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Virtual nonholonomic constraints: A geometric approach[☆]

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ABSTRACT

Virtual constraints are relations imposed in a control system that become invariant via feedback instead of real physical constraints acting on the system. Nonholonomic systems are mechanical systems with non-integrable constraints on the velocities. In this work, we introduce the notion of *virtual nonholonomic constraints* in a geometric framework. More precisely, it is a controlled invariant distribution associated with an affine connection mechanical control system. We show the existence and uniqueness of a control law defining a virtual nonholonomic constraint and we characterize the trajectories of the closed-loop system as solutions of a mechanical system associated with an induced constrained connection. Moreover, we characterize the dynamics for nonholonomic systems in terms of virtual nonholonomic constraints, i.e., we characterize when can we obtain nonholonomic dynamics from virtual nonholonomic constraints.

1. Introduction

Virtual constraints are relations on the configuration variables of a control system which are imposed through feedback control and the action of actuators, instead of through physical connections such as gears or contact conditions with the environment. The class of virtual holonomic constraints became popular in applications to biped locomotion where it was used to express a desired walking gait (see for instance (Chevallereau et al., 2009, 2018; La Hera et al., 2013; Razavi et al., 2016)), as well as for motion planning to search for periodic orbits and its employment in the technique of transverse linearization to stabilize such orbits (Consolini & Maggiore, 2013; Consolini et al., 2010; Freidovich et al., 2008; Mohammadi et al., 2018; Nielsen & Maggiore, 2008; Shiriaev et al., 2010; Westerberg et al., 2009).

Virtual nonholonomic constraints are a class of virtual constraints that depend on velocities rather than only on the configurations of the system. Those constraints were introduced in Griffin and Grizzle (2015, 2017) to design a velocity-based swing foot placement in bipedal robots. In particular, this class of virtual constraints has been used in Hamed and Ames (2019), Horn and Gregg (2021), Horn et al. (2018, 2020) to encode velocity-dependent stable walking gaits via momenta conjugate to the unacatuated degrees of freedom of legged robots and prosthetic legs.

From a theoretical perspective, virtual constraints extend the application of zero dynamics to feedback design (see for instance Isidori (2013) and Westervelt et al. (2018)). In particular, the class of virtual holonomic constraints applied to mechanical systems has seen rich advances in theoretical foundations and applications in the last decade (see Čelikovský (2015), Čelikovský and Anderle (2016, 2017), Čelikovskỳ and Anderle (2018), Consolini and Costalunga (2015), Consolini et al. (2018), Maggiore and Consolini (2012), Mohammadi et al. (2013, 2017, 2018, 2015)). Nevertheless there is a lack of a rigorous definition and qualitative description for the class of virtual nonholonomic constraints in contrast with the holonomic situation. The recent work (Moran-MacDonald, 2021) demonstrates a first approach to defining rigorously virtual nonholonomic constraints, but the nonlinear nature of the constraints makes difficult a thorough mathematical analysis. In this work, we provide a formal definition of linear virtual nonholonomic constraints, i.e., constraints that are linear in the velocities. This particular case includes most of the examples of nonholonomic constraints in the literature of nonholonomic systems (see Bloch (2003) and Neimark and Fufaev (2004) for instance). Our definition is based on the invariance

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property under the closed-loop system and coincides with the one of Moran-MacDonald (2021), in the linear case.

In particular, a virtual nonholonomic constraint is described by a non-integrable distribution on the configuration manifold of the system for which there is a feedback control making it invariant under the flow of the closed-loop system. We provide sufficient conditions for the existence and uniqueness of such a feedback law defining the virtual nonholonomic constraint and we also characterize the trajectories of the closed-loop system as solutions of a mechanical system associated with an induced constrained connection. Moreover, we are able to produce nonholonomic dynamics by imposing virtual nonholonomic constraints on a mechanical control system. This last result allows controlling the system to satisfy desired stability properties that are well known in the literature on nonholonomic systems, through the imposition of suitable virtual nonholonomic constraints. Moreover, it relates the geometric control contributions we make in the paper with the geometric mechanics literature about the stability of nonholonomic systems (see Bloch (2003) for instance).

The remainder of the paper is structured as follow. Section 2 introduces nonholonomic systems. We define virtual nonholonomic constraints in Section 3, where we provide sufficient conditions for the existence and uniqueness of a control law defining a virtual nonholonomic constraint, and provide examples and comparisons with the literature. In Section 4, we introduce a constrained connection to characterize the closed-loop dynamics as a solution of the mechanical system associated with such a constrained connection. In Section 5, we show that if the input distribution is orthogonal to the virtual nonholonomic constraint distribution then the constrained dynamics is precisely the noholonomic dynamics with respect to the original Lagrangian function. Conclusions are given in Section 6.

2. Preliminaries

Let Q be the configuration space of a mechanical system, a differentiable manifold with $\dim(Q) = n$, and with local coordinates denoted by $(q^1, \dots, q^n)$. In the following, we will define mechanical systems in the setting of arbitrary manifolds. In this setting, once the configuration manifold Q has been fixed, the proper phase space where the dynamics evolves is the tangent bundle of Q, denoted by TQ. The tangent bundle is the disjoint union of all tangent spaces T_qQ at all points q in the manifold Q, i.e.,

$$TQ = \bigcup_{q\in Q}\{(q,\dot q) | \dot q \in T_qQ\}.$$

The tangent bundle is also a manifold with twice the dimension of Q and local coordinates given by $(q^1, \dots, q^n, \dot q^1, \dots, \dot q^n)$. A typical element of the tangent bundle is a tangent vector to the manifold Q at one of its points. There is a canonical projection map, denoted by τ_Q, from TQ to the configuration manifold Q, sending each tangent vector v in TQ to the point at which v is tangent to. For instance, if v is tangent to Q at q, meaning that $v \in T_qQ$, then $\tau_Q(v) = q$. For convenience of the reader, we will often denote elements of the tangent bundle with a subscript indicating the point at which they are tangent, e.g., v_q would belong to T_qQ. In local coordinates, its expression is simply $\tau_Q(q^1, \dots, q^n, \dot q^1, \dots, \dot q^n) = (q^1, \dots, q^n)$. A local basis of each tangent space associated with the local coordinates on Q is denoted by $\left\{\frac{\partial}{\partial q^1}, \dots, \frac{\partial}{\partial q^n}\right\}$.

Before proceeding, we will recall the definition of Riemannian metric. A Riemannian metric is a generalization of the inner product on a vector space to arbitrary manifolds. In fact, one can describe it as an inner product in each tangent space T_qQ that varies smoothly with the base point q. In particular, since the metric will be an inner product on each tangent space, as will see below, it will be defined on the space $TQ \times_Q TQ$ composed of pairs of tangent vectors lying in the same tangent space. In this way, we avoid defining the inner product between two vectors that are tangent at different points. More precisely,

Definition 1. A Riemannian metric $\mathscr{G}$ on a manifold Q is a $(0, 2)$-tensor, i.e., a bilinear map $\mathscr{G} : TQ \times_Q TQ \to \mathbb{R}$, satisfying the following properties:

(i) symmetric: $\mathscr{G}(v_q, w_q) = \mathscr{G}(w_q, v_q)$ for all $q \in Q$ and v_q, $w_q \in T_qQ$.
(ii) non-degenerate: $\mathscr{G}(v_q, w_q) = 0$ for all $w_q \in TQ$ if and only if $v_q = 0$.
(iii) positive-definite: $\mathscr{G}(v_q, v_q) \geqslant 0$, with equality holding only if $v_q = 0$.

Accordingly, if $\mathscr{G}$ is a Riemannian metric then the pair $(Q, \mathscr{G})$ is called a Riemannian manifold.

If $(q^1, \dots, q^n)$ are local coordinates on Q, then the local expression of the Riemannian metric $\mathscr{G}$ is

$$\mathscr{G} = \mathscr{G}_{ij} dq^i \otimes dq^j, \quad \text{with } \mathscr{G}_{ij} = \mathscr{G}\left(\frac{\partial}{\partial q^i}, \frac{\partial}{\partial q^j}\right).$$

Let $C^\infty(Q)$ denote the set of smooth function on Q and $\mathfrak{X}(Q)$ denote the set of smooth vector fields on Q, i.e, smooth maps from Q to TQ satisfying the requirement that $X(q) \in T_qQ$. If $X, Y \in \mathfrak{X}(Q)$, then $[X, Y] \in \mathfrak{X}(Q)$ denotes the standard Lie bracket of vector fields. Below, we will use the fact that vector fields $X \in \mathfrak{X}(Q)$ act on functions $f \in C^\infty(Q)$. If the local expression of X is $X = X^i\frac{\partial}{\partial q^i}$ then, in coordinates $X(f) = X^i\frac{\partial f}{\partial q^i}$.

Next, we introduce the concept of a linear connection on a manifold which is essentially a means of consistently implementing directional derivatives of a vector field along another in any manifold. In general terms, a linear connection on a manifold Q is any map of the form $\nabla : \mathfrak{X}(Q) \times \mathfrak{X}(Q) \to \mathfrak{X}(Q)$ which is $C^\infty(Q)$-linear on the first factor, $\mathbb{R}$-linear in the second factor, and if we denote the image of $X, Y \in \mathfrak{X}(Q)$ by $\nabla_X Y$, then ∇ satisfies the Leibniz rule $\nabla_X(fY) = X(f) \cdot Y + f \cdot \nabla_X Y$ for every $f \in C^\infty(Q)$. Connections are locally characterized by the Chrystoffel symbols which are real-valued functions on Q given by

$$\nabla_{\frac{\partial}{\partial q^i}}\frac{\partial}{\partial q^j} = \Gamma^k_{ij}\frac{\partial}{\partial q^k}.$$

Thus if X and Y are vector fields locally given by $X = X^i\frac{\partial}{\partial q^i}$ and $Y = Y^i\frac{\partial}{\partial q^i}$, then

$$\nabla_X Y = \left(X^i\frac{\partial Y^k}{\partial q^i} + X^iY^j\Gamma^k_{ij}\right)\frac{\partial}{\partial q^k}.$$

However, in any Riemannian manifold, there is a distinguished linear connection called the *Levi-Civita connection*. Given a Riemannian metric $\mathscr{G}$, there is a unique connection $\nabla^{\mathscr{G}} : \mathfrak{X}(Q) \times \mathfrak{X}(Q) \to \mathfrak{X}(Q)$ satisfying the following two additional properties:

(i) $[X, Y] = \nabla^{\mathscr{G}}_X Y - \nabla^{\mathscr{G}}_Y X$ (symmetry)
(ii) $X(\mathscr{G}(Y, Z)) = \mathscr{G}(\nabla^{\mathscr{G}}_X Y, Z) + \mathscr{G}(Y, \nabla^{\mathscr{G}}_X Z)$ (compatibility of the metric).

Given a linear connection, we might extend the notion of directional derivative of a vector field along another to the concept of derivative of a vector field along a curve. Sometimes this notion is called covariant derivative. The covariant derivative of a vector field $X \in \mathfrak{X}(Q)$ along a curve $q : I \to Q$, where I is an interval of $\mathbb{R}$, is given by the local expression

$$\nabla_{\dot q}X(t) = \left(\dot X^k(t) + \dot q^i(t)X^j(t)\Gamma^k_{ij}(q(t))\right)\frac{\partial}{\partial q^k}.$$

Note that to compute the covariant derivative along the curve $q(t)$, as one can guess by its local expression, it is enough to consider a curve of tangent vectors $X(t)$ which are tangent to $q(t)$ for every $t \in I$. In particular, we can compute $\nabla_{\dot{q}}\dot{q}$ which is usually called the covariant acceleration of the curve $q(t)$. This is related with the last concept in this section which might be familiar to the reader: once a connection ∇ has been chosen, a geodesic is a curve satisfying the equation $\nabla_{\dot{q}}\dot{q} = 0$.

2.1. Nonholonomic mechanical systems

Next, consider mechanical systems where the dynamics is described by a Lagrangian function $L : TQ \to \mathbb{R}$ of the type

$$L(v_q) = \frac{1}{2}\mathscr{G}(v_q, v_q) - V(q), \tag{1}$$

with $v_q \in T_qQ$, where $\mathscr{G}$ denotes a Riemannian metric on Q representing the kinetic energy of the systems, T_qQ, the tangent space at the point q of Q, and $V : Q \to \mathbb{R}$ is a (smooth) potential function.

The trajectories $q : I \to Q$ of a mechanical Lagrangian determined by a Lagrangian function as in (1) satisfy Euler–Lagrange equations which in turn are equivalent to the following equation which can be seen as a Riemmanian version of Newton's second law:

$$\nabla^{\mathscr{G}}_{\dot{q}}\dot{q} + \mathrm{grad}_{\mathscr{G}} V(q(t)) = 0. \tag{2}$$

Observe that if the potential function vanishes, then the trajectories of the mechanical system are just the geodesics with respect to the connection $\nabla^{\mathscr{G}}$. Here, the vector field $\mathrm{grad}_{\mathscr{G}} V \in \mathfrak{X}(Q)$ is characterized by

$$\mathscr{G}(\mathrm{grad}_{\mathscr{G}} V, X) = dV(X), \quad \text{for every } X \in \mathfrak{X}(Q).$$

Most nonholonomic systems have linear constraints on velocities, so these are precisely the ones that we will consider. Linear constraints on the velocities (or Pfaffian constraints) are locally given by equations of the form

$$\mu_i(q)\dot{q}^i = 0, \quad i = 1, \dots, m, \ \text{ with } m < n \tag{3}$$

depending in general, on the configurations and velocities of the system (see Bloch (2003) for instance). Above and throughout the paper, we will use the Einstein summation convention: repeated indices appearing twice, first as a superscript and then as a subscript or vice-versa, must be summed over, for instance, $\mu_i\dot{q}^i$ means $\sum_i \mu_i\dot{q}^i$.

From a geometric point of view, these constraints are defined by a nonintegrable regular distribution $\mathscr{D}$ on Q of constant rank $(n - m)$. A rank $(n - m)$ distribution on a manifold Q is the assignment of a subspace $\mathscr{D}_q$ of T_qQ with constant dimension $n - m$ to each point $q \in Q$. We denote by $\mathscr{D}$ the collection of all such subspaces $\mathscr{D}_q$ at all point q of the manifold. The annihilator of $\mathscr{D}$, denoted by $\mathscr{D}^o$, is locally given at each point of Q by $\mathscr{D}^o_q = \mathrm{span}\left\{\mu^a(q) = \mu^a_i dq^i \,;\, 1 \le a \le m\right\}$, where μ^a are linearly independent differential one-forms at each point of Q. We further denote by $\Omega^1(Q)$ the set of differential one-forms on Q.

Example 1. The simplest example of a distribution is that of an integrable distribution such as all tangent vectors in $\mathbb{R}^3$ with vanishing third coordinate, i.e., $\dot{z} = 0$. However, observe that these vectors are all tangent to the planes defined by $z =$ constant. In the other hand, the distribution defined by the constraint $\dot{z} - y\dot{x} = 0$ cannot be written as the tangent plane to some plane or surface of $\mathbb{R}^3$. So, this is an example of a non-integrable distribution.

Denote by $\tau_{\mathscr{D}} : \mathscr{D} \to Q$ the canonical projection from $\mathscr{D}$ to Q which is defined to be the restriction of the canonical projection τ_Q to the distribution $\mathscr{D}$. Locally, it is given by $\tau_{\mathscr{D}}(q^1, \dots, q^n, \dot{q}^1, \dots, \dot{q}^n) = (q^1, \dots, q^n)$, and denote by $\Gamma(\tau_{\mathscr{D}})$ the set of sections of $\tau_{\mathscr{D}}$, that is, $Z \in \Gamma(\tau_{\mathscr{D}})$ if $Z : Q \to \mathscr{D}$ satisfies $(\tau_{\mathscr{D}} \circ Z)(q) = q$.

Now, assume the Lagrangian system is subjected to nonholonomic constraints given by (3).

Definition 2. A *nonholonomic mechanical system* on a smooth manifold Q is given by the triple $(\mathscr{G}, V, \mathscr{D})$, where $\mathscr{G}$ is a Riemannian metric on Q, representing the kinetic energy of the system, $V : Q \to \mathbb{R}$ is a smooth function representing the potential energy, and $\mathscr{D}$ a regular distribution on Q describing the nonholonomic constraints.

Using the Riemannian metric $\mathscr{G}$ we can define two complementary orthogonal projectors $\mathscr{P}: TQ \to \mathscr{D}$ and $\mathscr{Q}: TQ \to \mathscr{D}^\perp$, with respect to the tangent bundle orthogonal decomposition $\mathscr{D} \oplus \mathscr{D}^\perp = TQ$.

In the presence of a constraint distribution $\mathscr{D}$, Eq. (2) must be slightly modified as follows. Consider the *nonholonomic connection* $\nabla^{nh} : \mathfrak{X}(Q) \times \mathfrak{X}(Q) \to \mathfrak{X}(Q)$ defined by (see Bullo and Lewis (2005) for instance)

$$\nabla^{nh}_X Y = \nabla^{\mathscr{G}}_X Y + (\nabla^{\mathscr{G}}_X \mathscr{Q})(Y). \tag{4}$$

Then, the trajectories for the nonholonomic mechanical system associated with the Lagrangian (1) and the distribution $\mathscr{D}$ must satisfy the following equation

$$\nabla^{nh}_{\dot{q}}\dot{q} + \mathscr{P}(\mathrm{grad}_{\mathscr{G}} V(q(t))) = 0. \tag{5}$$

3. Virtual nonholonomic constraints

Next, we present the rigorous construction of virtual nonholonomic constraints. On contrary to the case of standard nonholonomic constraints of the form (3), the concept of virtual constraint is always associated with a controlled system, rather than with the distribution defined by the constraints.

Given the Riemannian metric $\mathscr{G}$ on Q, we can use its nondegeneracy property to define the musical isomorphism $\flat : \mathfrak{X}(Q) \to \Omega^1(Q)$ defined by $\flat(X)(Y) = \mathscr{G}(X, Y)$ for any $X, Y \in \mathfrak{X}(Q)$. Also, denote by $\sharp : \Omega^1(Q) \to \mathfrak{X}(Q)$ the inverse musical isomorphism, i.e., $\sharp = \flat^{-1}$.

Given an external force $F^0 : TQ \to T^*Q$ and a control force $F : TQ \times U \to T^*Q$ of the form

$$F(q, \dot{q}, u) = \sum_{a=1}^{m} u_a f^a(q) \tag{6}$$

where $f^a \in \Omega^1(Q)$ with $m < n$, $U \subset \mathbb{R}^m$ the set of controls and $u_a \in \mathbb{R}$ with $1 \le a \le m$ the control inputs, consider the associated mechanical control system of the form

$$\nabla^{\mathscr{G}}_{\dot{q}(t)}\dot{q}(t) = Y^0(q(t), \dot{q}(t)) + u_a(t)Y^a(q(t)), \tag{7}$$

with $Y^0 = \sharp(F^0)$ and $Y^a = \sharp(f^a)$ the corresponding force vector fields.

Hence, q is the trajectory of a vector field of the form

$$\Gamma(v_q) = G(v_q) + u_a(Y^a)^V_{v_q}, \tag{8}$$

where G is the vector field determined by the unactuated forced mechanical system

$$\nabla^{\mathscr{G}}_{\dot{q}(t)}\dot{q}(t) = Y^0(q(t), \dot{q}(t))$$

and where the vertical lift of a vector field $X \in \mathfrak{X}(Q)$ to TQ is defined by

$$X^V_{v_q} = \left.\frac{d}{dt}\right|_{t=0} (v_q + tX(q)).$$

Definition 3. The distribution $\mathscr{F} \subseteq TQ$ generated by the vector fields $\sharp(f_i)$ is called the *input distribution* associated with the mechanical control system (7).

Now we will define the concept of virtual nonholonomic constraint.

Definition 4. A *virtual nonholonomic constraint* associated with the mechanical control system (7) is a controlled invariant distribution $\mathscr{D} \subseteq TQ$ for that system, that is, there exists a control function $\hat{u} : \mathscr{D} \to \mathbb{R}^m$ such that the solution of the closed-loop system satisfies $\phi_t(\mathscr{D}) \subseteq \mathscr{D}$, where $\phi_t : TQ \to TQ$ denotes its flow.

Remark 1. A particular example of mechanical control system appearing in applications is determined by a mechanical Lagrangian function $L : TQ \to \mathbb{R}$. In this case, the control system is given by the controlled Euler–Lagrange equations, i.e.,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = F(q, \dot{q}, u). \tag{9}$$

If the curve $q : I \to Q$ is a solution of the controlled Euler Lagrange equations (9), it may be shown that it satisfies the mechanical equation (see Bullo and Lewis (2005) for instance)

$$\nabla^{\mathscr{G}}_{\dot{q}(t)}\dot{q}(t) + \operatorname{grad}_{\mathscr{G}} V(q(t)) = u_a(t)Y^a(q(t)). \tag{10}$$

These are the equations of a mechanical control system as in (7), where the force field Y^0 is simply given by $-\operatorname{grad}_{\mathscr{G}} V(q(t))$. In this case, we call (10) a controlled Lagrangian system. $\diamond$

3.1. Relation with previous definitions of virtual nonholonomic constraints

In previous works, virtual nonholonomic constraints appeared under different definitions. The most general one, comprising every single other as a particular case, is given in Moran-MacDonald (2021) where a virtual nonholonomic constraint is a set of the form

$$\mathscr{M} = \{(q, p) \in Q \times \mathbb{R}^n \mid h(q, p) = 0\},$$

for which there exists a control law making it invariant under the flow of the closed-loop controlled Hamiltonian equations. This constraint might be rewritten using the cotangent bundle T^*Q and h might be seen as a function $h : T^*Q \to \mathbb{R}^m$. In addition, h should satisfy rank $dh(q, p) = m$ for all $(q, p) \in \mathscr{M}$.

Our definition falls under this general definition, for the particular case where the function h is linear on the fibers, i.e., a linear function on the momenta p_i. In order to see it, we must rewrite the virtual nonholonomic constraints and the control system on the cotangent bundle.

Indeed, consider the Hamiltonian function $H : T^*Q \to \mathbb{R}$ obtained from a Lagrangian function in the following way

$$H(q, p) = p\dot{q}(q, p) - L(q, \dot{q}(q, p)),$$

where $\dot{q}(q, p)$ is a function of (q, p) given by the inverse of the Legendre transformation

$$p = \frac{\partial L}{\partial \dot{q}}.$$

The controlled Hamiltonian equations are given by

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q} + F^0(q, \dot{q}(q, p)) + u_a f^a(q),$$

where F^0 is an external force map. Now, any distribution $\mathscr{D} \subseteq TQ$ might be defined as the set

$$\mathscr{D} = \{(q, \dot{q}) \in TQ \mid \mu^a(q)(\dot{q}) = 0\},$$

where μ^a with $1 \leqslant a \leqslant m$ are m linearly independent one-forms. The cotangent version of the distribution is the set

$$\tilde{\mathscr{M}} = \{(q, p) \mid \mu^a(q)(\dot{q}(q, p)) = 0\}.$$

Therefore, we set

$$h(q, p) = (\mu^1(q)(\dot{q}(q, p)), \dots, \mu^m(q)(\dot{q}(q, p))).$$

We just have to check if rank $dh = m$. Note that each component of h is linear on fibers if the Lagrangian function (and thus, the corresponding Hamiltonian function) is of mechanical type, i.e., $L = \dot{q}^T M\dot{q} - V(q)$, where M is the mass matrix and it represents the Riemannian metric on coordinates, then the Legendre transform is just $p = M\dot{q}$ and its inverse is $\dot{q} = M^{-1}p$. Therefore,

$$h(q, p) = (\mu^1 M^{-1}p, \dots, \mu^m M^{-1}p).$$

Hence, the submatrix of the Jacobian formed by the partial derivatives with respect to the momenta p are formed by the rows

$$M^{-1}\mu^1, \dots, M^{-1}\mu^m,$$

which are linearly independent. Thus this submatrix has rank m and this implies that the Jacobian matrix dh has rank greater than m. However, since it is formed by m rows, the rank of dh must be exactly m and $\tilde{\mathscr{M}}$ is a virtual nonholonomic constraint according to Moran-MacDonald (2021) if there is a control law making it invariant.

Remark 2. In the case that the mechanical control system is described by a mechanical Lagrangian function, our definition of virtual nonholonomic constraint coincides with the one given in Moran-MacDonald (2021) when we view it in the cotangent bundle. The requirement that the mechanical control system comes from a mechanical Lagrangian is not necessary in order to have equivalence of both definitions but it is at least necessary that we have some way of pushing forward the constraints to the cotangent bundle. This property is usually the regularity of the Lagrangian function, which amounts to having the Legendre transformation as a local diffeomorphism between TQ and T^*Q.

Remark 3. Note that the definition of virtual nonholonomic constraints provided in Griffin and Grizzle (2015) and Horn et al. (2020) is the Lagrangian version of the virtual nonholonomic constraint given in Moran-MacDonald (2021) by using the Legendre transformation to translate momenta constraints to velocity constraints.

3.2. Examples

Example 2. Consider in $SE(2) \cong \mathbb{R}^2 \times \mathbb{S}^1$ the mechanical Lagrangian function

$$L(x, y, \theta, \dot{x}, \dot{y}, \dot{\theta}) = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{I\dot{\theta}^2}{2}$$

together with the control force

$$F(x, y, \theta, \dot{x}, \dot{y}, \dot{\theta}, u) = u(\sin\theta dx - \cos\theta dy + d\theta).$$

The corresponding controlled Lagrangian system is

$$m\ddot{x} = u\sin\theta, \quad m\ddot{y} = -u\cos\theta, \quad I\ddot{\theta} = u$$

and, as we will show, it has the following virtual nonholonomic constraint

$$\sin\theta\dot{x} - \cos\theta\dot{y} = 0.$$

The input distribution $\mathscr{F}$ is generated just by one vector field

$$Y = \frac{\sin\theta}{m}\frac{\partial}{\partial x} - \frac{\cos\theta}{m}\frac{\partial}{\partial y} + \frac{1}{I}\frac{\partial}{\partial \theta},$$

while the virtual nonholonomic constraint is the distribution $\mathscr{D}$ defined as the set of tangent vectors $v_q \in T_qQ$ where $\mu(q)(v) = 0$, with $\mu = \sin\theta dx - cos\theta dy$. Thus, we may write it as

$$\mathscr{D} = \text{span}\left\{X_1 = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y},\ X_2 = \frac{\partial}{\partial\theta}\right\}.$$

We may check that $\mathscr{D}$ is controlled invariant for the controlled Lagrangian system above. In fact, the control law

$$\hat{u}(x, y, \theta, \dot{x}, \dot{y}, \dot{\theta}) = -m\dot{\theta}(\cos\theta\dot{x} + \sin\theta\dot{y})$$

makes the distribution invariant under the closed-loop system, since in this case, the dynamical vector field arising from the controlled Euler–Lagrange equations given by

$$\Gamma = \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + \dot{\theta}\frac{\partial}{\partial\theta} + \frac{\hat{u}\sin\theta}{m}\frac{\partial}{\partial\dot{x}} - \frac{\hat{u}\cos\theta}{m}\frac{\partial}{\partial\dot{y}} + \frac{\hat{u}}{I}\frac{\partial}{\partial\dot{\theta}}$$

is tangent to $\mathscr{D}$. This is deduced from the fact that $\Gamma(\sin\theta\dot{x} - \cos\theta\dot{y}) = 0$. ◇

Example 3. Consider in $\mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{S}^1$ the mechanical Lagrangian function

$$L(x, y, \theta, \varphi, \dot{x}, \dot{y}, \dot{\theta}, \dot{\varphi}) = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{I\dot{\theta}^2}{2} + \frac{J\dot{\varphi}^2}{2}$$

together with the control force

$$\begin{aligned} F(x, y, \theta, \varphi, \dot{x}, \dot{y}, \dot{\theta}, \dot{\varphi}, u) =& u_1(dx - \cos\varphi d\theta + d\varphi) \\ &+ u_2(dy - \sin\varphi d\theta + d\varphi). \end{aligned}$$

The controlled Lagrangian system is then

$$m\ddot{x} = u_1,\ m\ddot{y} = u_2,\ I\ddot{\theta} = -u_1\cos\varphi - u_2\sin\varphi,\ J\ddot{\varphi} = u_1 + u_2.$$

The virtual nonholonomic constraints associated to this system are defined by the following equations

$$\dot{x} = \dot{\theta}\cos\varphi, \quad \dot{y} = \dot{\theta}\sin\varphi.$$

Therefore, the input distribution $\mathscr{F}$ is the set

$$\begin{aligned} \mathscr{F} = \text{span}\ \Big\{\ Y^1 =& \frac{1}{m}\frac{\partial}{\partial x} - \frac{\cos\varphi}{I}\frac{\partial}{\partial\theta} + \frac{1}{J}\frac{\partial}{\partial\varphi}, \\ Y^2 =& \frac{1}{m}\frac{\partial}{\partial y} - \frac{\sin\varphi}{I}\frac{\partial}{\partial\theta} + \frac{1}{J}\frac{\partial}{\partial\varphi}\ \Big\}, \end{aligned}$$

and the constraint distribution $\mathscr{D}$ is defined by the 1-forms $\mu^1 = dx - \cos\varphi d\theta$ and $\mu^2 = dy - \sin\varphi d\theta$, thus

$$\mathscr{D} = \left\{X_1 = \cos\varphi\frac{\partial}{\partial x} + \sin\varphi\frac{\partial}{\partial y} + \frac{\partial}{\partial\theta},\ X_2 = \frac{\partial}{\partial\varphi}\right\}.$$

We may verify, using a similar argument as Example 2, that $\mathscr{D}$ is in fact controlled invariant under the control law

$$\hat{u}_1 = -m\dot{\theta}\dot{\varphi}\sin\varphi, \quad \hat{u}_2 = m\dot{\theta}\dot{\varphi}\cos\varphi. \quad ◇$$

Example 4. Let us see an example of a mechanical control system which is not a Lagrangian system. Consider again the mechanical control system proposed in Example 2 but now with an additional damping term determined by the vector field $Y^0 = -\frac{\gamma}{m}(\dot{x}dx + \dot{y}dy)$, where $\gamma > 0$ is a damping constant. The mechanical control system has the following equations of motion

$$m\ddot{x} = u\sin\theta - \gamma\dot{x}, \quad m\ddot{y} = -u\cos\theta - \gamma\dot{y}, \quad I\ddot{\theta} = u.$$

It is not difficult to check that the control law

$$\hat{u}(x, y, \theta, \dot{x}, \dot{y}, \dot{\theta}) = -m\dot{\theta}(\cos\theta\dot{x} + \sin\theta\dot{y})$$

still makes the distribution invariant under the flow of the closed-loop system. ◇

3.3. Existence and uniqueness of a feedback control making the constraints invariant

It is often very useful if we have conditions under which we are guaranteed that a distribution $\mathscr{D}$ is controlled invariant for the controlled Lagrangian system (10). The next result not only states the existence of a control function making $\mathscr{D}$ invariant, but it also states that it is unique. In the following, two distributions $\mathscr{A}_1$ and $\mathscr{A}_2$ on the manifold Q are said to be transversal if they are complementary, in the sense that $TQ = \mathscr{A}_1 \oplus \mathscr{A}_2$.

Theorem 1. *If the distribution $\mathscr{D}$ and the control input distribution $\mathscr{F}$ are transversal, then there exists a unique control function making the distribution a virtual nonholonomic constraint associated with the mechanical control system (7).*

Proof. Suppose that $TQ = \mathscr{D} \oplus \mathscr{F}$ and that trajectories of the control system (7) may be written as the integral curves of the vector field Γ defined by (8). For each $v_q \in \mathscr{D}_q$, we have that

$$\Gamma(v_q) \in T_{v_q}(TQ) = T_{v_q}\mathscr{D} \oplus \text{span}\left\{(Y^a)^V_{v_q}\right\},$$

with $Y^a = \sharp(f^a)$. Using the uniqueness decomposition property arising from transversality, we conclude there exists a unique vector $\tau^*(v_q) = (\tau^*_1(v_q), \ldots, \tau^*_m(v_q)) \in \mathbb{R}^m$ such that

$$\Gamma(v_q) = G(v_q) + \tau^*_a(v_q)(Y^a)^V_{v_q} \in T_{v_q}\mathscr{D},$$

where Γ and G are as in Eq. (8). If $\mathscr{D}$ is defined by m constraints of the form $\phi^b(v_q) = 0$, $1 \leq b \leq m$, then the condition above may be rewritten as

$$d\phi^b(G(v_q) + \tau^*_a(v_q)(Y^a)^V_{v_q}) = 0,$$

which is equivalent to

$$\tau^*_a(v_q)d\phi^b((Y^a)^V_{v_q}) = -d\phi^b(G(v_q)).$$

Note that, the equation above is a linear equation of the form $A(v_q)\tau = b(v_q)$, where $b(v_q)$ is the vector $(-d\phi^1(G(v_q)), \ldots, -d\phi^m(G(v_q))) \in \mathbb{R}^m$ and $A(v_q)$ is the $m \times m$ matrix with entries $A^b_a(v_q) = d\phi^b((Y^a)^V_{v_q}) = \mu^b(q)(Y^a)$, where the last equality may be deduced by computing the expressions in local coordinates. That is, if $(q^i\dot{q}^i)$ are natural bundle coordinates for the tangent bundle, then

$$\begin{aligned} d\phi^b((Y^a)^V_{v_q}) &= \left(\frac{\partial\mu^b_i}{\partial q^j}\dot{q}^i dq^j + \mu^b_i d\dot{q}^i\right)\left(Y^{a,k}\frac{\partial}{\partial\dot{q}^k}\right) \\ &= \mu^b_i Y^{a,i} = \mu^b(q)(Y^a). \end{aligned}$$

In addition, $A(v_q)$ has full rank, since its columns are linearly independent. In fact suppose that

$$c_1\begin{bmatrix}\mu^1(Y^1)\\ \vdots \\ \mu^m(Y^1)\end{bmatrix} + \cdots + c_m\begin{bmatrix}\mu^1(Y^m)\\ \vdots \\ \mu^m(Y^m)\end{bmatrix} = 0,$$

which is equivalent to

$$\begin{bmatrix}\mu^1(c_1Y^1 + \cdots + c_mY^m)\\ \vdots \\ \mu^m(c_1Y^1 + \cdots + c_mY^m)\end{bmatrix} = 0.$$

However, by transversality we have $\mathscr{D} \cap \mathscr{F} = \{0\}$ which implies that $c_1Y^1 + \cdots + c_mY^m = 0$. Since $\{Y_i\}$ are linearly independent we conclude that $c_1 = \cdots = c_m = 0$ and A has full rank. But, since A is an $m \times m$ matrix, and $\mathscr{D}$ is a regular distribution, it must be invertible. Therefore, there is a unique vector $\tau^*(v_q)$ satisfying the matrix equation and $\tau^* : \mathscr{D} \to \mathbb{R}^m$ is smooth since it is the solution of a matrix equation depending smoothly on v_q. □

Remark 4. Note that in Examples 2 and 3, the constraint distribution $\mathscr{D}$ and the control input distribution $\mathscr{F}$ are transversal. Thus the control laws obtained in there are unique by Theorem 1. $\diamond$

The transversality condition is essential in order to have existence and uniqueness of the control law making the constraint distribution control invariant. If they are not transversal then a control law making $\mathscr{D}$ control invariant may not exist or may not be unique as we will see in the next examples.

Example 5 (*Non-existence*)**.** Consider the Lagrangian function L and the distribution $\mathscr{D}$ given in Example 2, but now let the control force be

$$F(x, y, \theta, \dot{x}, \dot{y}, \dot{\theta}, u) = u(\cos\theta dx + \sin\theta dy),$$

so that the controlled Lagrangian system is now

$$m\ddot{x} = u\cos\theta, \quad m\ddot{y} = u\sin\theta, \quad I\ddot{\theta} = 0.$$

Note that, in this case, the control input distribution $\mathscr{F}$ is generated by the vector field $Y = \frac{\cos\theta}{m}\frac{\partial}{\partial x} + \frac{\sin\theta}{m}\frac{\partial}{\partial y}$. Hence, $\mathscr{F} \subseteq \mathscr{D}$.

Suppose that a control law $\hat{u}$ making the distribution control invariant exists. Differentiating the constraints, we get

$$\cos\theta\dot{x} + \sin\theta\ddot{x} + \sin\theta\dot{y} - \cos\theta\ddot{y} = 0,$$

and substituting by the closed-loop system we get

$$0 = \cos\theta\dot{x} + \frac{\hat{u}\sin\theta\cos\theta}{m} + \sin\theta\dot{y} - \frac{\hat{u}\sin\theta\cos\theta}{m},$$

which is satisfied only when $\cos\theta\dot{x} + \sin\theta\dot{y} = 0$. Therefore, there is no control law $\hat{u}$ making the distribution control invariant. $\diamond$

Example 6 (*Non-Uniqueness*)**.** Consider again the situation given in Example 2 but now with the control force

$$\begin{aligned} F(x, y, \theta, \dot{x}, \dot{y}, \dot{\theta}, u) = {} & u_1(\sin\theta dx - \cos\theta dy + d\theta) \\ & + u_2(\sin\theta dx - \cos\theta dy). \end{aligned}$$

In this case, we have that $TQ = \mathscr{D} + \mathscr{F}$ but $\mathscr{D} \cap \mathscr{F} \neq \{0\}$. Two examples of control laws making $\mathscr{D}$ control invariant are

$$\hat{u}_1 = -m\dot{\theta}(\cos\theta\dot{x} + \sin\theta\dot{y}), \quad \hat{u}_2 = 0$$

and

$$\hat{u}_1 = 0, \quad \hat{u}_2 = -m\dot{\theta}(\cos\theta\dot{x} + \sin\theta\dot{y}). \quad \diamond$$

4. The induced constrained connection

From now on suppose that the distribution $\mathscr{D}$ describing the virtual nonholonomic constraints and the input distribution $\mathscr{F}$ are transversal. Therefore, the projections $P_{\mathscr{F}} : TQ \to \mathscr{F}$ and $P_{\mathscr{D}} : TQ \to \mathscr{D}$ associated to the direct sum are well-defined.

The *induced constrained connection* associated to the distribution $\mathscr{D}$ and the input distribution $\mathscr{F}$ is given by

$$\overset{c}{\nabla}_X Y = \nabla^{\mathscr{G}}_X Y + (\nabla^{\mathscr{G}}_X P_{\mathscr{F}})(Y), \tag{11}$$

where $\nabla^{\mathscr{G}}$ is the Levi-Civita connection associated to the Riemannian metric $\mathscr{G}$. The induced constrained connection is a linear connection on Q with the special property that $\mathscr{D}$ is geodesically invariant for $\overset{c}{\nabla}$, i.e., if a geodesic of $\overset{c}{\nabla}$ starts on $\mathscr{D}$ then it stays in $\mathscr{D}$ for all time (see Lewis (1998)).

We have the following useful lemma that we will use later on.

Lemma 1. *If $X, Y \in \Gamma(\tau_{\mathscr{D}})$ then*

$$\overset{c}{\nabla}_X Y = P_{\mathscr{D}}(\nabla^{\mathscr{G}}_X Y).$$

Proof. If $X, Y \in \Gamma(\tau_{\mathscr{D}})$ we have that

$$\begin{aligned} \overset{c}{\nabla}_X Y &= \nabla^{\mathscr{G}}_X Y + (\nabla^{\mathscr{G}}_X P_{\mathscr{F}})(Y) \\ &= \nabla^{\mathscr{G}}_X Y + \nabla^{\mathscr{G}}_X (P_{\mathscr{F}}(Y)) - P_{\mathscr{F}}(\nabla^{\mathscr{G}}_X Y), \end{aligned}$$

where we have used the definition of covariant derivative of a map of the form $T : TQ \to TQ$ in the last equality. Noting that $P_{\mathscr{F}}(Y) = 0$ since Y is a section of $\tau_{\mathscr{D}}$, we conclude that $\overset{c}{\nabla}_X Y = P_{\mathscr{D}}(\nabla^{\mathscr{G}}_X Y)$. $\square$

The last lemma implies in particular that $\overset{c}{\nabla}$ is well-defined as a connection on sections of $\tau_{\mathscr{D}}$ in the sense that the restriction $\overset{c}{\nabla}|_{\Gamma(\tau_{\mathscr{D}})\times\Gamma(\tau_{\mathscr{D}})}$ takes values also on $\Gamma(\tau_{\mathscr{D}})$. However, as the following lemma shows the constrained connection is not symmetric, in general.

Lemma 2. *If the constrained connection $\overset{c}{\nabla}$ is symmetric then the constraint distribution $\mathscr{D}$ is integrable.*

Proof. The torsion of the constrained connection is given by

$$T^c(X, Y) = \overset{c}{\nabla}_X Y - \overset{c}{\nabla}_Y X - [X, Y].$$

Suppose that $X, Y \in \Gamma(\tau_{\mathscr{D}})$. In this case

$$\begin{aligned} T^c(X, Y) &= P_{\mathscr{D}}(\nabla^{\mathscr{G}}_X Y - \nabla^{\mathscr{G}}_Y X) - [X, Y] \\ &= P_{\mathscr{D}}([X, Y]) - [X, Y] \\ &= -P_{\mathscr{F}}([X, Y]), \end{aligned}$$

where we used the fact that $\nabla^{\mathscr{G}}$ is symmetric in the first equality. It is clear now that if $\overset{c}{\nabla}$ is symmetric then $[X, Y]$ must be a section of $\mathscr{D}$, which implies that $\mathscr{D}$ is integrable. $\square$

Remark 5. Lemma 2 has been also shown in Lewis (1998), however we provided here an alternative simple proof in order to keep the discussion as much self-contained as possible. $\diamond$

In the following, we characterize the closed-loop dynamics as solutions of the mechanical system associated with the induced constrained connection.

Theorem 2. *A curve $q : I \to Q$ is a trajectory of the closed-loop system for the Lagrangian control system (10) making $\mathscr{D}$ invariant if and only if it satisfies*

$$\overset{c}{\nabla}_{\dot{q}(t)}\dot{q}(t) + P_{\mathscr{D}}(grad_{\mathscr{G}} V(q(t))) = 0. \tag{12}$$

Proof. If $q : I \to Q$ is a trajectory of the closed-loop system for (10) with $\dot{q}(t) \in \mathscr{D}_{q(t)}$ then it satisfies

$$\nabla^{\mathscr{G}}_{\dot{q}(t)}\dot{q}(t) + \text{grad}_{\mathscr{G}} V(q(t)) = \hat{u}_a(t)Y^a(q(t)),$$

where $\hat{u} : \mathscr{D} \to \mathbb{R}^m$ is the unique control law making $\mathscr{D}$ invariant. Attending to the fact that $\dot{q}(t) \in \mathscr{D}_{q(t)}$ we have that

$$\begin{aligned} \overset{c}{\nabla}_{\dot{q}(t)}\dot{q}(t) &= P_{\mathscr{D}}(\nabla^{\mathscr{G}}_{\dot{q}(t)}\dot{q}(t)) \\ &= -P_{\mathscr{D}}(\text{grad}_{\mathscr{G}} V(q(t))) + P_{\mathscr{D}}(\hat{u}_a(t)Y^a(q(t))) \\ &= -P_{\mathscr{D}}(\text{grad}_{\mathscr{G}} V(q(t))), \end{aligned}$$

where we have used Lemma 1 in the first equality and $P_{\mathscr{D}}(Y^a) = 0$ in the last one.

Conversely, if the curve q satisfies (12), we have

$$P_{\mathscr{D}}(\nabla^{\mathscr{G}}_{\dot{q}(t)}\dot{q}(t) + \text{grad}_{\mathscr{G}} V(q(t))) = 0,$$

where we used Lemma 1. Since $\ker P_{\mathscr{D}} = \mathscr{F}$, there exist $u = (u_1, \ldots, u_m) \in \mathbb{R}^m$ such that

$$\nabla^{\mathscr{G}}_{\dot{q}(t)}\dot{q}(t) + \text{grad}_{\mathscr{G}} V(q(t)) = u_a Y^a.$$

By Theorem 1, we conclude that $u = \hat{u}$, since the control law making $\mathscr{D}$ invariant is unique. $\square$

Remark 6. Suppose $\mathscr{D}$ is an integrable distribution and assume $\mathscr{C}$ is a maximal integrable manifold of $\mathscr{D}$. If $\overset{h}{\nabla}$ denotes the holonomic connection on $\mathscr{C}$ defined in Consolini and Costalunga (2015) (see also Consolini et al. (2018)), as

$$\overset{h}{\nabla}_X Y = P_{\mathscr{D}}(\nabla^{\mathscr{G}}_X Y), \quad X, Y \in \mathfrak{X}(\mathscr{C}),$$

then Lemma 1 implies that the two connections are the same when $\overset{c}{\nabla}$ is restricted to vector fields on $\mathscr{C}$. $\diamond$

4.1. The constrained connection in coordinates

In this section we will compute the Christoffel symbols of the induced connection. Given any coordinate chart (q^i) on Q the Christoffel symbols are determined by the values of the connection taken over the standard basis of the tangent space $\{\frac{\partial}{\partial q^1}, \dots, \frac{\partial}{\partial q^n}\}$. It is not difficult to prove the following useful expression

$$\overset{c}{\nabla}_{\frac{\partial}{\partial q^i}} \frac{\partial}{\partial q^j} = P_{\mathscr{D}}\left(\nabla^{\mathscr{G}}_{\frac{\partial}{\partial q^i}} \frac{\partial}{\partial q^j}\right) + \nabla^{\mathscr{G}}_{\frac{\partial}{\partial q^i}}\left(P_{\mathscr{F}}\left(\frac{\partial}{\partial q^j}\right)\right).$$

Example 7. Consider once again the control system given in Example 2. The Levi-Civita connection $\nabla^{\mathscr{G}}$ associated with this system has vanishing Christoffel symbols. Considering the coordinates $q = (x, y, \theta)$ on $SE(2)$, we have that

$$\overset{c}{\nabla}_{\frac{\partial}{\partial q^i}} \frac{\partial}{\partial q^j} = \nabla^{\mathscr{G}}_{\frac{\partial}{\partial q^i}}\left(P_{\mathscr{F}}\left(\frac{\partial}{\partial q^j}\right)\right).$$

Note that the natural coordinate vector fields for $SE(2)$ may be decomposed in a unique way, under the direct sum $\mathscr{D} \oplus \mathscr{F}$, and this decomposition is given by

$$\frac{\partial}{\partial x} = \cos\theta X_1 - \frac{m\sin\theta}{I} X_2 + m\sin\theta Y,$$
$$\frac{\partial}{\partial y} = \sin\theta X_1 + \frac{m\cos\theta}{I} X_2 - m\cos\theta Y,$$
$$\frac{\partial}{\partial \theta} = X_2.$$

Hence, we obtain the following non-vanishing Christoffel symbols for the constrained connection $\overset{c}{\nabla}$

$$\Gamma^x_{\theta x} = 2\sin\theta\cos\theta, \qquad \Gamma^x_{\theta y} = \sin^2\theta - \cos^2\theta,$$
$$\Gamma^y_{\theta x} = \sin^2\theta - \cos^2\theta, \qquad \Gamma^y_{\theta y} = -2\sin\theta\cos\theta,$$
$$\Gamma^\theta_{\theta x} = \frac{m\cos\theta}{I}, \qquad \Gamma^\theta_{\theta y} = \frac{m\sin\theta}{I}.$$

If we introduce the coordinates $q = (x, y, \theta, \varphi)$ in Example 3 and following the same reasoning we get

$$P_{\mathscr{F}}\left(\frac{\partial}{\partial x}\right) = \frac{IJm + Jm^2\sin^2(\varphi)}{L(\varphi)} Y^1 - \frac{Jm^2\sin(\varphi)\cos(\varphi)}{L(\varphi)} Y^2$$
$$P_{\mathscr{F}}\left(\frac{\partial}{\partial y}\right) = \frac{Im - Jm^2\sin(\varphi)\cos(\varphi)}{L(\varphi)} Y^1 + \frac{-Im + Jm^2\cos^2(\varphi)}{L(\varphi)} Y^2$$
$$P_{\mathscr{F}}\left(\frac{\partial}{\partial \theta}\right) = \frac{-IJm\cos(\varphi) - Im\sin(\varphi)}{L(\varphi)} Y^1 + \frac{Im\sin(\varphi)}{L(\varphi)} Y^2$$
$$P_{\mathscr{F}}\left(\frac{\partial}{\partial \varphi}\right) = \frac{-IJ - Jm\sin^2(\varphi)}{L(\varphi)} Y^1 + \frac{Jm\sin(\varphi)\cos(\varphi)}{L(\varphi)} Y^2,$$

with $L(\varphi) = -I + Jm\cos^2(\varphi) - m\sin^2(\varphi) + m\sin(\varphi)\cos(\varphi)$. In addition, the non-vanishing Christoffel symbols are given in the Appendix.

5. Existence of a nonholonomic Lagrangian structure for the dynamics on $\mathscr{D}$

The next proposition shows that if the input distribution is orthogonal to the virtual nonholonomic constraint distribution then the constrained dynamics is precisely the nonholonomic dynamics with respect to the original Lagrangian function.

Proposition 1. *If the input distribution $\mathscr{F}$ is orthogonal to the virtual constraint distribution $\mathscr{D}$ with respect to the metric $\mathscr{G}$, then the trajectories of the constrained mechanical system (12) are the nonholonomic equations of motion.*

Proof.
If $\mathscr{F} = \mathscr{D}^\perp$, then the projectors $P_{\mathscr{D}}$ and $\mathscr{P}$ coincide (as well as the projectors $P_{\mathscr{F}}$ and $\mathscr{Q}$). Thus, the constrained connection $\overset{c}{\nabla}$ is precisely the nonholonomic connection ∇^{nh}. This implies that the trajectories of the constrained connection are nonholonomic trajectories. $\square$

Remark 7. The fact that $\mathscr{F} = \mathscr{D}^\perp$ is independent of the chosen metric. Once you fix the control force F and let the control input distribution be obtained using the musical isomorphism $\sharp$ as in Section 3, then $\mathscr{F}$ is orthogonal to $\mathscr{D}$ if and only if $f^a \in \mathscr{D}^o$, for $a = 1, \dots, m$. $\diamond$

Although the orthogonal condition $\mathscr{F} = \mathscr{D}^\perp$ is sufficient in order for the constrained dynamics to be the nonholonomic dynamics, it is not necessary as the following result shows.

Proposition 2. *Suppose there exists a modified potential function $\tilde{V}$ satisfying*

$$\mathscr{P}(\mathrm{grad}_{\mathscr{G}}\tilde{V}) = P_{\mathscr{D}}(\mathrm{grad}_{\mathscr{G}}V). \tag{13}$$

Then the nonholonomic trajectories with respect to $(\mathscr{G}, \tilde{V}, \mathscr{D})$ coincide with the constrained dynamics (12) if and only if $\nabla^{\mathscr{G}}_X \mathscr{Q}(X) = \nabla^{\mathscr{G}}_X P_{\mathscr{F}}(X)$ for all $X \in \Gamma(\mathscr{D})$.

Proof. It is not difficult to see that $\nabla^{\mathscr{G}}_X \mathscr{Q}(X) = \nabla^{\mathscr{G}}_X P_{\mathscr{F}}(X)$ if and only if the two connections satisfy $\overset{c}{\nabla}_X X = \nabla^{nh}_X X$. Therefore, the equation

$$\overset{c}{\nabla}_{\dot{q}(t)}\dot{q}(t) + P_{\mathscr{D}}(\mathrm{grad}_{\mathscr{G}}V(q(t))) = 0$$

holds if and only if

$$\nabla^{nh}_{\dot{q}(t)}\dot{q}(t) + \mathscr{P}(\mathrm{grad}_{\mathscr{G}}\tilde{V}(q(t))) = 0$$

also holds.

Conversely, if the trajectory $q(t)$ satisfies both equation, then

$$\nabla^{nh}_{\dot{q}(t)}\dot{q}(t) = \overset{c}{\nabla}_{\dot{q}(t)}\dot{q}(t)$$

is also satisfied. Using tensoriality of the difference tensor

$$D(X, Y) = \overset{c}{\nabla}_X Y - \nabla^{nh}_X Y,$$

we may evaluate D point-wise so that

$$D(X_q, X_q) = (\overset{c}{\nabla}_X X - \nabla^{nh}_X X)(q).$$

Choosing the trajectory $q(t)$ with initial point q and initial velocity $X_q \in \mathscr{D}_q$, which is always possible thanks to the existence and uniqueness theorem for ODE, we deduce that $D(X_q, X_q) = 0$ for any $X_q \in \mathscr{D}_q$. Hence, $D(X, X) = 0$ which is equivalent to $\nabla^{\mathscr{G}}_X \mathscr{Q}(X) = \nabla^{\mathscr{G}}_X P_{\mathscr{F}}(X)$. $\square$

In the absence of a potential function, i.e., $V = 0$, the nonholonomic trajectories coincide with the constrained dynamics if and only if $\nabla^{\mathcal{G}}_X \mathcal{Q}(X) = \nabla^{\mathcal{G}}_X P_{\mathcal{F}}(X)$ for any $X \in \Gamma(\tau_{\mathcal{D}})$.

Note that the previous characterization of when both dynamics have the same trajectories may be equivalently written as

$$\mathcal{P}(\nabla^{\mathcal{G}}_X X) = P_{\mathcal{D}}(\nabla^{\mathcal{G}}_X X) \text{ or } \mathcal{Q}(\nabla^{\mathcal{G}}_X X) = P_{\mathcal{F}}(\nabla^{\mathcal{G}}_X X)$$

for any $X \in \Gamma(\tau_{\mathcal{D}})$.

Corollary 1. *If the geodesic vector field associated with $\nabla^{\mathcal{G}}$ is tangent to $\mathcal{D}$, then the nonholonomic trajectories coincide with the constrained geodesics and they are both the geodesics of $\nabla^{\mathcal{G}}$ with initial velocity in $\mathcal{D}$.*

Proof. We just have to establish that the geodesic vector field associated with $\nabla^{\mathcal{G}}$ is tangent to $\mathcal{D}$ if and only if $\nabla^{\mathcal{G}}_X X \in \Gamma(\tau_{\mathcal{D}})$ for every $X \in \Gamma(\tau_{\mathcal{D}})$. Then this is equivalent to $\mathcal{Q}(\nabla^{\mathcal{G}}_X X) = 0$ and also to $P_{\mathcal{F}}(\nabla^{\mathcal{G}}_X X) = 0$. Hence, by the previous result, the geodesics with initial velocity in $\mathcal{D}$ of ∇^{nh} coincide with the geodesics with initial velocity in $\mathcal{D}$ of $\overset{c}{\nabla}$.

Now, $\nabla^{\mathcal{G}}_X X \in \Gamma(\tau_{\mathcal{D}})$ for every $X \in \Gamma(\tau_{\mathcal{D}})$ if and only if $\mathcal{D}$ is geodesically invariant with respect to $\nabla^{\mathcal{G}}$ (see Lewis (1998), Theorem 5.4). Using standard results on differential geometry, $\mathcal{D}$ is geodesically invariant with respect to $\nabla^{\mathcal{G}}$ if and only if the geodesic vector field associated with $\nabla^{\mathcal{G}}$ is tangent to $\mathcal{D}$. □

Remark 8. One important feature of the theory of virtual holonomic constraints presented in Consolini et al. (2018) is that if the induced connection has the same trajectories as the Levi-Civita connection with respect to the induced metric on the constraint submanifold $\mathcal{C} \subseteq Q$, then the two connections are the same. However, its argument relies on the fact that the induced connection is symmetric. Therefore, the result does not follow in the nonholonomic case whenever the distribution is not integrable.

The next example illustrates Proposition 1.

Example 8. Consider the Chaplygin sleigh, a celebrated example of a nonholonomic mechanical system evolving on the configuration manifold $SE(2)$ with Lagrangian function as in Example 2 but now we consider the control force

$$F(x, y, \theta, \dot{x}, \dot{y}, \dot{\theta}, u) = u(\sin\theta dx - \cos\theta dy).$$

The corresponding controlled Lagrangian system is

$$m\ddot{x} = u\sin\theta, \quad m\ddot{y} = -u\cos\theta, \quad I\ddot{\theta} = 0.$$

The input distribution $\mathcal{F}$ is generated just by one vector field

$$Y = \frac{\sin\theta}{m}\frac{\partial}{\partial x} - \frac{\cos\theta}{m}\frac{\partial}{\partial y},$$

while the virtual nonholonomic constraint is the same distribution $\mathcal{D}$ as in Example 2. We may check that the control law

$$\hat{u}(x, y, \theta, \dot{x}, \dot{y}, \dot{\theta}) = -m\dot{\theta}(\cos\theta\dot{x} + \sin\theta\dot{y})$$

makes the distribution invariant under the closed-loop system. In addition, by Proposition 1 the resulting system is precisely the nonholonomic equation (5) for the Chaplygin system, since the input distribution spanned by Y is orthogonal to the virtual nonholonomic constraints. ⋄

Remark 9. There are plenty of ways to impose a virtual nonholonomic constraint on a mechanical control system in order to obtain a nonholonomic system. In the last example, one could choose the control force to be

$$F(x, y, \theta, \dot{x}, \dot{y}, \dot{\theta}, u_1, u_2) = u_1\sin\theta dx + u_2\cos\theta dy$$

and the corresponding controlled Lagrangian system would be

$$m\ddot{x} = u_1\sin\theta, \quad m\ddot{y} = u_2\cos\theta, \quad I\ddot{\theta} = 0.$$

Then, the control law

$$\hat{u}_1(x, y, \theta, \dot{x}, \dot{y}, \dot{\theta}) = -m\dot{\theta}(\cos\theta\dot{x} + \sin\theta\dot{y}), \quad \hat{u}_2 = -\hat{u}_1$$

makes the closed-loop system coincide again with the nonholonomic equations for the Chaplygin system. Note that the input distribution is now generated by the vector fields $Y^1 = \frac{\sin\theta}{m}\frac{\partial}{\partial x}$ and $Y^2 = \frac{\cos\theta}{m}\frac{\partial}{\partial y}$. Since they do not generate a transversal distribution to $\mathcal{D}$, we should not expect the control law to be unique. ⋄

Remark 10.

Under the conditions of Proposition 1, certain mechanical control systems may be driven to desired stable trajectories by imposing virtual nonholonomic constraints and using the proper control force.

For instance, for the mechanical control system appearing in Example 8, we may drive the system to an asymptotically stable trajectory characterized by $\dot{\theta} = 0$. Indeed, by defining the variables $v = \dot{x}\cos\theta + \dot{y}\sin\theta$ and $\omega = \dot{\theta}$, the equations of motion of the Chaplygin sleigh might be written as

$$\dot{\omega} = -\frac{ma}{I + ma^2}v\omega, \quad \dot{v} = a\omega^2,$$

for which the points with $\omega = 0$ are equilibria. Moreover, from a stability analysis we deduce that the system exhibits asymptotic stability.

However, not every nonholonomic system exhibits asymptotically stable behavior. As discussed in e.g. Zenkov et al. (1998) one may have a stable (but not asymptotically stable) dynamics or a mix of stable and asymptotically dynamics. Therefore, the applicability of our method is largely related to which kind of trajectories you wish to obtain. Thus, when we are given a mechanical control system satisfying the conditions of Proposition 1, we should first examine the qualitative properties of the associated nonholonomic system. Typical behavior includes asymptotic stability, periodic or quasi-periodic orbits and conservation of first integrals such as the energy or the nonholonomic momentum. In a wide class of examples, virtual nonholonomic constraints enable us to use energy–momentum methods from Zenkov et al. (1998) to decide when it is possible to obtain stable or asymptotically stable trajectories. ⋄

6. Conclusions

We introduced virtual nonholonomic constraints for mechanical control systems evolving on differentiable manifolds by using an affine connection formalism. We have shown the existence and uniqueness of a control law allowing one to define a virtual nonholonomic constraint and we have characterized the trajectories of the closed-loop system as solutions of a mechanical system associated with an induced constrained connection. In addition, we have characterized the dynamics of nonholonomic systems with linear constraints on the velocities in terms of virtual nonholonomic constraints. In a future work, we would like to extend the results of this paper to nonlinear constraints in order to gain further insight into the nonlinear nonholonomic virtual constraints defined in Moran-MacDonald (2021) and Čelikovský et al. (2021). In this direction, it would be interesting to impose the energy of the mechanical system as the nonlinear virtual nonholonomic constraint and check if it is possible to design a control keeping the energy constant. Moreover, it would also be interesting to study conditions under which the closed-loop system obtained from Theorem 1 is equivalent to a nonholonomic system in the same spirit of the approach followed in Ricardo and

Respondek (2010). Two control systems on a manifold Q of the form

$$\dot{q} = G(q) + u_a Y^a(q),$$

where G and Y^a are vector fields on Q, are S-equivalent if there exists a diffeomorphism $\phi : Q \to Q$ such that both their drift vector fields G and control vector fields Y^a are ϕ-related. Then, we may define a control system to be equivalent to a nonholonomic system if it is S-equivalent to a mechanical control system for which there exists a control law making its trajectories nonholonomic trajectories. Equivalence is a less restrictive condition than the relation with nonholonomic systems provided in this work. Hence, in principle, it is easier to impose a control law making a control system equivalent to a nonholonomic mechanical system. Though it is a weaker condition, equivalent systems still share the same qualitative behavior such as stability properties, periodic orbits, etc.

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Appendix. Christoffel symbols with constrained connection for Example 3

The following are the non-vanishing Christoffel symbols:

$$\Gamma^x_{\varphi x} = \frac{2Jm\sin\varphi\cos\varphi}{L} - \frac{(IJ + Jm\sin^2\varphi)L'}{L^2},$$

$$\Gamma^y_{\varphi x} = \frac{Jm(\sin^2\varphi - \cos^2\varphi)}{L} + \frac{Jm\sin\varphi\cos\varphi L'}{L^2},$$

$$\Gamma^\theta_{\varphi x} = \frac{Jm\sin\varphi}{L} + \frac{Jm\cos\varphi L'}{L^2},$$

$$\Gamma^\varphi_{\varphi x} = \frac{m^2(2\sin\varphi\cos\varphi + \sin^2\varphi - \cos^2\varphi)}{L}, - \frac{m(I + m\sin^2\varphi - m\sin\varphi\cos\varphi)L'}{L^2},$$

$$\Gamma^x_{\varphi y} = \frac{Jm(\sin^2(\phi) - \cos^2(\phi))}{L} + \frac{(I - Jm\sin(\phi)\cos(\phi))\,L'}{L^2},$$

$$\Gamma^y_{\varphi y} = -\frac{2Jm\sin(\phi)\cos(\phi)}{L} + \frac{\left(-I + Jm\cos^2(\phi)\right)L'}{L^2},$$

$$\Gamma^\theta_{\varphi y} = \frac{2Jm^2\sin^2(\phi)\cos(\phi)}{IL} - \frac{\left(-Im + Jm^2\cos^2(\phi)\right)\cos(\phi)}{IL} - \frac{\left(-Im + Jm^2\cos^2(\phi)\right)L'\sin(\phi)}{IL^2} + \frac{\left(Im - Jm^2\sin(\phi)\cos(\phi)\right)\sin(\phi)}{IL} - \frac{\left(Im - Jm^2\sin(\phi)\cos(\phi)\right)L'\cos(\phi)}{IL^2} - \frac{\left(Jm^2\sin^2(\phi) - Jm^2\cos^2(\phi)\right)\cos(\phi)}{IL},$$

$$\Gamma^\varphi_{\varphi y} = \frac{\left(Jm^2\cos^2(\phi) - Jm^2\sin(\phi)\cos(\phi)\right)L'}{JL^2} + \frac{m^2(\sin^2(\phi) - \cos^2(\phi) - 2\sin(\phi)\cos(\phi))}{L},$$

$$\Gamma^x_{\varphi\theta} = \frac{IJ\sin(\phi) - I\cos(\phi)}{L} + \frac{(-IJ\cos(\phi) - I\sin(\phi))\,L'}{L^2},$$

$$\Gamma^y_{\varphi\theta} = \frac{I\cos(\phi)}{L} + \frac{IL'\sin(\phi)}{L^2},$$

$$\Gamma^\theta_{\varphi\theta} = -\frac{(2+2J)m\sin(\phi)\cos(\phi)}{L} - \frac{mL'\sin^2(\phi)}{L^2}, + \frac{m(\cos^2(\phi) - \sin^2(\phi))}{L} + \frac{(Jm\cos(\phi) + m\sin(\phi))\,L'\cos(\phi)}{L^2},$$

$$\Gamma^\varphi_{\varphi\theta} = \frac{Im\cos(\phi)}{JL} + \frac{ImL'\sin(\phi)}{JL^2} + \frac{IJm\sin(\phi) - Im\cos(\phi)}{JL} + \frac{(-IJm\cos(\phi) - Im\sin(\phi))\,L'}{JL^2},$$

$$\Gamma^x_{\varphi\varphi} = -\frac{2J\sin(\phi)\cos(\phi)}{L} + \frac{\left(-IJ - Jm\sin^2(\phi)\right)L'}{mL^2},$$

$$\Gamma^y_{\varphi\varphi} = \frac{J(\cos^2(\phi) - \sin^2(\phi))}{L} + \frac{JL'\sin(\phi)\cos(\phi)}{L^2},$$

$$\Gamma^\theta_{\varphi\varphi} = \frac{Jm\sin^3(\phi)}{IL} - \frac{JmL'\sin^2(\phi)\cos(\phi)}{IL^2}, + \frac{\left(-IJ - Jm\sin^2(\phi)\right)\sin(\phi)}{IL} - \frac{\left(-IJ - Jm\sin^2(\phi)\right)L'\cos(\phi)}{IL^2},$$

$$\Gamma^\varphi_{\varphi\varphi} = -\frac{m\sin^2(\phi)}{L} - \frac{2\,m\sin(\phi)\cos(\phi)}{L} + \frac{m\cos^2(\phi)}{L}, + \frac{mL'\sin(\phi)\cos(\phi)}{L^2} + \frac{\left(-IJ - Jm\sin^2(\phi)\right)L'}{JL^2}.$$

References

Bloch, A. M. (2003). Nonholonomic mechanics. In *Nonholonomic mechanics and control* (pp. 207–276). Springer.

Bullo, F., & Lewis, A. (2005). *Geometric control of mechanical systems, Vol. 49*. Springer Science & Business Media.

Čelikovský, S. (2015). Flatness and realization of virtual holonomic constraints in Lagrangian systems. *IFAC-PapersOnLine*, *48*(13), 25–30.

Čelikovský, S., & Anderle, M. (2016). On the collocated virtual holonomic constraints in Lagrangian systems. In *2016 American control conference* (pp. 6030–6035). IEEE.

Čelikovský, S., & Anderle, M. (2017). Collocated virtual holonomic constraints in Hamiltonian formalism and their application in the underactuated walking. In *2017 11th Asian control conference* (pp. 192–197). IEEE.

Čelikovskỳ, S., & Anderle, M. (2018). On the Hamiltonian approach to the collocated virtual holonomic constraints in the underactuated mechanical systems. In *AETA 2017-Recent advances in electrical engineering and related sciences: Theory and application* (pp. 554–568). Springer.

Čelikovský, S., Anderle, M., & Vyhlídal, T. (2021). Virtual nonholonomic constraints to damp the varying length pendulum swing. In *2021 60th IEEE conference on decision and control* (pp. 3893–3900). IEEE.

Chevallereau, C., Grizzle, J. W., & Shih, C.-L. (2009). Asymptotically stable walking of a five-link underactuated 3-D bipedal robot. *IEEE Transactions on Robotics*, *25*(1), 37–50.

Chevallereau, C., Razavi, H., Six, D., Aoustin, Y., & Grizzle, J. (2018). Self-synchronization and self-stabilization of 3D bipedal walking gaits. *Robotics and Autonomous Systems*, *100*, 43–60.

Consolini, L., & Costalunga, A. (2015). Induced connections on virtual holonomic constraints. In *2015 54th IEEE conference on decision and control* (pp. 139–144). IEEE.

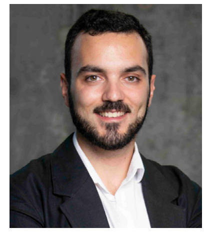
Consolini, L., Costalunga, A., & Maggiore, M. (2018). A coordinate-free theory of virtual holonomic constraints. *Journal of Geometric Mechanics*, *10*(4), 467–502.

Consolini, L., & Maggiore, M. (2013). Control of a bicycle using virtual holonomic constraints. *Automatica*, *49*(9), 2831–2839.

Consolini, L., Maggiore, M., Nielsen, C., & Tosques, M. (2010). Path following for the PVTOL aircraft. *Automatica*, *46*(8), 1284–1296.

Freidovich, L., Robertsson, A., Shiriaev, A., & Johansson, R. (2008). Periodic motions of the pendubot via virtual holonomic constraints: Theory and experiments. *Automatica*, *44*(3), 785–791.

Griffin, B., & Grizzle, J. (2015). Nonholonomic virtual constraints for dynamic walking. In *2015 54th IEEE conference on decision and control* (pp. 4053–4060). IEEE.
Griffin, B., & Grizzle, J. (2017). Nonholonomic virtual constraints and gait optimization for robust walking control. *International Journal of Robotics Research*, *36*(8), 895–922.
Hamed, K. A., & Ames, A. D. (2019). Nonholonomic hybrid zero dynamics for the stabilization of periodic orbits: Application to underactuated robotic walking. *IEEE Transactions on Control Systems Technology*, *28*(6), 2689–2696.
Horn, J. C., & Gregg, R. D. (2021). Nonholonomic virtual constraints for control of powered prostheses across walking speeds. *IEEE Transactions on Control Systems Technology*.
Horn, J. C., Mohammadi, A., Hamed, K. A., & Gregg, R. D. (2018). Hybrid zero dynamics of bipedal robots under nonholonomic virtual constraints. *IEEE Control Systems Letters*, *3*(2), 386–391.
Horn, J. C., Mohammadi, A., Hamed, K. A., & Gregg, R. D. (2020). Nonholonomic virtual constraint design for variable-incline bipedal robotic walking. *IEEE Robotics and Automation Letters*, *5*(2), 3691–3698.
Isidori, A. (2013). *Nonlinear control systems*. Springer Science & Business Media.
La Hera, P. X. M., Shiriaev, A. S., Freidovich, L. B., Mettin, U., & Gusev, S. V. (2013). Stable walking gaits for a three-link planar biped robot with one actuator. *IEEE Transactions on Robotics*, *29*(3), 589–601.
Lewis, A. D. (1998). Affine connections and distributions with applications to nonholonomic mechanics. *Reports on Mathematical Physics*, *42*(1–2), 135–164.
Maggiore, M., & Consolini, L. (2012). Virtual holonomic constraints for Euler–Lagrange systems. *IEEE Transactions on Automatic Control*, *58*(4), 1001–1008.
Mohammadi, A., Maggiore, M., & Consolini, L. (2013). When is a lagrangian control system with virtual holonomic constraints lagrangian? *IFAC Proceedings Volumes*, *46*(23), 512–517.
Mohammadi, A., Maggiore, M., & Consolini, L. (2017). On the Lagrangian structure of reduced dynamics under virtual holonomic constraints. *ESAIM. Control, Optimisation and Calculus of Variations*, *23*(3), 913–935.
Mohammadi, A., Maggiore, M., & Consolini, L. (2018). Dynamic virtual holonomic constraints for stabilization of closed orbits in underactuated mechanical systems. *Automatica*, *94*, 112–124.
Mohammadi, A., Rezapour, E., Maggiore, M., & Pettersen, K. Y. (2015). Maneuvering control of planar snake robots using virtual holonomic constraints. *IEEE Transactions on Control Systems Technology*, *24*(3), 884–899.
Moran-MacDonald, A. (2021). *Energy injection for mechanical systems through the method of Virtual Nonholonomic Constraints* (Ph.D. thesis), University of Toronto (Canada).
Neimark, J. I., & Fufaev, N. A. (2004). *Dynamics of nonholonomic systems, Vol. 33*. American Mathematical Soc..
Nielsen, C., & Maggiore, M. (2008). On local transverse feedback linearization. *SIAM Journal on Control and Optimization*, *47*(5), 2227–2250.
Razavi, H., Bloch, A., Da, X., & Ijspeert, A. (2016). Symmetric virtual constraints for periodic walking of legged robots. In *2016 IEEE 55th conference on decision and control* (pp. 7520–7526). IEEE.
Ricardo, S., & Respondek, W. (2010). When is a control system mechanical? *Journal of Geometric Mechanics*, *2*(3), 265.
Shiriaev, A. S., Freidovich, L. B., & Gusev, S. V. (2010). Transverse linearization for controlled mechanical systems with several passive degrees of freedom. *IEEE Transactions on Automatic Control*, *55*(4), 893–906.
Westerberg, S., Mettin, U., Shiriaev, A. S., Freidovich, L. B., & Orlov, Y. (2009). Motion planning and control of a simplified helicopter model based on virtual holonomic constraints. In *2009 International conference on advanced robotics* (pp. 1–6). IEEE.
Westervelt, E. R., Grizzle, J. W., Chevallereau, C., Choi, J. H., & Morris, B. (2018). *Feedback control of dynamic bipedal robot locomotion*. CRC Press.
Zenkov, V., Bloch, A. M., & Marsden, J. E. (1998). The energy-momentum method for stability of nonholonomic systems. *Dynamics Stability Systems*, *13*, 123–165.



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