AN EXPLORATION OF HOW COLLEGE STUDENTS THINK ABOUT PARENTHESES IN THE CONTEXT OF ALGEBRAIC SYNTAX

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In this paper we explore how college students across different courses appeared to interpret the meaning of parentheses or brackets in the context of algebraic syntax. This work was influenced by theories of computational vs structural thinking, and also considered the extent to which students' definitions, computational work, and explanations appeared to be consistent with specific normative definitions of parentheses. In analyzing student work, several categories of students' conceptions emerged, which may be helpful in diagnosing which conceptions may be more productive or problematic as students progress through algebra. For students who appear to conceptualize parentheses as a cue to non-normative procedures, several categories of procedures were found, which could have implications for instruction.

Keywords: algebra and algebraic thinking; cognition

Parentheses, or brackets, play a critical role in symbolic mathematics. However, how students think about parentheses in various mathematical contexts is significantly understudied. While some studies examine how parentheses may or may not cue the correct order of operations during arithmetic tasks in primary school or whether adding extra parentheses helps students to better "see" algebraic structure (e.g., Gunnarsson et al., 2016), there appears to be no systematic research which explores which kinds of meanings students have for brackets or parentheses, particularly in a wider array of content domains which include substantial symbolic algebra. In this paper we aim to address this gap. Though extensive data collection with college students in a wide range of classes, we explore students' meanings for parentheses and generate several common categories of conceptions which students may hold.

Literature Review

Research into student's use of parentheses in algebra has been limited, and what does exist has focused primarily on what students do with parentheses when calculating rather than how students conceptualize parentheses. In the context of structure sense, Hoch and Dreyfus (2004) found that secondary students tend to use structural approaches to solve algebraic equations more when an equation uses more parentheses than when less parentheses are present. In interviews, students' use of parentheses were mixed, where some students approached problems by first 'opening' parentheses, while others preferred maintaining parentheses, finding the symbol helpful and saying "with parentheses, it's easier to see" (p.3-54). Similar findings have been reported with middle school students as well (e.g., Linchevski & Livneh, 1999), where some students at times operated on parentheses within expressions as if they are not present (Gunnarsson et al., 2016), while others operated in markedly different ways when working with expressions within parentheses (Banerjee & Subramaniam, 2005). Given that the presence of parentheses may change one's view of a mathematical expression, we think it is worthwhile to explore students' meanings of parentheses.

Theoretical Framework

Tall and Vinner (1981) describe an individual's understanding of a concept in terms of their concept image, or the "total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (p. 152). Within this structure, they describe one's *personal* concept definition as the "form of words that a learner uses for [their] own explanation of [their] (evoked) concept image" (p. 152). One's personal concept definition is idiosyncratic to the learner, and is contrasted with the *formal* concept definition, which is "a concept definition which is accepted by the mathematical community at large" (p.152). In the examples that follow, when we refer to a student's definition, we are referring to their personal concept definition, although our model is a second order model, as we do not have direct access to student's personal concept images or definitions.

In the context of this paper, we consider the role that parentheses play in the syntax of algebraic expressions and equations in which the symbols are used to indicate an operation which is prioritized over other adjacent operations; we call this the ***grouping role*** of parentheses. This role of parentheses is pervasive throughout symbolic representations across domains, but we focus on algebra examples in this paper. We note that this grouping role is distinct from other roles which parentheses may play, such as denoting intervals, ordered pairs, sets, etc.

Students may conceptualize parentheses from either a computational or a structural view, just as they may conceptualize algebraic syntax more generally as computational[1] or structural (Sfard, 1995). In a ***computational view***, students conceptualize parentheses as cuing a particular calculation or procedure, which may be normative or non-normative. In a ***structural view***, students conceptualize parentheses as demarcating a particular unified sub-expression which can be treated as an object in and of itself. For example, a student with a computational view may normatively "see" the expression $2(5 + 3)$ as indicating that one should first add $5 + 3$ and then multiply 2 by the result. A student with a structural view is able to "see" this also as 2 times whatever the result of $5 + 3$ might be, without actually performing the addition of $5 + 3$ and replacing it with the single number 8 first; instead the substring $(5 + 3)$ can be seen as an object itself. In an arithmetic example such as this one, the structural view appears less important, but in an algebraic expression like $2(x + 3)$, the affordances of a structural view suddenly becomes more apparent and, at times, necessary, for example, for many cases of *u*-substitution in upper level classes where one must have a structural view of certain substrings as objects.

Edwards and Ward (2004) distinguished between two sorts of definitions: those that are created through experiences with the term (an *extracted* definition), and those that are expressed in an explicit well-defined way (a *stipulated* definition). When students have a computational view of parentheses, this may be stipulated and normative or non-normative (which may have been extracted from their experiences with common types of tasks in which parentheses occur, rather than taken from stipulated normative definitions). For some students, parentheses may only cue the normative set of procedures stipulated by the order of operations, while for others, parentheses may cue a particular procedure, regardless of whether that procedure is appropriate (e.g., taking whatever is outside the parentheses and multiplying it by each "thing" inside the parentheses, even when the structure of the algebraic expression with which they are working is not in line with the distributive property). We summarize this in the Figure 1, where we see learners' conceptions varying along a continuum, where conceptions to the right are more generalizable and transferable to a wider range of problems than those to the left. We note that an individual's conceptions of parentheses are not fixed and can further vary back and forth along

1 Sfard (1995) refers to 'operational' views, but we use the term computational here.

this continuum, even within a single problem.

Extracted computational view: Learners conceptualize brackets as a cue to a procedure, but that procedure is not consistent in all contexts with the stipulated order of operations and other symbolic conventions (e.g., to multiply regardless of actual operations).

Stipulated normative computational view: Learners are able to conceptualize brackets as a cue to follow the stipulated order of operations (i.e., simplify what is inside the innermost brackets first).

Structural view: Learners are able to conceptualize brackets as demarcating a unified sub-expression within a larger expression or equation, which can be thought of as an object in and of itself. This may be the reification of the process of the stipulated order of operations.

Figure 1: Types of Conceptions of Parentheses

Methods

Data for this study includes students' written responses to open-ended questions and transcripts from stimulated recall interviews about similar multiple-choice questions. Open-ended responses were collected from 124 students at an urban community college in 18 different courses, from developmental elementary algebra (similar to Algebra I in high school) to linear algebra. Open-ended questions were about what parentheses means to the students (both in and out of context), as well as algebra problems where students were asked to simplify expressions that included parentheses. Students' open-ended responses were analyzed using thematic analysis (Braun & Clarke, 2006). This analysis was influenced by an initial theoretical stance which was particularly attuned to noticing similarities in patterns of students' responses with theory on extracted and stipulated definitions (Edwards & Ward, 2004) and computational and structural (Sfard, 1995) conceptions. Analysis of students' work led to a more nuanced emergent coding scheme of students' responses for definitions of parentheses. We select students' work to illustrate the resulting coding scheme in this work and draw on interview transcripts to further expound on these patterns.

Results

In this section we present several vignettes to illustrate the different ways in which students appeared to be conceptualizing parentheses in the context of algebra expressions and equations. We note that categorizations of student work here are second order models: we cannot actually know what a student is thinking—we only categorize what they conveyed through their work.

Different non-normative computational conceptions of parentheses

It was common for parentheses to cue various non-normative procedures, though they tended to fall mostly into one of three categories: "Multiplication", "do first", and "ease of reading".

"Multiplication" conception of parentheses. The most common response among students at all levels when asked what parentheses mean in mathematics was "multiplication". We note that technically parentheses do not indicate multiplication; they simply co-occur often with concatenation, which actually indicates multiplication. However, during data collection in both interviews and open-ended questions, we found that students rarely recognized this distinction. This may be an extracted definition taken from their many experiences with tasks in which parentheses and multiplication co-occur, or it may even sometimes be inaccurately stipulated by instructors (we have observed instructors in classrooms using phrases like "parentheses mean to multiply"). So while not technically mathematically normative, this view can be a very rational reaction to existing instructional practices. However, this conception may lead to incorrect computational results when the operation next to the parentheses is not multiplication, as well as a fundamental misunderstanding of the role that parentheses are intended to play in algebraic expressions and equations. Consider, for example, the work from Alpha (see Figure 2) who was enrolled in an elementary algebra course (a non-credit course similar in content to Algebra I).

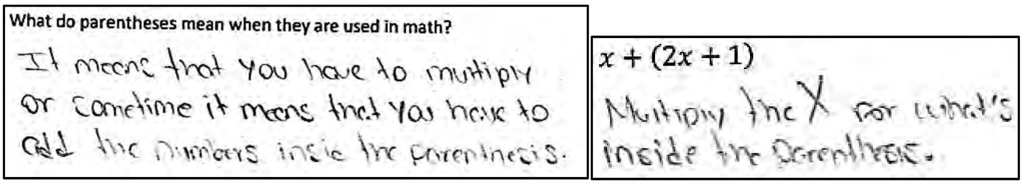


Figure 2. Alpha's Parentheses Definition and Evoked Interpretation in Computation

Alpha's definition of parentheses (Figure 2, left) shows two meanings. Firstly, they tell us that parentheses tell us to multiply; but they also say that parentheses tell us that we have to add the numbers inside the parentheses, which we understand to be a kind of "do first" interpretation of parentheses (see next section). Alpha further provides an example of how the interpretation of parentheses as multiplication can be problematic during computation on the right in Figure 2, where they multiply even though addition is the operation between x and $(2x + 1)$.

"Do first" conception of parentheses. The second meaning Alpha's definition (Figure 2, left) may be referring to is a potentially normative definition of parentheses-- it is not completely clear from their wording, but they may be referring to the precedence of parentheses in the order of operations. We describe this thinking more in the next section, but focus now on some ways in which students applied this definition non-normatively. We begin by considering the work of a student whom we will call Gamma, who was enrolled in a course for students training to be elementary school teachers (prerequisites for this course include elementary and intermediate algebra, but not precalculus). In this excerpt, Gamma is attempting to simplify $3 - 2 \cdot (8 - 3^2)$.

> So, I try in my head, so they break it down, PEMDAS, so parentheses first, so I did like eight minus three is five and I know that's going to leave me with five squared. So, I just left that as it is and looked over here and seen what has five to the second exponent. And realized, you know, okay that's the same because that would be my next step is to solve the parentheses.... Because PEMDAS I solved what's in the parentheses first and then looked at the exponents and then that's pretty much how I saw it.

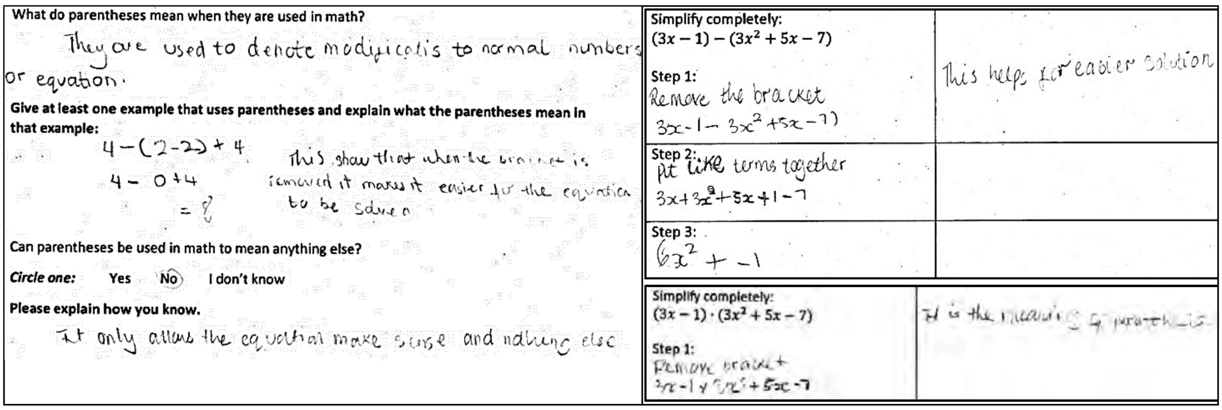
Gamma appears to be thinking of the parentheses as something which cues work inside them to be done first, and because exponents come after parentheses, they do $8 - 3$ first. We classify this as an extracted view, because their notion of "doing the parentheses first" is cuing an incorrect extracted definition of the order of operations, in which things inside the parentheses are combined together in order to "get rid of" the parentheses before doing the exponent that is inside the parentheses itself. Here Gamma appears to be interpreting the parentheses as a cue to a particular (incorrect) computational action.

In other work, Gamma is able to use this conception of parentheses to simplify an algebraic expression correctly, although they still use a computational approach. In the next excerpt, they were asked about what was being multiplied by 2 in the expression $2 \cdot (3x - 5)$. We note that even though the question was intended to prompt an object approach (by thinking of $(3x - 5)$ as the object being multiplied by 2), Gamma still gives a computational explanation focused on "solving", even though that is not what has been asked.

When I think of parentheses it's something that has to be done first. In this particular problem, I feel like you have to distribute because you have x there. So, it's like you can't solve $3x$ minus five. I don't think you could just like get an answer from that. You have to solve what's inside of the parentheses. So, what's in the parentheses is $3x$ minus five. So, in order for me to solve that I must distribute the two that's outside into that equation. I just think of PEMDAS, you have got to do parentheses first, and then exponents, and down the line. So, I just look at a question, I know I have to do something with the parentheses first.

Gamma's notion of parentheses here appears to be a "do first" conception—they appear to be correctly connecting their notion of the meaning of the parentheses to the stipulated order of operations. However, it is not completely clear whether this "do first" notion is fully well-defined and in line with stipulated definitions. Would Gamma distribute in other problems inappropriately, for example if the two were an exponent of the parentheses instead of a coefficient? We cannot be sure since Gamma was not interviewed about such questions. During data collection, we saw many students who seemed to convey a "do first" definition of brackets/parentheses, but who performed computational work that violated the order of operations in some cases and not in others. This suggests that we may want to be more attentive to how students interpret the "do first" conceptualization of parentheses as they move from arithmetic expressions (in which what is inside the parentheses can be simplified) to algebra (where what is inside the parentheses may not be able to be simplified).

"Ease of reading" conception of parentheses. Now we consider some examples in which students appear to have interpreted parentheses as unnecessary symbols which can be removed; sometimes they justify this by explaining that parentheses are just there to make expressions or equations "easier to read". We note that this is related to the "bracket ignoring" procedure identified by Gunnarsson, et al. (2016) with middle school students. First we consider work from an elementary algebra student whom we call Tau (Figure 3).


What do parentheses mean when they are used in math?

They are used to denote modificatis to normal numbers or equation.

Give at least one example that uses parentheses and explain what the parentheses mean in that example:

4 − (2 − 2) + 4
4 − 0 + 4
= 8

This show that when the bracket is removed it makes it easier for the equation to be solved

Can parentheses be used in math to mean anything else?

Circle one: Yes No I don't know

Please explain how you know.

It only allows the equation make sense and nothing else

Simplify completely:
$(3x - 1) - (3x^2 + 5x - 7)$

Step 1:
Remove the bracket
3x − 1 − 3x² + 5x − 7)

This helps for easier solution

Step 2:
Put like terms together
3x + 3x² + 5x + 1 − 7

Step 3:
6x² + −1

Simplify completely:
$(3x - 1) \cdot (3x^2 + 5x - 7)$

Step 1:
Remove bracket
3x − 1 × 3x² + 5x − 7

It is the meaning of parenthesis

Figure 3. Tau's Definition of Parentheses and Computational work

Tau appears to conceptualize parentheses as something that makes it "easier for the equation to be solved" or "allows the equation to make sense". It is not entirely clear what they mean in this case—for example, in the arithmetic example that they give in Figure 3, their calculations appear to be correct. They "remove" the parentheses in that case by simplifying what is inside them first. However, this seems not to generalize in the same way to algebra cases. When we look at how they dealt with parentheses in practice when solving standard algebra tasks, we see that their first step was often to remove the parentheses before proceeding (see right of Figure 3), similarly to erasing the parenthetical symbols. Tau even justify their work by writing "Remove the bracket. This helps with easier solution" and "Remove bracket…It is the meaning of parentheses." We suspect that Tau has extracted from their prior mathematical experiences that it is desirable to find valid mathematical ways to "get rid of the bracket", this process involves simply removing the bracket, and parentheses are superfluous.

This may also be related to the "do first" conception of parentheses, as students may feel that they must remove the parentheses in order to do any computation with what is inside. Some students may remove the parentheses using a valid transformation (e.g., the distributive property, or expanding multiplication of two polynomials) and other students may remove the parentheses using an invalid transformation, like Tau. This approach was not limited only to elementary algebra students. In Figure 4, we see similar work from a Calculus I student, Zeta.

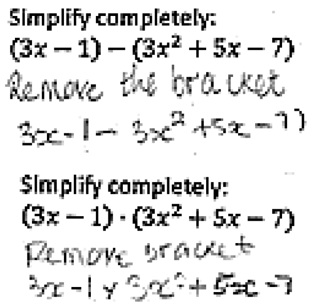


Figure 4. Zeta's Work in which They Remove Parentheses Arbitrarily

Combinations of different extracted and stipulated computational views.

So far we have seen some examples of different types of non-normative extracted conceptions of parentheses, but there were also students who showed a mix of normative and non-normative meanings for parentheses. One such example comes from a student, Epsilon, enrolled in the second semester of a one-year math course for future elementary school teachers (the prerequisites for the first semester of the course were elementary and intermediate algebra, but not precalculus). In an interview, Epsilon was asked which part was being subtracted, or taken away, in the expression $3x^2 - 2(x^2 + 1)$:

Interviewer:	What do the parentheses mean here in this expression?
Epsilon:	Multiplication.
Interviewer:	Do you know what the order of operations is?
Epsilon:	PEMDAS.
Interviewer:	How does the order of operations help you to understand what is being subtracted in this expression?
Epsilon:	So first, you have to deal with the parentheses. So, then that means that everything with the parentheses is like together in a box.

Epsilon, like many students, thinks of multiplication first when they are asked what parentheses mean. When asked directly about how the order of operations relates to what they are doing in the problem, they exhibit the "do first" notion of parentheses (we can't tell from this context whether it is normative or not), pointing out that "first, you have to deal with the parentheses". But they immediately follow that up with a structural grouping interpretation of parentheses, saying “that everything with the parentheses is like together in a box”. It is unclear to us how Epsilon is interpreting parentheses to mean multiplication here, but when they cite a “do first” conception of parentheses they immediately link it to a grouping conception, where the “do first” conception is a justification (“then that means”) for the grouping conception. While we cannot be certain, we suspect that Epsilon has reified the order-of-operations process into the grouping object, which would explain how their “do first” and grouping conceptions are linked.

Some students may use the language “parentheses mean multiplication”, but then go on to only describe a correct alternative meaning, at least in the context of Algebra I course content; this may be a case in which they are incorrectly describing the meaning of the parentheses symbol, but are correctly performing computation with parentheses, so while logically incorrect, this imprecision of language may be less problematic in this case. For example, consider the next excerpt from Epsilon, in which they discuss their meanings of parentheses when asked whether $\sqrt{(3xy)^2}$ and $\left(\sqrt{3xy}\right)^2$ represent the same operations on the same things in the same order:

> [Parentheses] don't really mean like multiplication per say in this instance. So, in the first one the parentheses is the expression of $3xy$ to the second power. So, it's distributing the- Well, in both instance it's distributing the second power, but it's just where it's placed. I'm sorry, I don't know if I'm making any sense, but because of where the parentheses are placed it just changes the meaning.
>
> So, here [pointing to first expression] you have to deal with the parentheses first. So, in the first equation, you have to distribute the second power to the $3xy$ and then take the square root. And then in the second equation, you have to take the square root first of $3xy$ and then square it and those would result in different answers probably.

In this explanation, Epsilon says that parentheses mean multiplication, but then they say that this is not necessarily the correct interpretation in this case. Here instead they treat the parentheses as a cue to distribute the exponent. Their work is correct, but it is unclear if this notion of parentheses cuing distributing might lead them to distribute the exponent inappropriately in other cases (e.g., if instead of $3xy$ in the parentheses, there were a multi-term expression). As with Gamma, we had no questions on the set of questions on which Epsilon was interviewed which would allow us to see if Epsilon might distribute incorrectly in other cases. So while Epsilon’s work here appears to be based on stipulated definitions, and on the surface appears to be less extracted and less problematic than Alpha’s multiplication conception in terms of how it may impact their computational work, we cannot be sure whether it is actually completely grounded in standard stipulated definitions or whether their approach here might prove more problematic on other questions. Compared to Alpha, Epsilon at least recognizes that parentheses do not always “mean” multiplication, and therefore Epsilon may be more receptive to rethinking their initial statement that parentheses themselves “mean” multiplication. However, more detailed analysis of Epsilon’s thinking is necessary if we are to understand whether it aligns with normative meanings for parentheses and is just being expressed in an ill-defined way, or if it actually conflicts with normative meanings.

We note also that Epsilon’s explanations in this excerpt are entirely computational: e.g., they describe the process of “distributing” the square to $3xy$. This is a common pattern where students appear to rely on the notion of distributing rather than thinking of the whole $3xy$ as a single object being squared. There is not necessarily anything wrong with the way that Epsilon has explained this; however, being able to conceptualize $(3xy)$ as an object itself might be essential in other contexts, so it might be important to further assess the extent to which Epsilon is able to do this when needed. For example, we note that Epsilon struggles to explain their thinking here; it is possible that if they were able to describe this using a structural view (e.g., "in the first expression only $3xy$ is being squared, but in the second expression $\sqrt{3xy}$ is being squared"), it might be easier for them to provide a justification for their thinking.

Grouping view of parentheses

We now consider one more example in which a student appears to be conceptualizing parentheses structurally as a grouping mechanism. The following excerpt is a Calculus I student Theta's response after being asked what the result would be of substituting $2y$ in for x into the expression $2x^2 - 7x + 3$.

> If you're putting something in for something else, I always usually keep parentheses around it to make sure that I maintain whatever structure the original function had... especially in this one [the first term] because x is being multiplied, and squared, the parentheses really make a difference because if we don't have them, we could get a different, the wrong answer…. you don't need [the parentheses in $-7(2y)$] as long as you make sure you multiply the -7 by 2.

Theta mentions structure explicitly and explains how removing the parentheses gives the expression a different syntactic meaning, which in the case of $2(2y)^2$ will result in a "different result" but in the case of $7(2y)$ will not yield a different result, seemingly because of the order of operations (although they do not use the word "order of operations" explicitly). Theta's explanation shows how they appear to have linked their structural view of the parentheses to their conception of the order of operations, suggesting that it is a reification of this process.

Implications and Conclusion

This paper is an initial exploration of how students may conceptualize brackets and parentheses and adds to the literature by providing hypothetical models of what students may attribute to parentheses or brackets. We make no recommendations about how students should be taught about parentheses, however we think that these students' responses do suggest that they extract many non-normative meanings of parentheses in algebraic syntax and this may impact students' computational work. This suggests that it may be important for instructors and curricula writers to think carefully about whether their language or examples may be encouraging students to think of parentheses as "meaning multiplication", as an instruction to "do something" (where what should be done and why may be ill-defined or non-normative), or as something that is always superfluous. We suspect that simple phrases like "do the parentheses first" are being interpreted by students in a number of ways and may need to be articulated more clearly to students. It may be helpful to provide a variety of examples where parentheses are and are not extraneous to support students in coming to more normative understanding of the role that parentheses are intended to play in algebraic expressions and equations. In addition, it may be important for future research to explore in more detail how a learner comes to reify the normative "do first" view of parentheses into a grouping conception of parentheses, in which they are able to "see" the substring inside the parentheses as a unified subexpression. Thus, further research which explores the prevalence of these various views among different student groups as well as its relationship to computational work could help to improve instruction.

Acknowledgments

The research reported in this paper was supported by the National Science Foundation (grant no. DUE-1760491). The opinions expressed here are those of the authors and do not represent those of the granting agency

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