathfrak{l}_m/\mathfrak{l}_{m+1} \geq \mathbf{N}/2$, then

$$\mathcal{W}^+_{\text{ext},m}(X^1 \cup X^2 \cup \cdots \cup X^{\mathbf{M}}, Y^1 \cup Y^2 \cup \cdots \cup Y^{\mathbf{M}}) \gg_{\mathbf{K}} \mathbf{M} \gg \mathbf{K}$$

by Lemma 10.5. Therefore, $\mathcal{F}^+_{\text{ext},m}(X^1 \cup X^2 \cup \cdots \cup X^{\mathbf{M}}, Y^1 \cup Y^2 \cup \cdots \cup Y^{\mathbf{M}})$ contains a parabolic external level m rectangle of width $\sqrt{\mathbf{K}}$ and the regularization happens within the orbit of such rectangle. This implies (10.2).

The combinatorial threshold (10.2) implies that for every interval $I \subset \partial Z$ regular rel $\widehat{Z}^m$ with $\mathfrak{l}_m \geq |I| \geq \mathfrak{l}_{m+1}$, there is a grounded rel $\widehat{Z}^m$ interval $I_{\text{grnd}} \subset I$ such that $I \setminus I_{\text{grnd}}$ is within $2\mathbf{N}$ level $m+1$ combinatorial intervals. Therefore, the condition "grounded rel $\widehat{Z}^m$" in Lemmas 10.3 and 10.4 can be replaced with "regular rel $\widehat{Z}^m$" by possibly increasing $\mathbf{K}$.

11. Mother Hedgehogs and uniform quasi-conformality of $\widehat{Z}$

Recall that we are considering eventually golden-mean rotations numbers θ, (10.1).

Theorem 11.1. *There is an absolute constant* $\mathbf{K} \gg 1$ *such that the pseudo-Siegel disk* $\widehat{Z}_f = \widehat{Z}^{-1}$ *in Theorem 10.2 is* $\mathbf{K}$*-qc.*

Recall that a hull $Q \subset \mathbb{C}$ is a compact connected full set. The *Mother Hedgehog* [Chi] for a neutral polynomial f_θ is an invariant hull containing both the fixed point 0 and the critical point $c_0(f)$.

Theorem 11.2. *Any neutral quadratic polynomial $f = f_\theta$, $\theta \notin \mathbb{Q}$, has a Mother Hedgehog $H_f \ni c_0(f)$ such that $f\colon H_f \to H_f$ is a homeomorphism.*

11.0.1. *Outline of the section.* Since $\widehat{Z} = \widehat{Z}^{-1}$ is obtained from a qc disk $\overline{Z}$ by adding finitely many fjords bounded by hyperbolic geodesics in $\widehat{\mathbb{C}} \setminus \overline{Z}$, the resulting pseudo-Siegel disk $\widehat{Z}$ is a qc disk. To show that $\widehat{Z}$ is uniformly $\mathbf{K}$-qc, we will introduce a nest of tilings on $\partial\widehat{Z}$ as follows:

$$\mathcal{T}(\widehat{Z}) \coloneqq \text{Projection}_{\widehat{Z}}(\mathfrak{D}) \cup \bigcup_{\beta_i^m \subset \partial \widehat{Z}^m} \mathcal{T}(\beta_i^m), \tag{11.1}$$

where:

- $\text{Projection}_{\widehat{Z}}(\mathfrak{D})$ is the projection onto $\widehat{Z}$ the nest of diffeo-tilings $\mathfrak{D} = [\mathfrak{D}_n]_{n\ge -1}$ (2.8), where intervals completely submerged into $\widehat{Z}$ are removed;
- $\mathcal{T}(\beta_i^m) = [\mathcal{T}_n(\beta_i^m)]_{n\ge m+1}$ is an appropriate nest of tilings on dams, see §11.3.

The combinatorial threshold $\mathbf{N}$ will imply that $\mathcal{T}(\widehat{Z})$ has $2\mathbf{N}$-bounded combinatorics: each level-n interval consists of at most $2\mathbf{N}$ intervals of level $n+1$. Using Theorem 10.1, we will show that $\mathcal{T}(\widehat{Z})$ has uniformly bounded outer geometry: neighboring intervals in $\mathcal{T}^n(\widehat{Z})$ have comparable outer harmonic measures. And using Theorem 5.12, we will show that $\mathcal{T}(\widehat{Z})$ has uniformly bounded inner geometry. This will conclude that $\widehat{Z}$ is uniformly $\mathbf{K}$-qc as a result of quasisymmetric welding, see Lemma 11.3.

Let us comment on the construction of the $\mathcal{T}(\beta_i^m)$. Every dam β_i^m connects two points in CP_{m+1}, call them x and y. For every $n \ge m+1$, we can consider four level-n intervals of $\mathfrak{D}_n$ adjacent to x, y; we call these intervals the *nth foundation* of β_i^m. Our estimates imply that intervals in the nth and $(n+1)$th foundations have comparable outer harmonic measures. This fact allows us to introduce a nest of tilings $\mathcal{T}(\beta_i^m) = [\mathcal{T}_n(\beta_i^m)]_{n\ge m+1}$ comparable with the foundations of β_i^m on all levels $\ge m+1$. We view β_i^m as a sole interval in $\mathcal{T}_{m+1}(\beta_i^m)$. Since every dam β_i^m is protected by a wide rectangle $\mathcal{X}_i^m$ (Assumption 6), different dams are geometrically faraway and their nests of tilings do not interact much in (11.1).

Theorem 11.2 follows from Theorem 11.1 by taking Hausdorff limits of bounded-type Siegel disks:

Proof of Theorem 11.2 using Theorem 11.1. For every $\theta \in \mathbb{R} \setminus \mathbb{Z}$, consider a sequence of eventually golden-mean rotation numbers θ_n converging to θ. Let $Z_{\theta_n} \subset \widehat{Z}_{\theta_n}$ be the Siegel disk and a $\mathbf{K}$-qc pseudo-Siegel disk of f_{θ_n}. By passing to a subsequence, we can assume that the Z_{θ_n} have a Hausdorff limit $H_{f_\theta} = H_\theta$ and the $\widehat{Z}_{\theta_n}$ have a qc limit $\widehat{Z}_\theta$. We obtain that H_θ is f_θ invariant, $c_0, \alpha \in H_\theta \subset \widehat{Z}_\theta$, and $f_\theta\colon \widehat{Z}_\theta \to f_\theta\left(\widehat{Z}_\theta\right)$ is a homeomorphism. Therefore, $f_\theta \mid H_\theta$ is a homeomorphism. □

11.1. **Nests of tilings.** We will use notations similar to [L2, §15.1]. Consider a closed qc disk $D \subset \mathbb{C}$. Let $\mathcal{T} = (\mathcal{T}_n)_{n\ge m}$ be a system of finite partitions of ∂D into finitely many closed intervals such that $\mathcal{T}_{n+1}$ is a refinement of $\mathcal{T}_n$. We say that $\mathcal{T}$ is a *nest of tilings* if

- the maximal diameter of intervals in $\mathcal{T}_n$ tends to 0 as $n \to \infty$, and
- every interval in $\mathcal{T}_n$ for $n \geq m$ decomposes into at least two intervals of $\mathcal{T}_{n+2}$.

Similarly, a nest of tilings is defined for a closed qc arc. (In the second condition, we require $\mathcal{T}_{n+2}$ instead of $\mathcal{T}_{n+1}$ because of Lemma 2.3.)

We say that a nest of tilings $\mathcal{T}$ has *M-bounded combinatorics* if every interval of $\mathcal{T}_n$ consists of at most $M > 1$ intervals of $\mathcal{T}_{n+1}$.

For an interval $I \in \mathcal{T}_n$, let $I_\ell, I_\rho \in \mathcal{T}_n$ be two its neighboring intervals. We denote by $[3I]^c$ the closure of $\partial D \setminus (I_\ell \cup I \cup I_\rho)$. We set

$$[3I]^c \coloneqq \overline{\Gamma \setminus (I \cup I_\ell \cup I_\rho)},$$

and we define:

- $\mathcal{F}^-_{3,\mathcal{T}}(I)$ to be the family of curves in D connecting I and $[3I]^c$;
- $\mathcal{F}^+_{3,\mathcal{T}}(I)$ to be the family of curves in $\widehat{\mathbb{C}} \setminus D$ connecting I and $[3I]^c$;
- $\mathcal{W}^\pm_{3,\mathcal{T}}(I) = \mathcal{W}\big(\mathcal{F}^\pm_{3,\mathcal{T}}(I)\big)$.

We say that a nest of tilings $\mathcal{T}$ has *essentially C-bounded outer geometry* if for every $I \in \mathcal{T}$ we have $\mathcal{W}^+_3(I) \leq C$. If moreover, $\mathcal{T}$ has M-bounded combinatorics, then we say that $\mathcal{T}$ has *(C, M)-bounded outer geometry*. Similarly, bounded and essentially bounded inner geometries are defined.

Lemma 11.3. *For every pair C, M, there is a $K_{C,M} > 1$ such that the following holds. Let D be a closed qc disk and $\mathcal{T}$ be a nest of tilings of ∂D. If $\mathcal{T}$ has (C, M)-bounded inner and outer geometries, then D is a $K_{C,M}$ qc disk.*

Proof. Assume that $\mathcal{T} = [\mathcal{T}^n]_{n\geq -1}$. Then there are at least 4 intervals in $\mathcal{T}_3$. Let us choose base points $u \in \operatorname{int} D$ and $v \in \widehat{\mathbb{C}} \setminus D$ such that the inner and outer harmonic measure of every $I \in \mathcal{T}_3$ with respect to u and v is less than 1/3.

Consider conformal maps $h_- \colon (\operatorname{int} D, u) \to (\mathbb{D}, 0)$ and $h_+ \colon (\widehat{\mathbb{C}} \setminus D, v) \to (\mathbb{D}, 0)$ and define

$$\mathcal{T}^-_n \coloneqq h_{-,*}(\mathcal{T}_n) \qquad \text{and} \qquad \mathcal{T}^+_n \coloneqq h_{+,*}(\mathcal{T}_n)$$

to be the induced partitions on $\mathbb{S}^1 = \partial\mathbb{D}$. The assumptions on the harmonic measures and the width imply that the diameter of every $I \in \mathcal{T}^-_n \cup \mathcal{T}^+_n$ is comparable to the diameters of two neighboring intervals in the same tiling – see the estimates in Lemma 2.5. Therefore, $h_+ \circ h_-^{-1}$ is quasisymmetric with the dilatation bounded in terms C and M. The curve ∂D is a $K_{C,M}$-qc circle as the result of a qc welding. □

11.2. **Estimates for $\mathfrak{D}_n$.** *Let us for the rest of this section view* $\mathbf{K}$ *in Theorems 10.1 and 10.2 as* $\mathbf{K} = O(1)$. In particular, the main estimate in Theorem 10.1 takes form $\mathcal{W}^+_3(I) \leq \mathbf{K} = O(1)$ for every combinatorial interval $I \subset \partial Z$. We will need the following estimates:

Lemma 11.4. *For every diffeo-tiling §2.1.6 $\mathfrak{D}_m$ consisting of at least 4 intervals and every interval $I_j \in \mathfrak{D}_m$ the following holds. Write $L_j \coloneqq I_{j-1} \cup I_j \cup I_{j+1}$, and let $I^m_j, L^{m,c}_j$ be the projections of I_j, L^c_j onto $\partial\widehat{Z}^m$. Then*

(I) $\mathcal{W}^+_{3,\mathfrak{D}_m}(I_j) \coloneqq \mathcal{W}^+_Z\left(I_j, L^c_j\right) \asymp 1$;

(II) $\mathcal{W}_{3,\widehat{\mathfrak{D}}^m_m}(I_j) = \mathcal{W}\left(\mathcal{F}_{3,\widehat{\mathfrak{D}}^m_m}(I_j)\right) \coloneqq \mathcal{W}_{\widehat{Z}^m}\left(I^m_j, L^{m,c}\right) \asymp 1$

(where $\widehat{\mathfrak{D}}^m_m$ denotes the projection of $\mathfrak{D}_m$ onto $\widehat{Z}^m$);

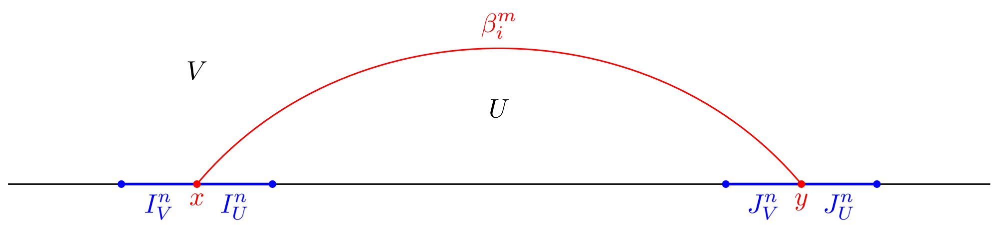


FIGURE 28. Intervals $I_U^n, I_V^n, J_U^n, J_V^n$ form the foundation of β_i^m.

(III) *for an interval $V \in \mathfrak{D}_{m+n}$ such that $V \subset I_j$ and V is attached to one of the endpoints of I_j, we have $\mathcal{W}^+_Z(V, L^c) \asymp_n 1$.*

Proof. It follows from Lemmas 10.3 and 10.4 by splitting I_j as in §10.1.1 that $\mathcal{W}^+_{3,\mathfrak{D}_m}(I_j) \succeq 1$. Since this holds for all the I_j, we obtain Statement (I).

Lemma 6.8 reduces Statement (II) to Lemmas 10.3, 10.4.

Choose a point $w \in \widehat{\mathbb{C}} \setminus \overline{Z}$ such that the intervals $I_{j-1}, I_j, I_{j+1}, L^c_j$ have comparable harmonic measures in $\widehat{\mathbb{C}} \setminus \overline{Z}$ with respect to w. Let $V_n \in \mathfrak{D}_{m+n}, V_0 = I_j$ be a sequence of nested intervals so that V_n is attached to one of the endpoints of I_j. We **claim** that V_n and V_{n-1} have comparable harmonic measures in $(\widehat{\mathbb{C}} \setminus \overline{Z}, w)$; this will imply Statement (III).

Proof of the claim. If $\mathfrak{l}_{m+n} \asymp \mathfrak{l}_{m+n-1}$, then the claim follows from Statement (I).

Assume that $\mathfrak{l}_{m+n-1} \gg \mathfrak{l}_{m+n}$. Let $V'_n \subset V_{n-1} \setminus V_n$ be the interval in $\mathfrak{D}_{m+n}$ attached to another endpoint of V_{n-1}. It follows from $\mathcal{W}^+(X^1, Y^1) \succeq 1$ (in Lemma 10.5) and Statement (I) that $\mathcal{W}^+(V_n, V'_n) \asymp 1$. This implies the claim. □

□

For an interval $I_j \subset \mathfrak{D}_m$, let $\mathcal{R}^+_{\text{dual}}(I_j)$ be the geodesic rectangle (see A.1.12) in $\widehat{\mathbb{C}} \setminus Z$ between I_{j-1} and I_{j+1}; i.e., $\partial^{h,0}\mathcal{R}^+_{\text{dual}}(I_j) = I_{j-1}$, $\partial^{h,1}\mathcal{R}^+_{\text{dual}}(I_j) = I_{j+1}$, and the vertical sides of $\mathcal{R}^+_{\text{dual}}(I_j) = I_{j-1}$ are the hyperbolic geodesics of $\widehat{\mathbb{C}} \setminus \overline{Z}$. It follows from Lemma 11.4, (I) that

$$\mathcal{W}\left(\mathcal{R}^+_{\text{dual}}(I_j)\right) \asymp 1. \tag{11.2}$$

11.3. **Nest of tiling of dams.** Consider a dam $\beta_i^m \subset \partial \widehat{Z}$, and assume that it connects x and y. We recall from Assumption 7 that $x, y \in \mathrm{CP}_{m+1}$. Let us denote by $V = V(\beta_i^m) \ni \infty$ the unbounded component of $\widehat{\mathbb{C}} \setminus (\overline{Z} \cup \beta_i^m)$ and by $U = U(\beta_i^m) \not\ni \infty$ the bounded component of $\widehat{\mathbb{C}} \setminus (\overline{Z} \cup \beta_i^m)$. For every $n \geq m+1$, we specify, see Figure 28:

- I_V^n to be the interval in $\mathfrak{D}_n$ (see §2.1.6) adjacent to x such that $I_V^n \subset \partial V$,
- I_U^n to be the interval in $\mathfrak{D}_n$ adjacent to x such that $I_U^n \subset \partial U$,
- J_V^n to be the interval in $\mathfrak{D}_n$ adjacent to y such that $J_V^n \subset \partial V$,
- J_U^n to be the interval in $\mathfrak{D}_n$ adjacent to y such that $J_U^n \subset \partial U$.

We will refer to $I_U^n, I_V^n, J_U^n, J_V^n \in \mathfrak{D}_n$ as the *nth foundation* of β_i^m.

We say that intervals $A_1, A_2, \dots, A_s \subset \partial S^1$ with pairwise disjoint interiors are *harmonically comparable with respect to* $\mathbb{D}$ if $\mathcal{W}^-_{\mathbb{D}}(I,J) \asymp 1$ for every pair of non-adjacent intervals

$$I, J \in \{A_j, j \le s\} \cup \{\text{connected components of } S^1 \setminus \bigcup_{j=1}^{s} A_j\}.$$

In other words, all the A_j as well as all their complementary intervals have comparable inner harmonic measures with respect to a certain point in $\mathbb{D}$. Similarly, the harmonic comparison is defined for intervals of ∂Z rel $\widehat{\mathbb{C}} \setminus \overline{Z}$. The following lemma is a consequence of Lemma 11.4, Estimates (I) and (III).

Lemma 11.5. *For every* β_i^m *and every* $n > m+1$, *we have:*

- $I_U^{m+1}, I_V^{m+1}, J_U^{m+1}, J_V^{m+1}$ *are harmonically comparable with respect to* $\widehat{\mathbb{C}} \setminus \overline{Z}$,
- $I_U^n, I_V^n, (I_U^{n-2} \cup I_V^{n-2})^c$ *are harmonically comparable with respect to* $\widehat{\mathbb{C}} \setminus \overline{Z}$,
- $J_U^n, J_V^n, (J_U^{n-2} \cup J_V^{n-2})^c$ *are harmonically comparable with respect to* $\widehat{\mathbb{C}} \setminus \overline{Z}$.

□

Lemma 11.6. *There is an absolute* $\mathbf{C} > 0$ *such that for every* β_i^m *there is a nest of tilings* $\mathcal{T}(\beta_i^m) = (\mathcal{T}_n)_{n \ge m+1}$ *with* 10*-bounded combinatorics such that the following properties hold.*

For $V = V(\beta_i^m)$ *as above and every interval* $I \in \mathcal{T}_n \cup \{I_V^n, J_V^n\}$, $n \ge m+1$, *let*

- I_-^V, I_+^V *be two neighboring intervals of* I *in*
$$\mathcal{T}_n(\beta_i^m) \ \cup \ \{\textit{intervals of } \mathfrak{D}_n \textit{ that are in } \partial V\};$$
- $\mathcal{R}^+_{\text{dual},V}(I_j)$ *be the geodesic rectangle (similar to* (11.2)*) in* V *between* I_-^V, I_+^V;

For $U = U(\beta_i^m)$ *and every interval* $I \in \mathcal{T}_n \cup \{I_U^n, J_U^n\}$, $n \ge m+1$, *let*

- I_-^U, I_+^U *be two neighboring intervals of* I *in*
$$\mathcal{T}_n(\beta_i^m) \ \cup \ \{\textit{intervals of } \mathfrak{D}_n \textit{ that are in } \partial U\};$$
- $\mathcal{F}^-_{3,U}(I)$ *be the family of curves in* U *connecting* I *to* $\partial U \setminus (I_-^U \cup I \cup I_+^U)$.

Then

(A) $\mathcal{W}(\mathcal{R}^+_{\text{dual},V}(I_j)) \asymp 1$;

(B) $\mathcal{W}^-_{3,U}(I) = \mathcal{W}(\mathcal{F}^-_{3,U}(I)) \asymp 1$.

Proof. Consider

$$S^1 = \{z : \ |z| = 1\} \subset \mathbb{C} \qquad \text{and} \qquad X = S^1 \cup [-1,1] \subset \mathbb{C}.$$

And let us consider a conformal map $h \colon \widehat{\mathbb{C}} \setminus Z \to \overline{\mathbb{D}}$ mapping x and y to -1 and 1 respectively. Since β_i^m is a hyperbolic geodesic, we have

$$h(\beta_i^m) = [-1,1] \qquad \text{and} \qquad h(\partial Z \cup \beta_i^m) = X.$$

Let $h_*(\mathfrak{D}_n)$ be the pushforward of the diffeo-tiling $\mathfrak{D}_n, n > m$ onto S^1 by h. By Lemma 11.4, any two neighboring intervals in $h_*(\mathfrak{D}_n)$ have comparable diameters (uniformly over n). And by Lemma 11.5, the following diameters are comparable:

- of $h(I_U^{m+1}), h(I_V^{m+1}), h(J_U^{m+1}), h(J_V^{m+1})$;
- of $h(I_U^n), h(I_V^n), h(I_U^{n+1}), h(I_V^{n+1})$ for $n \ge m+1$;
- of $h(J_U^n), h(J_V^n), h(J_U^{n+1}), h(J_V^{n+1})$ for $n \ge m+1$.

For $n \geq m+1$, let ℓ_I^n be the hyperbolic geodesic of $\mathbb{D}$ connecting the endpoints of $h\left(I_U^n \cup I_V^n\right)$, and let x_I^n be the intersection of ℓ_I^n with $[-1,1] \subset X$. We define $I^n := [-1, x_I^n] \subset [-1,1]$.

Similarly, let ℓ_J^n be the hyperbolic geodesic of $\mathbb{D}$ connecting the endpoints of $h\left(J_U^n \cup J_V^n\right)$, and let x_J^n be the intersection of ℓ_J^n with $[-1,1] \subset X$. We define $J^n := [x_J^n, 1] \subset [-1,1]$. By construction, the following diameters are comparable:

- of $I^{m+1}, J^{m+1}, h(I_U^{m+1}), h(I_V^{m+1}), h(J_U^{m+1}), h(J_V^{m+1})$;
- of $I^n, I^{n+1}, h(I_U^n), h(I_V^n), h(I_U^{n+1}), h(I_V^{n+1})$ for $n \geq m+1$;
- of $J^n, J^{n+1}, h(J_U^n), h(J_V^n), h(J_U^{n+1}), h(J_V^{n+1})$ for $n \geq m+1$.

We can now easily extend $\{I^n, J^n\}_{n \geq m+1}$ to a tiling of $[-1,1]$ and then pull it back under h to a required tiling of β_i^m. ☐

11.4. **Nest of tilings on $\partial \widehat{Z}$.** For $m \geq 0$, consider the diffeo-tiling $\mathfrak{D}_m$ of ∂Z. Since level $n \leq m$ dams land at CP_{n+1} (Assumption 7), every interval $T \in \mathfrak{D}_m$ is either regular rel $\widehat{Z}$ or is inside a reclaimed fjord of generation $n < m$. We denote by $\mathfrak{D}'_m$ the set of regular rel $\widehat{Z}$ intervals in $\mathfrak{D}_m$. And we defined $\widehat{\mathfrak{D}}'_m$ to be the set of projections of intervals in $\mathfrak{D}'_m$ onto $\widehat{Z}$. We define

$$\mathcal{T}(\widehat{Z}) := \left[\widehat{\mathfrak{D}}'_n\right]_{n \geq -1} \cup \bigcup_{\beta_i^n} \mathcal{T}(\beta_i^n), \tag{11.3}$$

The following proposition combined with Lemma 11.3 implies Theorem 11.1.

Proposition 11.7. *There is an absolute $\mathbf{C} \gg 1$ such that for every eventually golden-mean rotation number the nest of tilings $\mathcal{T}(\widehat{Z})$ has $(2\mathbf{N}, \mathbf{C})$-bounded inner and outer geometries.*

Proof. By construction, $\mathcal{T}(\widehat{Z})$ has $2\mathbf{N}$ bounded combinatorics. Consider an interval $X \in \mathcal{T}^n(\widehat{Z})$. We need to show that $\mathcal{W}^{\pm}_{3,\mathcal{T}(\widehat{Z})}(X) = O(1)$, where $\mathcal{W}^{\pm}_{3,\mathcal{T}(\widehat{Z})}$ are defined in §11.1. We write

- $X^\bullet := X$ if X is within a dam β_i^m, $m < n$;
- $X^\bullet$ to be the projection of X onto ∂Z if X is an interval in $\partial \widehat{Z}^n$.

Set either

- $\mathcal{R} := \mathcal{R}^+_{\mathrm{dual},V}(X^\bullet)$ as in Lemma 11.6 if $X \in \left[\mathcal{T}_n \cup \{I_V^n, J_V^n\}\right](\beta_i^m)$ for some dam β_i^m;
- or, otherwise, $\mathcal{R} := \mathcal{R}^+_{\mathrm{dual}}(X^\bullet)$ as in (11.2).

In both cases, we have $\mathcal{W}(\mathcal{R}) \asymp 1$. Let $\mathcal{R}^n$ be the restriction (as in §5.2.4) of $\mathcal{R}$ onto $\widehat{\mathbb{C}} \setminus \widehat{Z}^n$. By (5.12), we have $\mathcal{W}(\mathcal{R}^n) \asymp 1$. Since the curves in $\mathcal{F}^+_{3,\mathcal{T}(\widehat{Z})}(X)$ cross $\mathcal{R}^n$, we obtain

$$\mathcal{W}^+_{3,\mathcal{T}(\widehat{Z})}(X) = O(1).$$

To show $\mathcal{W}^-_{3,\mathcal{T}(\widehat{Z})}(X) = O(1)$, we will use the monotonicity of the width under the embeddings $\widehat{Z}^n, U(\beta_i^m) \subset \widehat{Z} = \widehat{Z}^{-1}$. Consider several cases.

Assume first that $X \subset \partial \widehat{Z}^n$ and X is not a neighbor of any dam β_i^m, $m < n$. Then $\mathcal{W}^-_{3,\mathcal{T}(\widehat{Z})}(X) \leq \mathcal{W}^-_{3,\widehat{\mathfrak{D}}^n_n(\widehat{Z}^n)}(X) = O(1)$ by Theorem 5.12, where $\widehat{\mathfrak{D}}^n_n$ is the projection of $\mathfrak{D}_n$ onto $\widehat{Z}^n$.

In the remaining case, we have $X^\bullet \in \mathcal{T}_n \cup \{I^n_V, J^n_V\}$, $n \geq m+1$ as in Lemma 11.6 for a dam β^m_i. If X is in the interior of β^m_i, then $\mathcal{W}^-_{3,\mathcal{T}(\widehat{Z})}(X) \leq \mathcal{W}^-_{3,U}(X) = O(1)$ by Lemma 11.6, (B) because $\mathcal{F}^-_{3,\mathcal{T}(\widehat{Z})}(X)$ overflows $\mathcal{F}^-_{3,U}(X)$; i.e. because $U(\beta^m_i) \subset \widehat{Z}$.

Assume finally that X touches one of the endpoints of β^m_i. If $X \subset \beta^m_i$, then $\mathcal{F}^-_{3,\mathcal{T}(\widehat{Z})}(X)$ overflows

$$\mathcal{F}^-_{3,U}(X) \cup \mathcal{F}_{3,\widehat{\mathfrak{D}}^n_n}\left(\left[I^n_U(\beta^m_i)\right]^n\right) \cup \mathcal{F}_{3,\widehat{\mathfrak{D}}^n_n}\left(\left[J^n_U(\beta^m_i)\right]^n\right)$$

(see Lemma 11.4, (II); here $[\ \ ]^n$ denotes the projection onto $\widehat{Z}^n$); otherwise $X \in \left[\{I^n_V, J^n_V\}\right](\beta^m_i)$ and $\mathcal{F}^-_{3,\mathcal{T}(\widehat{Z})}(X)$ overflows

$$\mathcal{F}_{3,\widehat{\mathfrak{D}}^n_n}(X) \cup \mathcal{F}^-_{3,U}\left(\left[I^n_U(\beta^m_i)\right]^n\right) \cup \mathcal{F}^-_{3,U}\left(\left[J^n_U(\beta^m_i)\right]^n\right).$$

Lemma 11.4, (II) and Lemma 11.6, (B) complete the proof. □

Appendix A. Degeneration of Riemann surfaces

Consider a compact Riemann surface $S \Subset \mathbb{C}$ with boundary. We assume that ∂S consists of finitely many components. In this subsection, we will recall basic tools to detect degenerations of S, see [A, L2, KL1] for details. The discussion can be adjusted for open Riemann surfaces in $\mathbb{C}$ by considering their Caratheodory boundaries.

A.1. **Rectangles and Laminations.**

Given two disjoint intervals $I, J \subset \partial S$ on the boundary of a Riemann surface, we denote by

- $\mathcal{F}_S(I,J)$ the family of curves in S connecting I and J:

(A.1) $$\mathcal{F}_S(I,J) \coloneqq \{\gamma\colon [0,1] \to S \mid \gamma(0) \in I,\ \ \gamma(1) \in J\};$$

- $\mathcal{W}_S(I,J) = \mathcal{W}(\mathcal{F}_S(I,J))$ the extremal width between I, J – the modulus of the family $\mathcal{F}_S(I,J)$.

We will often write $\mathcal{F}^-(I,J) = \mathcal{F}_S(I,J)$ and $\mathcal{W}^-(I,J) = \mathcal{W}_S(I,J)$ when the surface S is fixed.

A.1.1. *Rectangles.* A *Euclidean rectangle* is a rectangle $E_x \coloneqq [0,x] \times [0,1] \subset \mathbb{C}$, where:

- $(0,0), (x,0), (x,1), (0,1)$ are four vertices of E_x,
- $\partial^h E_x = [0,x] \times \{0,1\}$ is the horizontal boundary of E_x,
- $\partial^{h,0} E_x \coloneqq [0,x] \times \{0\}$ is the *base* of E_x,
- $\partial^{h,1} E_x \coloneqq [0,x] \times \{1\}$ is the *roof* of E_x,
- $\partial^v E_x = \{0,x\} \times [0,1]$ is the *vertical* boundary of E_x,
- $\partial^{v,\ell} E_x \coloneqq \{0\} \times [0,1]$, $\partial^{v,\rho} E_x \coloneqq \{x\} \times [0,1]$ is the *left* and *right vertical* boundaries of E_x;
- $\mathcal{F}(E_x) \coloneqq \{\{t\} \times [0,1] \mid t \in [0,x]\}$ is the *vertical foliation* of E_x,
- $\mathcal{F}^{\text{full}}(E_x) \coloneqq \{\gamma\colon [0,1] \to E_x \mid \gamma(0) \in \partial^{h,0} E_x,\ \gamma(1) \in \partial^{h,1} E_x\}$ is the *full family of curves* in E_x;
- $\mathcal{W}(E_x) = \mathcal{W}(\mathcal{F}(E_x)) = \mathcal{W}(\mathcal{F}^{\text{full}}(E_x)) = x$ is the *width* of E_x,
- $\operatorname{mod}(E_x) = 1/\mathcal{W}(E_x) = 1/x$ the extremal length of E_x.

By a *(topological) rectangle in* $\mathbb{C}$ we mean a closed Jordan disk $\mathcal{R}$ together with a conformal map $h\colon \mathcal{R} \to E_x$ into the standard rectangle E_x. The vertical foliation $\mathcal{F}(\mathcal{R})$, the full family $\mathcal{F}^{\text{full}}(\mathcal{R})$, the base $\partial^{h,0}\mathcal{R}$, the roof $\partial^{h,1}\mathcal{R}$, the vertices of $\mathcal{R}$, and other objects are defined by pulling back the corresponding objects of E_x. Equivalently, a rectangle $\mathcal{R} \subset \mathbb{C}$ is a closed Jordan disk together with four marked vertices on $\partial\mathcal{R}$ and a chosen base between two vertices.

A *genuine subrectangle* of E_x is any rectangle of the form $E' = [x_1, x_2] \times [0,1]$, where $0 \le x_1 < x_2 \le x$; it is identified with the standard Euclidean rectangle $[0, x_2 - x_1] \times [0,1]$ via $z \mapsto z - x_1$. A genuine subrectangle of a topological rectangle is defined accordingly.

A *subrectangle* of a rectangle $\mathcal{R}$ is a topological rectangle $\mathcal{R}_2 \subset \mathcal{R}$ such that $\partial^{h,0}\mathcal{R}_2 \subset \mathcal{R}$ and $\partial^{h,1}\mathcal{R}_2 \subset \mathcal{R}$. By monotonicity: $\mathcal{W}(\mathcal{R}_2) \le \mathcal{W}(\mathcal{R})$.

Assume that $\mathcal{W}(E_x) > 2$. The *left and right* 1*-buffers* of E_x are defined

$$B_1^\ell \coloneqq [0,1] \times [0,1] \quad \text{and} \quad B_1^\rho \coloneqq [x-1, x] \times [0,1]$$

respectively. We say that the rectangle

$$E_x^{\text{new}} \coloneqq [1, x-1] \times [0,1] = E_x \setminus \left(B_1^\ell \cup B_1^\rho\right)$$

is obtained from E_x by *removing* 1*-buffers.* If $\mathcal{W}(E_x) \le 2$, then we set $E_x^{\text{new}} \coloneqq \emptyset$. Similarly, buffers of any width are defined.

A.1.2. *Annuli.* A *closed annulus A of modulus* $1/x$ is a Riemann surface obtained from E_x by gluing its vertical boundaries:

$$A \coloneqq E_x/_{\partial^{v,\ell}E_x \ni (0,y) \sim (x,y) \in \partial^{v,\rho}E_x| \quad \forall y} \quad , \quad \mathcal{W}(A) = x, \quad \operatorname{mod}(A) \coloneqq 1/x.$$

Its interior $\operatorname{int}(A)$ is an *open annulus with modulus* x. The induced image of the vertical foliation $\mathcal{F}(E_x)$ is the *vertical foliation* $\mathcal{F}(A)$ *of* A. The width of $\mathcal{F}(A)$ is equal to the width of all the curves in A connecting its boundaries $\partial^{h,0}A, \partial^{h,1}A$ – the induced images of the horizontal boundaries $\partial^{h,0}E_x, \partial^{h,1}E_x$.

A.1.3. *Monotonicity and Grötzsch inequality.* We say a family of curves $\mathcal{S}$ *overflows* a family $\mathcal{G}$ if every curve $\gamma \in \mathcal{S}$ contains a subcurve $\gamma' \in \mathcal{G}$. We also say that

- a family of curves $\mathcal{F}$ *overflows* a rectangle $\mathcal{R}$ if $\mathcal{F}$ overflows $\mathcal{F}^{\text{full}}(\mathcal{R})$;
- a rectangle $\mathcal{R}_1$ overflows another rectangle $\mathcal{R}_2$ if $\mathcal{F}(\mathcal{R}_1)$ overflows $\mathcal{F}^{\text{full}}(\mathcal{R}_2)$.

If $\mathcal{F}$ overflows a family or a rectangle $\mathcal{G}$, then $\mathcal{G}$ is wider than $\mathcal{F}$:

$$\mathcal{W}(\mathcal{F}) \le \mathcal{W}(\mathcal{G}). \tag{A.2}$$

If $\mathcal{F}$ overflows both $\mathcal{G}_1, \mathcal{G}_2$, and $\mathcal{G}_1, \mathcal{G}_2$ are disjointly supported, then the *Grötzsch* inequality states:

$$\mathcal{W}(\mathcal{F}) \le \mathcal{W}(\mathcal{G}_1) \oplus \mathcal{W}(\mathcal{G}_2), \tag{A.3}$$

where $x \oplus y = (x^{-1} + y^{-1})^{-1}$ is the harmonic sum.

A.1.4. *Parallel Law.* For any families of curves $\mathcal{G}_1, \mathcal{G}_2$, we have:

$$\mathcal{W}(\mathcal{G}_1 \cup \mathcal{G}_2) \le \mathcal{W}(\mathcal{G}_1) + \mathcal{W}(\mathcal{G}_2). \tag{A.4}$$

If $\mathcal{G}_1, \mathcal{G}_2$ are disjointly supported, then

$$\mathcal{W}(\mathcal{G}_1 \cup \mathcal{G}_2) = \mathcal{W}(\mathcal{G}_1) + \mathcal{W}(\mathcal{G}_2). \tag{A.5}$$

A.1.5. *Restriction of families.* Consider a family of curves $\mathcal{G}$ connecting X and Y. And suppose

$$\widetilde{X} \supset X, \quad \widetilde{Y} \supset Y, \quad \widetilde{X} \cap \widetilde{Y} = \emptyset$$

are enlargements. Then every curve

$$[\gamma\colon [0,1] \to \widehat{\mathbb{C}}] \in \mathcal{G}$$

has a *unique first shortest* subcurve $\gamma' \subset \gamma$ connecting $\widetilde{X}$ and $\widetilde{Y}$: there is a minimal $t_1 \geq 0$ for which there is a $t_2 > t_1$ such that

$$\gamma\big((t_1, t_2)\big) \subset \mathbb{C} \setminus (\widetilde{X} \cup \widetilde{Y}), \qquad \text{and} \qquad \gamma(t_1) \in \widetilde{X}, \ \ \gamma(t_2) \in \widetilde{Y};$$

we set $\gamma' := \gamma \mid [t_1, t_2]$. Define $\mathcal{G}^{\text{new}}$ to be the family consisting of γ' for all $\gamma \in \mathcal{G}$. Since $\mathcal{G}$ overflows $\mathcal{G}^{\text{new}}$, we have (see §A.1.3):

$$\mathcal{W}(\mathcal{G}^{\text{new}}) \geq \mathcal{W}(\mathcal{G}).$$

Note that if $\mathcal{G}$ is a lamination, then so is $\mathcal{G}^{\text{new}}$.

Consider now the following generalization. For a lamination $\mathcal{G}$ and disjoint sets $X_1, X_2, \dots, X_m$ suppose that the following holds. Every curve $\gamma \in \mathcal{G}$ intersects all the X_i and it intersects X_i before intersecting any X_{i+j} for $j > 0$. Then every $\gamma \in \mathcal{G}$ contains disjoint subcurves $\gamma_1, \gamma_2, \dots, \gamma_{m-1}$ where γ_i is the first shortest subcurve between X_i and X_{i_1}. Setting $\mathcal{G}_i$ to be the set of γ_i over all $\gamma \in \mathcal{G}$, we obtain that $\mathcal{G}$ *overflows consequently* $\mathcal{G}_i$ and, by §A.1.3:

$$\mathcal{W}(\mathcal{G}) \leq \mathcal{W}(\mathcal{G}_1) \oplus \dots \oplus \mathcal{W}(\mathcal{G}_{m-1}).$$

Note that $\mathcal{G}_i$ are disjoint laminations.

A.1.6. *Canonical rectangles.* Consider a closed Jordan disk $D \subset \mathbb{C}$ together with disjoint intervals $I, J \subset \partial D$. We denote by $\mathcal{F}^-(I,J), \mathcal{F}^+(I,J), \mathcal{F}(I,J)$ the families of curves in D, $\widehat{\mathbb{C}} \setminus \operatorname{int} D$, $\widehat{\mathbb{C}} \setminus (I \cup J)$ connecting I, J. The widths of these families are denoted by $\mathcal{W}^-(I,J), \mathcal{W}^+(I,J), \mathcal{W}(I,J)$.

We can view D as a rectangle $\mathcal{R}$ with $\partial^v \mathcal{R} = I \cup J$. We call $\mathcal{R}$ the *canonical rectangle of* $\mathcal{F}^-(I,J)$; we have $\mathcal{W}(\mathcal{R}) = \mathcal{W}^-(I,J)$. Similarly, viewing $\widehat{\mathbb{C}} \setminus \operatorname{int} D$ as a rectangle $\mathcal{R}_2$ with $\partial^h \mathcal{R}_2 = I \cup J$, we obtain the *canonical rectangle* $\mathcal{R}_2$ of $\mathcal{F}^+(I,J)$; we have $\mathcal{W}^+_D(I,J) = \mathcal{W}(\mathcal{R}_2)$.

Observe that $A := \widehat{\mathbb{C}} \setminus (I \cup J)$ is an open annulus; its Caratheodory boundary consists of $I^- \cup I^+$ and $J^- \cup J^+$, where $I^-,\ J^-$ are the sides of I, J from $\operatorname{int} D$ while $I^+,\ J^+$ are the sides of I, J from $\widehat{\mathbb{C}} \setminus D$. The *vertical family* $\mathcal{H}$ of $\mathcal{F}(I,J)$ consists of vertical curves of A together with their landing points. We have $\mathcal{W}(\mathcal{H}) = \mathcal{W}(I,J)$.

A.1.7. *Innermost and outermost curves.* It will be convenient for us to use the following inner-outer order on vertical curves in rectangles. Consider a rectangle

$$\mathcal{R} \subset \widehat{\mathbb{C}}, \qquad \text{with} \quad \partial^h \mathcal{R} \subset \partial D,$$

where D is a closed Jordan disk, such that $\mathcal{R}$ is disjoint from a complementary interval $N \subset \partial D$ between $\partial^{h,0}\mathcal{R}, \partial^{h,1}\mathcal{R}$. Let N^- and N^+ be two sides of N from the inside and outside of D. Consider a set of vertical curves $\{\ell_i\}_i \subset \mathcal{F}(\mathcal{R})$. The *innermost* curve of $\{\ell_i\}_i$ is the curve ℓ_{inn} separating N^- from all remaining ℓ_i in $\widehat{\mathbb{C}} \setminus (\partial^h \mathcal{R} \cup N)$. The *outermost* curve of $\{\ell_i\}_i$ is the curve ℓ_{out} separating N^+ from all remaining ℓ_i in $\widehat{\mathbb{C}} \setminus (\partial^h \mathcal{R} \cup N)$.

A.1.8. *Laminations.* By a *lamination* $\mathcal{L}$ we mean a family of pairwise disjoint simple rectifiable arcs such that $\operatorname{supp}\mathcal{L}$ is measurable. A *sublamination* of $\mathcal{L}$ is any collection $\mathcal{H}$ of arcs from $\mathcal{G}$ such that $\operatorname{supp}\mathcal{H}$ is measurable.

Laminations naturally appear as restrictions of rectangles – see §A.1.5. Note that a restriction of a rectangle is usually not a rectangle as discussed in [KL1, §2.3]. For convenience, a lamination $\mathcal{G}$ can often be replaced by a rectangle $\mathcal{R}$ bounded by the left- and rightmost curves of $\mathcal{G}$; then $\mathcal{W}(\mathcal{R}) \geq \mathcal{W}(\mathcal{G})$.

All laminations in our paper will appear from rectangles using basic operations like restrictions and finite unions.

A.1.9. *Restrictions of sublaminations.* Consider a lamination or a rectangle $\mathcal{R}$, and let $\widetilde{\mathcal{S}}$ be a sublamination of $\mathcal{R}$ (or of $\mathcal{F}(\mathcal{R})$). Assume that $\widetilde{\mathcal{S}}$ overflows a lamination $\mathcal{S}$. Then we write

$$\mathcal{W}(\mathcal{R}|\mathcal{S}) \coloneqq \mathcal{W}(\widetilde{\mathcal{S}}) \qquad \text{(note that } \mathcal{W}(\mathcal{R}|\mathcal{S}) \leq \mathcal{W}(\mathcal{R})\text{)}. \tag{A.6}$$

A.1.10. *Splitting Rectangles.*

Lemma A.1. *Consider a Jordan disk D and let $I, J \subset \partial D$ be a pair of disjoint intervals. Consider an arc ℓ in the canonical rectangle of $\mathcal{F}^-_D(I,J)$, §A.1.6. Suppose ℓ splits I and J into I_1, I_2 and J_1, J_2 enumerated so that the pairs I_1, J_1 and I_2, J_2 are on the same side of ℓ. We denote by D_1 and D_2 connected components of $D \setminus \ell$ containing I_1, J_1 and I_2, J_2 on its boundaries respectively. Then*

$$\mathcal{W}^-_D(I_1,J_1) + \mathcal{W}^-_D(I_2,J_2) - 2 \leq \mathcal{W}^-_{D_1}(I_1,J_1) + \mathcal{W}^-_{D_2}(I_2,J_2) =$$
$$= \mathcal{W}^-_D(I,J) \leq \mathcal{W}^-_D(I_1,J_1) + \mathcal{W}^-_D(I_2,J_2).$$

Proof. The last inequality is immediate. Let

- $\mathcal{R}$ be the canonical rectangle of $\mathcal{F}^-_D(I,J)$;
- $\mathcal{R}_1, \mathcal{R}_2$ be the canonical rectangles of $\mathcal{F}^-_{D_1}(I_1,J_1), \mathcal{F}^-_{D_2}(I_2,J_2)$;
- $\widetilde{\mathcal{R}}_1, \widetilde{\mathcal{R}}_2$ be the canonical rectangles of $\mathcal{F}^-_D(I_1,J_1), \mathcal{F}^-_D(I_2,J_2)$.

Since ℓ splits $\mathcal{R}$ into $\mathcal{R}_1, \mathcal{R}_2$, we have

$$\mathcal{W}^-_{D_1}(I_1,J_1) + \mathcal{W}^-_{D_2}(I_2,J_2) = \mathcal{W}(\mathcal{R}_1) + \mathcal{W}(\mathcal{R}_2) = \mathcal{W}(\mathcal{R}) = \mathcal{W}^-_D(I,J).$$

By removing 1-buffers from $\widetilde{\mathcal{R}}_1, \widetilde{\mathcal{R}}_2$, we obtain new disjoint rectangles $\widetilde{\mathcal{R}}^{\text{new}}_1, \widetilde{\mathcal{R}}^{\text{new}}_2$; since $\widetilde{\mathcal{R}}^{\text{new}}_1 \sqcup \widetilde{\mathcal{R}}^{\text{new}}_2 \subset \mathcal{R}$, we have

$$\mathcal{W}(\widetilde{\mathcal{R}}^{\text{new}}_1) + \mathcal{W}(\widetilde{\mathcal{R}}^{\text{new}}_2) \leq \mathcal{W}(\mathcal{R}).$$

□

Lemma A.2. *Under the assumptions of Lemma A.1, let $\mathcal{G}_1$ be the family of curves in D connecting I to J such that every curve in $\mathcal{G}_1$ intersects D_1. Then*

$$\mathcal{W}^-_{D_1}(I_1,J_1) \leq \mathcal{W}(\mathcal{G}_1) \leq \mathcal{W}^-_{D_1}(I_1,J_1) + 2$$

Proof. As in the proof of Lemma A.1, let $\mathcal{R}$ be the canonical rectangle of $\mathcal{F}^-_D(I,J)$ and $\mathcal{R}_1$ be the canonical rectangle of $\mathcal{F}^-_{D_1}(I_1,J_1)$. Since $\mathcal{R}_1$ is a genuine subrectangle of $\mathcal{R}$, we can consider the genuine subrectangle $\mathcal{R}^+_1$ of $\mathcal{R}$ specified by

$$\mathcal{R}_1 \subset \mathcal{R}^+_1 \qquad \text{and} \qquad \mathcal{W}(\mathcal{R}^+_1) = \mathcal{W}(\mathcal{R}_1) + 1;$$

i.e. $\mathcal{R}_1$ is $\mathcal{R}^+_1$ minus its one 1-buffer $\mathcal{B}$. The width of curves in $\mathcal{G}_1$ crossing $\mathcal{B}$ is less than 1; the remaining curves of $\mathcal{G}_1$ are in $\mathcal{R}^+_1$. □

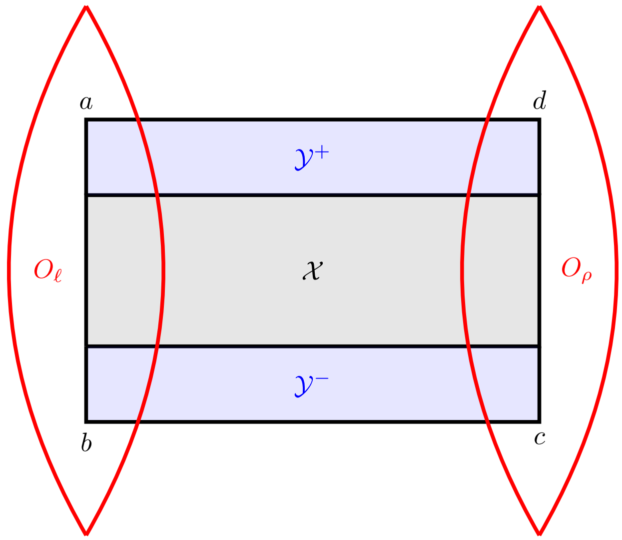


FIGURE 29. Assume a rectangle $\mathcal{Y}$ is a union $\mathcal{Y}^+ \cup \mathcal{X} \cup \mathcal{Y}^-$ such that $\mathcal{W}(\mathcal{Y}^+) \asymp \mathcal{W}(\mathcal{Y}^-) \asymp \operatorname{mod}(O_\ell \setminus [a,b]) \asymp \operatorname{mod}(O_\rho \setminus [c,d]) \asymp 1$. Then $\operatorname{mod}(O_\ell \cup \mathcal{Y} \cup O_\rho \setminus \mathcal{X}) \succeq 1$, see Lemma A.4.

A.1.11. *Enclosed annuli.* Let $A, B \subset \mathbb{C}$ be two closed annuli surrounding open disks U and V respectively. Assume that

- $U \cup V$ is an open topological disk;
- $A \cup U \cup B \cup V$ is a closed topological disk.

Then the *enclosed annulus* is

$$A \square B := \big(A \cup U \cup B \cup V\big) \setminus \big(U \cup V\big).$$

Lemma A.3. *If* $\operatorname{mod}(A), \operatorname{mod}(B) \geq 2\varepsilon$*, then* $\operatorname{mod}(A \square B) \geq \varepsilon$*.*

Proof. Let $\gamma \colon [0,1] \to A \square B$ be a vertical curve of the annulus $A \square B$. Assume first that $\gamma(0) \in \partial^{\text{inn}} A$. Then for some $t \in (0,1]$ we have $\gamma(t) \in \partial^{\text{out}} A$; i.e. γ crosses A. Similarly, if $\gamma(0) \in \partial^{\text{inn}} B$, then γ crosses B. By the Parallel Law §A.1.4, $\mathcal{W}(A \square B) \leq \mathcal{W}(A) + \mathcal{W}(B) \leq 1/\varepsilon$. □

Let us consider the following construction which will be used in §7. Suppose, see Figure 29:

- a rectangle $\mathcal{Y}$ is a union of its genuine subrectangles $\mathcal{Y}^+, \mathcal{X}, \mathcal{Y}^-$ with disjoint interiors, where $\mathcal{X}$ is between $\mathcal{Y}^+$ and $\mathcal{Y}^-$;
- closed disks O_ℓ, O_ρ contain $\partial^{h,0}\mathcal{Y} = [a,b]$ and $\partial^{h,1}\mathcal{Y} = [c,d]$ respectively;
- $O_\ell \cup \mathcal{Y} \cup O_\rho$ is a closed topological disk.

Lemma A.4. *For* O_ℓ*,* $\mathcal{Y} = \mathcal{Y}^+ \cup \mathcal{X} \cup \mathcal{Y}^-$*,* O_ρ *as above, if*

$$\mathcal{W}(\mathcal{Y}^+) \asymp \mathcal{W}(\mathcal{Y}^-) \asymp \operatorname{mod}(O_\ell \setminus [a,b]) \asymp \operatorname{mod}(O_\rho \setminus [c,d]) \asymp 1,$$

then $\operatorname{mod}(O_\ell \cup \mathcal{Y} \cup O_\rho \setminus \mathcal{X}) \succeq 1$*.*

Proof. Consider a vertical curve γ of the annulus $O_\ell \cup \mathcal{Y} \cup O_\rho \setminus \mathcal{X}$.

- If γ intersects $[a,b]$, then γ crosses the annulus $O_\ell \setminus [a,b]$.
- If γ intersects $[c,d]$, then γ crosses the annulus $O_\rho \setminus [c,d]$.
- If γ is disjoint from $[a,b]\cup[c,d]$, then γ crosses either $\mathcal{Y}^+$ or $\mathcal{Y}^-$.

By the Parallel Law §A.1.4, $\operatorname{mod}(O_\ell \cup \mathcal{Y} \cup O_\rho \setminus \mathcal{X}) \succeq 1$. □

A.1.12. *Geodesic Rectangles.* Let D be a closed Jordan disk, and consider two closed disjoint intervals $I, J \subset \partial D$. The *geodesic rectangle* $\mathcal{R}(I,J)$ in $\overline{D}$ is a rectangle such that

$$\partial^{h,0}\mathcal{R}(I,J) = I, \qquad \partial^{h,1}\mathcal{R}(I,J) = J,$$

and the vertical sides of $\mathcal{R}(I,J)$ are the hyperbolic geodesics of D.

Lemma A.5. *Let $\mathcal{R}$ with $\mathcal{W}(\mathcal{R}) > 1$ be a rectangle in $\overline{D}$ with $\partial^{h,0}\mathcal{R} = I$ and $\partial^{h,1}\mathcal{R} = J$. Let $\mathcal{R}^{\text{new}}$ be the rectangle obtained from $\mathcal{R}$ by removing two $1/2$-buffers on each side. Write $\partial^{h,0}\mathcal{R}^{\text{new}} = I^{\text{new}} \subset I$ and $\partial^{h,1}\mathcal{R}^{\text{new}} = J^{\text{new}} \subset J$. Then*

$$\mathcal{R} \supset \mathcal{R}(I^{\text{new}}, J^{\text{new}}) \qquad \textit{and} \qquad \mathcal{R}^{\text{new}} \subset \mathcal{R}(I,J),$$

where $\mathcal{R}(I^{\text{new}}, J^{\text{new}}), \mathcal{R}(I,J)$ are geodesic rectangles as above.

In particular, $\mathcal{R}$ can be replaced with a geodesic subrectangle $\mathcal{R}(I^{\text{new}}, J^{\text{new}})$ so that $\mathcal{W}(\mathcal{R}) - \mathcal{W}[\mathcal{R}(I^{\text{new}}, J^{\text{new}})] \le 1$.

Proof. We can assume that $\mathcal{R}' = D = E_x$, where E_x is a Euclidean rectangle, see §A.1.1. Then the lemma follows by appropriately applying the following **claim**: the hyperbolic geodesic $\gamma \subset E_x$ connecting $(0,0)$ and $(0,1)$ is within $E_{1/2}$ – the left $1/2$-buffer of E_x.

To prove the claim about γ, consider the right half-plan $\mathbb{C}_{>0} := \{z \mid \operatorname{Re} z > 0\}$. Then the hyperbolic geodesic $\widetilde{\gamma} \subset \mathbb{C}_{>0}$ connecting $(0,0)$ and $(0,1)$ is the semicircle orthogonal to the imaginary line; i.e. $\widetilde{\gamma} \subset E_{1/2}$. Since $E_x \subset \mathbb{C}_{>0}$, we also obtain that $\gamma \subset E_{1/2}$. □

It follows from Lemma A.5 that if $J \subset \partial D$ is a concatenation of subintervals $J_1 \# J_2$ and $I \subset \partial D$, then

$$\mathcal{W}^-(I,J) = \mathcal{W}^-(I,J_1) + \mathcal{W}^-(I,J_2) - O(1). \tag{A.7}$$

A.2. **Small overlapping of wide families.** Many arguments in the near-degenerate regime are based on the principle that wide families have a relatively small overlap.

A.2.1. *Non-Crossing Principle.* Consider a closed Jordan disk D and let

$$\mathcal{R}_1, \mathcal{R}_2 \subset D, \qquad \partial^h\mathcal{R}_1, \partial^h\mathcal{R}_2 \subset \partial D, \qquad \partial^h\mathcal{R}_1 \cap \partial^h\mathcal{R}_2 = \emptyset$$

be two rectangles. If $\mathcal{W}(\mathcal{R}_1), \mathcal{W}(\mathcal{R}_2) > 1$, then $\mathcal{R}_1, \mathcal{R}_2$ *do not cross-intersect*: there are vertical curves $\gamma_1 \in \mathcal{F}(\mathcal{R}_1)$ and $\gamma_2 \in \mathcal{F}(\mathcal{R}_2)$ with $\gamma_1 \cap \gamma_2 = \emptyset$. Indeed, assuming otherwise, we obtain

$$1/\mathcal{W}(\mathcal{R}_1) = \operatorname{mod}(\mathcal{R}_1) \ge \mathcal{W}(\mathcal{R}_2)$$

by monotonicity of the external length.

A.2.2. *Vertical boundaries.* The following lemma is a slight generalization of [KL1, Lemma 2.14].

Lemma A.6. *For every $\varepsilon > 0$ the following holds. Consider rectangles*

$$\mathcal{G},\quad \mathcal{R}_1, \mathcal{R}_2, \dots, \mathcal{R}_n \subset \widehat{\mathbb{C}}, \qquad \mathcal{W}(\mathcal{G}),\ \mathcal{W}(\mathcal{R}_i) > 8n + 2\varepsilon$$

such that the $\mathcal{R}_i$ are pairwise disjoint. Then after removing buffers of width at most $4n+\varepsilon$, we can assume that the new rectangles $\mathcal{G}^{\text{new}}, \mathcal{R}_1^{\text{new}}, \mathcal{R}_2^{\text{new}}, \dots, \mathcal{R}_n^{\text{new}}$ have disjoint vertical boundaries.

Proof. We need the following fact:

Lemma A.7 ([KL1, Lemma 2.13])**.** *Consider two laminations $\Lambda, \mathcal{G}$ such that Λ is a sublamination of the vertical foliation of a rectangle. If $\mathcal{W}(\Lambda) > \kappa$ and $\mathcal{W}(\mathcal{G}) \geq \kappa \geq 1$, then there is a curve $\ell \in \mathcal{G}$ that intersects less than $\frac{1}{\kappa}\mathcal{W}(\Lambda)$ of curves in Λ.*

Let $\mathcal{G}_\pm, \mathcal{R}_{\pm,n}$ be the buffers of width $4n+\varepsilon$ in $\mathcal{G}, \mathcal{R}_n$. Applying Lemma A.7, we can select vertical curves $\gamma_{-,n} \in \mathcal{R}_{-,n}, \gamma_{+,n} \in \mathcal{R}_{+,n}$ so that each $\gamma_{\pm,n}$ intersects less than $\frac{1}{4n}\mathcal{W}(\mathcal{G}_- \sqcup \mathcal{G}_+) = \frac{1}{2n}\mathcal{W}(\mathcal{G}_\pm)$ curves in $\mathcal{G}_- \sqcup \mathcal{G}_+$. Therefore, there are curves $\beta_- \in \mathcal{G}_-$, $\beta_+ \in \mathcal{G}_+$ that are disjoint from all the $\gamma_{\pm,n}$. We set $\beta_\pm, \gamma_{\pm,n}$ to be the vertical boundaries of $\mathcal{G}^{\text{new}}, \mathcal{R}_1^{\text{new}}, \mathcal{R}_2^{\text{new}}, \dots, \mathcal{R}_n^{\text{new}}$. □

A.2.3. *Crossing an annulus.* Let $A \subset \mathbb{C}$ be an annulus and $\mathcal{G}$ be a family of curves such that every its curve starts in the unbounded component U of $\mathbb{C} \setminus A$. Then at most $1/\operatorname{mod} A$ curves in $\mathcal{G}$ intersect the bounded component O of $\mathbb{C} \setminus A$. Indeed, every curve $\gamma \in \mathcal{G}$ intersecting O contains a subcurve γ' connecting the inner and outer boundaries of A. The width of such γ is at most $1/\operatorname{mod} A$.

Lemma A.8. *Let $D \subset \mathbb{C}$ be a closed Jordan disk and $A, \mu = \operatorname{mod} A$ be a closed topological annulus such that the bounded component O of $\mathbb{C} \setminus A$ intersects ∂D. Then for every rectangle*

$$\mathcal{R} \subset D \qquad \text{such that} \quad \partial^h \mathcal{R} \subset D \setminus (A \cup O),$$

after removing two $1/\mu$-buffers from $\mathcal{R}$, the new rectangle $\mathcal{R}^{\text{new}}$ is disjoint from O.

We will need the following topological property:

Lemma A.9. *Let $D \subset \mathbb{C}$ be a closed Jordan disk together with a rectangle*

$$\mathcal{R} \subset D, \qquad \partial^h \mathcal{R} \subset \partial D.$$

Let $O \subset \mathbb{C} \setminus \partial^h \mathcal{R}$ be a connected set intersecting $\partial D \setminus \partial^h \mathcal{R}$. If $\mathcal{R}$ intersects O, then the set of vertical curves in $\mathcal{R}$ intersecting O forms either one or two buffers of $\mathcal{R}$. If, moreover, O intersects exactly one component of $\partial D \setminus \partial^h \mathcal{R}$, then the set of vertical curves in $\mathcal{R}$ intersecting O forms a buffer of $\mathcal{R}$.

Proof. If $\gamma_1, \gamma_2 \in \mathcal{F}(\mathcal{R})$ are two curves disjoint from O, then all vertical curves of $\mathcal{R}$ between γ_1 and γ_2 are also disjoint from O – otherwise O would be enclosed by $\partial^h \mathcal{R} \cup \gamma_1 \cup \gamma_2$. Therefore, the set of vertical curves intersecting O form one or two buffers.

Assume there are two buffers. Then there will be a vertical curve $\gamma \in \mathcal{F}(\mathcal{R}) \setminus \partial^v \mathcal{R}$ that is disjoint from O. Since O is disjoint from $\gamma \cup \partial^h \mathcal{R}$ and since O intersects both $\partial^{v,\ell}\mathcal{R}, \partial^{v,\rho}\mathcal{R}$, the set O intersects both components of $\partial D \setminus \partial^h \mathcal{R}$. □

Proof of Lemma A.8. At most $1/\mu$ vertical curves in $\mathcal{R}$ can cross A and all such curves form one or two buffers of $\mathcal{R}$ by Lemma A.9. □

A.2.4. *Push-forwards.* Suppose $f\colon S_1 \to S_2$ is a branched covering between Riemann surfaces of degree d. Let $\mathcal{G}$ be a family of curves in S_1. Then, see [KL1, Lemma 4.3]:

$$\mathcal{W}(f[\mathcal{G}]) \geq \frac{1}{d}\mathcal{W}(\mathcal{G}). \tag{A.8}$$

Covering Lemma [KL1] (stated as Lemma 8.5) allows one to push-forward width of curves more efficiently.

Lemma A.10. *Suppose $g\colon A \to B$ is a covering between either two closed annuli or between punctured disks. Let $\mathcal{R} \subset A$, $\partial^h\mathcal{R} \subset \partial A$ be a rectangle in A such that g maps $\partial^{h,0}\mathcal{R}$ injectively onto $g\big(\partial^{h,0}\mathcal{R}\big)$. Then after removing two 1-buffers from $\mathcal{R}$, the map g is injective on the new rectangle $\mathcal{R}^{\mathrm{new}}$.*

Proof. Write $D \coloneqq \deg g$. Since g is a normal covering, g has a group of deck transformations; we denote by $\mathcal{R}_0 = \mathcal{R}, \mathcal{R}_1, \dots, \mathcal{R}_{D-1}$ the orbit of $\mathcal{R}$ under the group of deck transformations. Since $\partial^{h,0}\mathcal{R}_i$ are disjoint, all $\mathcal{R}_i^{\mathrm{new}}$ are disjoint. (The last claim can be easily checked by lifting the $\mathcal{R}_i$ to the universal cover.) Therefore, $g \mid \mathcal{R}_0^{\mathrm{new}}$ is injective. □

A.3. **Shift Argument.** If a rectangle $\mathcal{R}$ has a conformal shift $\mathcal{R}_1$ cross-intersecting $\mathcal{R}$, then $\mathcal{W}(\mathcal{R}) \leq 1$, see Figure 30. Often, a weaker condition is sufficient: $\partial^{h,0}\mathcal{R}$ is disjoint from $\partial^h\mathcal{R}_1$. Let us provide details. Consider a rectangle

$$\mathcal{R} \subset \mathbb{C}\setminus Z \qquad \text{such that} \qquad \partial^h\mathcal{R} \subset \partial Z$$

that has a conformal pullback or push-forward

$$\mathcal{R}_1 \coloneqq f^t(\mathcal{R}) \xleftarrow{1:1} \mathcal{R}, \qquad \mathcal{R}_1 \subset \mathbb{C}\setminus Z, \qquad \partial^h\mathcal{R}_1 \subset \partial Z$$

for $t \in \mathbb{Z}$. Assume next that there is an interval $T \subsetneq \partial Z$ containing $\lfloor \partial^h\mathcal{R}\rfloor \cup \lfloor \partial\mathcal{R}_1^h\rfloor$ such that

$$\partial^{h,0}\mathcal{R} < \partial^{h,1}\mathcal{R}, \qquad \partial^{h,0}\mathcal{R}_1 < \partial^{h,1}\mathcal{R}_1 \qquad \text{in} \quad T$$

and f^t maps $\partial^{h,0}\mathcal{R}, \partial^{h,1}\mathcal{R}, \lfloor\partial^h\mathcal{R}\rfloor$ onto $\partial^{h,0}\mathcal{R}_1, \partial^{h,1}\mathcal{R}_1, \lfloor\partial^h\mathcal{R}_1\rfloor$. We say that $\mathcal{R}, \mathcal{R}_1$ are *linked* if

$$\begin{array}{lllll} \text{either} & \partial^{h,0}\mathcal{R}_1 < \partial^{h,0}\mathcal{R} < \partial^{h,1}\mathcal{R}_1 & \text{or} & \partial^{h,0}\mathcal{R} < \partial^{h,0}\mathcal{R}_1 < \partial^{h,1}\mathcal{R} \\ \text{or} & \partial^{h,0}\mathcal{R}_1 < \partial^{h,1}\mathcal{R} < \partial^{h,1}\mathcal{R}_1 & \text{or} & \partial^{h,0}\mathcal{R} < \partial^{h,1}\mathcal{R}_1 < \partial^{h,1}\mathcal{R} \end{array} \tag{A.9}$$

holds.

Lemma A.11. *If $\mathcal{R}$ is linked to its conformal pullback or push-forward $\mathcal{R}_1$ as above, then $\mathcal{W}(\mathcal{R}) \leq 2$.*

Proof. Assume that $\partial^{h,0}\mathcal{R}_1 < \partial^{h,0}\mathcal{R} < \partial^{h,1}\mathcal{R}_1$ holds; the other cases are analogous. Assume that $\mathcal{W}(\mathcal{R}) > 2$. Let $\mathcal{R}^{\mathrm{new}}$ be the rectangle obtained from $\mathcal{R}$ by removing two 1-buffers on each side. Set $\mathcal{R}_1^{\mathrm{new}} \coloneqq f^t(\mathcal{R}^{\mathrm{new}}) \subset \mathcal{R}_1$. Since $\partial^{h,0}\mathcal{R}^{\mathrm{new}} \subset \partial^{h,0}\mathcal{R}$ is disjoint from $\partial^h\mathcal{R}_1^{\mathrm{new}} \subset \partial^h\mathcal{R}_1$, the new rectangles $\mathcal{R}^{\mathrm{new}}, \mathcal{R}_1^{\mathrm{new}}$ are disjoint. Since f^t maps $\partial^{h,0}\mathcal{R}^{\mathrm{new}}, \partial^{h,1}\mathcal{R}^{\mathrm{new}}, \lfloor\partial^h\mathcal{R}^{\mathrm{new}}\rfloor$ onto $\partial^{h,0}\mathcal{R}_1^{\mathrm{new}}, \partial^{h,1}\mathcal{R}_1^{\mathrm{new}}, \lfloor\partial^h\mathcal{R}_1^{\mathrm{new}}\rfloor$ and all the intervals are in T, we obtain

$$\partial^{h,0}\mathcal{R}_1 < \partial^{h,0}\mathcal{R} < \partial^{h,1}\mathcal{R}_1 < \partial^{h,1}\mathcal{R} \qquad \text{in} \quad T;$$

i.e., $\mathcal{R}^{\mathrm{new}}, \mathcal{R}_1^{\mathrm{new}}$ intersect, compare with Figure 30. This is a contradiction. □

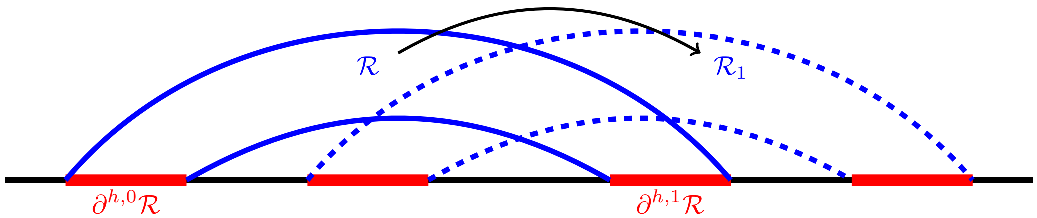


FIGURE 30. If a rectangle $\mathcal{R}$ cross-intersects its conformal image, then $\mathcal{W}(\mathcal{R}) \leq 1$.

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