What is College-Level Mathematics? A Proposed Framework for Generating Developmental Progressions in Mathematics up to and through College

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In this theoretical paper, our aim is to start a conversation about how "levels" in mathematics are operationalized and defined, with a specific focus on "college level". We approach this from the lens of developmental stages, using this to propose an initial framework for describing how learners might progress along a developmental continuum delineated by the kinds of reasoning/justification, generalization/abstraction, and types of conceptions that they hold, rather than by the particular computations learners are able to do, or the kinds of mathematical objects with which learners are engaging.

Keywords: college level; mathematical maturity; reasoning and justification; generalization and abstraction; conceptions

It is often assumed to be obvious whether a particular mathematics course is "college-level" or not; however, in practice, the transition point operationalized as "college-level" begins as early as Intermediate Algebra (Logue et al, 2016) and as late as Calculus I (Hsu & Gehring, 2016). In addition, determinations about which courses "count" as college-level are often based on syllabi that focus primarily on a list of computational skills on specific mathematical objects (e.g., linear equations, trinomials), rather than on how students reason with, justify, generalize or conceptualize mathematical ideas. Yet a conception of "level" that is driven more by the mathematical objects that are the focus of study, rather than *how* learners engage with those objects, contradicts many of the values of both mathematicians and mathematics educators about what high quality mathematics learning looks like. In addition, it further disadvantages students who may have strong higher-level thinking skills but who, for a variety of reasons, may not perform well on computational placement exams; currently, such students are often placed into non-credit bearing developmental courses that focus heavily on procedures, and which contribute to disparate impacts on college and economic outcomes (e.g., Bailey & Cho, 2010).

Mathematical learning is about more than just "content" (conceptualized as the specific mathematical objects of study or the particular procedures that students are expected to use): there are other skills, knowledge and practices that are important; yet much of this is left unarticulated in learning outcomes, and rarely used in the determination of course level. In this paper, we describe an initial framework for how we might define "college-level" mathematics, or more specifically, a spectrum of different "levels" of learning across the K-16+ mathematics curriculum as characterized by the kinds of reasoning, generalization, and conceptions that might describe developmental shifts or progressions in learning. This then includes a more focused discussion of how we might use such a framework to better articulate where the shift from K-12 to "college level" might occur in various mathematical domains.

Adolescent and Young Adult Development in Psychology and Neuroscience

It is known that the brain changes physically throughout the lifespan and that the last significant period of remodeling begins in adolescence and culminates in the early to mid-twenties. The parts of the brain most impacted by this last remodeling are those that control functions such as working memory, planning, and impulse control (Konrad et al, 2013). Based on this science, it is *developmentally appropriate* that college students should be able to interact with mathematical objects (e.g., algebraic expressions) in more sophisticated ways than younger students. Early educational literature, such as that of Piaget (1964), posited developmental stages for school-aged children that influence our expectations for what "grade level" means in subjects like math, reading, and writing. Psychology and neuroscience research acknowledge how understudied adolescents and young adults are and posit that much still remains to be learned about how brain development might impact behavior and learning (Blakemore, 2012; Shanmugan & Satterthwaite, 2016). In her survey of the field of adolescent brain imaging, Blakemore (2012) speculated about how changes in brain structure could make signal processing more efficient, which we speculate could have a direct impact on mathematics learning. This work in neuroscience compliments frameworks like that proposed by Erik Erikson (1994) for ongoing psychosocial development into and continuing through adulthood. Yet while it is known from neuroscience and developmental psychology research that college students differ developmentally from younger students, mathematics education frameworks have tended to ignore this when describing domains such as algebra that may be learned by students of widely varying ages. This paper seeks to explore how we might begin to conceptualize developmental stages as impacting how the same mathematical "objects" might be studied at different levels, with particular focus on what it might mean to do mathematics at the college level.

Proposed Mathematical Maturity Framework

In this paper we aim to problematize and redefine the term *mathematical maturity*. This term has been used in both research and practice to describe a kind of developmental progression like the one we hope to focus on here. However, this term has also been used in ways that are often vague and ill-defined; that provide deficit framings of students (e.g., "students can't take linear algebra before calculus because they don't have the mathematical maturity for the course"); and that describe binary destinations (e.g., students either have "mathematical maturity" or they don't) of what we conceptualize as a continuous life-long process of growth.

Mathematical maturity is a term used widely and often without formal definition within undergraduate mathematics education research and practice (Braun, 2019; Lew, 2019). In some instances, the completion of a specific course is used as an operational definition for the sake of a study, but even in those cases it is generally clarified that it is not the course *content* but a set of skills and increasing sophistication in how one approaches mathematics that is being referenced (Faulkner, Earl, & Herman, 2019; Lew, 2019). Two recent studies sought to determine how those using the term "mathematical maturity" define it. Faulkner et al (2019) interviewed engineering faculty and Lew (2019) interviewed mathematics faculty about their use of the term. Common definitions provided between these studies included many types of reasoning, generalizing, and conceptualizing, including the ability to: make connections across mathematical topics; use symbolic representations; relate different representations to one another and recognize when they describe the same phenomenon or relationship; choose between different representations for the purpose of solving problems; and understand if a solution to a problem makes sense.

Definition of Mathematical Maturity

Here we offer a new conceptualization of mathematical maturity via a framework, which we see as a starting point for describing how learners might acquire higher-level mathematical thinking skills and practices as they develop over time. We anticipate that this definition will evolve over time, but we present this as a first step, to start a conversation about what it means to learn mathematics at different "levels". A framework of this sort could then be used to generate new courses which could allow learners with missing "content knowledge" to nonetheless take a college-level, credit-bearing mathematics course that respects their different developmental stage compared to the age at which such content is traditionally introduced. For example, a course might require no algebra prerequisite, but allow students to engage with algebraic reasoning and justification at a higher level than would be expected in a typical K-12 algebra course.

We define ***mathematical maturity*** as a spectrum with no upper bound that describes the extent to which learners may acquire over time the ability to 1) reason and justify; 2) generalize and abstract; and 3) internalize particular conceptions of specific mathematical objects that occur throughout the curriculum. We conceptualize the process of developing mathematical maturity as a combination of physiological development and the outcome of particular mathematical experiences acquired over time. We now briefly present a framework for describing mathematical maturity which we have synthesized from existing research literature (Figure 1).

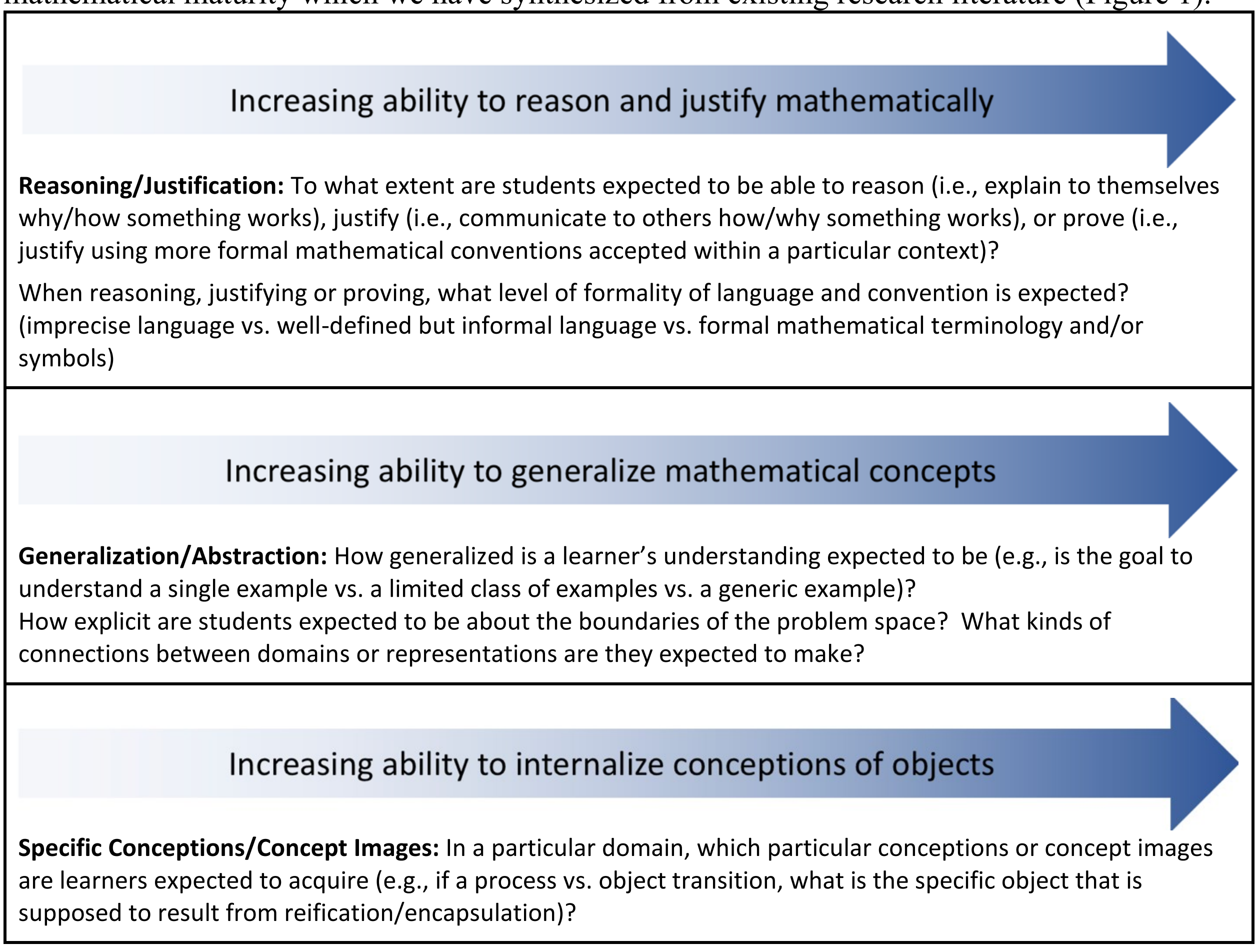

Figure 2. Mathematical Maturity Framework for Describing Developmental Progression through K-16+ Mathematical Curriculum, Three Possible Dimensions

We contrast this approach with traditional conceptualization of "level" which have tended to focus on computations on particular mathematical objects as the primary feature which determines the "level" of a course (Figure 2).

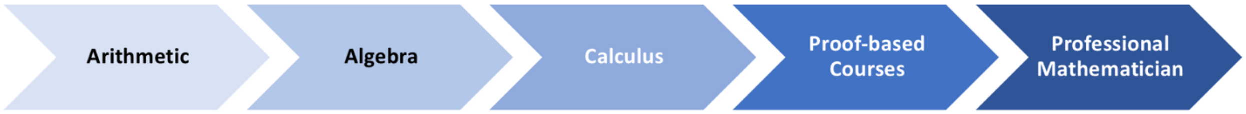

Figure 1: Informal model of progress through mathematics as currently conceived

In contrast, we consider which features are most relevant to determining the "level" at which the **same mathematical object might be learned at different points** in a students' K-16+ learning trajectory. We now describe each of the three dimensions of the framework in more detail.

Review of Literature from which the Proposed Framework was Drawn

Reasoning/Justification

One of the topics that is often discussed in the literature as a way of distinguishing whether students have "learned" mathematics, is the extent to which and ways in which students are able to reason or justify in mathematics. One of the major formal transitions in this area is when students are expected to generate mathematical proofs in college; substantial research has documented student difficulties with this. One explanation for this is that while students have experienced instruction focused on specific "content", students often do not come out of these courses with clear understanding of more general mathematical skills and practices, such as what constitutes mathematical proof (Selden & Selden, 2008). But formal proof simply describes one end of a much longer spectrum of skills, perspectives and practices. There have been extensive calls for students in K-12 to learn to reason and justify, long before the introduction of formal proof (e.g., National Governors Association Center for Best Practices [NGA] & Council of Chief State School Officers [CCSSO], 2010). And the processes of reasoning and justifying have been identified as critical mathematical skills that students may often not acquire during "standard" computational instruction (e.g., Mata-Pereira & da Ponte, 2017; Ball & Bass, 2003). Thus, reasoning and justification describes a core skill that is critical across mathematical domains.

Generalization/Abstraction

Another feature of mathematical "level" that often arises in the literature is the extent to which students are able to generalize about mathematical objects. To date, most of the research on generalization has been in the realms of early algebra (Carraher et al., 2008), pattern-forming (Amit & Neria, 2008), and linearity (Ellis, 2007) (see Ellis et al. 2022 for a more complete list). Notably, Ellis et al. (2022) is the first to consider students' generalizing activity across multiple domains and grade levels, ranging from middle school to undergraduates. Through their extensions of Ellis' (2007) taxonomy for categorizing types of generalization, they identified three main types: *relating*, *forming*, and *extending*. This framework enabled the researchers to identify what they considered to be generative (i.e., productive or useful) generalizing activity, and discuss how generalizing was both independent and dependent of the mathematical domain. We see this research area, and the RFE framework in particular, as having great potential for helping to illuminate our eventual taxonomy of delineating college-level mathematics.

Strongly related to generalization is the notion of abstraction. Abstraction may be defined as the processes that lead learners to grasp deeper understandings of mathematical structures, such as the underlying structure behind a vector space (Dreyfus, 2020, p. 13). Abstraction may also be thought of as a vertical reorganization of existing knowledge, or as a reconceptualization of information (as opposed to a de-construction). Many researchers have considered abstraction as part of a student's cognitive development, such as Piaget's ideas of empirical and reflective abstraction (Dubinsky, 2002), Thompson's processes and objects (1985), APOS theory (Asiala et al, 1997), Sfard's reification (1991), and Tall's structural abstraction (2013).

Conceptions/Concept Images

A third feature that often arises in studies of learners' progression through mathematical levels is the extent to which learners have particular conceptions about mathematical objects or concepts. One example that has been widely discussed is the transition documented by process-to-object theories (Sfard, 1991; Dubinsky, 1991; Gray & Tall, 1994), in which learners are theorized to conceptualize certain entities first as a process, and then later to reify/encapsulate that process into an object which can then subsequently be acted upon by even higher-order processes. For example, the expression $2x$ could be conceptualized as a process representing that 2 and x should be multiplied together. Later, a learner may conceptualize $2x$ as an object itself, representing the process of multiplying 2 by x or the result of multiplying 2 by x, without actually carrying out computation. Then $2x$ can be acted on by even higher-order processes, for example adding it to another object, $3x$, to obtain the result $5x$.

While students may switch back and forth between process and object conceptualizations, the ability to utilize an object conception is typically considered to be further along the developmental spectrum than using process conceptions alone (e.g., Sfard, 1991). Many higher-level mathematics courses also require object conceptions: for example, while arithmetic is rooted in a process conception of numerical computation, algebra requires that these same calculations be reified or encapsulated into objects (i.e., expressions/equations that can themselves be transformed using higher-order processes). Similarly, as algebra becomes more complex, students may be required to reify the process of the order of operations on algebraic expressions into subexpressions as objects (i.e., substrings of expression/equations that must be treated as unified objects); for example, this kind of higher-order structuring is necessary in order to perform function composition, u-substitution, or the chain rule in calculus. In fact, we can envision a larger progression in which one process is reified into an object, which is acted upon by higher-order processes which are themselves reified into an object, which is itself acted upon by even higher-order processes, etc. This progression has tended to be studied as individual shifts for one particular entity going from a process to an object, rather than discussed as a larger progression with many different shifts; however, original process-to-object theories precisely pointed out how reified objects became the focus of yet higher-order processes (Sfard, 1991), thus implying the existence of a larger progression containing many layers of more and more complex reified objects. Process to object views are likely not the only kinds of conceptual shifts that are expected of students as they progress through the mathematics curriculum; we present them here only as one example of how a key characteristic that determines the "level" of a mathematics course is the set of particular conceptions that learners are expected to internalize.

Brief Illustrative Example: The Distributive Property

In order to illustrate some of the affordances of the Mathematical Maturity Framework, we present one example of how this framework could be used to map out learning goals for the same object (the distributive property) at different stages of the K-16+ trajectory (Figure 3). The distributive property is first encountered in 3rd-5th grade, but is also the subject of study throughout the K-16+ curriculum. Currently, much focus is on which objects learners are expected to transform using the distributive property, yet research has documented extensive difficulties that students have in using the distributive property appropriately at many different levels (e.g., Malle, 1993; Schüler-Meyer, 2017). This may be because instruction often focuses on computation divorced from reasoning and justification. However, reconceptualizing the distributive property as a learning object by thinking about the types of reasoning, generalization, and conceptions students might use, could help us to shift our conceptions of how we determine the mathematical "level" of a particular course.

Algebra I (8-12th grade): Learners are expected to understand the symbolic representation $a(b+c) = ab + ac$ as a pattern in which a, b and c of the property represent objects (simple terms that are the product of a number and variable(s)). They are expected to conceptualize simple subexpressions such as px and qy as reified objects representing generic unknown numbers. The property is seen to hold because it represents two processes that produce the same numerical output for every possible numeric input from the domain into the expression which is being "transformed" by the property. They may or may not be expected to use an area model of multiplication to reason about or justify this idea, but they are not expected to prove the property, nor necessarily to generalize to algebraic objects with other forms.
Intro college-level algebra (lower-level undergraduate): Learners are expected to conceptualize the distributive property as a one-to-one mapping of specific reified subexpressions to a, b and c, respectively, in the property. Thus a, b, and c are seen as representing generic algebraic subexpressions and the specific reified subexpressions which are being mapped to variables in the property are seen as representing generic numerical values. The property is seen to hold because it is the process of replacing one expression with another equivalent one (substitution equivalence). The property is understood to be generalized to a generic number of terms, and students are able to describe this clearly but somewhat informally, and to justify why this is the case using an area model of multiplication. Reasoning and justification are expected, with well-defined language and some limited formal symbolism and terminology, but proof is not expected.
Abstract algebra (upper-level undergraduate or graduate): Multiplication and addition are conceptualized as abstract binary operations on an (often abstract) set, with a, b and c in the property representing generic set elements. Operations are defined axiomatically. The distributive law itself has been reified into an object: a property which a given set and pair of operations may or may not have. Learners are expected to prove, using formal mathematical terminology and symbolism, whether or not the left or right (or both) distributive properties hold for a given set with a given pair of operations (or for a larger class of sets with pairs of operations).

Figure 3. Examples of how levels of understanding of the distributive property might differ (as described by student learning goals) in high school versus lower/upper-level college, even when the objects which are the focus of the distributive property (algebraic expressions) are similar

The descriptions in Figure 3 are just one example of how we might describe learning outcomes which depict different levels of learning for a common mathematical "object". For students in the theoretical "intro college level" algebra class described in Figure 3, no extensive prerequisite proficiency in algebraic computation is required–however, once the learner starts working with the distributive property, higher-level reasoning/justification, generalization/abstraction, and reified objects are expected to be used and learned as a part of the curriculum. This is just one brief example of how using the Mathematical Maturity Framework as a tool for developing and describing learning in college-level mathematics may help us to shift our focus from computations on specific objects, to how students are learning to reason about, generalize and conceptualize specific key mathematical ideas as they progress along their mathematical learning trajectories up to and through college.

Conclusion

Our aim in presenting this framework is to shift our discussion of "college-level" mathematics (and levels in mathematics more generally) away from a focus on specific computations or particular mathematical objects, and towards a focus on reasoning, generalizing, and particular conceptual shifts. This reconceptualization can be particularly important from an equity perspective, since conflating computational skills with reasoning ability can be particularly detrimental to some of the most marginalized students. One example of this is developmental mathematics in college. The term "developmental" is often used to describe courses that are not "college-level", but this definition is ill-defined in the research literature (Wladis et al, 2022) and may be circular (e.g., a course is developmental because it does not earn college credit and does not earn credit because it is not "college-level"). Most students in college are in fact re-taking mathematics which they already took in high school (e.g., 70% of those who attended college the year after graduating HS had already taken at least one math course above Algebra II [IES, HLS, 2009], and 52% of students who enroll in Calculus I in college have taken calculus previously (Sadler & Sonnert, 2016), yet what makes some of the courses that students repeat in college "college-level" and others not is unclear. We see this as a critical equity issue in mathematics: many college students (particularly those from more marginalized groups) are labeled "developmental" in college (with both stigma and practical barriers attached to these labels) because they are deemed "not ready for college-level work", yet what it means to be ready for college-level work is not well-defined.

The transition from high school to college is also not the only transition point in mathematics learning in college that has been documented to be difficult for students. For example, many students struggle with the transition to proof in undergraduate mathematics (Selden & Selden, 2008). One reason the observed difficulty with many transition points into and through college mathematics may be that as a community we have not yet clearly enough articulated the specific goals of instruction, as defined in terms of particular types of higher-order thinking skills such as reasoning, generalization, and specific conceptions of mathematical concepts; nor have we adequately described on a larger-grained scale how we might expect students to progress through these developmental stages as they mature mathematically. This paper is an attempt to start a conversation about the potential of reframing "college-level" classifications based on specific high-level thinking skills, rather than organizing it around the specific mathematical objects to be studied or the particular calculations to be made. Our hope is that this will lead to more productive and equitable ways of teaching and assessing students in college, and across the K-16+ mathematics spectrum.

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