Joint Moment Generating Function of Ages of Information in Networks

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***Abstract*—In this paper, we study a general setting of status updating systems in which a set of source nodes provide status updates about some physical process(es) to a set of monitors. The freshness of information available at each monitor is quantified in terms of the Age of Information (AoI), and the vector of AoI processes at the monitors (or equivalently the *age vector*) models the continuous state of the system. While the *marginal* distributional properties of each AoI process have been studied for a variety of settings using the stochastic hybrid system (SHS) approach, we lack a counterpart of this approach to systematically study their *joint* distributional properties. Developing such a framework is the main contribution of this paper. In particular, we model the discrete state of the system as a finite-state continuous-time Markov chain, and describe the coupled evolution of the continuous and discrete states of the system by a piecewise linear SHS with linear reset maps. Using the notion of tensors, we first derive first-order linear differential equations for the temporal evolution of both the joint moments and the joint moment generating function (MGF) for an arbitrary set of age processes. We then characterize the conditions under which the derived differential equations are asymptotically stable. The generality of our framework is demonstrated by recovering several existing results as special cases. Finally, we apply our framework to derive the stationary joint MGF in a multi-source updating system under the non-preemptive in service queueing discipline.**

***Index Terms*—Age of information, queueing systems, communication networks, stochastic hybrid systems.**

I. Introduction

The ongoing massive deployment of the Internet of Things (IoT) will enable many critical real-time status updating systems that fundamentally rely on the timely delivery of status updates [1]. The authors of [2] introduced the concept of AoI which provides a rigorous way of quantifying the freshness of information at a *destination node* as a result of receiving status updates over time from a *transmitter node*. In particular, for a single-source queueing-theoretic model in which status updates are generated randomly at a transmitter with a single source of information, the AoI at the destination was defined in [2] as the following random process: $x(t) = t - u(t)$, where $u(t)$ is the generation time instant of the latest status update received at the destination by time t.

Following [2], the average value of AoI or peak AoI (a related metric based on the peak values of AoI over time) has been extensively analyzed in single-source systems under several queueing disciplines [3]–[5]. Meanwhile, the characterization of the average AoI in multi-source systems (where the transmitter has multiple sources of information) is quite challenging, and hence the prior work in this direction is relatively sparse [6]–[9]. Further, a handful of recent works have aimed to characterize the stationary distribution (or some distributional properties) of AoI/peak AoI in single-source [10]–[13] or multi-source [14] systems. Note that the analyses of the above works studying multi-source system settings (i.e., there are multiple AoI or age processes in the system) have been limited to the characterization of the marginal distributional properties of each source's AoI process.

The analyses of the above works were mainly based on identifying the properties of AoI sample functions and applying geometric arguments, which often involve tedious calculations of joint moments. Motivated by this, the authors of [15] and [16] have developed a SHS-based framework for characterizing the marginal distributional properties of each AoI process in a network with multiple AoI processes. The results of [15] and [16] have then been applied to characterize the marginal distributional properties of AoI under a variety of queueing disciplines [17]–[22]. However, a systematic approach to the joint analysis of AoI processes is an open problem. In this paper, we develop an SHS-based general framework to facilitate the analysis of the joint distributional properties of an arbitrary set of AoI processes through the characterization of their joint stationary moments and MGFs. Therefore, this paper is a joint distributional counterpart of [16].

Contributions. An SHS-based general framework is presented in this paper to allow the characterization of the stationary joint moments and joint MGFs of an arbitrary set of AoI processes in networks. We first derive first-order linear differential equations for the temporal evolution of both the joint moments and the joint MGFs. We then characterize the conditions for asymptotic stability of the differential equations, which in turn enables the characterization of the stationary joint moments and joint MGFs. Afterwards, we apply our framework to derive the stationary joint MGF for a multi-source updating system under the non-preemptive in service queueing discipline in closed-form. An interesting insight obtained from our analysis is that the existence of the stationary joint first moments guarantees the existence of the stationary joint higher-order moments and MGFs. Our analytical findings also reveal that for a two-source updating system, there exists a threshold value of server utilization above which the two age processes are positively correlated.

Notations. A set $\mathsf{X} \in \mathbb{R}^n$ has a cardinality of $|\mathsf{X}| = n$ and its j-th element is denoted by $\mathsf{X}(j)$. A vector $\mathbf{x} \in \mathbb{R}^{1\times n}$ is a $1 \times n$ row vector with $[\mathbf{x}]_j$ or $[\mathbf{x}]_{\mathsf{K}=\{j\}}$ denoting its j-th

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element. A matrix $\mathbf{X} \in \mathbb{R}^{n_1 \times n_2}$ has (i,j)-th element $[\mathbf{X}]_{i,j}$ or $[\mathbf{X}]_{\mathsf{K}=\{i,j\}}$, and j-th column $[\mathbf{X}]_j$. The vectors $\mathbf{0}_n$ and $\mathbf{1}_n$ contain all zeros and ones in $\mathbb{R}^{1\times n}$, the vector $\mathrm{Tra}(\mathbf{x})$ or $\mathbf{x}^{\mathrm{T}}$ is the transpose of $\mathbf{x}$, the matrix $\mathrm{Tra}(\mathbf{X})$ or $\mathbf{X}^{\mathrm{T}}$ is the transpose of $\mathbf{X}$, and $\mathbf{I}_n$ is the $n \times n$ identity matrix. Whenever subscript n is dropped, the dimensions of $\mathbf{0}$, $\mathbf{1}$, and $\mathbf{I}$ will be clear from the context. The Kronecker delta function $\delta_{i,j}$ equals 1 if $i = j$ and 0 otherwise. The vector $\mathbf{e}_i$ denotes the i-th Cartesian unit vector satisfying $[\mathbf{e}_i]_j = \delta_{i,j}$. A tensor is a multi-dimensional array whose order defines the number of its dimensions. For instance, a vector is a one-dimensional array or first-order tensor, and a matrix is a two-dimensional array or second-order tensor. An I-th order tensor $\mathcal{X} \in \mathbb{R}^{n_1\times n_2 \times \cdots \times n_I}$ has K-th element $[\mathcal{X}]_{\mathsf{K}}$, where $|\mathsf{K}| = I$ and $1 \le \mathsf{K}(j) \le n_j$ for all $1 \le j \le I$. The product of the tensor $\mathcal{X}$ and a matrix along its j-th dimension is denoted by $\times_j$ and known as the j-mode product. In particular, the j-mode product of $\mathcal{X}$ and a matrix $\mathbf{X} \in \mathbb{R}^{m\times n_j}$ is represented as: $\mathcal{Y} = \mathcal{X} \times_j \mathbf{X}$, where $\mathcal{Y} \in \mathbb{R}^{n_1\times\cdots\times n_{j-1}\times m \times n_{j+1}\times\cdots\times n_I}$. For a process $\mathbf{x}(t)$, $\mathbf{X}(t)$ or $\mathcal{X}(t)$, $\dot{\mathbf{x}}(t)$, $\dot{\mathbf{X}}(t)$ or $\dot{\mathcal{X}}(t)$ denote the derivative $\mathrm{d}\mathbf{x}(t)/\mathrm{d}t$, $\mathrm{d}\mathbf{X}(t)/\mathrm{d}t$ or $\mathrm{d}\mathcal{X}(t)/\mathrm{d}t$. For a scalar function $f(\cdot)$ and a vector $\mathbf{x}$, $f(\mathbf{x}) = [f(x_1)\ f(x_2)\ \cdots\ f(x_n)]$. For integers $m \le n, m : n$ is the set $\{m, m+1, \cdots, n\}$, and $\mathsf{X}(m:n) = \{\mathsf{X}(m), \mathsf{X}(m+1), \cdots, \mathsf{X}(n)\}$. The set of all permutations of a set X is denoted by $\mathcal{P}(\mathsf{X})$, and the set of all subsets of a set X is denoted by 2^{X}. The indicator function $\mathbf{1}(\cdot)$ is 1 if the condition inside the brackets is satisfied and 0 otherwise.

II. System Model and Problem Statement

A. Network Model

We consider a general setting of status updating systems where a set of source nodes provide status updates about some physical process(es) to a set of monitors. The freshness of information available at each monitor is quantified in terms of AoI. The AoI processes (or equivalently the age processes) in the system are modeled using the row vector $\mathbf{x}(t) = [x_1(t)\ \cdots\ x_n(t)]$, which is also referred to as the continuous state of the system. In particular, $x_j(t)$ is the age process at monitor j, which may refer to a node, a position in a queue, or a server in a multi-server system. Further, the discrete state of the system is modeled using a finite-state continuous-time Markov chain $q(t) \in \mathcal{Q} = \{0, \cdots, q_{max}\}$, where $\mathcal{Q}$ is the discrete state space. This Markov chain governs the dynamics of the system discrete state, e.g., $q(t)$ may describe the system occupancy with respect to the status updates generated by each source node. In the graphical representation of the Markov chain $q(t)$, each state $q \in \mathcal{Q}$ is a node and each transition l is a directed edge (q_l, q_l') with fixed transition rate $\lambda^{(l)}(q(t)) = \lambda^{(l)}\delta_{q_l,q(t)}$, where the Kronecker delta function $\delta_{q_l,q(t)}$ ensures that transition l occurs only in state q_l. We denote the set of all transitions $\{l\}$ by $\mathcal{L}$, and the sets of incoming and outgoing transitions for state $\bar{q} \in \mathcal{Q}$ by $\mathcal{L}'_{\bar{q}} = \{l \in \mathcal{L} : q'_l = \bar{q}\}$ and $\mathcal{L}_{\bar{q}} = \{l \in \mathcal{L} : q_l = \bar{q}\}$, respectively.

B. An SHS Formulation and Problem Statement

The coupled evolution of the continuous state $\mathbf{x}(t)$ and the discrete state $q(t)$ is modeled using a piecewise linear SHS with linear reset maps [16]. In particular, when a transition l occurs in the Markov chain $q(t)$, $\mathbf{x}$ is reset to $\mathbf{x}'$ according to a reset map matrix $\mathbf{A}_l$ as $\mathbf{x}' = \mathbf{x}\mathbf{A}_l$. Further, in the absence of a transition in $q(t)$, each element in the age vector $\mathbf{x}(t)$ grows at a unit rate with time (which yields piecewise linear age processes over time), i.e., $\dot{\mathbf{x}}(t) \triangleq \frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = \mathbf{1}$. To capture the temporal evolution of the age processes in our system setup, it is sufficient to assume that $\mathbf{A}_l$ is a binary matrix with no more than a single 1 in a column. Since column $[\mathbf{A}_l]_j$ determines the value that will be assigned to x'_j, we have two different cases given the assumed structure of $\mathbf{A}_l$. In the first case, $[\mathbf{A}_l]_j = \mathbf{0}^{\mathrm{T}}$ and so $x'_j = 0$, whereas the second case corresponds to $[\mathbf{A}_l]_j = \mathbf{e}_i^{\mathrm{T}}$ where x'_j is reset to x_i. Different from ordinary continuous-time Markov chains, an inherent feature of SHS is the possibility of having self-transitions in the Markov chain $q(t)$ modeling the system discrete state. In particular, although a self-transition keeps $q(t)$ unchanged, it causes a change in the continuous state $\mathbf{x}(t)$. Further, there may be multiple transitions between any two states in $\mathcal{Q}$ such that their associated reset map matrices are different.

For the above SHS formulation, our prime objective in this paper is to develop a framework that allows analyzing the joint distributional properties of an arbitrary set of AoI processes in the age vector $\mathbf{x}(t)$. Formally, we aim at characterizing the stationary joint moments and joint MGFs for an arbitrary set $\mathsf{K} \subseteq 1:n$ of age processes, which are respectively of the following forms: $\lim_{t\to\infty} \mathbb{E}\Big[\prod_{j=1}^{|\mathsf{K}|} x_{\mathsf{K}(j)}^{[\mathbf{m}]_j}(t)\Big]$ and $\lim_{t\to\infty} \mathbb{E}\Big[\exp\Big[\sum_{j=1}^{|\mathsf{K}|}[\mathbf{s}]_j x_{\mathsf{K}(j)}(t)\Big]\Big]$, where the length of vector $\mathbf{m}$ or $\mathbf{s}$ is $|\mathsf{K}|$, $[\mathbf{m}]_j \in \{0, 1, 2, \cdots\}$, and $[\mathbf{s}]_j \in \mathbb{R}, \forall j \in 1 : |\mathsf{K}|$. When $|\mathsf{K}| = 1$, the problem at hand reduces to the one studied in [16], where the goal was to characterize the marginal distributional properties of each age element in $\mathbf{x}(t)$.

III. Joint Analysis of Age Processes in Networks

A. Differential Equations for the Temporal Evolution of the Joint Moments and Joint MGFs

In order to characterize the temporal evolution of the joint moments and joint MGFs for a set K of age processes, $\mathbb{E}\Big[\prod_{j=1}^{|\mathsf{K}|} x_{\mathsf{K}(j)}^{[\mathbf{m}]_j}(t)\Big]$ and $\mathbb{E}\Big[\exp\Big[\sum_{j=1}^{|\mathsf{K}|}[\mathbf{s}]_j x_{\mathsf{K}(j)}(t)\Big]\Big]$, it is useful to define the following quantities that express different forms of correlation between $q(t)$ and the processes in $\mathbf{x}(t)$:

$$v_{\bar{q},\mathsf{K}}^{(\mathbf{m})}(t) = \mathbb{E}\Bigg[\prod_{j=1}^{|\mathsf{K}|} x_{\mathsf{K}(j)}^{[\mathbf{m}]_j}(t)\delta_{\bar{q},q(t)}\Bigg], \tag{1}$$

$$v_{\bar{q},\mathsf{K}}^{(\mathbf{s})}(t) = \mathbb{E}\Bigg[\exp\Bigg[\sum_{j=1}^{|\mathsf{K}|}[\mathbf{s}]_j x_{\mathsf{K}(j)}(t)\Bigg]\delta_{\bar{q},q(t)}\Bigg], \tag{2}$$

for all states $\bar{q} \in \mathcal{Q}$ and $\mathsf{K} \subseteq \{1, 2, \cdots, N\}$. To see this, note that we have

$$\mathbb{E}\Big[\prod_{j=1}^{|\mathsf{K}|} x_{\mathsf{K}(j)}^{[\mathbf{m}]_j}(t)\Big] = \sum_{\bar{q}\in\mathcal{Q}} \mathbb{E}\Big[\prod_{j=1}^{|\mathsf{K}|} x_{\mathsf{K}(j)}^{[\mathbf{m}]_j}(t)\delta_{\bar{q},q(t)}\Big] = \sum_{\bar{q}\in\mathcal{Q}} v_{\bar{q},\mathsf{K}}^{(\mathbf{m})}(t), \tag{3}$$

$$\mathbb{E}\Big[\exp\Big[\sum_{j=1}^{|\mathsf{K}|}[\mathbf{s}]_j x_{\mathsf{K}(j)}(t)\Big]\Big] = \sum_{\bar{q}\in\mathcal{Q}}\mathbb{E}\Big[\exp\Big[\sum_{j=1}^{|\mathsf{K}|}[\mathbf{s}]_j x_{\mathsf{K}(j)}(t)\Big] \times \delta_{\bar{q},q(t)}\Big] = \sum_{\bar{q}\in\mathcal{Q}} v_{\bar{q},\mathsf{K}}^{(\mathbf{s})}(t). \tag{4}$$

Thus, according to (3) and (4), characterizing the temporal evolution of $v_{\bar{q},\mathsf{K}}^{(\mathbf{m})}(t)$ and $v_{\bar{q},\mathsf{K}}^{(\mathbf{s})}(t)$ directly characterizes the temporal evolution of $\mathbb{E}\left[\prod_{j=1}^{|\mathsf{K}|} x_{\mathsf{K}(j)}^{[\mathbf{m}]_j}(t)\right]$ and $\mathbb{E}\left[\exp\Big[\sum_{j=1}^{|\mathsf{K}|}[\mathbf{s}]_j x_{\mathsf{K}(j)}(t)\Big]\right]$, respectively. Some key notes about the notations in (1) and (2) are provided next. First, $v_{\bar{q},\mathsf{K}}^{(\mathbf{1})}(t)$ may generally refer to $v_{\bar{q},\mathsf{K}}^{(\mathbf{m})}(t)|_{\mathbf{m}=\mathbf{1}}$ or $v_{\bar{q},\mathsf{K}}^{(\mathbf{s})}(t)|_{\mathbf{s}=\mathbf{1}}$. To eliminate this conflict, the convention that $v_{\bar{q},\mathsf{K}}^{(\mathbf{i})}$, for any set of integers $\{[\mathbf{i}]_j \geq 1\}_{j\in 1:|\mathsf{K}|}$, refers to $v_{\bar{q},\mathsf{K}}^{(\mathbf{m})}$ at $\mathbf{m} = \mathbf{i}$ is maintained here. Further, we have $v_{\bar{q},\mathsf{K}}^{(\mathbf{m})}(t)|_{\mathbf{m}=\mathbf{0}} = v_{\bar{q},\mathsf{K}}^{(\mathbf{s})}(t)|_{\mathbf{s}=\mathbf{0}} = \mathbb{E}[\delta_{\bar{q},q(t)}] = \mathbb{P}[q(t) = \bar{q}]$, i.e., $v_{\bar{q},\mathsf{K}}^{(\mathbf{0})}(t) = \mathbb{P}[q(t) = \bar{q}]$ (the probability that $q(t)$ is equal to $\bar{q}$) regardless of K. For $\bar{q} \in \mathcal{Q}$, we define $\mathbf{v}_{\bar{q}}^{(0)}(t) \in \mathbb{R}^{1\times n}$ as:

$$[\mathbf{v}_{\bar{q}}^{(0)}(t)]_k = v_{\bar{q}}^{(0)}(t) = \mathbb{P}[q(t) = \bar{q}], \forall k \in 1 : n. \tag{5}$$

It will also be useful in our subsequent analysis and exposition to define the following tensors in $\mathbb{R}^{n_1\times n_2\times\cdots\times n_{|\mathsf{K}|}}(n_j = n, \forall j \in 1 : |\mathsf{K}|)$ containing the scalars in (1) and (2): $\Big[\mathcal{V}_{\bar{q},|\mathsf{K}|}^{(\mathbf{m})}(t)\Big]_{\mathsf{K}} = v_{\bar{q},\mathsf{K}}^{(\mathbf{m})}(t)$ and $\Big[\mathcal{V}_{\bar{q},|\mathsf{K}|}^{(\mathbf{s})}(t)\Big]_{\mathsf{K}} = v_{\bar{q},\mathsf{K}}^{(\mathbf{s})}(t), \forall \bar{q} \in \mathcal{Q}$. In other words, the tensors $\mathcal{V}_{\bar{q},|\mathsf{K}|}^{(\mathbf{m})}(t)$ and $\mathcal{V}_{\bar{q},|\mathsf{K}|}^{(\mathbf{s})}(t)$ contain the scalars $\{v_{\bar{q},\mathsf{M}}^{(\mathbf{m})}(t)\}_{\mathsf{M}\subseteq 1:n, |\mathsf{M}|=|\mathsf{K}|}$ and $\{v_{\bar{q},\mathsf{M}}^{(\mathbf{s})}(t)\}_{\mathsf{M}\subseteq 1:n, |\mathsf{M}|=|\mathsf{K}|}$, respectively. For instance, $\mathcal{V}_{\bar{q},1}^{(\mathbf{m})}(t)$ and $\mathcal{V}_{\bar{q},2}^{(\mathbf{m})}(t)$ can be respectively expressed as:

$$\mathcal{V}_{\bar{q},1}^{(\mathbf{m})}(t) = [v_{\bar{q},\{1\}}^{(\mathbf{m})}(t)\ v_{\bar{q},\{2\}}^{(\mathbf{m})}(t)\ \cdots\ v_{\bar{q},\{n\}}^{(\mathbf{m})}(t)], \quad \forall \bar{q} \in \mathcal{Q}, \tag{6}$$

$$\mathcal{V}_{\bar{q},2}^{(\mathbf{m})}(t) = \begin{bmatrix} v_{\bar{q},\{1,1\}}^{(\mathbf{m})}(t) & v_{\bar{q},\{1,2\}}^{(\mathbf{m})}(t) & \dots & v_{\bar{q},\{1,n\}}^{(\mathbf{m})}(t) \\ v_{\bar{q},\{2,1\}}^{(\mathbf{m})}(t) & v_{\bar{q},\{2,2\}}^{(\mathbf{m})}(t) & \dots & v_{\bar{q},\{2,n\}}^{(\mathbf{m})}(t) \\ \vdots & \vdots & \ddots & \vdots \\ v_{\bar{q},\{n,1\}}^{(\mathbf{m})}(t) & v_{\bar{q},\{n,2\}}^{(\mathbf{m})}(t) & \dots & v_{\bar{q},\{n,n\}}^{(\mathbf{m})}(t) \end{bmatrix}. \tag{7}$$

The following Lemma shows that $\{v_{\bar{q},\mathsf{K}}^{(\mathbf{m})}(t)\}_{\bar{q}\in\mathcal{Q}}$ and $\{v_{\bar{q},\mathsf{K}}^{(\mathbf{s})}(t)\}_{\bar{q}\in\mathcal{Q}}$ obey a system of first-order ordinary differential equations.

Lemma 1. *For state $\bar{q} \in \mathcal{Q}$ in the piecewise linear SHS with linear reset maps under consideration,*

$$\dot{v}_{\bar{q},\mathsf{K}}^{(\mathbf{m})}(t) = \sum_{j=1}^{|\mathsf{K}|}[\mathbf{m}]_j v_{\bar{q},\mathsf{K}}^{(\mathbf{m}-\mathbf{e}_j)}(t) + \sum_{l\in\mathcal{L}'_{\bar{q}}}\lambda^{(l)}\Big[\mathcal{V}_{\bar{q}_l,|\mathsf{K}|}^{(\mathbf{m})}(t) \times_1 \mathbf{A}_l \cdots \times_{|\mathsf{K}|} \mathbf{A}_l\Big]_{\mathsf{K}} - v_{\bar{q},\mathsf{K}}^{(\mathbf{m})}(t)\sum_{l\in\mathcal{L}_{\bar{q}}}\lambda^{(l)}, \tag{8}$$

$$\dot{v}_{\bar{q},\mathsf{K}}^{(\mathbf{s})}(t) = \Big[\sum_{j=1}^{|\mathsf{K}|}[\mathbf{s}]_j - \sum_{l\in\mathcal{L}_{\bar{q}}}\lambda^{(l)}\Big]v_{\bar{q},\mathsf{K}}^{(\mathbf{s})}(t) + \sum_{l\in\mathcal{L}'_{\bar{q}}}\lambda^{(l)}\Big[\mathcal{V}_{\bar{q}_l,|\mathsf{K}|}^{(\mathbf{s})}(t) \times_1 \mathbf{A}_l \times_2 \mathbf{A}_l \cdots \times_{|\mathsf{K}|} \mathbf{A}_l\Big]_{\mathsf{K}} + c_{\bar{q},\mathsf{K}}(t), \tag{9}$$

such that $c_{\bar{q},\mathsf{K}}(t)$ is defined as

$$c_{\bar{q},\mathsf{K}}(t) = \sum_{l\in\mathcal{L}'_{\bar{q}}}\lambda^{(l)}\sum_{\mathsf{Z}\in 2^{\mathsf{K}}\setminus\varnothing}\mathbf{1}(\mathsf{Z}_l = \mathsf{Z})\times \Big[\mathcal{V}_{\bar{q}_l,|\mathsf{K}\setminus\mathsf{Z}_l|}^{(\mathbf{s}')}(t) \times_1 \mathbf{A}_l \times_2 \mathbf{A}_l \cdots \times_{|\mathsf{K}\setminus\mathsf{Z}_l|} \mathbf{A}_l\Big]_{\mathsf{K}\setminus\mathsf{Z}_l}, \tag{10}$$

where the set $\mathsf{Z}_l = \{j \in \mathsf{K} : [\mathbf{A}_l]_j = \mathbf{0}^{\mathrm{T}}\}$, the vector $\mathbf{s}' = \big[[\mathbf{s}]_{\mathsf{I}_l(1)}\ [\mathbf{s}]_{\mathsf{I}_l(2)}\ \cdots\ [\mathbf{s}]_{\mathsf{I}_l(|\mathsf{K}\setminus\mathsf{Z}_l|)}\big]$, and the set I_l contains the indices of the elements of $\mathsf{K}\setminus\mathsf{Z}_l$ inside K. When $\mathsf{Z}_l = \mathsf{K}$, we also define:

$$\Big[\mathcal{V}_{\bar{q}_l,|\mathsf{K}\setminus\mathsf{Z}_l|}^{(\mathbf{s}')}(t) \times_1 \mathbf{A}_l \times_2 \mathbf{A}_l \cdots \times_{|\mathsf{K}\setminus\mathsf{Z}_l|} \mathbf{A}_l\Big]_{\mathsf{K}\setminus\mathsf{Z}_l} = v_{\bar{q}_l}^{(0)}(t). \tag{11}$$

Proof: The proof is given in [23, Appendix A] (the extended journal version of this paper), and is omitted here due to space limitations. ■

It is worth noting that (8) and (9) in Lemma 1 can be expressed in a tensor form as

$$\dot{\mathcal{V}}_{\bar{q},|\mathsf{K}|}^{(\mathbf{m})}(t) = \sum_{j=1}^{|\mathsf{K}|}[\mathbf{m}]_j\mathcal{V}_{\bar{q},|\mathsf{K}|}^{(\mathbf{m}-\mathbf{e}_j)}(t) + \sum_{l\in\mathcal{L}'_{\bar{q}}}\lambda^{(l)}\Big[\mathcal{V}_{\bar{q}_l,|\mathsf{K}|}^{(\mathbf{m})}(t) \times_1 \mathbf{A}_l \cdots \times_{|\mathsf{K}|} \mathbf{A}_l\Big] - \mathcal{V}_{\bar{q},|\mathsf{K}|}^{(\mathbf{m})}(t)\sum_{l\in\mathcal{L}_{\bar{q}}}\lambda^{(l)}, \tag{12}$$

$$\dot{\mathcal{V}}_{\bar{q},|\mathsf{K}|}^{(\mathbf{s})}(t) = \Big[\sum_{j=1}^{|\mathsf{K}|}[\mathbf{s}]_j - \sum_{l\in\mathcal{L}_{\bar{q}}}\lambda^{(l)}\Big]\mathcal{V}_{\bar{q},|\mathsf{K}|}^{(\mathbf{s})}(t) + \sum_{l\in\mathcal{L}'_{\bar{q}}}\lambda^{(l)}\Big[\mathcal{V}_{\bar{q}_l,|\mathsf{K}|}^{(\mathbf{s})}(t) \times_1 \mathbf{A}_l \times_2 \mathbf{A}_l \cdots \times_{|\mathsf{K}|} \mathbf{A}_l\Big] + \mathcal{C}_{\bar{q},|\mathsf{K}|}(t), \tag{13}$$

where $\mathcal{C}_{\bar{q},|\mathsf{K}|}(t) \in \mathbb{R}^{n_1\times n_2\times\cdots\times n_{|\mathsf{K}|}}(n_j = n, \forall j \in 1 : |\mathsf{K}|)$ such that $[\mathcal{C}_{\bar{q},|\mathsf{K}|}(t)]_{\mathsf{K}} = c_{\bar{q},\mathsf{K}}(t)$. In order to clearly see that Lemma 1 characterizes the trajectories of $v_{\bar{q},\mathsf{K}}^{(\mathbf{m})}(t)$ and $v_{\bar{q},\mathsf{K}}^{(\mathbf{s})}(t)$ over time, it is useful to first state the following Corollaries.

Corollary 1. *When $\mathsf{K} = \{k\}$, $\mathbf{m} = [m_1]$ and $\mathbf{s} = [s_1]$, the system of first-order ordinary differential equations in Lemma 1 reduces to:*

$$\dot{v}_{\bar{q},\{k\}}^{([m_1])}(t) = m_1 v_{\bar{q},\{k\}}^{([m_1-1])}(t) + \sum_{l\in\mathcal{L}'_{\bar{q}}}\lambda^{(l)}\big[\mathbf{v}_{\bar{q}_l}^{(m_1)}(t)\mathbf{A}_l\big]_k - v_{\bar{q},\{k\}}^{([m_1])}(t)\sum_{l\in\mathcal{L}_{\bar{q}}}\lambda^{(l)}, \tag{14}$$

$$\dot{v}^{([s_1])}_{\bar{q},\{k\}}(t) = \Big[s_1 - \sum_{l\in\mathcal{L}_{\bar{q}}} \lambda^{(l)}\Big] v^{([s_1])}_{\bar{q},\{k\}}(t) + \sum_{l\in\mathcal{L}'_{\bar{q}}} \lambda^{(l)} \Big[\mathbf{v}^{(s_1)}_{\bar{q}_l}(t)\mathbf{A}_l + \mathbf{1}\big(\mathsf{Z}_l = \{k\}\big)\mathbf{v}^{(0)}_{\bar{q}_l}(t)\Big]_k. \tag{15}$$

Proof: The expressions in (14) and (15) directly follow from Lemma 1 along with noting that $\mathcal{V}^{(\mathbf{m})}_{\bar{q}_l,|\mathsf{K}|}(t)$ and $\mathcal{V}^{(\mathbf{s})}_{\bar{q}_l,|\mathsf{K}|}(t)$ are vectors in $\mathbb{R}^{1\times n}$ when $|\mathsf{K}| = 1$, and hence we define $\mathbf{v}^{(m_1)}_{\bar{q}_l}(t) = \mathcal{V}^{([m_1])}_{\bar{q}_l,1}(t)$ and $\mathbf{v}^{(s_1)}_{\bar{q}_l}(t) = \mathcal{V}^{([s_1])}_{\bar{q}_l,1}(t)$. ■

Corollary 2. *When $\mathsf{K} = \{k_1, k_2\}$, $\mathbf{m} = [m_1\ m_2]$ and $\mathbf{s} = [s_1\ s_2]$, the system of first-order ordinary differential equations in Lemma 1 reduces to:*

$$\dot{v}^{([m_1\ m_2])}_{\bar{q},\{k_1,k_2\}}(t) = m_1 v^{([m_1-1\ m_2])}_{\bar{q},\{k_1,k_2\}}(t) + m_2 v^{([m_1\ m_2-1])}_{\bar{q},\{k_1,k_2\}}(t) + \sum_{l\in\mathcal{L}'_{\bar{q}}} \lambda^{(l)} \big[\mathbf{A}_l^{\mathrm{T}} \mathbf{V}^{(m_1,m_2)}_{\bar{q}_l}(t)\mathbf{A}_l\big]_{k_1,k_2} - v^{([m_1\ m_2])}_{\bar{q},\{k_1,k_2\}}(t) \sum_{l\in\mathcal{L}_{\bar{q}}} \lambda^{(l)}, \tag{16}$$

$$\dot{v}^{([s_1\ s_2])}_{\bar{q},\{k_1,k_2\}}(t) = \Big[s_1 + s_2 - \sum_{l\in\mathcal{L}_{\bar{q}}} \lambda^{(l)}\Big] v^{([s_1\ s_2])}_{\bar{q},\{k_1,k_2\}}(t) + \sum_{l\in\mathcal{L}'_{\bar{q}}} \lambda^{(l)} [\mathbf{A}_l^{\mathrm{T}} \mathbf{V}^{(s_1,s_2)}_{\bar{q}_l}(t)\mathbf{A}_l]_{k_1,k_2} + c_{\bar{q},\{k_1,k_2\}}(t), \tag{17}$$

where $c_{\bar{q},\{k_1,k_2\}}(t)$ is given by

$$c_{\bar{q},\{k_1,k_2\}}(t) = \sum_{l\in\mathcal{L}'_{\bar{q}}} \lambda^{(l)} \Big[\mathbf{1}\Big(\mathsf{Z}_l = \{k_2\}\Big)\big[\mathbf{v}^{(s_1)}_{q_l}(t)\mathbf{A}_l\big]_{k_1} + \mathbf{1}\Big(\mathsf{Z}_l = \{k_1\}\Big)\big[\mathbf{v}^{(s_2)}_{q_l}(t)\mathbf{A}_l\big]_{k_2} \mathbf{1}\Big(\mathsf{Z}_l = \{k_1,k_2\}\Big) v^{(0)}_{q_l}(t)\Big], \tag{18}$$

Proof: The expressions in (16) and (17) directly follow from Lemma 1 along with noting that $\mathcal{V}^{(\mathbf{m})}_{\bar{q}_l,|\mathsf{K}|}(t)$ and $\mathcal{V}^{(\mathbf{s})}_{\bar{q}_l,|\mathsf{K}|}(t)$ are matrices in $\mathbb{R}^{n\times n}$ when $|\mathsf{K}| = 2$, and hence we define $\mathbf{V}^{(m_1,m_2)}_{\bar{q}_l}(t) = \mathcal{V}^{([m_1\ m_2])}_{\bar{q}_l,2}(t)$ and $\mathbf{V}^{(s_1,s_2)}_{\bar{q}_l}(t) = \mathcal{V}^{([s_1\ s_2])}_{\bar{q}_l,2}(t)$. ■

Remark 1. *Note that the system of differential equations in Corollary 1 is identical to the one derived in [16, Lemma 1] for the temporal evolution of the marginal moments and MGF of each age element in $\mathbf{x}(t)$. Further, the system of differential equations in Corollary 2 is identical to the one derived in [24, Lemma 1] for the temporal evolution of the joint moments and MGF of any two arbitrary age processes in $\mathbf{x}(t)$.*

We are now ready to elaborate on the use of Lemma 1 to obtain the trajectories of $\mathcal{V}^{(\mathbf{m})}_{\bar{q},|\mathsf{K}|}(t)$ and $\mathcal{V}^{(\mathbf{s})}_{\bar{q},|\mathsf{K}|}(t)$ for an arbitrary set K starting from a given initial condition at $t = 0$. We start this discussion with the case of $|\mathsf{K}| = 2$ for which the trajectories can be characterized using Corollaries 1 and 2. When $|\mathsf{K}| = 2$ and for all $\bar{q} \in \mathcal{Q}$, we observe from Corollary 2 that $\mathcal{V}^{([m_1\ m_2])}_{\bar{q},2}(t) = \mathbf{V}^{(m_1,m_2)}_{\bar{q}}(t)$ and $\mathcal{V}^{([s_1\ s_2])}_{\bar{q},2}(t) = \mathbf{V}^{(s_1,s_2)}_{\bar{q}}(t)$ can be evaluated using (16) and (17), respectively. In particular, we note from (16) that in order to compute $\mathbf{V}^{(1,1)}_{\bar{q}}(t)$, we need to first compute $\mathbf{V}^{(0,1)}_{\bar{q}}(t) = \mathrm{Tra}\big(\mathbf{V}^{(1,0)}_{\bar{q}}(t)\big)$ and $\mathbf{V}^{(1,0)}_{\bar{q}}(t) = \big[\mathrm{Tra}\big(\mathbf{v}^{(1)}_{\bar{q}}(t)\big)\ \mathrm{Tra}\big(\mathbf{v}^{(1)}_{\bar{q}}(t)\big)\ \cdots\ \mathrm{Tra}\big(\mathbf{v}^{(1)}_{\bar{q}}(t)\big)\big]$ by using (14) in Corollary 1 to evaluate $\mathbf{v}^{(1)}_{\bar{q}}$. From (14), we note that $\mathbf{v}^{(1)}_{\bar{q}}$ is obtained from $\mathbf{v}^{(0)}_{\bar{q}}(t)$, which can be computed from [16, Lemma 1] as:

$$\dot{\mathbf{v}}^{(0)}_{\bar{q}}(t) = \sum_{l\in\mathcal{L}'_{\bar{q}}} \lambda^{(l)} \mathbf{v}^{(0)}_{q_l}(t) - \mathbf{v}^{(0)}_{\bar{q}}(t) \sum_{l\in\mathcal{L}_{\bar{q}}} \lambda^{(l)}, \forall \bar{q} \in \mathcal{Q}. \tag{19}$$

Afterwards, $\mathbf{V}^{(2,1)}_{\bar{q}}(t)$ is computed from $\mathbf{V}^{(2,0)}_{\bar{q}}(t) = \big[\mathrm{Tra}\big(\mathbf{v}^{(2)}_{\bar{q}}(t)\big)\ \mathrm{Tra}\big(\mathbf{v}^{(2)}_{\bar{q}}(t)\big)\ \cdots\ \mathrm{Tra}\big(\mathbf{v}^{(2)}_{\bar{q}}(t)\big)\big]$ and $\mathbf{V}^{(1,1)}_{\bar{q}}(t)$ such that $\mathbf{v}^{(2)}_{\bar{q}}(t)$ can be evaluated from $\mathbf{v}^{(1)}_{\bar{q}}(t)$ using (14). The process can be repeated to compute $\mathbf{V}^{(m_1,m_2)}_{\bar{q}}(t)$ for the desired $m_1, m_2 \geq 2$ using $\mathbf{V}^{(m_1-1,m_2)}_{\bar{q}}(t)$ and $\mathbf{V}^{(m_1,m_2-1)}_{\bar{q}}(t)$ evaluated in the previous steps. Further, by inspecting the structure of $c_{\bar{q},\{k_1,k_2\}}(t)$ in (18), we note that $\mathbf{V}^{(s_1,s_2)}_{\bar{q}}(t)$ can be computed from $\mathbf{v}^{(s_k)}_{\bar{q}}(t)$ and $\mathbf{v}^{(0)}_{\bar{q}}(t)$, where $\mathbf{v}^{(s_k)}_{\bar{q}}(t)$ can be evaluated from $\mathbf{v}^{(0)}_{\bar{q}}(t)$ using (15). Now, one can clearly see from Lemma 1 that $\mathcal{V}^{([m_1\ m_2\ m_3])}_{\bar{q},3}(t)$ can be computed from $\mathcal{V}^{([m_1\ m_2])}_{\bar{q},2}(t) = \mathbf{V}^{(m_1,m_2)}_{\bar{q}}(t)$, and $\mathcal{V}^{([s_1\ s_2\ s_3])}_{\bar{q},3}(t)$ can be computed from $\mathcal{V}^{([s_1\ s_2])}_{\bar{q},2}(t) = \mathbf{V}^{(s_1,s_2)}_{\bar{q}}(t)$ and $\mathcal{V}^{([s_1])}_{\bar{q},1}(t) = \mathbf{v}^{(s_1)}_{\bar{q}}(t)$. Thus, through the repeated application of Lemma 1, we can evaluate $\mathcal{V}^{(\mathbf{m})}_{\bar{q},|\mathsf{K}|}$ and $\mathcal{V}^{(\mathbf{s})}_{\bar{q},|\mathsf{K}|}$ for an arbitrary set K with $|\mathsf{K}| \geq 3$.

B. Stationary Joint Moments and Joint MGFs

While Lemma 1 holds for any collection of reset map matrices $\{\mathbf{A}_l\}_{l\in\mathcal{L}}$, the set of differential equations in Lemma 1 can be unstable for some choices of $\{\mathbf{A}_l\}_{l\in\mathcal{L}}$. Thus, it is essential to investigate the conditions under which the differential equations in Lemma 1 are stable. While there are several notions of stability including Lyapunov, Lagrange, and exponential stability, we are interested here in the asymptotic stability under which $v^{(\mathbf{m})}_{\bar{q},\mathsf{K}}(t)$ and $v^{(\mathbf{s})}_{\bar{q},\mathsf{K}}(t)$ respectively converge to the limits $\bar{v}^{(\mathbf{m})}_{\bar{q},\mathsf{K}}$ and $\bar{v}^{(\mathbf{s})}_{\bar{q},\mathsf{K}}$ as $t \to \infty$. The limiting values can then be evaluated as the solution of the equations resulting from setting the derivatives in Lemma 1 to zero. To clearly see why we are concerned about the asymptotic stability in this paper, recall that our prime objective is to characterize the stationary joint moments and joint MGFs: $\lim_{t\to\infty} \mathbb{E}\Big[\prod_{j=1}^{|\mathsf{K}|} x^{[\mathbf{m}]_j}_{\mathsf{K}(j)}(t)\Big]$ and $\lim_{t\to\infty} \mathbb{E}\Big[\exp\Big[\sum_{j=1}^{|\mathsf{K}|} [\mathbf{s}]_j x_{\mathsf{K}(j)}(t)\Big]\Big]$. Under the asymptotic stability, these quantities can simply be evaluated from (3) and (4) as

$$\lim_{t\to\infty} \mathbb{E}\Big[\prod_{j=1}^{|\mathsf{K}|} x^{[\mathbf{m}]_j}_{\mathsf{K}(j)}(t)\Big] = \sum_{\bar{q}\in\mathcal{Q}} \lim_{t\to\infty} v^{(\mathbf{m})}_{\bar{q},\mathsf{K}}(t) = \sum_{\bar{q}\in\mathcal{Q}} \bar{v}^{(\mathbf{m})}_{\bar{q},\mathsf{K}}, \tag{20}$$

$$\lim_{t\to\infty} \mathbb{E}\Big[\exp\Big[\sum_{j=1}^{|\mathsf{K}|} [\mathbf{s}]_j x_{\mathsf{K}(j)}(t)\Big]\Big] = \sum_{\bar{q}\in\mathcal{Q}} \lim_{t\to\infty} v^{(\mathbf{s})}_{\bar{q},\mathsf{K}}(t) = \sum_{\bar{q}\in\mathcal{Q}} \bar{v}^{(\mathbf{s})}_{\bar{q},\mathsf{K}}. \tag{21}$$

We now proceed to characterizing the conditions under which the differential equations in Lemma 1 are asymptotically stable. Let us first recall the asymptotic stability theorem for linear systems. The linear system

$$\dot{\mathbf{v}}(t) = \mathbf{v}(t)\mathbf{P},\ \mathbf{v}(0) = \mathbf{v}_0 \tag{22}$$

is asymptotically stable if and only if the eigenvalues of $\mathbf{P}$ have strictly negative real parts. Thus, according to (22), it is always useful to write the differential equations at hand in a vector form to test the asymptotic stability. For all $\bar{q} \in \mathcal{Q}$, let $\bar{\mathbf{v}}_{\bar{q}}^{(0)}$, $\bar{\mathcal{V}}_{\bar{q},|\mathsf{K}|}^{(\mathbf{m})}$ and $\bar{\mathcal{V}}_{\bar{q},|\mathsf{K}|}^{(\mathbf{s})}$ respectively denote the limiting values of $\mathbf{v}_{\bar{q}}^{(0)}(t)$, $\mathcal{V}_{\bar{q},|\mathsf{K}|}^{(\mathbf{m})}(t)$ and $\mathcal{V}_{\bar{q},|\mathsf{K}|}^{(\mathbf{s})}(t)$, when $t \to \infty$. Clearly, $\bar{\mathbf{v}}_{\bar{q}}^{(0)}$, $\bar{\mathcal{V}}_{\bar{q},|\mathsf{K}|}^{(\mathbf{m})}$ and $\bar{\mathcal{V}}_{\bar{q},|\mathsf{K}|}^{(\mathbf{s})}$ are the fixed points of (19), (8) and (9), respectively, which can be obtained after setting the derivatives to zero. The next theorem characterizes the conditions for asymptotic stability for the differential equations in Lemma 1, which in turn enables the characterization of stationary joint moments and joint MGFs for an arbitrary set K.

Theorem 1. *If the Markov chain $q(t)$ is ergodic with stationary distribution $\{\bar{\mathbf{v}}_{\bar{q}}^{(0)} > \mathbf{0} : \bar{q} \in \mathcal{Q}\}$, and there exist positive fixed points $\{\bar{\mathcal{V}}_{\bar{q},|\mathsf{Z}|}^{(\mathbf{1}_{|\mathsf{Z}|})} : \bar{q} \in \mathcal{Q}\}_{|\mathsf{Z}| \in \{1,2,\cdots,|\mathsf{K}|\}}$ of (8), then:*

- *(i) For all $\bar{q} \in \mathcal{Q}$, $v_{\bar{q},\mathsf{K}}^{(\mathbf{m})}(t)$ converges to $\bar{v}_{\bar{q},\mathsf{K}}^{(\mathbf{m})}$ satisfying*

$$\bar{v}_{\bar{q},\mathsf{K}}^{(\mathbf{m})} \sum_{l \in \mathcal{L}_{\bar{q}}} \lambda^{(l)} = \sum_{j=1}^{|\mathsf{K}|} [\mathbf{m}]_j \bar{v}_{\bar{q},\mathsf{K}}^{(\mathbf{m}-\mathbf{e}_j)} + \sum_{l \in \mathcal{L}'_{\bar{q}}} \lambda^{(l)} \left[\bar{\mathcal{V}}_{\bar{q}_l,|\mathsf{K}|}^{(\mathbf{m})} \times_1 \mathbf{A}_l \times_2 \mathbf{A}_l \cdots \times_{|\mathsf{K}|} \mathbf{A}_l\right]_{\mathsf{K}}. \quad (23)$$

- *(ii) There exists $s_0 > 0$ such that for all $\mathbf{s} \in \mathcal{S} = \{\mathbf{s} : \sum_{j=1}^{|\mathsf{K}|} [\mathbf{s}]_j < s_0\}$ and $\bar{q} \in \mathcal{Q}$, $v_{\bar{q},\mathsf{K}}^{(\mathbf{s})}(t)$ and $c_{\bar{q},\mathsf{K}}(t)$ respectively converge to $\bar{v}_{\bar{q},\mathsf{K}}^{(\mathbf{s})}$ and $\bar{c}_{\bar{q},\mathsf{K}}$ satisfying*

$$\bar{v}_{\bar{q},\mathsf{K}}^{(\mathbf{s})} \sum_{l \in \mathcal{L}_{\bar{q}}} \lambda^{(l)} = \bar{v}_{\bar{q},\mathsf{K}}^{(\mathbf{s})} \sum_{j=1}^{|\mathsf{K}|} [\mathbf{s}]_j + \sum_{l \in \mathcal{L}'_{\bar{q}}} \lambda^{(l)} \left[\bar{\mathcal{V}}_{\bar{q}_l,|\mathsf{K}|}^{(\mathbf{s})} \times_1 \mathbf{A}_l \times_2 \mathbf{A}_l \cdots \times_{|\mathsf{K}|} \mathbf{A}_l\right]_{\mathsf{K}} + \bar{c}_{\bar{q},\mathsf{K}}, \quad (24)$$

where $\bar{c}_{\bar{q},\mathsf{K}}$ is given by

$$\bar{c}_{\bar{q},\mathsf{K}} = \sum_{l \in \mathcal{L}'_{\bar{q}}} \lambda^{(l)} \sum_{\mathsf{Z} \in 2^{\mathsf{K}} \setminus \varnothing} \mathbf{1}(\mathsf{Z}_l = \mathsf{Z}) \times \left[\bar{\mathcal{V}}_{\bar{q}_l,|\mathsf{K} \setminus \mathsf{Z}_l|}^{(\mathbf{s}')} \times_1 \mathbf{A}_l \times_2 \mathbf{A}_l \cdots \times_{|\mathsf{K} \setminus \mathsf{Z}_l|} \mathbf{A}_l\right]_{\mathsf{K} \setminus \mathsf{Z}_l}. \quad (25)$$

Proof: The proof is given in [23, Appendix B], and is omitted here due to space limitations. ■

Theorem 1 is a generalization of [16, Theorem 1] which was focused on the characterization of the marginal stationary moments and MGFs, i.e., the fixed points of the differential equations in Corollary 1. In particular, when $|\mathsf{K}| = 1$, Theorem 1 directly reduces to [16, Theorem 1]. A useful insight provided by Theorem 1 is that the existence of the stationary joint first moments guarantees the existence of the stationary joint higher-order moments and MGFs. It is worth emphasizing that the generality of Theorem 1 lies in the fact that it allows the investigation of the stationary joint moments and MGFs for an arbitrary set of age processes under any arbitrary queueing discipline. This opens the door for the use of Theorem 1 to study the joint analysis of age processes in networks for different queueing disciplines/status updating system settings in the literature, which have only been analyzed in terms of the marginal moments and MGFs until now.

The exact number of linear equations (that need to be solved to find the joint MGF using Theorem 1) indeed depends on the queueing discipline under consideration (which determines the set of transitions $\mathcal{L}$). In fact, to find the joint MGF of an arbitrary set K of AoI processes using Theorem 1, one needs to solve a maximum of $\bar{N} = |\mathcal{Q}| \times \sum_{i=0}^{|\mathsf{K}|-1} \binom{n}{|\mathsf{K}|-i}$ linear equations. Note that $\binom{n}{|\mathsf{K}|}$ represents all the possible combinations of the age processes that can be included in the set K. It is also worth noting that the solution of the $\bar{N}$ equations not only characterizes the joint MGFs of all possible combinations of $|\mathsf{K}|$ age processes from the age vector $\mathbf{x}(t)$ but also characterizes: 1) the joint MGFs of all possible combinations of $|\mathsf{K}| - i$ age processes where $i \in [1, |\mathsf{K}| - 2]$, and 2) the marginal MGFs of all age processes in the system.

IV. Analysis of the Stationary Joint MGF in Multi-source Updating Systems

In this section, we use Theorem 1 to analyze the stationary joint MGF of the age processes in a multi-source status updating system, where a transmitter monitors N physical processes, and sends its measurements to a destination in the form of status updates. As shown in Fig. 1, the transmitter consists of N sources and a single server; each source generates status updates about one physical process, and the server delivers the status updates generated from the sources to the destination. Status updates generated by the i-th source are assumed to follow a Poisson process with rate λ_i. Further, the time needed by the server to send a status update is assumed to be a rate μ exponential random variable. Let $\rho = \frac{\lambda}{\mu}$ denote the server utilization factor, where $\lambda = \sum_{j=1}^{N} \lambda_j$. Further, we define $\lambda_{\mathsf{Z}} = \sum_{j=1}^{|\mathsf{Z}|} \lambda_{\mathsf{Z}(j)}, \lambda_{-\mathsf{Z}} = \sum_{j=1,\, j \notin \mathsf{Z}}^{N} \lambda_{\mathsf{Z}(j)}, \rho_{\mathsf{Z}} = \frac{\lambda_{\mathsf{Z}}}{\mu}$ and $\rho_{-\mathsf{Z}} = \frac{\lambda_{-\mathsf{Z}}}{\mu}$. Thus, we have $\rho_i = \frac{\lambda_i}{\mu}$, $\lambda_{-i} = \sum_{j=1,\, j \neq i}^{N} \lambda_j$, and $\rho_{-i} = \frac{\lambda_{-i}}{\mu}$. We derive the joint MGF of an arbitrary set $\mathsf{K} \subseteq \{1, 2, \cdots, N\}$ of age processes (associated with the observed N physical processes) at the destination under the last-come-first-served with no preemption in service (LCFS-NP) queueing discipline[1]. Under this queueing discipline, a new arriving status update at the transmitter (from any of the sources) enters service upon its arrival if the server is idle (i.e., there is no status update in service); otherwise, the new arriving status update is discarded.

Using the notations of the SHS framework, the continuous state $\mathbf{x}(t)$ is given by $\mathbf{x}(t) = [x_0(t)\ x_1(t)\ \cdots x_N(t)]$, where $x_i(t), i \in 1 : N$, represents the value of the source i's AoI at the destination node, and $x_0(t)$ is the age of the status update in service. Further, the discrete state space is given by $\mathcal{Q} = \{0, 1, \cdots, N\}$, where $q(t) = 0$ indicates that the system is empty and hence the server is idle, and $q(t) = i, i \in 1 : N$, indicates that the server is serving a status update generated from the i-th source. Further, the continuous-time Markov chain modeling the system discrete state $q(t) \in \mathcal{Q}$ is depicted in Fig. 2. Table I presents the set of transitions $\mathcal{L}$

[1] We also analyze source-agnostic and source-aware preemptive in service queueing disciplines in the extended journal version of this paper [23].

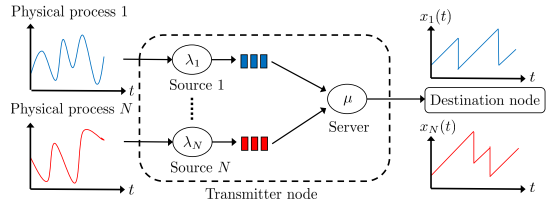

Fig. 1. An illustration of a multi-source status updating system.

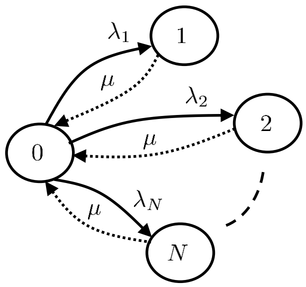

Fig. 2. Markov chain modeling $q(t)$ under the LCFS-NP queueing discipline.

TABLE I
TRANSITIONS OF THE LCFS-NP QUEUEING DISCIPLINE IN FIG. 2 $(2 \leq i \leq N)$.

l	$q_l \to q'_l$	$\lambda^{(l)}$	$\mathbf{x}\mathbf{A}_l$	$\mathbf{A}_l$
1	$0 \to 1$	λ_1	$[0\ x_1\ x_2\ \cdots\ x_N]$	$\begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$
2	$1 \to 0$	μ	$[0\ x_0\ x_2\ \cdots\ x_N]$	$\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$
$2i-1$	$0 \to i$	λ_i	$[0\ x_1\ x_2\ \cdots\ x_N]$	$[\mathbf{0}^T_{N+1}\ \mathbf{e}^T_2\ \mathbf{e}^T_3\ \cdots\ \mathbf{e}^T_{N+1}]$
$2i$	$i \to 0$	μ	$[0\ x_1\ x_2\ \cdots\ x_{i-1}\ x_0\ \cdots\ x_N]$	$[\mathbf{0}^T_{N+1}\ \mathbf{e}^T_2\ \mathbf{e}^T_3\ \cdots\ \mathbf{e}^T_i\ \mathbf{e}^T_1\ \cdots\ \mathbf{e}^T_{N+1}]$

and their impact on the values of both $q(t)$ and $\mathbf{x}(t)$. Before proceeding into evaluating $\bar{v}^{(\mathbf{s})}_{\bar{q},\mathsf{K}}$, $\forall \bar{q} \in \mathcal{Q}$, satisfying (24), we first describe the set of transitions as follows:

$l = 2i - 1$: This transition occurs if there is a new arriving status update of source i at the transmitter node when the server is idle. Note that the age of this new arriving status update is 0 and it does not have any impact on the AoI processes of the N sources at the destination. Thus, as a result of this transition, the age process x_0 in the updated age vector is reset to 0 (i.e., $[\mathbf{x}\mathbf{A}_{2i-1}]_1 = 0$) whereas the other age processes remain the same.

$l = 2i$: This transition occurs when the source i's status update in service is delivered to the destination. Thus, as a result of this transition, the source i's AoI is reset to the age of the status update received at the destination whereas the AoI values of the other sources do not change. In addition, since the system becomes empty after the occurrence of this transition, the first element of the age vector $\mathbf{x}(t)$ becomes irrelevant. Following the convention of [15], we set the value corresponding to such irrelevant elements in the updated age value to 0, and thus we observe that $[\mathbf{x}\mathbf{A}_{2i}]_1 = 0$.

Using Table I, we are now ready to derive $\{\bar{v}^{(\mathbf{s})}_{\bar{q},\mathsf{K}}\}_{\bar{q}\in\mathcal{Q}}$ satisfying (24), from which the stationary joint MGF of set

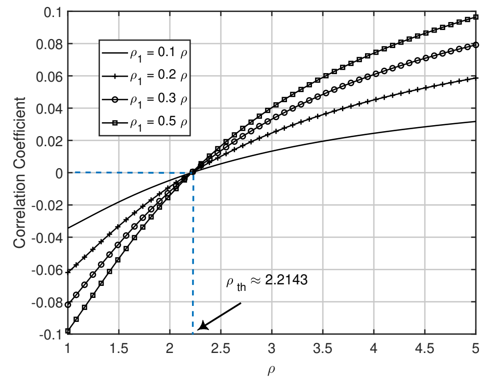

Fig. 3. Correlation coefficient of the two AoI processes $x_1(t)$ and $x_2(t)$ as a function of ρ under the LCFS-NP queueing discipline when $N = 2$.

K is characterized in the following theorem.

Theorem 2. *Under the LCFS-NP queueing discipline and for $\mathbf{s} = [s_{\mathsf{K}(1)}\ s_{\mathsf{K}(2)}\ \cdots\ s_{\mathsf{K}(|\mathsf{K}|)}]$, the stationary joint MGF of a set $\mathsf{K} \subseteq \{1, 2, \cdots, N\}$ of age processes is given by*

$$\overset{\text{NP}}{M}(\mathbf{s}) = \sum_{\bar{q}\in\mathcal{Q}} \bar{v}^{(\mathbf{s})}_{\bar{q},\mathsf{K}} = \mu^{|\mathsf{K}|} \left(\prod_{i=1}^{|\mathsf{K}|} \lambda_{\mathsf{K}(i)} \right) \left(\frac{\mu}{\lambda + \mu} \right) \times \left(1 + \frac{\lambda}{\mu - \sum_{j=1}^{|\mathsf{K}|} s_{\mathsf{K}(j)}} \right) \sum_{\mathsf{P}\in\mathcal{P}(\mathsf{K})} \frac{1}{C(\mathsf{P})}, \tag{26}$$

where $C(\mathsf{P}) = \prod_{i=1}^{|\mathsf{P}|} c_{\mathsf{P}(i:|\mathsf{P}|)}$ such that c_{Z} is defined for a set $\mathsf{Z} \subseteq \{1, 2, \cdots, N\}$ as

$$c_{\mathsf{Z}} = \left(\lambda - \sum_{j=1}^{|\mathsf{Z}|} s_{\mathsf{Z}(j)} \right) \left(\mu - \sum_{j=1}^{|\mathsf{Z}|} s_{\mathsf{Z}(j)} \right) - \mu \sum_{j=1, j\notin\mathsf{Z}}^{N} \lambda_j. \tag{27}$$

Proof: See Appendix A. ∎

Corollary 3. *Under the LCFS-NP queueing discipline, the marginal stationary MGF of source k's AoI is given by*

$$\overset{\text{NP}}{M}(\bar{s}_k) = \frac{\rho_k (1 + \rho - \bar{s}_k)}{(1 + \rho)(1 - \bar{s}_k)\left[(1 - \bar{s}_k)(\rho - \bar{s}_k) - \rho_{-k}\right]}, \tag{28}$$

where $k \in 1 : N$, $\rho_{-k} = \frac{\sum_{j=1,\, j\neq k}^{N} \lambda_j}{\mu}$ and $\bar{s}_k = \frac{s_k}{\mu}$.

Proof: This result follows from Theorem 2 by setting $\mathsf{K} = \{k\}$. ∎

Corollary 4. *For $k_1, k_2 \in 1 : N$, the stationary joint MGF of the two AoI processes $x_{k_1}(t)$ and $x_{k_2}(t)$ under the LCFS-NP queueing discipline is given by (29) [at the top of the next page], where $\rho_{-\{k_1,k_2\}} = \frac{\sum_{j=1,\, j\notin\{k_1,k_2\}}^{N} \lambda_j}{\mu}$.*

Proof: This result follows from Theorem 2 by setting $\mathsf{K} = \{k_1, k_2\}$. ∎

Proposition 1. *For $k_1, k_2 \in 1 : N$, the correlation coefficient of $x_{k_1}(t)$ and $x_{k_2}(t)$ under the LCFS-NP queueing discipline is given by (30) [at the top of the next page].*

Proof: The expression in (30) follows from the fact that the joint MGF in (29) can be used to obtain the sta-

$$\overset{\rm NP}{M}(\bar{s}_{k_1},\bar{s}_{k_2})=\frac{\rho_{k_1}\rho_{k_2}\big[1+\rho-(\bar{s}_{k_1}+\bar{s}_{k_2})\big]}{(1+\rho)\Big[\big[\rho-(\bar{s}_{k_1}+\bar{s}_{k_2})\big]\big[1-(\bar{s}_{k_1}+\bar{s}_{k_2})\big]-\rho_{-\{k_1,k_2\}}\Big]\Big[1-(\bar{s}_{k_1}+\bar{s}_{k_2})\Big]}\sum_{i\in\{k_1,k_2\}}\frac{1}{(1-\bar{s}_i)(\rho-\bar{s}_i)-\rho_{-i}}. \tag{29}$$

$$\overset{\rm NP}{\rm Cor}=\frac{\rho\left(\rho_{k_1}+\rho_{k_2}\right)\left[\rho_{k_1}\rho_{k_2}\left(\rho+2\right)-\rho_{-\{k_1,k_2\}}\left(1+\rho\right)^2\right]-2\rho_{k_1}\rho_{k_2}\left(1+\rho\right)^2}{\left(\rho_{k_1}+\rho_{k_2}\right)\prod_{i\in\{k_1,k_2\}}\sqrt{(1+\rho)^2[\rho^2+2\rho_{-i}+1]+\rho_i^2\rho(\rho+2)}}. \tag{30}$$

tionary (m_1,m_2)-th moment as: $\lim\limits_{t\to\infty}\mathbb{E}\big[x_{k_1}^{m_1}(t)x_{k_2}^{m_2}(t)\big]=\left.\frac{\partial^{m_1+m_2}\left[\overset{\rm NP}{M}(\bar{s}_{k_1},\bar{s}_{k_2})\right]}{\partial\bar{s}_{k_1}^{m_1}\partial\bar{s}_{k_2}^{m_2}}\right|_{\bar{s}_{k_1}=0,\bar{s}_{k_2}=0}$. ■

Corollary 5. *When $N=2$, the correlation coefficient of the two AoI processes $x_1(t)$ and $x_2(t)$ under the LCFS-NP queueing discipline is given by*

$$\overset{\rm NP}{\rm Cor}=\frac{\rho_1\rho_2\big[\rho^3-2(2\rho+1)\big]}{\rho\prod_{i=1}^2\sqrt{(1+\rho)^2[\rho^2+2\rho_{-i}+1]+\rho_i^2\rho(\rho+2)}}. \tag{31}$$

Proof: This result follows from Proposition 1 by setting $\rho=\rho_1+\rho_2$. ■

Remark 2. *From Proposition 1, we note that the two age processes $x_{k_1}(t)$ and $x_{k_2}(t)$ may be positively correlated under the LCFS-NP queueing discipline. Further, when $N=2$, one can deduce from Corollary 5 that there exists a threshold value $\rho_{\rm th}\approx 2.2143$ of ρ above which the two age processes $x_1(t)$ and $x_2(t)$ are positively correlated under the LCFS-NP queueing discipline, as shown in Fig. 3. This follows from the fact that the term $[\rho^3-2(2\rho+1)]$ in (31) is monotonically increasing for $\rho>\frac{2}{\sqrt{3}}$, and it equals to zero at $\rho\approx 2.2143$.*

V. Conclusion

In this paper, we developed an SHS-based general framework to facilitate the study of joint distributional properties of an arbitrary set of AoI/age processes in a network through the characterization of their joint stationary moments and MGFs. We demonstrated the generality of our framework by recovering several existing results as its special cases. An interesting insight obtained from our analysis is that the existence of the stationary joint first moments guarantees the existence of the stationary joint higher-order moments and MGFs. As an application of our framework, we obtained the stationary joint MGF for a multi-source updating system under the non-preemptive in service queueing discipline in closed-form. Our derived expressions demonstrated that for a two-source status updating system, there exists a threshold value of server utilization above which the two age processes are positively correlated.

The generality of our analytical framework stems from the fact that it allows one to understand the joint distributional properties for an arbitrary set of AoI processes in a broad range of system settings under any arbitrary queueing discipline. This, in turn, opens the door for the use of our framework in the future to investigate the stationary joint moments and MGFs of age processes for a variety of queuing disciplines/status updating systems that have only been analyzed in terms of the marginal moments and MGFs until now.

Appendix

A. Proof of Theorem 2

Using the set of transitions in Table I and (24) in Theorem 1, $\bar{v}_{0,\{k_1\}}^{([s_{k_1}])}$ can be expressed as

$$\left(\lambda-s_{k_1}\right)\bar{v}_{0,\{k_1\}}^{([s_{k_1}])}=\mu\left(\bar{v}_{k_1,\{0\}}^{([s_{k_1}])}+\sum_{\bar{q}=1,\bar{q}\notin\{k_1\}}^{N}\bar{v}_{\bar{q},\{k_1\}}^{([s_{k_1}])}\right), \tag{32}$$

where $\bar{v}_{k_1,\{0\}}^{([s_{k_1}])}$ and $\bar{v}_{\bar{q},\{k_1\}}^{([s_{k_1}])}$ are given by

$$\left(\mu-s_{k_1}\right)\bar{v}_{k_1,\{0\}}^{([s_{k_1}])}=\lambda_{k_1}\bar{v}_0^{(0)}, \tag{33}$$

$$\left(\mu-s_{k_1}\right)\bar{v}_{\bar{q},\{k_1\}}^{([s_{k_1}])}=\lambda_{\bar{q}}\bar{v}_{0,\{k_1\}}^{([s_{k_1}])}, \tag{34}$$

where $\bar{q}\in 1:N$. Substituting (33) and (34) into (32), we get

$$\bar{v}_{0,\{k_1\}}^{([s_{k_1}])}\overset{\rm (a)}{=}\frac{\mu\lambda_{k_1}\bar{v}_0^{(0)}}{c_{\{k_1\}}}, \tag{35}$$

where $k_1\in 1:N$ and step (a) follows from defining c_{Z} for a set $\mathsf{Z}\subseteq\{1,2,\cdots,N\}$ in (27) as

$$c_{\mathsf{Z}}=\left(\lambda-\sum_{j=1}^{|\mathsf{Z}|}s_{\mathsf{Z}(j)}\right)\left(\mu-\sum_{j=1}^{|\mathsf{Z}|}s_{\mathsf{Z}(j)}\right)-\mu\sum_{j=1,j\notin\mathsf{Z}}^{N}\lambda_j.$$

Now, using (35), one can evaluate $\bar{v}_{0,\{k_1,k_2\}}^{([s_{k_1}\ s_{k_2}])}$. In particular, from (24), $\bar{v}_{0,\{k_1,k_2\}}^{([s_{k_1}\ s_{k_2}])}$ can be expressed as

$$[\lambda-(s_{k_1}+s_{k_2})]\bar{v}_{0,\{k_1,k_2\}}^{([s_{k_1}\ s_{k_2}])}=\mu\Big[\bar{v}_{k_1,\{0,k_2\}}^{([s_{k_1}\ s_{k_2}])}+\bar{v}_{k_2,\{k_1,0\}}^{([s_{k_1}\ s_{k_2}])}+\sum_{\bar{q}=1,\bar{q}\notin\{k_1,k_2\}}^{N}\bar{v}_{\bar{q},\{k_1,k_2\}}^{([s_{k_1}\ s_{k_2}])}\Big], \tag{36}$$

where

$$[\mu-(s_{k_1}+s_{k_2})]\bar{v}_{k_1,\{0,k_2\}}^{([s_{k_1}\ s_{k_2}])}=\lambda_{k_1}\bar{v}_{0,\{k_2\}}^{([s_{k_2}])}, \tag{37}$$

$$[\mu-(s_{k_1}+s_{k_2})]\bar{v}_{k_2,\{k_1,0\}}^{([s_{k_1}\ s_{k_2}])}=\lambda_{k_2}\bar{v}_{0,\{k_1\}}^{([s_{k_1}])}, \tag{38}$$

$$[\mu-(s_{k_1}+s_{k_2})]\bar{v}_{\bar{q},\{k_1,k_2\}}^{([s_{k_1}\ s_{k_2}])}=\lambda_{\bar{q}}\bar{v}_{0,\{k_1,k_2\}}^{([s_{k_1}\ s_{k_2}])}, \tag{39}$$

where $\bar{q}\in 1:N$. Thus, $\bar{v}_{0,\{k_1,k_2\}}^{([s_{k_1}\ s_{k_2}])}$ can be rewritten as

$$\bar{v}_{0,\{k_1,k_2\}}^{([s_{k_1}\ s_{k_2}])}\overset{\rm (a)}{=}\frac{\mu\left(\lambda_{k_1}\bar{v}_{0,\{k_2\}}^{([s_{k_2}])}+\lambda_{k_2}\bar{v}_{0,\{k_1\}}^{([s_{k_1}])}\right)}{c_{\{k_1,k_2\}}},$$

$$\overset{(b)}{=} \mu^2 \lambda_{k_1} \lambda_{k_2} \bar{v}_0^{(0)} \left(\frac{1}{c_{\{k_1,k_2\}} c_{\{k_1\}}} + \frac{1}{c_{\{k_1,k_2\}} c_{\{k_2\}}} \right),$$

$$\overset{(c)}{=} \mu^2 \lambda_{k_1} \lambda_{k_2} \bar{v}_0^{(0)} \sum_{\mathsf{P} \in \mathcal{P}(\{k_1,k_2\})} \frac{1}{C(\mathsf{P})}, \tag{40}$$

where step (a) follows from substituting (37)-(39) into (36) along with the fact that $c_{\{k_1,k_2\}} = [\lambda - (s_{k_1} + s_{k_2})][\mu - (s_{k_1} + s_{k_2})] - \mu \sum_{j=1, j \notin \{k_1,k_2\}}^{N} \lambda_j$, step (b) follows from substituting (35) into (40), and step (c) follows from defining $C(\mathsf{P})$ as follows

$$C(\mathsf{P}) = \prod_{i=1}^{|\mathsf{P}|} c_{\mathsf{P}(i:|\mathsf{P}|)} =$$

$$\prod_{i=1}^{|\mathsf{P}|} \left[\left(\lambda - \sum_{j=i}^{|\mathsf{P}|} s_{\mathsf{P}(j)} \right) \left(\mu - \sum_{j=i}^{|\mathsf{P}|} s_{\mathsf{P}(j)} \right) - \mu \sum_{j=1, j \notin \mathsf{P}(i:|\mathsf{P}|)}^{N} \lambda_j \right]. \tag{41}$$

In order to clearly see how $\bar{v}_{0,\mathsf{K}}^{(\mathbf{s})}$ can be obtained for an arbitrary set $\mathsf{K} \subseteq \{1, 2, \cdots, N\}$, where $\mathbf{s} = [s_{\mathsf{K}(1)} \ s_{\mathsf{K}(2)} \ \cdots \ s_{\mathsf{K}(|\mathsf{K}|)}]$, it will be useful to further derive $\bar{v}_{0,\{k_1,k_2,k_3\}}^{([s_{k_1} \ s_{k_2} \ s_{k_3}])}$ using (40). From (24), $\bar{v}_{0,\{k_1,k_2,k_3\}}^{([s_{k_1} \ s_{k_2} \ s_{k_3}])}$ can be expressed as: $\bar{v}_{0,\{k_1,k_2,k_3\}}^{([s_{k_1} \ s_{k_2} \ s_{k_3}])} =$

$$\frac{\mu \left(\lambda_{k_1} \bar{v}_{0,\{k_2,k_3\}}^{([s_{k_2} \ s_{k_3}])} + \lambda_{k_2} \bar{v}_{0,\{k_1,k_3\}}^{([s_{k_1} \ s_{k_3}])} + \lambda_{k_3} \bar{v}_{0,\{k_1,k_2\}}^{([s_{k_1} \ s_{k_2}])} \right)}{c_{\{k_1,k_2,k_3\}}},$$

$$\overset{(a)}{=} \mu^3 \lambda_{k_1} \lambda_{k_2} \lambda_{k_3} \bar{v}_0^{(0)} \sum_{\mathsf{P} \in \mathcal{P}(\{k_1,k_2,k_3\})} \frac{1}{C(\mathsf{P})}, \tag{42}$$

where step (a) follows from substituting (40) into (42) along with the fact that

$$\frac{1}{c_{\{k_1,k_2,k_3\}}} \left(\sum_{\mathsf{P} \in \mathcal{P}(\{k_2,k_3\})} \frac{1}{C(\mathsf{P})} + \sum_{\mathsf{P} \in \mathcal{P}(\{k_1,k_3\})} \frac{1}{C(\mathsf{P})} + \sum_{\mathsf{P} \in \mathcal{P}(\{k_1,k_2\})} \frac{1}{C(\mathsf{P})} \right) = \sum_{\mathsf{P} \in \mathcal{P}(\{k_1,k_2,k_3\})} \frac{1}{C(\mathsf{P})}.$$

By inspecting the expressions of $\bar{v}_{0,\{k_1\}}^{([s_{k_1}])}$, $\bar{v}_{0,\{k_1,k_2\}}^{([s_{k_1} \ s_{k_2}])}$ and $\bar{v}_{0,\{k_1,k_2,k_3\}}^{([s_{k_1} \ s_{k_2} \ s_{k_3}])}$ in (35), (40) and (42), respectively, one can see that repeated application of (24) gives $\bar{v}_{0,\mathsf{K}}^{(\mathbf{s})}$ for an arbitrary set $\mathsf{K} \subseteq \{1, 2, \cdots, N\}$ as

$$\bar{v}_{0,\mathsf{K}}^{(\mathbf{s})} = \mu^{|\mathsf{K}|} \left(\prod_{i=1}^{|\mathsf{K}|} \lambda_{\mathsf{K}(i)} \right) \bar{v}_0^{(0)} \sum_{\mathsf{P} \in \mathcal{P}(\mathsf{K})} \frac{1}{C(\mathsf{P})}. \tag{43}$$

Thus, the stationary joint MGF of a set $\mathsf{K} \subseteq \{1, 2, \cdots, N\}$ of age processes is given by

$$\overset{\mathrm{NP}}{M}(\mathbf{s}) = \sum_{\bar{q} \in \mathcal{Q}} \bar{v}_{\bar{q},\mathsf{K}}^{(\mathbf{s})} \overset{(a)}{=} \left(1 + \frac{\lambda}{\mu - \sum_{j=1}^{|\mathsf{K}|} s_{\mathsf{K}(j)}} \right) \bar{v}_{0,\mathsf{K}}^{(\mathbf{s})},$$

where step (a) follows from expressing $\bar{v}_{\bar{q},\mathsf{K}}^{(\mathbf{s})}$, $\bar{q} \in \mathcal{Q}/\{0\}$, as a function of $\bar{v}_{0,\mathsf{K}}^{(\mathbf{s})}$ using (24). The final expression of $\overset{\mathrm{NP}}{M}(\mathbf{s})$ in (26) is obtained from (43) along with noting that $\bar{v}_0^{(0)} = \mu/(\lambda + \mu)$. This completes the proof. ∎

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