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A NON-COMMUTATIVE F. & M. RIESZ THEOREM

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ABSTRACT. We extend results on complex analytic measures on the complex unit circle to a non-commutative multivariate setting. Identifying continuous linear functionals on a certain self-adjoint subspace of the Cuntz–Toeplitz C^*-algebra, the *free disk operator system*, with non-commutative (NC) analogues of complex measures, we refine a previously developed Lebesgue decomposition for positive NC measures to establish an NC version of the Frigyes and Marcel Riesz Theorem for "analytic" measures, i.e. complex measures with vanishing positive moments. The proof relies on novel results on the order properties of positive NC measures that we develop and extend from classical measure theory.

KEYWORDS: *Non-commutative disc algebra, F. and M. Riesz Theorem, operator systems, Cuntz and Cuntz–Toeplitz algebras, non-commutative measure theory.*

MSC (2020): 47C05, 46L52, 46L51, 46L07, 47A13, 47L80.

INTRODUCTION

The Riesz–Markov theorem identifies any finite and regular Borel measure on the complex unit circle with a bounded linear functional on the Banach space of continuous functions. By the Weierstrass approximation theorem, the Banach space of continuous functions on the circle is the supremum norm-closure of the linear span of the disk algebra and its conjugate algebra. The disk algebra is often defined as the unital Banach algebra of analytic functions in the complex unit disk with continuous boundary values. This is completely isometrically isomorphic to the unital norm-closed operator algebra generated by the shift operator, $S = M_z$, of multiplication by the independent variable z on H^2. Here, H^2 denotes the *Hardy space*, the Hilbert space of analytic functions in the complex unit disk that have square-summable Taylor series coefficients at 0, equipped with the ℓ^2-inner product of these coefficients.

An immediate non-commutative (NC) multivariate generalization of H^2 is then $\mathbb{H}^2_d$, the *NC Hardy space* or *full Fock space*, which consists of square-summable power series in several formal NC variables $\mathfrak{z} = (\mathfrak{z}_1, \ldots, \mathfrak{z}_d)$. Elements of $\mathbb{H}^2_d$ are power series,

$$h(\mathfrak{z}) = \sum_{\omega \in \mathbb{F}^d} \widehat{h}_\omega \mathfrak{z}^\omega,$$

with square-summable coefficients $\widehat{h}_\omega \in \mathbb{C}$. The *free monoid*, $\mathbb{F}^d$, is the set of all words, $\omega = i_1 \cdots i_n$, $1 \leqslant i_k \leqslant d$, in the d letters $\{1, \ldots, d\}$. This is a monoid with product given by concatenation of words and the unit $\varnothing$ is the empty word containing no letters. Given any $\omega = i_1 \cdots i_n \in \mathbb{F}^d$, the *free monomial* $\mathfrak{z}^\omega$ is $\mathfrak{z}_{i_1} \cdots \mathfrak{z}_{i_n}$ and $\mathfrak{z}^\varnothing = 1$, viewed as a constant NC function. As in the classical setting, left multiplication $L_k := M^L_{\mathfrak{z}_k}$ by any of the d independent NC variables defines an isometry on $\mathbb{H}^2_d$, and we call these isometries the *left free shifts*. These collectively play the role of the shift in this NC Hardy space theory.

The NC analogues of the disk algebra, the continuous functions (equivalently, the *disk operator system*, the supremum norm-closed linear span of the disk algebra and its conjugates) and positive measures are then the *free disk algebra*, $\mathbb{A}_d := \mathrm{Alg}\{I, L_1, \ldots, L_d\}^{-\|\cdot\|}$, the *free disk system*,

$$\mathscr{A}_d := (\mathbb{A}_d + \mathbb{A}^*_d)^{-\|\cdot\|},$$

and *NC measures*, i.e. bounded linear functionals on the free disk system. Here and after, given a set $\mathcal{S}$ of operators on a Hilbert space, we set $\mathcal{S}^* := \{A^* : A \in \mathcal{S}\}$. For the Banach space dual, we use $\dagger$. In particular, the Banach space of all NC measures is denoted by $\mathscr{A}^\dagger_d$ and the cone of positive NC measures by $(\mathscr{A}_d)^\dagger_+$. In [16, 17], the first two named authors constructed the Lebesgue decomposition of any positive NC measure with respect to a canonical NC Lebesgue measure and showed that the sets of absolutely continuous and singular NC measures are positive and hereditary cones, in syzygy with classical measure theory.

The isometric left free shifts $L_k = M^L_{\mathfrak{z}_k}$ on $\mathbb{H}^2_d$ have pairwise orthogonal ranges and it follows that the linear map $L := (L_1, \ldots, L_d) : \mathbb{H}^2_d \otimes \mathbb{C}^d \to \mathbb{H}^2_d$ is an isometry from several copies of $\mathbb{H}^2_d$ into itself. Such an isometry is called a *row isometry*. By the classical Wold decomposition theorem, any isometry on Hilbert space decomposes as the direct sum of a *pure isometry*, i.e. an isometry unitarily equivalent to copies of the shift on H^2, and a unitary operator. There is an exact analogue of the Wold decomposition theorem for row isometries, established by G. Popescu; any row isometry decomposes as the direct sum of a pure row isometry, unitarily equivalent to several copies of the left free shift and a surjective or *Cuntz* row isometry, the row analogue of a unitary operator [27]. The C^*-algebra $\mathscr{E}_d = C^*\{I, L_1, \ldots, L_d\}$ and its quotient, $\mathscr{O}_d$, by the compact operators are the celebrated *Cuntz–Toeplitz* and *Cuntz algebras*, respectively. When $d \geqslant 2$, these are universal C^*-algebras for row isometries and are important objects in C^*-algebra theory: any $*$-representation π of $\mathscr{E}_d$ determines, and is uniquely determined by, the row isometry $\Pi := \pi(L) = (\pi(L_1), \ldots, \pi(L_d))$. For more on the representations of these algebras, see [4] and [28].

The Lebesgue–von Neumann–Wold decomposition of an isometry on Hilbert space further splits the unitary direct summand into the direct sum of a unitary with absolutely continuous spectral measure (with respect to Lebesgue measure) and a singular unitary. In [23], M. Kennedy extended this decomposition for single isometries to row isometries. A new feature in this multivariable theory is that the Cuntz direct summand of any row isometry generally splits as the direct sum of three different types which we call *absolutely continuous (AC) Cuntz*, *von Neumann-type* (called *singular* in [23]) and *dilation-type*.

A Gelfand–Naimark–Segal (GNS) construction applied to the free disk algebra and any positive NC measure μ produces a Hilbert space $\mathbb{H}^2_d(\mu)$ and a row isometry Π_μ acting on $\mathbb{H}^2_d(\mu)$. Any cyclic row isometry (or $*$-representation of $\mathscr{E}_d$) can be obtained as the GNS row isometry of a positive NC measure, as recorded in Lemma 1.2. The role of normalized Lebesgue measure on the circle is played by the so-called vacuum state of the Fock space, $m(L^\omega) = \langle 1, L^\omega 1\rangle_{\mathbb{H}^2}$ where 1 is identified with the *vacuum vector* $\mathfrak{z}^\varnothing$. We call the vacuum state, m, *non-commutative (NC) Lebesgue measure*. Observe that if we identify normalized Lebesgue measure on the complex unit circle with a positive linear functional, m, on the norm-closure of the linear span of the disk algebra and its conjugate algebra, via the Riesz–Markov Theorem, then $m(S^k) = \langle 1, S^k 1\rangle_{H^2}$. Thus, our definition of NC Lebesgue measure recovers normalized Lebesgue measure when $d = 1$. In [16, 17], the first two named authors constructed the Lebesgue decomposition of any positive NC measure, $\mu \in (\mathscr{A}_d)^\dagger_+$, $\mu = \mu_{\mathrm{ac}} + \mu_{\mathrm{s}}$, where $\mu_{\mathrm{ac}}, \mu_{\mathrm{s}} \in (\mathscr{A}_d)^\dagger_+$ are absolutely continuous and singular, respectively, with respect to NC Lebesgue measure [17, Corollary 8.12, Corollary 8.13]. In particular, it is shown that the sets of absolutely continuous and singular NC measures are positive cones that are *hereditary* in the sense that if $\mu, \lambda \in (\mathscr{A}_d)^\dagger_+$ and λ is absolutely continuous or singular, then $\mu \leqslant \lambda$ implies that μ is also absolutely continuous or singular, respectively, in parallel with classical measure theory. In our NC Lebesgue decomposition, μ is absolutely continuous if and only if its GNS row isometry, Π_μ, is the direct sum of pure and AC Cuntz row isometries, while μ is singular if and only if Π_μ is the direct sum of von Neumann-type and dilation-type row isometries [17, Corollary 8.12, Corollary 8.13].

In this paper we further refine the NC Lebesgue decomposition of [17] by proving that the sets of dilation-type and von Neumann-type NC measures (NC measures whose GNS row isometries are dilation or von Neumann-type) are both positive hereditary cones in Theorem 2.15. This yields, in Corollary 2.16, a refined Kennedy–Lebesgue decomposition of any positive NC measure. We then apply this decomposition to obtain analogues of classical results due to Frigyes and Marcel Riesz characterizing analytic (complex) NC measures, which are those bounded linear functionals on the free disk system that annihilate $\{L^\beta : \beta \in \mathbb{F}^d \backslash \{\varnothing\}\}$. In particular, in Theorem 4.2, we show that a complex NC measure is analytic if and only if its absolutely continuous, singular, dilation and von Neumann parts are each also analytic. This result has some overlap with [24, Theorem 4.1], where the objects of study are bounded linear functionals on the weak* closure of $\mathbb{A}_d$ that annihilate a two-sided ideal, in contrast to our focus here. We apply Theorem 4.2 in Theorem 4.9 to establish an analogue of the classical F&M Riesz analytic measure theorem. The traditional theorem states that analytic measures are absolutely continuous [13, Chapter 4], [30, Theorem 5.5], [12, Theorem 6.14] (compare with the original [32]). We remark that the NC F&M Riesz theorem obtained here is related, but not equivalent to, one previously developed in [3, Theorem A] using different techniques. Both Theorem 4.9 of the present paper and Theorem A of [3] conclude that an analytic NC measure need not be absolutely continuous, however the results of this paper and those of [3] describe the obstruction in different ways. We discuss the relationship between these two papers in Remark 4.10.

1. BACKGROUND AND NOTATION

We borrow notation from [16, 17]. Throughout, given h, h' in a Hilbert space $\mathcal{H}$, we denote the inner product of h and h' by $\langle h, h' \rangle$, with $\langle \cdot, \cdot \rangle$ being conjugate linear in the first argument and linear in the second. The *Fock space* or *NC Hardy space* is

$$\mathbb{H}^2_d := \Big\{ h(\mathfrak{z}) = \sum_{\omega \in \mathbb{F}^d} \widehat{h}_\omega \mathfrak{z}^\omega : \widehat{h}_\omega \in \mathbb{C}, \sum_{\omega \in \mathbb{F}^d} |\widehat{h}_\omega|^2 < +\infty \Big\},$$

equipped with the ℓ^2-inner product of the power series coefficients $\widehat{h}_\omega$. Elements of the NC Hardy space can be viewed as free non-commutative functions in the *NC unit row-ball*, $\mathbb{B}^d_{\mathbb{N}}$, of all finite-dimensional strict row contractions:

$$\mathbb{B}^d_{\mathbb{N}} = \bigsqcup_{n=1}^{\infty} \mathbb{B}^d_n; \quad \mathbb{B}^d_n := \{ Z = (Z_1, \dots, Z_d) \in \mathbb{C}^{n\times n} \otimes \mathbb{C}^{1\times d} : ZZ^* = Z_1 Z_1^* + \cdots + Z_d Z_d^* < I_n \};$$

see, for example, [1, 16, 17, 19, 29, 33] for more details on non-commutative function theory.

The *left free shift* $L = (L_1, \dots, L_d)$ is the row isometry on the full Fock space $\mathbb{H}^2_d$ whose component operators act by left multiplication by the independent variables, $L_k = M^L_{\mathfrak{z}_k}$. The free or NC disk algebra is $\mathbb{A}_d := \mathrm{Alg}\{I, L_1, \dots, L_d\}^{-\|\cdot\|}$, the *free disk system* is $\mathscr{A}_d := (\mathbb{A}_d + \mathbb{A}_d^*)^{-\|\cdot\|}$. This is a self-adjoint unital norm-closed subspace of operators, i.e. an operator system. A (complex) NC measure is a bounded linear functional on the free disk system. The set of all complex NC measures is denoted by $\mathscr{A}_d^\dagger$, and the positive NC measures by $(\mathscr{A}_d)^\dagger_+$. We remark that any $\mu \in (\mathscr{A}_d)^\dagger_+$ is uniquely determined by the moments $(\mu(L^\alpha))_{\alpha \in \mathbb{F}^d}$, and thus we often describe a given positive NC measure μ by specifying its moments when positivity is clear from the context.

Given $a \in \mathbb{A}_d$, we write $a = a(L) = M^L_a$ for the operator of left multiplication by $a(\mathfrak{z})$ on $\mathbb{H}^2_d$, where

$$a(\mathfrak{z}) = \sum_{\omega \in \mathbb{F}^d} \widehat{a}_\omega \mathfrak{z}^\omega,$$

is the NC function determined by a. The partial Cesàro sums of the series for $a(L)$ converge in the strong operator topology to $a(L) = M^L_{a(\mathfrak{z})}$, as shown in [11, Lemma 1.1]. More generally, given a row isometry Π, we write $a \mapsto a(\Pi)$ for the unique representation of $\mathbb{A}_d$ satisfying $\mathfrak{z}^\alpha(\Pi) = \Pi^\alpha$ for all $\alpha \in \mathbb{F}^d$.

The free disk system has the *semi-Dirichlet property* [8]:

$$\mathbb{A}_d^* \mathbb{A}_d \subseteq (\mathbb{A}_d + \mathbb{A}_d^*)^{-\|\cdot\|}.$$

The semi-Dirichlet property enables one to apply a Gelfand–Naimark–Segal (GNS)-type construction to $(\mu, \mathbb{A}_d)$, where $\mu \in (\mathscr{A}_d)^\dagger_+$ is any positive NC measure. One obtains a GNS–Hilbert space $\mathbb{H}^2_d(\mu)$ as the completion of the free disk algebra, modulo vectors of zero length, with respect to the pre-inner product:

$$(a_1, a_2) \mapsto \mu(a_1^* a_2).$$

The set of equivalence classes $\{a + N_\mu : a \in \mathbb{A}_d\}$, where $N_\mu = \{a \in \mathbb{A}_d : \mu(a^* a) = 0\}$, is norm-dense in $\mathbb{H}^2_d(\mu)$. This construction provides a representation $\pi_\mu : \mathbb{A}_d \to \mathcal{L}(\mathbb{H}^2_d)$, where

$$\pi_\mu(a)(a' + N_\mu) := aa' + N_\mu.$$

When $d \geqslant 2$, it follows from [28, Theorem 3.1] that π_μ is a unital completely isometric isomorphism of $\mathbb{A}_d$ onto the unital, norm-closed operator algebra $\mathbb{A}_d(\Pi_\mu)$, where $\Pi_{\mu;k} := \pi_\mu(L_k)$ and

$$\mathbb{A}_d(\Pi_\mu) := \mathrm{Alg}\{I, \Pi_{\mu;1}, \ldots, \Pi_{\mu;d}\}^{-\|\cdot\|},$$

so that the image of the left free shifts,

$$\Pi_\mu := (\Pi_{\mu;1}, \ldots, \Pi_{\mu;d}) : \mathbb{H}^2_d(\mu) \otimes \mathbb{C}^d \to \mathbb{H}^2_d(\mu)\,,$$

defines a row isometry. We call Π_μ the *GNS row isometry of* μ, acting on the GNS space $\mathbb{H}^2_d(\mu)$. The original positive NC measure $\mu \in (\mathscr{A}_d)^\dagger_+$ then has the spacial representation,

$$\mu(L^\omega) = \langle I + N_\mu, \Pi^\omega_\mu (I + N_\mu)\rangle_{\mathbb{H}^2_d(\mu)}.$$

If $\mu, \lambda \in (\mathscr{A}_d)^\dagger_+$ and $\mu \leqslant \lambda$, then the map

$$a + N_\lambda \;\mapsto\; a + N_\mu, \quad a \in \mathbb{A}_d,$$

extends by continuity to a contraction $E_{\mu,\lambda} : \mathbb{H}^2_d(\lambda) \to \mathbb{H}^2_d(\mu)$ with dense range. In this case, setting $D_{\mu,\lambda} := E^*_{\mu,\lambda} E_{\mu,\lambda}$, we have

$$\mu(L^\omega) = \langle I + N_\lambda, D_{\mu,\lambda}\, \Pi^\omega_\lambda (I + N_\lambda)\rangle,$$

and $D_{\mu,\lambda} \geqslant 0$ can be viewed as the "NC Radon–Nikodym derivative" of μ with respect to λ.

REMARK 1.1. The NC Radon–Nikodym derivative $D_{\mu,\lambda}$, as described above, is Π_λ*-Toeplitz* (also written λ-Toeplitz) in the sense that

$$\Pi^*_{\lambda;j} D_{\mu,\lambda} \Pi_{\lambda;k} = \delta_{j,k} D_{\mu,\lambda} \ ,$$

where $\delta_{j,k} = 1$ when $j = k$ and 0 otherwise. Recall that a bounded linear operator, T, on the Hardy space, H^2, is called *Toeplitz* if T is the compression, $T_f := P_{H^2} M_f|_{H^2}$, of the bounded multiplication operator, M_f, on $L^2(\partial\mathbb{D})$ to H^2 for some $f \in L^\infty(\partial\mathbb{D})$. A theorem of Brown and Halmos identifies the Toeplitz operators as the set of all bounded linear operators, $T \in \mathscr{L}(H^2)$, satisfying

$$S^* T S = T,$$

where $S = M_z$ is the shift [2, Theorem 6].

We refer to $E_{\mu,\lambda}$ as the *co-embedding* determined by the inequality $\mu \leqslant \lambda$, as its adjoint is injective. Given $\mu, \nu, \lambda \in (\mathscr{A}_d)^\dagger_+$ satisfying $\mu \leqslant \nu \leqslant \lambda$, it follows that

$$E_{\mu,\nu} E_{\nu,\lambda} = E_{\mu,\lambda}.$$

We remark that that $E^*_{\mu,\lambda}$ is unitarily equivalent to an embedding of NC reproducing kernel Hilbert spaces; see [16, Lemma 3], [17].

We now record the fact that any cyclic row isometry is unitarily equivalent to the GNS row isometry of a positive NC measure. The proof is straightforward and thus omitted.

LEMMA 1.2. *Let Π be a cyclic row isometry on a Hilbert space $\mathcal{H}$ with a cyclic vector x. Define a positive* NC *measure $\mu \in (\mathscr{A}_d)^\dagger_+$ by setting $\mu(L^\omega) = \langle x, \Pi^\omega x \rangle$. The map $U_x : \mathbb{H}^2_d(\mu) \to \mathcal{H}$ defined by $U_x\, a + N_\mu = a(\Pi)x$, $a \in \mathbb{A}_d$, extends to a surjective isometry that intertwines Π_μ and Π.*

Let $\mathfrak{L}^\infty_d := \mathrm{Alg}\{I, L_1, \dots, L_d\}^{-\mathrm{weak}^*}$ denote the *left free analytic Toeplitz algebra* or the *Free Hardy Algebra*. From a result of Davidson–Pitts [11, Corollary 2.12], it follows that

$$\mathfrak{L}^\infty_d = \mathrm{Alg}\{I, L_1, \dots, L_d\}^{-\mathrm{WOT}},$$

the closure of $\mathrm{Alg}\{I, L_1, \dots, L_d\}$ in the weak operator topology (WOT). That is, $\mathfrak{L}^\infty_d$ is a *free semigroup algebra*, the unital WOT-closed operator algebra generated by a row isometry [5]. The algebra $\mathfrak{L}^\infty_d$ can also be identified with the left multiplier algebra of $\mathbb{H}^2_d$, viewed as a non-commutative reproducing kernel Hilbert space (RKHS) [1, 19, 33]. We remark that this left multiplier algebra is equal to the unital Banach algebra, $\mathbb{H}^\infty_d$, of all free NC functions in the NC unit row-ball $\mathbb{B}^d_{\mathbb{N}}$ that are uniformly bounded in supremum norm [29, 33]. The NC or *free Toeplitz system* is

$$\mathscr{T}_d := (\mathfrak{L}^\infty_d + (\mathfrak{L}^\infty_d)^*)^{-\mathrm{weak}^*} = \mathscr{A}_d^{-\mathrm{weak}^*}.$$

We also use the right free shift $R = (R_1, \dots, R_d)$, the row isometry of right multiplications $R_k := M^R_{\mathfrak{z}_k}$ by the independent NC variables on the Fock space. The *right free analtyic Toeplitz algebra* is

$$\mathfrak{R}_d^\infty := \mathrm{Alg}\{I, R_1, \ldots, R_d\}^{-\mathrm{WOT}}.$$

1.1. STUCTURE OF GNS ROW ISOMETRIES. By [23, Theorem 6.5], any row isometry, Π, on a separable Hilbert space, $\mathcal{H}$, can be decomposed as the direct sum of four types of row isometries:

$$\Pi = \Pi_L \oplus \Pi_{\mathrm{ACC}} \oplus \Pi_{\mathrm{dil}} \oplus \Pi_{\mathrm{vN}}. \tag{1.1}$$

We call this decomposition the *Kennedy–Lebesgue–von Neumann–Wold* decomposition of Π. Here, Π_L is *pure type L* or simply, *type L*, if it is unitarily equivalent to an ampliation of L. The remaining three types are all Cuntz, i.e. surjective, row isometries. A Cuntz row isometry is the multi-variable or "row" analogue of a unitary operator and we sometimes call a Cuntz row isometry a Cuntz unitary. We also call any pure type L row isometry a *pure row isometry*. The summand Π_{ACC} is *absolutely continuous Cuntz* (or *AC Cuntz* or *ACC*), meaning that Π_{ACC} is a Cuntz row isometry and the free semigroup algebra it generates is completely isometrically isomorphic and weak* homeomorphic to $\mathfrak{L}_d^\infty$. Both type L and ACC row isometries are *absolutely continuous (AC)* or *weak* continuous*, meaning that Π is an AC row isometry if and only if the representation $\pi : \mathbb{A}_d \to \mathscr{L}(\mathcal{H})$, defined by $L_k \overset{\pi}{\mapsto} \Pi_k$, extends to a weak* continuous representation of $\mathfrak{L}_d^\infty$. The summand Π_{vN} is *totally singular*, or of *von Neumann-type*, if it has no weak* continuous restriction to a non-trivial invariant subspace. For $d \geqslant 2$, it follows from [23, Theorem 5.1] that a row isometry is of von Neumann-type if and only if the free semigroup algebra it generates is self-adjoint and hence a von Neumann algebra. The leftover piece, Π_{dil}, is of *dilation-type*. That is, a row isometry, Π, is of dilation-type if it has no direct summand of the previous three types, L, ACC or vN. A row isometry containing only dilation-type and von Neumann-type summands is said to be *singular*, in keeping with the terminology established in [17]. We remark that, when $d = 1$, the dilation-type summand is absent, see Section 1.3.

REMARK 1.3. By [23, Proposition 6.2], any dilation-type row isometry Π has a block upper triangular decomposition,

$$\Pi \simeq \begin{pmatrix} L \otimes I & * \\ 0 & T \end{pmatrix},$$

so that Π has a pure restriction to an invariant subspace and Π is the minimal row isometric dilation of its compression, T, to the orthogonal complement of this invariant space. This motivates the name, dilation-type. Since Π is of Cuntz-type, T is necessarily a non-isometric row co-isometry [27, Proposition 2.5].

REMARK 1.4. It is implicit in the results of [23] that the above Kennedy–Lebesgue–von Neumann–Wold decomposition of any row isometry, Π, on $\mathcal{H}$, is unique. Indeed, the Wold decomposition of Π, $\Pi = \Pi_L \oplus \Pi_{\mathrm{Cuntz}}$ on $\mathcal{H} =$

$\mathcal{H}_L \oplus \mathcal{H}_{\mathrm{Cuntz}}$ is unique by [27, Theorem 1.3]. Next, there is a unique maximal Π_{Cuntz}-reducing subspace, $\mathcal{H}_{\mathrm{ACC}}$, so that the restriction, $\Pi_{\mathrm{ACC}} := \Pi_{\mathrm{Cuntz}}|_{\mathcal{H}_{\mathrm{ACC}}}$, is absolutely continuous, so that $\Pi_{\mathrm{Cuntz}} = \Pi_{\mathrm{ACC}} \oplus \Pi'$ on $\mathcal{H}_{\mathrm{ACC}} \oplus \mathcal{H}'$. Then, Π' can be decomposed by setting $\mathcal{H}'_{WC} \subseteq \mathcal{H}'$ to be the largest Π'-invariant subspace so that $\Pi'|_{\mathcal{H}'_{WC}}$ is weak* continuous and then $\mathcal{H}_{\mathrm{dil}} \subseteq \mathcal{H}'$ is defined as the smallest Π'-reducing subspace containing $\mathcal{H}'_{WC}$. Finally, $\Pi_{\mathrm{dil}} := \Pi'|_{\mathcal{H}_{\mathrm{dil}}}$, $\mathcal{H}_{\mathrm{vN}} := \mathcal{H}' \ominus \mathcal{H}_{\mathrm{dil}}$ and $\Pi_{\mathrm{vN}} := \Pi'|_{\mathcal{H}_{\mathrm{vN}}}$.

1.2. STRUCTURE PROJECTIONS. We form the set of labels

$$\mathrm{Types} = \{\mathrm{L}, \mathrm{Cuntz}, \mathrm{ac}, \mathrm{s}, \mathrm{ACC}, \mathrm{dil}, \mathrm{vN}, \mathrm{all}\}.$$

For a given row isometry Π, if we write "Π is type $\mathfrak{t}$", then we mean "Π is pure type L" when $\mathfrak{t} = \mathrm{L}$, "$\Pi$ is Cuntz-type" when $\mathfrak{t} = \mathrm{Cuntz}$, "$\Pi$ is absolutely continuous" when $\mathfrak{t} = \mathrm{ac}$, "$\Pi$ is singular" when $\mathfrak{t} = \mathrm{s}$, "$\Pi$ is absolutely continuous Cuntz" when $\mathfrak{t} = \mathrm{ACC}$, "$\Pi$ is dilation-type" when $\mathfrak{t} = \mathrm{dil}$, and "$\Pi$ is von Neumann-type" when $\mathfrak{t} = \mathrm{vN}$. We include the trivial type, $\mathfrak{t} = \mathrm{all}$. If Π is of type all this simply means that Π can be any row isometry.

DEFINITION 1.5. Let $\mathfrak{t} \in \mathrm{Types}$. A positive NC measure $\mu \in (\mathscr{A}_d)^\dagger_+$ is said to be *type* $\mathfrak{t}$ if its GNS row isometry Π_μ is type $\mathfrak{t}$. A positive NC measure, $\mu \in (\mathscr{A}_d)^\dagger_+$, is *weak* continuous*, if it has a weak* continuous extension to $\mathscr{T}_d = \mathscr{A}_d^{-\mathrm{weak}^*}$.

EXAMPLE 1.6. Let $m \in (\mathscr{A}_d)^\dagger_+$ be given by $m(b) = \langle 1, b1 \rangle$, $b \in \mathscr{A}_d$. We call this positive NC measure, m, *NC Lebesgue measure*, and note that it is absolutely continuous, as Π_m is unitarily equivalent to L. NC Lebesgue measure is so named because it plays the role of normalized Lebesgue measure in the NC measure theory used here and in [16, 17].

A positive NC measure, $\mu \in (\mathscr{A}_d)^\dagger_+$, is absolutely continuous if and only if it is weak* continuous [17]. We provide an example of such an NC measure in Example 2.18, while an example of dilation-type is found in Section 5.

Let $\mathfrak{t} \in \mathrm{Types} \setminus \{\mathrm{Cuntz}\}$ and consider a row isometry Π. By the Kennedy–Lebesgue–von Neumann–Wold decomposition, there is an orthogonal projection $P_\mathfrak{t}$ that commutes with Π such that Π restricted to the range of $P_\mathfrak{t}$ is the type $\mathfrak{t}$ summand of Π. In the case of a GNS row isometry Π_μ, we write $P_{\mu;\mathfrak{t}}$. Given a positive NC measure μ, we denote by $\mu_\mathfrak{t}$ the positive NC measure satisfying

$$\mu_\mathfrak{t}(L^\beta) = \langle I + N_\mu, P_{\mu;\mathfrak{t}} \Pi_\mu^\beta (I + N_\mu) \rangle, \quad \beta \in \mathbb{F}^d.$$

One may readily verify that $E_{\mu_\mathfrak{t},\mu}$ is a co-isometry satisfying

$$E^*_{\mu_\mathfrak{t},\mu} E_{\mu_\mathfrak{t},\mu} = P_{\mu;\mathfrak{t}}.$$

Note that $E^*_{\mu_\mathfrak{t},\mu}$ satisfies

$$E^*_{\mu_\mathfrak{t},\mu}(a + N_{\mu_\mathfrak{t}}) = P_{\mu;\mathfrak{t}}(a + N_\mu).$$

Because $P_{\mu;\mathfrak{t}}$ is reducing for Π_μ, it follows that $E^*_{\mu_\mathfrak{t},\mu}\Pi^\beta_{\mu_\mathfrak{t}} = \Pi^\beta_\mu E^*_{\mu_\mathfrak{t},\mu}$ for all words β. From this, we see that $\Pi_{\mu_\mathfrak{t}}$ is unitarily equivalent to the restriction of Π to $\operatorname{Ran} P_{\mu;\mathfrak{t}}$. Therefore, the GNS row isometry of $\mu_\mathfrak{t}$ is type $\mathfrak{t}$, and thus $\mu_\mathfrak{t}$ is type $\mathfrak{t}$.

There is an additional projection associated with any row isometry Π, and that is the free semigroup algebra structure projection Q of Π. With $\mathfrak{S}(\Pi) := \operatorname{Alg}\{I, \Pi_1, \dots, \Pi_d\}^{-\mathrm{WOT}}$ denoting the free semigroup algebra of Π, we denote by Q the largest projection in $\mathfrak{S}(\Pi)$ so that $Q\mathfrak{S}(\Pi)Q$ is self-adjoint [9, Structure Theorem 2.6]. It has the following properties. First, $\mathfrak{S}(\Pi)$ has the decomposition

$$\mathfrak{S}(\Pi) = \mathrm{vN}(\Pi)Q + Q^\perp\mathfrak{S}(\Pi)Q^\perp,$$

where $\mathrm{vN}(\Pi)$ denotes the von Neumann algebra generated by $\{\Pi_1, \dots, \Pi_d\}$. When $Q \neq I$,

$$Q^\perp\mathfrak{S}(\Pi)Q^\perp = \mathfrak{S}(\Pi)Q^\perp$$

is completely isometrically isomorphic and weak* homeomorphic to $\mathfrak{L}^\infty_d$. Here and elsewhere, $P^\perp = I - P$ whenever P is an orthogonal projection.

The structure projection is related to the subspace of all *weak* continuous vectors* for a row isometry Π. A vector $x \in \mathcal{H}$ is *weak* continuous* if the linear functional $\ell_x \in (\mathscr{A}_d)^\dagger_+$, defined by $\ell_x(L^\alpha) := \langle x, \Pi^\alpha x\rangle$, is weak* continuous [7]. A bounded operator $X : \mathbb{H}^2_d \to \mathcal{H}$ is an *intertwiner for Π* if

$$XL^\alpha = \Pi^\alpha X, \quad \alpha \in \mathbb{F}^d.$$

The following theorem combines results of Davidson–Li–Pitts and Kennedy to characterize the set, $\mathrm{WC}(\Pi)$, consisting of all weak* continuous vectors of Π in terms of bounded intertwiners. For $\mu \in (\mathscr{A}_d)^\dagger_+$, we also write $\mathrm{WC}(\mu) = \mathrm{WC}(\Pi_\mu)$.

THEOREM 1.7 (Davidson–Li–Pitts, Kennedy). *Let Π be a row isometry on $\mathcal{H}$.*
(i) *If $x, y \in \mathrm{WC}(\Pi)$, then the linear functional $\ell_{x,y} : \mathbb{A}_d \to \mathbb{C}$ satisfying*

$$\ell_{x,y}(L^\alpha) = \langle x, \Pi^\alpha y\rangle_{\mathcal{H}}, \quad \alpha \in \mathbb{F}^d$$

is weak continuous.*
(ii) $\mathrm{WC}(\Pi)$ *is a closed Π-invariant subspace, and*

$$\mathrm{WC}(\Pi) = \{Xh : h \in \mathbb{H}^2_d,\ X \text{ an intertwiner}\}.$$

(iii) *If Q is the structure projection of Π, then*

$$\mathrm{WC}(\Pi) = \operatorname{Ran} Q^\perp.$$

Proof. Items (i) and (ii) are directly from [7, Theorem 2.7].

For item (iii), we note the following. The second dual $\mathbb{A}_d^{\dagger\dagger}$ of $\mathbb{A}_d$ is a free semi-group algebra, and thus there exists a structure projection $\mathfrak{q}$ for $\mathbb{A}_d^{\dagger\dagger}$. Let $\widehat{\pi}$ denote the weak* continuous representation of $\mathscr{E}_d^{\dagger\dagger}$ determined by π. By [7, Proposition 5.2], $\widehat{\pi}(\mathfrak{q})^\perp = Q_{\mathrm{WC}}$, where Q_{WC} denotes the projection onto the closed subspace $\mathrm{WC}(\Pi)$. In comments following [7, Proposition 5.2], it is shown that $\widehat{\pi}(\mathfrak{q}) = Q$ if and only if π is "regular", meaning that the $a \mapsto \pi(a)|_{\mathrm{WC}(\Pi)}$

and $a \mapsto \pi(a)|_{\operatorname{Ran} Q^\perp}$ coincide. By [7, Theorem 3.4] and [23, Corollary 4.17], we see that π is always regular. ∎

1.3. MEASURE THEORY AND ISOMETRIES IN ONE VS. SEVERAL NC VARIABLES. As described in the introduction, positive NC measures and their GNS row isometries are canonical and, in some sense, minimal non-commutative multivariate analogues of positive regular finite Borel measures on the complex unit circle and the (isometric) restriction of multiplication by the independent variable to the closure of the analytic polynomials, respectively. There are, however, several key differences between the one and several-variable theories, and this NC measure theory exhibits some new phenomena that have no direct single-variable analogues.

Firstly, in one-variable, the disk operator system, $\mathscr{A}_1 := (\mathbb{A}_1 + \mathbb{A}_1^*)^{-\|\cdot\|}$, is completely isometrically isomorphic to the commutative C^*-algebra, $\mathscr{C}(\partial\mathbb{D})$, of continuous functions on the circle, which is C^*-isomorphic to the quotient of the Toeplitz algebra, $\mathscr{E}_1$, by the compact operators. For $d \geqslant 2$, the free disk system, $\mathscr{A}_d$, is not completely isometrically isomorphic to the quotient of $\mathscr{E}_d$ by the compacts and, moreover, $\mathscr{A}_d$ is not a C^*-algebra, it is an operator system, a norm-closed and self-adjoint subspace of operators. Another point of difference is that for $d \geqslant 2$, if $\mu \in (\mathscr{A}_d)^\dagger_+$ is any positive NC measure, then the GNS representation π_μ implements a completely isometric isomorphism between the free disk algebra, $\mathbb{A}_d$, and the unital, norm-closed operator algebra generated by the GNS row isometry $\Pi_\mu = \pi_\mu(L)$ [28, Theorem 3.1]. This is not true if $d = 1$. Indeed, if μ is Lebesgue measure on the right half circle, then multiplication by $a(z) := z - 1$ has norm 2 in $\mathbb{A}_1$, but $\pi_\mu(a)$ has norm $\sqrt{2}$ in $\mathscr{L}(H^2(\mu))$.

The multi-variable Kennedy–Lebesgue–von Neumann–Wold decomposition of row isometries also enjoys several novel features that have no classical analogue in the Lebesgue–von Neumann–Wold decomposition of an isometry. If $V \in \mathscr{L}(\mathcal{H})$ is an isometry, then its Wold decomposition is $V_0 \oplus U$ on $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}'$ where V_0 is a *pure isometry*, i.e. unitarily equivalent to several copies of the unilateral shift on H^2, and U is unitary on $\mathcal{H}'$. The unitary direct summand further decomposes, via spectral theory, as $U = U_{\mathrm{ac}} \oplus U_{\mathrm{s}}$, where U_{ac} has absolutely continuous spectral measure and U_{s} has singular spectral measure (with respect to Lebesgue measure). As described in [23, Section 2], by Wermer's Theorem, it is possible for the weak*-closed unital operator algebra $H^\infty(V) := \mathrm{Alg}\{I, V\}^{-\mathrm{weak}^*}$ of an isometry V to be self-adjoint [34]. Let $n \in \mathbb{N} \cup \{+\infty\} \cup \{0\}$ be the multiplicity of the pure part V_0 of V, i.e. $V_0 \simeq S \otimes I_{\mathbb{C}^n}$, and let $\mu_{\mathrm{ac}}, \mu_{\mathrm{s}}$ be scalar, positive measures equivalent to the spectral measures of U_{ac} and U_{s}, respectively. Then, provided that either (i) $n \neq 0$ or (ii) Lebesgue measure is mutually absolutely continuous with μ_{ac}, then

$$H^\infty(V) \simeq H^\infty(V_0 \oplus U_{\mathrm{ac}}) \oplus L^\infty(U_{\mathrm{s}}, \mu_{\mathrm{s}}),$$

where $L^\infty(U_s, \mu_s)$ is the commutative von Neumann algebra obtained by applying the $L^\infty(\mu_s)$ functional calculus to U_s. If neither of these conditions hold, then $H^\infty(V)$ is self-adjoint, i.e. a von Neumann algebra. For example, if $n = 0$, $U_s = 0$ and μ_{ac} is Lebesgue measure restricted to the upper or lower half-circle, then normalized Lebesgue measure, m, is not absolutely continuous with respect to μ_{ac}, so that $H^\infty(V) = H^\infty(U_{ac}) = L^\infty(U_{ac}, \mu_{ac})$ is self-adjoint; see [23, Example 2.2]. This is in contrast to the multivariate setting where $\mathbb{H}^\infty_d(V) := \mathrm{Alg}\{I, V_1, \ldots, V_d\}^{-\mathrm{WOT}}$ is self-adjoint for a row isometry $V = (V_1, \ldots, V_d)$, if and only if V has no weak* continuous restriction to a non-trivial invariant subspace, i.e. if and only if V is of von Neumann-type; see the previous Subsection 1.1. (There is a slight difference here in that in one variable, if $V \in \mathscr{L}(\mathcal{H})$ is an isometry, we define $H^\infty(V) := \mathrm{Alg}(I, V)^{-\mathrm{weak}^*}$, while if $V : \mathcal{H} \otimes \mathbb{C}^d \to \mathcal{H}$ is a row isometry, we define $\mathbb{H}^\infty_d(V) := \mathrm{Alg}(I, V_1, \ldots, V_d)^{-\mathrm{WOT}}$ since the WOT and weak* closures of $\mathfrak{L}^\infty_d$ coincide.) Row isometries of von Neumann-type are, at this time, poorly understood. There is essentially only one known example of a von Neumann row isometry whose free semigroup algebra is equal to all of $\mathscr{L}(\mathcal{H})$ [6, 31]. In particular, it is not known whether or not there exist von Neumann-type row isometries whose free semigroup algebras are infinite von Neumann algebras of type II or III.

There is also no single-variable analogue of a dilation-type row isometry. In the Lebesgue–von Neumann–Wold decomposition of an isometry, $V = V_0 \oplus U_{ac} \oplus U_s$, the unitary direct summand, $U_{ac} \oplus U_s$ acts on $\mathcal{H}'_{ac} \oplus \mathcal{H}'_s$ where the subspaces $\mathcal{H}'_{ac}$ and $\mathcal{H}'_s$ are *reducing* for V and U. In contrast, the Cuntz unitary part U of a row isometry V decomposes as $U = U_{ACC} \oplus U_{dil} \oplus U_{vN}$, where U_{dil} is the dilation-type direct summand. Any dilation-type row isometry necessarily has a weak* continuous restriction to the non-trivial *invariant but not reducing* subspace of its weak* continuous vectors. Indeed, by definition, if V is a row isometry of dilation-type, then there is no non-trivial reducing subspace for V so that the restriction of V to this subspace is AC. Since a dilation-type row isometry V has, again by definition, a weak* continuous restriction to a non-trivial invariant subspace, it is not immediately obvious as to whether such a row isometry should be called "absolutely continuous" or "singular". However, as shown in [16, 17], any positive NC measure $\mu \in (\mathscr{A}_d)^\dagger_+$ has a unique Lebesgue-type decomposition $\mu = \mu_{ac} + \mu_s$, $\mu_{ac}, \mu_s \in (\mathscr{A}_d)^\dagger_+$, where μ_{ac} is the maximal positive NC measure bounded above by μ that extends weak* continuously from the free disk system, $\mathscr{A}_d$, to its weak* closure, $\mathscr{T}_d$, and μ_s is singular in the sense that the only positive and absolutely continuous NC measure dominated by μ_s is the identically zero NC measure. Moreover, the sets of positive AC and singular NC measures are both positive and hereditary cones and a positive NC measure is singular in this sense if and only if its GNS row isometry decomposes as the direct sum of von Neumann-type and dilation-type row isometries [17, Corollary 8.13]. These results justify the definition of a singular row isometry as the direct sum of von Neumann and dilation-type row isometries.

Furthermore, and finally, the natural NC analogues of the singular Clark measures of large classes of inner functions are of dilation-type. Here, recall that there is, essentially, a bijection between contractive analytic functions in the complex unit disk and positive measures on the unit circle. (To be precise, two contractive analytic functions b_1, b_2 correspond to the same positive measure if and only if $\frac{1+b_1}{1-b_1}$ and $\frac{1+b_2}{1-b_2}$ differ by an imaginary constant.) If a positive measure, μ, corresponds to a contractive analytic function, $b \in [H^\infty]_1$, $\mu = \mu_b$ is called the *Clark* or *Aleksandrov–Clark measure* of b. As a corollary to Fatou's Theorem, μ_b is singular with respect to Lebesgue measure if and only if b is inner. In [16, Corollary 3] we extended one half of this corollary to the NC setting — if a contractive left multiplier, $b \in [\mathbb{H}^\infty_d]_1$, is inner, then its NC Clark measure, $\mu_b \in (\mathscr{A}_d)^\dagger_+$, is singular. The converse is currently an open problem, in general, although we have been able to show that the converse holds in the case where $b = \mathfrak{b} \in [\mathbb{H}^\infty_d]_1$ is a *non-commutative rational function* [18, Corollary 6]. Here, an NC rational expression, is, essentially, any well-defined expression obtained by applying the operations of multiplication, summation and inversion to free polynomials [21]. In [20, Corollary 1], it is proved that the NC Clark measure of any inner NC rational left multiplier of the free Hardy space, $\mathfrak{b} \in [\mathbb{H}^\infty_d]_1$ is always purely of dilation-type. See also Section 5 of this paper, which shows that the NC Clark measure of $\mathfrak{b}(Z) = Z_1$ is of dilation-type. It would be of significant interest to characterize, in general, when an inner left multiplier of the free Hardy space has an NC Clark measure purely of dilation or von Neumann-type, respectively.

2. CONVEX AND ORDER STRUCTURE OF NC MEASURES

If $0 \leqslant \mu \leqslant \lambda$ are positive NC measures, it is natural to ask whether the contractive co-embedding $E_{\mu,\lambda} : \mathbb{H}^2_d(\lambda) \to \mathbb{H}^2_d(\mu)$ intertwines the various structure projections of μ and λ. That is, do we generally have that $E_{\mu,\lambda} P_{\lambda;\mathfrak{t}} = P_{\mu;\mathfrak{t}} E_{\mu,\lambda}$, where $\mathfrak{t} \in$ Types? For $\mathfrak{t} \in \{\mathrm{ac}, \mathrm{dil}, \mathrm{vN}\}$, we provide affirmative answers. By [17, Corollary 8.11] the sets of absolutely continuous (AC) and singular positive NC measures are positive hereditary cones. It is therefore also natural to ask whether the sets of von Neumann-type and dilation-type NC measures are also positive hereditary cones. In Theorem 2.15 at the end of this section, we provide a positive answer to this question.

DEFINITION 2.1. Let $\mu, \lambda \in (\mathscr{A}_d)^\dagger_+$. We say that $\mathfrak{t} \in$ Types is a *hereditary* type if $\mu \leqslant \lambda$ and λ being type $\mathfrak{t}$ together imply that μ is type $\mathfrak{t}$. A positive sub-cone $\mathscr{P}_0$ of positive cone $\mathscr{P}$ is *hereditary* if $p_0 \in \mathscr{P}_0$, $p \in \mathscr{P}$ and $p \leqslant p_0$ imply that $p \in \mathscr{P}_0$. We say that $\mathfrak{t}$ *determines a hereditary cone* if the set of type $\mathfrak{t}$ positive NC measures form a hereditary cone.

LEMMA 2.2. *Let $\lambda, \mu \in (\mathscr{A}_d)^\dagger_+$ with $\mu \leqslant \lambda$. If $c \in \mathscr{A}_d$ is positive semi-definite, then*
$$E^*_{\mu,\lambda}\pi_\mu(c)E_{\mu,\lambda} \leqslant \pi_\lambda(c).$$

Proof. By [14, Lemma 4.6] the cone of "positive finite sums of squares" of free polynomials, i.e. elements of the form
$$\sum_{j=1}^N p_j(L)^* p_j(L), \quad p_j \in \mathbb{C}\{\mathfrak{z}_1, \dots, \mathfrak{z}_d\},$$
is norm-dense in the cone of positive elements of the free disk system, $\mathscr{A}_d$. Hence, to prove the claim, it suffices to show that
$$E^*_{\mu,\lambda}\pi_\mu(p(L)^*p(L))E_{\mu,\lambda} \leqslant \pi_\lambda(p(L)^*p(L)),$$
for any $p \in \mathbb{C}\{\mathfrak{z}_1, \dots, \mathfrak{z}_d\}$. This is easily verified:
$$\begin{aligned} E^*_{\mu,\lambda}\pi_\mu(p(L)^*p(L))E_{\mu,\lambda} &= E^*_{\mu,\lambda}\pi_\mu(p(L))^*\pi_\mu(p(L))E_{\mu,\lambda} \\ &= \pi_\lambda(p(L))^* E^*_{\mu,\lambda}E_{\mu,\lambda}\pi_\lambda(p(L)) \\ &\leqslant \pi_\lambda(p(L))^*\pi_\lambda(p(L)) = \pi_\lambda(p(L)^*p(L)). \quad \blacksquare \end{aligned}$$

PROPOSITION 2.3. *Let $\mathfrak{t} \in$ Types.*

(i) *If $\mu, \lambda \in (\mathscr{A}_d)^\dagger_+$ are such that $E_{\mu,\lambda}P_{\lambda;\mathfrak{t}} = P_{\mu;\mathfrak{t}}E_{\mu,\lambda}P_{\lambda;\mathfrak{t}}$ and λ is of type $\mathfrak{t}$, then μ is also of type $\mathfrak{t}$. In particular, if this formula holds for all $\mu, \lambda \in (\mathscr{A}_d)^\dagger_+$ such that $\mu \leqslant \lambda$, then $\mathfrak{t}$ is a hereditary-type.*

(ii) *If $E_{\mu,\lambda}P_{\lambda;\mathfrak{t}} = P_{\mu;\mathfrak{t}}E_{\mu,\lambda}$ whenever $\mu, \lambda \in (\mathscr{A}_d)^\dagger_+$ are such that $\mu \leqslant \lambda$, then $\mathfrak{t}$ determines a hereditary cone.*

(iii) *Suppose that $\mathfrak{t}, \mathfrak{u}$ are types and $\lambda, \mu \in (\mathscr{A}_d)^\dagger_+$ are such that $P^\perp_{\lambda;\mathfrak{t}} = P_{\lambda;\mathfrak{u}}$ and similarly for μ. If $\mu_\mathfrak{t} \leqslant \lambda_\mathfrak{t}$ and $\mu_\mathfrak{u} \leqslant \lambda_\mathfrak{u}$, then $E_{\mu,\lambda}P_{\lambda;\mathfrak{t}} = P_{\mu;\mathfrak{t}}E_{\mu,\lambda}$.*

(iv) *Suppose that $\mu, \lambda \in (\mathscr{A}_d)^\dagger_+$, $\mu \leqslant \lambda$, $\mathfrak{t}$ is a type and $P_{\mu;\mathfrak{t}}E_{\mu,\lambda}P_{\lambda;\mathfrak{t}} = P_{\mu;\mathfrak{t}}E_{\mu,\lambda}$. If μ is of type $\mathfrak{t}$ then $\mu \leqslant \lambda_\mathfrak{t}$.*

Proof. (i) Suppose $\mu \leqslant \lambda$ and λ is type $\mathfrak{t}$. Then $P_{\lambda;\mathfrak{t}} = I$ and thus $E_{\mu,\lambda} = P_{\mu;\mathfrak{t}}E_{\mu,\lambda}$, and therefore
$$\begin{aligned} \mu(L^\beta) &= \langle E_{\mu,\lambda}(I + N_\lambda), \Pi^\beta_\mu E_{\mu,\lambda}(I + N_\lambda)\rangle \\ &= \langle E_{\mu,\lambda}(I + N_\lambda), P_{\mu;\mathfrak{t}}\Pi^\beta_\mu E_{\mu,\lambda}(I + N_\lambda)\rangle = \mu_\mathfrak{t}(L^\beta) \end{aligned}$$
for each $\beta \in \mathbb{F}^d$. Thus, μ is type $\mathfrak{t}$.

(ii) The hereditary property follows from (i). To see that $\mathfrak{t}$ determines a cone, suppose μ, ν are type $\mathfrak{t}$. Clearly, $\mu, \nu \leqslant \mu + \nu$. Then,
$$I = E^*_{\mu,\mu+\nu}E_{\mu,\mu+\nu} + E^*_{\nu,\mu+\nu}E_{\nu,\mu+\nu},$$
$P_{\mu;\mathfrak{t}} = I$ and $P_{\nu;\mathfrak{t}} = I$. Thus,
$$P_{\mu+\nu;\mathfrak{t}} = (E^*_{\mu,\mu+\nu}E_{\mu,\mu+\nu} + E^*_{\nu,\mu+\nu}E_{\nu,\mu+\nu})P_{\mu+\nu;\mathfrak{t}}$$

$$= E^*_{\mu,\mu+\nu}P_{\mu;\mathfrak{t}}E_{\mu,\mu+\nu} + E^*_{\nu,\mu+\nu}P_{\nu;\mathfrak{t}}E_{\nu,\mu+\nu}$$
$$= E^*_{\mu,\mu+\nu}E_{\mu,\mu+\nu} + E^*_{\nu,\mu+\nu}E_{\nu,\mu+\nu} = I.$$

Therefore, $(\mu+\nu)_{\mathfrak{t}} = \mu+\nu$ is type $\mathfrak{t}$.

(iii) Define $U_\mu : \mathbb{H}^2_d(\mu) \to \mathbb{H}^2_d(\mu_{\mathfrak{t}}) \oplus \mathbb{H}^2_d(\mu_{\mathfrak{u}})$ by setting $U_\mu h = E_{\mu_{\mathfrak{t}},\mu}h \oplus E_{\mu_{\mathfrak{u}},\mu}h$ for $h \in \mathbb{H}^2_d(\mu)$. Then it follows from comments following Definition 1.5 that U_μ is a surjective isometry. The surjective isometry $U_\lambda : \mathbb{H}^2_d(\lambda) \to \mathbb{H}^2_d(\lambda_{\mathfrak{t}}) \oplus \mathbb{H}^2_d(\lambda_{\mathfrak{u}})$ is defined similarly. We note that, with respect to this direct sum decomposition,

$$\begin{aligned} U_\mu E_{\mu,\lambda} &= \begin{bmatrix} E_{\mu_{\mathfrak{t}},\mu} \\ E_{\mu_{\mathfrak{u}},\mu} \end{bmatrix} E_{\mu,\lambda} = \begin{bmatrix} E_{\mu_{\mathfrak{t}},\lambda} \\ E_{\mu_{\mathfrak{u}},\lambda} \end{bmatrix} = \begin{bmatrix} E_{\mu_{\mathfrak{t}},\lambda_{\mathfrak{t}}}E_{\lambda_{\mathfrak{t}},\lambda} \\ E_{\mu_{\mathfrak{u}},\lambda_{\mathfrak{u}}}E_{\lambda_{\mathfrak{u}},\lambda} \end{bmatrix} \\ &= \begin{bmatrix} E_{\mu_{\mathfrak{t}},\lambda_{\mathfrak{t}}} & 0 \\ 0 & E_{\mu_{\mathfrak{u}},\lambda_{\mathfrak{u}}} \end{bmatrix} \begin{bmatrix} E_{\lambda_{\mathfrak{t}},\lambda} \\ E_{\lambda_{\mathfrak{u}},\lambda} \end{bmatrix} = \begin{bmatrix} E_{\mu_{\mathfrak{t}},\lambda_{\mathfrak{t}}} & 0 \\ 0 & E_{\mu_{\mathfrak{u}},\lambda_{\mathfrak{u}}} \end{bmatrix} U_\lambda. \end{aligned}$$

Thus,

$$E_{\mu,\lambda}U^*_\lambda = U^*_\mu \begin{bmatrix} E_{\mu_{\mathfrak{t}},\lambda_{\mathfrak{t}}} & 0 \\ 0 & E_{\mu_{\mathfrak{u}},\lambda_{\mathfrak{u}}} \end{bmatrix}.$$

Let $C_\mu : \mathbb{H}^2_d(\mu) \to \mathbb{H}^2_d(\mu_{\mathfrak{t}}) \oplus \mathbb{H}^2_d(\mu_{\mathfrak{u}})$ be defined by $C_\mu h = E_{\mu_{\mathfrak{t}},\mu}h \oplus 0$, with C_λ similarly defined. Then

$$\begin{aligned} P_{\mu;\mathfrak{t}}E_{\mu,\lambda} &= C^*_\mu U_\mu E_{\mu,\lambda} = C^*_\mu \begin{bmatrix} E_{\mu_{\mathfrak{t}},\lambda_{\mathfrak{t}}} & 0 \\ 0 & E_{\mu_{\mathfrak{u}},\lambda_{\mathfrak{u}}} \end{bmatrix} U_\lambda = E^*_{\mu_{\mathfrak{t}},\mu}E_{\mu_{\mathfrak{t}},\lambda_{\mathfrak{t}}}E_{\lambda_{\mathfrak{t}},\lambda} \quad \text{and} \\ E_{\mu,\lambda}P_{\lambda;\mathfrak{t}} &= E_{\mu,\lambda}U^*_\lambda C_\lambda = U^*_\mu \begin{bmatrix} E_{\mu_{\mathfrak{t}},\lambda_{\mathfrak{t}}} & 0 \\ 0 & E_{\mu_{\mathfrak{u}},\lambda_{\mathfrak{u}}} \end{bmatrix} C_\lambda = E^*_{\mu_{\mathfrak{t}},\mu}E_{\mu_{\mathfrak{t}},\lambda_{\mathfrak{t}}}E_{\lambda_{\mathfrak{t}},\lambda}. \end{aligned}$$

Therefore, $P_{\mu;\mathfrak{t}}E_{\mu,\lambda} = E_{\mu,\lambda}P_{\lambda;\mathfrak{t}}$.

(iv) Since μ is type $\mathfrak{t}$, we have $P_{\mu;\mathfrak{t}} = I$ and so $E_{\mu,\lambda}P_{\lambda;\mathfrak{t}} = E_{\mu,\lambda}$. Let $c \in \mathscr{A}_d$ be positive semi-definite. By Lemma 2.2, we have

$$E^*_{\mu,\lambda}\pi_\lambda(c)E_{\mu,\lambda} \leqslant \pi_\lambda(c),$$

and so

$$\begin{aligned} \mu(c) &= \langle I+N_\lambda, E^*_{\mu,\lambda}\pi_\mu(c)E_{\mu,\lambda}(I+N_\lambda)\rangle \\ &= \langle P_{\lambda;\mathfrak{t}}(I+N_\lambda), E^*_{\mu,\lambda}\pi_\mu(c)E_{\mu,\lambda}P_{\lambda;\mathfrak{t}}(I+N_\lambda)\rangle \\ &\leqslant \langle I+N_\lambda, P_{\lambda;\mathfrak{t}}\pi_\lambda(c)(I+N_\lambda)\rangle = \lambda_{\mathfrak{t}}(c). \end{aligned}$$

That is, $\mu \leqslant \lambda_{\mathfrak{t}}$. ∎

LEMMA 2.4. *Suppose that* $\mathfrak{t}, \mathfrak{u}, \mathfrak{w} \in$ Types *and that* $E_{\mu,\lambda}P_{\lambda;\mathfrak{t}} = P_{\mu;\mathfrak{t}}E_{\mu,\lambda}$ *for all* $\mu, \lambda \in (\mathscr{A}_d)^\dagger_+$ *of type* $\mathfrak{w}$ *for which* $\mu \leqslant \lambda$. *Further assume that* $P^\perp_{\nu;\mathfrak{t}} = P_{\nu;\mathfrak{u}}$ *for all* $\nu \in (\mathscr{A}_d)^\dagger_+$ *of type* $\mathfrak{w}$. *Then the following assertions hold:*

(i) *if* $\nu_1, \nu_2, \mu \in (\mathscr{A}_d)^\dagger_+$, *of type* $\mathfrak{w}$, *are such that* $\nu_1 + \nu_2 = \mu$ *and* ν_1 *and* ν_2 *are type* $\mathfrak{t}$ *and* $\mathfrak{u}$, *respectively, then* $\nu_1 = \mu_{\mathfrak{t}}$ *and* $\nu_2 = \mu_{\mathfrak{u}}$;

(ii) *for any* $\mu, \lambda \in (\mathscr{A}_d)^\dagger_+$ *of type* $\mathfrak{w}$, *one has* $(\mu+\lambda)_{\mathfrak{t}} = \mu_{\mathfrak{t}} + \lambda_{\mathfrak{t}}$.

Proof. (i) Plainly $\nu_1 \leqslant \mu$ and $\nu_2 \leqslant \mu$. It follows from Proposition 2.3(iv) that $\nu_1 \leqslant \mu_{\mathfrak{t}}$ and $\nu_2 \leqslant \mu_{\mathfrak{u}}$ since

$$E_{\mu,\lambda}P_{\lambda;\mathfrak{u}} = E_{\mu,\lambda} - E_{\mu,\lambda}P_{\lambda;\mathfrak{t}} = E_{\mu,\lambda} - P_{\mu;\mathfrak{t}}E_{\mu,\lambda} = (I - P_{\mu;\mathfrak{t}})E_{\mu,\lambda} = P_{\mu;\mathfrak{u}}E_{\mu,\lambda}.$$

For any positive semi-definite $c \in \mathscr{A}_d$, set

$$\delta_1 = \mu_{\mathfrak{t}}(c) - \nu_1(c), \quad \delta_2 = \mu_{\mathfrak{u}}(c) - \nu_2(c).$$

Note that δ_1, δ_2 are non-negative real numbers. As

$$0 = \nu_1(c) + \nu_2(c) - \mu(c) = -(\delta_1 + \delta_2),$$

it follows that $\delta_1 = \delta_2 = 0$. As every element of $\mathscr{A}_d$ is a linear combination of positive semi-definite elements, assertion (i) is proved.

(ii) It follows from Proposition 2.3(ii) that $\mu_{\mathfrak{t}} + \lambda_{\mathfrak{t}}$ is type $\mathfrak{t}$ and $\mu_{\mathfrak{u}} + \lambda_{\mathfrak{u}}$ is type $\mathfrak{u}$. As $(\mu_{\mathfrak{t}} + \lambda_{\mathfrak{t}}) + (\mu_{\mathfrak{u}} + \lambda_{\mathfrak{u}}) = \mu + \lambda$, it follows from (i) that

$$\mu_{\mathfrak{t}} + \lambda_{\mathfrak{t}} = (\mu + \lambda)_{\mathfrak{t}}. \quad \blacksquare$$

REMARK 2.5. Let $\mathfrak{t}, \mathfrak{u}, \mathfrak{w} \in$ Types be such that $P_{\mu;\mathfrak{w}} = P_{\mu;\mathfrak{t}} + P_{\mu;\mathfrak{u}}$ for all $\mu \in (\mathscr{A}_d)^{\dagger}_{+}$. It follows from Proposition 2.3 that the following assertions are equivalent:

(i) $E_{\mu,\lambda}P_{\lambda;\mathfrak{t}} = P_{\mu;\mathfrak{t}}E_{\mu,\lambda}$ whenever $\mu, \lambda \in (\mathscr{A}_d)^{\dagger}_{+}$ are of type $\mathfrak{w}$ and $\mu \leqslant \lambda$.

(ii) $\mu_{\mathfrak{t}} \leqslant \lambda_{\mathfrak{t}}$ and $\mu_{\mathfrak{u}} \leqslant \lambda_{\mathfrak{u}}$ whenever $\mu, \lambda \in (\mathscr{A}_d)^{\dagger}_{+}$ are of type $\mathfrak{w}$ and $\mu \leqslant \lambda$.

Indeed, that (ii) implies (i) is precisely Proposition 2.3(iii). In the other direction, we first note that $\mu_{\mathfrak{t}} \leqslant \lambda$ and $\mu_{\mathfrak{u}} \leqslant \lambda$. Assume (i). Since $\mu_{\mathfrak{t}}$ and $\mu_{\mathfrak{u}}$ are type $\mathfrak{t}$ and $\mathfrak{u}$, respectively, it then follows from Proposition 2.3(iv) that $\mu_{\mathfrak{t}} \leqslant \lambda_{\mathfrak{t}}$ and $\mu_{\mathfrak{u}} \leqslant \lambda_{\mathfrak{u}}$. In particular, (i) and (ii) hold in the case where $\mathfrak{w} =$ all, in which case our starting assumption is that $P_{\mu;\mathfrak{w}} = I = P_{\mu;\mathfrak{t}} + P_{\mu;\mathfrak{u}}$.

PROPOSITION 2.6. *Suppose that* $\gamma, \lambda \in (\mathscr{A}_d)^{\dagger}_{+}$ *and* $\gamma \leqslant \lambda$. *Let* $E := E_{\gamma,\lambda}$: $\mathbb{H}^2_d(\lambda) \to \mathbb{H}^2_d(\gamma)$ *be the contractive co-embedding. Then* $EP_{\lambda_{\mathrm{ac}}} = P_{\gamma_{\mathrm{ac}}}E$ *and* $EP_{\lambda_{\mathrm{s}}} = P_{\gamma_{\mathrm{s}}}E$. *That is,* $\mathfrak{t} = \mathrm{ac}$ *and* $\mathfrak{u} = \mathrm{s}$ *are hereditary-types and determine positive hereditary cones.*

Proof. By [17, Corollary 8.8], if $\gamma = \gamma_{\mathrm{ac}} + \gamma_{\mathrm{s}}$ and $\lambda = \lambda_{\mathrm{ac}} + \lambda_{\mathrm{s}}$ are the NC Lebesgue decompositions of γ, λ, then $\gamma_{\mathrm{s}} \leqslant \lambda_{\mathrm{s}}$. Since $\lambda = \gamma + (\lambda - \gamma) \geqslant \gamma$, it follows from [17, Corollary 8.14] that $\lambda_{\mathrm{ac}} = \gamma_{\mathrm{ac}} + (\lambda - \gamma)_{\mathrm{ac}} \geqslant \gamma_{\mathrm{ac}}$. Thus, $\lambda_{\mathrm{ac}} \geqslant \gamma_{\mathrm{ac}}$ as well. The proposition now follows from Proposition 2.3. $\blacksquare$

COROLLARY 2.7. *With* $\gamma \leqslant \lambda$ *as in Proposition* 2.5, *if* $D = E^*_{\gamma,\lambda}E_{\gamma,\lambda}$, *then* $DP_{\lambda;\mathrm{ac}} = P_{\lambda;\mathrm{ac}}D$.

In the next lemma, recall that if Q_λ is the structure projection of Π_λ, then $Q^{\perp}_{\lambda} = Q_{\lambda;\mathrm{WC}}$ is the projection onto $\mathrm{WC}(\Pi_\lambda)$ by Theorem 1.7.

LEMMA 2.8. *Suppose* $\mu, \lambda \in (\mathscr{A}_d)^{\dagger}_{+}$ *with* $\mu \leqslant \lambda$. *Let* Q_λ *and* Q_μ *be the structure projections of* Π_λ *and* Π_μ*, respectively.*

Then, $E_{\mu,\lambda}Q^{\perp}_{\lambda} = Q^{\perp}_{\mu}E_{\mu,\lambda}Q^{\perp}_{\lambda}$.

Proof. Set $E = E_{\mu,\lambda}$. Let $h \in \mathbb{H}^2_d(\lambda)$ be a WC vector of Π_λ. By Theorem 1.7 there is an intertwiner $X : \mathbb{H}^2_d \to \mathbb{H}^2_d(\lambda)$ and a vector $g \in \mathbb{H}^2_d$ such that $Xg = h$. As EX intertwines Π_λ and Π_μ, it follows that $Eh = EXg$ is a WC vector of Π_μ. Thus, $EQ^\perp_\lambda = Q^\perp_\mu EQ^\perp_\lambda$. ∎

Recall that a vector, $h \in \mathcal{H}$, is said to be a *wandering vector* for a row isometry $V : \mathcal{H} \otimes \mathbb{C}^d \to \mathcal{H}$, if

$$\langle V^\alpha h, V^\omega h\rangle = \delta_{\alpha,\omega}\|h\|^2,$$

and that the closed linear span of all wandering vectors for V is $\operatorname{Ran} V^\perp$. If x is a unit wandering vector for V, then

$$\mathcal{H}_x := \bigvee_{\omega\in\mathbb{F}^d} V^\omega x,$$

is V-invariant and the linear map, $U_x : \mathcal{H}_x \to \mathbb{H}^2_d$, defined by $U_x V^\omega x := L^\omega 1$ is an onto isometry intertwining V and L, $U_x V^\omega = L^\omega U_x$ [27].

LEMMA 2.9. *Let Π be a row isometry on a Hilbert space $\mathcal{H}$ and set*

$$\mathcal{H}_0 := \bigvee_{\beta,\gamma\in\mathbb{F}^d} \Pi^\beta \Pi^{\gamma *}\mathrm{WC}(\Pi).$$

Then $\mathcal{H}_0$ is Π-reducing and the restriction of Π to $\mathcal{H}_0^\perp$ is the von Neumann-type summand of Π.

Proof. By a result of M. Kennedy, a row isometry Π is of von Neumann-type if and only if it has no wandering vectors [22, Corollary 4.13]. Specifically, any pure or AC Cuntz row isometry has wandering vectors. Since a dilation-type row isometry has a pure type L restriction to the non-trivial invariant subspace of its weak* continuous vectors, it also has wandering vectors. It is clear that any wandering vector for Π belongs to $\mathrm{WC}(\Pi)$, and it is clear that $\mathcal{H}_0$ and hence $\mathcal{H}_0^\perp$ is Π-reducing. By construction $\mathcal{H}_0^\perp \cap \mathrm{WC}(\Pi) = \{0\}$ so that Π restricted to $\mathcal{H}_0^\perp$ has no wandering vectors and is hence of von Neumann-type. That is, $\mathcal{H}_0^\perp \subset \operatorname{Ran} P_{\mathrm{vN}}$. Let $h \in \operatorname{Ran} P_{\mathrm{vN}}$, let $v \in \mathbb{H}^2_d$, and let $X : \mathbb{H}^2_d \to \mathcal{H}$ be an intertwiner. Then, for any words β, γ, we have

$$\langle \Pi^\beta \Pi^{\gamma *} Xv, h\rangle = \langle \Pi^\beta \Pi^{\gamma *} Xv, P_{\mathrm{vN}} h\rangle = \langle \Pi^\beta \Pi^{\gamma *} P_{\mathrm{vN}} X v, h\rangle.$$

Since $P_{\mathrm{vN}} Xv \in \mathrm{WC}(\Pi) \cap \operatorname{Ran} P_{\mathrm{vN}}$, we have $P_{\mathrm{vN}} Xv = 0$ and thus $h \in \mathcal{H}_0^\perp$. Therefore, $\mathcal{H}_0^\perp = \operatorname{Ran} P_{\mathrm{vN}}$. ∎

REMARK 2.10. If Π is of dilation-type on $\mathcal{H}$, then $\mathrm{WC}(\Pi)$ is Π-invariant but cannot contain any Π-reducing subspace. Thus, $\mathcal{H}_0 = \mathcal{H}$ for dilation-type row isometries.

REMARK 2.11. By Theorem 1.7, we have $\mathrm{WC}(\Pi)$ equal to the range of $Q^{\perp}$, where Q is the structure projection of Π. Suppose Π and Ξ are unitarily equivalent row isometries. That is, there exists a surjective isometry, U, such that $U\Pi^{\alpha} = \Xi^{\alpha}U$ for each word α. It then follows from Lemma 2.8 that $UP_{\Pi;\mathrm{vN}} = P_{\Xi;\mathrm{vN}}U$.

The following fact is well-known and can be found in [17, Lemma 8.9].

LEMMA 2.12. *Let Π and Ξ be row isometries on Hilbert spaces $\mathcal{H}, \mathcal{J}$, respectively, and suppose that Π is a Cuntz unitary. If $X : \mathcal{H} \to \mathcal{J}$ is a bounded linear map satisfying*

$$X\Pi^{\alpha} = \Xi^{\alpha}X, \quad \alpha \in \mathbb{F}^d,$$

then

$$\Pi^{\alpha}X^* = X^*\Xi^{\alpha}, \quad \alpha \in \mathbb{F}^d,$$

and X^*X is in the commutant of the von Neumann algebra of Π. Similarly, XX^* is in the commutant of the von Neumann algebra of Ξ.

LEMMA 2.13. *Suppose $\mu, \lambda \in (\mathscr{A}_d)^{\dagger}_{+}$ satisfy $\mu \leqslant \lambda$. If λ is of von Neumann-type, then so is μ. That is, $\mathfrak{t} = \mathrm{vN}$ is a hereditary-type.*

Proof. The set of all positive singular NC measures is a positive hereditary cone so that μ is necessarily singular. It follows that $\mu = \mu_{\mathrm{dil}} + \mu_{\mathrm{vN}}$. Suppose that $x \in \mathrm{WC}(\mu)$. By Theorem 1.7, there is a bounded intertwiner $X : \mathbb{H}^2_d \to \mathbb{H}^2_d(\mu)$ and a vector $f \in \mathbb{H}^2_d$ so that $Xf = x$. Since $\mu \leqslant \lambda$, the co-embedding $E : \mathbb{H}^2_d(\lambda) \to \mathbb{H}^2_d(\mu)$ is contractive and $E\Pi^{\alpha}_{\lambda} = \Pi^{\alpha}_{\mu}E$ for any word α. By Lemma 2.11, we also have that $E^*\Pi^{\alpha}_{\mu} = \Pi^{\alpha}_{\lambda}E^*$, so that $Y := E^*X : \mathbb{H}^2_d \to \mathbb{H}^2_d(\lambda)$ is an intertwiner:

$$YL^{\alpha} = E^*XL^{\alpha} = E^*\Pi^{\alpha}_{\mu}X = \Pi^{\alpha}_{\lambda}Y.$$

Setting $y = Yf \in \mathbb{H}^2_d(\lambda)$, we see that y is in the range of a bounded interwtiner and thus in $\mathrm{WC}(\lambda)$. Because λ is of von Neumann-type, we have $y \in \mathrm{WC}(\lambda) = \{0\}$. Since E^* is injective, we have $x = 0$. It follows that $\mathrm{WC}(\mu) = \{0\}$, and thus $\mu_{\mathrm{dil}} = 0$. We conclude that μ is of von Neumann-type. ∎

The next lemma and Proposition 2.3(i) imply that $\mathfrak{t} = \mathrm{dil}$ is also a hereditary-type.

LEMMA 2.14. *Suppose $\mu, \lambda \in (\mathscr{A}_d)^{\dagger}_{+}$ satisfy $\mu \leqslant \lambda$. Then,*

$$E_{\mu,\lambda}P_{\lambda;\mathrm{dil}} = P_{\mu;\mathrm{dil}}E_{\mu,\lambda}P_{\lambda;\mathrm{dil}}.$$

Proof. We know that $P_{\lambda;\mathrm{dil}} \leqslant P_{\lambda;\mathrm{s}}$, that $EP_{\lambda;\mathrm{s}} = P_{\mu;\mathrm{s}}E$ and that $E\Pi^{\beta}_{\lambda} = \Pi^{\beta}_{\mu}E$ for every word β. Using Remark **??**, we assume, without loss of generality, that both μ and λ are singular. The GNS row isometry of any singular NC measure is Cuntz, and thus Π_{λ}, Π_{μ} are Cuntz. Note that $\mathrm{WC}(\lambda) \subset \operatorname{Ran} P_{\lambda;\mathrm{dil}}$ is $\Pi_{\lambda;\mathrm{dil}}$-invariant and that $\operatorname{Ran} P_{\lambda;\mathrm{dil}}$ is the smallest Π_{λ}-reducing subspace of $\mathbb{H}^2_d(\lambda)$ which contains $\mathrm{WC}(\lambda)$ by Lemma 2.8. Let $x \in \mathbb{H}^2_d(\lambda_{\mathrm{dil}})$, and denote by π_{λ} and π_{μ} the GNS representations of $\mathscr{E}_d$ induced by λ and μ, respectively. Since $x \in \mathbb{H}^2_d(\lambda_{\mathrm{dil}})$, it belongs to Ran $P_{\lambda;\mathrm{dil}}$, and thus by Lemma 2.8 there is a sequence of operators $A_1, A_2, \ldots \in \mathscr{E}_d$ and a $y \in \mathrm{WC}(\lambda)$ such that

$$x = \lim_n \pi_\lambda(A_n)y.$$

Since Π_λ is Cuntz, we can again apply Lemma 2.11 to find that

$$Ex = \lim_n \pi_\mu(A_n)Ey.$$

Because $y \in \mathrm{WC}(\lambda)$, it follows from Lemma 2.7 that $Ey \in \mathrm{WC}(\mu)$. Thus, $Ex \in$ Ran $P_{\mu;\mathrm{dil}}$ by Lemma 2.8 again. ∎

LEMMA 2.15. *Let $\lambda \in (\mathscr{A}_d)^\dagger_+$. If $x \in$ Ran $P_{\lambda;\mathrm{vN}}$, then the positive NC measure λ_x determined by $\lambda_x(L^\alpha) = \langle x, \Pi^\alpha_\lambda x\rangle_\lambda$, $\alpha \in \mathbb{F}^d$, is of von Neumann-type.*

Proof. Let $\mathcal{H}_x$ denote the cyclic subspace of $\mathbb{H}^2_d(\lambda)$ generated by x. Note that there is a surjective isometry $U : \mathbb{H}^2_d(\lambda_x) \to \mathcal{H}_x$ such that $U(a + N_{\lambda_x}) = \pi_\lambda(a)x$ for each $a \in \mathbb{A}_d$. Because λ is von Neumann-type, for any given word β, there exists a net $(a_\gamma)_\gamma$ in $\mathbb{A}_d$ such that $(\Pi^\beta_\lambda)^*$ is the weak* limit of $(\pi_\lambda(a_\gamma))_\gamma$. As $U^*\pi_\lambda(a)U = \pi_{\lambda_x}(a)$ for any $a \in \mathbb{A}_d$, it follows that $(\Pi^\beta_{\lambda_x})^*$ is the weak* limit of $(\pi_{\lambda_x}(a_\gamma))_\gamma$. This shows that the weak* closure of $\pi_{\lambda_x}(\mathbb{A}_d)$ is self-adjoint, and thus Π_{λ_x} is of von Neumann-type. ∎

LEMMA 2.16. *Let $\mu, \nu \in (\mathscr{A}_d)^\dagger_+$. If $\nu \leqslant \mu$ and ν is of von Neumann-type, then $\nu \leqslant \mu_{\mathrm{vN}}$.*

Proof. By Proposition 2.5, $E_{\nu,\mu}P_{\mu;\mathrm{ac}} = P_{\nu;\mathrm{ac}}E_{\nu,\mu}$. By Lemma **??**, $E_{\nu,\mu}P_{\mu;\mathrm{dil}} = P_{\nu;\mathrm{dil}}E_{\nu,\mu}P_{\mu;\mathrm{dil}}$. As ν is of von Neumann-type, we know that $P_{\nu;\mathrm{dil}} = 0 = P_{\nu;\mathrm{ac}}$, and thus

$$E_{\nu,\mu} = E_{\nu,\mu}(P_{\mu;\mathrm{ac}} + P_{\mu;\mathrm{dil}} + P_{\mu;\mathrm{vN}}) = E_{\nu,\mu}P_{\mu;\mathrm{vN}}.$$

Thus, for any positive semi-definite $c \in \mathscr{A}_d$, we have

$$\begin{aligned}\nu(c) &= \langle I + N_\nu, \pi_\nu(c)(I + N_\nu)\rangle = \langle I + N_\mu, E^*_{\nu,\mu}\pi_\nu(c)E_{\nu,\mu}(I + N_\mu)\rangle \\ &= \langle P_{\mu;\mathrm{vN}}(I + N_\mu), E^*_{\nu,\mu}\pi_\nu(c)E_{\nu,\mu}P_{\mu;\mathrm{vN}}(I + N_\mu)\rangle.\end{aligned}$$

By Lemma 2.2, we have $E^*_{\nu,\mu}\pi_\nu(c)E_{\nu,\mu} \leqslant \pi_\mu(c)$, and thus

$$\nu(c) \leqslant \langle P_{\mu;\mathrm{vN}}(I + N_\mu), \pi_\mu(c)P_{\mu;\mathrm{vN}}(I + N_\mu)\rangle = \mu_{\mathrm{vN}}(c).$$

That is, $\nu \leqslant \mu_{\mathrm{vN}}$. ∎

THEOREM 2.17. *Suppose $\mu, \lambda \in (\mathscr{A}_d)^\dagger_+$ satisfy $\mu \leqslant \lambda$. Let $E : \mathbb{H}^2_d(\lambda) \to \mathbb{H}^2_d(\mu)$ denote the contractive co-embedding. Then, $EP_{\lambda;\mathrm{vN}} = P_{\mu;\mathrm{vN}}E$, and $EP_{\lambda;\mathrm{dil}} = P_{\mu;\mathrm{dil}}E$ and the sets of positive NC measures of dilation and von Neumann-type are positive hereditary cones.*

Proof. Assume first that μ and λ are singular. Set $x = P_{\lambda;\mathrm{vN}}E^*(I+N_\mu) \in \mathbb{H}^2_d(\lambda)$. Plainly, $\lambda_x \leqslant \mu \leqslant \lambda$, where, as before, $\lambda_x(L^\omega) := \langle x, \Pi^\omega_\lambda x\rangle_\lambda$. By Lemma 2.13, we see that λ_x is of von Neumann-type, and thus by Lemma 2.14 we see that $\lambda_x \leqslant \mu_{\mathrm{vN}}$. Then, for any $a \in \mathbb{A}_d$,

$$\langle (a+N_\mu), EP_{\lambda;\mathrm{vN}}E^*(a+N_\mu)\rangle = \lambda_x(a^*a) \leqslant \mu_{\mathrm{vN}}(a^*a) = \langle a+N_\mu, P_{\mu;\mathrm{vN}}(a+N_\mu)\rangle,$$

whence

$$EP_{\lambda;\mathrm{vN}}E^* \leqslant P_{\mu;\mathrm{vN}}.$$

Let Q be the projection onto the range of $EP_{\lambda;\mathrm{vN}}$. Applying the Douglas factorization lemma then yields $\mathrm{Ran}\, EP_{\lambda;\mathrm{vN}} \subseteq \mathrm{Ran}\, P_{\mu;\mathrm{vN}}$. In particular, $Q \leqslant P_{\mu;vN}$, and it follows that

$$P_{\mu;\mathrm{vN}}EP_{\lambda;\mathrm{vN}} = P_{\mu;\mathrm{vN}}QEP_{\lambda;\mathrm{vN}} = QEP_{\lambda;\mathrm{vN}} = EP_{\lambda;\mathrm{vN}}. \tag{2.1}$$

As μ and λ are assumed singular, we have $P_{\lambda;\mathrm{dil}} = P^{\perp}_{\lambda;\mathrm{vN}}$ (and similarly for μ). Then,

$$\begin{aligned} P_{\mu;\mathrm{vN}}E &= P_{\mu;\mathrm{vN}}E(P_{\lambda;\mathrm{dil}} + P_{\lambda;\mathrm{vN}}) \\ &= \underbrace{P_{\mu;\mathrm{vN}}P_{\mu;\mathrm{dil}}}_{=0} EP_{\lambda;\mathrm{dil}} + P_{\mu;\mathrm{vN}}EP_{\lambda;\mathrm{vN}} \quad \text{(by Lemma \textbf{??})} \\ &= EP_{\lambda;\mathrm{vN}} \quad \text{(by equation (\textbf{??}))}. \end{aligned}$$

As $P_{\lambda;\mathrm{dil}} = I - P_{\lambda;\mathrm{vN}}$, the theorem is proved in the case where λ and μ are singular.

In the general case, where λ and μ are not necessarily singular, we note that $\mu \leqslant \lambda$ implies $\mu_{\mathrm{s}} \leqslant \lambda_{\mathrm{s}}$, thus $E_{\mu_{\mathrm{s}},\lambda_{\mathrm{s}}}P_{\lambda_{\mathrm{s}};\mathrm{vN}} = P_{\mu_{\mathrm{s}};\mathrm{vN}}E_{\mu_{\mathrm{s}},\lambda_{\mathrm{s}}}$. As seen in the proof of Proposition 2.3, there are unitary intertwiners $U_\mu : \mathbb{H}^2_d(\mu) \to \mathbb{H}^2_d(\mu_{\mathrm{ac}}) \oplus \mathbb{H}^2_d(\mu_{\mathrm{s}})$ and $U_\lambda : \mathbb{H}^2_d(\lambda) \to \mathbb{H}^2_d(\lambda_{\mathrm{ac}}) \oplus \mathbb{H}^2_d(\lambda_{\mathrm{s}})$ such that

$$U_\mu E U_\lambda^* = \begin{bmatrix} E_{\mu_{\mathrm{ac}},\lambda_{\mathrm{ac}}} & 0 \\ 0 & E_{\mu_{\mathrm{s}},\lambda_{\mathrm{s}}} \end{bmatrix}.$$

Since $\mu_{\mathrm{s}} = \mu_{\mathrm{vN}} + \mu_{\mathrm{dil}}$ and $\mathbb{H}^2_d(\mu_{\mathrm{s}}) \simeq \mathbb{H}^2_d(\mu_{\mathrm{vN}}) \oplus \mathbb{H}^2_d(\mu_{\mathrm{dil}})$, it follows that $U_\mu P_{\mu;\mathrm{vN}} U_\mu^* = 0 \oplus P_{\mu_{\mathrm{s}};\mathrm{vN}}$ and a similar formula holds for λ. Thus,

$$U_\mu P_{\mu;\mathrm{vN}} E U_\lambda^* = (U_\mu P_{\mu;\mathrm{vN}} U_\mu^*)(U_\mu E U_\lambda^*) = (U_\mu E U_\lambda^*)(U_\lambda P_{\lambda;\mathrm{vN}} U_\lambda^*) = U_\mu E P_{\lambda;\mathrm{vN}} U_\lambda^*.$$

That is, $P_{\mu;\mathrm{vN}}E = EP_{\lambda;\mathrm{vN}}$. As $P_{\mu;\mathrm{ac}}E = EP_{\lambda;\mathrm{ac}}$, we have

$$P_{\mu;\mathrm{dil}}E = (I - P_{\mu;\mathrm{ac}} - P_{\mu;\mathrm{vN}})E = E(I - P_{\lambda;\mathrm{ac}} - P_{\lambda;\mathrm{vN}}) = EP_{\lambda;\mathrm{dil}}.$$

It now follows from Proposition 2.3(ii) that the dilation-type and von Neumann-type positive NC measures form hereditary cones. ∎

The following result refines the NC Lebesgue decomposition of [17, Section 8] by further decomposing any positive and singular NC measure into positive dilation-type and von Neumann-type NC measures.

COROLLARY 2.18 (NC Kennedy–Lebesgue decomposition). *Any positive* NC *measure $\mu \in (\mathscr{A}_d)^\dagger_+$ has a unique NC Kennedy–Lebesgue decomposition,*

$$\mu = \mu_{\mathrm{ac}} + \mu_{\mathrm{dil}} + \mu_{\mathrm{vN}},$$

where $\mu_{\mathrm{ac}}, \mu_{\mathrm{dil}}, \mu_{\mathrm{vN}} \in (\mathscr{A}_d)^\dagger_+$ are positive non-commutative measures of absolutely continuous-type, dilation-type and von Neumann-type, respectively. In particular, if ν_1, ν_2, ν_3 are, respectively, absolutely continuous, dilation-type and von Neumann-type positive NC *measures, and $\mu = \nu_1 + \nu_2 + \nu_3$, then*

$$\nu_1 = \mu_{\mathrm{ac}}, \quad \nu_2 = \mu_{\mathrm{dil}} \quad \textit{and} \quad \nu_3 = \mu_{\mathrm{vN}}.$$

The absolutely continuous, dilation-type and von Neumann-type positive NC *measures each form a positive hereditary cone. Moreover, if $\mathfrak{t} \in \{\mathrm{ac}, \mathrm{dil}, \mathrm{vN}\}$, then for any $\nu, \lambda \in (\mathscr{A}_d)^\dagger_+$*

$$(\nu + \lambda)_{\mathfrak{t}} = \nu_{\mathfrak{t}} + \lambda_{\mathfrak{t}}.$$

Proof. It is already known from [17] that the absolutely continuous positive NC measures form a hereditary cone and the fact that the dilation- and von Neumann-type positive NC measures form hereditary cones was proven in Theorem 2.15. The additivity of $(\cdot)_{\mathfrak{t}}$ follows from Theorem 2.15 and Lemma 2.4(ii). That $\nu_1 = \lambda_{\mathrm{ac}}$ follows from Proposition 2.5, leaving $\nu_2 + \nu_3 = \lambda_{\mathrm{s}}$. From this and Lemma 2.4(i), we have $\nu_3 = (\lambda_{\mathrm{s}})_{\mathrm{vN}} = \lambda_{\mathrm{vN}}$, the second equality following from Lemma 2.8. It now follows that $\nu_2 = \lambda_{\mathrm{dil}}$. ∎

PROPOSITION 2.19. *Suppose $\mu, \lambda \in (\mathscr{A}_d)^\dagger_+$ with $\mu \leqslant \lambda$. Let $P_\lambda := P_{\lambda;\mathrm{Cuntz}}$ be the Π_λ-reducing projection onto the support of its Cuntz direct summand. Then, $E_{\mu,\lambda} P_\lambda = P_\mu E_{\mu,\lambda} P_\lambda$. In particular, if λ is Cuntz-type, then μ is Cuntz-type, and $\mathfrak{t} =$* Cuntz *is a hereditary-type.*

Proof. It follows from Popescu's Wold decomposition theorem, [27, Theorem 1.3], that the range of P_λ is the set of all $x \in \mathbb{H}^2_d(\lambda)$ so that for any non-negative integer, N, there exist $\{x_\alpha : \alpha \in \mathbb{F}^d, |\alpha| = N\} \subset \mathbb{H}^2_d(\lambda)$ such that

$$x = \sum_{|\alpha|=N} \Pi_\lambda^\alpha x_\alpha.$$

Set $E = E_{\mu,\lambda}$. Then,

$$Ex = \sum_{|\alpha|=N} \Pi_\mu^\alpha E x_\alpha,$$

for any non-negative integer N. It follows that $EP_\lambda = P_\mu E P_\lambda$, from which the remaining claim follows on application of Proposition 2.3(i). ∎

EXAMPLE 2.20. In contrast to the results of Corollary 2.16, the set of Cuntz-type positive NC measures is not a cone. Denote by $U_\pm$ the unitary operator of co-ordinate multiplication $f \mapsto \zeta f$ on the Hilbert space $\chi_{\partial\mathbb{D}_\pm} L^2(\partial\mathbb{D})$, where $\chi_{\partial\mathbb{D}_\pm}$ is the characteristic function of $\partial\mathbb{D}_\pm$, the upper half-unit circle (when $+$) or lower half-unit circle (when $-$). Consider $\ell^2(\mathbb{F}^2)$ and let P denote the projection onto the closed span of $\{e_{1^k} : k = 0, 1, 2, \ldots\}$, the standard basis vectors containing only the "letter" 1. Let $\mathscr{S}$ denote the weak*-closed operator system generated by $\{L^\alpha|_{\mathrm{Ran}(P)} : \alpha \in \mathbb{F}^2\}$. Then $\Psi : \mathscr{T}_2 \to \mathscr{S}$, given by $\Psi(x) := Px|_{\mathrm{Ran}(P)}$ for $x \in \mathscr{T}_2$, is unital, weak* continuous, and positive. Evidently, $\mathscr{S}$ is weak*-weak* homeomorphic and isometrically linearly $*$-isomorphic to $L^\infty(\partial\mathbb{D})$; call the map implementing this isomorphism $\Theta : \mathscr{S} \to L^\infty(\partial\mathbb{D})$. Since the spectral measures for $U_\pm$ are absolutely continuous with respect to Lebesgue measure, the map $x \mapsto \Theta(\Psi(x))(U_\pm)$ is defined and weak*-continuous, where for $f \in L^\infty(\partial\mathbb{D})$, the operator $f(U_\pm)$ is given by the usual functional calculus for unitary operators. Thus, the functional $\lambda_\pm \in (\mathscr{A}_2^\dagger)_+$ given by

$$\lambda_\pm(x) = \langle \chi_{\partial\mathbb{D}_\pm}, \Theta(\Psi(x))(U_\pm)\chi_{\partial\mathbb{D}_\pm}\rangle, \quad x \in \mathscr{A}_2,$$

is absoutely continuous. Note that

$$\lambda_\pm(L^\alpha) = \begin{cases} \int_{\partial\mathbb{D}_\pm} \zeta^k m(\mathrm{d}\theta) & \alpha = 1^k, \\ 0 & \alpha \in \mathbb{F}^2 2\mathbb{F}^2, \end{cases} \qquad \alpha \in \mathbb{F}^2$$

where m is normalized Lebesgue measure on $\partial\mathbb{D}$. In particular, $\lambda_+ + \lambda_- = m$, where here, m denotes NC Lebesgue measure, the vacuum state, $x \mapsto \langle e_\varnothing, xe_\varnothing\rangle$. We now show that $\lambda_\pm$ are both Cuntz. In what follows, it will suffice to consider the case of $+$ (rather than $\pm$ simultaneously). First, let $\mathfrak{C}$ denote the norm-closed span of $\{e_\varnothing\} \cup \{e_\alpha : \alpha \in \mathbb{F}^2 2\}$, and set $\mathfrak{H} = \chi_{\partial\mathbb{D}_+} L^2(\partial\mathbb{D})$ and $\mathfrak{K} = \mathfrak{H} \otimes \mathfrak{C}$. Define the pair of operators $Z = (Z_1, Z_2)$ over $\mathfrak{K}$ by setting

$$Z_1(h \otimes e_\alpha) = \begin{cases} Uh \otimes e_\varnothing & \alpha = \varnothing, \\ h \otimes e_{1\alpha} & \alpha \in \mathbb{F}^2 2, \end{cases} \qquad Z_2(h \otimes e_\alpha) = h \otimes e_{2\alpha}.$$

It is readily verified by direct computation that Z is a Cuntz row isometry. An application of Lavrentiev's Theorem shows that U_+ has $\chi_{\partial\mathbb{D}_+}$ as a cyclic vector, and thus Z has $\chi_{\partial\mathbb{D}_+} \otimes e_\varnothing$ as a cyclic vector. Set

$$\nu(x) = \langle \chi_{\partial\mathbb{D}_+} \otimes e_\varnothing, \pi_\nu(x)(\chi_{\partial\mathbb{D}_+} \otimes e_\varnothing)\rangle, \quad x \in \mathscr{A}_2.$$

Note that $\nu \in (\mathscr{A}_2^\dagger)_+$. Since $Z^\alpha(h \otimes e_\varnothing) \perp \mathfrak{H} \otimes e_\varnothing$ when α contains the letter 2, we find that $\nu(L^\alpha) = 0$ for $\alpha \in \mathbb{F}^2 2\mathbb{F}^2$, while

$$\nu(L_1^k) = \langle \chi_{\partial\mathbb{D}_+} \otimes e_\varnothing, Z_1^k(\chi_{\partial\mathbb{D}_+} \otimes e_\varnothing)\rangle = \langle \chi_{\partial\mathbb{D}_+}, U_+^k \chi_{\partial\mathbb{D}_+}\rangle$$

for $k \in \mathbb{N} \cup \{0\}$. As both λ_+ and ν are positive, it follows that $\lambda_+ = \nu$. In particular, for $a, b \in \mathbb{A}_2$, we have

$$\lambda_+(b^*a) = \nu(b^*a) = \langle \pi_Z(b)(\chi_{\partial\mathbb{D}_+} \otimes e_\varnothing), \pi_Z(a)(\chi_{\partial\mathbb{D}_+} \otimes e_\varnothing)\rangle,$$

from which it follows that there exists a surjective isometry $W : \mathbb{H}_2^2(\lambda_+) \to \mathfrak{K}$ satisfying

$$W(L^\alpha + N_{\lambda_+}) = Z^\alpha(\chi_{\partial\mathbb{D}_+} \otimes e_\varnothing).$$

We now see that Π_{λ_+} is unitarily equivalent to Z, and thus Π_{λ_+} is Cuntz.

REMARK 2.21. The example above is interesting for additional reasons. First, since $\lambda_+ \leqslant \lambda_+ + \lambda_- = m$, we see that the pure type L positive NC measures are not hereditary. Next, one can show that Z is unitarily equivalent to a direct integral of atomic dilation-type row isometries, in a manner similar to [11, Example 3.3]. However, Π_{λ_+}, and thus Z, must be absolutely continuous. A similar phenomenon can be observed on comparison of [11, Example 3.3] and [7, Example 2.11]; an absolutely continuous row isometry decomposed as a direct integral of dilation-type row isometries. Now, translating this to functionals, these examples demonstrate that the dilation-type, Cuntz, and singular families of positive NC measures all fail to be weak*-closed. Compare this with what happens for $d = 1$, where we may write, for example, Lebesgue measure as weak* limit of linear combinations of Dirac masses.

3. COMPLEX NC MEASURES

Our goal for the remainder of the paper is to apply the preceding results to study analytic (and complex) NC measures.

DEFINITION 3.1. An NC measure $\mu \in \mathscr{A}_d^\dagger$ is *absolutely continuous* (AC) if it has a weak* continuous extension to the free Toeplitz system, $\mathscr{T}_d = \mathscr{A}_d^{-\text{weak}^*}$.

In [11, Theorem 2.10], Davidson–Pitts show that any bounded linear functional on $\mathbb{A}_d$ that extends weak* continuously to $\mathbb{A}_d^{-\text{weak}^*} = \mathfrak{L}_d^\infty$ is a vector functional, for $d \geqslant 2$. The next lemma shows that their proof extends to our setting.

LEMMA 3.2. *Any absolutely continuous* NC *measure $\mu \in \mathscr{A}_d^\dagger$, for $d \geqslant 2$, is a vector functional. That is, if $\mu \in \mathscr{A}_d^\dagger$ is absolutely continuous, then there exist $f, g \in \mathbb{H}_d^2$ such that*

$$\mu(b) = \langle f, bg \rangle, \quad b \in \mathscr{A}_d.$$

Proof. By general considerations, μ can be extended (with generally $\varepsilon > 0$ increase in norm) to a weak* continuous linear functional, $\widehat{\mu}$, acting on $\mathscr{L}(\mathbb{H}_d^2)$. Indeed, since $\mathscr{T}_d = \mathscr{A}_d^{-\text{weak}^*}$ is weak*-closed, it can be identified with the annihilator $\mathscr{S}^\perp$ of some norm-closed subspace $\mathscr{S}$ of the trace-class operators $\mathrm{Tr}(\mathbb{H}_d^2)$ on $\mathbb{H}_d^2$. Here, we recall that $\mathrm{Tr}(\mathbb{H}_d^2)$ is the pre-dual of $\mathscr{L}(\mathbb{H}_d^2)$ [26, Corollary 2.4.11]. Thus, for any $K \in \mathscr{S}$ and any $a \in \mathscr{T}_d$, we have $\mathrm{tr}(Ka) = 0$. If $q : \mathrm{Tr}(\mathbb{H}_d^2) \to \mathrm{Tr}(\mathbb{H}_d^2)/\mathscr{S}$ is the quotient map, then $q^\dagger : (\mathrm{Tr}(\mathbb{H}_d^2)/\mathscr{S})^\dagger \to \mathscr{L}(\mathbb{H}_d^2)$ can be identified with the inclusion map of $\mathscr{T}_d$ into $\mathscr{L}(\mathbb{H}_d^2)$ [26, Proposition 2.4.13]. That is, the pre-dual of $\mathscr{T}_d$ is isomorphic to $\mathrm{Tr}(\mathbb{H}_d^2)/\mathscr{S}$ and linear functionals on $\mathscr{A}_d$ which extend weak* continuously to $\mathscr{T}_d$ can be identified with this quotient space. It follows that we can identify μ with the equivalence class $K + \mathscr{S}$ for some $K \in \mathrm{Tr}(\mathbb{H}_d^2)$. Hence, for any $S \in \mathscr{S}$ and $a \in \mathscr{T}_d$,

$$\mathrm{tr}((K+S)a) = \mathrm{tr}(Ka) = \mu(a).$$

Since

$$\|\mu\| = \inf_{S \in \mathscr{S}} \|K + S\|_{\mathrm{Tr}(\mathbb{H}_d^2)},$$

there exists, given $\varepsilon > 0$, an $S' \in \mathscr{S}$ so that

$$\|K + S'\|_{\mathrm{Tr}(\mathbb{H}^2_d)} \leqslant \|\mu\| + \varepsilon.$$

Set $K' = K + S'$. Then $\widehat{\mu} : \mathscr{L}(\mathbb{H}^2_d) \to \mathbb{C}$ given by

$$\widehat{\mu}(A) = \mathrm{tr}(AK'), \quad A \in \mathscr{L}(\mathbb{H}^2_d),$$

is a weak* continuous extension of μ to $\mathscr{L}(\mathbb{H}^2_d)$ with norm $\|\widehat{\mu}\| \leqslant \|\mu\| + \varepsilon$. The trace-class operator K' has a singular-value decomposition

$$K' = \sum_{k=1}^{\infty} s_k \langle x_k, \cdot \rangle y_k; \quad x_k, y_k \in \mathbb{H}^2_d, \quad s_k \geqslant 0.$$

Choose any sequence of words $\omega_k \in \mathbb{F}^d$ so that $\mathrm{Ran}\, R^{\omega_k} \perp \mathrm{Ran}\, R^{\omega_j}$ for $k \neq j$. For example, one can choose the words $\omega_{k+1} = 2^k 1$ for $k \in \mathbb{N}$. Then,

$$x := \sum_{k=1}^{\infty} s_k^{1/2} R^{\omega_k} x_k \quad \text{and} \quad y := \sum_{k=1}^{\infty} s_k^{1/2} R^{\omega_k} y_k$$

both converge to elements in $\mathbb{H}^2_d$. In what follows, $\delta_{k,j} = 0$ when $k \neq j$ and $\delta_{j,j} = 1$ for all j. For any $a_1, a_2 \in \mathbb{A}_d$, we have

$$\begin{aligned}
\langle x, a_1^* a_2 y \rangle &= \sum_{k,j=1}^{\infty} s_k^{1/2} s_j^{1/2} \langle R^{\omega_k} x_k, a_1^* a_2 R^{\omega_j} y_j \rangle_{\mathbb{H}^2_d} \\
&= \sum_{k,j=1}^{\infty} s_k^{1/2} s_j^{1/2} \langle x_k, a_1^* \underbrace{R^{\omega_k *} R^{\omega_j}}_{=\delta_{k,j} I} a_2 y_j \rangle_{\mathbb{H}^2_d} \\
&= \sum_{k=1}^{\infty} s_k \langle x_k, a_1^* a_2 y_k \rangle_{\mathbb{H}^2_d} = \mathrm{tr}(a_1^* a_2 K) = \widehat{\mu}(a_1^* a_2) = \mu(a_1^* a_2).
\end{aligned}$$

The lemma now follows from the fact that $\mathbb{A}_d^* \mathbb{A}_d$ is norm dense in $\mathscr{A}_d$. ∎

Given $\mu \in \mathscr{A}_d^\dagger$, define $\mu^* \in \mathscr{A}_d^\dagger$ by

$$\mu^*(b) = \overline{\mu(b^*)}, \quad b \in \mathscr{A}_d.$$

We also set

$$\mathrm{Re}\, \mu = \frac{1}{2}(\mu + \mu^*) \quad \text{and} \quad \mathrm{Im}\, \mu = \frac{1}{2\mathrm{i}}(\mu - \mu^*).$$

COROLLARY 3.3. *Any absolutely continuous* NC *measure* $\mu \in \mathscr{A}_d^\dagger$ *can be decomposed as* $\mu = (\mu_1 - \mu_2) + \mathrm{i}(\mu_3 - \mu_4)$ *where each* $\mu_k \geqslant 0$ *is* AC. *In particular,* $\mu_1 + \mu_2 + \mu_3 + \mu_4$ *is* AC.

Proof. Applying Lemma 3.2 to μ, we obtain vectors $x, y \in \mathbb{H}^2_d$ such that $\mu(b) = \langle x, by \rangle$ for $b \in \mathscr{A}_d$. Set $\lambda := (1/2)(m_x + m_y)$, where,

$$m_h(b) := \langle h, bh \rangle, \quad b \in \mathscr{A}_d, \ h \in \mathbb{H}^2_d.$$

Then we observe that, for any positive semi-definite $c \in \mathscr{A}_d$,

$$2\lambda(c) \pm 2\mathrm{Re}\,\mu(c) = m_{x\pm y}(c) \geqslant 0,$$

and similarly,

$$2\lambda(c) \mp 2\mathrm{Im}\,\mu(c) = m_{x\pm \mathrm{i}y}(c) \geqslant 0.$$

Therefore, we can set $\mu_1 = \lambda + \mathrm{Re}\,\mu$, $\mu_2 = \lambda - \mathrm{Re}\,\mu$, $\mu_3 = \lambda + \mathrm{Im}\,\mu$, and $\mu_4 = \lambda - \mathrm{Im}\,\mu$. ∎

3.1. GENERAL WITTSTOCK DECOMPOSITION. Any $\mu \in \mathscr{A}_d^\dagger$ can be written as a linear combination of four positive NC measures,

$$\mu = (\mu_1 - \mu_2) + \mathrm{i}(\mu_3 - \mu_4); \quad \mu_k \in (\mathscr{A}_d)_+^\dagger,$$

where

$$\mathrm{Re}\,\mu = \frac{\mu + \mu^*}{2} = \mu_1 - \mu_2, \quad \text{and} \quad \mathrm{Im}\,\mu = \frac{\mu - \mu^*}{2\mathrm{i}} = \mu_3 - \mu_4.$$

This also works for operator-valued NC measures, i.e. operator-valued completely bounded maps on the free disk system, by the Wittstock decomposition theorem [25, Theorem 8.5], [35].

DEFINITION 3.4. If $\vec{\lambda} := (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in (\mathscr{A}_d^\dagger)_+^4$ is such that $\mu = (\lambda_1 - \lambda_2) + \mathrm{i}(\lambda_3 - \lambda_4)$, then we call $\vec{\lambda}$ a *Wittstock decomposition of* μ. The set of all Wittstock decompositions of μ is denoted by $\mathscr{W}(\mu)$. Given a Wittstock decomposition $\vec{\lambda} \in \mathscr{W}(\mu)$, the *total variation of* μ *with respect to* $\vec{\lambda}$ is

$$|\vec{\lambda}| := \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \geqslant 0.$$

REMARK 3.5. The total variation $|\vec{\lambda}|$ defined above for $\vec{\lambda} \in \mathscr{W}(\mu)$ is not uniquely determined by μ. Indeed, if $\vec{\lambda} = (\lambda_k)_{k=1}^4$ is any Wittstock decomposition of μ, then so is $\vec{\mu}$ where $\mu_1 = \lambda_1 + \gamma$, $\mu_2 = \lambda_2 + \gamma$ and $\lambda_3 = \mu_3$, $\lambda_4 = \mu_4$, for any $\gamma \in (\mathscr{A}_d)_+^\dagger$. This is a rather trivial example of non-uniqueness and in this case $|\vec{\mu}| = |\vec{\lambda}| + 2\gamma \geqslant |\vec{\lambda}|$. However, even if $\vec{\mu}, \vec{\lambda} \in \mathscr{W}(\mu)$ are two different Wittstock decompositions of $\mu \in \mathscr{A}_d^\dagger$ so that $|\vec{\lambda}| \leqslant |\vec{\mu}|$, this need not imply that $\mu_k \geqslant \lambda_k$ for each k. That is, one could have, for example, that $\mu_1 - \mu_2 = \lambda_1 - \lambda_2$ without having $\mu_1 = \lambda_1 + \gamma$ and $\mu_2 = \lambda_2 + \gamma$ for some $\gamma \in (\mathscr{A}_d)_+^\dagger$.

PROPOSITION 3.6. *Let $\mu \in \mathscr{A}_d^\dagger$. Then, μ is absolutely continuous if and only if there exists a $\vec{\mu} \in \mathscr{W}(\mu)$ such that $|\vec{\mu}|$ is absolutely continuous.*

Proof. If μ is absolutely continuous, then we can apply Corollary 3.3. Conversely, suppose that $\vec{\mu} = (\mu_1, \mu_2, \mu_3, \mu_4) \in \mathscr{W}(\mu)$ is such that $|\vec{\mu}|$ is absolutely continuous. Recall that the absolutely continuous NC measures form a hereditary cone. As $\mu_j \leqslant |\vec{\mu}|$, it follows that μ_j is absolutely continuous for each j. Thus, μ is then absolutely continuous. ∎

DEFINITION 3.7. Let $\mathfrak{t} \in \{\mathrm{ac}, \mathrm{s}, \mathrm{dil}, \mathrm{vN}\}$. A complex NC measure $\mu \in \mathscr{A}_d^\dagger$ is type $\mathfrak{t}$, if there exists a Wittstock decomposition $\vec{\mu} = (\mu_k)_{k=1}^4 \in \mathscr{W}(\mu)$ such that $|\vec{\mu}| = \mu_1 + \mu_2 + \mu_3 + \mu_4$ is type $\mathfrak{t}$.

REMARK 3.8. Since $|\vec{\mu}| \geqslant \mu_k$ for each $1 \leqslant k \leqslant 4$, where $\vec{\mu} = (\mu_k)_k$ is a Wittstock decomposition of μ, and the sets of AC, singular, dilation-type and von Neumann-type NC measures are positive hereditary cones, it follows that if $|\vec{\mu}|$ is one of these four types, then so is each μ_k. It follows that $\vec{\mu}$ cannot, for example, be both absolutely continuous and von Neumann-type without also being $(0, 0, 0, 0)$.

LEMMA 3.9. *Let* $\mathfrak{t}, \mathfrak{u} \in \{\mathrm{ac}, \mathrm{s}, \mathrm{dil}, \mathrm{vN}\}$, *and let* $\mu \in \mathscr{A}_d^\dagger$.

(i) *If* $(\mu_j)_{j=1}^4, (\lambda_j)_{j=1}^4 \in \mathscr{W}(\mu)$, *then*

$$\lambda_{1;\mathfrak{u}} - \lambda_{2;\mathfrak{u}} = \mu_{1;\mathfrak{u}} - \mu_{2;\mathfrak{u}}, \quad \lambda_{3;\mathfrak{u}} - \lambda_{4;\mathfrak{u}} = \mu_{3;\mathfrak{u}} - \mu_{4;\mathfrak{u}}.$$

(ii) *Suppose* $(\nu_\mathfrak{t})_\mathfrak{u} = 0$ *for all* $\nu \in (\mathscr{A}_d)_+^\dagger$. *If* μ *is type* $\mathfrak{t}$ *and* $(\lambda_j)_{j=1}^4 \in \mathscr{W}(\mu)$, *then*

$$\lambda_{1;\mathfrak{u}} = \lambda_{2;\mathfrak{u}}, \quad \textit{and} \quad \lambda_{3;\mathfrak{u}} = \lambda_{4;\mathfrak{u}}.$$

Proof. (i) By separating the real and imaginary parts, we obtain

$$\lambda_1 - \lambda_2 = \mu_1 - \mu_2, \quad \lambda_3 - \lambda_4 = \mu_3 - \mu_4.$$

Thus

$$\lambda_1 + \mu_2 = \mu_1 + \lambda_2, \quad \lambda_3 + \mu_4 = \mu_3 + \lambda_4.$$

Applying Corollary 2.16, we obtain

$$\lambda_{1;\mathfrak{u}} + \mu_{2;\mathfrak{u}} = \mu_{1;\mathfrak{u}} + \lambda_{2;\mathfrak{u}}, \quad \lambda_{3;\mathfrak{u}} + \mu_{4;\mathfrak{u}} = \mu_{3;\mathfrak{u}} + \lambda_{4;\mathfrak{u}},$$

from which (i) easily follows.

(ii) Since μ is type $\mathfrak{t}$, there exists $(\mu_1, \mu_2, \mu_3, \mu_4) \in \mathscr{W}(\mu)$, where each μ_j is type $\mathfrak{t}$. Thus, $(\mu_j)_\mathfrak{t} = \mu_j$ whence $\mu_{j;\mathfrak{u}} = (\mu_{j;\mathfrak{t}})_\mathfrak{u} = 0$ for each j. Assertion (ii) now follows on applying (i). ∎

By the preceeding lemma, the condition $(\nu_\mathfrak{t})_\mathfrak{u} = 0$ is symmetric in $\mathfrak{t}, \mathfrak{u}$ and it occurs when $(\mathfrak{t}, \mathfrak{u}) \in \{(\mathrm{ac}, \mathrm{s}), (\mathrm{ac}, \mathrm{dil}), (\mathrm{ac}, \mathrm{vN}), (\mathrm{dil}, \mathrm{vN})\}$, i.e. whenever $P_{\nu;\mathfrak{t}} P_{\nu;\mathfrak{u}} = 0$ for every $\nu \in (\mathscr{A}_d)_+^\dagger$.

Any $\mu \in \mathscr{A}_d^\dagger$ with Wittstock decomposition $\mu = (\mu_1 - \mu_2) + \mathrm{i}(\mu_3 - \mu_4)$ has a corresponding Lebesgue decomposition

$$\mu = \mu_{\mathrm{ac}} + \mu_{\mathrm{s}},$$

where

$$\mu_{\mathrm{ac}} = (\mu_{1;\mathrm{ac}} - \mu_{2;\mathrm{ac}}) + \mathrm{i}(\mu_{3;\mathrm{ac}} - \mu_{4;\mathrm{ac}}),$$

and similarly for μ_s and $\mu_\mathrm{s} = \mu_{\mathrm{dil}} + \mu_{\mathrm{vN}}$. More generally, for $\mathfrak{t} \in$ Types, we set

$$\mu_\mathfrak{t} := (\mu_{1;\mathfrak{t}} - \mu_{2;\mathfrak{t}}) + \mathrm{i}(\mu_{3;\mathfrak{t}} - \mu_{4;\mathfrak{t}}).$$

We remark that Lemma 3.9(i) assures us that this is well-defined.

3.2. GNS FORMULA. Let $\mu \in \mathscr{A}_d^\dagger$ and let $\vec{\mu} = (\mu_k)_{k=1}^4$ be a Wittstock decomposition of μ. Since $|\vec{\mu}| \geqslant \mu_k$, there exists a corresponding contractive co-embedding $E_k : \mathbb{H}_d^2(|\vec{\mu}|) \to \mathbb{H}_d^2(\mu_k)$ for $k \in \{1,2,3,4\}$. Then, given $a_1, a_2 \in \mathbb{A}_d$,

$$\mu_k(a_1^* a_2) = \langle I + N_{|\vec{\mu}|}, \pi_{|\vec{\mu}|}(a_1)^* D_k \pi_{|\vec{\mu}|}(a_2)(I + N_{|\vec{\mu}|})\rangle; \quad D_k := E_k^* E_k.$$

It then follows that

$$\mu(a_1^* a_2) = \langle I + N_{|\vec{\mu}|}, \pi_{|\vec{\mu}|}(a_1)^* T_{\vec{\mu}} \pi_{|\vec{\mu}|}(a_2)(I + N_{|\vec{\mu}|})\rangle, \tag{3.1}$$

where

$$T_{\vec{\mu}} := (D_1 - D_2) + \mathrm{i}(D_3 - D_4)$$

is a $|\vec{\mu}|$-Toeplitz operator, i.e.

$$\pi_{|\vec{\mu}|}(L_k^*) T_{\vec{\mu}} \pi_{|\vec{\mu}|}(L_j) = \delta_{k,j} T_{\vec{\mu}}.$$

4. ANALYTIC NC MEASURES

Set, $\mathbb{A}_d^{(0)} = L_1\mathbb{A}_d + L_2\mathbb{A}_d + \cdots + L_d\mathbb{A}_d$.

DEFINITION 4.1. We say that an NC measure, $\mu \in \mathscr{A}_d^\dagger$, is *analytic* if it annihilates $\mathbb{A}_d^{(0)}$.

The following is an analogue of [13, Corollary 2, Chapter 4]. Compare also with the structure of [24, Theorem 4.1], which applies to the quotient of the weak* closure of $\mathbb{A}_d$ with respect to a weak*-closed two-sided ideal.

THEOREM 4.2. *If $\mu \in \mathscr{A}_d^\dagger$ is analytic, then each of μ_{ac}, μ_{s}, μ_{dil} and μ_{vN} are also analytic.*

Proof. Let $\vec{\mu} = (\mu_1, \mu_2, \mu_3, \mu_4) \in \mathscr{W}(\mu)$. As in Section 3.2, if $\lambda := |\vec{\mu}|$, then

$$\mu(L^\alpha) = \langle I + N_\lambda, T_{\vec{\mu}} \Pi_\lambda^\alpha (I + N_\lambda)\rangle_\lambda, \quad \alpha \in \mathbb{F}^d,$$

where

$$T_{\vec{\mu}} = (D_1 - D_2) + \mathrm{i}(D_3 - D_4); \quad D_k = E_k^* E_k$$

and each $E_k : \mathbb{H}_d^2(\lambda) \to \mathbb{H}_d^2(\mu_k)$ is the contractive co-embedding arising from $\mu_k \leqslant |\vec{\mu}|$. By Proposition 2.5, Theorem 2.15 and Corollary 2.6, it further follows that if $\mathfrak{t} \in \{\mathrm{ac}, \mathrm{s}, \mathrm{dil}, \mathrm{vN}\}$, then

$$\mu_{\mathfrak{t}}(L^\alpha) = \langle I + N_\lambda, T_{\vec{\mu};\mathfrak{t}} \Pi_\lambda^\alpha (I + N_\lambda)\rangle, \quad \alpha \in \mathbb{F}^d,$$

where $T_{\vec{\mu};\mathfrak{t}} = T_{\vec{\mu}} P_{\lambda;\mathfrak{t}} = P_{\lambda;\mathfrak{t}} T_{\vec{\mu}}$. In particular, since $\mu|_{\mathbb{A}_d^{(0)}} \equiv 0$, we can find a sequence of NC polynomials $(a_n)_n$ in $\mathbb{A}_d$ such that $a_n + N_\lambda \to P_{\lambda;\mathfrak{t}}(I + N_\lambda)$. Then, for any $\alpha \neq \varnothing$,

$$\begin{aligned} 0 = \lim_{n\to\infty} \mu(L^\alpha a_n) &= \langle I + N_\lambda, T_{\vec{\mu}} \Pi_\lambda^\alpha P_{\lambda;\mathfrak{t}}(I + N_\lambda)\rangle_\lambda \\ &= \langle I + N_\lambda, T_{\vec{\mu}} P_{\lambda;\mathfrak{t}} \Pi_\lambda^\alpha (I + N_\lambda)\rangle_\lambda = \mu_{\mathfrak{t}}(L^\alpha). \end{aligned}$$

This proves that μ_{t} also annihilates $\mathbb{A}_d^{(0)}$ for any $\mathrm{t} \in \{\mathrm{ac}, \mathrm{s}, \mathrm{dil}, \mathrm{vN}\}$. ∎

If $\mu \in \mathscr{A}_d^\dagger$ is of *Cuntz-type*, meaning that it has a Wittstock decomposition $(\mu_k)_{k=1}^4$ whose corresponding total variation $\mu_1 + \mu_2 + \mu_3 + \mu_4$ is Cuntz-type, then each μ_k is of Cuntz-type by Proposition 2.17.

LEMMA 4.3. *If $\mu \in \mathscr{A}_d^\dagger$ is Cuntz-type and analytic, then μ also annihilates $\mathbb{A}_d$.*

Proof. Let $\vec{\mu} \in \mathscr{W}(\mu)$ and set $\lambda := |\vec{\mu}|$. By hypothesis, λ is Cuntz. Since Π_λ is Cuntz, $I + N_\lambda$ is the limit of a sequence of equivalence classes of $a_n \in \mathbb{A}_d^{(0)}$, as follows from [15, Theorem 6.4].

With $T_{\vec{\mu}}$ as in Section 3.2, we have

$$\mu(I) = \lim_{n\to\infty} \langle I + N_\lambda, T_{\vec{\mu}}(a_n + N_\lambda)\rangle = \lim_{n\to\infty} \mu(a_n) = 0. \quad ∎$$

REMARK 4.4. At this point, the proof of the classical F&M Riesz theorem, as presented in [13, Chapter 4], is straightforward. Given any complex measure μ obeying the above assumptions we have that

$$\int_{\partial\mathbb{D}} a(\zeta)\mu_{\mathrm{s}}(\mathrm{d}\zeta) = 0,$$

for any a in the disk algebra $\mathcal{A}(\mathbb{D})$. We then consider the complex measure $\mu_{\mathrm{s}}^{(\overline{\zeta})}(\mathrm{d}\zeta) := \overline{\zeta}\mu_{\mathrm{s}}(\mathrm{d}\zeta)$. This is again a complex singular measure that annihilates $\mathcal{A}(\mathbb{D})^{(0)} = \{f \in \mathcal{A}(\mathbb{D}) : f(0) = 0\}$. By the above lemma μ also annihilates $\mathcal{A}(\mathbb{D})$. In particular it annihilates 1, so that by construction

$$\int_{\partial\mathbb{D}} \zeta^k \mu_{\mathrm{s}}(\mathrm{d}\zeta) = 0; \quad k \in \{-1, 0, 1, 2, \dots\}.$$

Proceeding inductively, we conclude that all moments of μ_{s} vanish so that $\mu_{\mathrm{s}} \equiv 0$. In the NC setting, this argument breaks down for singular NC measures of dilation-type (see Section 5), and thus we follow a different approach.

Given $a \in \mathbb{A}_d$, note that $x \mapsto a^*xa$ is norm continuous and positive on the algebra of bounded operators, and if $x \in \mathscr{A}_d$, then $a^*xa \in \mathscr{A}_d$ by the semi-Dirichlet property. Thus, $x \mapsto a^*xa$ is a continuous positive endomorphism of $\mathscr{A}_d$.

LEMMA 4.5. *Let $a_0 \in \mathbb{A}_d$ and $\lambda \in (\mathscr{A}_d)_+^\dagger$. Define $\mu \in (\mathscr{A}_d)_+^\dagger$ by setting $\mu(b) = \lambda(a_0^* b a_0)$ for $b \in \mathscr{A}_d$. If λ is absolutely continuous, then so is μ. If λ is von Neumann-type, then so is μ.*

Proof. It is an elementary exercise to produce an isometry $V : \mathbb{H}_d^2(\mu) \to \mathbb{H}_d^2(\lambda)$ satisfying

$$V(a + N_\mu) = aa_0 + N_\lambda, \quad a \in \mathbb{A}_d.$$

We see immediately that $\operatorname{Ran} V$ is a closed Π_λ-invariant subspace of $\mathbb{H}_d^2(\lambda)$ and that $\Pi_\lambda|_{\operatorname{Ran} V}$ is unitarily equivalent to Π_μ.

Suppose that λ is absolutely continuous. Then every element of $\mathbb{H}^2_d(\lambda)$ is a weak* continuous vector. In particular, $V(I+N_\mu)$ is weak* continuous. For any $b \in \mathscr{A}_d$, we note that $V^*\pi_\lambda(b)V = \pi_\mu(b)$, and thus

$$\mu(b) = \langle I+N_\mu, \pi_\mu(b)(I+N_\mu)\rangle = \langle V(I+N_\mu), \pi_\lambda(b)V(I+N_\mu)\rangle.$$

Therefore, μ is absolutely continuous.

Now suppose instead that λ is of von Neumann-type. Let $\mathfrak{W}_\mu$ and $\mathfrak{W}_\lambda$ denote the weak* closures of $\pi_\mu(\mathbb{A}_d)$ and $\pi_\lambda(\mathbb{A}_d)$, respectively. Clearly, $\operatorname{Ran} V$ is $\mathfrak{W}_\lambda$ invariant. As Π_λ is of von Neumann-type, it follows that $\mathfrak{W}_\lambda$ is a von Neumann algebra, and thus $\operatorname{Ran} V$ is Π_λ-reducing. In particular, $\mathfrak{W}_\lambda|_{\operatorname{Ran} V}$ is a von Neumann algebra. As $\mathfrak{W}_\lambda|_{\operatorname{Ran} V}$ is unitarily equivalent to $\mathfrak{W}_\mu$, we see that Π_μ is of von Neumann-type. Thus, μ is of von Neumann-type. ∎

Note that $X \mapsto L_j^*X$ is a contractive linear map on $\mathscr{L}(\mathbb{H}^2_d)$. As $L_j^*(\mathbb{A}_d + \mathbb{A}_d^*) \subset \mathbb{A}_d + \mathbb{A}_d^*$, it follows by continuity that $b \mapsto L_j^*b$ is a contractive linear endomorphism of $\mathscr{A}_d$.

DEFINITION 4.6. Let $\lambda \in \mathscr{A}_d^\dagger$ and $k = 1, 2, \ldots, d$. Define $\lambda^{(k)} \in \mathscr{A}_d$ by

$$\lambda^{(k)}(b) = \lambda(L_k^*b), \quad b \in \mathscr{A}_d.$$

REMARK 4.7. If $\lambda \in (\mathscr{A}_d)_+^\dagger$ is of dilation-type, it can happen that $\lambda^{(k)}$ is not of dilation-type and hence the classical proof as described in Remark 4.4 breaks down. The next section, Section 5 provides an example of a dilation-type NC measure, $\xi \in (\mathscr{A}_d)_+^\dagger$, so that $\Xi^{[2]} := \xi \circ \mathrm{Ad}_{L_2^*, L_2} \in (\mathscr{A}_d)_+^\dagger$ is weak* continuous and such that $\xi^{(2)} = \xi(L_2^*(\cdot)) \in \mathscr{A}_d^\dagger$ is analytic but not weak* continuous, see Proposition 5.4.

PROPOSITION 4.8. *Let $\mu \in \mathscr{A}_d^\dagger$ and $k \in \{1, 2, \ldots, d\}$. If μ is absolutely continuous, then $\mu^{(k)}$ is absolutely continuous. If μ is of von Neumann-type, then $\mu^{(k)}$ is of von Neumann-type.*

Proof. Let $\mathrm{t} \in \{\mathrm{ac}, \mathrm{vN}\}$, and let μ be of type t. First assume that μ is positive. For each $b \in \mathscr{A}_d$, set

$$\phi_1(b) = \frac{1}{2}\mu((I+L_k)^*b(I+L_k)), \quad \phi_2(b) = \frac{1}{2}\mu((I-L_k)^*b(I-L_k)),$$
$$\phi_3(b) = \frac{1}{2}\mu((I+\mathrm{i}L_k)^*b(I+\mathrm{i}L_k)), \quad \text{and} \quad \phi_4(b) = \frac{1}{2}\mu((I-\mathrm{i}L_k)^*b(I-\mathrm{i}L_k)).$$

Then,

$$\phi_1(b) - \phi_2(b) + \mathrm{i}(\phi_3(b) - \phi_4(b)) = \frac{1}{2}\mu(L_k^*b + bL_k) + \frac{\mathrm{i}}{2}\mu(-\mathrm{i}L_k^*b + \mathrm{i}bL_k) = \mu^{(k)}(b),$$

for all $b \in \mathscr{A}_d$ and so $(\phi_1, \phi_2, \phi_3, \phi_4) \in \mathscr{W}(\mu^{(k)})$. Because μ is type t, it follows from Lemma 4.5 that each ϕ_j is type t, and thus $\mu^{(k)}$ is type t.

Now consider the general case of $\mu \in \mathscr{A}_d^\dagger$. Since μ is type t, there exists a $(\mu_1, \mu_2, \mu_3, \mu_4) \in \mathscr{W}(\mu)$ such that each μ_j is of type t. It follows that $\mu_j^{(k)}$ is type t for each j, k. Let $(\phi_{j,\ell})_{\ell=1}^4$ be a Wittstock decomposition of μ_j where each $\phi_{j,\ell}$ is type t. Then

$$\begin{aligned}\mu^{(k)} &= \mu_1^{(k)} - \mu_2^{(k)} + \mathrm{i}(\mu_3^{(k)} - \mu_4^{(k)}) \\ &= \phi_{1,1} - \phi_{1,2} + \mathrm{i}(\phi_{1,3} - \phi_{1,4}) - (\phi_{2,1} - \phi_{2,2} + \mathrm{i}(\phi_{2,3} - \phi_{2,4})) \\ &\qquad + \mathrm{i}(\phi_{3,1} - \phi_{3,2} + \mathrm{i}(\phi_{3,3} - \phi_{3,4})) - \mathrm{i}(\phi_{4,1} - \phi_{4,2} + \mathrm{i}(\phi_{4,3} - \phi_{4,4})) \\ &= (\phi_{1,1} + \phi_{2,2} + \phi_{3,4} + \phi_{4,3}) - (\phi_{1,2} + \phi_{2,1} + \phi_{3,3} + \phi_{4,4}) \\ &\qquad + \mathrm{i}(\phi_{1,3} + \phi_{2,4} + \phi_{3,1} + \phi_{4,2}) - \mathrm{i}(\phi_{4,1} + \phi_{3,2} + \phi_{2,3} + \phi_{1,4}).\end{aligned}$$

As each $\phi_{j,\ell}$ is of type t, it follows that $\mu^{(k)}$ has a Wittstock decomposition $\vec{\psi}$ such that $|\vec{\psi}|$ is of type t, and therefore $\mu^{(k)}$ is of type t. ∎

The main result of this section is the following analogue of the F&M Riesz analytic measure theorem. It follows from this that if $d \geqslant 2$, an analytic NC measure will be AC if and only if it has no dilation part. We demonstrate in Proposition 5.4 that an analytic linear functional on $\mathscr{A}_d$ need not extend weak* continuously to $\mathscr{T}_d$, in contrast to the classical result.

THEOREM 4.9 (NC F&M Riesz Theorem). *Every analytic* NC *measure,* $\mu \in \mathscr{A}_d^\dagger$*, for* $d \geqslant 2$*, has vanishing von Neumann part.*

Proof. By Theorem 4.2 and Lemma 4.3, if $\mu \in \mathscr{A}_d^\dagger$ annihilates $\mathbb{A}_d^{(0)}$, then μ_{vN} annihilates $\mathbb{A}_d$. By Proposition 4.8, for any $k \in \{1, 2, \ldots, d\}$, we see that $\mu_{\mathrm{vN}}^{(k)}$ is of von Neumann-type. Since μ_{vN} annihilates $\mathbb{A}_d$, we have that $\mu_{\mathrm{vN}}^{(k)}$ annihilates $\mathbb{A}_d^{(0)}$. By Lemma 4.3 again, $\mu_{\mathrm{vN}}^{(k)}$ annihilates $\mathbb{A}_d$ so that

$$0 = \mu_{\mathrm{vN}}^{(k)}(I) = \mu_{\mathrm{vN}}(L_k^*).$$

Proceeding inductively we obtain that

$$\mu_{\mathrm{vN}}(L^{\alpha *}) = 0,$$

for any $\alpha \in \mathbb{F}^d$ and we conclude that $\mu_{\mathrm{vN}} \equiv 0$. ∎

REMARK 4.10. An NC F&M Riesz theorem was previously obtained in [3, Theorem A], by R. Clouâtre and the second two named authors of the present paper, using different techniques. Both Theorem 4.9 and [3, Theorem A] disallow the presence of von Neumann-type summands in their corresponding models of analytic linear functionals on $\mathscr{A}_d$, though what this means in these two cases is different. Both papers find that analytic linear functionals on $\mathscr{A}_d$ need not have weak* continuous extensions to the weak* closure, $\mathscr{T}_d$, of $\mathscr{A}_d$, as we detail in the next section.

We also remark that the results of [3] and the current paper describe the obstruction to weak* continuous extension in different ways.

To compare the two sets of results, fix $\lambda \in \mathscr{A}_d^\dagger$ and suppose $\lambda(\mathbb{A}_d) = \{0\}$, as this is the definition of analyticity used in [3]. There, the free disk system is viewed as embedded, completely isometrically, inside the Cuntz algebra, $\mathscr{O}_d$, via the quotient map $q : \mathscr{E}_d \to \mathscr{O}_d$ whose kernel is the compact operators. Let Λ be an extension of λ to $\mathscr{O}_d$. In [3, Theorem A], it is proved that there exists a $*$-representation $\pi : \mathscr{O}_d \to \mathscr{L}(\mathcal{H})$, a $\pi(\mathscr{O}_d)$-cyclic vector $h_1 \in \mathcal{H}$ and a vector $h_2 \in \mathcal{H}$ such that

$$\Lambda(x) = \langle h_2, \pi(x)h_1 \rangle, \quad x \in \mathscr{O}_d,$$

$\|\Lambda\| = \|h_1\|^2 = \|h_2\|^2$, and the restriction of $\pi(L) := (\pi(L_1), \ldots, \pi(L_d))$ to the norm closure of $\pi(\mathbb{A}_d)h_1$ is unitarily equivalent to L. The triple (π, h_1, h_2) is referred to as a "Riesz representation" of the functional Λ. The particular form of the representation implies that $\pi(L)$ has no von Neumann-type summand, as is noted in [3, Section 4]. We note that

$$\lambda(b) = \langle h_2, \pi(q(b))h_1 \rangle, \quad b \in \mathscr{A}_d.$$

The representation π and the vector h_1 are such that $|\widehat{\Lambda}|(f) = \langle h_1, \widehat{\pi}(f)h_1 \rangle$, where $\widehat{\Lambda}$ and $\widehat{\pi}$ denote their weak* continuous extensions to the second dual of $\mathscr{O}_d$ and $|\widehat{\Lambda}|$ is the "radial" part of the polar decomposition a normal linear functional. In the present paper, we find that there exists a Wittstock decomposition $\vec{\lambda}$ of λ and a $\Pi_{|\vec{\lambda}|}$-Toeplitz operator T such that

$$\lambda(a_2^* a_1) = \langle I + N_{|\vec{\lambda}|}, \pi_{|\vec{\lambda}|}(a_2)^* T \pi_{|\vec{\lambda}|}(a_1)(I + N_{|\vec{\lambda}|}) \rangle, \quad a_1, a_2 \in \mathbb{A}_d,$$

with $\Pi_{|\vec{\lambda}|} = (\pi_{|\vec{\lambda}|}(L_1), \ldots, \pi_{|\vec{\lambda}|}(L_d))$ having no von Neumann-type summand. One will recall that we assume in Theorem 4.9 that $\lambda \in \mathscr{A}_d^\dagger$ annihilates $\mathbb{A}_d^{(0)}$, not $\mathbb{A}_d$. However, this is equivalent to the analyticity assumptions of [3, Theorem A], since a vector functional applied to a $*$-representation of $\mathscr{O}_d$ annihilates $\mathbb{A}_d^{(0)}$ if and only if it annihilates $\mathbb{A}_d$; see Lemma 4.3.

As another point of contrast, our total variation $|\vec{\lambda}| = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$ is not uniquely determined by λ. From the polar decomposition $\widehat{\Lambda}(x) = |\widehat{\Lambda}|(v^* x)$, we may readily produce a Wittstock decomposition $\vec{v}$ of λ, but it is not generally of the types that have proven useful in this paper, nor of course is $|\vec{v}|$ equal to $|\widehat{\Lambda}|$. We also remark that, unless the GNS row isometry of $|\vec{\lambda}|$ is Cuntz, $|\vec{\lambda}|$ need not have a unique positive Kreĭn–Arveson extension to $\mathscr{E}_d$; see Proposition 4.11 below.

For yet another point of contrast, it follows from [3, Corollary 5.2] that λ can be analytic and admit a weak* continuous extension to $\mathscr{A}_d$, with $\pi(L)$ having a dilation-type summand. In the representation of this paper, λ admits a weak* continuous extension to $\mathscr{T}_d$ precisely when $\Pi_{|\vec{\lambda}|}$ is absolutely continuous (for some $\vec{\lambda} \in \mathscr{W}(\lambda)$). Despite these differences, we can collect some necessary and sufficient conditions for the absolute continuity of an analytic λ. By combining Proposition 3.6, Lemma 3.2 and [3, Theorem B], we see that the following conditions are equivalent, where we suppose $\lambda \in \mathscr{A}_d^\dagger$ satisfies $\lambda(\mathbb{A}_d^{(0)}) = \{0\}$:

(i) λ extends weak* continuously to $\mathscr{T}_d$;

(ii) there exists a $\vec{\lambda} \in \mathscr{W}(\lambda)$ such that $|\vec{\lambda}|$ is absolutely continuous;

(iii) there exist $f, g \in \mathbb{H}_d^2$ such that $\lambda(b) = \langle f, bg \rangle$, $b \in \mathscr{A}_d$;

(iv) the weak* continuous extension of λ to the second dual of $\mathscr{A}_d$ annihilates $\{\widehat{a}^*\mathfrak{q} - \mathfrak{q}\widehat{a}^* : a \in \mathbb{A}_d\}$, where $\mathfrak{q}$ is the free semi-group structure projection of the second dual of $\mathbb{A}_d$.

It should be noted that the equivalences (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) do not require analyticity of λ. For (iv), we remark that λ extends weak* continuously to $\mathscr{T}_d$ if and only if $b \mapsto \lambda(b) - \lambda(I)m(b)$ does, and so the different conditions for analyticity from these two papers do not present any difficulties here.

PROPOSITION 4.11. *If $\lambda \in (\mathscr{A}_d)_+^\dagger$ is Cuntz-type, then λ has a unique positive Kreĭn–Arveson extension Λ to the Cuntz–Toeplitz algebra, $\mathscr{E}_d$.*

Proof. Let $\Lambda : \mathscr{E}_d \to \mathbb{C}$ be any Kreĭn–Arveson (positive) extension of λ. Apply the GNS construction to $(\Lambda, \mathscr{E}_d)$ to obtain a GNS Hilbert space $L^2(\Lambda)$ and a $*$-representation π_Λ satisfying

$$\Lambda(x) = \langle I + N_\Lambda, \pi_\Lambda(x)(I + N_\Lambda) \rangle; \quad x \in \mathscr{E}_d.$$

By construction, $I + N_\Lambda$ is cyclic for the GNS row isometry Π_Λ.

For any $a \in \mathbb{A}_d$,

$$\|a + N_\lambda\|^2 = \lambda(a^*a) = \Lambda(a^*a) = \|a + N_\Lambda\|^2.$$

Thus, there is an isometry $V : \mathbb{H}_d^2(\lambda) \to L^2(\Lambda)$ determined by $V(a + N_\lambda) = a + N_\Lambda$, $a \in \mathbb{A}_d$. Plainly, $V\Pi_{\lambda;k} = \Pi_{\Lambda;k}V$ for each k.

Next, we claim that $V\mathbb{H}_d^2(\lambda)$ is Π_Λ-reducing and $\Pi_\Lambda|_{V\mathbb{H}_d^2(\lambda)}$ is unitarily equivalent to Π_λ. Indeed, Π_λ is Cuntz, and so any given element $x \in \mathbb{H}_d^2(\lambda)$ is the norm-limit of vectors $x_n + N_\lambda$, where $x_n \in \mathbb{A}_d^{(0)}$, as shown in [15, Theorem 6.4]. Hence, for any $z \in \mathscr{E}_d$ and $k = 1, 2, \ldots, d$,

$$\begin{aligned}
\langle z + N_\Lambda, \Pi_{\Lambda;k}^* V x \rangle &= \lim_{n\to\infty} \langle z + N_\Lambda, \Pi_{\Lambda;k}^* V(x_n + N_\lambda) \rangle = \lim_{n\to\infty} \langle L_k z + N_\Lambda, x_n + N_\Lambda \rangle \\
&= \lim_{n\to\infty} \langle z + N_\Lambda, (L_k^* x_n) + N_\Lambda \rangle \\
&= \lim_{n\to\infty} \langle z + N_\Lambda, V\Pi_{\lambda;k}^*(x_n + N_\lambda) \rangle = \langle z + N_\Lambda, V\Pi_{\lambda;k}^* x \rangle.
\end{aligned}$$

That is, $\Pi_{\Lambda;k}^* V = V\Pi_{\lambda;k}^*$ for each k, whence $V\pi_\lambda(y) = \pi_\Lambda(y)V$ for each $y \in \mathscr{E}_d$.

Finally, note that $I + N_\Lambda = V(I + N_\lambda)$. Since $\pi_\Lambda(\mathscr{E}_d)(I + N_\Lambda)$ is dense in $L^2(\Lambda)$ and $\pi_\lambda(\mathscr{E}_d)(I + N_\lambda)$ is dense in $\mathbb{H}_d^2(\lambda)$, it follows that $V\mathbb{H}_d^2(\lambda) = L^2(\Lambda)$, showing that V is in fact a surjective isometry. Therefore, for any $z \in \mathscr{E}_d$,

$$\Lambda(z) = \langle I + N_\lambda, \pi_\lambda(z)(I + N_\lambda) \rangle_{\mathbb{H}_d^2(\lambda)}. \quad \blacksquare$$

5. A DILATION-TYPE EXAMPLE

Recall that there is, in essence, a bijection between positive finite regular Borel measures on the circle and the set of Herglotz functions in the disk, i.e. analytic functions in the complex unit disk with positive semi-definite real part. This correspondence extends to positive NC measures and non-commutative (left) Herglotz functions in $\mathbb{B}^d_{\mathbb{N}}$, $\mu \leftrightarrow H_\mu$; see [14, 15]. A fractional linear transformation, the so-called *Cayley transform*, then implements a bijection between the left NC Schur class of contractive NC functions in $\mathbb{B}^d_{\mathbb{N}}$ and the left NC Herglotz class. If $\mu \in (\mathscr{A}_d)^\dagger_+$ is the (essentially) unique NC measure corresponding to the contractive NC function $b \in [\mathbb{H}^\infty_d]_1$, we write $\mu = \mu_b$, and μ_b is called the NC Clark measure of b; see [16, Section 3] for details.

By [16, Corollary 3], if $b \in [\mathbb{H}^\infty_d]_1$ is *inner*, i.e. an isometric left multiplier, then its NC Clark measure is singular, so that its GNS representation $\Pi_b := \Pi_{\mu_b}$ is a Cuntz row isometry which can be decomposed as the direct sum of a dilation-type row isometry and a von Neumann-type row isometry [17].

Classically, any sum of Dirac point masses is singular with respect to Lebesgue measure on the circle. Motivated by this, consider the positive linear functional $\xi \in (\mathscr{A}_2^\dagger)_+$ defined by

$$\xi(L^\alpha) = \begin{cases} 0 & 2 \in \alpha, \\ 1 & 2 \notin \alpha, \end{cases} \quad \alpha \in \mathbb{F}^2,$$

and $\xi(I) = 1$. Here, $2 \notin \alpha$ is used to indicate that α does not contain the "letter" 2. One may think of ξ as a "Dirac point mass" at the point $(1, 0) \in \partial\mathbb{B}^2_1$, where $\mathbb{B}^2_1$ is the first level of the NC unit ball $\mathbb{B}^2_{\mathbb{N}}$. Setting $Z := (1, 0)$, we note that $\xi(L^\alpha) = Z^\alpha$ for all words α. Since Z is a row contraction, it follows from results of Popescu that the map ξ extends to a positive linear functional on $\mathscr{A}_2$ [28, Theorem 2.1].

Before continuing, we remark that the example of this section is related to [3, Example 2], which is itself related to atomic representations of [11]. This example is also a special case of [10, Example 5.1]. However, we here choose to start from the linear functional, ξ, rather than the functional's representation as a vector functional on a representation, to emphasize the NC function theory associated with the NC measure.

CLAIM 5.1. *$L_2 + N_\xi$ is a wandering vector for Π_ξ and Π_ξ has vanishing von Neumann part.*

Proof. Note that any wandering vector for Π_ξ is always a weak* continuous vector. Indeed, if w is wandering for Π_ξ, then

$$\xi_w(L^\alpha) = \langle w, \Pi^\alpha_\xi w\rangle_{\mathbb{H}^2_2(\xi)} = \|w\|^2_{\mathbb{H}^2_2(\xi)} \delta_{\alpha,\varnothing} = \|w\|^2 m(L^\alpha),$$

is a constant multiple of NC Lebesgue measure, and hence is absolutely continuous.

To see that $L_2 + N_\xi$ is wandering for Π_ξ, note that

$$\langle L_2 + N_\xi, \Pi_\xi^\alpha (L_2 + N_\xi)\rangle = \xi(L_2^* L^\alpha L_2) = \delta_{\alpha,\varnothing}.$$

However $L_2 + N_\xi$ is also $*$-cyclic for Π_ξ since $\pi_\xi(L_2)^*(L_2 + N_\xi) = I + N_\xi$, which is cyclic for Π_ξ. This means that the smallest reducing subspace that contains the weak* continuous vector $L_2 + N_\xi$ is all of $\mathbb{H}^2_2(\xi)$, so that $\mathbb{H}^2_2(\xi_{\mathrm{vN}}) = \{0\}$ by Lemma 2.8. ∎

In what follows, if $\omega = i_1 \cdots i_n \in \mathbb{F}^d$, $i_k \in \{1, \dots, d\}$, is any word, then we set $\omega^{\mathrm{t}} := i_n \cdots i_1$. This letter reversal map is an involution on the free monoid.

CLAIM 5.2. *The* NC *measure ξ is the* NC *Clark measure of $b_\xi(Z) = Z_1$, $Z \in \mathbb{B}^2_{\mathbb{N}}$; a left-inner* NC *function. Thus, Π_ξ is purely of dilation-type.*

Proof. Given $Z \in \mathbb{B}^2_n$, let $Z \otimes L^* := Z_1 \otimes L_1^* + \cdots + Z_d \otimes L_d^*$. The (left) Herglotz function, H_ξ of ξ is

$$\begin{aligned} H_\xi(Z) &= (\mathrm{id}_n \otimes \xi)((I_n \otimes I_{\mathbb{H}^2_d} + Z \otimes L^*)(I_n \otimes I_{\mathbb{H}^2_d} - Z \otimes L^*)^{-1}) \\ &= 2\sum_\alpha Z^\alpha \xi(L^{\alpha^{\mathrm{t}}})^* - I_n = 2\sum_{k=0}^{\infty} Z_1^k - I_n = 2(I-Z_1)^{-1} - I_n = (I+Z_1)(I-Z_1)^{-1}. \end{aligned}$$

It follows that the Cayley transform,

$$b_\xi(Z) := (H_\xi(Z) - I)(H_\xi(Z) + I)^{-1},$$

of H_ξ is $b_\xi(Z) = Z_1$, which is inner. By [16, Corollary 3], Π_ξ is the direct sum of a dilation-type and a von Neumann-type row isometry and the previous claim shows that the von Neumann part vanishes. ∎

CLAIM 5.3. *For any word α such that $2 \in \alpha$, the vector $L^\alpha + N_\xi$ is weak* continuous. In particular, the closed span of $\{L^\alpha + N_\xi : 2 \in \alpha\}$ is contained in* $\mathrm{WC}(\Pi_\xi)$.

The proof below uses the concept of the NC Herglotz space of NC Cauchy transforms with respect to a positive NC measure, see [16, Section 3.8, Lemma 5.2]. The NC Herglotz space, $\mathscr{H}^+(H_\mu)$ of any positive NC measure, $\mu \in (\mathscr{A}_d)^\dagger_+$, is a non-commutative reproducing kernel Hilbert space (NC-RKHS) of NC functions in the NC unit row-ball [14, 15, 16, 17]. The details of this construction will not be relevant or needed for our purposes here. It will suffice to remark that if $\mu \in (\mathscr{A}_d)^\dagger_+$ is a positive NC measure, then there is an onto and isometric linear map, $\mathscr{C}_\mu : \mathbb{H}^2_d(\mu) \to \mathscr{H}^+(H_\mu)$, the *free Cauchy transform*.

Proof of Claim 5.3. Given any $\beta \in \mathbb{F}^2$ so that $2 \in \beta$, the vector $L^\beta + N_\xi$ is a WC vector if and only if

$$\xi_\beta(L^\alpha) := \langle L^\beta + N_\xi, \Pi_\xi^\alpha\, (L^\beta + N_\xi)\rangle_\xi = \xi(L^{\beta *} L^\alpha L^\beta),$$

is an absolutely continuous (weak* continuous) NC measure. The free Cauchy transform of $I + N_{\xi_\beta} \in \mathbb{H}^2_2(\xi_\beta)$, is then,

$$h(Z) := \sum_{\alpha \in \mathbb{F}^2} Z^\alpha \xi(L^{\beta *} L^{\alpha *} L^\beta).$$

The Taylor coefficients $(h_\alpha)_\alpha$ of h vanish if α, β are not comparable. Thus, $h_\alpha \neq 0$ if and only if $\alpha = \beta\gamma$ or if $\beta = \alpha\gamma$. Since $\beta \in \mathbb{F}^2$ is fixed, there are only finitely many words α such that $\beta = \alpha\gamma$. On the other hand if $\alpha = \beta\gamma$, then

$$h_\alpha = h_{\beta\gamma} = \xi(L^{\beta *} L^{\gamma *}) = 0,$$

since $2 \in \beta$. This proves that h has at most finitely many non-zero Taylor coefficients and so $h \in \mathbb{C}\{\mathfrak{z}_1, \mathfrak{z}_2\} \subseteq \mathbb{H}^2_2$ and $I + N_{\xi_\beta}$ is a weak* continuous vector for Π_{ξ_β}. Indeed, since $\mathscr{C}_{\xi_\beta}(I + N_{\xi_\beta}) \in \mathbb{H}^2_2$, $I + N_{\xi_\beta}$ is a *weak* analytic vector* for ξ_β in the sense of [17, Definition 8.2] and is hence a weak* continuous vector by [17, Corollary 8.3].

Since this vector is cyclic and $\mathrm{WC}(\xi_\beta)$ must be Π_{ξ_β}-invariant, we see that $\mathrm{WC}(\Pi_{\xi_\beta}) = \mathbb{H}^2_2(\xi_\beta)$. Therefore, ξ_β is weak* continuous and $L^\beta + N_\xi$ is a weak* continuous vector for Π_ξ. ∎

PROPOSITION 5.4. *The positive* NC *measure defined by* $\Xi(L^\alpha) := \xi(L_2^* L^\alpha L_2)$ *is equal to* NC *Lebesgue measure, m, and hence is weak* continuous. The* NC *measure* $\gamma := \xi^{(2)*} \in \mathscr{A}_2^\dagger$*, i.e.* $c \mapsto \xi(cL_2)$*, annihilates the* NC *disk algebra* $\mathbb{A}_2$*, but is not weak* continuous.*

Proof. The vector $L_2 + N_\xi$ is a unit wandering vector for the dilation-type positive NC measure ξ. Hence, $\Xi(L^\alpha) = \delta_{\alpha,\varnothing} = m(L^\alpha)$. Consider the sequence $(L_1^k)^* L_2^* \in \mathscr{A}_2$. This converges weak* to 0, and yet,

$$\gamma(L_1^{k*} L_2^*) = \overline{\xi(L_1^k)} = 1,$$

which of course cannot converge to 0. ∎

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