Stochastic Geometry Analysis of Localizability in Vision-Based Geolocation Systems

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***Abstract*—This paper employs stochastic geometry tools to rigorously analyze the performance of vision-based geolocation systems. Despite significant algorithmic advances in vision-based positioning, its mathematical underpinnings have not been explored in depth, which is the main objective of this paper. Due to limitations in sensor resolution, the level of detail in prior information, and computational resources, we may not be able to differentiate between landmarks that are similar in appearance, such as trees, lampposts, and bus stops. While one cannot accurately determine the absolute target position using a single non-unique landmark, it is possible to obtain an approximate position fix if the target can see multiple landmarks whose geometric placement on the map is unique. Modeling the locations of these indistinguishable landmarks as a Poisson point process (PPP), we develop a fundamentally new approach to analyze *localizability* in this setting. We define localizability as the ability of the target to determine the correct set of indistinguishable landmarks around it from the visual information. Our analysis reveals that the *localizability probability* approaches one when the landmark intensity tends to infinity, which means that error-free localization is achievable in this limiting regime.**

***Index Terms*—Vision-based localization, localizability, stochastic geometry, Poisson point process.**

I. Introduction

Vision-based positioning systems utilize visual information obtained from a variety of vision sensors, such as cameras, Lidar, and radar, to estimate the unknown position and orientation of the target. However, when performing global positioning using vision data, we may not achieve the same level of accuracy as the widely used wireless-based approaches because of two intertwined reasons. First, the resolution of the vision sensors might be limited, which might make similar geo-tagged landmarks (such as trees or lampposts) appearing in the vision data indistinguishable from each other. Second, a prohibitively large amount of prior information and computational resources are needed to store and process unique identifiers corresponding to every landmark appearing on the map. As a result, landmarks may not necessarily be unique, which is the setting of interest in this paper. In other words, similar landmarks might appear at multiple places on the map, which makes it challenging to determine the exact global location of the target. That said if a target can see multiple landmarks, the exact geometric setting or pattern of these *visible* landmarks observed by the target might not appear frequently on the map, which can be used to aid global positioning. In many practical applications, such as navigation, it is reasonable to assume that the target can obtain range measurements to the visible landmarks (e.g., from the depth analysis of the images), which we term as the *range vector*. Using tools from stochastic geometry, we comprehensively analyze localizability using this range vector. *It should be noted that this perspective of vision-based positioning, probabilistic problem formulation, and the corresponding stochastic geometry analysis are all reported for the first time in this paper.*

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A. Related Works

Given its interdisciplinary nature, there are three lines of work that are relevant to this paper: vision-based positioning, localizability analysis, and stochastic geometry.

Traditionally, vision-based positioning is treated as a *retrieval task*, which estimates the unknown location of a query image using locations of the most similar geo-tagged images in the database [1]. The vision information in the database is preprocessed and represented as several invariant features using the Scale-Invariant Feature Transform [2], which extracts local invariant features, such as bag-of-words [3], [4], VLAD [5], and graphs [6], [7]. The computation to retrieve the location for the query image is reduced by representing vision information in the whole image as a compressed vector. In recent years, a variety of convolutional neural networks (CNNs) have been proposed to track the temporal sequence of location and orientation observations of the camera, such as PoseNet [8], MapNet [9], CamNet [10], to name a few. Their performance is evaluated on relatively small-scale datasets, such as 7-Scenes and Oxford RobotCar datasets. Another related line of work is cross-view matching, which performs large-scale localization by matching ground images to geo-tagged satellite images. Several variants of CNNs are proposed to estimate the orientation and location [11]–[13].

Despite the advances in the algorithmic treatment of this problem, localizability has not yet been explored rigorously in vision-based positioning. Not surprisingly, in wireless-based positioning, localizability has been studied from different perspectives, such as using graph theory in [14], the Cramér Rao Lower Bound (CRLB) in [15], [16], and stochastic geometry in [17]–[23]. Since our work is based on stochastic geometry, we now discuss the prior art focusing on its applications to localization. In wireless-based positioning, stochastic geometry provides the framework to analyze key localization metrics, including localizability, by modeling the location of anchors and blockages as point processes, e.g., see [17]–[23]. These analyses are useful in identifying key factors influencing localization, evaluating the impact of these factors on the network performance, and suggesting guidelines for optimizing localization algorithms.

Another less obvious direction of research that is relevant to this work is the information-theoretic analyses of point process models. For instance, [24] studies capacity and error exponents of stationary point processes by considering points in the process as codewords and random displacement of points as additive noise. This work shows that error-free communication is achievable when the rate does not exceed the Poltyrev capacity. Even though one can draw parallels between our work and this specific direction of research (e.g., our range measurements are analogous to codewords), the line of questioning (inspired by vision-based localization) and mathematical development are fundamentally different. Nevertheless, this general connection makes this approach relevant to the communication theory community as well.

B. Contributions and Outcomes

The main contribution of this paper is to propose a fundamentally new perspective to vision-based localization based on stochastic geometry, mathematically formulate the problem of *localizability* within this framework, and perform rigorous mathematical analysis of this metric (which quantifies our ability to obtain a global position fix based on only the vision data). Specifically, in our system model, we consider indistinguishable landmarks (such as lampposts that all appear similar in the vision data) and model their locations as a PPP. To enhance the realism of the model, we include a visibility model and incorporate additive noise into the range measurements. We represent the point pattern of visible landmarks using range vectors and define the target as localizable if the measured ranges are unique. This means that range vectors observed from other locations on the map need to be at least a certain distance away from the range vector observed at the target location. We extensively analyze this setup by deriving *conditional localizability probability*, which quantifies the probability of correctly determining the set of landmarks that correspond to the range vector observed at the target. To characterize localizability across the whole map, we study *localizability probability* by taking the expectations of conditional localizability probability over the joint distribution of the range vector. Some interesting observations and connections can be made from the result. For instance, as the landmark intensity tends to infinity, the localizability probability tends to one. This observation is analogous to having codewords with infinite lengths in communication systems. The implication is that error-free localization is achievable in this limiting scenario.

II. System Model

This section constructs the statistical model for the landmarks, formulates the system model, and presents the problem statement.

Given that the landmarks are indistinguishable from one another, we model them as points on the map. The maps

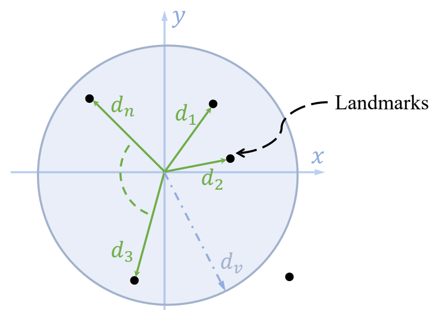

Fig. 1. An illustration of the system model. Black points are landmarks. Landmarks in the blue ball $\mathbf{b}(\mathbf{x}_0, d_v)$ are visible to the target located at $\mathbf{x}_0 = \mathbf{0}$. Green lines denote the ranges. The target measures ranges in the clockwise order starting from the true north (y-axis).

of various locations are considered realizations of the point process $\Phi = \{\mathbf{x}_i\}$, where $\mathbf{x}_i$ denotes the locations of the landmarks. We define a landmark as *visible* to the target if the distance between them, denoted as d, is less than a certain threshold d_v. Specifically, landmarks within the ball $B_{\mathbf{x}_0} = \mathbf{b}(\mathbf{x}_0, d_v)$ are considered *visible* to the target, where d_v represents the *maximum visibility distance* and $\mathbf{x}_0$ is the unknown target location on the map. As mentioned before, we assume Φ is a PPP, which allows us to leverage the mathematical properties of PPP in our analysis. *It is worth noting that the landmark locations have been shown to follow a PPP in some vision-based localization settings [25].*

The point representation of landmarks adopted in this paper is similar to the graphical representation of landmarks discussed in prior works, such as [7]. In addition to providing a complementary probabilistic approach to this problem, our work also offers an alternate definition for the weights of edges in these graphs through the ranges to landmarks measured by vision sensors.

A. Measurements

We assume that all the visible landmarks in $B_{\mathbf{x}_0}$ are detected by the target using visual information. The target then measures the ranges to all detected landmarks. Details of the specific methods for measuring the ranges are not required to derive the localizability in our work. Without loss of generality, we assume that the target measures the ranges to the visible landmarks in a clockwise order, starting from a fixed orientation such as the true north. As illustrated in Fig. 1, there are $k = 4$ visible landmarks within visible distance d_v. Ranges to these landmarks are obtained and recorded as a *range vector* in the aforementioned order, represented as

$$\mathbf{d}_0 = [d_{0,1}, d_{0,2}, \ldots, d_{0,k}], \tag{1}$$

where k is the realization of random variable $N = N(B_{\mathbf{x}_0}) = \#\{\Phi \cap B_{\mathbf{x}_0}\}$, which gives the number of visible landmarks.

Similar to the distances between codewords in communication systems, we define the distance between two range vectors obtained at two different locations as

$$d_p(\mathbf{d}_1, \mathbf{d}_2) = \begin{cases} \|\mathbf{d}_1 - \mathbf{d}_2\|_\infty\,, & \dim(\mathbf{d}_1) = \dim(\mathbf{d}_2) \\ \infty, & \dim(\mathbf{d}_1) \neq \dim(\mathbf{d}_2) \end{cases}, \quad (2)$$

where $\|\cdot\|_\infty$ is the infinity norm of a vector.

B. Problem Formulation

From each candidate location $\mathbf{x}$ on the map, we can refer to the map and its corresponding visible landmark pattern $\varphi \cap B_{\mathbf{x}}$, which provides the true range vector $\mathbf{d}$ if the target were to be placed at $\mathbf{x}$. Here φ is the realization of the point process Φ and serves as the map. We assume that the actual measurements obtained by the target are noisy, which we denote as $\mathbf{r}$, and model the measurement error $\mathbf{n} = \mathbf{r} - \mathbf{d}$ as additive noise, bounded as $\|\mathbf{n}\|_\infty < \epsilon/2$. To determine the target location, we compare the measured range vector $\mathbf{r}$ with true range vectors corresponding to different locations on the map. If $\mathbf{d}_i$ and $\mathbf{d}_j$ are two true range vectors for locations $\mathbf{x}_i$ and $\mathbf{x}_j$, we will say that the actual (observed) measurements at these locations are indistinguishable if

$$d_p(\mathbf{d}_i, \mathbf{d}_j) \leq \epsilon. \quad (3)$$

One can also think of these range vectors as codewords in communications, where one can achieve error-free communication if the noise is bounded by half of the minimum distance between the codewords. In the same way, to achieve error-free localization in the above setup, any true range vectors $\mathbf{d}$ of other locations on the map should be at least ϵ distance away from the true range vector $\mathbf{d}_0$ of the target location $\mathbf{x}_0$.

Definition 1 (*Localizability Probability*)**.** *We define* localizability *in this work as the ability to identify the correct set of landmarks that are associated with the range vector observed at the target. In this work, we mathematically characterize "localizability probability", denoted by P_{Loc}, that is the probability that the range vector observed at an arbitrary location in the network is distinguishable from the range vector observed at the target (as per* (3)*). Whenever convenient, we will present results for the "non-localizability probability", denoted by $P_{\text{N-Loc}} = 1 - P_{\text{Loc}}$.*

Remark 1. *Intuitively, the range vectors obtained around the target location $\mathbf{x}_0$ will result in indistinguishable range vectors as per* (3)*. This is exactly what we want since we are just interested in identifying the correct set of landmarks that correspond to the true range vector. Interestingly, this does not pose any technical issues in our analysis since we sample the candidate target locations from an infinite plane. Therefore, even if candidate locations around the true location (countably infinite because of the continuous space) will yield similar range vectors, the area of such region will be bounded. Therefore, the probability that we will ever sample a candidate location from around the target location $\mathbf{x}_0$ is zero. Consequently, we do not need to put any additional constraints on our analysis to capture the above intuition about localizability.*

In this paper, we first introduce the concept of *conditional localizability probability*, denoted by $P_{\text{C,Loc}}$, which represents the probability of correctly identifying the set of landmarks associated with the range vector $\mathbf{d}_0$. Mathematically, we define $P_{\text{C,Loc}}$ as

$$P_{\text{C,Loc}} = \mathbb{P}[d_p(\mathbf{D}, \mathbf{d}_0) > \epsilon \mid \mathbf{d}_0, k] \quad (4)$$

$$= \mathbf{E}\{\mathbf{1}(d_p(\mathbf{D}, \mathbf{d}_0) > \epsilon) \mid \mathbf{d}_0, k\}\,, \quad (5)$$

where $\mathbf{D}$ is the range vector corresponding to an arbitrary location $\mathbf{x}$, and k is the number of visible landmarks at ground truth location $\mathbf{x}_0$. Note that there is no need to explicitly exclude $\mathbf{x}_0$ from the candidate locations since the probability of $\mathbf{x}_0$ being selected as a candidate location is zero. The expectation in (4) is computed over all potential locations on the map. Since the landmark locations are modeled as a PPP, due to the ergodic property, the spatial average of range vectors across all candidate locations is equivalent to the ensemble average obtained by considering the candidate target location at the origin. In other words, to derive the conditional localizability probability, we can fix the location $\mathbf{x}$ at the origin and compute the expectation over different realizations of Φ.

While the above result is conditional on a specific range vector $\mathbf{d}_0$, the localizability on the whole map is also of interest, which we obtain by averaging over all possible range vectors $\mathbf{d}_0$, i.e., taking the expectation over $\mathbf{d}_0$ and $N(B_{\mathbf{0}})$. Now, we define *localizability probability* on the whole map as

$$P_{\text{Loc}} = \mathbb{P}[d_p(\mathbf{D}, \mathbf{d}_0) > \epsilon] = \mathbf{E}\{\mathbb{P}[d_p(\mathbf{D}, \mathbf{d}_0) > \epsilon | \mathbf{d}_0, k]\}. \quad (6)$$

III. Localizability Probability Analysis

In this section, we first study key mathematical constructs that are involved in the analysis. The following lemmas provide statistical results necessary for the analysis of localizability probability. First, we derive the distribution of ranges conditioned on the number of visible landmarks in the following lemma.

Lemma 1. *The distribution of the distance D from an arbitrary point to the origin, conditioned on the number of points $N(B_{\mathbf{x}_0}) = k$, $(k > 0)$, is*

$$f_D(d) = \frac{2d}{d_v^2} \delta(0 \leq d \leq d_v)\,. \quad (7)$$

Proof: The result is a direct consequence that locations of points in a PPP are uniformly distributed when conditioned on the number of points. ∎

Remark 2. *Because PPP is a stationary point process, $f_D(d)$ is invariant to the target location $\mathbf{x}_0$.*

Using Lemma 1, we provide the probability that the range measurement D_i is within distance ϵ to $d_{0,i}$ in the next lemma.

Lemma 2. *Conditioned on the number of points, the probability that the distance between the range measurement D_i and $d_{0,i}$ is smaller than ϵ is given below.*

When $0 \le \epsilon < \frac{d_v}{2}$,

$$\mathbb{P}[|D_i - d_{0,i}| \le \epsilon \mid N_0 = k] = \begin{cases} \frac{(d_{0,i}+\epsilon)^2}{d_v^2}, & 0 \le d_{0,i} < \epsilon, \\ \frac{4\epsilon d_{0,i}}{d_v^2}, & \epsilon \le d_{0,i} < d_v - \epsilon, \\ 1 - \frac{(d_{0,i}-\epsilon)^2}{d_v^2}, & d_v - \epsilon \le d_{0,i} \le d_v, \end{cases} \tag{8}$$

where $N_0 = N(B_{\mathbf{0}}) = \#\{\Phi \cap \mathbf{b}(\mathbf{0}, d_v)\}$ *is the number of landmarks that are visible from the origin.*

When $\frac{d_v}{2} \le \epsilon < d_v$,

$$\mathbb{P}[|D_i - d_{0,i}| \le \epsilon \mid N_0 = k] = \begin{cases} \frac{(d_{0,i}+\epsilon)^2}{d_v^2}, & 0 \le d_{0,i} < d_v - \epsilon, \\ 1, & d_v - \epsilon \le d_{0,i} < \epsilon, \\ 1 - \frac{(d_{0,i}-\epsilon)^2}{d_v^2}, & \epsilon \le d_{0,i} \le d_v, \end{cases} \tag{9}$$

When $d_v < \epsilon$,

$$\mathbb{P}[|D_i - d_{0,i}| \le \epsilon \mid N_0 = k] = 1. \tag{10}$$

Proof: By definition and Lemma 1, we have

$$\begin{aligned} \mathbb{P}[|D_i - d_{0,i}| \le \epsilon \mid N_0 = k] &= \int_{d_{0,i}-\epsilon}^{d_{0,i}+\epsilon} f_D(d)\,\mathrm{d}d \\ &= \int_{\max\{d_{0,i}-\epsilon,0\}}^{\min\{d_{0,i}+\epsilon,d_v\}} \frac{2d}{d_v^2}\,\mathrm{d}d, \end{aligned} \tag{11}$$

where the last equation follows from the fact that $f_D(d)$ only has non-zero values when $d \in [0, d_v]$. The result in Lemma 2 is derived by considering different values of ϵ and $d_{0,i}$. ■

Further, we derive conditional localizability probability, $P_{\mathrm{C,Loc}}$, which is the probability that the distance between $\mathbf{D}$ and a given $\mathbf{d}_0$ is greater than ϵ. The result is presented in the following lemma.

Lemma 3. *Given a* range vector $\mathbf{d}_0 \in \mathbb{R}^k$, *the conditional localizability probability is*

$$\begin{aligned} P_{\mathrm{C,Loc}} &= 1 - \mathbb{P}[d_p(\mathbf{D}, \mathbf{d}_0) \le \epsilon \mid \mathbf{d}_0, k] \\ &= 1 - \frac{m^k}{k!} e^{-m} \left\{ \prod_{i=1}^{k} \mathbb{P}[|D_i - d_{0,i}| \le \epsilon \mid N = k] \right\}. \end{aligned} \tag{12}$$

Proof: To calculate $P_{\mathrm{C,Loc}}$, we can consider the probability of the complementary event, i.e., making an error in determining the correct set of landmarks that correspond to the observed range vector, is

$$\begin{aligned} &\mathbb{P}[d_p(\mathbf{D}, \mathbf{d}_0) \le \epsilon \mid \mathbf{d}_0, k] \\ &\overset{(a)}{=} \mathbb{P}[d_p(\mathbf{D}, \mathbf{d}_0) \le \epsilon, \dim(\mathbf{D}) = k \mid \mathbf{d}_0] && (13) \\ &\overset{(b)}{=} \mathbb{P}[d_p(\mathbf{D}, \mathbf{d}_0) \le \epsilon, N(B_{\mathbf{x}}) = k \mid \mathbf{d}_0] && (14) \\ &\overset{(c)}{=} \mathbb{P}\left[\max_{i \in \{N\}} \{|D_i - d_{0,i}|\} \le \epsilon \mid N = k, \mathbf{d}_0\right] \times \mathbb{P}[N = k \mid \mathbf{d}_0] && (15) \\ &\overset{(d)}{=} \mathbb{P}[|D_1 - d_{0,1}| \le \epsilon, \ldots, |D_k - d_{0,k}| \le \epsilon \mid N = k] \times \mathbb{P}[N = k \mid \mathbf{d}_0] && (16) \\ &\overset{(e)}{=} \left\{ \prod_{i=1}^{k} \mathbb{P}[|D_i - d_{0,i}| \le \epsilon \mid N = k] \right\} \cdot \mathbb{P}[N = k], && (17) \end{aligned}$$

where in (a) we leverage the fact that the event $E_2 = \{\dim(\mathbf{D}) = k\}$ contains $E_1 = \{d_p(\mathbf{D}, \mathbf{d}_0) \le \epsilon\}$ where the condition $\mathbf{d}_0$ includes the dimension information of the range vector; in (b) the dimension of the range vector is equal to the number of visible landmarks; in (c) $N = N(B_{\mathbf{x}})$ is the number of visible landmarks within the visibility distance d_v. Additionally, we employ the definition of the infinity norm to characterize the maximum distance between the elements of $\mathbf{D}$ and $\mathbf{d}_0$; in (d) we utilize the i-th element of the range vector $\mathbf{d}_0$ to impose a constraint on D_i; and in (e) we use the motion-invariance property of the PPP, which means that range measurements from all orientations are independent and identical to each other.

The second term in (17) is the probability that k points of Φ lie in $N(B_{\mathbf{x}})$, which is

$$\mathbb{P}[N = k] = \frac{m^k}{k!} e^{-m}, \tag{18}$$

where $m = \Lambda(B_{\mathbf{x}}) = \int_{B_{\mathbf{x}}} \lambda\,\mathrm{d}\mathbf{x} = \lambda \pi d_v^2$, and λ is the intensity of landmarks. This completes the proof. ■

Lemma 3 provides the expression of conditional localizability probability, which characterizes the probability of correctly identifying the set of landmarks associated with $\mathbf{d}_0$. To evaluate the localizability probability of the target across the entire map, we now derive the joint probability density function of range vector $\mathbf{D}_0$ observed from the origin and the number of visible landmarks $N(B_{\mathbf{0}})$. The result is presented in the following lemma.

Lemma 4. *The joint probability density function of* $\mathbf{D}_0$ *and* N_0 *is*

$$f_{\mathbf{D}_0,N_0}(\mathbf{d}_0, k) = \frac{m^k}{k!} e^{-m} \prod_{i=1}^{k} \left\{ \frac{2d_{0,i}}{d_v^2} \delta(0 \le d_{0,i} \le d_v) \right\}. \tag{19}$$

Proof: Because of the motion-invariance property of PPP, the distribution of the number of points in $B_{\mathbf{x}}$ is invariant with respect to the location $\mathbf{x}$. As a result, the distribution of N_0 is identical to N, given in (18). By definition, the mixed joint probability is

$$f_{\mathbf{D}_0,N_0}(\mathbf{d}_0, k) = f_{\mathbf{D}_0|N_0}(\mathbf{d}_0|k)\,\mathbb{P}[N_0 = k]. \tag{20}$$

Because PPP is motion-invariant, the distributions of range measurements at all orientations are independent and identical. Hence, we can write

$$f_{\mathbf{D}_0|N_0}(\mathbf{d}_0|k) = \prod_{i=1}^{k} \{f_D(d_{0,i})\} \tag{21}$$

$$= \prod_{i=1}^{k} \left\{ \frac{2d_{0,i}}{d_v^2} \delta(0 \le d_{0,i} \le d_v) \right\}. \quad (22)$$

This completes the proof. ∎

Using the previous lemmas and the definition in (1), we now present the main result of this paper in the following theorem.

Theorem 1. *The localizability probability P_{Loc} of the target on the map with landmark intensity λ is*

$$\begin{aligned} P_{\mathrm{Loc}} &= \mathbb{P}[d_p(\mathbf{D}, \mathbf{d}_0) > \epsilon] \\ &= \begin{cases} 1-e^{-2m} I_0\Big(2m \cdot \sqrt{\frac{8d_v^3\epsilon - 6d_v^2\epsilon^2+\epsilon^4}{3d_v^4}}\Big), & 0 \le \epsilon < d_v, \\ 1-e^{-2m} I_0(2m), & \epsilon \ge d_v, \end{cases} \end{aligned} \quad (23)$$

where $m = \lambda\pi d_v^2$ is defined in (18).

Proof: Using the definition in (1) and Lemma 4, we can write the expectation as

$$\begin{aligned} &\mathbb{P}[d_p(\mathbf{D}, \mathbf{d}_0) > \epsilon] \\ &= \sum_{k=0}^{\infty} \left\{ \int_0^{\mathbf{d}_v} \mathbb{P}[d_p(\mathbf{D}, \mathbf{d}_0) > \epsilon \mid \mathbf{d}_0, k] \right. \\ &\qquad \left. f_{\mathbf{D}_0|N_0}(\mathbf{d}_0|k)\, \mathbb{P}[N_0 = k]\, \mathrm{d}\mathbf{d}_0 \right\}. \end{aligned} \quad (24)$$

When $0 \le \epsilon < d_v$, using equation (8), (9) in Lemma 2 and Lemma 3, the result is

$$P_{\mathrm{Loc}} = 1 - \sum_{k=0}^{\infty} \left\{ \left(\frac{8d_v^3\epsilon - 6d_v^2\epsilon^2 + \epsilon^4}{3d_v^4} \right)^k \cdot \frac{m^{2k}}{(k!)^2} e^{-2m} \right\} \quad (25)$$

$$= 1 - e^{-2m} \cdot I_0\left(2m \cdot \sqrt{\frac{8d_v^3\epsilon - 6d_v^2\epsilon^2 + \epsilon^4}{3d_v^4}} \right), \quad (26)$$

where $I_0(\cdot)$ is the modified Bessel function of the first kind. It should be noted that although equations (8) and (9) may appear different, the resulting integrals yield identical results.

When $d_v \le \epsilon$, using equation (8), the result is

$$P_{\mathrm{Loc}} = 1 - \sum_{k=0}^{\infty} \left\{ \left(\frac{m^k}{k!} e^{-m} \right)^2 \right\} = 1 - e^{-2m} I_0(2m). \quad (27)$$

This completes the proof. ∎

Remark 3. *When the noise level is significantly high and $\epsilon \ge d_v$, P_{Loc} cannot be improved by obtaining range measurements. This can be inferred from equation* (27)*, showing that the probability of $N = N_0 = k$ dominates. The range measurements are too noisy to help localize the target.*

Theorem 1 provides the analytical expression of localizability probability representing the probability of identifying the correct set of landmarks. A natural question arises regarding the performance of localizability probability as the landmark intensity λ approaches infinity. This question is addressed in the following proposition.

Proposition 1. *As the landmark intensity λ tends to infinity, the localizability probability P_{Loc} approaches one regardless of the value of ϵ. Equivalently, $P_{\mathrm{N-Loc}} = 1 - P_{\mathrm{Loc}}$ approaches zero as λ tends to infinity.*

Proof: The asymptotic expansions of the modified Bessel functions of the first kind can be derived using the results provided in [26]. As $z \to \infty$ with fixed v, we have

$$I_v(z) = \frac{e^z}{(2\pi z)^{\frac{1}{2}}} \sum_{k=0}^{\infty} (-1)^k \left(\frac{a_k(v)}{z^k} \right), \quad (28)$$

where $a_k(v) =$

$$\begin{cases} 1, & k = 0, \\ \frac{(4v^2-1^2)(4v^2-3^2)\dots(4v^2-(2k-1)^2)}{k!8k}, & k \ne 0, \end{cases} \quad (29)$$

Let $v = 0$ and use (28), we can calculate (23) when $\lambda \to \infty$

$$\begin{aligned} &\lim_{\lambda\to\infty} P_{\mathrm{N-Loc}} = \lim_{\lambda\to\infty} 1 - P_{\mathrm{Loc}} && (30) \\ &= \lim_{\lambda\to\infty} 1 - \mathbb{P}[d_p(\mathbf{D}, \mathbf{d}_0) > \epsilon] \\ &= \lim_{m\to\infty} e^{-2m} \cdot \frac{e^{2\alpha m}}{(4\pi\alpha m)^{\frac{1}{2}}} \left(1 + O\left(2\alpha m^{-\frac{1}{2}} \right) \right) && (31) \\ &= \lim_{m\to\infty} e^{-2(1-\alpha)m} (4\pi\alpha m)^{-\frac{1}{2}} \left(1 + O\left(m^{-\frac{1}{2}} \right) \right) = 0, && (32) \end{aligned}$$

where $m = \lambda\pi d_v^2$ and

$$\alpha = \begin{cases} \sqrt{\frac{8d_v^3\epsilon - 6d_v^2\epsilon^2+\epsilon^4}{3d_v^4}}, & 0 \le \epsilon < d_v, \\ 1, & \epsilon \ge d_v, \end{cases} \quad (33)$$

which is bounded by $0 \le \alpha \le 1$. This completes the proof. ∎

IV. Numerical Results

In this section, we verify our analytical results by comparing the theoretical non-localizability probability $P_{\mathrm{N-Loc}}$, which is the complement of P_{Loc}, derived in Theorem 1, with the result obtained from Monte-Carlo simulations. We consider a range of landmark intensities $m = \lambda\pi d_v^2 \in [2, 10]$. The maximum visibility distance d_v is set to 50 meters, and we consider values of ϵ from the set $\{1, 5, 10, 20, 40, 60\}$ meters. In Fig. 2, we present the plot of the theoretical $P_{\mathrm{N-Loc}}$, alongside the results obtained from Monte Carlo simulations. We chose $P_{\mathrm{N-Loc}}$ over P_{Loc} to focus on the "errors" for an easier visualization of the result. The perfect match between the curves confirms the accuracy of our analysis. As depicted in Fig. 2, it is evident that $P_{\mathrm{N-Loc}}$ increases with increasing values of ϵ. Remarkably, when the noise level is significantly high, such that $\epsilon \ge d_v$, the model essentially reduces to matching the number of visible landmarks, as discussed in Remark 3. Additionally, we observe that $P_{\mathrm{N-Loc}}$ decreases as the landmark intensity λ increases. Proposition 1 establishes that $P_{\mathrm{N-Loc}}$ approaches zero as λ tends to infinity, indicating the feasibility of error-free localization.

V. Conclusion

In this paper, we have presented a new and tractable statistical model using stochastic geometry to rigorously analyze the concept of *localizability* in vision-based positioning. Modeling the locations of the landmarks as a PPP, we assumed that the

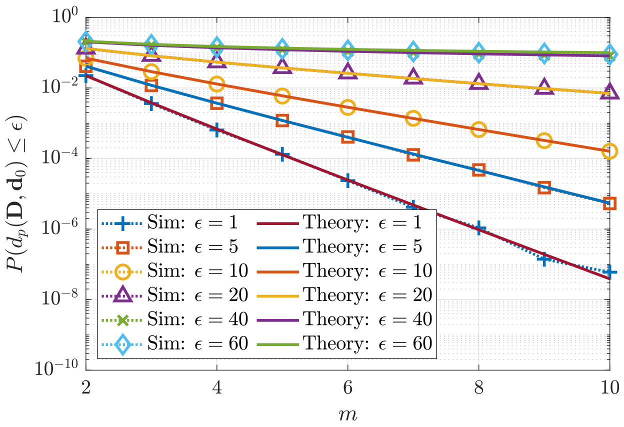

Fig. 2. The plot of non-localizability probability $P_{\mathrm{N-Loc}} = 1 - P_{\mathrm{Loc}}$ with visibility distance $d_v = 50\mathrm{m}$.

unknown location of the target is encoded in terms of the range vector observed at that location. One of the key findings is that the localizability probability, as defined in this paper, approaches one as the landmark intensity approaches infinity. Our work provides valuable insights into understanding the limitations and challenges associated with vision-based localization in the presence of indistinguishable landmarks.

Building upon this research, there are two promising lines of work for future extensions. Firstly, an information-theoretic analysis can be conducted to explore the fundamental limits of localization accuracy in vision-based systems, considering factors such as sensor resolution, environmental conditions, and landmark type. This would provide a deeper understanding of localization using indistinguishable landmarks and guide the design of optimal localization strategies. The potential connections to codewords in the communication system are also valuable to explore, possibly along the lines of the work in [24] on capacity and error exponents of point processes. In terms of practical applications, there is significant interest in designing specific algorithms for vision-based positioning that can operate effectively with limited visual information and computational resources. We have initiated this study in a recent work [27], where we used pairwise constraints to identify the correct landmark combination. While range vector representations of point patterns can achieve error-free localization, they may not necessarily be optimal. Therefore, it would be valuable to explore optimal schemes for representing point patterns, ensuring the retention of maximum location information. *Overall, this paper makes the very first attempt to connect stochastic geometry, localization, and computer vision, which could be potentially a new direction of research.*

References

[1] A. R. Zamir, A. Hakeem, L. Van Gool, M. Shah, and R. Szeliski, *Introduction to large-scale visual geo-localization*. Springer, 2016.

[2] D. G. Lowe, "Distinctive image features from scale-invariant keypoints," *Intl. Journal of Computer Vision*, vol. 60, pp. 91–110, 2004.

[3] G. Csurka, C. Dance, L. Fan, J. Willamowski, and C. Bray, "Visual categorization with bags of keypoints," in *Proc., European Conference on Computer Vision (ECCV)*, 2004.

[4] J. Philbin, O. Chum, M. Isard, J. Sivic, and A. Zisserman, "Object retrieval with large vocabularies and fast spatial matching," in *Proc., IEEE Conf. on Computer Vision and Pattern Recognition (CVPR)*, 2007.

[5] R. Arandjelovic and A. Zisserman, "All about VLAD," in *Proc., IEEE Conf. on Computer Vision and Pattern Recognition (CVPR)*, 2013.

[6] V. Shankar, J. Zhang, J. Chen, C. Dinh, M. Clements, and A. Zakhor, "Approximate subgraph isomorphism for image localization," *Electronic Imaging*, vol. 2016, no. 15, pp. 1–9, 2016.

[7] S. Verde, T. Resek, S. Milani, and A. Rocha, "Ground-to-aerial viewpoint localization via landmark graphs matching," *IEEE Signal Processing Letters*, vol. 27, pp. 1490–1494, 2020.

[8] A. Kendall, M. Grimes, and R. Cipolla, "PoseNet: A convolutional network for real-time 6-DOF camera relocalization," in *Proc., IEEE Intl. Conf. on Computer Vision (ICCV)*, 2015.

[9] S. Brahmbhatt, J. Gu, K. Kim, J. Hays, and J. Kautz, "Geometry-aware learning of maps for camera localization," in *Proc., IEEE Conf. on Computer Vision and Pattern Recognition (CVPR)*, 2018.

[10] M. Ding, Z. Wang, J. Sun, J. Shi, and P. Luo, "CamNet: Coarse-to-fine retrieval for camera re-localization," in *Proc., IEEE Intl. Conf. on Computer Vision (ICCV)*, 2019.

[11] S. Zhu, T. Yang, and C. Chen, "Revisiting street-to-aerial view image geo-localization and orientation estimation," in *Proc., IEEE/CVF Winter Conf. on Applications of Computer Vision*, 2021.

[12] ——, "VIGOR: Cross-view image geo-localization beyond one-to-one retrieval," in *Proc., IEEE Conf. on Computer Vision and Pattern Recognition (CVPR)*, 2021.

[13] W. Hu, Y. Zhang, Y. Liang, Y. Yin, A. Georgescu, A. Tran, H. Kruppa, S.-K. Ng, and R. Zimmermann, "Beyond geo-localization: Fine-grained orientation of street-view images by cross-view matching with satellite imagery," in *Proc., ACM Intl. Conference on Multimedia*, 2022.

[14] Z. Yang and Y. Liu, "Understanding node localizability of wireless Ad Hoc and sensor networks," *IEEE Trans. on Mobile Computing*, vol. 11, no. 8, pp. 1249–1260, 2012.

[15] T. Jia and R. M. Buehrer, "A new Cramer-Rao lower bound for TOA-based localization," in *Proc., IEEE MILCOM*, 2008.

[16] L. Gui, M. Yang, H. Yu, J. Li, F. Shu, and F. Xiao, "A Cramer-Rao lower bound of CSI-based indoor localization," *IEEE Trans. on Veh. Technology*, vol. 67, no. 3, pp. 2814–2818, 2017.

[17] C. E. O'Lone, H. S. Dhillon, and R. M. Buehrer, "A statistical characterization of localization performance in wireless networks," *IEEE Trans. on Wireless Commun.*, vol. 17, no. 9, pp. 5841–5856, Sep. 2018.

[18] ——, "A mathematical justification for exponentially distributed NLOS bias," in *Proc., IEEE Globecom*, 2019.

[19] J. Schloemann, H. S. Dhillon, and R. M. Buehrer, "A tractable metric for evaluating base station geometries in cellular network localization," *IEEE Wireless Commun. Letters*, vol. 5, no. 2, pp. 140–143, 2016.

[20] ——, "A tractable analysis of the improvement in unique localizability through collaboration," *IEEE Trans. on Wireless Commun.*, vol. 15, no. 6, pp. 3934–3948, 2016.

[21] S. Aditya, H. S. Dhillon, A. F. Molisch, and H. M. Behairy, "A tractable analysis of the blind spot probability in localization networks under correlated blocking," *IEEE Trans. on Wireless Commun.*, vol. 17, no. 12, pp. 8150–8164, Dec. 2018.

[22] S. Aditya, H. S. Dhillon, A. F. Molisch, R. M. Buehrer, and H. M. Behairy, "Characterizing the impact of SNR heterogeneity on time-of-arrival-based localization outage probability," *IEEE Trans. on Wireless Commun.*, vol. 18, no. 1, pp. 637–649, Jan. 2019.

[23] H. Elsawy, W. Dai, M.-S. Alouini, and M. Z. Win, "Base station ordering for emergency call localization in ultra-dense cellular networks," *IEEE Access*, vol. 6, pp. 301–315, 2018.

[24] V. Anantharam and F. Baccelli, "Capacity and error exponents of stationary point processes under random additive displacements," *Advances in Applied Probability*, vol. 47, no. 1, pp. 1–26, 2015.

[25] J. Rohde, J. E. Stellet, H. Mielenz, and J. M. Zöllner, "Localization accuracy estimation with application to perception design," in *IEEE Intl. Conf. on Robotics and Automation (ICRA)*, 2016.

[26] F. W. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, *NIST handbook of mathematical functions*. Cambridge University Press, 2010.

[27] H. Hu, H. S. Dhillon, and R. M. Buehrer, "Landmark-based localization using range measurements: A stochastic geometry perspective," in *IEEE/IFIP WiOpt Workshops*, Aug. 2023.