Moment Generating Function of the Age of Gossip in Networks

Mohamed A. Abd-Elmagid and Harpreet S. Dhillon

***Abstract*—We study a general setting of gossip networks in which a source node forwards its measurements (in the form of status updates) about some observed physical process to a set of monitoring nodes according to independent Poisson processes. Further, each monitoring node sends status updates about its information status (about the process observed by the source) to the other monitoring nodes according to independent Poisson processes. We quantify the freshness of information available at each monitoring node in terms of Age of Information (AoI). While this setting has been analyzed in a handful of prior works, the focus has been on characterizing the average (i.e., marginal first moment) of each age process. In contrast, our analysis is focused on understanding the distributional properties of the AoI processes through the characterization of their stationary marginal and joint moment generating functions (MGFs). In particular, for the serially and parallelly-connected gossip network topologies, we derive closed-form expressions for marginal/joint higher-order statistics of age processes, such as the variance of each age process and the correlation coefficients between all possible pairwise combinations of age processes. Our analytical results demonstrate the importance of incorporating the higher-order moments of age processes in the implementation/optimization of age-aware gossip networks rather than just relying on their average values.**

I. Introduction

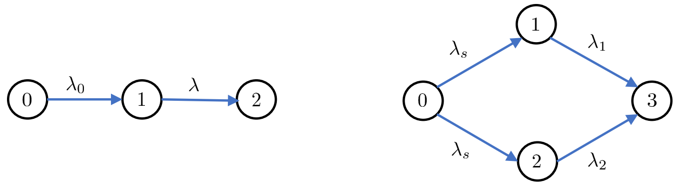
Timely delivery of status updates is crucial for enabling the operation of many emerging Internet of Things (IoT)-based real-time status updating systems [1]. The concept of AoI was introduced in [2] to quantify the freshness of information available at some node about a physical process, as a result of status update receptions over time. In particular, for a single-source of information queueing-theoretic model in which status updates about a single physical process are generated randomly at a *transmitter node* and are then sent to a *destination node* through a single server, the AoI at the destination was defined in [2] as the following random process: $x(t) = t - u(t)$, where $u(t)$ is the generation time instant of the latest status update received at the destination by time t. The authors of [3], [4] developed a stochastic hybrid system (SHS)-based framework to analyze the marginal distributional properties of each AoI process (in a network with multiple AoI processes) through the characterization of its stationary marginal moments and MGF. Further, by using the notion of tensors, the authors of [5] and [6] generalized the analysis of [3] and [4], and developed an SHS-based general framework that facilitates the analysis of the joint distributional properties of an arbitrary set of AoI processes in a network through the characterization of their stationary joint moments and MGFs. Note that the framework of [4] has been applied to characterize the marginal distributional properties of AoI under a variety of system settings/queueing disciplines [7]–[10] (see [11] for a comprehensive book and [12] for a recent survey).

The frameworks of [3]–[6] are not applicable to the age analysis in classes of status updating systems that cannot be modeled by an SHS with *linear reset maps*. A popular class of such systems is the gossip-based status updating systems where each node in the network randomly shares its information status over time with the other nodes [13], [14]. As a result, there have been a handful of recent efforts developing new SHS-based methods that are suitable for the age analysis in such gossip networks [15]–[20]. However, these analyses have been limited to the characterization of the stationary marginal first moment (average value) of each age process in the network. In this paper, our analysis is focused on the characterization of marginal/joint higher-order statistics of age processes, such as the variance of each age process and the correlation coefficients between all possible pairwise combinations of age processes. In particular, we apply our SHS-based methods developed in [21] (the extended journal version of this paper) to derive closed-form expressions for marginal/joint higher-order statistics of age processes in serially-connected and parallel-connected gossip network topologies. We further characterize the structural properties of these higher-order statistics in terms of their convexity and monotonicity with respect to the status updating rates, and provide asymptotic results showing their behaviors when each of the status updating rates becomes small/large. A key insight drawn from our analysis is that it is crucial to incorporate the higher-order moments of age processes in the implementation of age-aware gossip networks rather than just relying on their average values.

II. System Model

A source node (referred to as node 0) provides its measurements about some observed physical process to a set of nodes $\mathcal{N} = \{1, 2, \cdots, N\}$ in the form of status updates. In particular, all the nodes in $\mathcal{N}$ are tracking the age of the process observed by the source, and the status updates sent by node 0 to node $j \in \mathcal{N}$ are assumed to follow an independent Poisson process with rate λ_{0j}. Besides, node $i \in \mathcal{N}$ sends updates about its status of information (about the process observed by the source) to each node $j \in \mathcal{N} \setminus \{i\}$ according to an independent Poisson process with rate λ_{ij}. When $\lambda_{ij} > 0$, we say that nodes i and j are connected to each other. The freshness of status of information available at each node is quantified in terms of AoI. Let $x_i(t)$ denote the AoI process

The authors are with Wireless@VT, Department of ECE, Virginia Tech, Blacksburg, VA. Email: {maelaziz, hdhillon}@vt.edu. The support of the U.S. NSF (Grants CNS-1814477 and CNS-1923807) is gratefully acknowledged.



(a) (b)

Fig. 1. (a) A serially-connected network setting, (b) a parallelly-connected network setting.

(or equivalently the age process) at node $i \in \mathcal{N}$. Assuming that node 0 always maintains a fresh status of information about the observed physical process, the age/AoI at node $j \in \mathcal{N}$ is reset to zero whenever it receives a status update from node 0. Further, when node $j \in \mathcal{N}$ receives a status update from node $i \in \mathcal{N}\backslash\{j\}$ at time t, its age $x_j(t)$ is reset to the age of node i only if $x_i(t)$ is smaller than $x_j(t)$. To summarize, when node $j \in \mathcal{N}$ receives a status update from node $i \in \{0\} \cup \mathcal{N}$, the age at node $k \in \mathcal{N}$ is updated as

$$x'_k(t) = \begin{cases} 0, & \text{if } \ i = 0 \ \text{ and } \ k = j, \\ \min\left[x_j(t), x_i(t)\right], & \text{if } \ i \in \mathcal{N} \ \text{ and } \ k = j, \\ x_k(t), & \text{otherwise.} \end{cases} \quad (1)$$

For an arbitrary set $S \subseteq \mathcal{N}$, define $x_S(t) = \min_{i \in S} x_i(t)$ as the age/AoI process associated with S (or simply the age/AoI of S). The stationary marginal MGF and stationary m-th moment of the AoI process $x_S(t)$ are denoted by $\bar{v}_S^{(n)}$ and $\bar{v}_S^{(m)}$, respectively. Note that $\bar{v}_S^{(1)}$ may generally refer to $\bar{v}_S^{(n)}|_{n=1}$ or $\bar{v}_S^{(m)}|_{m=1}$. To eliminate this conflict, the convention that $\bar{v}_S^{(i)}$, for an integer i, refers to $\bar{v}_S^{(m)}$ at $m = i$ is maintained here.

III. MGF Analysis of Age in Gossip Networks

In the extended journal version of this paper [21], we develop SHS-based methods that allow the characterization of the stationary marginal and joint MGFs of the age processes in the general setting of gossip networks described in Section II. In this paper, we apply the methods developed in [21] to understand the distributional properties of age processes in the two canonical gossip network topologies depicted in Fig. 1, i.e., the serially and parallelly-connected network topologies. In particular, we first derive closed form expressions for the stationary marginal and joint MGFs of the AoI processes in each network topology. We then use the derived MGFs expressions to obtain marginal/joint higher-order statistics of age processes.

A. Serially-Connected Networks

Theorem 1. *For the serially-connected network in Fig. 1a, the stationary marginal MGFs of the AoI processes at nodes 1 and 2 are respectively given by*

$$\bar{v}_{\{1\}}^{(n)} = \frac{\lambda_0}{\lambda_0 - n}, \quad (2)$$

$$\bar{v}_{\{2\}}^{(n)} = \frac{\lambda_0 \lambda}{(\lambda_0 - n)(\lambda - n)}. \quad (3)$$

Additionally, the stationary joint MGF of the two AoI processes at nodes 1 and 2 is given by

$$\bar{v}_{\{2\},\{1\}}^{(n_1,n_2)} = \frac{\lambda_0 \lambda}{\lambda_0 + \lambda - (n_1 + n_2)} \Big(\frac{\lambda_0}{(\lambda_0 - n_1)(\lambda - n_1)} + \frac{1}{\lambda_0 - (n_1 + n_2)} \Big). \quad (4)$$

Proof: These results are obtained by applying [21, Theorem 1] and [21, Theorem 2] to the serially-connected network in Fig. 1a. The detailed proof is omitted due to space limitations, and can be found in [21, Appendix C]. ■

Proposition 1. *For the serially-connected network in Fig. 1a, the first moment, second moment, and variance of the AoI process at each node are given by*

$$\bar{v}_{\{1\}}^{(1)} = \lambda_0^{-1}, \ \ \bar{v}_{\{1\}}^{(2)} = 2\lambda_0^{-2}, \ \ \mathrm{var}\left[x_1\left(t\right)\right] = \lambda_0^{-2}, \quad (5)$$

$$\bar{v}_{\{2\}}^{(1)} = \frac{1}{\lambda_0} + \frac{1}{\lambda}, \ \ \bar{v}_{\{2\}}^{(2)} = 2\left(\frac{1}{\lambda_0^2} + \frac{1}{\lambda_0\lambda} + \frac{1}{\lambda^2}\right),$$
$$\mathrm{var}\left[x_2\left(t\right)\right] = \frac{1}{\lambda_0^2} + \frac{1}{\lambda^2}. \quad (6)$$

Further, the correlation coefficient between the AoI processes at nodes 1 and 2 can be expressed as

$$\mathrm{cor}\left[x_1(t), x_2(t)\right] = \frac{\lambda^2}{(\lambda_0 + \lambda)\sqrt{\lambda_0^2 + \lambda^2}}. \quad (7)$$

Proof: These results are obtained using the derivatives of the MGF expressions derived in Theorem 1. ■

Remark 1. *Note that the expressions of the stationary marginal MGFs in Theorem 1 and the stationary marginal moments in Proposition 1 match their corresponding expressions for preemptive line networks analyzed in [4].*

Remark 2. *Note that the stationary moments and variance of the age process at node 1 in (5) are univariate functions of λ_0. This happens since node 1 is directly connected to node 0. This argument will also apply to the expressions derived for the age processes at nodes 1 and 2 in the parallelly-connected network in Fig. 1b.*

Remark 3. *Note that the stationary moments and variance of the age process at node 2 in (6) are invariant to exchanging λ and λ_0. These quantities are also jointly convex functions in (λ_0, λ), where the minimum value (zero) of each function is achieved at $\lambda_0 = \lambda = \infty$. Further, for a given λ or λ_0, each quantity in (6) is a monotonically non-increasing function with respect to λ_0 or λ. This can also be observed from Fig. 2.*

Remark 4. *For a given λ, $\mathrm{cor}[x_1(t), x_2(t)]$ in (7) monotonically decreases as a function of λ_0 form $\lim_{\lambda_0 \to 0} \mathrm{cor}[x_1(t), x_2(t)] = 1$ until it approaches $\lim_{\lambda_0 \to \infty} \mathrm{cor}[x_1(t), x_2(t)] = 0$. On the other hand, for a given λ_0, $\mathrm{cor}[x_1(t), x_2(t)]$ monotonically increases as* *a function of λ form $\lim_{\lambda\to 0}\mathrm{cor}[x_1(t),x_2(t)] = 0$ until it approaches $\lim_{\lambda\to\infty}\mathrm{cor}[x_1(t),x_2(t)] = 1$. This can also be observed from Fig. 3.*

B. Parallelly-connected Networks

Theorem 2. *For the parallelly-connected network in Fig. 1b, the stationary marginal MGFs of the AoI processes at nodes 1, 2 and 3 are given by (8) and (9) [at the top of the next page]. Additionally, the stationary joint MGF of the two AoI processes at nodes 1 and 3 is given by (10) [at the top of the next page].*

Proof: These results are obtained by applying [21, Theorem 1] and [21, Theorem 2] to the parallelly-connected network in Fig. 1b. The detailed proof is omitted due to space limitations, and can be found in [21, Appendix E]. ■

Proposition 2. *For the parallelly-connected network in Fig. 1b, the first moment, second moment, and variance of the AoI process at each node are given by*

$$\bar{v}^{(1)}_{\{1\}} = \bar{v}^{(1)}_{\{2\}} = \lambda_s^{-1},\ \ \bar{v}^{(2)}_{\{1\}} = \bar{v}^{(2)}_{\{2\}} = 2\lambda_s^{-2},$$
$$\mathrm{var}\left[x_1\left(t\right)\right] = \mathrm{var}\left[x_2\left(t\right)\right] = \lambda_s^{-2}, \tag{11}$$

$$\bar{v}^{(1)}_{\{3\}} = \frac{\beta}{2\lambda_s\left(\lambda_s+\lambda_1\right)\left(\lambda_s+\lambda_2\right)\left(\lambda_1+\lambda_2\right)}, \tag{12}$$

$$\bar{v}^{(2)}_{\{3\}} = \frac{\sum_{i=0}^{6}\gamma_i\lambda_s^i}{2\lambda_s^2\left(\lambda_1+\lambda_2\right)^2\left(\lambda_s+\lambda_1\right)^2\left(\lambda_s+\lambda_2\right)^2}, \tag{13}$$

$$\mathrm{var}\left[x_3(t)\right] = \frac{\sum_{i=0}^{6}\eta_i\lambda_s^i}{4\lambda_s^2\left(\lambda_1+\lambda_2\right)^2\left(\lambda_s+\lambda_1\right)^2\left(\lambda_s+\lambda_2\right)^2}, \tag{14}$$

where

$$\beta = 2\lambda_s\left(\lambda_s+\lambda_1\right)\left(\lambda_s+\lambda_2\right)+\lambda_1\left(2\lambda_s+\lambda_2\right)\left(\lambda_s+\lambda_1\right)+$$
$$\lambda_2\left(2\lambda_s+\lambda_1\right)\left(\lambda_s+\lambda_2\right),$$
$$\gamma_6 = 4,\ \gamma_5 = 12\left(\lambda_1+\lambda_2\right),\gamma_4 = 4\left[4\left(\lambda_1+\lambda_2\right)^2+\lambda_1\lambda_2\right],$$
$$\gamma_3 = 12\left(\lambda_1+\lambda_2\right)^3,\ \gamma_2 = \left(\lambda_1+\lambda_2\right)^2\left[4\left(\lambda_1+\lambda_2\right)^2+\lambda_1\lambda_2\right],$$
$$\gamma_1 = 3\lambda_1\lambda_2\left(\lambda_1+\lambda_2\right)^3,\gamma_0 = \lambda_1^2\lambda_2^2\left(\lambda_1+\lambda_2\right)^2,\ \eta_6 = 4,$$
$$\eta_5 = 8\left(\lambda_1+\lambda_2\right),\ \eta_4 = 8\left[\left(\lambda_1+\lambda_2\right)^2+\lambda_1\lambda_2\right],$$
$$\eta_3 = 4\left(\lambda_1+\lambda_2\right)\left(2\lambda_1^2+3\lambda_1\lambda_2+2\lambda_2^2\right),$$
$$\eta_2 = 2\left(\lambda_1+\lambda_2\right)^2\left(2\lambda_1^2+\lambda_1\lambda_2+2\lambda_2^2\right),$$
$$\eta_1 = 2\lambda_1\lambda_2\left(\lambda_1+\lambda_2\right)^3,\eta_0 = \lambda_1^2\lambda_2^2\left(\lambda_1+\lambda_2\right)^2.$$

Further, the correlation coefficient between the AoI processes at nodes 1 and 3 can be expressed as: $\mathrm{cor}\left[x_1(t),x_3(t)\right] =$

$$\frac{\lambda_1\left(\lambda_1+\lambda_2\right)}{2\left(\lambda_s+\lambda_1+\lambda_2\right)\left(2\lambda_s+\lambda_1\right)\left(\lambda_s+\lambda_2\right)\sqrt{\sum_{i=0}^{6}\delta_i\lambda_s^i}}\times$$
$$\Big[8\lambda_s^4+\lambda_s^3\left(12\lambda_1+7\lambda_2\right)+2\lambda_s^2\left(\lambda_1+2\lambda_2\right)\left(2\lambda_1+\lambda_2\right)+$$
$$\lambda_s\lambda_2\left(3\lambda_1^2+5\lambda_1\lambda_2+\lambda_2^2\right)+\lambda_1\lambda_2^2\left(\lambda_1+\lambda_2\right)\Big], \tag{15}$$

where

$$\delta_6 = 4,\ \delta_5 = 8\left(\lambda_1+\lambda_2\right),\ \delta_4 = 8\left[\left(\lambda_1+\lambda_2\right)^2+\lambda_1\lambda_2\right],$$
$$\delta_3 = 4\left(\lambda_1+\lambda_2\right)\left(2\lambda_1^2+3\lambda_1\lambda_2+2\lambda_2^2\right),$$
$$\delta_2 = 2\left(\lambda_1+\lambda_2\right)^2\left(2\lambda_1^2+\lambda_1\lambda_2+2\lambda_2^2\right),$$
$$\delta_1 = 2\lambda_1\lambda_2\left(\lambda_1+\lambda_2\right)^3,\delta_0 = \lambda_1^2\lambda_2^2\left(\lambda_1+\lambda_2\right)^2.$$

Remark 5. *When λ_1 or λ_2 is zero, the parallelly-connected network reduces to the serially-connected network with a single path from node 0 to node 3. Thus, in that case, the stationary moments and variance of the age process at node 3 reduce to the corresponding expressions associated with the age process at node 2 in the serially-connected network such that λ_0 and λ are replaced by λ_s and λ_1 or λ_2. On the other hand, when λ_1 and λ_2 approach ∞, we have: $\lim_{\lambda_1\to\infty,\lambda_2\to\infty}\bar{v}^{(1)}_{\{3\}} = \frac{1}{2\lambda_s}$, $\lim_{\lambda_1\to\infty,\lambda_2\to\infty}\bar{v}^{(2)}_{\{3\}} = \frac{1}{2\lambda_s^2}$, and $\lim_{\lambda_1\to\infty,\lambda_2\to\infty}\mathrm{var}[x_3(t)] = \frac{1}{4\lambda_s^2}$. Note that the stationary moments and variance of $x_3(t)$ reduce to the ones associated with $x_{\{1,2\}}(t)$.*

Remark 6. *Note that the stationary moments and variance of the age process at node 3 in (12)-(14) are invariant to exchanging λ_1 and λ_2. Further, for a given (λ_s,λ_2), (λ_s,λ_1) or (λ_1,λ_2), each quantity in (12)-(14) is a monotonically non-increasing function with respect to λ_1, λ_2 or λ_s. This can also be observed from Fig. 2.*

Remark 7. *For the same status updating rate out of node 0 (i.e., $\lambda_0 = 2\lambda_s$) and $\lambda = \lambda_1 = \lambda_2$, one can compare the achievable age performance at node 3 in the parallelly-connected network with the achievable age performance at node 2 in the serially-connected network using Propositions 1 and 2 as follows*

$$\bar{v}^{(1)}_{\{2\}} - \bar{v}^{(1)}_{\{3\}} = \frac{\lambda_0}{2\lambda\left(\lambda_0+2\lambda\right)}, \tag{16}$$

$$\bar{v}^{(2)}_{\{2\}} - \bar{v}^{(2)}_{\{3\}} = \frac{3\lambda_0^2+4\left(\lambda^2+2\lambda_0\lambda\right)}{2\lambda^2\left(\lambda_0+2\lambda\right)^2}, \tag{17}$$

$$\mathrm{var}[x_2(t)] - \mathrm{var}[x_3(t)] = \frac{3\lambda_0\left(\lambda_0+4\lambda\right)}{4\lambda^2\left(\lambda_0+2\lambda\right)^2}. \tag{18}$$

By inspecting (16)-(18), one can see that these are positive quantities for any choice of values of (λ_0,λ). This certainly indicates that node 3 in the parallelly-connected network achieves a better age performance than the one achievable by node 2 in the serially-connected network. The improvement in the age performance at node 3 results from the existence of two status updating paths from node 0 to node 3, as opposed to only a single path from node 0 to node 2 in the serially-connected network. Further, each quantity in (16)-(18) is a monotonically decreasing function of λ for a given λ_0 such that its value approaches zero as $\lambda\to\infty$. This can also be observed from Fig. 2.

$$\bar{v}_{\{1\}}^{(n)} = \bar{v}_{\{2\}}^{(n)} = \frac{\lambda_s}{\lambda_s - n}, \tag{8}$$

$$\bar{v}_{\{3\}}^{(n)} = \frac{\lambda_s(2\lambda_s - n)\Big[\lambda_1\left(\lambda_s + \lambda_1 - n\right) + \lambda_2\left(\lambda_s + \lambda_2 - n\right)\Big] + 2\lambda_s\lambda_1\lambda_2\left(2\lambda_s + \lambda_1 + \lambda_2 - 2n\right)}{\left(2\lambda_s - n\right)\left(\lambda_1 + \lambda_2 - n\right)\left(\lambda_s + \lambda_1 - n\right)\left(\lambda_s + \lambda_2 - n\right)}. \tag{9}$$

$$\begin{aligned}\bar{v}_{\{3\},\{1\}}^{(n_1,n_2)} =& \frac{\sum_{i=1}^{4}\alpha_i(n_1,n_2)}{\left[\lambda_s + \lambda_1 + \lambda_2 - (n_1+n_2)\right]\left[2\lambda_s + \lambda_1 - (n_1+n_2)\right]\left[2\lambda_s - (n_1+n_2)\right]\left[\lambda_s + \lambda_2 - (n_1+n_2)\right]} \\ &\times \frac{1}{\left(\lambda_s - n_2\right)\left(\lambda_1 + \lambda_2 - n_1\right)\left(2\lambda_s - n_1\right)\left(\lambda_s + \lambda_2 - n_1\right)\left(\lambda_s + \lambda_1 - n_1\right)},\end{aligned} \tag{10}$$

where

$$\begin{aligned}\alpha_1(n_1,n_2) =& \lambda_s^2\left(\lambda_s - n_2\right)\left[\lambda_s + \lambda_2 - (n_1+n_2)\right]\left[2\lambda_s + \lambda_1 - (n_1+n_2)\right]\left[2\lambda_s - (n_1+n_2)\right] \\ &\times \Big[\left(2\lambda_2 - n_1\right)\big[\lambda_1\left(\lambda_s + \lambda_1 - n_1\right) + \lambda_2\left(\lambda_s + \lambda_2 - n_1\right)\big] + 2\lambda_1\lambda_2\left(2\lambda_s + \lambda_1 + \lambda_2 - 2n_1\right)\Big],\end{aligned}$$

$$\begin{aligned}\alpha_2(n_1,n_2) = \lambda_s^2\lambda_2\left(\lambda_1 + \lambda_2 - n_1\right)\left(2\lambda_s - n_1\right)\left(\lambda_s + \lambda_1 - n_1\right)\left(\lambda_s + \lambda_2 - n_1\right)\left[2\lambda_s + \lambda_1 - (n_1+n_2)\right] \\ \times\left[\lambda_s + \lambda_1 + \lambda_2 - (n_1+n_2)\right],\end{aligned}$$

$$\alpha_3(n_1,n_2) = \lambda_s^2\lambda_2\left(\lambda_1 + \lambda_2 - n_1\right)\left(\lambda_s + \lambda_2 - n_1\right)\left(2\lambda_s + 2\lambda_1 - n_1\right)\left(\lambda_s - n_2\right)\left[\lambda_s + \lambda_2 - (n_1+n_2)\right]\left[2\lambda_s - (n_1+n_2)\right],$$

$$\begin{aligned}\alpha_4(n_1,n_2) =& \lambda_s\lambda_1\left(\lambda_s - n_2\right)\left(\lambda_1 + \lambda_2 - n_1\right)\left(2\lambda_s - n_1\right)\left(\lambda_s + \lambda_2 - n_1\right)\left(\lambda_s + \lambda_1 - n_1\right) \\ &\times \Big[\big[2\lambda_s + \lambda_1 - (n_1+n_2)\big]\big[2\lambda_s + \lambda_2 - (n_1+n_2)\big] + \lambda_2\big[\lambda_s + \lambda_2 - (n_1+n_2)\big]\Big].\end{aligned}$$

Remark 8. *Due to symmetry in the configuration of the parallelly-connected network, note that the correlation coefficient between $x_2(t)$ and $x_3(t)$ (i.e., $\mathrm{cor}[x_2(t), x_3(t)]$) can be obtained by replacing λ_1 and λ_2 with λ_2 and λ_1, respectively, in (15). Further, for a given (λ_1, λ_2), $\mathrm{cor}[x_1(t), x_3(t)]$ monotonically decreases as a function of λ_s from $\lim_{\lambda_s \to 0} \mathrm{cor}[x_1(t), x_3(t)] = \frac{1}{2}$ until it approaches $\lim_{\lambda_s \to \infty} \mathrm{cor}[x_1(t), x_3(t)] = 0$. On the other hand, for a given (λ_s, λ_2), $\mathrm{cor}[x_1(t), x_3(t)]$ monotonically increases as a function of λ_1 from $\lim_{\lambda_1 \to 0} \mathrm{cor}[x_1(t), x_3(t)] = 0$ until it approaches $\lim_{\lambda_1 \to \infty} \mathrm{cor}[x_1(t), x_3(t)] = \frac{4\lambda_s^2 + 3\lambda_s\lambda_2 + \lambda_2^2}{2(\lambda_s+\lambda_2)\sqrt{4\lambda_s^2 + 2\lambda_s\lambda_2 + \lambda_2^2}}$. Finally, for a given (λ_s, λ_1), one can deduce the following asymptotic results: $\lim_{\lambda_2 \to 0} \mathrm{cor}[x_1(t), x_3(t)] = \frac{\lambda_1^2}{(\lambda_s+\lambda_1)\sqrt{\lambda_s^2 + \lambda_1^2}}$ and $\lim_{\lambda_2 \to \infty} \mathrm{cor}[x_1(t), x_3(t)] = \frac{\lambda_1(\lambda_s+\lambda_1)}{2(2\lambda_s+\lambda_1)\sqrt{4\lambda_s^2 + 2\lambda_s\lambda_1 + \lambda_1^2}}$. Clearly, when $\lambda_2 = 0$, there will be only a single status updating path from node 0 to node 3 (through node 1), and hence we observe that $\mathrm{cor}[x_1(t), x_3(t)]$ reduces to the same expression of $\mathrm{cor}[x_1(t), x_2(t)]$ in (7) for the serially-connected network after replacing λ_0 and λ with λ_s and λ_1, respectively. Some of the above insights can also be visualized from Fig. 3.*

Remark 9. *From Propositions 1 and 2, one can see that the standard deviation of $x_1(t)$ (i.e., $\sqrt{\mathrm{var}[x_1(t)]}$) is equal to its average value $\bar{v}_{\{1\}}^{(1)}$. Additionally, the standard deviations of the age processes at the other nodes are relatively large with respect to their average values (which is also demonstrated numerically in Figs. 4 and 5). This key insight promotes the importance of incorporating the higher-order moments of age processes in the implementation/optimization of age-aware gossip networks rather than just relying on the average values of the age processes (as has been done in the existing literature so far). This insight also demonstrates the need of the development of Theorems 1 and 2 in [21], which allow the characterization of the marginal/joint MGFs of different age processes in the network that can then be used to evaluate the marginal/joint higher-order moments.*

IV. Conclusion

In this paper, we derived the stationary marginal and joint MGFs in serially and parallelly-connected gossip network topologies. Using the derived MGF expressions, we obtained closed-form expressions for the following quantities: i) the stationary marginal first and second moments of each age process, ii) the variance of each age process, and iii) the correlation coefficients between all possible pairwise combinations of the age processes. We further characterized the structural properties of these quantities in terms of their convexity and monotonicity with respect to the status updating rates, and provided asymptotic results showing their behaviors when each of the status updating rates becomes small/large. Our

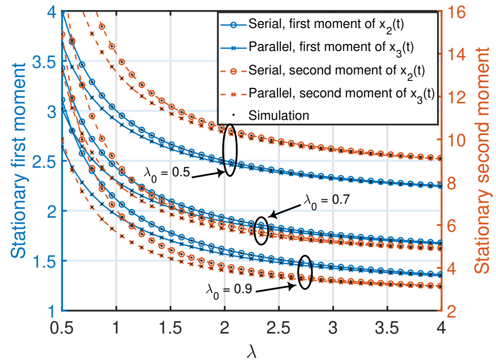


Fig. 2. Stationary first and second moments of age processes in the serially and parallelly-connected network topologies. We set $\lambda_s = 0.5\lambda_0$ and $\lambda = \lambda_1 = \lambda_2$.

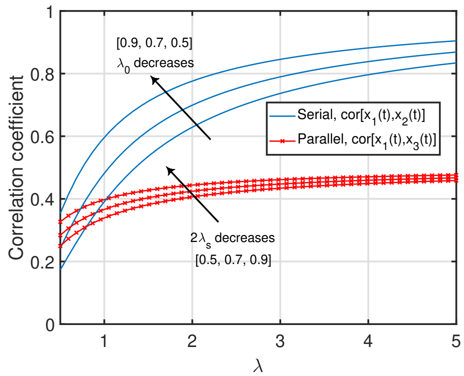


Fig. 3. Correlation coefficients between age processes in the serially and parallelly-connected network topologies.

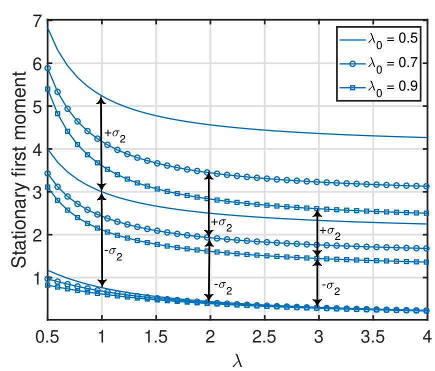


Fig. 4. Variance of $x_2(t)$ in the serially-connected network setting. We denote the standard deviation of $x_2(t)$ by σ_2.

analytical findings highlight the importance of incorporating the higher-order moments of age processes in the implementation/optimization of age-aware gossip networks rather than just relying on their average values (as has been done in the existing literature so far).

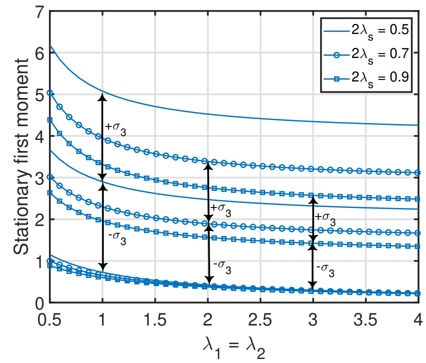


Fig. 5. Variance of $x_3(t)$ in the parallelly-connected network setting. We denote the standard deviation of $x_3(t)$ by σ_3.

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