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Big Ramsey degrees in universal inverse limit structures

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Abstract

We build a collection of topological Ramsey spaces of trees giving rise to universal inverse limit structures, extending Zheng's work for the profinite graph to the setting of Fraïssé classes of finite ordered binary relational structures with the Ramsey property. This work is based on the Halpern-Läuchli theorem, but different from the Milliken space of strong subtrees. Based on these topological Ramsey spaces and the work of Huber-Geschke-Kojman on inverse limits of finite ordered graphs, we prove that for each such Fraïssé class, its universal inverse limit structure has finite big Ramsey degrees under finite Baire-measurable colorings. For such Fraïssé classes satisfying free amalgamation as well as finite ordered tournaments and finite partial orders with a linear extension, we characterize the exact big Ramsey degrees.

Keywords Universal inverse limit structure · Fraïssé class · Big Ramsey degree · Tree · Topological Ramsey space

Mathematics Subject Classification Primary 05C15 · 05D10 · Secondary 05C69 · 05C05 · 03E02

1 Introduction

Structural Ramsey theory originated at the beginning of the 1970's in a series of papers (see [24]). Given structures $\boldsymbol{A}$ and $\boldsymbol{B}$, let $\binom{\boldsymbol{B}}{\boldsymbol{A}}$ denote the set of all copies of $\boldsymbol{A}$ in $\boldsymbol{B}$. We write $\boldsymbol{C} \longrightarrow (\boldsymbol{B})^{\boldsymbol{A}}_{l,m}$ to denote the following property: For every finite coloring $c : \binom{\boldsymbol{C}}{\boldsymbol{A}} \longrightarrow l$, there is $\boldsymbol{B}' \in \binom{\boldsymbol{C}}{\boldsymbol{B}}$ such that c takes no more than m colors on $\binom{\boldsymbol{B}'}{\boldsymbol{A}}$. Let

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$\mathcal{K}$ be a class of structures. The (small) Ramsey degree of $\boldsymbol{A}$ in $\mathcal{K}$ is the smallest positive integer m, if it exists, such that for every $\boldsymbol{B} \in \mathcal{K}$ and every positive integer $l \geq 2$ there exists $\boldsymbol{C} \in \mathcal{K}$ such that $\boldsymbol{C} \longrightarrow (\boldsymbol{B})^{\boldsymbol{A}}_{l,m}$. The class $\mathcal{K}$ is said to be a *Ramsey class* if the Ramsey degree of every $A \in \mathcal{K}$ is 1. Ramsey classes are the main topic of interest of structural Ramsey theory. Many Ramsey classes are known. Examples relevant for our presentation include the classes of finite ordered graphs, finite ordered k-clique free graphs with $k \geq 3$, finite ordered oriented graphs, finite ordered tournaments, and finite partial orders with a linear extension.

Given an infinite structure $\boldsymbol{S}$ and a finite substructure $\boldsymbol{A}$, the *big Ramsey degree* of $\boldsymbol{A}$ in $\boldsymbol{S}$ is the smallest positive integer m, if it exists, such that $\boldsymbol{S} \longrightarrow (\boldsymbol{S})^{\boldsymbol{A}}_{l,m}$ for every $l \geq 2$. Research on big Ramsey degrees has gained recent momentum due to the seminal paper of Kechris et al. in [15], and the results by Zucker in [31] connecting big Ramsey degrees for countable structures with topological dynamics, answering a question in [15].

The history of big Ramsey degrees for countably infinite structures has its beginnings in an example of Sierpiński, who constructed a 2-coloring of pairs of rationals such that every subset forming a dense linear order retains both colors. Later, Galvin proved that for every finite coloring of pairs of rationals, there is a subset forming a dense linear order on which the coloring takes no more than two colors, thus proving that the big Ramsey degree for pairs of rationals is exactly two. This line of work has developed over the decades, notably with Laver proving upper bounds for all finite sets of rationals, and culminating in Devlin's calculations of the exact big Ramsey degrees for finite sets of rationals in [6].

The area of big Ramsey degrees on countably infinite structures has seen considerable growth in the past two decades, beginning notably with Sauer's proof in [28] that every finite graph has finite big Ramsey degree in the Rado graph, which is the Fraïssé limit of the class of all the finite graphs, and the immediately following result of Laflamme et al. in [16] characterizing the exact big Ramsey degrees of the Rado graph. Other recent work on big Ramsey degrees of countable structures include ultrametric spaces (Nguyen Van Thé [25]), the dense local order (Laflamme et al. [17]), the ultrahomogeneous k-clique free graphs (Dobrinen, [7, 8]), and, very recently, the following: [2, 3, 5, 14, 18, 19, 32]. For more background in this area, we refer the reader to the excellent Habilitation of Nguyen Van Thé [26] and a more recent expository paper of the first author [9].

Results on big Ramsey degrees for uncountable structures are even more sparse than for countable structures. Ramsey theorems for perfect sets mark a beginning of this line of inquiry, and most of these theorems have at their core either the Milliken theorem ([20]), or the Halpern-Läuchli theorem ([12]) on which Milliken's theorem is based. For example, Blass proved in [4] the following partition theorem for perfect sets of $\mathbb{R}$, which was conjectured by Galvin (see [10]), who proved it for $n \leq 3$.

Theorem 1.1 (Blass [4]) *For every perfect subset P of $\mathbb{R}$ and every finite continuous coloring of $[P]^n$, there is a perfect set $Q \subseteq P$ such that $[Q]^n$ has at most $(n-1)!$ colors.*

In the proof of this theorem, Blass defined patterns for finite subsets of a perfect tree T such that for every finite continuous coloring of finite subsets of the nodes in T, one can make all subsets with a fixed pattern monochromatic by going to a perfect subtree. Todorčević (see [29], Corollary 6.47) provided a simpler proof of Blass' theorem using the Milliken space, as the perfect trees in Blass' argument can be replaced by strong subtrees.

Given a set X, a subset Y is called an n-subset of X if Y is a subset of X of size n. Let $[X]^n = \{Y \subseteq X : |Y| = n\}$ be the set of all n-subsets of X. For a graph G, let $V(G)$ denote its vertex set and $E(G) \subseteq [V(G)]^2$ denote its edge relation, that is, $E(G)$ is an irreflexive and symmetric binary relation. An inverse limit of finite ordered graphs is called *universal* if every inverse limit of finite ordered graphs order-embeds continuously into it. Geschke (see [11]) proved the existence of a universal inverse limit graph. Moreover, Huber-Geschke-Kojman (see [13]) gave the definition of a universal inverse limit graph with no mention of an inverse system.

Definition 1.2 (*Huber-Geschke-Kojman* [13]). A universal inverse limit of finite ordered graphs is a triple $G = \langle V, E, < \rangle$, such that the following conditions hold.

(1) V is a compact subset of $\mathbb{R}\backslash\mathbb{Q}$, $E \subseteq [V]^2$, and $<$ is the restriction of the standard order on $\mathbb{R}$ to V.
(2) (Modular profiniteness) For every pair of distinct vertices $u, v \in V$, there is a partition of V to finitely many closed intervals such that
   (a) u, v belong to different intervals from the partition;
   (b) for every interval I in the partition, for all $x \in V\backslash I$ and for all $y, z \in I$, $(x, y) \in E$ if and only if $(x, z) \in E$.
(3) (Universality) Every nonempty open interval of V contains induced copies of all finite ordered graphs.

Based on the way Blass proved Theorem 1.1 by partitioning finite subsets into patterns in [4], Huber, Geschke and Kojman proved in [13] the following partition theorem for universal inverse limits of finite ordered graphs by partitioning the isomorphism class of finite ordered graph H into $T(H)$ many strong isomorphism classes, called types. This theorem tells us that the universal inverse limit graphs have finite big Ramsey degrees under finite Baire-measurable colorings.

Theorem 1.3 (Huber-Geschke-Kojman [13]). *For every finite ordered graph H there is $T(H) < \omega$ such that for every universal inverse limit graph G, and for every finite Baire-measurable coloring of the set $\binom{G}{H}$ of all copies of H in G, there is a closed copy G' of G in G such that the set $\binom{G'}{H}$ of all copies of H in G' has at most $T(H)$ colours.*

The following notation will be used throughout. The set of natural numbers, $\{0, 1, 2, 3, \ldots\}$, will be denoted by ω. Let $\omega^{<\omega}$ be the set of all finite sequences of natural numbers. Let $\subseteq$ denote the initial segment relation. For an element $s \in \omega^{<\omega}$, let $|s|$ denote the length of s. We call a downward closed subset T of $\omega^{<\omega}$ a tree, ordered by $\subseteq$. Every element t of a tree T is called a node. Given a tree T, let $[T]$ be the set of all infinite branches of T, i.e., $[T] = \{x \in \omega^\omega : (\forall n < \omega)\ x \restriction n \in T\}$, where $x \restriction n$ is its initial segment of length n. T' is called a subtree of T if $T' \subseteq T$ and T' is a tree. For a tree T and $t \in T$, s is called an immediate successor of t if s is a minimal element of T above t. The set of immediate successors of t in T is denoted by $\text{succ}_T(t)$. Let T_t be the set of all nodes in T comparable to t, i.e., $T_t = \{s \in T : t \subseteq s \vee s \subseteq t\}$. For $n \in \omega$, we let $T(n) = \{t \in T : |t| = n\}$.

In order to state the results of Huber-Geschke-Kojman and of Zheng, we need to introduce the following notation and structures. Let R denote the Rado graph, i.e., the unique (up to isomorphism) countable universal homogeneous graph. We assume that the set of vertices of R is just the set ω of natural numbers. For $n \in \omega$, let R_n be the induced subgraph of R on $\{0, \ldots, n\}$.

Definition 1.4 ([13]) Let $T_{\max} \subseteq \omega^{<\omega}$ be the nonempty tree such that for each $t \in T_{\max}$,

$$\text{succ}_{T_{\max}}(t) = \{t^\frown\langle 0\rangle, t^\frown\langle 1\rangle, \ldots, t^\frown\langle |t|\rangle\}.$$

For $t \in T_{\max}$, we define G_t to be the ordered graph on the vertex set $\text{succ}_{T_{\max}}(t)$ with lexicographical ordering, such that G_t is isomorphic to $\mathrm{R}_{|t|}$.

Note that $[T_{\max}]$ is a subset of ω^ω. Given $x, y \in [T_{\max}]$ with $x \neq y$, let $x \cap y \in \omega^{<\omega}$ be the common initial segment of x and y, i.e. $x \cap y = x \restriction \min\{n : x(n) \neq y(n)\}$. The tree $T_{\max}$ and the ordered graphs G_t $(t \in T_{\max})$ induce an ordered graph $G_{\max}$ on the vertex set $[T_{\max}]$, ordered lexicographically, with the edge relation defined as follows. For $x, y \in [T_{\max}]$, $(x, y) \in E(G_{\max})$ if and only if $(x \restriction (|x \cap y| + 1), y \restriction (|x \cap y| + 1)) \in E(G_{x \cap y})$. Suppose that T is a subtree of $T_{\max}$ and $t \in T$. Let G_t^T denote the induced subgraph of G_t on the vertex set $\text{succ}_T(t)$. We define $G(T)$ to be the induced subgraph of $G_{\max}$ on $[T]$. A subtree T of $T_{\max}$ is called a $G_{\max}$-*tree* if for every finite ordered graph H and every $t \in T$, there is $s \in T$ with $t \subseteq s$ such that H embeds into G_s^T. In particular, $T_{\max}$ is a $G_{\max}$-tree.

Let $(\mathcal{R}, \leq, r)$ be a triple satisfying the following: $\mathcal{R}$ is a nonempty set, $\leq$ is a quasi-ordering on $\mathcal{R}$, and $r : \mathcal{R} \times \omega \longrightarrow \mathcal{AR}$ is a mapping giving us the sequence $(r_n(\cdot) = r(\cdot, n))$ of approximation mappings, where

$$\mathcal{AR} = \{r_n(A) : A \in \mathcal{R} \text{ and } n \in \omega\}.$$

For $a \in \mathcal{AR}$ and $A \in \mathcal{R}$,

$$[a, A] = \{B \in \mathcal{R} : (B \leq A) \wedge (\exists n)(r_n(A) = a)\}.$$

The topology on $\mathcal{R}$ is given by the basic open sets $[a, A]$. This topology is called the *Ellentuck topology* on $\mathcal{R}$. Given the Ellentuck topology on $\mathcal{R}$, the notions of nowhere dense, and hence of meager are defined in the natural way. Thus, we may say that a subset $\mathcal{X}$ of $\mathcal{R}$ has the *property of Baire* iff $\mathcal{X} = \mathcal{O} \triangle \mathcal{M}$ for some Ellentuck open set $\mathcal{O} \subseteq \mathcal{R}$ and Ellentuck meager set $\mathcal{M} \subseteq \mathcal{R}$.

Definition 1.5 ([29]) A subset $\mathcal{X}$ of $\mathcal{R}$ is *Ramsey* if for every $\emptyset \neq [a, A]$, there is a $B \in [a, A]$ such that $[a, B] \subseteq \mathcal{X}$ or $[a, B] \cap \mathcal{X} = \emptyset$. $\mathcal{X} \subseteq \mathcal{R}$ is *Ramsey null* if for every $\emptyset \neq [a, A]$, there is a $B \in [a, A]$ such that $[a, B] \cap \mathcal{X} = \emptyset$.

A triple $(\mathcal{R}, \leq, r)$ is a *topological Ramsey space* if every property of Baire subset of $\mathcal{R}$ is Ramsey and if every meager subset of $\mathcal{R}$ is Ramsey null.

In [30], Zheng constructed a collection of topological Ramsey spaces of trees. For each type τ of finite ordered graphs, the space $(\mathcal{G}_\infty(\tau), \leq, r)$ consists of $G_{\max}$-trees of a particular shape. The new spaces $\mathcal{G}_\infty(\tau)$ not only depend on the fact that the class of finite ordered graphs is the Ramsey class, but also, similarly to the Milliken space, are based on the Halpern-Läuchli theorem. Moreover, she presented an application of the topological Ramsey spaces $\mathcal{G}_\infty(\tau)$ to inverse limit graph theory. Similarly to how Todorčević proved Blass' Theorem 1.1, Zheng used the new spaces $\mathcal{G}_\infty(\tau)$ to prove the following Theorem 1.6 (Theorem 3.1 in [13]), which is a key step to show the above Theorem 1.3 in [13].

Theorem 1.6 (Theorem 3.1 in [13]) *Let T be an arbitrary $G_{\max}$-tree. For every type τ of a finite induced subgraph of $G_{\max}$, and for every continuous coloring $c : \binom{G(T)}{\tau} \longrightarrow 2$, there is a $G_{\max}$-subtree S of T such that c is constant on $\binom{G(S)}{\tau}$.*

In this paper, we extend Zheng's methods to build a collection of topological Ramsey spaces of trees in the setting of Fraïssé classes of finite ordered structures with finitely many binary relations satisfying the Ramsey property. Based on these topological Ramsey spaces and the work of Huber-Geschke-Kojman on inverse limits of finite ordered graphs, we prove the following theorem. Here, $\boldsymbol{F_{\max}}$ is a universal limit structure encoded in a particular way on the set of infinite branches of a certain finitely branching tree $T_{\max}$ (see Definitions 2.3 and 2.4).

Theorem 1.7 *Let $\mathcal{K}$ be a Fraïssé class, in a finite signature, of finite ordered binary relational structures with the Ramsey property. For every $\boldsymbol{H} \in \mathcal{K}$, there is a finite number $T(\boldsymbol{H}, \boldsymbol{F_{\max}})$ such that for every universal inverse limit structure $\boldsymbol{G}$, for every finite Baire-measurable coloring of the set $\binom{\boldsymbol{G}}{\boldsymbol{H}}$ of all copies of $\boldsymbol{H}$ in $\boldsymbol{G}$, there is a closed copy $\boldsymbol{G'}$ of $\boldsymbol{G}$ contained in $\boldsymbol{G}$ such that the set $\binom{\boldsymbol{G'}}{\boldsymbol{H}}$ of all copies of $\boldsymbol{H}$ in $\boldsymbol{G'}$ has no more than $T(\boldsymbol{H}, \boldsymbol{F_{\max}})$ colors.*

This means that for each such Fraïssé class, its universal inverse limit structures have finite big Ramsey degrees under finite Baire-measurable colorings. For the following classes, we characterize the big Ramsey degrees in terms of types.

Theorem 1.8 *Let $\mathcal{K}$ be a Fraïssé class in a finite binary relational signature such that one of the following hold:*

(1) $\mathcal{K}$ is an ordered expansion of a free amalgamation class;
(2) $\mathcal{K}$ is the class of finite ordered tournaments;
(3) $\mathcal{K}$ is the class of finite partial orders with a linear extension.

Let $\boldsymbol{G}$ be a universal inverse limit structure for $\mathcal{K}$ contained in $\boldsymbol{F_{\max}}$. Then for each $\boldsymbol{H} \in \mathcal{K}$, each type representing $\boldsymbol{H}$ in $\boldsymbol{G}$ persists in each closed subcopy of $\boldsymbol{G}$. It follows that the big Ramsey degree $T(\boldsymbol{H}, \boldsymbol{F_{\max}})$ for finite Baire-measurable colorings of $\binom{\boldsymbol{F_{\max}}}{\boldsymbol{H}}$ is exactly the number of types in $T_{\max}$ representing a copy of $\boldsymbol{H}$.

2 Ordered binary relational Fraïssé classes and $F_{\max}$-trees

Let us first review some basic facts of the Fraïssé theory for finite ordered binary relational structures which are necessary to this paper. More general background on Fraïssé theory can be found in [15].

We shall call $L = \{<, R_0, \ldots, R_{k-1}\}$ an *ordered binary relational signature* if it consists of the order relation symbol $<$ and finitely many binary relation symbols R_ℓ, $\ell < k$ for some $k < \omega$. A structure for L is of the form $\boldsymbol{A} = \langle A, <^{\boldsymbol{A}}, R_0^{\boldsymbol{A}}, \ldots, R_{k-1}^{\boldsymbol{A}}\rangle$, where $A \neq \emptyset$ is the universe of $\boldsymbol{A}$, $<^{\boldsymbol{A}}$ is a linear ordering of A, and each $R_\ell^{\boldsymbol{A}} \subseteq A \times A$. An embedding between structures $\boldsymbol{A}, \boldsymbol{B}$ for L is an injection $\pi : A \longrightarrow B$ such that for any two $a, a' \in A$, $a <^{\boldsymbol{A}} a' \iff \pi(a) <^{\boldsymbol{B}} \pi(a')$ and for each $\ell < k$, $(a_1, a_2) \in R_\ell^{\boldsymbol{A}} \iff (\pi(a_1), \pi(a_2)) \in R_\ell^{\boldsymbol{B}}$. If π is the identity, we say that $\boldsymbol{A}$ is a substructure of $\boldsymbol{B}$. An isomorphism is an onto embedding. We write $\boldsymbol{A} \leq \boldsymbol{B}$ if $\boldsymbol{A}$ can be embedded in $\boldsymbol{B}$ and $\boldsymbol{A} \cong \boldsymbol{B}$ if $\boldsymbol{A}$ is isomorphic to $\boldsymbol{B}$.

A class $\mathcal{K}$ of finite structures is called *hereditary* if $\boldsymbol{A} \leq \boldsymbol{B} \in \mathcal{K}$ implies $\boldsymbol{A} \in \mathcal{K}$. It satisfies the *joint embedding property* if for any $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{K}$, there is $\boldsymbol{C} \in \mathcal{K}$ with $\boldsymbol{A} \leq \boldsymbol{C}$ and $\boldsymbol{B} \leq \boldsymbol{C}$. We say that $\mathcal{K}$ satisfies the *amalgamation property* if for any embeddings $f : \boldsymbol{A} \longrightarrow \boldsymbol{B}$, $g : \boldsymbol{A} \longrightarrow \boldsymbol{C}$ with $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathcal{K}$, there is $\boldsymbol{D} \in \mathcal{K}$ and embeddings $r : \boldsymbol{B} \longrightarrow \boldsymbol{D}$ and $s : \boldsymbol{C} \longrightarrow \boldsymbol{D}$, such that $r \circ f = s \circ g$. A class of finite structures $\mathcal{K}$ is called a *Fraïssé class* if it is hereditary, satisfies joint embedding and amalgamation, contains only countably many structures, up to isomorphism, and contains structures of arbitrarily large (finite) cardinality. A Fraïssé class satisfies the *free amalgamation property* (or *has free amalgamation*) if $\boldsymbol{D}$, r, and s in the amalgamation property can be chosen so that $r[B] \cap s[C] = r \circ f[A] = s \circ g[A]$, and $\boldsymbol{D}$ has no additional relations on its universe other than those inherited from $\boldsymbol{B}$ and $\boldsymbol{C}$.

Let $\boldsymbol{A}$ be a structure for L. For each $X \subseteq A$, there is a smallest substructure containing X, called the substructure generated by X. A substructure is called finitely generated if it is generated by a finite set. A structure is locally finite if all its finitely generated substructures are finite. The age of $\boldsymbol{A}$, $\mathrm{Age}(\boldsymbol{A})$ is the class of all finitely generated structures in L which can be embedded in $\boldsymbol{A}$. We call $\boldsymbol{A}$ *ultrahomogeneous* if every isomorphism between finitely generated substructures of $\boldsymbol{A}$ can be extended to an automorphism of $\boldsymbol{A}$. A locally finite, countably infinite, ultrahomogeneous structure is called a *Fraïssé structure*.

There is a canonical one-to-one correspondence between Fraïssé classes of finite structures and Fraïssé structures, discovered by Fraïssé. If $\boldsymbol{A}$ is a Fraïssé structure, then $\mathrm{Age}(\boldsymbol{A})$ is a Fraïssé class of finite structures. Conversely, if $\mathcal{K}$ is a Fraïssé class of relational structures, then there is a unique Fraïssé structure, called the Fraïssé limit of $\mathcal{K}$, denoted by $\mathrm{Flim}(\mathcal{K})$, whose age is exactly $\mathcal{K}$.

Definition 2.1 Let $\mathcal{K}$ be a Fraïssé class of finite ordered binary relational structures. We say that $\mathcal{K}$ satisfies the *Ramsey property* if $\mathcal{K}$ is a Ramsey class. That is, for each $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{K}$ such that $\boldsymbol{A} \leq \boldsymbol{B}$ and for every positive integer $l \geq 2$, there exists $\boldsymbol{C} \in \mathcal{K}$ such that $\boldsymbol{C} \longrightarrow (\boldsymbol{B})_l^{\boldsymbol{A}}$.

Given an ordered binary relational signature $L = \{<, R_0, \ldots, R_{k-1}\}$, let L^- denote $\{R_0, \ldots, R_{k-1}\}$. An L^--structure $\boldsymbol{A}$ is called *irreducible* if for any two elements

$x, y \in A$, there is some relation $R \in L^-$ such that either $R^{\boldsymbol{A}}(x, y)$ or $R^{\boldsymbol{A}}(y, x)$ holds. Given a set $\mathcal{F}$ of finite L^--structures, let $\mathrm{Forb}(\mathcal{F})$ denote the class of finite L^--structures $\boldsymbol{A}$ such that no member of $\mathcal{F}$ embeds into $\boldsymbol{A}$. It is well-known that a Fraïssé class in signature L^- has free amalgamation if and only if it is of the form $\mathrm{Forb}(\mathcal{F})$ for some set $\mathcal{F}$ of finite irreducible L^--structures. It follows from results of Nešetřil and Rödl in [21, 22] that all Fraïssé classes in signature L for which the L^--reduct has free amalgamation has the Ramsey property.

For $k \geq 3$, a graph G is called *k-clique free* if for any k vertices in G, there is at least one pair with no edge between them; in other words, no k-clique embeds into G as an induced subgraph. An oriented graph $G = \langle V(G), E(G)\rangle$ is a relational structure, where $V(G)$ denotes its vertex set and $E(G) \subseteq V(G) \times V(G)$ denotes its directed edge relation, that is, $E(G) \subseteq V(G) \times V(G)$ is an irreflexive binary relation such that for all $x, y \in V(G)$, $(x, y) \in E(G)$ implies $(y, x) \notin E(G)$. A tournament G is an oriented graph such that for all $x \neq y$, either $(x, y) \in E(G)$ or $(y, x) \in E(G)$. A partial order with a linear extension is a structure $\boldsymbol{P} = \langle P, <^{\boldsymbol{P}}, R^{\boldsymbol{P}}\rangle$ where $R^{\boldsymbol{P}}$ is a partial ordering on P, $<^{\boldsymbol{P}}$ is a linear ordering on P, and whenever $x \neq y$ and $R^{\boldsymbol{P}}(x, y)$ holds, then also $x <^{\boldsymbol{P}} y$ holds.

Example 2.2 Let $\mathcal{OG}$, $\mathcal{OG}_k$, $\mathcal{OOG}$, $\mathcal{OT}$, and $\mathcal{OPO}$ denote the Fraïssé classes of all finite ordered graphs, finite ordered k-clique free graphs ($k \geq 3$), finite ordered oriented graphs, finite ordered tournaments, and finite partial orders with a linear extension, respectively. Each of these classes has the Ramsey property.

The Ramsey property for $\mathcal{OG}$, $\mathcal{OG}_k$, $\mathcal{OOG}$, and $\mathcal{OT}$, are special cases of a theorem of Nešetřil-Rödl ([21, 22]); the Ramsey property for that $\mathcal{OG}$ and $\mathcal{OT}$ follow from independent work of Abramson and Harrington in [1]. The Ramsey property for $\mathcal{OPO}$ was announced by Nešetřil and Rödl in [23], and the first proof was published by Paoli, Trotter, and Walker in [27].

Let $\mathcal{K}$ be a Fraïssé class of finite ordered binary relational structures with the Ramsey property. We may assume the universe of $\mathrm{Flim}(\mathcal{K})$ is ω, so that the universe is well-ordered. For $n \in \omega$, let $\mathrm{Flim}(\mathcal{K})_n$ be the initial segment of $\mathrm{Flim}(\mathcal{K})$ on $\{0, \ldots, n\}$.

Definition 2.3 Let $\mathcal{K}$ be a Fraïssé class of finite ordered binary relational structures with the Ramsey property, and let $T_{\max} \subseteq \omega^{<\omega}$ be the nonempty tree such that for each $t \in T_{\max}$,

$$\mathrm{succ}_{T_{\max}}(t) = \{t^\frown\langle 0\rangle, t^\frown\langle 1\rangle, \ldots, t^\frown\langle |t|\rangle\}.$$

For $t \in T_{\max}$, we define $\boldsymbol{F_t} \in \mathcal{K}$ to have universe $F_t := \mathrm{succ}_{T_{\max}}(t)$, ordered by the lexicographical ordering, such that $\boldsymbol{F_t}$ is isomorphic to $\mathrm{Flim}(\mathcal{K})_{|t|}$.

Definition 2.4 Let $\mathcal{K}$ be a Fraïssé class of finite ordered binary relational structures with the Ramsey property. The tree $T_{\max}$ and $\boldsymbol{F_t} \in \mathcal{K}$ ($t \in T_{\max}$) induce a structure $\boldsymbol{F_{\max}}$ on the universe $F_{\max} := [T_{\max}]$, ordered lexicographically, with the binary relations $R_\ell^{\boldsymbol{F_{\max}}}$, $\ell < k$ (where k is the cardinality of the signature L), as follows:

$$\forall\, x, y \in [T_{\max}],\ (x, y) \in R_\ell^{\boldsymbol{F_{\max}}} \iff (x \restriction (|x \cap y| + 1), y \restriction (|x \cap y| + 1)) \in R_\ell^{\boldsymbol{F_{x\cap y}}}.$$

Lemma 2.5 *Let $\mathcal{K}$ be a Fraïssé class of finite ordered binary relational structures with the Ramsey property. If $\boldsymbol{F} \in \mathcal{K}$ and $t \in T_{\max}$, then there is $s \in T_{\max}$ with $t \subseteq s$ such that $\boldsymbol{F}$ embeds into $\boldsymbol{F_s}$.*

Proof Since $\boldsymbol{F_t}$ is isomorphic to $\mathrm{Flim}(\mathcal{K})_{|t|}$, it suffices to prove that each $\boldsymbol{F} \in \mathcal{K}$ embeds into $\mathrm{Flim}(\mathcal{K})$ on universe ω. Now the age of $\mathrm{Flim}(\mathcal{K})$ is exactly $\mathcal{K}$. So each $\boldsymbol{F} \in \mathcal{K}$ embeds into $\mathrm{Flim}(\mathcal{K})$ on universe ω. □

Definition 2.6 (1) Suppose that T is a subtree of $T_{\max}$ and $t \in T$. Let $\boldsymbol{F_t^T}$ denote the induced substructure of $\boldsymbol{F_t}$ on the universe $F_t^T := \mathrm{succ}_T(t)$. We define $\boldsymbol{F(T)}$ to be the induced substructure of $\boldsymbol{F_{\max}}$ on the universe $F(T) := [T]$.

(2) Let $\mathcal{K}$ be a Fraïssé class of finite ordered binary relational structures with the Ramsey property. A subtree T of $T_{\max}$ is called an $\boldsymbol{F_{\max}}$*-tree* if for every $\boldsymbol{F} \in \mathcal{K}$ and every $t \in T$, there is $s \in T$ with $t \subseteq s$ such that $\boldsymbol{F}$ embeds into $\boldsymbol{F_s^T}$. In particular, $T_{\max}$ is an $\boldsymbol{F_{\max}}$-tree.

Definition 2.7 Let $\mathcal{K}$ be a Fraïssé class of finite ordered binary relational structures with the Ramsey property. A sequence $(T_j)_{j\in\omega}$ is a *fusion sequence* with witness $(m_j)_{j\in\omega}$ if the following hold:

(1) $(m_j)_{j\in\omega}$ is a strictly increasing sequence of natural numbers.
(2) For all $j, l \in \omega$, if $j < l$, then T_l is an $\boldsymbol{F_{\max}}$-subtree of T_j such that $T_j(m_j) = T_l(m_j)$.
(3) For every $\boldsymbol{F} \in \mathcal{K}$, every $j \in \omega$, and every $t \in T_j(m_j)$, there is $l > j$ such that t has an extension s in T_l such that $|s| < m_l$ and $\boldsymbol{F}$ embeds into $\boldsymbol{F_s^{T_l}}$.

One can check that if $(T_j)_{j\in\omega}$ is a fusion sequence witnessed by $(m_j)_{j\in\omega}$, then the fusion $\bigcap_{j\in\omega} T_j = \bigcup_{j\in\omega}(T_j \cap \omega^{\leq m_j})$ is an $\boldsymbol{F_{\max}}$-tree.

3 Types

Definition 3.1 Let T be an $\boldsymbol{F_{\max}}$-tree and $\boldsymbol{F}$ a finite induced substructure of $\boldsymbol{F(T)}$. We define $\triangle(\boldsymbol{F})$ and $\boldsymbol{F}^\vee$ as follows:

$$\triangle(\boldsymbol{F}) = \max\{|x \cap y| : x, y \in F \wedge x \neq y\},$$
$$\boldsymbol{F}^\vee = \{x \restriction (\triangle(\boldsymbol{F}) + 1) : x \in F\}.$$

Example 3.2 Let $\mathcal{K} = \mathcal{OG}$ and $\boldsymbol{H} \in \mathcal{K}$ as in Fig. 1, where $H = \{x, y, z\}$, $x = 0000\ldots$, $y = 0100\ldots$, and $z = 0111\ldots$. Then $\triangle(\boldsymbol{H}) = 2$ and $\boldsymbol{H}^\vee = \{u, v, w\}$, where $u = 000$, $v = 010$ and $w = 011$.

Definition 3.3 Let $\boldsymbol{F}$ and $\boldsymbol{F'}$ be finite induced substructure of $\boldsymbol{F_{\max}}$. We say $\boldsymbol{F}$ and $\boldsymbol{F'}$ are *strongly isomorphic* if there exists an isomorphism $\varphi : \boldsymbol{F} \longrightarrow \boldsymbol{F'}$ such that $\forall \{x_0, y_0\}, \{x_1, y_1\} \in [F]^2$,

$$|x_0 \cap y_0| \leq |x_1 \cap y_1| \iff |\varphi(x_0) \cap \varphi(y_0)| \leq |\varphi(x_1) \cap \varphi(y_1)|.$$

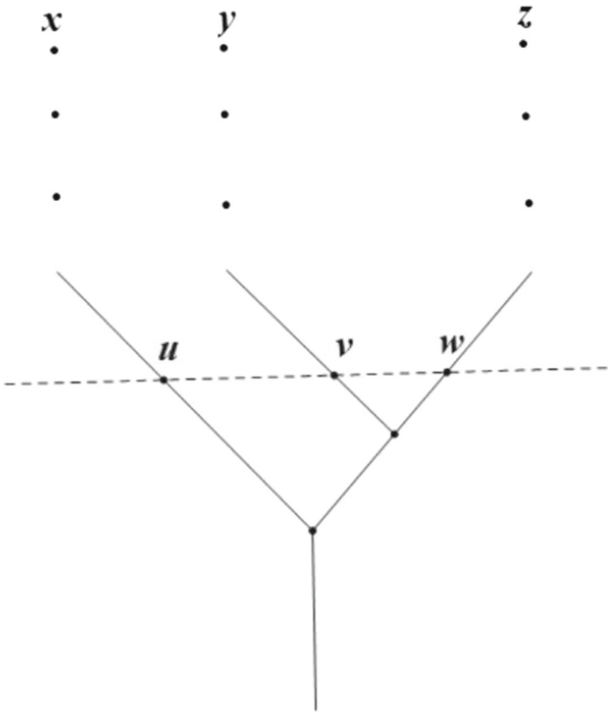


Fig. 1 $\triangle(\boldsymbol{H})$ and $\boldsymbol{H}^{\vee}$

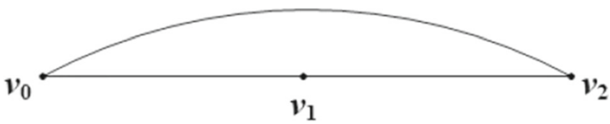


Fig. 2 $\boldsymbol{H}$

Clearly, strong isomorphism is an equivalence relation. By a *type* we mean a strong isomorphism equivalence class. In particular, there are only finitely many types inside an isomorphism class.

Suppose that $\boldsymbol{F}$ and $\boldsymbol{H}$ are structures. Let $\binom{\boldsymbol{F}}{\boldsymbol{H}}$ be the set of all induced substructures $\boldsymbol{H'}$ of $\boldsymbol{F}$ isomorphic to $\boldsymbol{H}$. If $\boldsymbol{F}$ is an induced substructure of $\boldsymbol{F}_{\mathbf{max}}$ and τ is a type, we let $\binom{\boldsymbol{F}}{\tau}$ be the set of all induced substructures of $\boldsymbol{F}$ of type τ.

Example 3.4 Let $\mathcal{K} = \mathcal{OG}$ and $\boldsymbol{H} \in \mathcal{K}$ be as in Fig. 2, where $H = \{v_0, v_1, v_2\}$. Then there are 3 types for $\boldsymbol{H}$ as in Fig. 3.

Let $\boldsymbol{H} \in \mathcal{K}$ be as in Fig. 4. Then there are 2 types for $\boldsymbol{H}$ as in Fig. 5.
Let $\boldsymbol{H} \in \mathcal{K}$ be as in Fig. 6. Then there is only 1 type for $\boldsymbol{H}$ as in Fig. 7.

Example 3.5 Let $\mathcal{K} = \mathcal{OT}$ and $\boldsymbol{F} \in \mathcal{K}$ be as in Fig. 8. Then there are 3 types for $\boldsymbol{F}$ as in Fig. 9.

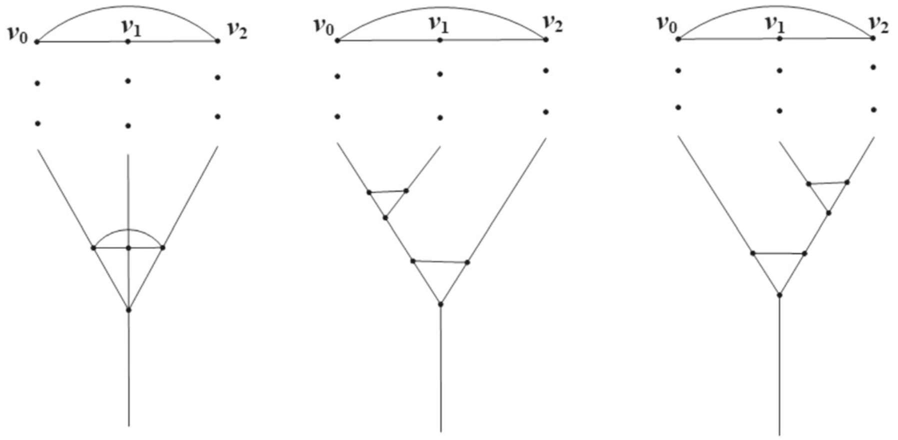


Fig. 3 3 types for ***H***

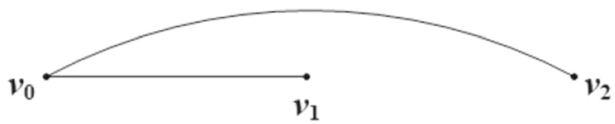


Fig. 4 ***H***

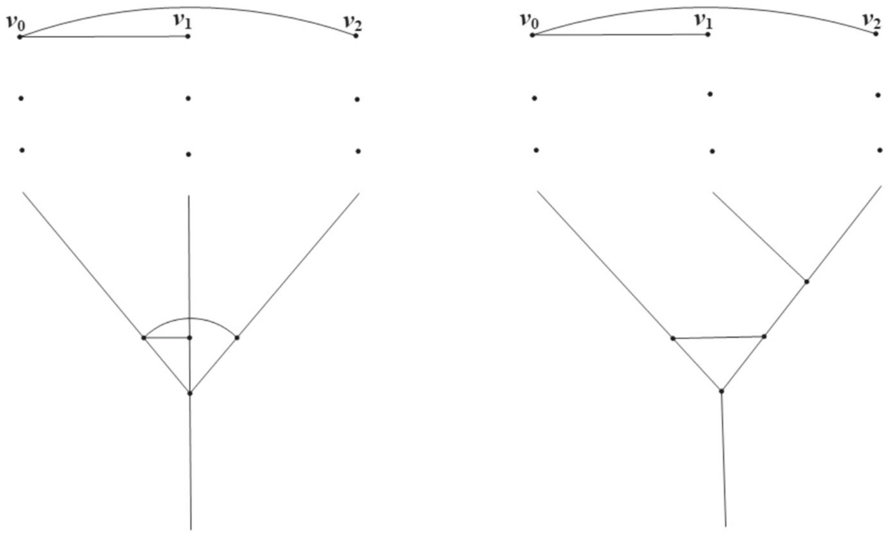


Fig. 5 2 types for ***H***

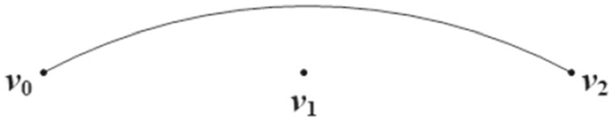
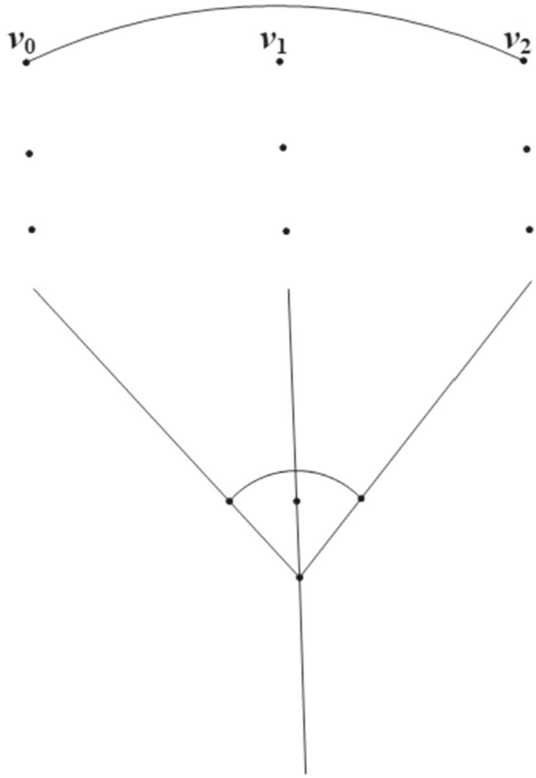


Fig. 6 H

Fig. 7 1 type for H



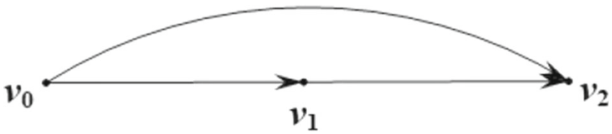


Fig. 8 F

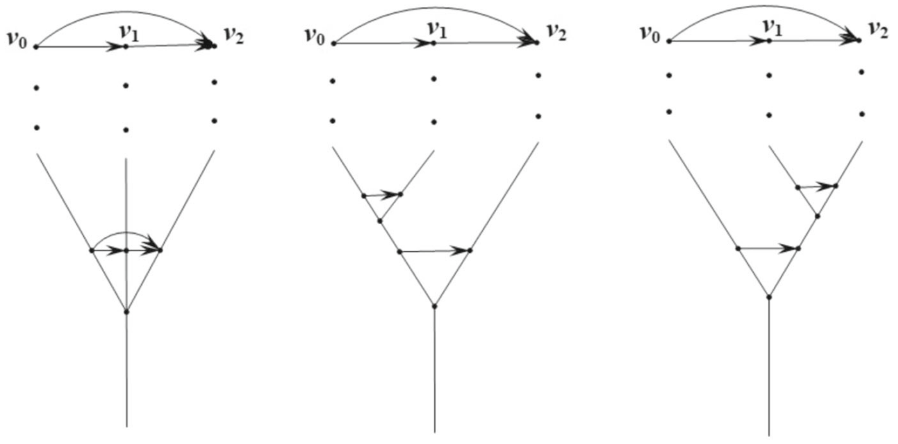


Fig. 9 3 types for $\boldsymbol{F}$

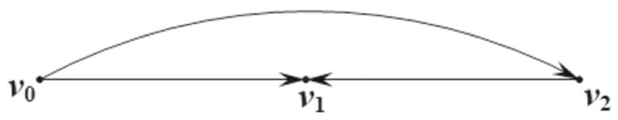


Fig. 10 $\boldsymbol{F}$

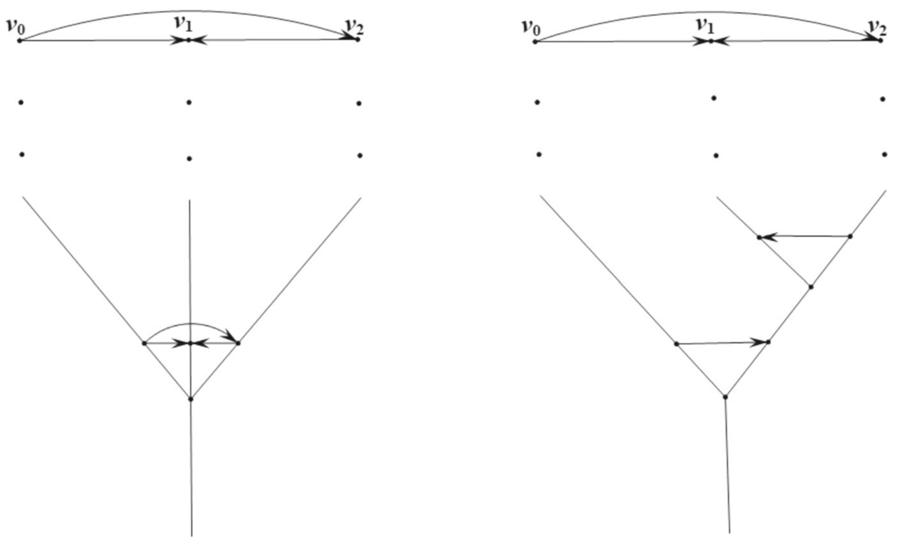


Fig. 11 2 types for $\boldsymbol{F}$

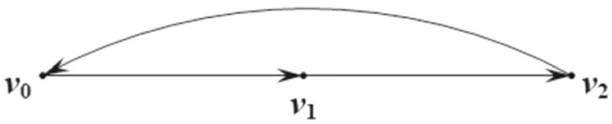
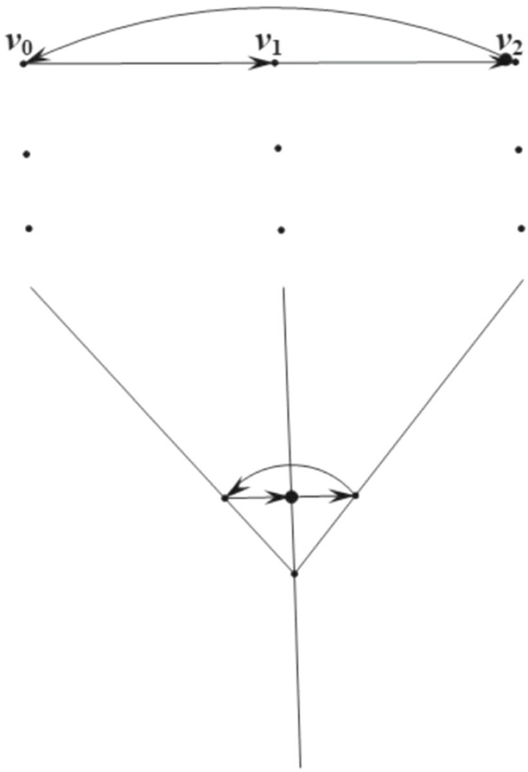


Fig. 12 ***F***

Fig. 13 1 type for ***F***



If $\boldsymbol{F} \in \mathcal{K}$ is as in Fig. 10, then there are 2 types for $\boldsymbol{F}$ as in Fig. 11.

Let $\boldsymbol{F} \in \mathcal{K}$ be as in Fig. 12.

Then there is only 1 type for $\boldsymbol{F}$ as in Fig. 13.

Example 3.6 Suppose that $\mathcal{K} = \mathcal{OPO}$ and $\boldsymbol{H} \in \mathcal{K}$ as in Fig. 14.

Here

denotes that $R(v_i, v_j)$, where R is a partial order. Then there are 3 types for $\boldsymbol{H}$ as in Fig. 15.

Let $\boldsymbol{H} \in \mathcal{K}$ be as in Fig. 16. Then there are 2 types for $\boldsymbol{H}$ as in Fig. 17.

Let $\boldsymbol{H} \in \mathcal{K}$ be as in Fig. 18. Then there is only 1 type for $\boldsymbol{H}$ as in Fig. 19.

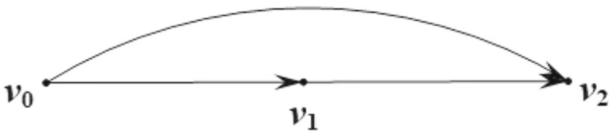


Fig. 14 $\boldsymbol{H}$

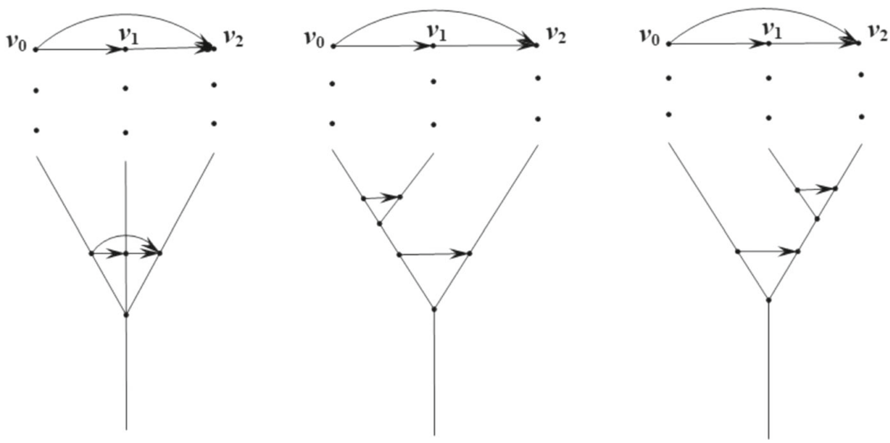


Fig. 15 3 types for $\boldsymbol{H}$

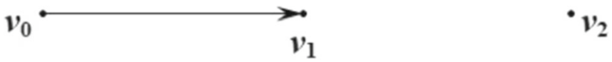


Fig. 16 $\boldsymbol{H}$

4 Topological Ramsey spaces for coding inverse limit structures for finitely many binary relations

This section is essentially work of Zheng from Sect. 3 in [30]. Her work is straightforwardly extended from the context of finite ordered graphs to the broader context of Fraïssé classes of finite ordered binary relational structures satisfying the Ramsey property. We include it in this paper for the reader's convenience, making a few modifications.

Let $\mathcal{K}$ be a Fraïssé class of finite ordered binary relational structures satisfying the Ramsey property, with signature $L = \{<, R_0, \ldots, R_{k-1}\}$, where each R_ℓ ($\ell < k$) is a binary relation. We fix a type τ and build a topological Ramsey space $\mathcal{F}_\infty(\tau)$. We may denote this space by $\mathcal{F}_\infty$ when the type is clear from the context. Let $m+1$ be the number of elements for a finite ordered structure in τ. Let $\{\boldsymbol{F_i} : i < \omega\}$ enumerate the

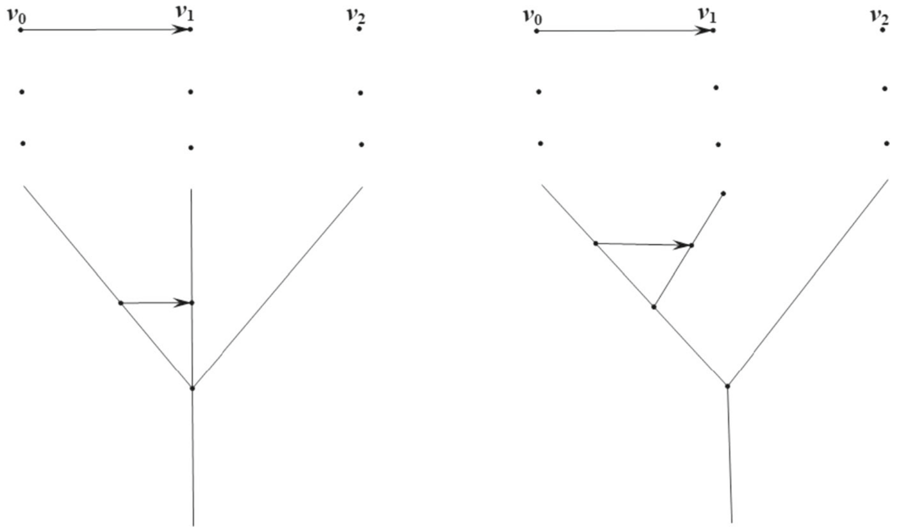


Fig. 17 2 types for $\boldsymbol{H}$

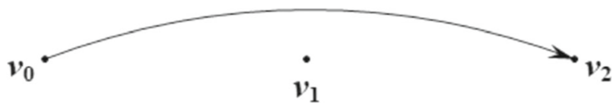


Fig. 18 $\boldsymbol{H}$

set of all finite structures in $\mathcal{K}$, up to isomorphism, labelled so that for every $i < j < \omega$, $|F_i| \leq |F_j|$.

Let S be a tree. A node $t \in S$ is a *splitting node* if $|\text{succ}_S(t)| > 1$. We say S is *skew* if S has at most one splitting node at each level, i.e.,

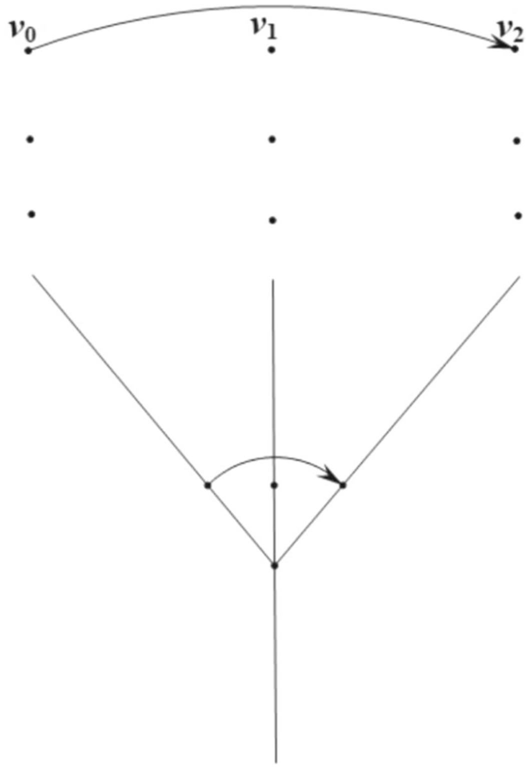
$$\forall\, n \in \omega,\ \ |\{t \in S \cap \omega^n : |\text{succ}_S(t)| > 1\}| \leq 1.$$

Notice that if S is a skew tree and $i > 0$ is given, then each node $t \in S$ for which $\boldsymbol{F}_t^S \cong \boldsymbol{F}_i$ is a splitting node, since the structure $\boldsymbol{F}_i$ has universe of size at least two. Thus, any two nodes in the set $\{t \in S : \boldsymbol{F}_t^S \cong \boldsymbol{F}_i\}$ have different lengths, so the nodes in this set can be enumerated in order of increasing length. This will be useful in part (iii) of (2) in the next definition.

Definition 4.1 Let τ be a type and $m+1$ be the number of elements for a finite ordered structure in τ. We define the space $(\mathcal{F}_\infty, \leq, r)$ as follows.

Let S be a member of $\mathcal{F}_\infty$ if S is a skew subtree of $T_{\max}$ and when we enumerate the set of splitting nodes $\{s \in S : |\text{succ}_S(s)| > 1\}$ as $\{s_i\}_{i<\omega}$ in the order of length,

Fig. 19 1 type for $\boldsymbol{H}$



(1) There is a finite structure $\boldsymbol{F} \in \tau$ such that

$$\boldsymbol{F}^{\vee} = S \cap \omega^{|s_{m-1}|+1};$$

(2) For all $i > 0$,

(i) $\forall\, s \in S \cap \omega^{>|s_{m-1}|}, \forall\, u \in \mathrm{succ}_S(s)$,

$$\boldsymbol{F}^{\boldsymbol{S}}_{\boldsymbol{s}} \cong \boldsymbol{F_i} \Rightarrow \exists!\, t \in S\ (u \subseteq t \wedge \boldsymbol{F}^{\boldsymbol{S}}_{\boldsymbol{t}} \cong \boldsymbol{F_{i+1}});$$

(ii) For every pair $s,\ t \in S \cap \omega^{>|s_{m-1}|}$,

$$\boldsymbol{F}^{\boldsymbol{S}}_{\boldsymbol{s}} \cong \boldsymbol{F_i} \wedge \boldsymbol{F}^{\boldsymbol{S}}_{\boldsymbol{t}} \cong \boldsymbol{F_{i+1}} \Rightarrow |s| < |t|;$$

(iii) If $\{t \in S : \boldsymbol{F}^{\boldsymbol{S}}_{\boldsymbol{t}} \cong \boldsymbol{F_i}\}$ is enumerated in order of increasing length as $\{t_j\}_j$, then there is $l < \omega$ such that $\{t_j \restriction l\}_j$ is strictly increasing in lexicographical ordering.

When we say that $\{s_i\}_{i<\omega}$ is the set of splitting nodes in S, we tacitly assume that the length $|s_i|$ is strictly increasing in i.

For $S, U \in \mathcal{F}_\infty$, we write $S \le U$ if and only if $S \subseteq U$. For $l < \omega$ and $S \in \mathcal{F}_\infty$ with the set of splitting nodes $\{s_i\}_{i<\omega}$, we define the finite approximation $r_l(S)$ as follows: let

$$r_0(S) = \emptyset \text{ and } r_{l+1}(S) = S \cap \omega^{\le |s_l|+1}.$$

We specify a few more definitions that are often used in topological Ramsey spaces. Let $\mathcal{F}_{<\infty}$ denote the set of all finite approximations, i.e.

$$\mathcal{F}_{<\infty} = \{r_l(S) : S \in \mathcal{F}_\infty \wedge l \in \omega\}.$$

For $a, b \in \mathcal{F}_{<\infty}$, let $a \le_{\text{fin}} b$ if $a \subseteq b$. Let $|a| = n$ if there is $S \in \mathcal{F}_\infty$ with $r_n(S) = a$. For $a, b \in \mathcal{F}_{<\infty}$, we write $a \sqsubseteq b$ if there are $l < p < \omega$ and $S \in \mathcal{F}_\infty$ such that $a = r_l(S)$ and $b = r_p(S)$.

For $a \in \mathcal{F}_{<\infty}$ and $S \in \mathcal{F}_\infty$, $\text{depth}_S(a) = \min\{n : a \le_{\text{fin}} r_n(S)\}$, where by convention, $\min \emptyset = \infty$. We equip the space $\mathcal{F}_\infty$ with the Ellentuck topology, with basic open sets of the form

$$[a, S] = \{X \in \mathcal{F}_\infty : (X \le S) \wedge (\exists l)(r_l(X) = a)\},$$

for $a \in \mathcal{F}_{<\infty}$ and $S \in \mathcal{F}_\infty$. For $l < \omega$, let

$$\begin{aligned} \mathcal{F}_l &= \{r_l(X) : X \in \mathcal{F}_\infty\}, \\ [l, S] &= [r_l(S), S], \text{ and} \\ r_l[a, S] &= \{r_l(X) : X \in [a, S]\}. \end{aligned}$$

The height of an element $a \in \mathcal{F}_{<\infty}$ is $\text{height}(a) = \max_{s \in a} |s|$. In general, $|a| \le \text{height}(a)$.

Now we show that $(\mathcal{F}_\infty, \le, r)$ is a topological Ramsey space by proving that $\mathcal{F}_\infty$ is closed as a subspace of $(\mathcal{F}_{<\infty})^\omega$, and satisfies the axioms **(A1)–(A4)** as defined in pages 93–94 of [29]. It is straightforward to check **(A1)–(A3)**. Moreover, $\mathcal{F}_\infty$ is a closed subset of $(\mathcal{F}_{<\infty})^\omega$ when we identify $S \in \mathcal{F}_\infty$ with $(r_n(S))_{n<\omega} \in (\mathcal{F}_{<\infty})^\omega$ and equip $\mathcal{F}_{<\infty}$ with the discrete topology and $(\mathcal{F}_{<\infty})^\omega$ with the product topology.

Definition 4.2 Let $T \subseteq \omega^{<\omega}$ be a (downwards closed) finitely branching tree with no terminal nodes, and let N an infinite subset of ω. A set U is called a *strong subtree* of $\bigcup_{n\in N} T \cap \omega^n$ if there is an infinite set $M \subseteq N$ such that the following conditions hold.

(1) $U \subseteq \bigcup_{n\in M} T \cap \omega^m$ and $U \cap \omega^m \neq \emptyset$ for all $m \in M$. In this case, we say that M witnesses that U is a strong subtree.
(2) If $m_1 < m_2$ are two successive elements of M and if $u \in U \cap \omega^{m_1}$, then every immediate successor of u in $\bigcup_{n\in N} T \cap \omega^n$ has exactly one extension in $U \cap \omega^{m_2}$.

Theorem 4.3 (Halpern-Läuchli [12]). *For each $i < d$, let $T_i \subseteq \omega^{<\omega}$ be a finitely branching tree with no terminal nodes, let $N \in [\omega]^\omega$, where d is any positive integer,* *and let* $c : \bigcup_{n\in N} \prod_{i<d} T_i \cap \omega^n \longrightarrow p$ *be a finite coloring, where* p *is any positive integer. Then there is an infinite subset* $M \subseteq N$ *and infinite strong subtrees* $U_i \subseteq \bigcup_{n\in N} T_i \cap \omega^n$ *witnessed by* M *such that* c *is monochromatic on* $\bigcup_{m\in M} \prod_{i<d} U_i \cap \omega^m$.

We observe that every $S \in \mathcal{F}_\infty$ is an $\boldsymbol{F_{\mathbf{max}}}$-tree and every $\boldsymbol{F_{\mathbf{max}}}$-tree contains some $S \in \mathcal{F}_\infty$ as a subtree. This is because for all $n < \omega$, we can find infinitely many $m \in \omega$ such that $\boldsymbol{F_n}$ embeds into $\boldsymbol{F_m}$ (in fact, this holds for all but finitely many $m \in \omega$.) We define $(\boldsymbol{A_n})_{n<\omega}$ to be a sequence of finite structures in $\mathcal{K}$ such that for each $n < \omega$, the finite structures $\boldsymbol{F_0}, \ldots, \boldsymbol{F_n}$ embed into $\boldsymbol{A_n}$.

Now we will prove **(A4)** in Lemma 4.4 For $\mathcal{K} = \mathcal{OG}$, Zheng proved in [30] that $(\mathcal{F}_\infty, \le, r)$ is a topological Ramsey space. So the axiom **(A4)** holds for finite ordered graphs. It should be pointed out that the proof of Lemma 4.4 follows from Zheng's proof for finite ordered graphs. However, for the convenience of the reader, we also present the proof here.

Lemma 4.4 *The axiom* **(A4)** *holds for* $(\mathcal{F}_\infty, \le, r)$*, i.e. for* $a \in \mathcal{F}_{<\infty}$ *and* $S \in \mathcal{F}_\infty$*, if* $\mathrm{depth}_S(a) < \infty$ *and* $\mathcal{O} \subseteq \mathcal{F}_{|a|+1}$*, then there is* $U \in [depth_S(a), S]$ *such that* $r_{|a|+1}[a, U] \subseteq \mathcal{O}$ *or* $r_{|a|+1}[a, U] \cap \mathcal{O} = \emptyset$*.*

Proof Let $m_0 = \mathrm{height}(a)$. Then a has $|a|$ many splitting nodes, and each element in $\mathcal{F}_{|a|+1}$ has $|a| + 1$ many splitting nodes. In particular, we can find $u \in a \cap \omega^{m_0}$ and $j < \omega$ such that for every $b \in \mathcal{F}_{|a|+1}$ with $a \sqsubseteq b$, there is a unique splitting node $t \in b_u$, and $\boldsymbol{F_t^b} \cong \boldsymbol{F_j}$. Let $a \cap \omega^{m_0}$ be enumerated as $u = v_0, v_1, \ldots, v_d$.

Step 1 Let us construct a subtree $X \subseteq S$ along with a strictly increasing sequence $(m_l)_{l<\omega}$ starting with $m_0 = \mathrm{height}(a)$ such that the following conditions hold for every $l < \omega$ and every $s \in X \cap \omega^{m_l}$:

(a) X_s has a unique splitting node t of length in $[m_l, m_{l+1})$.
(b) For the t from condition (a), $\boldsymbol{F_t^X} \cong \boldsymbol{A_{j+l}}$. Let $t \in S_u$. The set of tuples of *nephews* of t is defined to be

$$\{(t_1, \ldots, t_d) : t_i \in X_{v_i} \cap \omega^{|t|+1} \text{ for } 1 \le i \le d\}.$$

Suppose $(t_i)_{1\le i\le d}$ is a tuple of nephews of t. Each finite structure $\boldsymbol{F} \in \binom{\boldsymbol{F_t^X}}{\boldsymbol{F_j}}$ together with $(t_i)_{1\le i\le d}$ determines an element $b_{\boldsymbol{F},(t_i)} \in \mathcal{F}_{|a|+1}$, where the set of $\subseteq$-maximal nodes in $b_{\boldsymbol{F},(t_i)}$ is $F \cup \{t_i : 1 \le i \le d\}$. With this notion of $b_{\boldsymbol{F},(t_i)}$ defined, we can state another requirement for X.

(c) For every $t \in X$ and every tuple $(t_i)_{1\le i\le d}$ of nephews of t, the set $\left\{b_{\boldsymbol{F},(t_i)} : \boldsymbol{F} \in \binom{\boldsymbol{F_t^X}}{\boldsymbol{F_j}}\right\}$ is either included or disjoint from $\mathcal{O}$.

We recursively construct sets $X(m_l) \subseteq \omega^{m_l}$. Then X will be the downward closure of $\bigcup_{l<\omega} X(m_l)$ and $X \cap \omega^{m_l} = X(m_l)$. Start with $X(m_0) = \{v_i : i \le d\}$. Assume we have constructed $X(m_l)$. The number of extensions in $X(m_{l+1})$ for each $s \in X(m_l)$ is prescribed. In particular, for each $t \supseteq u$ in $X(m_{l+1})$, the set of (tuples of) nephews of t will be finite and of the same size, independent of t. Let this size be $k < \omega$. Since $\mathcal{K}$ is a Fraïssé class of finite ordered binary relational structures with the Ramsey property, there is a finite ordered structure $\boldsymbol{H} \in \mathcal{K}$ such that $\boldsymbol{H} \longrightarrow (\boldsymbol{A_{j+l}})^{\boldsymbol{F_j}}_{2^k}$ Since S is an $\boldsymbol{F}_{\mathbf{max}}$-tree, for each $s \in X(m_l)$ extending v_i $(1 \leq i \leq d)$, s has an extension $t(s) \in S$ such that there is $\boldsymbol{F(s)} \in \binom{\boldsymbol{F^S_{t(s)}}}{\boldsymbol{A_{j+l}}}$. On the other hand, for each $s \in X(m_l)$ extending u, s has an extension $t(s) \in S$ such that $\boldsymbol{H}$ embeds into $\boldsymbol{F^S_{t(s)}}$. For each tuple $(t_i)_{1 \leq i \leq d}$ of nephews of $t(s)$, there is a natural coloring $c : \binom{\boldsymbol{F^S_{t(s)}}}{\boldsymbol{F_j}} \longrightarrow 2$ depending on whether $b_{\boldsymbol{F},(t_i)}$ is in $\mathcal{O}$. Thus, there are at most k many 2-colorings. These colorings can be encoded in a single 2^n-coloring of $\binom{\boldsymbol{F^S_{t(s)}}}{\boldsymbol{F_j}}$. Then there is $\boldsymbol{H(s)} \in \binom{\boldsymbol{F^S_{t(s)}}}{\boldsymbol{A_{j+l}}}$ such that the set $\binom{\boldsymbol{H(s)}}{\boldsymbol{F_j}}$ is monochromatic. Let

$$m_{l+1} = \max\{|t(s)| : s \in X(m_l)\} + 1.$$

Suppose that $X(m_{l+1}) \subseteq S \cap \omega^{m_{l+1}}$ has the property that every node in $(\bigcup_{s \in D} F(s)) \cup (\bigcup_{s \in X(m_l) \setminus D} H(s))$ has a unique extension in $X(m_{l+1})$, where $D = \{s \in X(m_l) : s$ extends $v_i,\ 1 \leq i \leq d\}$. Thus X satisfies (a), (b) and (c). This finishes the construction of X and $(m_l)_{l<\omega}$.

Step 2 We use the Halpern-Läuchli theorem to shrink X to an $\boldsymbol{F}_{\mathbf{max}}$-tree T such that the set $\{b \in \mathcal{F}_{|a|+1} : b \subseteq T\}$ is either included in or disjoint from $\mathcal{O}$. We define a coloring

$$c : \bigcup_{l<\omega} \prod_{i \leq d} X_{v_i} \cap \omega^{m_l} \longrightarrow 2$$

as follows: Let $c(v_0, v_1, \dots, v_d) = 1$. Suppose $0 < l < \omega$ and $(t_i)_{i \leq d} \in X_{v_i} \cap \omega^{m_l}$. Since S is a skew tree, so is X. Let

$$sl(t_0) = \max\{|t| : t \sqsubseteq t_0 \text{ and } \mathrm{succ}_X(t) > 1\} + 1.$$

Then $(t_i \restriction sl(t_0))_{1 \leq i \leq d}$ is a tuple of nephews for $t_0 \restriction sl(t_0) - 1$. Let

$$c(t_0, t_1, \dots, t_d) = \begin{cases} 1, & \left\{ b_{\boldsymbol{F},(t_i \restriction sl(t_0))} : \boldsymbol{F} \in \binom{\boldsymbol{F^X_{t_0 \restriction sl(t_0)-1}}}{\boldsymbol{F_j}} \right\} \subseteq \mathcal{O}, \\ 0, & \left\{ b_{\boldsymbol{F},(t_i \restriction sl(t_0))} : \boldsymbol{F} \in \binom{\boldsymbol{F^X_{t_0 \restriction sl(t_0)-1}}}{\boldsymbol{F_j}} \right\} \cap \mathcal{O} = \emptyset. \end{cases}$$

By (c), this colouring is well-defined. By Theorem 4.3, there are a strictly increasing sequence $(n_j)_{j<\omega} \subseteq (m_l)_{l<\omega}$ and strong subtrees

$$Y_i \subseteq \bigcup_{l<\omega} X_{v_i} \cap \omega^{m_l}$$

witnessed by $(n_j)_{j<\omega}$ such that c is monochromatic on $\bigcup_{j<\omega} \prod_{i \leq d} Y_i \cap \omega^{n_j}$. Let T be the downward closure of $\bigcup_{i \leq d} Y_i$.

Claim 4.5 *If $l < \omega$, $i \le d$, $s \in Y_i \cap \omega^{m_l}$ and t corresponds to s as in (a) and (b), then $t \in T$ and $\boldsymbol{F}_t^{\boldsymbol{T}} = \boldsymbol{F}_t^{\boldsymbol{X}}$.*

Proof Since $s \in Y_i \cap \omega^{m_l}$, there is some $p < \omega$ such that $n_p = m_l$. As Y_i is a strong subtree of $\bigcup_{l'<\omega} X_{v_i} \cap \omega^{m_{l'}}$, every immediate successor of s in $\bigcup_{l'<\omega} X_{v_i} \cap \omega^{m_{l'}}$ has exactly one extension in $Y_i \cap \omega^{n_{p+1}}$. It follows from the construction of X that every node in $\mathrm{succ}_X(t)$ has exactly one extension in $X \cap \omega^{m_{l+1}}$. Moreover, the immediate successors of s in $\bigcup_{l'<\omega} X_{v_i} \cap \omega^{m_{l'}}$ are precisely the extensions in $X \cap \omega^{m_{l+1}}$ of nodes in $\mathrm{succ}_X(t)$. Then t is in the downward closure of Y_i, and thus $t \in T$. So $\boldsymbol{F}_t^{\boldsymbol{T}} = \boldsymbol{F}_t^{\boldsymbol{X}}$. □

Then it is straightforward to check that T is an $\boldsymbol{F}_{\mathbf{max}}$-tree such that the set $\{b \in \mathcal{F}_{|a|+1} : b \subseteq T\}$ is either included in or disjoint from $\mathcal{O}$. By the observation before this lemma, we can further shrink T to $U \in \mathcal{F}_\infty$ satisfying the conclusion of the lemma. □

Theorem 4.6 *For each Fraïssé class of finite ordered binary relational structures with the Ramsey property and each type τ, the space $(\mathcal{F}_\infty(\tau), \le, r)$ is a topological Ramsey space.*

5 Finite big Ramsey degrees for ordered binary relational universal inverse limit structures

Let $\mathcal{K}$ be a Fraïssé class of finite ordered binary relational structures satisfying the Ramsey property, with signature $L = \{<, R_0, \ldots, R_{k-1}\}$ where each R_i, $i < k$, is a binary relation. In this section, we prove that for each such $\mathcal{K}$, the universal inverse limit structure has finite big Ramsey degrees under finite Baire-measurable colorings. The proofs in this section are straightforward via the topological Ramsey spaces from Theorem 4.6 (which is based on work of Zheng in [30]) and the work of Huber-Geschke-Kojman on inverse limits of finite ordered graphs in [13].

Definition 5.1 Let $\mathcal{K}$ be a Fraïssé class of finite ordered binary relational structures with the Ramsey property. A *universal inverse limit* of finite ordered structures in $\mathcal{K}$ is a triple $\boldsymbol{G} = \langle G, <^{\boldsymbol{G}}, R_0^{\boldsymbol{G}}, \ldots, R_{k-1}^{\boldsymbol{G}}\rangle$, such that the following conditions hold.

1. G is a compact subset of $\mathbb{R}\backslash\mathbb{Q}$ without isolated points, $<^{\boldsymbol{G}}$ is the restriction of the standard order on $\mathbb{R}$ to G, and $R_i^{\boldsymbol{G}} \subseteq [G]^2$ for each $i < k$.
2. For every pair of distinct elements $u, v \in G$, there is a partition of G to finitely many closed intervals such that
   (a) u, v belong to different intervals from the partition;
   (b) For every interval I in the partition, for all $x \in G\backslash I$ and for all $y, z \in I$, $(x, y) \in R_i^{\boldsymbol{G}}$ if and only if $(x, z) \in R_i^{\boldsymbol{G}}$, for each $i < k$.
3. Every nonempty open interval of G contains induced copies of all finite ordered structures in $\mathcal{K}$.

For every $\boldsymbol{F}_{\mathbf{max}}$-tree T, it can be seen from Definition 2.6 that $\boldsymbol{F(T)}$ is a universal inverse limit structure. So it follows from the universality that we can consider colourings of finite induced substructures of $\boldsymbol{F(T)}$.

Definition 5.2 Let τ be a type and $\boldsymbol{H} \in \tau$. The $\mathcal{F}_\infty$-envelope of $\boldsymbol{H}$ is

$$\mathcal{C}_{\boldsymbol{H}} = \{U \in \mathcal{F}_\infty : (\exists\, l)(r_l(U) = \downarrow \boldsymbol{H}^\vee)\},$$

where $\downarrow H^\vee = \{a \in \omega^{<\omega} : (\exists\, x \in H)(a \subseteq x \restriction (\triangle(\boldsymbol{H}) + 1))\}$.

Lemma 5.3 *Let τ be a type and $T \in \mathcal{F}_\infty$. Define a map $c_1 : [\emptyset, T] \longrightarrow \left\{\boldsymbol{H}^\vee : \boldsymbol{H} \in \binom{\boldsymbol{F(T)}}{\tau}\right\}$ as follows:*

$$\forall\, U \in [\emptyset, T],\ \textit{if}\ U \in \mathcal{C}_{\boldsymbol{H}},\ c_1(U) = \boldsymbol{H}^\vee.$$

Then c_1 is well-defined and continuous, where we equip the range with the discrete topology.

Proof Let $m+1$ be the number of elements for each $\boldsymbol{H} \in \tau$. Then $\downarrow \boldsymbol{H}^\vee$ has m splitting nodes. Thus

$$\forall\, U \in \mathcal{F}_\infty, \forall\, l, (r_l(U) = \downarrow \boldsymbol{H}^\vee \Rightarrow l = m).$$

Let $U, V \in [\emptyset, T]$. Then there are $\boldsymbol{H}, \boldsymbol{K} \in \tau$ such that $U \in \mathcal{C}_{\boldsymbol{H}}$ and $V \in \mathcal{C}_{\boldsymbol{K}}$. If $U = V$, then $\downarrow \boldsymbol{H}^\vee = r_m(U) = r_m(V) = \downarrow \boldsymbol{K}^\vee$, and thus $\boldsymbol{H}^\vee = \boldsymbol{K}^\vee$. So c_1 is well-defined.

Suppose that $\boldsymbol{H} \in \tau$ and $U \in (c_1)^{-1}(\boldsymbol{H}^\vee)$. We have that $U \in \mathcal{C}_{\boldsymbol{H}}$. Then the set $[m, U]$ is an open set containing U and $[m, U] \subseteq (c_1)^{-1}(\boldsymbol{H}^\vee)$. Thus c_1 is continuous. □

We equip ω^ω with the first-difference metric topology, which has basic open sets of the form $[s] = \{x \in \omega^\omega : s \subseteq x\}$ for $s \in \omega^{<\omega}$. For $n \in \omega$, let $[\boldsymbol{F}_{\mathbf{max}}]^n$ denote the set of all induced substructures of $\boldsymbol{F}_{\mathbf{max}}$ of size n.

Definition 5.4 For $n \geq 1$, we define a topology on $[\boldsymbol{F}_{\mathbf{max}}]^n$ as follows: A set $\mathcal{U} \subseteq [\boldsymbol{F}_{\mathbf{max}}]^n$ is open if for all $\boldsymbol{H} \in \mathcal{U}$, there are open neighborhoods $U_1, \ldots, U_n$ of the elements of H such that all $\boldsymbol{H'} \in [\boldsymbol{F}_{\mathbf{max}}]^n$ that have exactly one vertex in each U_i are also in $\mathcal{U}$. This topology is separable and induced by a complete metric. A coloring of n-tuples from $[\boldsymbol{F}_{\mathbf{max}}]^n$ is continuous if it is continuous with respect to this topology.

Lemma 5.5 *Let τ be a type and $T \in \mathcal{F}_\infty$. For every continuous coloring $c : \binom{\boldsymbol{F(T)}}{\tau} \longrightarrow 2$, there exists an $\boldsymbol{F}_{\mathbf{max}}$-subtree S of T such that c depends only on $\boldsymbol{H}^\vee$, i.e., for $\boldsymbol{H}, \boldsymbol{K} \in \binom{\boldsymbol{F(S)}}{\tau}$, if $\boldsymbol{H}^\vee = \boldsymbol{K}^\vee$, then $c(\boldsymbol{H}) = c(\boldsymbol{K})$.*

Proof For $\boldsymbol{H} \in \binom{\boldsymbol{F(T)}}{\tau}$, by definition of $\triangle(\boldsymbol{H})$, the map $x \longmapsto x \restriction (\triangle(\boldsymbol{H}) + 1))$ is a bijection from the universe H of $\boldsymbol{H}$ onto $\boldsymbol{H}^\vee$. Let $t_1, \ldots, t_l$ denote the elements of $\boldsymbol{H}^\vee$. For all $\bar{x} = (x_1, \ldots, x_l) \in [T_{t_1}] \times \cdots \times [T_{t_l}]$, the induced substructure

$\overline{\boldsymbol{H}}$ of $\boldsymbol{F(T)}$ on the set $\{x_1, \ldots, x_l\}$ is isomorphic to $\boldsymbol{H}$. By the continuity of c, for all such x there are open neighborhoods $x_1 \in U_1^{\bar{x}}, \ldots, x_l \in U_l^{\bar{x}}$ such that for all $(y_1, \ldots, y_l) \in U_1^{\bar{x}} \times \cdots \times U_l^{\bar{x}}$ for the induced substructure $\boldsymbol{H'}$ of $\boldsymbol{F(T)}$ on the vertices $y_1, \ldots, y_l$, we have $c(\overline{\boldsymbol{H}}) = c(\boldsymbol{H'})$.

We may assume that the $U_i^{\bar{x}}$ are basic open sets, i.e., sets of the form $[T_r]$ for some $r \in T$. Since the space $[T_{t_1}] \times \cdots \times [T_{t_l}]$ is compact, there is a finite set $A \subseteq [T_{t_1}] \times \cdots \times [T_{t_l}]$ such that

$$[T_{t_1}] \times \cdots \times [T_{t_l}] = \bigcup_{\bar{x} \in A} \prod_{i=1}^{l} U_i^{\bar{x}}.$$

Hence there is $m \in \omega$ such that for all induced substructures $\boldsymbol{H'}$ of $\boldsymbol{F(T)}$ with $\boldsymbol{H'} \upharpoonright (\triangle(\boldsymbol{H})+1) = \boldsymbol{H}^{\vee}$, the color $c(\boldsymbol{H'})$ only depends on $\boldsymbol{H'} \upharpoonright m$, where m is the maximal length of the r's with $[T_r] = U_i^{\bar{x}}$ for some $\bar{x} \in F$ and $i \in \{1, \ldots, l\}$.

Since for each $m \in \omega$, there are only finitely many sets of the form $\boldsymbol{H} \upharpoonright m$, where $H \in \binom{\boldsymbol{F(T)}}{\tau}$, there is a function $f : \omega \longrightarrow \omega$ such that for every finite induced substructure $\boldsymbol{H}$ of $\boldsymbol{F(T)}$ with $\triangle(\boldsymbol{H})+1 = n$, the color $c(\boldsymbol{H})$ only depends on $\boldsymbol{H} \upharpoonright f(n)$. Now let S be an $\boldsymbol{F}_{\mathbf{max}}$-subtree of T such that whenever $s \in S$ is a splitting node of S of length n, then S has no splitting node t whose length is in the interval $(n, f(n)]$. Now for all $H \in \binom{\boldsymbol{F(S)}}{\tau}$, the color $c(\boldsymbol{H})$ only depends on $\boldsymbol{H}^{\vee}$. □

Theorem 5.6 *Let T be an $\boldsymbol{F}_{\mathbf{max}}$-tree. For every type τ of a finite induced substructure of $\boldsymbol{F}_{\mathbf{max}}$, and every continuous coloring $c : \binom{\boldsymbol{F(T)}}{\tau} \longrightarrow 2$, there is an $\boldsymbol{F}_{\mathbf{max}}$-subtree S of T such that c is monochromatic on $\binom{\boldsymbol{F(S)}}{\tau}$.*

Proof We can shrink T and assume $T \in \mathcal{F}_{\infty}$. By Lemma 5.5, c depends only on $\boldsymbol{H}^{\vee}$. We may think of c as a map as follows:

$$c : \left\{\boldsymbol{H}^{\vee} : \boldsymbol{H} \in \binom{\boldsymbol{F(T)}}{\tau}\right\} \longrightarrow 2.$$

Define $\bar{c} : [\emptyset, T] \longrightarrow 2$ by $\bar{c} = c \circ c_1$. By Lemma 5.3, $\bar{c}$ is also a continuous map. By Theorem 4.6, there is some $S \leq T$ such that $\bar{c}$ is monochromatic on $[\emptyset, S]$. Suppose that $\boldsymbol{H} \in \binom{\boldsymbol{F(S)}}{\tau}$. Then the universe H of $\boldsymbol{H}$ is contained in $[S]$, so $\boldsymbol{H}^{\vee} \subseteq S$. Hence there is a $U \in [\emptyset, S]$ such that $U \in \mathcal{C}_{\boldsymbol{H}}$, and thus, $c(\boldsymbol{H}) = \bar{c}(U)$. Therefore c is monochromatic on $\binom{\boldsymbol{F(S)}}{\tau}$. □

The next lemma is a straightforward extension of Lemma 3.8 in [13].

Lemma 5.7 *Let T be an $\boldsymbol{F}_{\mathbf{max}}$-tree. For every type τ of a finite induced substructure of $\boldsymbol{F}_{\mathbf{max}}$, and every Baire-measurable coloring $c : \binom{\boldsymbol{F(T)}}{\tau} \longrightarrow 2$, there is an $\boldsymbol{F}_{\mathbf{max}}$-subtree S of T such that c is continuous on $\binom{\boldsymbol{F(S)}}{\tau}$.*

Proof Since $c : \binom{\boldsymbol{F(T)}}{\tau} \longrightarrow 2$ is Baire-measurable, $c^{-1}(0)$ and $c^{-1}(1)$ have the property of Baire. Then there exist open sets U, V in $\binom{\boldsymbol{F(T)}}{\tau}$ and meager sets M, N

in $\binom{\boldsymbol{F(T)}}{\tau}$ such that

$$c^{-1}(0) = U \triangle M \text{ and } c^{-1}(1) = V \triangle N,$$

where $\triangle$ denotes the symmetric difference. Let $(N_n)_{n\in\omega}$ be a sequence of closed nowhere dense subsets of $\binom{\boldsymbol{F(T)}}{\tau}$ such that $M \cup N \subseteq \bigcup_{n\in\omega} N_n$. We would like to construct an $\boldsymbol{F}_{\mathbf{max}}$-subtree S of T such that $\binom{\boldsymbol{F(S)}}{\tau}$ is disjoint from $\bigcup_{n\in\omega} N_n$. In this case, we have

$$c^{-1}(0) \cap \binom{\boldsymbol{F(S)}}{\tau} = U \cap \binom{\boldsymbol{F(S)}}{\tau} \text{ and } c^{-1}(1) \cap \binom{\boldsymbol{F(S)}}{\tau} = V \cap \binom{\boldsymbol{F(S)}}{\tau}.$$

It follows that c is continuous on $\binom{\boldsymbol{F(S)}}{\tau}$. In order to find an $\boldsymbol{F}_{\mathbf{max}}$-subtree S that is disjoint from $\bigcup_{n\in\omega} N_n$, we construct a fusion sequence $(T_j)_{j\in\omega}$ of $\boldsymbol{F}_{\mathbf{max}}$-subtrees of T with witness a strictly increasing sequence $(m_j)_{j\in\omega}$ of natural numbers. Put $S = \bigcap_{j\in\omega} T_j$. Then S is an $\boldsymbol{F}_{\mathbf{max}}$-subtree of T.

Suppose T_j and m_j have already been chosen. We assume that for all $t \in T_j(m_j)$ and all $s \in T$ with $t \subseteq s$, we have $s \in T_j$. For a certain $t \in T_j(m_j)$, we have to find a splitting node s with $t \subseteq s$ such that for a certain finite ordered structure $\boldsymbol{H}$, $\boldsymbol{H}$ embeds into $\boldsymbol{F}_{\boldsymbol{s}}^{\boldsymbol{T_{j+1}}}$. Since T_j is an $\boldsymbol{F}_{\mathbf{max}}$-tree, there is $m > m_j$ and an extension s of t with $|s| < m$ such that $\boldsymbol{H}$ embeds into $\boldsymbol{F}_{\boldsymbol{s}}^{\boldsymbol{T_j}}$.

Suppose that $\boldsymbol{H}$ is a finite substructure of $\boldsymbol{F(T_j)}$ of type τ such that $\triangle(\boldsymbol{H}) < m$. We list elements of H as $t_0, t_1, \dots, t_p$. The set $\boldsymbol{H} \restriction m$ determines an open subset O of $\binom{\boldsymbol{F(T)}}{\tau}$. Since $\bigcup_{n\le j} N_n$ is closed and nowhere dense in $\binom{\boldsymbol{F(T)}}{\tau}$, O contains a nonempty open subset that is disjoint from $\bigcup_{n\le j} N_n$. It follows that for $i \in \{0, 1, \dots, p\}$, $t_i \restriction m$ has an extension $s_i \in T_j$ such that the open subset of $\binom{\boldsymbol{F(T)}}{\tau}$ determined by $s_0, s_1, \dots, s_p$ is disjoint from $\bigcup_{n\le j} N_n$. We may assume that $s_0, s_1, \dots, s_p$ have the same length $m_{j+1} > m$.

Let $X \subseteq T_j(m_{j+1})$ be a set that contains exactly one extension of every element of $T_j(m_j)$ and in particular the elements $s_0, s_1, \dots, s_p$. Let

$$T_{j+1} = \{t \in T_j : \exists s \in X (s \subseteq t \vee t \subseteq s)\}.$$

Then T_{j+1} is an $\boldsymbol{F}_{\mathbf{max}}$-tree. Whenever $\boldsymbol{H'}$ is a finite substructure of $\boldsymbol{F(T_{j+1})}$ of type τ with $\boldsymbol{H'} \restriction m = \boldsymbol{H} \restriction m$, then $\boldsymbol{H'} \restriction m_{j+1} = \{s_0, s_1, \dots, s_p\}$. In particular, $\boldsymbol{H'} \notin \bigcup_{n\le j} N_n$. This finishes the recursive definition of the sequences $(T_j)_{j\in\omega}$ and $(m_j)_{j\in\omega}$.

Finally, let $S = \bigcap_{j\in\omega} T_j$. One can check that S is an $\boldsymbol{F}_{\mathbf{max}}$-tree. Let $n \in \omega$ and let $\boldsymbol{H}$ be a finite substructure of $\boldsymbol{F(S)}$ of type τ. Then there is $j \in \omega$ such that $\triangle(\boldsymbol{H}) < m_j$. We can choose $j \ge n$. Note that $S(m_j) = T(m_j)$. Since $S \subseteq T_{j+1}$, by the construction of T_{j+1}, $\boldsymbol{H} \notin \bigcup_{n\le j} N_n$. In particular, $H \notin N_n$. This shows that $\binom{\boldsymbol{F(S)}}{\tau}$ is disjoint from $\bigcup_{n\in\omega} N_n$. It follows that c is continuous on $\binom{\boldsymbol{F(S)}}{\tau}$. □

Theorem 5.8 *Let T be an $\boldsymbol{F}_{\mathbf{max}}$-tree. For every type τ of a finite induced substructure of $\boldsymbol{F}_{\mathbf{max}}$ and every Baire-measurable coloring $c : \binom{\boldsymbol{F(T)}}{\tau} \longrightarrow 2$, there is an $\boldsymbol{F}_{\mathbf{max}}$-subtree S of T such that c is constant on $\binom{\boldsymbol{F(S)}}{\tau}$.*

Proof By Lemma 5.7, there is an $\boldsymbol{F}_{\mathbf{max}}$-subtree U of T such that c is continuous on $\binom{\boldsymbol{F(U)}}{\tau}$. By Theorem 5.6, there is an $\boldsymbol{F}_{\mathbf{max}}$-subtree S of U such that c is constant on $\binom{\boldsymbol{F(S)}}{\tau}$. □

Theorem 5.9 *Let $\mathcal{K}$ be a Fraïssé class, in a finite signature, of finite ordered binary relational structures with the Ramsey property. For every $\boldsymbol{H} \in \mathcal{K}$, there is a finite number $T(\boldsymbol{H}, \boldsymbol{F}_{\mathbf{max}})$ such that the following holds: For every universal inverse limit structure $\boldsymbol{G}$ and for each finite Baire-measurable coloring of the set $\binom{\boldsymbol{G}}{\boldsymbol{H}}$ of all copies of $\boldsymbol{H}$ in $\boldsymbol{G}$, there is a closed copy $\boldsymbol{G}'$ of $\boldsymbol{G}$ contained in $\boldsymbol{G}$ such that the set $\binom{\boldsymbol{G}'}{\boldsymbol{H}}$ of all copies of $\boldsymbol{H}$ in $\boldsymbol{G}'$ has no more than $T(\boldsymbol{H}, \boldsymbol{F}_{\mathbf{max}})$ colors. In particular, $T(\boldsymbol{H}, \boldsymbol{F}_{\mathbf{max}})$ is at most the number of types associated to $\boldsymbol{H}$.*

Proof We list all types for $\boldsymbol{H}$ as $\tau_0, \tau_1, \ldots, \tau_{m-1}$. Since $\boldsymbol{G}$ is a universal inverse limit structure, $\boldsymbol{F}_{\mathbf{max}}$ embeds continuously into it. Let c be a finite Baire-measurable coloring of the set $\binom{\boldsymbol{G}}{\boldsymbol{H}}$. Now work with the tree $T_{\max}$ coding $\boldsymbol{F}_{\mathbf{max}}$. Iterating Theorem 5.8, there are $\boldsymbol{F}_{\mathbf{max}}$-trees $T_{\max} \geq T_{\tau_0} \geq \cdots \geq T_{\tau_{m-1}}$ so that for each $i < m$, c is constant on $\binom{\boldsymbol{F(T_{\tau_i})}}{\tau_i}$. We take $\boldsymbol{G}' = \boldsymbol{F(T_{\tau_{m-1}})}$. Then $\boldsymbol{G}'$ is a closed copy of $\boldsymbol{G}$ contained in $\boldsymbol{G}$, and the set $\binom{\boldsymbol{G}'}{\boldsymbol{H}}$ of all copies of $\boldsymbol{H}$ in $\boldsymbol{G}'$ has no more than m colors. □

6 Exact big Ramsey degrees for some ordered binary relational inverse limit structures

In this section, we find the exact big Ramsey degrees in the inverse limit structures $\boldsymbol{F}_{\mathbf{max}}$ of the following Fraïssé classes in an ordered binary relational signature: Free amalgamation classes, the class of finite ordered tournaments $\mathcal{OT}$, and the class of finite partial orders with a linear extension $\mathcal{OPO}$. We shall do so by first showing in Lemmas 6.1, 6.4, and 6.7 that for any finite substructure $\boldsymbol{H}$ of $\boldsymbol{F}_{\mathbf{max}}$, there is a larger finite structure $\overline{\boldsymbol{H}}$ containing $\boldsymbol{H}$ as an induced substructure such that any copy of $\overline{\boldsymbol{H}}$ in the inverse limit structure $\boldsymbol{F}_{\mathbf{max}}$ must have exactly one meet in the tree $T_{\max}$. Then in Theorem 6.10, we shall prove by induction on number of splitting nodes that each type persists in any subcopy of $\boldsymbol{F}_{\mathbf{max}}$. This proves that the exact big Ramsey degree for a given finite substructure $\boldsymbol{H}$ of $\boldsymbol{F}_{\mathbf{max}}$ is exactly the number of types τ representing $\boldsymbol{H}$ in the tree $T_{\max}$.

Given a structure $\boldsymbol{G} \leq \boldsymbol{F}_{\mathbf{max}}$, let $T_{\boldsymbol{G}} = \{x \restriction n : x \in \boldsymbol{G}, n \in \omega\}$. Then $T_{\boldsymbol{G}}$ is a subtree of $T_{\max}$. Given a tree T, its *stem*, denoted stem(T), is the minimal splitting node in T.

Lemma 6.1 *Let $\mathcal{K}$ be any Fraïssé class with free amalgamation in an ordered binary relational signature. Then for each $\boldsymbol{H} \in \mathcal{K}$, there is a structure $\overline{\boldsymbol{H}} \in \mathcal{K}$ containing a copy of $\boldsymbol{H}$, where $\overline{\boldsymbol{H}}$ has the following property: Given a universal inverse limit structure $\boldsymbol{G}$ for $\mathcal{K}$ contained in $\boldsymbol{F}_{\mathbf{max}}$, every copy $\overline{\boldsymbol{I}}$ of $\overline{\boldsymbol{H}}$ in $\boldsymbol{G}$ has induced a subtree $T_{\overline{\boldsymbol{I}}}$ of $T_{\boldsymbol{G}}$ such that the type of $T_{\overline{\boldsymbol{I}}}$ has exactly one splitting node. It follows that the immediate successors of stem$(T_{\overline{\boldsymbol{I}}})$ in $T_{\overline{\boldsymbol{I}}}$ have a copy of $\overline{\boldsymbol{I}}$.*

Proof Let $\mathcal{K}$ be as in the hypotheses, let $\boldsymbol{G}$ be the universal inverse limit for $\mathcal{K}$ contained in $\boldsymbol{F}_{\mathbf{max}}$, and fix $\boldsymbol{H}$ a finite substructure of $\boldsymbol{G}$. Then $\boldsymbol{H}$ is in $\mathcal{K}$. Let m be the size of the universe of $\boldsymbol{H}$. We construct a finite ordered structure $\overline{\boldsymbol{H}} \in \mathcal{K}$ of size $2m + 1$, containing a copy of $\boldsymbol{H}$ as a substructure on the odd indexed vertices, as follows. Let R denote the binary relation symbol R_0 in the signature of $\mathcal{K}$.

(1) Let $\overline{H} = \{v_0, v_1, v_2, \dots, v_{2m}\}$.
(2) $\overline{\boldsymbol{H}} \restriction \{v_1, v_3, \dots, v_{2m-1}\}$ is isomorphic to $\boldsymbol{H}$.
(3) For $i \in \{0, 2, \dots, 2m-2\}$, $R^{\overline{\boldsymbol{H}}}(v_i, v_{i+2})$ holds. If $R^{\overline{\boldsymbol{H}}}$ is a symmetric relation, then also $R^{\overline{\boldsymbol{H}}}(v_{i+2}, v_i)$ holds; otherwise, $\neg R^{\overline{\boldsymbol{H}}}(v_{i+2}, v_i)$ holds.
(4) No other relations are added to $\overline{\boldsymbol{H}}$.

Then $\overline{\boldsymbol{H}}$ contains a copy of $\boldsymbol{H}$. Now we check that $\overline{\boldsymbol{H}}$ satisfies the property in this lemma. Let $\overline{\boldsymbol{I}}$ be a copy of $\overline{\boldsymbol{H}}$ in $\boldsymbol{G}$ with $\overline{I} = \{u_0, u_1, u_2, \dots, u_{2m}\}$.

Claim 6.2 *Let $\boldsymbol{J}$ be the induced substructure of $\overline{\boldsymbol{I}}$ on universe $J = \{u_0, u_2, \dots, u_{2m}\}$. Then the associated subtree $T_{\boldsymbol{J}}$ of $T_{\boldsymbol{G}}$ has exactly one splitting node.*

Proof Without loss of generality, it suffices to prove that $u_0 \cap u_2 = u_2 \cap u_4$. Assume to the contrary that $u_0 \cap u_2 \neq u_2 \cap u_4$. Then either $u_0 \cap u_2 \subset u_2 \cap u_4$ or else $u_2 \cap u_4 \subset u_0 \cap u_2$, where $\subset$ denotes proper subset. If $u_0 \cap u_2 \subset u_2 \cap u_4$ (see Fig. 20), then $u_0 \cap u_2 = u_0 \cap u_4$; let s denote this node and let $l - 1$ denote its length. Then $u_2 \restriction l = u_4 \restriction l$ is a successor of s, and the relation $R(u_0 \restriction l, u_4 \restriction l)$ holds in $T_{\max}$ since $R(u_0 \restriction l, u_2 \restriction l)$ holds in $T_{\max}$. Hence, $R^{\boldsymbol{I}}(u_0, u_4)$ holds. Similarly, if $u_2 \cap u_4 \subset u_0 \cap u_2$, then $u_0 \cap u_4 = u_2 \cap u_4$, and it follows that $R^{\boldsymbol{I}}(u_0, u_4)$ holds (see Fig. 20). But this contradicts the fact that $R^{\boldsymbol{I}}(u_0, u_4)$ does not hold, by (3) in the defnition of $\overline{\boldsymbol{H}}$ above. Therefore, it must be the case that $u_0 \cap u_2 = u_2 \cap u_4$. Thus, every copy of $\boldsymbol{J}$ has induced subtree $T_{\boldsymbol{J}}$ in $T_{\boldsymbol{G}}$ with exactly one splitting node. □

Claim 6.3 *The subtree $T_{\overline{\boldsymbol{I}}}$ of $T_{\boldsymbol{G}}$ induced by $\overline{\boldsymbol{I}}$ has exactly one splitting node.*

Proof By Claim 6.2, without loss of generality, it suffices to prove that $u_0 \cap u_1 = u_1 \cap u_2$. Assume that $u_0 \cap u_1 \neq u_1 \cap u_2$. Then either $u_0 \cap u_1 \subset u_1 \cap u_2$ or $u_1 \cap u_2 \subset u_0 \cap u_1$. Since $u_0 < u_1 < u_2$ in the linear order $<$ on the universe of $\boldsymbol{F}_{\mathbf{max}}$, it follows that the set $\{u_0, u_1, u_2\}$ has type equal to one of the two types in Fig. 21 (the solid lines). Since $R^{\overline{\boldsymbol{I}}}(u_0, u_2)$ holds, at least one of $R^{\overline{\boldsymbol{I}}}(u_0, u_1)$ or $R^{\overline{\boldsymbol{I}}}(u_1, u_2)$ holds (the dashed lines in Fig. reff21). But this contradicts the fact that $\neg R^{\overline{\boldsymbol{I}}}(u_0, u_1)$ and $\neg R^{\overline{\boldsymbol{I}}}(u_1, u_2)$ hold, by (4) in the definition of $\overline{\boldsymbol{H}}$. Thus, the induced subtree $T_{\overline{\boldsymbol{I}}}$ has exactly one splitting node. □

By Claim 6.3, the induced subtree $T_{\overline{\boldsymbol{I}}}$ of $T_{\boldsymbol{G}}$ has exactly one splitting node, namely its stem. It follows from the definition of the relations on $\boldsymbol{F}_{\mathbf{max}}$ that the immediate successors of stem$(T_{\overline{\boldsymbol{I}}})$ in $T_{\overline{\boldsymbol{I}}}$ is isomorphic to $\overline{\boldsymbol{H}}$. □

Next, we prove a similar lemma for the class of finite ordered tournaments, $\mathcal{OT}$. The proof is similar to the previous lemma, the main difference being that the construction of $\overline{\boldsymbol{H}}$ must take into account the fact that any two vertices of a tournament must have some directed edge relation between them.

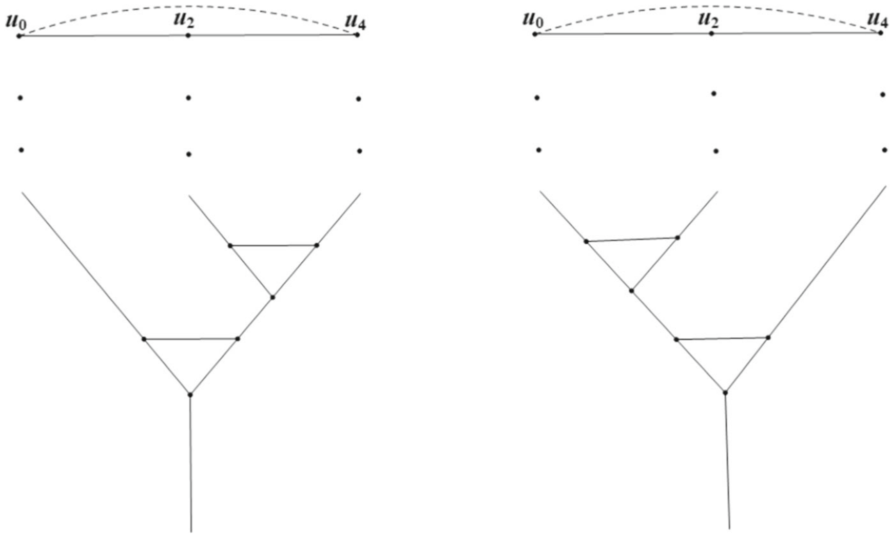


Fig. 20 2 types for $\boldsymbol{J} \restriction \{u_0, u_2, u_4\}$ with $u_0 \cap u_2 \neq u_2 \cap u_4$

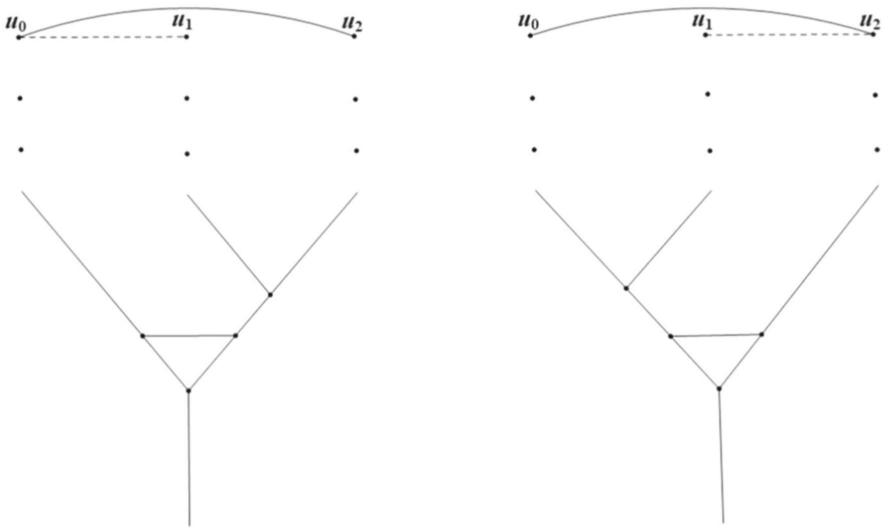


Fig. 21 2 types for $\overline{\boldsymbol{I}} \restriction \{u_0, u_1, u_2\}$ with $u_0 \cap u_1 \neq u_1 \cap u_2$

Lemma 6.4 *Let $\mathcal{K}$ be $\mathcal{OT}$. Then for each $\boldsymbol{H} \in \mathcal{K}$, there is a structure $\overline{\boldsymbol{H}} \in \mathcal{K}$ containing a copy of $\boldsymbol{H}$ and $\overline{\boldsymbol{H}}$ has the following property: Given a universal inverse limit structure $\boldsymbol{G}$ for $\mathcal{K}$ contained in $\boldsymbol{F}_{\mathbf{max}}$, every copy $\overline{\boldsymbol{I}}$ of $\overline{\boldsymbol{H}}$ in $\boldsymbol{G}$ has induced a subtree $T_{\overline{\boldsymbol{I}}}$ of $T_{\boldsymbol{G}}$ such that the type of $T_{\overline{\boldsymbol{I}}}$ has exactly one splitting node. It follows that the immediate successors of stem($T_{\overline{\boldsymbol{I}}}$) in $T_{\overline{\boldsymbol{I}}}$ have a copy of $\overline{\boldsymbol{I}}$.*

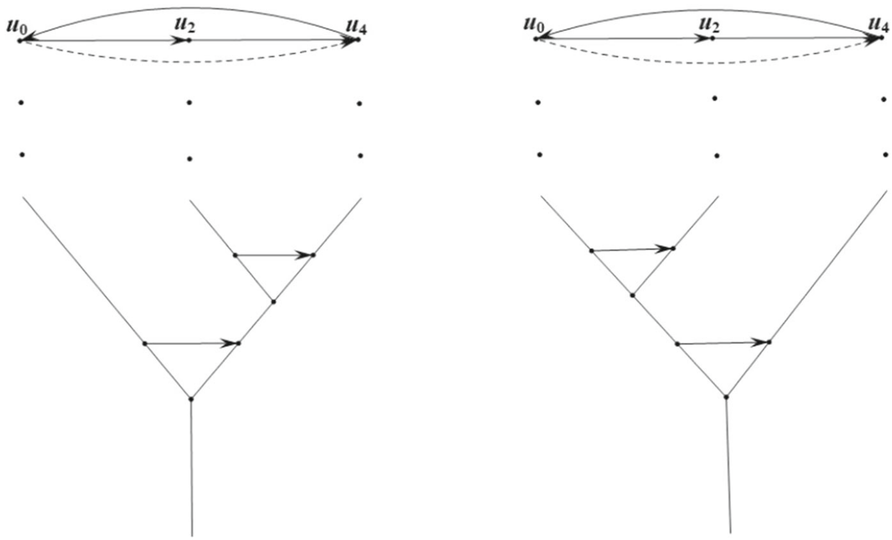


Fig. 22 2 types for $\boldsymbol{J} \upharpoonright \{u_0, u_2, u_4\}$ with $u_0 \cap u_2 \neq u_2 \cap u_4$

Proof Fix any $\boldsymbol{H} \in \mathcal{K}$, and let m be the size of the universe of $\boldsymbol{H}$. Recall that the relation R here is a directed edge. We construct a finite ordered structure $\overline{\boldsymbol{H}} \in \mathcal{K}$ containing a copy of $\boldsymbol{H}$ as an induced substructure as follows:

(1) Let $\overline{H} = \{v_0, v_1, v_2, \ldots, v_{2m}\}$.
(2) $\overline{\boldsymbol{H}} \upharpoonright \{v_1, v_3, \ldots, v_{2m-1}\}$ is isomorphic to $\boldsymbol{H}$.
(3) For $i, j \in \{0, 2, \ldots, 2m\}$ with $i < j$, if $j = i + 2$ then $R^{\overline{\boldsymbol{H}}}(v_i, v_j)$ holds.
(4) For $i, j \in \{0, 2, \ldots, 2m\}$ with $i < j$, if $j \neq i + 2$ then $R^{\overline{\boldsymbol{H}}}(v_j, v_i)$ holds.
(5) For all $i \in \{0, 2, \ldots, 2m\}$ and $j \in \{1, 3, \ldots, 2m-1\}$, if $i < j$ then $R^{\overline{\boldsymbol{H}}}(v_j, v_i)$ holds.
(6) For all $i \in \{0, 2, \ldots, 2m\}$ and $j \in \{1, 3, \ldots, 2m-1\}$, if $j < i$ then $R^{\overline{\boldsymbol{H}}}(v_i, v_j)$ holds.

By (2), $\overline{\boldsymbol{H}}$ contains a copy of $\boldsymbol{H}$. Now we check that $\overline{\boldsymbol{H}}$ satisfies the property in this lemma. Let $\overline{\boldsymbol{I}}$ be any copy of $\overline{\boldsymbol{H}}$ in $\boldsymbol{G}$, say with universe $\overline{I} = \{u_0, u_1, u_2, ..., u_{2m}\}$.

Claim 6.5 *Let $\boldsymbol{J}$ be the induced substructure of $\overline{\boldsymbol{I}}$ on universe $J = \{u_0, u_2, ..., u_{2m}\}$. Then the associated subtree $T_{\boldsymbol{J}}$ of $T_{\boldsymbol{G}}$ has exactly one splitting node.*

Proof Without loss of generality, it suffices to prove that $u_0 \cap u_2 = u_2 \cap u_4$. Assume that $u_0 \cap u_2 \neq u_2 \cap u_4$. Then $u_0 \cap u_2 \subset u_2 \cap u_4$ or $u_2 \cap u_4 \subset u_0 \cap u_2$. Since $R^{\boldsymbol{J}}(u_0, u_2)$ and $R^{\boldsymbol{J}}(u_2, u_4)$ hold, it follows that $R^{\boldsymbol{J}}(u_0, u_4)$ holds (see the dashed arrows in Fig. 22). This contradicts the fact that $R^{\boldsymbol{J}}(u_4, u_0)$ holds, by (4) in the definition of $\overline{\boldsymbol{H}}$ (solid arrows from u_4 to u_0 in Fig. 22). Thus, $u_0 \cap u_2 = u_2 \cap u_4$. It follows that $T_{\boldsymbol{J}}$ has exactly one splitting node, namely its stem. □

Claim 6.6 *The subtree $T_{\overline{\boldsymbol{I}}}$ of $T_{\boldsymbol{G}}$ induced by $\overline{\boldsymbol{I}}$ has exactly one splitting node.*

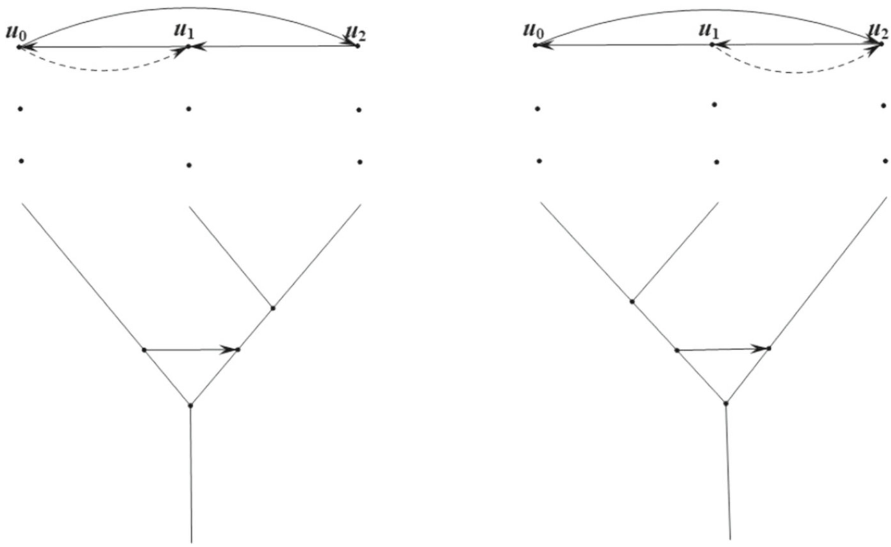


Fig. 23 2 types for $\overline{\boldsymbol{I}} \restriction \{u_0, u_1, u_2\}$ with $u_0 \cap u_1 \neq u_1 \cap u_2$

Proof By Claim 6.5, without loss of generality, it suffices to prove that $u_0 \cap u_1 = u_1 \cap u_2$. Assume that $u_0 \cap u_1 \neq u_1 \cap u_2$. Then either $u_0 \cap u_1 \subset u_1 \cap u_2$ or $u_1 \cap u_2 \subset u_0 \cap u_1$. Since $u_0 < u_1 < u_2$ in the linear order $<$ on the universe of $\boldsymbol{F}_{\mathbf{max}}$, it follows that the set $\{u_0, u_1, u_2\}$ has type equal to one of the two types in Fig. 23 (the solid arrows). Since $R^{\overline{\boldsymbol{I}}}(u_0, u_2)$, either $R^{\overline{\boldsymbol{I}}}(u_1, u_2)$ or $R^{\overline{\boldsymbol{I}}}(u_0, u_1)$ holds (see the dashed arrows in Fig. 23). This contradicts the facts that $R^{\overline{\boldsymbol{I}}}(u_1, u_0)$ and $R^{\overline{\boldsymbol{I}}}(u_2, u_1)$ hold, by (5) and (6) of the definition of $\overline{\boldsymbol{H}}$. Thus, the induced subtree $T_{\overline{\boldsymbol{I}}}$ has exactly one splitting node. □

By Claim 6.6, $\overline{\boldsymbol{H}}$ satisfies the property in this lemma. □

Finally, we prove a similar lemma for the class of finite partial orders with a linear extension, $\mathcal{OPO}$.

Lemma 6.7 *Let $\mathcal{K}$ be $\mathcal{OPO}$. Then for each $\boldsymbol{H} \in \mathcal{K}$, there is a structure $\overline{\boldsymbol{H}} \in \mathcal{K}$ containing a copy of $\boldsymbol{H}$ and $\overline{\boldsymbol{H}}$ has the following property: Given a universal inverse limit structure $\boldsymbol{G}$ for $\mathcal{K}$ contained in $\boldsymbol{F}_{\mathbf{max}}$, every copy $\overline{\boldsymbol{I}}$ of $\overline{\boldsymbol{H}}$ in $\boldsymbol{G}$ has induced a subtree $T_{\overline{\boldsymbol{I}}}$ of $T_{\boldsymbol{G}}$ such that the type of $T_{\overline{\boldsymbol{I}}}$ has exactly one splitting node. It follows that the immediate successors of $stem(T_{\overline{\boldsymbol{I}}})$ in $T_{\overline{\boldsymbol{I}}}$ have a copy of $\overline{\boldsymbol{I}}$.*

Proof Fix any $\boldsymbol{H} \in \mathcal{K}$, and let m be the size of the universe of $\boldsymbol{H}$. If $\boldsymbol{H}$ has universe of size one, then there is nothing to prove, so assume that the universe of $\boldsymbol{H}$ has size $m \geq 2$. The relation R here is a partial order, where $R(v, w)$ denotes that v is R-less than or equal to w. We construct a finite ordered structure $\overline{\boldsymbol{H}} \in \mathcal{K}$ containing a copy of $\boldsymbol{H}$ as an induced substructure as follows:

(1) Let $\overline{H} = \{v_0, v_1, v_2, \ldots, v_{2m+2}\}$.
(2) $\overline{\boldsymbol{H}} \restriction \{v_2, v_4, \ldots, v_{2m}\}$ is isomorphic to $\boldsymbol{H}$.
(3) $R(v_0, v_3)$ and $R(v_{2m-1}, v_{2m+2})$ hold.

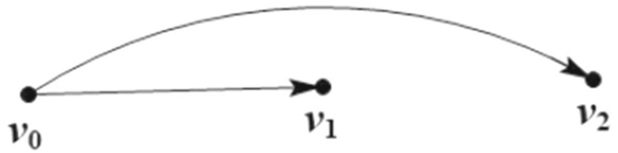


Fig. 24 $\boldsymbol{H} \in \mathcal{OPO}$ with 3 vertices

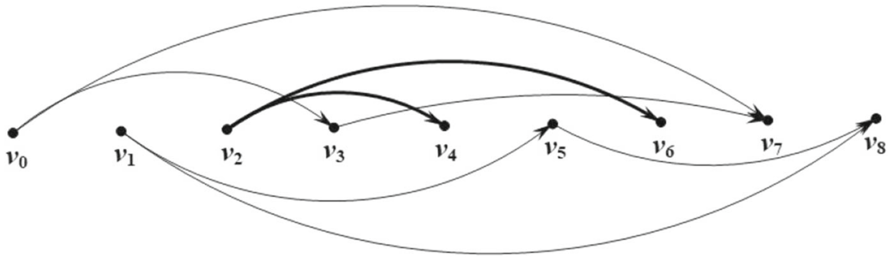


Fig. 25 $\overline{\boldsymbol{H}} \in \mathcal{OPO}$ with 9 vertices

(4) For each $i \in \{1, 3, \ldots, 2m-3\}$, $R^{\overline{\boldsymbol{H}}}(v_i, v_{i+4})$ holds.
(5) The R-relations above are closed under transitivity of R, and no other R relations are added.

By (2), $\overline{\boldsymbol{H}}$ contains a copy of $\boldsymbol{H}$. Note that there are no R-relations between any vertices in $\{v_2, v_4, \ldots, v_{2m}\}$ and any vertices $\{v_0, v_1, v_3, v_5, \ldots, v_{2m+1}, v_{2m}\}$. For example, if $\boldsymbol{H} \in \mathcal{OPO}$ with 3 vertices as in Fig. 24, then $\overline{\boldsymbol{H}}$ has 9 vertices as in Fig. 25. The copy of $\boldsymbol{H}$ in $\overline{\boldsymbol{H}}$ is on vertices $\{v_2, v_4, v_6\}$.

Now we check that $\overline{\boldsymbol{H}}$ satisfies the property in this lemma. Let $\overline{\boldsymbol{I}}$ be any copy of $\overline{\boldsymbol{H}}$ in $\boldsymbol{G}$, say with universe $\overline{I} = \{u_0, u_1, u_2, \ldots, u_{2m+2}\}$.

Claim 6.8 *Let $\boldsymbol{J}$ be the induced substructure of $\overline{\boldsymbol{I}}$ on universe $J = \{u_0, u_1, u_3, \ldots, u_{2m-1}, u_{2m+1}, u_{2m+2}\}$. Then the associated subtree $T_{\boldsymbol{J}}$ of $T_{\boldsymbol{G}}$ has exactly one splitting node.*

Proof It suffices to prove that any three successive vertices in J have the same meet. Without loss of generality, it suffices to prove that $u_0 \cap u_1 = u_1 \cap u_3$, as the same argument shows that for any $0 \leq i \leq m-2$, $u_{2i+1} \cap u_{2i+3} = u_{2i+3} \cap u_{2i+5}$, and that $u_{2m-1} \cap u_{2m+1} = u_{2m+1} \cap u_{2m+2}$.

Assume that $u_0 \cap u_1 \neq u_1 \cap u_3$. Then $u_0 \cap u_1 \subset u_1 \cap u_3$ or $u_1 \cap u_3 \subset u_0 \cap u_1$. Note that $R^{\boldsymbol{J}}(u_0, u_3)$ holds and $\neg R^{\boldsymbol{J}}(u_0, u_1)$ and $\neg R^{\boldsymbol{J}}(u_1, u_3)$ hold. If $u_0 \cap u_1 \subset u_1 \cap u_3$ then $R^{\boldsymbol{J}}(u_0, u_1)$, a contradiction. If $u_1 \cap u_3 \subset u_0 \cap u_1$, then $R^{\boldsymbol{J}}(u_1, u_3)$, also a contradiction (see Fig. 26). Thus, $u_0 \cap u_1 = u_1 \cap u_3$. It follows that $T_{\boldsymbol{J}}$ has exactly one splitting node, namely its stem. □

Claim 6.9 *The subtree $T_{\overline{\boldsymbol{I}}}$ of $T_{\boldsymbol{G}}$ induced by $\overline{\boldsymbol{I}}$ has exactly one splitting node.*

Proof We will first prove that for all $0 \leq i \leq m-2$, $u_{2i+1} \cap u_{2i+2} = u_{2i+2} \cap u_{2i+5}$, and that $u_{2m-1} \cap u_{2m} = u_{2m} \cap u_{2m+2}$. Since the argument is the same for each of

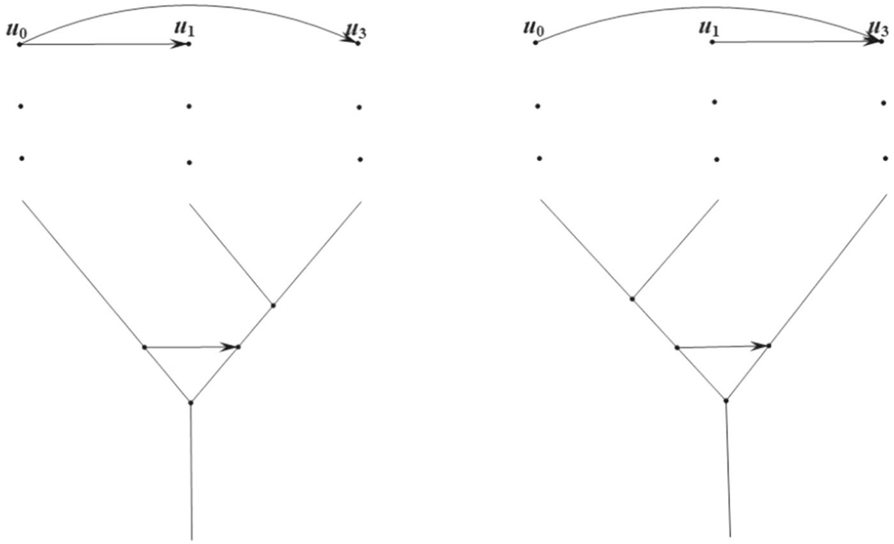


Fig. 26 2 types for $\boldsymbol{J} \restriction \{u_0, u_1, u_3\}$ with $u_0 \cap u_1 \neq u_1 \cap u_3$

these cases, it suffices to prove that $u_1 \cap u_2 = u_2 \cap u_5$. In fact, the same argument as that in Claim 6.8 applies here, for the structure $\overline{\boldsymbol{I}}$ restricted to the vertices $\{u_1, u_2, u_5\}$ is isomorphic to the one in the proof of Claim 6.8: $R^{\overline{\boldsymbol{I}}}(u_1, u_5)$, $\neg R^{\overline{\boldsymbol{I}}}(u_1, u_2)$, and $\neg R^{\overline{\boldsymbol{I}}}(u_2, u_5)$. Since the linear order $<$ extends $R^{\overline{\boldsymbol{I}}}$, $\neg R^{\overline{\boldsymbol{I}}}(u_2, u_1)$, $\neg R^{\overline{\boldsymbol{I}}}(u_5, u_1)$, and $\neg R^{\overline{\boldsymbol{I}}}(u_5, u_2)$ all hold. Therefore, $u_1 \cap u_2 = u_2 \cap u_5$ (see Fig. 27).

Given any three vertices $u_{2i+1}, u_{2i+2}, u_{2i+3}$, where $0 \leq i \leq m-1$, let $w = u_{2i+5}$ if $i < m-1$ and $w = u_{2m+2}$ if $i = m-1$. By the above argument,

$$u_{2i+1} \cap u_{2i+2} = u_{2i+2} \cap w = u_{2i+1} \cap w.$$

By Claim 6.8,

$$u_{2i+1} \cap u_{2i+3} = u_{2i+3} \cap w = u_{2i+1} \cap w.$$

Therefore,

$$u_{2i+1} \cap u_{2i+2} = u_{2i+1} \cap w = u_{2i+2} \cap u_{2i+3}.$$

Hence, also $u_{2i+1} \cap u_{2i+2} = u_{2i+1} \cap u_{2i+3}$. Thus, any three successive vertices in $\overline{\boldsymbol{I}}$ have a common meet, meaning that $\overline{\boldsymbol{I}}$ has exactly one splitting node. □

By Claim 6.9, $\overline{\boldsymbol{H}}$ satisfies the property in this lemma. □

The next theorem shows that the upper bounds proved in Theorem 5.9 are exact for ordered binary relational free amalgamation classes, for ordered tournaments, and for partial orders with a linear extension.

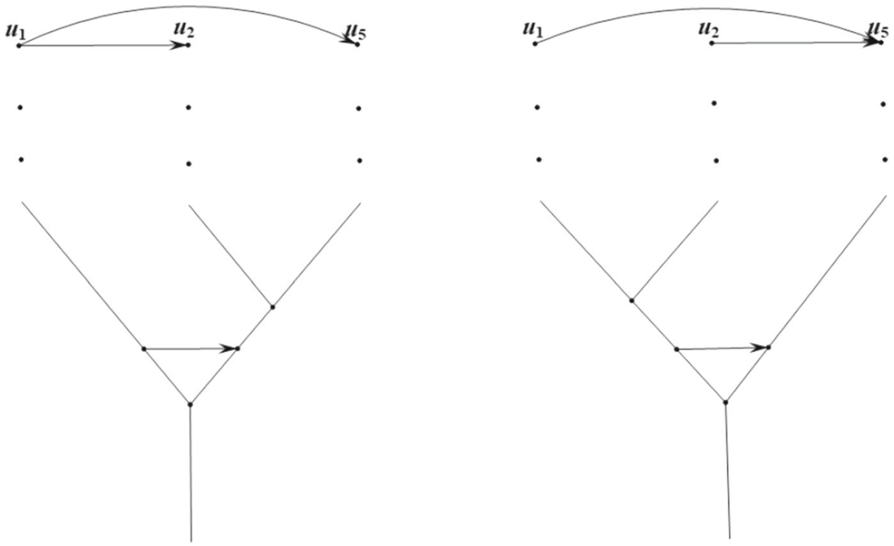


Fig. 27 2 types for $\overline{\boldsymbol{I}} \upharpoonright \{u_1, u_2, u_5\}$ with $u_1 \cap u_2 \neq u_2 \cap u_5$

Theorem 6.10 *Let $\mathcal{K}$ be any Fraïssé class in an ordered binary relational signature such that either $\mathcal{K}$ has free amalgamation or $\mathcal{K}$ is one of $\mathcal{OT}$ or $\mathcal{OPO}$. Let $\boldsymbol{G}$ be a universal inverse limit structure for $\mathcal{K}$ contained in $\boldsymbol{F}_{\mathbf{max}}$. Then for each $\boldsymbol{H} \in \mathcal{K}$, each type representing $\boldsymbol{H}$ in $\boldsymbol{G}$ persists in each closed subcopy of $\boldsymbol{G}$. It follows that the big Ramsey degree $T(\boldsymbol{H}, \boldsymbol{F}_{\mathbf{max}})$ for finite Baire-measurable colorings of $\binom{\boldsymbol{F}_{\mathbf{max}}}{\boldsymbol{H}}$ is exactly the number of types in $T_{\max}$ representing a copy of $\boldsymbol{H}$.*

Proof Let $\boldsymbol{G} \subseteq \boldsymbol{F}_{\mathbf{max}}$ be a universal inverse limit structure for $\mathcal{K}$. We will prove by induction on the number of splitting nodes that every type τ for each finite structure in $\mathcal{K}$ persists in $\boldsymbol{G}$.

Suppose τ is a type for a structure in $\mathcal{K}$ which has no splitting nodes. Then τ codes a single element, so there is a copy of τ in $T_{\boldsymbol{G}}$.

Now assume that $n \geq 1$ and for each type τ with less than n many splitting nodes, the type τ appears in $T_{\boldsymbol{G}}$. For the induction step, let τ be a type for a structure in $\mathcal{K}$ with exactly n splitting nodes. Let s denote the splitting node of longest length in τ, and let $\sigma = \tau \upharpoonright |s|$. By the induction hypothesis, there is a copy of σ in $T_{\boldsymbol{G}}$. Then there is a subtree U of $T_{\boldsymbol{G}}$ such that U has type σ. Let $\varphi : \sigma \longrightarrow U$ be the strong isomorphism from σ to U, and let $u = \varphi(s)$.

Suppose that $\boldsymbol{F}$ is the finite structure in $\mathcal{K}$ at the immediate successors of s in τ. Let $\overline{\boldsymbol{F}} \in \mathcal{K}$ be the structure containing a copy of $\boldsymbol{F}$ satisfying the properties in Lemma 6.1, 6.4, or 6.7, respectively. Since $\boldsymbol{G}$ is a universal inverse limit structure for $\mathcal{K}$, there is a copy $\overline{\boldsymbol{H}}$ of $\overline{\boldsymbol{F}}$ in the open interval N_u. Taking $t = \text{stem}(T_{\overline{\boldsymbol{H}}})$, then $\text{succ}_{T_{\overline{\boldsymbol{H}}}}(t)$ contains a copy of $\overline{\boldsymbol{F}}$, and thus $\text{succ}_{T_{\overline{\boldsymbol{H}}}}(t)$ contains a copy $\boldsymbol{H}$ of $\boldsymbol{F}$. Let X denote the set of nodes in $\text{succ}_{T_{\overline{\boldsymbol{H}}}}(t)$ forming the universe of $\boldsymbol{H}$. Let Y be a set of nodes $\{y_z : z \in U \setminus \{u\}\}$ of length $|t| + 1$ such that for each $z \in U \setminus \{u\}$, $y_z \supseteq z$. Then $U \cup \{t\} \cup X \cup Y$ is a copy of τ. Hence, τ persists in $\boldsymbol{G}$.

Now given a structure $\boldsymbol{H} \in \mathcal{K}$, let $\tau_0, \ldots, \tau_{m-1}$ list the collection of all types for copies of $\boldsymbol{H}$ in $\boldsymbol{F}_{\mathbf{max}}$. Let $c : \binom{\boldsymbol{F}_{\mathbf{max}}}{\boldsymbol{H}} \longrightarrow m$ be defined by $c(\boldsymbol{J}) = i$ if and only if $T_{\boldsymbol{J}}$ has type τ_i. Note that c is in fact continuous, hence Baire-measurable. By the above argument, there is a substructure $\boldsymbol{G}$ of $\boldsymbol{F}_{\mathbf{max}}$ which is again a universal inverse limit structure with the property that for each $i < m$, there is a copy of $\boldsymbol{H}$ in $\boldsymbol{G}$ with type τ_i. Therefore, all colors $i < m$ persist in $\boldsymbol{G}$. Therefore, $T(\boldsymbol{H}, \boldsymbol{F}_{\mathbf{max}}) \geq m$.

By Theorem 5.9, we know that $T(\boldsymbol{H}, \boldsymbol{F}_{\mathbf{max}}) \leq m$ for finite Baire-measurable colorings. Therefore, $T(\boldsymbol{H}, \boldsymbol{F}_{\mathbf{max}})$ is exactly the number of types associated to $\boldsymbol{H}$. □

Remark 6.11 Theorem 6.10 characterizes the exact big Ramsey degrees under the finite Baire-measurable colorings for some ordered structures with one (non-order) binary relation. It seems likely that similar methods can be developed to characterize the exact big Ramsey degrees for all structures considered in this paper in terms of the number of types.

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References

1. Abramson, F.G., Harrington, L.: Models without indiscernibles. J. Symb. Log. **43**(3), 572–600 (1978)
2. Balko, M., Chodounský, D., Hubička, J., Konečný, M., Vena, L.: Big Ramsey degrees of 3-uniform hypergraphs are finite. Combinatorica (2022, to appear)
3. Balko, M., Chodounský, D., Dobrinen, N., Hubička, J., Konečný, M., Vena, L., Zucker, A.: Exact big Ramsey degrees via coding trees. pp. 97, submitted, arXiv:2110.08409 (2021)
4. Blass, A.: A partition theorem for perfect sets. Proc. Am. Math. Soc. **82**(2), 271–277 (1981)
5. Coulson, R., Dobrinen, N., Patel, R.: Fraïssé classes with simply characterized big Ramsey degrees. pp. 69, submitted, arXiv:2010.02034 (2020)
6. Devlin, D.: Some partition theorems and ultrafilters on ω. PhD thesis, Dartmouth College (1979)
7. Dobrinen, N.: The Ramsey theory of Henson graphs. J. Math. Log. (2022, to appear)
8. Dobrinen, N.: The Ramsey theory of the universal homogeneous triangle-free graph. J. Math. Log. **20**(2), 2050012 (2020)
9. Dobrinen, N.: Ramsey theory on infinite structures and the method of strong coding trees. In: Rezus, A. (ed.) Contemporary Logic and Computing, pp. 444–467. College Publications, London (2020)
10. Galvin, F.: Partition theorems for the real line. Not. Am. Math. Soc. **15**, 660 (1968)
11. Geschke, S.: Clopen graphs. Fundam. Math. **220**, 155–189 (2013)
12. Halpern, J.D., Läuchli, H.: A partition theorem. Trans. Am. Math. Soc. **124**, 360–367 (1966)
13. Huber, S., Geschke, S., Kojman, M.: Partitioning subgraphs of profinite ordered graphs. Combinatorica **39**(3), 659–678 (2019)
14. Hubička, J.: Big Ramsey degrees using parameter spaces. pp. 19, submitted, arXiv:2009.00967 (2020)
15. Kechris, A., Pestov, V., Todorčević, S.: Fraïssé limits, Ramsey theory, and topological dynamics of automorphism groups. Geom. Funct. Anal. **15**(1), 106–189 (2005)
16. Laflamme, C., Sauer, N.W., Vuksanovic, V.: Canonical partitions of universal structures. Combinatorica **26**(2), 183–205 (2006)
17. Laflamme, C., Nguyen Van Thé, L., Sauer, N.: Partition properties of the dense local order and a colored version of Milliken's theorem. Combinatorica **30**, 83–104 (2010)
18. Mašulović, D.: Finite big Ramsey degrees in universal structures. J. Comb. Theory Ser. A **170**, 105137 (2020)

19. Mašulović, D.: Ramsey degrees: big v. small. Eur. J. Comb. **95**, 103323 (2021)
20. Milliken, K.R.: A partition theorem for the infinite subtrees of a tree. Trans. Am. Math. Soc. **263**(1), 137–148 (1981)
21. Nešetřil, J., Rödl, V.: Partitions of finite relational and set systems. J. Comb. Theory Ser. A **22**(3), 289–312 (1977)
22. Nešetřil, J., Rödl, V.: Ramsey classes of set systems. J. Comb. Theory Ser. A **34**(2), 183–201 (1983)
23. Nešetřil, J., Rödl, V.: Combinatorial partitions of finite posets and lattices-Ramsey lattices. Algebra Univers. **19**(1), 106–119 (1984)
24. Nešetřil, J.: Ramsey theory. In: Graham, R.L., Grötschel, M., Lovász, L. (eds.) Handbook of Combinatorics, vol. 2, pp. 1331–1403. MIT Press, Cambridge (1995)
25. Nguyen Van Thé, L.: Big Ramsey degrees and divisibility in classes of ultrametric spaces. Can. Math. Bull. **51**, 412–423 (2008)
26. Nguyen Van Thé, L.: Structural Ramsey theory with the Kechris-Pestov-Todorcevic correspondence in mind. Habilitation thesis, Université d'Aix-Marseille, pp. 44 (2013)
27. Paoli, M., Trotter, W.T., Walker, J.W.: Graphs and orders in Ramsey theory and in dimension theory. In: Rival, I. (ed.) Graphs and Order. NATO AST Series, vol. 147, pp. 351–394. Springer, Dordrecht (1985)
28. Sauer, N.W.: Coloring subgraphs of the Rado graph. Combinatorica **26**(2), 231–253 (2006)
29. Todorčević, S.: Introduction to Ramsey Spaces. Annals of Mathematics Studies. Princeton University Press, Princeton (2010)
30. Zheng, Y.Y.: A collection of topological Ramsey spaces of trees and their application to profinite graph theory. Arch. Math. Log. **57**, 939–952 (2018)
31. Zucker, A.: Big Ramsey degrees and topological dynamics. Gr. Geom. Dyn. **13**(1), 235–276 (2019)
32. Zucker, A.: On big Ramsey degrees for binary free amalgamation classes. Adv. Math. **408**, 108585 (2022)