Reconceptualizing Algebraic Transformation as a Process of Substitution Equivalence

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In this theoretical paper, we describe how algebraic transformation could be reconceptualized as a process of substitution equivalence, and we discuss how this conceptualization affords mathematical justification of transformation processes. In particular, we describe a model which deconstructs the process of substitution equivalence into core subdomains which could be learned serially and then re-integrated, in order to make them accessible to students with lower prior knowledge in syntactic reasoning. Our aim in presenting this model is to start a conversation about what the core components of knowledge might be in order for students to reason about and justify algebraic transformation using symbolic representations.

Keywords: algebraic transformation; substitution equivalence; reasoning and justification; syntactic reasoning; cognitive load

Algebraic transformation has been identified as a core task of algebra (e.g., Kieran, 2004), yet students often struggle to transform algebraic expressions and equations correctly (e.g., Agoestanto et al., 2019; Dustin & Coleman, 2012). One reason may be that algebraic transformation is often taught procedurally. Without the connection to syntactic reasoning, students may not understand why certain transformations can be justified mathematically as preserving equivalence. For example, in the case of equations, students often do not realize that valid transformation produces an equation that has the same solution set as the original (e.g., Pilet, 2012, 2013). In this paper, we develop a theoretical model for how algebraic transformation could be reconceptualized as a process of substitution equivalence. Our model describes separate but related concepts necessary to use substitution equivalence to replace one expression or equation with an equivalent one during the problem-solving process. This model is the result of a decades-long design research experiment that we do not report on here; analysis of those data is the focus of other ongoing research (e.g., Wladis et al, 2022a, 2022b, 2022c, 2022d, 2022e). Instead, in this paper, our goal is to describe the theoretical model and its relationship to existing theory, including a discussion of its affordances in helping us to understand how learners might reason about and justify algebraic transformation.

Theoretical Framework: Generalizing Computational Versus Relational Views of the Equals Sign to Equivalence Relationships Preserved by Algebraic Transformation

Research on the equals sign distinguishes between whether students have a computational (the equals sign is a cue to compute what is on the left and put the answer on the right, which can lead to errors such as $2 + 4 = 6 + 2 = 8$) or relational (the equals sign represents a relationship between two equal quantities) conception of the equals sign (e.g., Stephens et al., 2013). We could similarly generalize this beyond the equals sign to any type of equivalence such as equivalent algebraic expressions or equations. For example, a ***computational view of equivalent algebraic expressions (equations)*** would describe a learner who sees transformation as a command to perform some sort of procedure on the expression (equation) to produce a resulting expression (equation), without realizing that there is an equivalence relationship between the original and resulting expressions (equations) (e.g., contributing to such errors as "cancelling" the $2x$ in the top and bottom of the expression $\frac{6x^2-2x}{2x}$ (e.g., Cunningham & Yacone, 2013) or to believing that $6x^2 - 2x$ could be interpreted to sometimes mean $6(x^2) - 2x$ and sometimes mean $(6x)^2 - 2x$, even though the two expressions do not produce equal outputs for most inputs of x). A ***relational view of equivalence of expression (equations)*** would view transformation as a way of replacing one expression (equation) with an equivalent one: thus, transformations are only permitted if they preserve equivalence (i.e., produce an equivalent expression [equation]). Further, reasoning about or justifying which transformations are allowed requires determining which preserve equivalence. The relational conception allows for justification of computation, whereas a purely computational conception does not. In particular, because the relational conception requires the notion of *replacing* one expression (equation) with an equivalent one, this is an example of substitution equivalence. Thus, algebraic transformation could be reconceptualized as a process of substitution equivalence, which could be viewed as a ***relational conception of algebraic transformation***. This conceptualization then also links the action of transformation directly to its justification: namely, whether it is equivalence-preserving.

A Motivating Example

In order to provide a concrete example of how algebraic transformation can be conceptualized as a process of substitution equivalence, we choose one common algebraic task:

Example 1. Simplify $\frac{6x^2-2x}{2x}$ completely.

We have chosen this particular task because student errors from employing a computational rather than relational view of transformation are common (e.g., invalid "cancelling" of $2x$); thus it is an illustration of some affordances of taking a relational approach. Here is one possible way that substitution equivalence could be used to begin to simplify this expression:

Step 1	$\frac{6x^2-2x}{2x} = \frac{6x^2}{2x} - \frac{2x}{2x}$	$\frac{6x^2-2x}{2x}$ has the form $\frac{a-b}{c}$ (where $c \neq 0$), so we can use the property $\frac{a-b}{c} = \frac{a}{c} - \frac{b}{c}$ (where $c \neq 0$) by substituting $a = 6x^2$, $b = 2x$ and $c = 2x$ into the property, which gives us $\frac{6x^2-2x}{2x} = \frac{6x^2}{2x} - \frac{2x}{2x}$.
Step 2	$= \frac{(2x)(3x)}{2x} - \frac{2x}{2x}$	Because of the generalized associative/commutative property of multiplication, we can perform multiplication using any order or grouping (as long as only multiplication is involved). So $6x^2 = 6 \cdot (x \cdot x) = (2x) \cdot (3x)$.

Figure 1. Example substitution equivalence justification for solution to Example 1

Components of a Relational View of Transformation

If a learner has no experience with reasoning syntactically, the details in steps 1 and 2 above would produce a cognitive load that is too high for a learner to understand or reproduce this solution. However, the explanations in steps 1 and 2 can be broken down in to more discrete knowledge elements (which we call subdomains) that could each potentially be learned separately, and then reintegrated later. By identifying these elements separately, we serve two goals: 1) this may allow us to better diagnose which specific conceptions are the cause of observed difficulties; and 2) this may allow us to teach substitution equivalence in smaller "chunks", limiting the cognitive load placed on learners. In Figure 2 we illustrate the model; we begin by describing each subdomain individually.

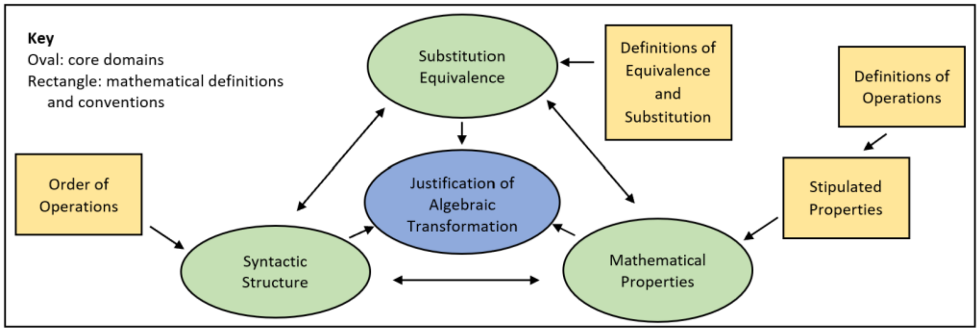

Figure 2. Model of Algebraic Transformation as Substitution Equivalence

Substitution Equivalence

To understand why $\frac{6x^2}{2x} - \frac{2x}{2x}$ can replace $\frac{6x^2-2x}{2x}$ in Step 1 above, a student must understand that any expression can be replaced with an equivalent one during the problem-solving process. Similarly, they must understand that if $a = 6x^2$, $b = 2x$ and $c = 2x$, then a, b and c can be replaced with $6x^2$, $2x$ and $2x$, respectively, in the property $\frac{a-b}{c} = \frac{a}{c} - \frac{b}{c}$, and the property will still be true because this is substitution of one subexpression for an equivalent one. During Step 2, they must also understand that $6x^2 = 6 \cdot (x \cdot x) = (2x) \cdot (3x)$ means that the subexpression $6x^2$ can be replaced by the equivalent subexpression $(2x) \cdot (3x)$, and the resulting expression will be equivalent to the original one. All three of these processes are facets of the concept of ***substitution equivalence***: that one mathematical object can be replaced with another during the problem-solving process if and only if those two objects are equivalent (Wladis et al, 2022a).

Equivalence: Having a conception of substitution equivalence requires some underlying definition of equivalence, which in this case could simply be a stipulated insertion equivalence definition of expressions (Zwetzschler & Prediger, 2014, e.g., two arithmetic expressions are equivalent if they produce the same result for every possible combination of variable values). Or it could be a different definition of equivalence (e.g., a generalized equivalence relation definition). Regardless of the definition, in order to use substitution equivalence, a student has to recognize some key characteristics of the concept of equivalence more generally: that equivalence is a *stable relationship between two specific objects, based on some well-defined criteria* (Wladis et al, 2022a, 2022b). In other words, equivalence is a relationship (not a computation) between two objects (it requires two things to be compared) that is well-defined (it requires an unambiguous set of criteria for determining whether those two things are equivalent), and finally: two things are either equivalent or not, and they stay that way (two objects are not equivalent sometimes and not others; they don't "become" equivalent with transformation—rather, transformation reveals a pre-existing equivalence relationship).

Substitution: Using substitution equivalence also requires a more general definition of substitution, in which it is conceptualized as the replacement of one unified subexpression with an equivalent unified subexpression. For example, if a student only conceptualizes "plugging a number in" for a letter as substitution, it becomes difficult to talk about replacing $6x^2$ with $(2x) \cdot (3x)$ in the expression $\frac{6x^2}{2x}$ to generate $\frac{(2x)\cdot(3x)}{2x}$ as a process of substitution equivalence.

Thus, the domain of substitution equivalence describes the extent a learner can conceptualize equivalence as a stable relationship between two objects that meet well-defined criteria, and the extent to which the learner understands that one object can be replaced with another during the problem-solving process if and only if the two are equivalent, based on a stipulated equivalence relationship. This is more than being able to execute more complex substitutions correctly; it includes the ability to justify transformations because they preserve equivalence, by describing the particular equivalence relationship that is preserved by that transformation.

Syntactic Structure

Understanding the concept of substitution equivalence alone is not enough to have a relational view of transformations: for example, in Steps 1 and 2 in Figure 2, in order to perform substitution correctly, it is necessary to identify the correct unified subexpressions that can be substituted out or in. Another skill that is necessary to tackle the task above is the ability to parse the intended meaning of the syntax $\frac{6x^2-2x}{2x}$ by recognizing which substrings of the expression can be treated as ***subexpressions***, or which substrings of the expression could have brackets placed around them without changing the syntactic meaning of the expression (i.e., the expression would still represent the same operations on the same objects in the same order). In this example, this would mean being able to recognize that this expression has the following syntactic meaning as it is currently written: $\frac{(6(x)^2)-(2x)}{(2x)}$, and that $6x^2$ and x^2 are both subexpressions, but $6x$ is not.

Just as substitution equivalence can be seen as a more structural than computational approach (i.e., as a relational versus a computational view), understanding syntactic structure can also be seen as a shift from thinking computationally to thinking structurally. This is related to what has been observed by Sfard (1991) and others (Asiala et al., 1997 Dienes, 1969, Dubinsky, 1991, Gray & Tall, 1994), when they observe that an expression like $\frac{6x^2-2x}{2x}$ can be viewed as a process of squaring x, then multiplying 6 by that result, then separately multiplying 2 by x, then taking that result away from the first result, then dividing that result by the result obtained after multiplying 2 by x again. Or, it can also be seen as an object that is a reification/encapsulation of a process: the anticipated final result of the process described above (whether or not one has actually carried that process out). We focus on this slightly differently by focusing on how and whether a learner is able to identify *subexpressions as objects*. This requires more than simply a reification of the process of computation, but rather, a reification of the process of the order of operations: in the computational view, a learner would conceptualize the process of the order of operations as telling us that x must first be squared, and then 6 must be multiplied afterwards by the result; in contrast, in the structural view, a learner would conceptualize the order of operations as being reified into a fixed structure where x^2 is conceptualized as a unified subobject (i.e., a subexpression) which is a part of the larger expression $6x^2$.

We call the domain that includes knowledge of how to parse algebraic symbolic representations and to identify subexpressions ***syntactic structure***. This includes not just the ability to normatively interpret the symbols, but also the ability to link that interpretation to a normative justification (i.e., by explaining how the order of operations and other stipulated conventions dictate which substrings are subexpressions). Our definition of syntactic structure is closely related to the notion of *surface structure* as defined by Kieran (1989) and others in linguistics (e.g., Chomsky, 1966)[1], and is also related to Malle's *Termstrukturen*, or "expression structuring" (1993). We discuss this in detail elsewhere (Wladis et al, 2022c).

Using Mathematical Properties, or "Form Mapping"

[1] Our use of the term *syntactic structure* should not be confused with Chomsky's use, which is different.

Substitution equivalence and syntactic structure alone are not enough to justify the transformation work in Figure 1. To use the property $\frac{a-b}{c} = \frac{a}{c} - \frac{b}{c}$ (when $c \neq 0$) to determine that $\frac{6x^2}{2x} - \frac{2x}{2x}$ is equal to $\frac{6x^2-2x}{2x}$ in Step 1, a learner must do several things. First, they must map one-to-one each subexpression in $\frac{6x^2-2x}{2x}$ to each variable in the "form" $\frac{a-b}{c}$ so every symbol in $\frac{6x^2-2x}{2x}$ gets mapped to a symbol with the same syntactic meaning in $\frac{a-b}{c}$, and the mapping preserves the relative order of all the subexpressions and symbols in the expression. Second, learners must use the form $\frac{a}{c} - \frac{b}{c}$ to map the same subexpression to the same variable in $\frac{a}{c} - \frac{b}{c}$ as they did in $\frac{a-b}{c}$. Thus, the ***using properties (or form mapping)*** domain describes the extent to which a learner can construct one-to-one mappings from a symbolic representation to a mathematical property so that every symbol is mapped to a symbol (or syntactic convention) with the same meaning in the property, and each variable in the property is mapped to a subexpression (with the same variables mapped to the same or equivalent subexpressions).

As with the other two subdomains, this also involves a shift from a process to object view: the student must shift from conceptualizing the use of properties as plugging in one particular set of values (or variables) into the property, to thinking of the property itself as a canonical representation of a particular existing structure in the expression which they are attempting to transform. The form mapping required to use the property on more complex expressions requires that the student be able to think structurally about subexpressions as objects in the expression that they are trying to transform, as well as the relationship between these various sub-objects, and whether this is the same relationship as the relationship between various variables in the property. They must also have a relational view of equivalence, as the property must be conceptualized as a statement about the relationship between the original expression and the transformed result. A student might also reify the process of substitution into the particular form mapping object itself. There are many different objects which the student could conceptualize (the form mapping itself; the property as a canonical representation of structural relationships, etc.); the key difference is that the student is doing more than simply "moving around" symbols in the expression in an attempt to produce a pattern that "looks like" the property.

This domain includes not just the ability to use properties to correctly transform one expression or equation into an equivalent one, but the ability to reason or justify how a particular structural mapping of subexpressions and symbols in the expression/equation to various variables and symbols in the property allows us to make an argument about the equivalence of the original expression/equation and the resulting expression equation.

Learning subdomains serially

The subdomains of substitution equivalence, syntactic structure, and using properties are all deeply interconnected, and are all necessary in order to conceptualize algebraic transformation as substitution equivalence. But they need not be learned all at once; the cognitive load of such a task is likely to be too demanding for learners with limited prior experience with syntactic reasoning. Thus, as a brief illustration, we demonstrate some ways in which aspects of these domains might initially be learned separately, or serially (and then later re-combined). In other contexts, this approach has been successful at improving student learning of complex ideas by reducing cognitive load (e.g., Pollock et al., 2002).

Syntactic structure and subexpressions

There are many ways that we could ask students questions that only draw on their knowledge of syntactic structure, and not require other types of complex and interrelated syntactic reasoning skills. For example, consider the following question, which limits the task not only to just identifying syntactic meanings, but also to identifying only *one* syntactic meaning at a time, significantly reducing the number of elements which must be held in working memory: “In the expression $\frac{6x^2-2x}{2x}$, what is being squared? Use the order of operations to justify your choice.” In other research, we have found that many college students identify $6x$ as the base of the exponent instead of x (Wladis et al, 2022c), often because they have extracted their notions of which subexpressions “look right” based on experience, rather than reifying them from the process of the order of operations (even when they can recite the order of operations correctly, or use it to calculate correctly with numbers). This suggests that it may be essential to tackle syntactic structure individually, before proceeding to other syntactic reasoning skills which may be more complex and interrelated, and all of which depend upon a student first being able to identify the “right” subexpressions in an expression to be transformed.

It may also be necessary to ask students whether there is more than one right answer to this question. In our research, we have encountered students at many levels who have explained that an expression can have multiple correct meanings, where different meanings provided by the student are not equivalent (Wladis et al, 2022c). Thus, another component of this subdomain is discussing with students that all expressions must be well-defined, with one unambiguous meaning. Students may not realize that this is a core tenet of mathematics.

We note that determining whether a student has an object or a pseudo-object conception of the order of operations may be difficult to determine when looking only at “standard” problem contexts. Students have created pseudo-object mental schema precisely because they appear to mimic the subexpression structurings of expressions and equations that “work” during situations seen during instruction (e.g., Aly, 2022; Erlwanger, 1973). Often it only becomes obvious that students’ justifications for choosing certain subexpressions are not mathematically valid when students are given more “non-standard” problems, or when students are asked directly how their choice of sub-expression relates to the order of operations (in our research, a common response, even from students in higher-level courses such as calculus was “it doesn’t relate to the order of operations” [Wladis et al, 2022c]). Giving students explicit instruction in syntactic structure may act to mitigate this issue that has been observed elsewhere in the literature.

Substitution Equivalence

As with the syntactic structure subdomain, the substitution equivalence domain (and by extension equivalence subdomain) can be thought of as an element which could be learned separately, to reduce the cognitive load of learning syntactic reasoning all at once. For example, in this particular problem, it might be important to find out if a learner understands that replacing $\frac{6x^2-2x}{2x}$ with $\frac{6x^2}{2x}-\frac{2x}{2x}$ during step 1 is a process of replacing an expression with another equivalent expression. Some students may have a computational rather than relational view of equivalence of expressions or equations, where they see $\frac{6x^2}{2x}-\frac{2x}{2x}$ as the result of “doing something” directly to $\frac{6x^2-2x}{2x}$ and do not see an equivalence relationship between the two expressions (or may not even conceptualize equivalence as a relationship between two things, but rather as a process of computation) (Wladis et al, 2022a, 2022b). This can be seen particularly clearly when we look at the substitution equivalence that the student needs to recognize in order to perform step 2. We could, for example, limit the cognitive load of that step almost exclusively to the process of substitution equivalence if we asked: "Suppose that $6x^2 = (2x) \cdot (3x)$. Use this fact to replace the expression $\frac{6x^2}{2x} - \frac{2x}{2x}$ with an equivalent expression, and to explain why the new expression is equivalent to $\frac{6x^2}{2x} - \frac{2x}{2x}$." As long as a student has enough understanding of syntactic structure to know that the numerator of a fraction is always a subexpression, and they understand the notion of substitution equivalence, this information should be sufficient for them to be able to replace $\frac{6x^2}{2x} - \frac{2x}{2x}$ with $\frac{(2x)(3x)}{2x} - \frac{2x}{2x}$ and to explain why the two expressions are equivalent. This can later be combined with more robust knowledge of identifying subexpressions to combine the two domains of substitution equivalence and syntactic structure, after students have had opportunities to master key conceptions in each subdomain separately.

Using Properties/Form Mapping

As with the syntactic structure and substitution equivalence domains, the using properties/form mapping domain can also be thought of as an element that could be learned separately, in order to reduce the cognitive load of learning syntactic reasoning all at once. For example, in this particular problem a student could be asked: "Let $a = 6x^2$, $b = 2x$ and $c = 2x$. Then use the property $\frac{a-b}{c} = \frac{a}{c} - \frac{b}{c}$ (where $c \neq 0$) to explain why $\frac{6x^2-2x}{2x} = \frac{6x^2}{2x} - \frac{2x}{2x}$." In this example, students need to have a basic idea of substitution equivalence, but they do *not* need to be able to identify the subexpressions of $\frac{6x^2-2x}{2x}$, as this has already been done for them. Thus, a task like this could be used to allow students to practice their knowledge in the domain of using properties, without yet requiring substantial knowledge of syntactic structure, and thus reducing the learner's cognitive load by allowing them to focus on fewer domains at a time.

After a leaner has had the opportunity to master substitution equivalence and syntactic structure, these separate conceptions that have been learned serially could be reintegrated with the using properties domain to complete questions like Example 1 without being given the specific values for *a*, *b* and *c*. Then, once the conceptions necessary to engage with these types of problems have been mastered, students could be given questions where they need to choose the particular property that could be fitted to the structure of a given expression (or equation); after that, they could be asked to select the property which serves a particular goal (e.g., producing an equivalent expression without parentheses, or with a particular form, etc.); and finally, after mastering each of these serialized tasks, they could progress to being asked to plan out the usage of a sequence of properties necessary to accomplish some larger goal.

Conclusion

In this theoretical paper we have aimed to identify and explore some necessary (but not necessarily sufficient) types of knowledge that are essential for students to be able to transform algebraic expressions and equations with understanding (by which we mean, to be able to reason about and justify these transformations in mathematically valid ways). We have framed this around the lens of substitution equivalence, with the aim of deconstructing complex knowledge structures into simpler component subdomains which could be learned serially before being reintegrated, to allow students with low prior knowledge in syntactic reasoning to build up this knowledge in ways that do not overload working memory. Our hope in presenting this model is to start a conversation about how we could more explicitly address reasoning and justification when teaching, and assessing learners' knowledge in, algebraic transformation.

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