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A Note on the Optimality of Balanced Truncation for a Class of Infinite Dimensional Systems

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Abstract— **Balanced truncation is a widespread model reduction method that uses balanced realizations for finite dimensional systems. The latter are state space realizations where the controllability and observability gramians are equal to the same diagonal positive matrix. In this paper, a generalization of balanced realization for a class of infinite dimensional LTI systems is employed to perform balanced truncation. It is shown that balanced truncation is optimal in the Hilbert-Schmidt sense if a particular time-varying balanced realization is used for the original LTI system. It appears that the use of time-varying balanced realizations to study the optimality and perform balanced model reduction for LTI systems is novel.**

I. Introduction

Balanced truncation is a simple yet widespread model reduction technique initiated in [2]. It has been studied extensively in many papers and books, for e.g., [3], [1], [4] and references therein. Balanced truncation has been extended to linear time varying systems in [6], nonlinear systems in [5], [8], [7], [9], and infinite dimensional systems in [13], [14], [10] to name a few.

Particular balanced realizations and their corresponding balanced truncations have been defined and studied for certain classes of infinite dimensional LTI systems in [13], [14], [10]. In this paper, for a certain class of infinite dimensional stable LTI systems defined in [13], [14], with impulse responses belonging to specific functional spaces. In particular, these systems correspond to Hilbert Schmidt Hankel operators. In this paper, *a particular time varying balanced realization based on the Schmidt pairs of the system Hankel operators is defined, and used to show that balanced truncation is in fact optimal in the sense of minimizing the Hilbert Schmidt norm of the approximation of the Hankel operator by finite rank linear operators. The results obtained obviously hold for stable finite dimensional LTI systems. The key idea is to define a new balanced linear time-varying realization for the LTI systems to show optimality*. The study of the optimality of balanced truncation is not new. In [2], it was stated that balanced truncation does not appear to be optimal in any sense. In [15] it was shown that, in general, balanced truncation is not optimal in the L^2-norm. In [16], it was shown that balanced approximation can lead to approximations that minimize various error norms, including the L^2-norm for specific finite dimensional LTI stable systems including a finite difference model for a parabolic partial differential equation (PDE). In contrast, it is shown here that balanced truncation is Hilbert Schmidt optimal if a particular time varying balanced realization is adopted. The optimality of balanced truncation was observed in [17], however an incorrect balanced realization was used to show optimality in the Hilbert-Schmidt sense. It appears that the use of time-varying balanced realizations to study the optimality and perform balanced model reduction for LTI systems is new.

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II. Infinite Dimensional Systems

For simplicity, we consider the class of stable linear infinite dimensional systems described by impulse response functions $h(\cdot)$ which act on input signals $u(t) \in \mathbb{R}^k$, and produce output signals $y(t) \in \mathbb{R}^m$, $t \in [0, \infty)$ defined by the following input-output convolution map,

$$\begin{aligned} P \quad &: L^2([0,\infty), \mathbf{R}^k) \longmapsto L^2([0,\infty), \mathbf{R}^m) \\ &u(t) \longmapsto Pu(t) := y(t) = \int_0^t h(t-\tau)u(\tau)d\tau \end{aligned} \tag{1}$$

where $L^2([0,\infty), \mathbf{R}^k)$ is the space of measurable and square integrable $\mathbf{R}^k$-valued functions.
The impulse response $h(\cdot)$ is assumed to belong to the

following spaces [13], [14]

$$\begin{array}{l} h(t) \in L^1 \cap L^2([0,\infty),\mathbb{R}^{k\times m}) \\ t^{\frac{1}{2}} h(t) \in L^2([0,\infty),\mathbb{R}^{k\times m}) \end{array} \tag{2}$$

Under these assumptions the corresponding Hankel operator, denoted Γ, and defined by

$$\begin{array}{rl} \Gamma & : L^2([0,\infty),\mathbf{R}^k) \longmapsto L^2([0,\infty),\mathbf{R}^m) \\ & u(t) \longmapsto \Gamma u(t) := \int_0^\infty h(t+\tau)u(\tau)d\tau \end{array} \tag{3}$$

is guaranteed to be a compact and a Hilbert Schmidt operator on $L^2([0,\infty),\mathbf{R}^k)$. In addition, Γ is a bounded linear operator on the space $\mathcal{C}^1([0,\infty);\mathbf{R}^k)$ of continuously differentiable functions under the norm [11]

$$\|f\|_1 := \|f\|_\infty + \int_0^\infty \left\| \frac{df(\tau)}{d\tau} \right\| d\tau \tag{4}$$

Under these conditions the operator $\Gamma^\star\Gamma$ is a compact and positive operator on $L^2([0,\infty),\mathbf{R}^k)$, where $\Gamma^\star$ is the adjoint operator of Γ [11], [14]. As a consequence $\Gamma^\star\Gamma$ has countable positive eigenvalues that can be ordered as follows: $\sigma_1^2 \geq \sigma_2^2 \geq \cdots \geq \sigma_i^2 \geq \cdots \geq 0$. σ_i's are known as the singular values of the operator Γ.

The corresponding Schmidt pairs, denoted (v_i, w_i) satisfy for $i = 1, 2, \cdots$, [11], [14]

$$\begin{array}{rcl} \Gamma v_i &=& \sigma_i w_i \\ \Gamma^\star w_i &=& \sigma_i v_i \end{array} \tag{5}$$

The Schmidt pairs satisfy the following identities

$$< v_i, v_j >_1 \ := \int_0^\infty v_j^\star(t) v_i(t) dt = \delta_{ij} \tag{6}$$

$$< w_i, w_j >_2 \ := \int_0^\infty w_j^\star(t) w_i(t) dt = \delta_{ij} \tag{7}$$

where $< \cdot,\cdot >_1$ and $< \cdot,\cdot >_2$ are the inner products in $L^2([0,\infty),\mathbf{R}^k)$ and $L^2([0,\infty),\mathbf{R}^m)$, respectively, and δ_{ij} is the Kronecker delta.

Note v_i's are the eigenvectors of the operator $\Gamma^\star\Gamma$ which is compact on $\mathcal{C}^1([0,\infty);\mathbf{R}^k) \cap L^2([0,\infty),\mathbf{R}^k)$, thus $v_i \in \mathcal{C}^1([0,\infty);\mathbf{R}^k) \cap L^2([0,\infty),\mathbf{R}^k)$ [11].
Likewise, w_i's are the eigenvectors of the operator $\Gamma\Gamma^\star$ which is compact on $\mathcal{C}^1([0,\infty);\mathbf{R}^m) \cap L^2([0,\infty),\mathbf{R}^m)$, thus $w_i \in \mathcal{C}^1([0,\infty);\mathbf{R}^m) \cap L^2([0,\infty),\mathbf{R}^m)$ [11], [14].

The polar decomposition of the Hankel operator yields [11], [13], [14]

$$\begin{array}{rcl} \Gamma u(t) &=& \int_0^\infty h(t+\tau)u(\tau)d\tau \\ &=& \sum_{i=1}^\infty \sigma_i w_i(t) \int_0^\infty v_i^\star(\tau)u(\tau)d\tau \end{array} \tag{8}$$

In the next section we propose a new balanced realization based solely on the Schmidt pairs. **The novelty about this realization is that it is a time varying state space realization although it represents an (infinite dimensional) linear time-invariant (LTI) system**. The results obviously apply to finite dimensional LTI systems.

III. A Time-Varying Balanced Realization for LTI Systems

Following [13], [14], a balanced realization for the system described by the impulse response $h(\cdot)$ on a Hilbert state space H is the triplet (C, A, B), where A, B, and C are linear operators

$$C : H \longmapsto \mathbf{R}^k, \ B : \mathbf{R}^m \longmapsto H \tag{9}$$

and the operator A is the infinitesimal generator of a strongly continuous semigroup, $T(t)$, on H such that the following identity holds [13], [14]

$$h(t) = CT(t)B, \ \text{ a.e. } t \tag{10}$$

Define the controllability operator $\mathcal{C}$ and observability operator $\mathcal{O}$ as follows

$$\begin{array}{rl} \mathcal{C}: & L^2([0,\infty),\mathbf{R}^k) \longmapsto H \\ & u(t) \longmapsto \mathcal{C}u := \int_0^\infty T(t)Bu(t)dt \end{array} \tag{11}$$

$$\begin{array}{rl} \mathcal{O}: & H \longmapsto L^2([0,\infty),\mathbf{R}^m) \\ & x(t) \longmapsto (\mathcal{O}x)(t) := CT(t)x(t) \end{array} \tag{12}$$

The operators $\mathcal{C}$ and $\mathcal{O}$ are well defined and bounded linear operators if the operators B and C are bounded and the semigroup $T(t)$ is exponentially stable [13], [14]. In addition the impulse response $h(t)$ defined in (10) satisfies the conditions in (2). Moreover, the Hankel operator Γ is given by the following identity [13], [14]

$$\Gamma = \mathcal{O} \cdot \mathcal{C} \tag{13}$$

The controllability gramian W_c and observability gramian W_o are then defined as follows [13], [14]

$$W_c := \mathcal{C}\mathcal{C}^\star \tag{14}$$

$$W_o := \mathcal{O}^\star\mathcal{O} \tag{15}$$

where $\mathcal{C}^\star$ and $\mathcal{O}^\star$ are the adjoint operators of $\mathcal{C}$ and $\mathcal{O}$, respectively.

A realization $(C, T(t), B)$ is a balanced realization for the Hankel operator Γ if the reachability and observability operators $\mathcal{C}$ and $\mathcal{O}$ are bounded, (13) holds, and the controllability and observability gramians W_c and W_o are equal to the same positive diagonal operator [13], [14].

We propose the following time varying realization for the impulse response $h(\cdot)$ defined on the Hilbert space $H = \ell^2$ of square summable sequences

$$\begin{aligned} T_{bij}(t) &:= 1 \text{ if } i = j \\ &:= 0 \text{ if } i \neq j \end{aligned} \tag{16}$$

$$B_b(t) := (\sqrt{\sigma_1}v_1^\star(t), \sqrt{\sigma_2}v_2^\star(t), \cdots, \sqrt{\sigma_i}v_i^\star(t), \cdots)^T \tag{17}$$

$$C_b(t) := (\sqrt{\sigma_1}w_1(t), \sqrt{\sigma_2}w_2(t), \cdots, \sqrt{\sigma_i}w_i(t), \cdots) \tag{18}$$

where $'T'$ in (17) denotes the transpose. Note $T_b = (T_{bij}) = (\delta_{ij})$ is simply the identity operator I and therefore satisfies all the properties of a semigroup.

From (11) the corresponding reachability operator is then

$$\begin{aligned} \mathcal{C}_b u &:= \int_0^\infty B_b u(t) dt \qquad (19) \\ &= \int_0^\infty \sum_i \sqrt{\sigma_i} v_i^\star(t) u_i(t) dt, \; u(t) = (u_i(t)) \end{aligned}$$

and from (12) the observability operator

$$\mathcal{O}_b x := C_b x \tag{20}$$

It is readily seen from (8) that $\Gamma = \mathcal{O}_b\mathcal{C}_b$, i.e., the Hankel operator Γ is realized by the new reachability and observability operators, and therefore admits $(C_b, T_b(t), B_b)$ as a state space realization.

Now let us show that (C_b, I, B_b) is a balanced realization for the infinite dimensional LTI system with impulse response $h(\cdot)$. The controllability gramian is given by

$$\begin{aligned} W_{cb} &:= \mathcal{C}_b\mathcal{C}_b^\star = \int_0^\infty B_b(t)B_b^\star(t)dt \\ &= \left(\int_0^\infty \sqrt{\sigma_i}\sqrt{\sigma_j} v_i^\star v_j(d) dt\right)_{i,j} \qquad (21) \\ &= \operatorname{diag}(\sigma_1, \sigma_2, \cdots, \sigma_k, \cdots) \qquad (22) \end{aligned}$$

where we used the orthonormality of the Schmidt vectors v_i and $\operatorname{diag}(\cdot)$ denotes the diagonal operator. Similarly,

$$\begin{aligned} W_{ob} &:= \mathcal{O}_b^\star\mathcal{O}_b = \int_0^\infty C_b^\star(t)C_b(t)dt \\ &= \left(\int_0^\infty \sqrt{\sigma_i}\sqrt{\sigma_j} w_i^\star w_j(d) dt\right)_{i,j} \qquad (23) \\ &= \operatorname{diag}(\sigma_1, \sigma_2, \cdots, \sigma_k, \cdots) \qquad (24) \end{aligned}$$

where we used the orthonormality of the Schmidt vectors w_i. Expressions (22) and (24) show that (C_b, I, B_b) is a balanced realization for Γ on ℓ^2, Moreover, it can be shown that since W_{cb} and W_{ob} are positive operators, this realization is approximately controllable and initially observable (see [14] for the definition). Details will be provided in a forthcoming journal paper.

IV. Optimality of the Time-Varying Balanced Truncation

Following balanced truncation in finite dimension where the state vector is truncated and the corresponding state matrices are truncated accordingly [2], [1], [3], we propose the following nth order truncations

$$T_{bn}(t) = \operatorname{diag}(1, 1, \cdots, 1) = I_{n\times n} \tag{25}$$

$$B_{bn}(t) = (\sqrt{\sigma}_1 v_1^\star(t), \sqrt{\sigma}_2 v_2^\star(t), \cdots, \sqrt{\sigma}_n v_n^\star(t))^T \tag{26}$$

$$C_{bn}(t) = (\sqrt{\sigma}_1 w_1(t), \sqrt{\sigma}_2 w_2(t), \cdots, \sqrt{\sigma}_n w_n(t)) \tag{27}$$

The truncated realization (27) corresponds to a finite dimensional LTI system. It is balanced since the corresponding controllability and observability gramians $W_{cbn} = W_{obn} = \operatorname{diag}(\sigma_1, \sigma_2, \cdots, \sigma_n)$ are diagonal. The impulse response $h_n(\cdot)$ of the truncated system (27) is given by

$$h_n(t) = C_{bn}(t)B_{bn}(t) \tag{28}$$

which shows that

$$\begin{aligned} &h_n(t) \in L^1 \cap L^2([0,\infty), \mathbb{R}^{k\times m}) \\ &t^{\frac{1}{2}} h_n(t) \in L^2([0,\infty), \mathbb{R}^{k\times m}) \end{aligned} \tag{29}$$

since $v_i \in \mathcal{C}^1([0,\infty); \mathbf{R}^k) \cap L^2([0,\infty), \mathbf{R}^k)$ and $w_i \in \mathcal{C}^1([0,\infty); \mathbf{R}^m) \cap L^2([0,\infty), \mathbf{R}^m)$. This shows that $h_n(\cdot)$ is stable, thus the balanced truncation (27) is also stable.

The corresponding Hankel operator Γ_n is defined by

$$
\begin{aligned}
(\Gamma_n u)(t) &= \int_0^\infty h_n(t+\tau)u(\tau)d\tau && (30)\\
&= \sum_{i=1}^{n} \sigma_i w_i(t) \int_0^\infty v_i^\star(\tau)u(\tau)d\tau && (31)
\end{aligned}
$$

The Hankel operator is compact, and in fact of finite rank n, and is also a Hilbert-Schmidt operator.

We will show next that the balanced truncation (27) is optimal in the Hilbert Schmidt sense. To see this define the Hilbert-Schmidt norm $\|\cdot\|_{\text{HS}}$ for the Hankel operator Γ as [12], [17]

$$
\begin{aligned}
\|\Gamma\|_{\text{HS}} &= \Big(\sum_i \sigma_i^2\Big)^{\frac{1}{2}}\\
&= \Big(\int_0^\infty \int_0^\infty tr(h(t+\tau)^\star h(t+\tau))d\tau dt\Big)^{\frac{1}{2}}
\end{aligned} \tag{32}
$$

where $tr(\cdot)$ denote the trace. From expression (8) Γ can be written in terms of its spectral factorization [17]

$$\Gamma = \sum_{i=1}^{\infty} \sigma_i v_i \otimes w_i \tag{33}$$

Next, consider the following optimal approximation problem

$$\mu_n := \min_{\text{rank}(\Upsilon)\le n<\infty} \|\Gamma - \Upsilon\|_{\text{HS}} \tag{34}$$

where the minimization is with respect to linear bonded operators Υ of rank at most n. It turns out that the minimizer in (34) exists and is unique since the optimization is posed in a Hilbert space (the space of Hilbert-Schmidt operators). By Hilbert-Schmidt theory [18], the optimal solution is given by the following expression

$$\mu_n := \min_{\text{rank}(\Upsilon)\le n<\infty} \|\Gamma - \Upsilon\|_{\text{HS}} = \|\Gamma - \Gamma_n\|_{\text{HS}} \tag{35}$$

where Γ_n is the Hankel operator defined by (30) and is realized by the impulse response $h_n(\cdot)$, and the balanced truncated system (25–27). Moreover, Γ_n has the spectral factorization [17]

$$\Gamma_n = \sum_{i=1}^{n} \sigma_i v_i \otimes w_i \tag{36}$$

Since Γ_n is realized by the balanced realization $(C_{bn}, I_{n\times n}, B_{bn})$, the latter is optimal in the sense of minimizing the Hilbert-Schmidt norm of the approximation of Γ by linear operators Υ of rank at most n.

The residual error is given by

$$
\begin{aligned}
\mu_n &= \Big\|\sum_{i=1}^{\infty} \sigma_i v_i \otimes w_i - \sum_{i=1}^{n} \sigma_i v_i \otimes w_i\Big\|_{\text{HS}} && (37)\\
&= \Big\|\sum_{i=n+1}^{\infty} \sigma_i v_i \otimes w_i\Big\|_{\text{HS}} = \Big(\sum_{n+1}^{\infty} \sigma_i^2\Big)^{\frac{1}{2}}
\end{aligned}
$$

To summarize, we have shown that if the time-varying balanced realization (16)–(18) is used to represent the impulse response of the infinite dimensional impulse response $h(\cdot)$ and realize the corresponding Hankel operator Γ, then balanced truncation is optimal in the sense of minimizing the Hilbert-Schmidt norm of the difference between the full order (infinite dimensional) Hankel operator Γ and an operator of rank at most n. This result has been alluded to in [17] but an incorrect LTI balanced realization was used. The results apply to stable finite dimensional LTI systems since the corresponding Hankel operators is finite rank and therefore always Hilbert-Schmidt.

V. Conclusion

In this paper, we have shown that for a class of infinite dimensional stable LTI systems that a particular time-varying balanced realization based on the Hankel singular values, and the Schmidt pairs can always be obtained to get a balanced realization although the original system is LTI. This particular balanced realization was used to obtain a reduced order model by a time-varying balanced truncation. The latter was used to show that balanced truncation is optimal in the Hilbert-Schmidt sense if this particular time-varying balanced realization is used. The results obtained in this paper apply to stable finite dimensional LTI systems. To the best of the our knowledge, the use of time-varying balanced realizations for LTI systems to study the optimality and perform model reduction is new.

We are currently working towards weakening some of the assumptions made in this paper, and providing full proofs in a forthcoming extended journal paper.

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