Efficient Sum of Squares-Based Verification and Construction of Control Barrier Functions by Sampling on Algebraic Varieties

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Abstract— **Safety is a critical property of control systems in vital applications such as manufacturing, energy, and autonomous vehicles. Control barrier functions have been proposed for safe control, however, verifying the safety guarantees of a given CBF and constructing CBFs to satisfy safety constraints are computationally challenging. In this paper, we propose a new approach to addressing these challenges, in which the global safety properties of CBFs are characterized based on a finite set of sample points. Specifically, we propose new algorithms for verifying CBFs for polynomial systems by solving a system of linear equalities and sum-of-squares constraints at a set of points sampled on an algebraic variety induced by the CBF. We extend this approach to high-order CBFs as well as systems with actuation constraints. Turning to the problem of constructing CBFs, we propose an algorithm that first selects a finite set of samples, and then computes a CBF such that the samples lie on the boundary of the safe region by solving a mixed-integer convex program. We prove that, if the number of samples is sufficiently large and a CBF exists, then our approach returns a function that satisfies necessary conditions of a CBF. We evaluate our approach on a linear cruise control scenario and a nonlinear quadrotor UAV, and find that both the verification and synthesis algorithms significantly outperform another state-of-the-art SOS-based algorithm.**

I. INTRODUCTION

Autonomous control systems in domains such as health care, autonomous vehicles, and manufacturing are expected to satisfy safety properties, which are often mapped to ensuring that the system state remains within a desired safe region for all time. Control Barrier Functions (CBFs) have recently emerged as promising approaches to safe control, due to their compatibility with a wide variety of control laws [1]. CBFs have been successfully deployed in areas ranging from robotic manipulation [2], vehicle cruise control [3] to space exploration [4], and have led to theoretical exploration of safe control under high-order dynamics [5], [6], actuation constraints [7], [8], [9], [10], and other properties.

Two important and interrelated research challenges in CBF-based control are constructing appropriate CBFs and verifying safety of given CBF-based controllers. A CBF defines a desired operating region for the system as the super level-set of the CBF. As the state approaches the boundary of the level set, a constraint is applied to the control to ensure that the system remains within the set. Certifying the CBF means guaranteeing that such a control input always exists for any state on the boundary of the level-set, and the super-level set is entirely within the safe set.

The problem of verifying that a given CBF ensures safety of a nonlinear system is NP-hard in general [11]. Two distinct approaches have been developed for CBF verification and synthesis. First, sampling-based methods check whether the CBF conditions can be satisfied at a discrete set of boundary points, and refine the definition of the CBF if safety cannot be guaranteed [8], [9], [12]. These methods are in general efficient but not complete. Second, methodologies have been proposed to map CBF feasibility conditions to sum-of-squares constraints, which can then be checked using semidefinite programming [1]. While these methods are complete, they become intractable for high-dimensional systems and may encounter numerical instability issues when solving the SOS program. CBF construction techniques, meanwhile, rely on local search to solve non-convex optimization problems, and are not guaranteed to find a CBF [6], [7], [13].

In this paper, we propose a novel approach to offline verify and synthesize CBFs with improved scalability. The key insight of our approach is that, for systems with polynomial dynamics, verifying a CBF is equivalent to ensuring that a given polynomial is nonnegative on an algebraic variety. This equivalence enables us to leverage recent results that have shown that it suffices to verify that the polynomial is equivalent to a sum-of-squares polynomial at a finite set of well-chosen sample points on the variety [14]. This reduces the complexity of the verification procedure from solving an optimization problem with multiple SOS constraints and variables, to solving an optimization problem with two SOS constraints and a set of linear equality constraints. We find that our approach generalizes to include high-order CBFs as well as safety verification under input constraints. We note that previous researchers [15] have applied the theorem in [14] to certify the Lyapunov condition for a *given* function by sampling the algebraic variety, but haven't demonstrate how to synthesize the function with these samples. In our work, apart from certifying the CBF verification, we further synthesize CBFs through another optimization program.

Turning to the problem of CBF synthesis, we propose a novel procedure in which we *first* choose a set of sample points, and then select a CBF that (i) is equal to zero on a sufficiently large subset of the sample points, and (ii) satisfies the aforementioned nonnegativity constraints. We prove that, if the set of sample points is a sufficiently dense cover of the safe region, then our approach returns a candidate function that satisfies necessary conditions of a CBF. We evaluate our

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verification and synthesis algorithms on two case studies, namely, a linear cruise control scenario and a 2D quadrotor model. For both case studies, we found that our algorithms reduce the dimensionality of the semidefinite programs that must be solved from over 100 using an existing algorithm to 7 using our proposed approach, leading to a significant reduction in computation time.

The rest of the paper is organized as follows. Section II reviews the related work. Section III presents background on algebraic varieties. Section IV presents the proposed verification procedure. Section V gives our proposed algorithm for constructing CBFs. Section VI containts simulation results. Section VII concludes the paper.

II. Related Work

Motivated by the growing prevalence of autonomous control systems in safety-critical applications, there has been heightened research interest in verifiably safe control [6]. Control barrier functions were first proposed in [3], and have since been generalized to multi-agent systems [16], time-delayed systems [17], and uncertain systems [18]. Applications of CBFs include [19], [20].

CBF-based control relies on selecting a control input that satisfies a linear constraint at each time instant, and hence could fail to guarantee safety if the constraint cannot be satisfied. High-order CBFs were proposed for safe control of high-degree systems in [5]. Techniques for constructing feasible CBFs under actuation constraints were presented in [11], [21], [7], [22]. The problem of verifying a given CBF has also been considered using such techniques in[6], [7], [23]. While our approach also considers the problem of verifying and constructing CBFs, we incorporate results on verifying nonnegativity of polynomials on algebraic varieties to reduce the complexity. A recent related work used sampling on varieties to verify Lyapunov function [15], however, this approach did not consider construction of such functions as in the present paper. The proposed method is based on sum-of-squares optimization showing great promise in Lyapunov function synthesis [24], invariant sets synthesis [25] and safety constrained controller synthesis [26].

Finally, algorithms have been proposed to construct CBFs (or Lyapunov functions) that are parametrized by neural networks [9], [27]. These techniques are orthogonal to our present effort, which considers polynomial CBFs. In particular, we observe that computationally efficient verification of neural network CBFs remains an open research problem.

III. Background on Algebraic Varieties

As we will see shortly in the next section, to verify that a function is a valid CBF, we will check conditions of the form

$$f(x) \geq 0 \; \forall x \in \{x : g_i(x) = 0\}, \tag{1}$$

where $f(x), g_i(x)$ are polynomials of x, namely a polynomial $(f(x))$ is always non-negative when some other polynomials $(g_i(x))$ vanish. In this section we introduce the mathematical tools to verify this condition (1).

A straightforward necessary condition for (1) is that on each $x^{(j)}$ sampled from the set $\{x : g_i(x) = 0\}$, $f(x^{(j)}) \geq 0$. Many of the previous works on CBF [8], [9] only check the polynomial non-negativity condition on finitely many samples. This naive sampling based approach, however, only verifies the CBF condition on the sampled states; it provides no guarantee on the infinitely many states that are not sampled. In this work, we seek a stronger approach, relying on sum-of-squares optimization, to certify the CBF condition for the infinitely many states.

A polynomial $p(x)$ is a sum-of-squares if there exist polynomials $q_1, \ldots, q_R$ such that

$$p(x) = \sum_{i=1}^{R} q_i(x)^2.$$

An ideal $\mathcal{I}$ of polynomials is a collection of polynomials such that, for any $f, g \in \mathcal{I}$ and any polynomial h, $(f + g) \in \mathcal{I}$ and $fh, gh \in \mathcal{I}$.

The algebraic variety of a set of polynomials $G = \{g_1, \ldots, g_L\}$ is defined by

$$\mathcal{V}(G) = \{x : g_i(x) = 0 \; \forall i = 1, \ldots, L\}.$$

We also let $\mathcal{V}[g_1, \ldots, g_L]$ denote the algebraic variety of $\{g_1, \ldots, g_L\}$. An algebraic variety $\mathcal{V}$ is *irreducible* if it cannot be written as $\mathcal{V}_1 \cup \mathcal{V}_2$ where $\mathcal{V}_1$ and $\mathcal{V}_2$ are algebraic varieties with $\mathcal{V}_i \subsetneq \mathcal{V}$ for $i = 1, 2$. Any algebraic variety $\mathcal{V}$ can be written as the union of a finite collection of irreducible varieties, which is denoted as the *irreducible decomposition* of $\mathcal{V}$. We denote $I(\mathcal{V})$ as the ideal $\{l(x) : l(x) = 0 \; \forall x \in \mathcal{V}\}$. The *coordinate ring* $\mathbb{R}[\mathcal{V}]$ is defined by $\mathbb{R}[\mathcal{V}] \triangleq \mathbb{R}[x] \setminus I(\mathcal{V})$, where $\mathbb{R}[x]$ denotes the ring of polynomials in x.

In what follows, we describe sufficient conditions for checking whether a polynomial f is nonnegative on an algebraic variety $\mathcal{V}(G)$. Since any SOS polynomial is nonnegative, a sufficient condition is the existence of an SOS polynomial F of degree at most $2d$ for some integer d such that $f(x) = F(x)$ for all $x \in \mathcal{V}(G)$. In [14], a procedure was described for verifying this sufficient condition that only requires checking a finite number of samples $\{x_1, \ldots, x_N\} \subseteq \mathcal{V}(G)$. In what follows, we briefly describe this approach.

Definition 1 ([14]): Let $\mathcal{V}$ be a variety, and let $\mathcal{R} = \mathbb{R}[\mathcal{V}]$. Let $\mathcal{L} \subseteq \mathcal{R}$ be a linear subspace, and let $Z \subseteq \mathcal{V}$ be a finite set. The tuple $(\mathcal{L}, Z)$ is poised if $q \in \mathcal{L}$ and $q(x) = 0$ for all $x \in Z$ imply that q is the zero polynomial.

We next present a connection between the poisedness property and verification of polynomial inequalities.

Theorem 1 ([14]): Let $\mathcal{V}$ be a variety and $\mathcal{R} = \mathbb{R}[\mathcal{V}]$. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$, and let $\mathcal{L}_{2d} \subseteq \mathcal{R}$ be a linear subspace. If F is an SOS polynomial with $f, F \in \mathcal{L}_{2d}$, $f(x) = F(x)$ for all $x \in Z$, and $(\mathcal{L}_{2d}, Z)$ is poised, then $f(x) \geq 0$ for all $x \in \mathcal{V}$.

Theorem 1 implies that, if we can construct a poised set of samples on $\mathcal{V}$, then nonnegativity of f on $\mathcal{V}$ can be verified by solving a convex program with one SOS variable and $|Z|$ linear equality constraints. Specifically, to prove that a polynomial $f(x)$ is nonnegative on a variety $\mathcal{V}$, it suffices to prove that there exists a positive semidefinite matrix S such that $f(x_i) = p(x_i)^T S p(x_i)$ where $p(x)$ is a vector of monomials and $p(x_i)$ is a vector of the monomials evaluated at x_i and the x_i's are sampled from $\mathcal{V}$.

A natural question is how to compute such a poised set. The next background results give sufficient conditions for poisedness and an algorithm for computing poised samples.

Lemma 1 ([14]): Let $\mathcal{L}$ be a linear subspace of the coordinate ring of the variety $\mathcal{V}$, and let $f_1(x), \ldots, f_R(x)$ span $\mathcal{L}$. Let $z_1, \ldots, z_N$ denote a set of samples, and suppose that the $R \times N$ matrix with i-th column $(f_1(z_i) \ \cdots \ f_R(z_i))^T$ is rank-deficient. Then $(\mathcal{L}, \{z_1, \ldots, z_N\})$ is poised for generic $z_1, \ldots, z_N$.

We denote the rank of the matrix defined in Lemma 1 as the *empirical dimension* of $z_1, \ldots, z_N$. Based on the lemma, a procedure for constructing a poised set of samples is as follows. First, we choose a spanning set for the linear subspace $\mathcal{L}$; as an example, the set of all monomials with a given degree bound could be used. Next, we select a sequence of samples in a probabilistic manner, terminating when the matrix defined in Lemma 1 is rank-deficient.

We refer to [14] for detailed proofs of the above.

IV. Sampling-Based Safety Verification

This section presents our sampling-based approach to safety verification. We formulate the problem and present verification algorithms for control barrier functions and high-order control barrier functions under input constraints.

A. System Model

We consider a control-affine dynamical system described by the nonlinear state-space model

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t), \tag{2}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathcal{U} \subseteq \mathbb{R}^m$ is the control input, $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a polynomial and $g : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$ is a matrix polynomial function. We assume that the system must satisfy a safety constraint, which is expressed as positive invariance of a set $\mathcal{C} \triangleq \{x : h(x) \geq 0\}$ for some polynomial $h : \mathbb{R}^n \rightarrow \mathbb{R}$. We remark later on extending our approach to cases where $\mathcal{C}$ is an intersection of sets of the form $\{x : h_i(x) \geq 0, i = 1, \ldots, r\}$.

B. Problem Formulation

We define a safe set as follows. We say that a set $\mathcal{D}$ is *safe* if (i) $\mathcal{D}$ is controlled positive invariant, i.e., there exists a control policy $\mu : \mathbb{R}^n \rightarrow \mathcal{U}$ such that $\mathcal{D}$ is positive invariant under dynamics $\dot{x} = f(x) + g(x)\mu(x)$ and (ii) $\mathcal{D} \subseteq \mathcal{C}$. We are particularly concerned with sets of the form $\mathcal{D} = \{x : b(x) \geq 0\}$ for some function $b : \mathbb{R}^n \rightarrow \mathbb{R}$.

If $\mathcal{D}$ as defined in this manner is safe for some function b, then the function b can be used to define a control barrier function (CBF)-based control policy. Such a policy selects a $u(t)$ satisfying

$$\frac{\partial b}{\partial x}(f(x(t)) + g(x(t))u(t)) \geq 0$$

whenever $b(x(t)) = 0$.

More generally, we can consider high-order CBFs (HOCBFs), which are defined as follows. We let $\psi_0(x) = b(x)$ for some function b, and define $\psi_i(x) = \dot{\psi}_{i-1}(x) + \alpha_i(\psi_{i-1}(x))$ for $i = 1, \ldots, r$, where α_i is a class-$\mathcal{K}$ function and $\frac{\partial \psi_i}{\partial x} g(x) = 0$ for $i < r$. An HOCBF-based policy selects u satisfying

$$L_g L_f^{r-1} b(x) u + \underbrace{L_f^r b(x) + O(b(x)) + \alpha_r(\psi_{r-1}(x))}_{\triangleq \Theta_r(x)} \geq 0$$

where $O(b(x))$ are the lower-order Lie derivatives of b. The problem studied in this section is stated as follows.

Problem 1: Given functions $\psi_0, \ldots, \psi_r : \mathbb{R}^n \rightarrow \mathbb{R}$ defined as above and a safe region $\mathcal{C} = \{x : h(x) \geq 0\}$, determine whether $\mathcal{D} = \{x : \psi_i(x) \geq 0, i = 0, \ldots, r\}$ is safe. As a special case, determine whether a set $\mathcal{D} = \{x : b(x) \geq 0\}$ is safe.

We assume that there exists an efficient algorithm to sample points from any algebraic variety $\mathcal{V}$. Furthermore, we assume that there is an efficient algorithm to compute the irreducible decomposition of a variety [28].

C. Verification of CBFs

We first consider the case where $\mathcal{U} = \mathbb{R}^m$ and the set $\mathcal{D} = \{x : b(x) \geq 0\}$. We have the following preliminary result.

Lemma 2: Let $\mathcal{U} = \mathbb{R}^m$. The set $\mathcal{D} = \{x : b(x) \geq 0\}$ is safe if and only if (a) for every x satisfying $b(x) = 0$, $\frac{\partial b}{\partial x} g(x) = 0$, we have $\frac{\partial b}{\partial x} f(x) \geq 0$, (b) there does not exist x with $h(x) = 0$ and $b(x) > 0$, (c) there exists $\overline{x}$ with $b(\overline{x}) > 0$ and $h(\overline{x}) > 0$.

Lemma 2 is a derived from [6] in a straight-forward manner. We omit the proof due to space limit. Lemma 2 presents sufficient and necessary conditions for CBF verification. The following result gives our sampling-based approach to CBF verification.

Proposition 1: Let $\mathcal{U} = \mathbb{R}^m$. Let $\mathcal{W}_1, \ldots, \mathcal{W}_s$ and $\mathcal{T}_1, \ldots, \mathcal{T}_v$ be the irreducible decompositions of the algebraic varieties $\mathcal{V}(b, \frac{\partial b}{\partial x} g_1(x), \ldots, \frac{\partial b}{\partial x} g_m(x))$ and $\mathcal{V}(h)$, respectively. Let $(L_1, Z_1), \ldots, (L_s, Z_s)$ and $(L_1', Z_1'), \ldots, (L_v', Z_v')$ be poised samples of $\mathcal{W}_1, \ldots, \mathcal{W}_s$ and $\mathcal{T}_1, \ldots, \mathcal{T}_v$. Suppose that there exist SOS polynomials $F_1(x)$ and $F_2(x)$ such that

$$\frac{\partial b}{\partial x} f(z) = F_1(z) \quad \forall z \in Z_i, i = 1, \ldots, s \tag{3}$$

$$-b(z) = F_2(z) \quad \forall z \in Z_i', i = 1, \ldots, v \tag{4}$$

Then $\mathcal{D}$ is safe.

Proof: Suppose that (3) holds. By Theorem 1, we have that $\frac{\partial b}{\partial x} f(x) \geq 0$ for all $x \in \mathcal{V}(b, \frac{\partial b}{\partial x} g_1(x), \ldots, \frac{\partial b}{\partial x} g_m(x))$, and hence condition (a) of Lemma 2 is satisfied. If (4) holds, then by Theorem 1, we have that $b(x) \leq 0$ for all x with $h(x) = 0$, implying that condition (b) of Lemma 2 is satisfied. We therefore conclude that $\mathcal{D}$ is safe. ■

Eq. (3) and (4) can be solved using SOS programming. We observe that (3)–(4) has $\sum_{i=1}^{s} |Z_i| + \sum_{j=1}^{v} |Z_j'|$ linear equality constraints and 2 sum-of-squares variables, in comparison with existing SOS-based CBF verification algorithms [6] that require two sum-of-squares variables, two sum-of-squares constraints, and $m+2$ polynomial variables. Moreover, the SOS polynomials in our sampling-based approach have lower degree, since the polynomial degrees in [6] are determined by the degrees of $\frac{\partial b}{\partial x}f(x)$ and $\frac{\partial b}{\partial x}g_1(x), \ldots, \frac{\partial b}{\partial x}g_m(x)$. We further observe that, if $\mathcal{C}$ is defined as $\mathcal{C} = \{x : h_i(x) \geq 0, i = 1, \ldots, L\}$ for some functions $h_1, \ldots, h_L$, then we can use the approach of Proposition 1 with one constraint of the form (4) for each h_i.

Note that the conditions (3) and (4) are stronger than simply ensuring that $\frac{\partial b}{\partial x}(z) \geq 0$ and $-b(x) \geq 0$ for all x in Z and Z', respectively. Indeed, we not only require that these quantities are nonnegative, but also require that they are equal to SOS polynomials $F_1(z)$ and $F_2(z)$, respectively, allowing us to use SOS optimizer.

We next turn to the problem of verifying an HOCBF.

Lemma 3: Let $\mathcal{U} = \mathbb{R}^m$. Let $\psi_0(x), \ldots, \psi_r(x), \Theta(x)$ be defined as in Section IV-B. Suppose that there exist SOS polynomials $\beta_0(x), \ldots, \beta_r(x), \gamma_0(x), \ldots, \gamma_r(x)$ such that

$$\Theta_r(x) - \sum_{i=0}^{r} \beta_i(x)\psi_i(x) \geq 0 \quad \forall x \in \mathcal{V}[\psi_r, \frac{\partial \psi_r}{\partial x}g_1(x), \ldots, \frac{\partial \psi_r}{\partial x}g_m(x)] \tag{5}$$

$$-\sum_{i=0}^{r} \gamma_i(x)\psi_i(x) \geq 0 \quad \forall x \in \mathcal{V}[h] \tag{6}$$

Then $\psi_0, \ldots, \psi_r$ defines an HOCBF.

Proof: By [6], $\psi_0, \ldots, \psi_r$ define an HOCBF if (a) there does not exist x with $\psi_i(x) \geq 0$, $i = 0, \ldots, r$, $\frac{\partial \psi_r}{\partial x}g_i(x) = 0$ for $i = 1, \ldots, m$, and $\Theta_r(x) < 0$, and (b) there does not exist x with $h(x) = 0$ and $\psi_i(x) \geq 0$ for $i = 0, \ldots, r$. We have that (5) is sufficient for condition (a), while (6) is sufficient for condition (b). ■

Based on Lemma 3, we have the following sampling-based approach.

Proposition 2: Let $\mathcal{W}_1, \ldots, \mathcal{W}_s$ and $\mathcal{T}_1, \ldots, \mathcal{T}_v$ be the irreducible decompositions of the algebraic varieties $\mathcal{V}[\psi_r, \frac{\partial \psi_r}{\partial x}g_1(x), \ldots, \frac{\partial \psi_r}{\partial x}g_m(x)]$ and $\mathcal{V}[h]$, respectively. Let $(L_1, Z_1), \ldots, (L_s, Z_s)$ and $(L_1', Z_1'), \ldots, (L_v', Z_v')$ be poised samples of the varieties $\mathcal{W}_1, \ldots, \mathcal{W}_s$ and $\mathcal{T}_1, \ldots, \mathcal{T}_v$. Suppose that there exist SOS polynomials $F_1, F_2, \beta_0, \ldots, \beta_r, \gamma_0, \ldots, \gamma_r$ such that

$$\Theta_r(x) - \sum_{i=0}^{r} \beta_i(x)\psi_i(x) = F_1(x) \quad \forall x \in Z_i, i = 1, \ldots, s \tag{7}$$

$$-\sum_{i=0}^{r} \gamma_i(x)\psi_i(x) = F_2(x) \quad \forall x \in Z_i', i = 1, \ldots, v \tag{8}$$

Then $\mathcal{D} = \{x : \psi_i(x) \geq 0, i = 0, \ldots, r\}$ is safe.

Proof: The result follows from Theorem 1 applied to the conditions (5)–(6). ■

We finally consider the impact of constraints on actuation. Suppose that $\mathcal{U} = \{u : A_u u \leq b_u\}$ for some $A_u \in \mathbb{R}^{p \times m}$ and $b_u \in \mathbb{R}^p$. We then have the following result on existence of HOCBFs for the actuation constrained system.

Lemma 4: Let $\psi_0(x), \ldots, \psi_r(x), \Theta_r(x)$ be defined as in Section IV-B. Let $\mathcal{U} = \{u : A_u u \leq b_u\}$. Suppose that there exist SOS polynomials $\beta_0(x,z), \ldots, \beta_{r-1}(x,z)$, $\gamma_0(x), \ldots, \gamma_r(x)$, where $z \in \mathbb{R}^{p+1}$, such that

$$\begin{pmatrix} \Theta_r(x) \\ b_u \end{pmatrix} z^2 - \sum_{i=0}^{r-1} \beta_i(x,z)\psi_i(x) \geq 0 \quad \forall (x,z) \in \mathcal{V}\left[\psi_r, \begin{pmatrix} -\frac{\partial \psi_r}{\partial x}g(x) \\ A_u \end{pmatrix}^T z^2\right] \tag{9}$$

$$-\sum_{i=0}^{r} \gamma_i(x)\psi_i(x) \geq 0 \quad \forall x \in \mathcal{V}[h] \tag{10}$$

where z^2 is a vector whose i-th element is equal to z_i^2. Then $\mathcal{D} = \{x : \psi_i(x) \geq 0, i = 0, \ldots, r\}$ is safe. In particular, if $r = 0$ and $\psi_0 = b$, and if

$$\begin{pmatrix} \frac{\partial b}{\partial x}f(x) \\ b_u \end{pmatrix} z^2 \geq 0 \quad \forall (x,z) \in \mathcal{V}\left[b, \begin{pmatrix} -\frac{\partial b}{\partial x}g(x) \\ A_u \end{pmatrix}^T z^2\right] \tag{11}$$

$$-b(x) \geq 0 \quad \forall x \in \mathcal{V}[h] \tag{12}$$

then $\mathcal{D} = \{x : b(x) \geq 0\}$ is safe.

Proof: The set $\mathcal{D} = \{x : \psi_i(x) \geq 0, i = 0, \ldots, r\}$ is positive invariant iff for any x with $\psi_r(x) = 0$ and $\psi_i(x) \geq 0$, $i = 0, \ldots, r-1$, there exists u satisfying

$$\begin{pmatrix} -\frac{\partial \psi_r}{\partial x}g(x) \\ A_u \end{pmatrix} u \leq \begin{pmatrix} \Theta_r(x) \\ b_u \end{pmatrix} \tag{13}$$

By Farkas's Lemma [29], Eq. (13) holds for some u iff there is no $y \in \mathbb{R}^{p+1}_{\geq 0}$ satisfying

$$\begin{pmatrix} \Theta_r(x) \\ b_u \end{pmatrix}^T y < 0, \quad \begin{pmatrix} -\frac{\partial \psi_r}{\partial x}g(x) \\ A_u \end{pmatrix}^T y = 0$$

Equivalently, the set $\mathcal{D}$ is positive invariant if and only if there do not exist $x \in \mathbb{R}^n$, $z \in \mathbb{R}^{p+1}$ such that $\psi_i(x) \geq 0$ for $i = 0, \ldots, r-1$, $\psi_r(x) = 0$,

$$\begin{pmatrix} \Theta_r(x) \\ b_u \end{pmatrix}^T z^2 < 0, \quad \begin{pmatrix} -\frac{\partial \psi_r}{\partial x}g(x) \\ A_u \end{pmatrix}^T z^2 = 0$$

where z^2 denotes the vector $(z_1^2 \cdots z_{p+1}^2)^T \in \mathbb{R}^{p+1}$. Hence (9) is a sufficient condition. The condition (10) implies that $\mathcal{D} \subseteq \mathcal{C}$.

In the case where $r = 0$ and we attempt to verify a CBF, the conditions (9) and (10) reduce to (11) and (12), respectively. ■

The following gives a sampling-based approach for verifying HOCBFs with input constraints.

Proposition 3: Let $\mathcal{U} = \{u : A_u u \leq b_u\}$. Let

$$\mathcal{W}_1 \cup \cdots \cup \mathcal{W}_s = \mathcal{V}\left[\psi_r, \begin{pmatrix} -\frac{\partial \psi_r}{\partial x}g(x) \\ A_u \end{pmatrix}^T z^2\right]$$

$$\mathcal{T}_1 \cup \cdots \cup \mathcal{T}_v = \mathcal{V}[h]$$

denote the irreducible decompositions. Let $(L_1, Z_1), \ldots, (L_s, Z_s)$ and $(L_1', Z_1'), \ldots, (L_v', Z_v')$ denote poised sets of samples for $\mathcal{W}_1, \ldots, \mathcal{W}_s$ and $\mathcal{T}_1, \ldots, \mathcal{T}_v$, respectively. Suppose that there exist SOS polynomials $\beta_0, \ldots, \beta_r$, $\gamma_0, \ldots, \gamma_r$, F_1, and F_2 such that

$$\begin{pmatrix} \Theta_r(x) \\ b_u \end{pmatrix} z^2 - \sum_{i=0}^{r-1} \beta_i(x,z)\psi_i(x) = F_1(x,z) \quad (14)$$

$$-\sum_{i=0}^{r} \gamma_i(x)\psi_i(x) = F_2(x) \quad (15)$$

where (14) holds for all $(x,z) \in Z_i, i = 1, \ldots, s$ and (15) holds for all $x \in Z_i', i = 1, \ldots, v$. Then $\mathcal{D} = \{x : \psi_i(x) \geq 0\}$ is safe.

Proof: Follows from Lemma 4 and Theorem 1. ∎

V. Sampling-Based Construction of CBFs

This section presents our sampling-based procedure for constructing CBFs. We first present our algorithm, and then identify conditions under which it results in a valid CBF.

A. Algorithm

The main idea of our algorithm is as follows. For a given candidate CBF, we can prove that it guarantees safety using the sampling-based methods of Section IV. When the CBF is not given, we instead select the samples first, and then attempt to find a CBF that satisfies the conditions of Section IV for a subset of the samples of sufficient size. This candidate CBF can then be verified using the procedure of Section IV.

Our approach is as follows. We first select a set of samples $\{y_1, \ldots, y_M\} \subseteq \mathcal{V}[h]$ such that, for each component $\mathcal{Y}_l$ of the irreducible decomposition of $\mathcal{V}[h]$, $\{y_1, \ldots, y_M\} \cap \mathcal{Y}_l$ is poised. We then select a set of samples $\{x_1, \ldots, x_N\} \subseteq \mathcal{C}$ in a probabilistic fashion (e.g., i.i.d. uniformly distributed on $\mathcal{C}$). We attempt to construct a CBF b by solving the mixed-integer convex program

$$\begin{array}{ll} \text{maximize} & \sum_{i=1}^N \alpha_i \\ \text{s.t.} & \alpha_i \in \{0,1\}, \ \forall i = 1, \ldots, N \\ & Q_1, Q_2 \succeq 0 \\ & -b(y_i) = p(y_i)^T Q_1 p(y_i) \ \forall i = 1, \ldots, M \\ & b(x_i) + Y(1-\alpha_i) \geq 0 \ \forall i = 1, \ldots, N \\ & b(x_i) - Y(1-\alpha_i) \leq 0 \ \forall i = 1, \ldots, N \\ & \frac{\partial b}{\partial x} g_j(x_i) + Y(1-\alpha_i) \geq 0 \ \forall i = 1, \ldots, N, \\ & \quad j = 1, \ldots, m \\ & \frac{\partial b}{\partial x} g_j(x_i) - Y(1-\alpha_i) \leq 0 \ \forall i = 1, \ldots, N, \\ & \quad j = 1, \ldots, m \\ & \frac{\partial b}{\partial x} f(x_i) - p(x_i)^T Q_2 p(x_i) + Y(1-\alpha_i) \geq 0 \\ & \quad \forall i = 1, \ldots, N \\ & \frac{\partial b}{\partial x} f(x_i) - p(x_i)^T Q_2 p(x_i) - Y(1-\alpha_i) \leq 0 \\ & \quad \forall i = 1, \ldots, N \end{array} \quad (16)$$

where Y is a sufficiently large number. The optimization variables of (16) are the coefficients of b, and the integer variables $\alpha_1, \ldots, \alpha_N$, and the positive semidefinite matrices Q_1 and Q_2. The first constraint ensures that the chosen polynomial b satisfies

$$-b(y_i) = p(y_i)^T Q_1 p(y_i)$$

for $i = 1, \ldots, N$, corresponding to Eq. (4).

The remaining constraints use the big-Y method to ensure that, for all i with $\alpha_i = 1$, the polynomial b and matrix Q_2 satisfy

$$b(x_i) = 0, \quad \frac{\partial b}{\partial x} g(x_i) = 0, \quad \frac{\partial b}{\partial x} f(x_i) = p(x_i)^T Q_2 p(x_i).$$

The objective function corresponds to selecting a candidate barrier function b that satisfies these constraints for as many sample points x_i as possible. The solution of (16) is the candidate barrier function.

After b has been computed, the approach of Section IV is used to verify that b is a valid barrier function. Note that we cannot conclude that b ensures safety based on the output of (16) alone, since there is no guarantee that the set of samples $X = \{x_i : \alpha_i = 1\}$ is poised with respect to the variety $\mathcal{V}[b, \frac{\partial b}{\partial x} g_1(x), \ldots, \frac{\partial b}{\partial x} g_m(x)]$. If the candidate barrier function fails the verification, then N is increased and (16) is recomputed.

B. Analysis of Sampling-Based CBF Algorithm

We next analyze our proposed algorithm. We first define a density property.

Definition 2: A set of points $\mathcal{X}$ is a *dense* (N,ϵ)*-cover* of a set $\mathcal{C}$ if, for any set of points $\{\hat{x}_1, \ldots, \hat{x}_N\} \subseteq \mathcal{C}$, there exists a one-to-one function $\pi : \{\hat{x}_1, \ldots, \hat{x}_N\} \rightarrow \mathcal{X}$ such that $||\hat{x}_i - \pi(\hat{x}_i)|| < \epsilon$ for all $i = 1, \ldots, N$.

Now, let P denote the number of coefficients of b, and let β denote the vector of coefficients of b. For given positive definite matrices Q_1 and Q_2, define

$$\begin{aligned} \gamma_i(\beta, x_1, \ldots, x_N) &= b(y_i) + p(y_i)^T Q_1 p(y_i), \\ &\quad i = 1, \ldots, M \\ \phi_{i1}(\beta, x_1, \ldots, x_N) &= b(x_i), \ i = 1, \ldots, N \\ \phi_{i2}(\beta, x_1, \ldots, x_N) &= \frac{\partial b}{\partial x} f(x_i) + p(x_i)^T Q_2 p(x_i), \\ &\quad i = 1, \ldots, N \\ \phi_{ij}(\beta, x_1, \ldots, x_N) &= \frac{\partial b}{\partial x} g_{j-2}(x), \\ &\quad i = 1, \ldots, N, j = 3, \ldots, (m+2) \end{aligned}$$

The following proposition describes the solution to (16) when a valid CBF exists and the set of samples is sufficiently dense with standard assumptions of invertibility in [30].

Proposition 4: Suppose that there is a polynomial $\hat{b}(x)$ that satisfies the criteria of a CBF with coefficient vector $\hat{\beta}$ and that there is a set of N poised samples $\hat{x}_1, \ldots, \hat{x}_N \in \mathcal{V}[\hat{b}, \frac{\partial \hat{b}}{\partial x} g_1(x), \ldots, \frac{\partial \hat{b}}{\partial x} g_m(x)]$. Suppose further that $N(m+2) + M \leq P$, and there exists an integer $q \leq P$ and a subset of coefficients $I = \{k_1, \ldots, k_q\} \subseteq \{1, \ldots, P\}$ such that

$$\begin{pmatrix} \frac{\partial \gamma_1}{\partial \beta_{k_1}} & \cdots & \frac{\partial \gamma_1}{\partial \beta_{k_q}} \\ \vdots & \vdots & \vdots \\ \frac{\partial \gamma_M}{\partial \beta_{k_1}} & \cdots & \frac{\partial \gamma_M}{\partial \beta_{k_q}} \\ \frac{\partial \phi_{11}}{\partial \beta_{k_1}} & \cdots & \frac{\partial \phi_{11}}{\partial \beta_{k_q}} \\ \vdots & \vdots & \vdots \\ \frac{\partial \phi_{N,m+2}}{\partial \beta_{k_1}} & \cdots & \frac{\partial \phi_{N,m+2}}{\partial \beta_{k_q}} \end{pmatrix}$$

is invertible at $(\hat{\beta}, \hat{x}_1, \ldots, \hat{x}_N)$. Then there is an $\epsilon > 0$ such that, if $\mathcal{X}$ is a dense (N, ϵ)-cover of $\{\hat{x}_1, \ldots, \hat{x}_N\}$ then there is a solution $b(x)$ to (16) with $|\{x_i \in \mathcal{X} : \alpha_i = 1\}| \geq N$. Moreover, if the variety $\mathcal{V}[b, \frac{\partial b}{\partial x} g_1(x), \ldots, \frac{\partial b}{\partial x} g_m(x)]$ is irreducible and the dimension of the set of degree-d polynomials of its coordinate ring is at most N, then the set $\mathcal{D} = \{x : b(x) \geq 0\}$ is safe.

Proof: The implicit function theorem implies that there is an open neighborhood $\mathcal{D}$ of $(\hat{\beta}, \hat{x}_1, \ldots, \hat{x}_N)$ and a polynomial $\theta(x_1, \ldots, x_N)$ such that the γ_i and ϕ_{ij} polynomials are zero if and only if $\beta = \theta(x_1, \ldots, x_N)$. Hence, if ϵ is sufficiently small so that $\pi(\hat{x}_i)$ lies in $\mathcal{D}$, then choosing $\beta = \theta(\pi(\hat{x}_1), \ldots, \pi(\hat{x}_N))$ will result in $\gamma_i(\beta, \pi(\hat{x}_1), \ldots, \pi(\hat{x}_N)) = 0$ for all $i = 1, \ldots, q$, and hence

$$\{\pi(\hat{x}_i) : i = 1, \ldots, N\} \subseteq \{x_j \in \mathcal{X} : \alpha_j = 1\}.$$

Suppose that N is bounded below by the dimension of $\mathcal{V}[b, \frac{\partial b}{\partial x} g(x)]$. Then the set of samples $\{\pi(\hat{x}_i) : i = 1, \ldots, N\}$ is generically poised and we have $\frac{\partial b}{\partial x} f(x_i) = Q_2(x_i)$ for $i = 1, \ldots, N$, implying that the system is safe by Proposition 1. ■

The following handles the case where $N(m+2)+M > P$.

Proposition 5: Suppose that $N(m+2) + M > P$ and there is a polynomial $\hat{b}(x)$ that satisfies the criteria of a CBF with coefficient vector $\hat{\beta}$ and that there is a set of $N + N'$ poised samples $\hat{x}_1, \ldots, \hat{x}_N, \ldots, \hat{x}_{N+N'} \in \mathcal{V}[\hat{b}, \frac{\partial b}{\partial x} g_1(x), \ldots, \frac{\partial b}{\partial x} g_m(x)]$ where

$$N' = \lceil \frac{N(m+2) + M - P}{n - (m+2)} \rceil$$

and the conditions $\hat{b}(\hat{x}_i) = 0$, $\frac{\partial \hat{b}}{\partial x} g_j(\hat{x}_i) = 0$, and $\frac{\partial \hat{b}}{\partial x} f(\hat{x}_i) = m(\hat{x}_i)^T Q_2 m(\hat{x}_i)$ hold for all $i = 1, \ldots, N + N'$. Suppose further that there exists a subset of coefficients $J \subseteq \{1, \ldots, N'n\}$ such that the matrix defined as in Proposition 4 is invertible. Then there is an $\epsilon > 0$ such that, if $\mathcal{X}$ is an (N, ϵ)-dense cover of $\{\hat{x}_1, \ldots, \hat{x}_{N+N'}\}$, then there is a solution to (16) with $|\{x_i \in \mathcal{X} : \alpha_i = 1\}| \geq N + N'$. Furthermore, if $\mathcal{V}[b, \frac{\partial b}{\partial x} g_1(x), \ldots, \frac{\partial b}{\partial x} g_m(x)]$ is irreducible and N is greater than the dimension of the coordinate ring of the variety, the region $\{x : b(x) \geq 0\}$ is safe.

The proof is similar to the proof of Proposition 4 and is omitted due to space constraints. We observe that, while the above results prove that a solution to (16) exists that satisfies the constraints (3)–(4) at a sufficient number of sample points, the procedures of the previous section must still be used to verify the candidate barrier function.

VI. Simulation Study

In this section, we evaluate our proposed CBF construction and verification with two case studies, namely, an adaptive cruise control system and a 2D quadrotor. The case studies are simulated on MacOS with M1 Pro chip and 32GB memory via Matlab with Yalmip [31], Gurobi [32], and SDPT3 [33]. Our case studies are open-source [1].

[1]Source Code: `https://github.com/HongchaoZhang-HZ/Sampling_CBF_Synthesis_Verification`.

A. Adaptive Cruise Control System

In this case study we consider an adaptive cruise control system with a linear dynamics [34]

$$\dot{x} = \begin{pmatrix} \dot{p} \\ \dot{v} \\ \dot{a} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -5 \end{pmatrix} \begin{pmatrix} p \\ v \\ a \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix} u,$$

where p, v, and a are the position, velocity, and acceleration of the vehicle with respect to the target vehicle, respectively, and $u \in \mathbb{R}$ is the acceleration control input. We choose $\{x : h(x) = 1 - p^2 - v^2 - a^2 \geq 0\}$ as the safe region.

1) Sampling-Based Construction of CBFs: We aim to construct a CBF candidate with degree 2 for the ACC linear system. We first sample the points i.i.d uniformly distributed on the unit sphere for $\{y_1, \ldots, y_{10}\} \subseteq \mathcal{V}[h]$. We then sample the points i.i.d uniformly distributed inside the unit sphere for $\{x_1, \ldots, x_6\} \subseteq \mathcal{C}$. We finally synthesize the CBF by solving Eq. (16) with $Q_1 \succeq 0$ and $Q_2 \succeq 0$ being relaxed as Scaled-Diagonally-Dominant-Sum-of-Squares (SDSOS) [35]. The values of the parameters are $M = 10, N = 6$, and $Y = 10$. The result is $b(x) = b_0 - x^T B_1 x$, where $b_0 = 0.0049$ and

$$B_1 = \begin{pmatrix} 0.0089 & 0.0197 & 6.5631 \times 10^{-4} \\ 0.0197 & 0.2255 & 0.0125 \\ 6.5631 \times 10^{-4} & 0.0125 & 0.0091 \end{pmatrix}.$$

The projection of the synthesized barrier function on the pv-plane is shown in Fig. 1. We find that the boundary of the constructed CBF, i.e., $b(x) = 0$ is contained in the safe set. The average running time over 5 runs of the construction program is $56.76s$ when the feasible barrier function is found in 20 iterations.

2) Sampling-Based Safety Verification: To evaluate proposed CBF verification, we verify containment and invariance properties of a given CBF candidate with degree 2, i.e. our previously generated CBF candidate. We sample on the varieties $\mathcal{V}(b, \frac{\partial b}{\partial x} g_1(x))$ and $\mathcal{V}(h)$ with $s = 6$ and $v = 10$, respectively until the set being poised.

We then verify the synthesized CBF by solving SDP formed in Eq. (3)–(4). As a comparison, we take the verification approach of [6] as a baseline. The baseline verifies the CBF by solving SOS programs formed in Eq. (4)–(5) in [6]. As shown in Table I, the dimensions of the SDP variables for the proposed containment and invariance verifications are 4, which are much less than the dimensions of the SDP variables for the baseline. The runtime performances are shown in Table II. Each entity in Table II is the average over 5 runs. The proposed method can verify the containment and invariance in $2.6766s$, which is 47% of the time for the baseline containment and invariance verifications.

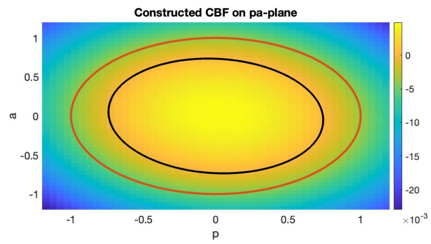

Fig. 1: Visualization of constructed CBF for the ACC system by projecting on $pa-$plane, with black and red lines denoting the boundaries of $b(x) = 0$ and $h(x) = 0$, respectively.

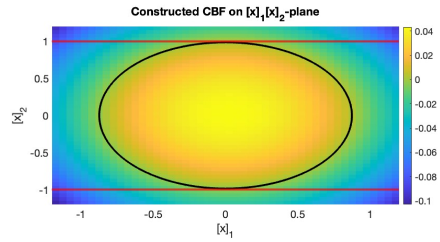

Fig. 2: Visualization of constructed CBF for the $2D$ Quadrotor system: the boundaries $b(x) = 0$ and $h(x) = 0$ are projected on $[x]_1[x]_2-$plane, denoted by black and red lines.

B. 2D Quadrotor System

In this case study, we consider a 2D Quadrotor system with nonlinear dynamics [36] given as

$$\begin{pmatrix}[\dot{x}]_1\\ [\dot{x}]_2\\ [\dot{x}]_3\\ [\dot{x}]_4\\ [\dot{x}]_5\\ [\dot{x}]_6\end{pmatrix} = \underbrace{\begin{pmatrix}[x]_4\\ [x]_5\\ [x]_6\\ 0\\ -g\\ 0\end{pmatrix}}_{f(x)} + \underbrace{\begin{pmatrix}0 & 0\\ 0 & 0\\ 0 & 0\\ -\frac{\sin [x]_3}{M_q} & -\frac{\sin [x]_3}{M_q}\\ \frac{\cos [x]_3}{M_q} & \frac{\cos [x]_3}{M_q}\\ \frac{L_r}{I_n} & -\frac{L_r}{I_n}\end{pmatrix}}_{g(x)} \begin{pmatrix}[u]_1\\ [u]_2\end{pmatrix},$$

where $[x]_1, [x]_2, [x]_3, [x]_4, [x]_5$, and $[x]_6$ are the offsets of horizontal position, vertical position, yaw angle, horizontal velocity, vertical velocity, and yaw velocity of the quadrotor corresponding to the equilibrium point $([x]_1^*, [x]_2^*, 0, 0, 0, 0)^T$, respectively, $u \in \mathbb{R}^2$ is the offset to the thrust control input corresponding to the equilibrium point $(M_q g/2, M_q g/2)^T$, $g = 9.81 m/s^2$ is the gravitational acceleration constant, $M_q = 0.486 kg$ is the mass of the quadrotor, $I_n = 0.00383 kg \cdot m^2$ is the moment of inertia, and $L_r = 0.25m$ is the length of rotor arm. We choose $\{x : h(x) = 1 - [x]_2^2 \geq 0\}$ as the safe region.

To make $g(x)$ polynomial, we incorporate the auxiliary variables [11] $z_1 = \sin [x]_3$ and $z_2 = \cos [x]_3$ with $z_1^2 + z_2^2 = 1$ into the system dynamics and the Lie derivative computations of $b(x)$ as $\frac{db}{d[x]_3} = \frac{db}{dz_1} z_1' + \frac{db}{dz_2} z_2' = \frac{db}{dz_1} z_2$.

1) Sampling-Based Construction of CBFs: We construct a CBF candidate with degree 2. We sample the points i.i.d uniformly distributed on $h(x) = 0$ within the range $(-1, -1, -0.1, -1, -1, -1)^T$ and $(1, 1, 0.1, 1, 1, 1)^T$ for $\{y_1, \ldots, y_{29}\} \subseteq \mathcal{V}[h]$. We sample the points i.i.d uniformly within the range $(-1, -1, -0.1, -1, -1, -1)^T$ and $(1, 1, 0.1, 1, 1, 1)^T$ for $\{x_1, \ldots, x_{29}\} \subseteq \mathcal{C}$. We solve Eq. (16) with $Q_1 \succeq 0$ and $Q_2 \succeq 0$ being relaxed as Diagonally-Dominant-Sum-of-Squares (DSOS) [35]. The values of the parameters are $M = 29, N = 15$, and $Y = 30$. The result is $b(x) = b_0 - x^T B_1 x$, where $b_0 = 0.043$ and $B_1 =$

$$\begin{pmatrix} 0.057 & 0 & 0.029 & 0 & 0.004 & -0.001\\ 0 & 0.045 & 0 & 0 & 0 & 0\\ 0.029 & 0 & 2.886 & 0.086 & 0 & -0.032\\ 0 & 0 & 0.086 & 0.003 & 0 & -0.001\\ 0.004 & 0 & 0 & 0 & 0.013 & -0.004\\ -0.001 & 0 & -0.032 & -0.001 & -0.004 & 0.001 \end{pmatrix}$$

The projection of the synthesized barrier function on the $[x]_1[x]_2-$plane is shown in Fig. 2. We can find that the boundary of the constructed CBF, i.e., $b(x) = 0$ is contained in the safe set. The average running time over 5 runs of the construction program is $145.33s$. Here we set a $120s$ time limit for solving Eq. (16) in each iteration due to the impact of the random samples on the barrier function construction.

2) Sampling-Based Safety Verification: We next verify containment and invariance properties of a given CBF candidate with degree 2, i.e. our previously generated CBF candidate. We sample sets of the varieties $\mathcal{V}(b, \frac{\partial b}{\partial x} g_1(x), \frac{\partial b}{\partial x} g_2(x))$ and $\mathcal{V}(h)$ with $s = 29$ and $v = 29$, respectively.

Based on Table I, the SDP variable dimensions of the proposed containment and invariance verifications are substantially lower, at 7, in contrast to those for the baseline method. The SDP variable dimensions for the baseline is usually greater than those with degree 2, since the baseline cannot verify the barrier function in 2 hours with polynomial degree 8. The runtime performance comparison between the proposed method and the verification algorithm of [6] is shown in Table II. The results indicate that the proposed method achieves significant improvements in both containment and invariance verification, within 10 seconds, compared to the baseline, which requires over 2 hours.

Dynamics	Linear		Nonlinear	
Method	SOS Baseline	Proposed Verification	SOS Baseline	Proposed Verification
containment verification	18	4	112	7
invariance verification	29	4	105	7

TABLE I: Dimension of SDP comparison of proposed CBF verification with SOS baseline with polynomial degree 2.

VII. Conclusions

This paper studied the problem of verifying and constructing control barrier functions for nonlinear systems. The key insight of our approach is that, for polynomial systems, it suffices to verify an SOS criterion at a finite set of sample points at the boundary of the safe region. This result enables us to develop algorithms for verifying CBFs that only require constructing a positive semidefinite matrix whose coefficients satisfy a set of linear equalities, and hence reducing the complexity compared to existing methods that

Dynamics	Linear		Nonlinear	
Methods	SOS ACC	Proposed ACC	SOS Quadrotor	Proposed Quadrotor
Sampling	NA	0.0336s	NA	6.3469s
Containment Verification	1.8070s	1.7988s	*Inf*	1.7382s
Invariance Verification	3.8711s	0.8423s	*Inf*	1.0872s
Total	5.6782s	2.6766s	*Inf*	9.3900s

TABLE II: Average runtime comparison of proposed CBF verification with SOS-based baseline based on 5 experiments for linear ACC system and nonlinear Quadrotor system. The value '*Inf*' means that the program cannot terminate within 2 hours with polynomial degree up to 8.

rely on solving large sum-of-squares programs. We extend our approach to high-order CBFs and CBFs with limits on actuation. Turning to the problem of synthesizing CBFs, we propose a novel heuristic in which we first select a set of samples, and then attempt to choose a CBF that satisfies the safety constraints at the chosen sample points by solving a mixed integer convex program.

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