Point-Based Value Iteration and Approximately Optimal Dynamic Sensor Selection for Linear-Gaussian Processes*

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***Abstract*— The problem of synthesizing an optimal sensor selection policy is pertinent to a wide variety of engineering applications, ranging from event detection to autonomous navigation. In this paper, we consider such a synthesis problem in the context of linear-Gaussian systems. Particularly, we formulate the optimal sensor selection problem in terms of a value iteration over the continuous space of covariance matrices. To obtain a computationally tractable solution, we subsequently formulate an approximate sensor selection problem, which is solvable through a point-based value iteration over a finite "mesh" of covariance matrices with a user-defined bounded trace. In addition, we provide theoretical guarantees bounding the suboptimality of the sensor selection policies synthesized through this approximate value iteration. Finally, we analyze the efficacy of our proposed method through a numerical example comparing our method to known results.**

I. Introduction

We consider the problem of strategic sensor selection for linear, time-invariant, Gaussian systems with a discounted cost over an infinite time horizon. The question of performing optimal state estimation for linear systems subject to constraints on the total allowable number of sensor activations is well studied, notably beginning with [1] in 1967. The problem has been shown to be NP-hard for different mean square error (MSE)-based objective functions in finite [2] and infinite [3] time horizons. Some popular existing approximation methods are semidefinite programming techniques and greedy algorithms exploiting submodularity. Reference [4] proposes a pruning approach based on algebraic redundancy.

In [5] it was shown that sensor selection for static systems (or dynamic systems without process noise) over finite time horizons is a mixed integer semidefinite programming problem (MISDP). The same classification was made for dynamic systems with process noise in [6]. As a result, many methods exist that utilize the SDP nature of the relaxed problem (corresponding to relaxing the Boolean selection variables to [0,1]). In [5], the relaxed problem was solved, the continuous selection variables rounded, and a local optimization scheme based on ordering and swapping applied to select a sensor subset satisfy the sensing constraint. In [6] and [7], the relaxed SDP was solved with an iterative, re-weighted l-1 constraint on the vector containing all scheduling variables to induce sparsity. The work in [8] did not solve the relaxed problem but rather approximated the MISDP via branch and bound.

The computational requirements for MISDP approaches have been found to be unrealistic for large-scale selection problems. Furthermore, rounding and sparsification techniques destroy performance guarantees for these solutions. These drawbacks motivated use of greedy approaches which are both more computationally efficient and hold the promise of proven performance when the objective function is submodular. It has been shown that mean square error is not in general submodular [9] in finite and infinite settings, so many works [10], [11], [12] consider surrogate objectives with proven submodularity. Performance guarantees though are only valid with respect to the surrogate objective. Other approaches maintain the original objective but prove performance via characterizing weak submodularity [13], [14] for certain special cases of linear-Gaussian systems over finite time horizons. Finally, a few works [9], [15] provide special cases of linear-Gaussian systems for which MSE is submodular, and sufficient conditions are given by which these cases can be established.

In this note, we consider an alternative approach based on standard point-based value iteration (PBVI) for partially observable Markov decision processes (POMDPs) [16]. At a high level, PBVI works by selecting a representative set of sample belief points (e.g. covariance matrices) and subsequently applying a value iteration procedure exclusively to these sample points. By considering only a representative sample, PBVI is able to avoid the curse of dimensionality typically associated with planning in POMDPs. Several works [17], [18], [19] exist which formulate scheduling or activations as a POMDP and apply some version of point-based value iteration. This work represents the belief space as the space of positive semidefinite covariance matrices. We propose an approximate dynamic programming algorithm that can be performed exclusively on a finite number of belief points, and we quantify the upper bound of the performance loss due to discretization as the size ϵ of the mesh grid (Theorem 1). Additionally, this method is shown to be massively parallelizable, and numerical experiments are provided that utilize GPUs.

Notation

We denote the set of all real numbers by $\mathbb{R}$ and the set of all integers by $\mathbb{Z}$. Furthermore, we denote the set of all positive semidefinite matrices of size $n{\times}n$ by $\mathbb{S}_+^n$. For a matrix $A{\in}\mathbb{R}^{n\times n}$, we denote its trace by $\text{Tr}(A){=}\sum_{i=1}^n a_{ii}$.

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II. Problem formulation

We consider a discrete-time system with linear dynamics given by

$$x_{t+1} = Ax_t + w_t, \;\; w_t \sim \mathcal{N}(0, W), \;\; t = 0, 1, 2, \cdots, \quad (1)$$

where $x_t \in \mathbb{R}^n$. The initial state distribution $x_0 \sim \mathcal{N}(0, P_0)$ is assumed to be known a priori. We assume that the matrix $A \in \mathbb{R}^{n\times n}$ is Schur stable; i.e., each of its eigenvalues λ_i satisfies $|\lambda_i(A)|<1$ for $i=1,\ldots,n$. We assume that, at each time step, a set of m sensors is available to make an observation about the underlying state of the system. The linear sensor measurements are made according to

$$y_t = Cx_t + v_t, \;\; v_t \sim \mathcal{N}(0, V), \;\; t = 1, 2, \ldots, \quad (2)$$

where the matrix $C \in \mathbb{R}^{n\times m}$. We interpret the i-th block row of (2) as the output of the i-th sensor. Due to the costs associated with operating each of the m sensors at each time step, it may be advantageous to select only a subset of these available sensors to make observations about the underlying state. If a subset of sensors $S_t \subseteq \{1, 2, \cdots, m\}$ is used at time step $t + 1$, then measurement equation becomes

$$y_{t+1} = C_{S_t} x_{t+1} + v_{t+1}, \;\; v_{t+1} \sim \mathcal{N}(0, V_{S_t}), \;\; t = 0, 1, 2, \cdots \quad (3)$$

as opposed to (2), where C_{S_t} and V_{S_t} are submatrices of C and V, respectively, formed by selecting the rows of C and the rows and columns of V corresponding to the index set S_t. Let $\{S_t\}_{t=0,1,\ldots}$ be the sequence of subsets of selected sensors. By the Kalman filter formula, the estimation error covariance

$$P_t = \mathbb{E}(x_t - \hat{x}_t)(x_t - \hat{x}_t)^\top, \;\; \hat{x}_t = \mathbb{E}(x_t | y_1, \cdots, y_t) \quad (4)$$

is a function of the set of selected sensors S_t, and is recursively computed according to

$$P_{t+1} = f(P_t, S_t) := \left((AP_tA^\top + W)^{-1} + C_{S_t}^\top V_{S_t}^{-1} C_{S_t}\right)^{-1} \quad (5)$$

for $t=0, 1, 2, \ldots$. Now, let $\pi:\mathbb{S}_+^n \to \{0, 1, \ldots, m\}$, $S_t=\pi(P_t)$ be a time-invariant sensor selection policy, and let Π denote the space of all such (Borel measurable) policies. We formulate our sensor selection problem as

$$\min_{\pi\in\Pi} \quad \sum_{t=0}^{\infty} \beta^t c(P_t, S_t), \quad (6)$$

where $c(P_t, S_t)$ is a cost function and $0\leq\beta<1$ is a discount factor weighing the relative importance of present and future costs. Particularly, the parameter β induces an exponential decay in the importance of measuring states at later time-steps, or equivalently, induces exponentially larger importance at earlier time-steps. Furthermore, due to the time-invariance of the state dynamics (5) and the discounted time-invariant cost, the assumption that the optimal policy is time-invariant is made without loss of generality [20]. In what follows, we assume that the cost function $c(P_t, S_t)$ is modeled according to

$$c(P_t, S_t) = \mathrm{Tr}(P_t) + g(S_t),$$

where $\mathrm{Tr}(P_t)$ measures the spread of the covariance matrix P_t and $g:2^{\{1,2,\cdots,m\}} \to [0, \infty)$ is a set function that maps subsets of active sensors to an associated cost for using those sensors. For instance, the set function $g(S_t)=|S_t|$ penalizes the number of sensors used simultaneously, whereas

$$g(S_t) = \begin{cases} 0 & \text{if } |S_t| = 1 \\ +\infty & \text{otherwise} \end{cases} \quad (7)$$

stipulates the use of exactly one sensor at each time step.

For computational tractability, we adopt a slight modification to the problem (6). To start with, define the set all positive semidefinite matrices in $\mathbb{R}^{n\times n}$ with trace at most γ as $\mathbb{S}_+^n(\gamma)$; i.e., $\mathbb{S}_+^n(\gamma):=\{P\in\mathbb{S}_+^n:\mathrm{Tr}(P)\leq\gamma\}$. Let $\Pi(\gamma)$ be the set of stationary sensor selection policies $S_t=\pi(P_t)$ for which the set $\mathbb{S}_+^n(\gamma)$ is invariant; i.e., over repeated sensor measurement updates, the covariance matrix remains in the set $\mathbb{S}_+^n(\gamma)$.

Assumption 1: The set $\Pi(\gamma)$ of policies π under which $\mathbb{S}_+^n(\gamma)$ is invariant is not empty.

In what follows, we study the modified problem:

$$\min_{\pi\in\Pi(\gamma)} \quad \sum_{t=0}^{\infty} \beta^t c(P_t, S_t). \quad (8)$$

Differing from the original problem in (6), the requirement that $P_t\in\mathbb{S}_+^n(\gamma)$ is included as a hard constraint for the problem in (8). We adopt problem (8) partly due to the fact that its bounded state space $P_t\in\mathbb{S}_+^n(\gamma)$ is more amenable for the analysis that follows in Sections III-V. Note that Assumption 1 is not restrictive, since under the assumption that A is Schur stable, there always exists a finite γ such that Assumption 1 holds.

III. Exact Value Iteration

In this section, we characterize the optimal solution to (8) using exact value iteration. To start with, define the space $B(\mathbb{S}_+^n(\gamma))$ of bounded functions on $\mathbb{S}_+^n(\gamma)$ by

$$B(\mathbb{S}_+^n(\gamma)) := \{J : \mathbb{S}_+^n(\gamma) \to \mathbb{R} : \|J\|_\infty < \infty\}$$

where $\|J\|_\infty:=\sup_{P\in\mathbb{S}_+^n(\gamma)} |J(P)|$ is the sup norm. Note that the space $B(\mathbb{S}_+^n(\gamma))$ equipped with the sup norm is complete [21]. Define the Bellman operator T by

$$(TJ)(P) := \min_S \{c(P, S) + \beta J(f(P, S))\}. \quad (9)$$

Proposition 1: Under Assumption 1, the following hold:

(i) $J\in B(\mathbb{S}_+^n(\gamma))$ implies that $TJ\in B(\mathbb{S}_+^n(\gamma))$.

(ii) For all $J, J'\in B(\mathbb{S}_+^n(\gamma))$, $\|TJ - TJ'\|_\infty\leq\beta\|J-J'\|_\infty$.

(iii) $\exists\ J^*\in B(\mathbb{S}_+^n(\gamma))$ satisfying Bellman's equation

$$J^*(P) = \min_S \{c(P, S) + \beta J^*(f(P, S))\}. \quad (10)$$

(iv) For every $J\in B(\mathbb{S}_+^n(\gamma))$, the sequence $J_k=T^kJ$ converges uniformly to J^*; i.e., $\lim_{k\to\infty} \|J_k - J^*\|_\infty=0$.

Proof: (i): Since Assumption 1 guarantees the existence of a sensor selection S such that $f(P, S)\in\mathbb{S}_+^n(\gamma)$, $(TJ)(P)=\min_S\{c(P, S)+\beta J(f(P, S))\}$ is finite for J bounded. Thus $J\in B(\mathbb{S}_+^n(\gamma))$ implies $TJ\in B(\mathbb{S}_+^n(\gamma))$.

(ii): Let $q{:=}\|J{-}J'\|_\infty{=}\sup_{P\in\mathbb{S}_+^n(\gamma)}|J(P){-}J'(P)|$. Then,

$$J(P) - q \leq J'(P) \leq J(P) + q \tag{11}$$

for every $P{\in}\mathbb{S}_+^n(\gamma)$. Applying T to each side of (21), and noticing that

$$\begin{aligned}&\min_S\{c(P,S) + \beta\,(J(f(P,S)) \pm q)\}\\ &\quad= \min_S\{c(P,S) + \beta\,(J(f(P,S)))\} \pm \beta q,\end{aligned}$$

we have

$$(TJ)(P) - \beta q \leq (TJ')(P) \leq (TJ)(P) + \beta q,$$

for every $P{\in}\mathbb{S}_+^n(\gamma)$. Thus, $\|TJ - TJ'\|_\infty{\leq}\beta\|J - J'\|_\infty$.

(iii) and (iv): Since the space $B(\mathbb{S}_+^n(\gamma))$ equipped with the sup norm is a complete metric space, and the operator T a contraction mapping, we can directly apply the Banach fixed point theorem [22] to obtain the desired results. ■

By Proposition 1-(iv), one can obtain J^* through a value iteration over $\mathbb{S}_+^n(\gamma)$. Once J^* is obtained, the optimal policy $\pi^*{\in}\Pi$ is then characterized according to

$$\pi^*(P) = \arg\min_S\{c(P,S) + \beta J^*(f(P,S))\}.$$

Unfortunately, the value iteration procedure described in (9) is not practical to implement as the function J must be evaluated everywhere in the continuous space $\mathbb{S}_+^n(\gamma)$.

IV. Approximate value iteration

Due to the impracticality of continuous state-space value iteration, we now seek an alternative solution method. This section presents a computationally tractable procedure to approximate J^*. Particularly, we will approximate these value functions through a value iteration procedure over a prespecified, finite set of sample covariance matrices.

A. Mesh grid on $\mathbb{S}_+^n(\gamma)$

Let $\epsilon{>}0$ be a fixed parameter that encodes the resolution of our gridding of the space $\mathbb{S}_+^n(\gamma)$. Smaller values of ϵ correspond to a finer resolution and vice versa for larger values of ϵ. We define our resulting mesh according to

$$\begin{aligned}\mathbb{M} &:= \{\epsilon P : P \in \mathbb{Z}^{n\times n}, P \succeq 0\},\\ \mathbb{M}(\gamma) &:= \{P \in \mathbb{M} : \mathrm{Tr}(P) \leq \gamma\}.\end{aligned}$$

Note that the set $\mathbb{M}(\gamma){\subset}\mathbb{S}_+^n(\gamma)$ has a finite number of elements and serves as a "mesh grid" on $\mathbb{S}_+^n(\gamma)$. Table I shows the number of elements $|\mathbb{M}(\gamma)|$ contained in this set depending on the parameters of the system [1]. As evident in Table I, $|\mathbb{M}(\gamma)|$ grows rapidly with n.

[1]Note that these values were obtained by a brute-force enumeration. The authors are unaware of an efficient method to construct $\mathbb{M}(\gamma)$.

	$\gamma = 10$	$\gamma = 20$	$\gamma = 30$	$\gamma = 40$
$n = 2$	312	2,261	7,416	17,349
$n = 3$	9,888	507,745	5,487,604	30,105,633
$n = 4$	217,905	133,895,766	–	–

TABLE I

Number of elements in $\mathbb{M}(\gamma)$ when $\epsilon = 1$.

B. Modified Bellman operator

We now introduce a modified value iteration which can be performed exclusively over the finite set $\mathbb{M}(\gamma)$. We will use this modified value iteration to approximate the exact value function J in (9). Define the quantizer $\Theta : \mathbb{S}_+^n{\to}\mathbb{M}$ by

$$\Theta(P) := \epsilon\mathrm{round}\left(\frac{1}{\epsilon}P\right) + \epsilon n I. \tag{12}$$

The extra $\epsilon n I$ term in (12) ensures that $\Theta(P)$ provides an upper bound in the sense that $\Theta(P){\succeq}P$ (note that this property will be formally proven in Lemma 1). Furthermore, this inequality is crucial for ensuring that the approximate value functions discussed in the sequel upper bound their corresponding exact values.

Remark 1: Alternatively, adopting a quantizer

$$\Theta'(P) = \arg\min_{Q\in\mathbb{M}}\{\mathrm{Tr}(Q) : Q \succeq P\} \tag{13}$$

provides a tighter bound than (12) in that $\Theta(P){\succeq}\Theta'(P){\succeq}P$. However, (13) is computationally difficult to implement. In the numerical studies in Section VI, we adopt

$$\Theta''(P) = \epsilon\mathrm{round}(\frac{1}{\epsilon}P + t^* I) \tag{14}$$

with $t^*{=}\min\{t{\in}\mathbb{R} : \epsilon\mathrm{round}(\frac{1}{\epsilon}P + tI) \succeq P\}$ which is easier to implement than (13) yet still provides a tighter upper bound than (12). For the ease of analysis, in what follows, we adopt the quantizer (12) in order to analyze the gap between the corresponding approximate and exact value iterations. The results of the analysis remain valid for the cases where (13) and (14) are alternatively incorporated. ■

Now, introduce a modified Bellman operator $\bar{T}$ defined by

$$(\bar{T}J)(P) := \min_S\{c(P,S) + \beta J(\Theta(f(P,S)))\}. \tag{15}$$

C. Modified value iteration and suboptimal sensor selection policy

For each $J{\in}B(\mathbb{S}_+^n(\gamma))$, the modified value iteration is defined by the sequence $\bar{J}_k{=}\bar{T}^k J$. In the following section, we provide a formal proof that the sequence of functions $\bar{J}_k$ converges uniformly to the unique solution $\bar{J}^*{\in}B(\mathbb{S}_+^n(\gamma))$ of the Bellman equation

$$\bar{J}^*(P) = \min_S\{c(P,S) + \beta\bar{J}^*(\Theta(f(P,S)))\}, \tag{16}$$

under an appropriate assumption (Proposition 2).

Notice that the modified value iteration $\bar{J}_k{=}\bar{T}^k J$ requires value updates only on the finite set $\mathbb{M}(\gamma)$. Denote by $\bar{J}_k|_{\mathbb{M}(\gamma)}{:}\mathbb{M}(\gamma){\to}\mathbb{R}$ the restriction of $\bar{J}_k$ to the set $\mathbb{M}(\gamma)$. Now, because each $\bar{J}_{k+1}$ is a function of $\bar{J}_k|_{\mathbb{M}(\gamma)}$ only, the modified value iteration can be performed exclusively on the finite set $\mathbb{M}(\gamma)$. Particularly, for each $P\in\mathbb{M}(\gamma)$ in each iteration, one can compute the value function update by solving

$$\bar{J}_{k+1}|_{\mathbb{M}(\gamma)}(P) = \min_S\{c(P,S) + \beta\bar{J}_k|_{\mathbb{M}(\gamma)}(\Theta(f(P,S)))\} \tag{17}$$

Through this procedure, the convergence of the value functions to their optimal values, i.e., $\lim_{k\to\infty}\bar{J}_k|_{\mathbb{M}(\gamma)}=\bar{J}^*|_{\mathbb{M}(\gamma)}$, follows directly from the uniform convergence of $\bar{J}_k$. Considering only the finite set $\mathbb{M}(\gamma)$, we additionally note that the optimal value function vector $\bar{J}^*|_{\mathbb{M}(\gamma)}$ can alternatively be obtained by solving a finite dimensional Linear Program (LP) [23]. Regardless, once the vector $\bar{J}^*|_{\mathbb{M}(\gamma)}$ is obtained, the solution $\bar{J}^*$ to (16) can be recovered as

$$\bar{J}^*(P) = \min_S\{c(P,S) + \beta\bar{J}^*|_{\mathbb{M}(\gamma)}(\Theta(f(P,S)))\}.$$

The suboptimal policy $\pi':\mathbb{S}_+^n(\gamma)\to 2^{\{1,2,\cdots,m\}}$ corresponding to the Bellman equation (16) is likewise obtained by

$$\pi'(P) = \arg\min_S\{c(P,S) + \beta\bar{J}^*(f(P,S))\}. \tag{18}$$

Let $J^{\pi'}:\mathbb{S}_+^n(\gamma)\to\mathbb{R}$ be the value function corresponding to the policy π' characterized as the unique solution to

$$J^{\pi'}(P) = c(P,\pi'(P)) + \beta J^{\pi'}(f(P,\pi'(P))).$$

We will establish that $\|J^*-J^{\pi'}\|_\infty\leq\|J^*-\bar{J}^*\|_\infty\leq\frac{2\epsilon n^2}{(1-\beta)^2}$ in the following section under an appropriate assumption based on the main results (Theorems 1 and 2 below). This bound implies that the performance gap between the optimal sensor selection policy and the suboptimal policy obtained by the approximate value iteration $\bar{J}_k = \bar{T}^k J_0$ can be made arbitrarily small by controlling the value of ϵ. Choosing a small ϵ is accompanied by the increase of $|\mathbb{M}(\gamma)|$ (Table I). Therefore, there exists a trade-off between the performance loss and the computational cost.

V. Analysis of modified value iteration

Although the value update using the modified Bellman operator (15) is straightforward to implement, its performance is difficult to analyze. For this purpose, we introduce a third Bellman operator $\Omega:\mathbb{S}_+^n\to\mathbb{S}_+^n$ defined by

$$\Omega(P) := P + 2\epsilon n I.$$

The following are basic properties of $\Theta(\cdot)$ and $\Omega(\cdot)$:

Lemma 1: For each $P\succeq 0$, we have $P\prec\Theta(P)\prec\Omega(P)$.

Proof: Define $R=\text{round}\left(\frac{1}{\epsilon}P\right)-\frac{1}{\epsilon}P$. By construction, R is symmetric and each entry R_{ij} satisfies $|R_{ij}|\leq\frac{1}{2}$. Consequently, the eigenvalues of R are bounded as

$$\max_i|\lambda_i(R)| \leq \sqrt{\text{Tr}(R^\top R)} = \sqrt{\sum_{i,j}R_{ij}^2} \leq \sqrt{\frac{n^2}{4}} = \frac{n}{2} < n.$$

Therefore,

$$\begin{aligned}\Theta(P) - P &= \epsilon\left(\text{round}\left(\frac{1}{\epsilon}P\right) - \frac{1}{\epsilon}P + nI\right)\\ &= \epsilon(R+nI)\succ 0.\\ \Omega(P)-\Theta(P) &= \epsilon\left(\frac{1}{\epsilon}P - \text{round}\left(\frac{1}{\epsilon}P\right) + nI\right)\\ &= \epsilon(-R+nI)\succ 0,\end{aligned}$$

where the positive definiteness of $(-R+nI)$ follows from our bound on the eigenvalues of R. ■

Finally, introduce a new Bellman operator $\hat{T}$ defined by

$$(\hat{T}J)(P) := \min_S\{c(P,S) + \beta J(\Omega(f(P,S)))\}. \tag{19}$$

for which we define $\hat{J}^*\in B(\mathbb{S}_+^n(\gamma))$ according to

$$\hat{J}^*(P) = \min_S\{c(P,S) + \beta\hat{J}^*(\Theta(f(P,S)))\}, \tag{20}$$

A. Convergence of value iteration

To proceed further, we make the following assumption.

Assumption 2: There exists $S\subseteq\{1,2,\cdots,N\}$ for each $P\in\mathbb{S}_+^n(\gamma)$ such that $\Omega(f(P,S))\in\mathbb{M}(\gamma)$.

Remark 2: Assumption 2 is stronger than Assumption 1. Note that while Assumption 2 is necessary for providing theoretical guarantees, numerical experiments indicate that the sensor selection policy synthesis proposed below often presents a favorable performance even without this assumption. ■

We have the following results regarding the Bellman operators $\bar{T}$ and $\hat{T}$ as defined by (15) and (19).

Proposition 2: Under Assumption 2, the following hold:

(i) $J\in B(\mathbb{S}_+^n(\gamma))$ implies that $\bar{T}J$ and $\hat{T}J\in B(\mathbb{S}_+^n(\gamma))$.

(ii) For all $J,J'\in B(\mathbb{S}_+^n(\gamma))$, $\|\bar{T}J-\bar{T}J'\|_\infty\leq\beta\|J-J'\|_\infty$ and $\|\hat{T}J-\hat{T}J'\|_\infty\leq\beta\|J-J'\|_\infty$.

(iii) $\exists\ \bar{J}^*$, $\hat{J}^*\in B(\mathbb{S}_+^n(\gamma))$ satisfying Bellman's equation (16) and (20), respectively.

(iv) For every $J\in B(\mathbb{S}_+^n(\gamma))$, define the sequences $\bar{J}_k=\bar{T}^kJ$ and $\hat{J}_k=\hat{T}^kJ$. Then, we have $\lim_{k\to\infty}\|\bar{J}_k-\bar{J}^*\|_\infty=0$ and $\lim_{k\to\infty}\|\hat{J}_k-\hat{J}^*\|_\infty=0$

Proof: Here we prove the aforementioned properties only for the operator $\hat{T}$, as those for $\bar{T}$ follow similarly.

(i): Since Assumption 2 guarantees the existence of a sensor selection S such that $\Omega(f(P,S))\in\mathbb{M}(\gamma)\subset\mathbb{S}_+^n(\gamma)$, $(\hat{T}J)(P)=\min_S\{c(P,S)+\beta J(\Omega(f(P,S)))\}$ is finite for J bounded. Thus $J\in B(\mathbb{S}_+^n(\gamma))$ implies $\hat{T}J\in B(\mathbb{S}_+^n(\gamma))$.

(ii): Let $q:=\|J-J'\|_\infty=\sup_{P\in\mathbb{S}_+^n(\gamma)}|J(P)-J'(P)|$. Then,

$$J(P)-q\leq J'(P)\leq J(P)+q \tag{21}$$

for every $P\in\mathbb{S}_+^n(\gamma)$. Applying $\hat{T}$ to each side of (21), and noticing that

$$\begin{aligned}&\min_S\{c(P,S)+\beta(J(\Omega(f(P,S)))\pm q)\}\\ &\qquad=\min_S\{c(P,S)+\beta J(\Omega(f(P,S)))\}\pm\beta q,\end{aligned}$$

we have

$$(\hat{T}J)(P)-\beta q\leq(\hat{T}J')(P)\leq(\hat{T}J)(P)+\beta q,$$

for every $P\in\mathbb{S}^n_+(\gamma)$. Thus, $\|\hat{T}J-\hat{T}J'\|_\infty\leq\beta\|J-J'\|_\infty$.

(iii) and (iv): Since the space $B(\mathbb{S}^n_+(\gamma))$ equipped with the sup norm is a complete metric space, and the operator $\hat{T}$ a contraction mapping, we can again apply the Banach fixed point theorem to obtain the desired results. ∎

B. Comparison of value iteration sequences

Let $J\in B(\mathbb{S}^n_+(\gamma))$ be an arbitrary constant function. In what follows, we analyze several basic properties of the three sequences $J_k=T^kJ$, $\bar{J}_k=\bar{T}^kJ$, and $\hat{J}_k=\hat{T}^kJ$.

Lemma 2: For each $k=0,1,2,...$ and for every $0\preceq P\preceq Q$, we have $J_k(P)\leq J_k(Q)$ and $\hat{J}_k(P)\leq\hat{J}_k(Q)$.

Proof: Here we prove only that $\hat{J}_k(P)\leq\hat{J}_k(Q)$, as the remaining inequality follows by a nearly identical argument. We prove the claim by induction. The claim trivially holds for $k=0$ since $\hat{J}_0$ is an arbitrary constant function by construction. Suppose that $\hat{J}_k(P)\leq\hat{J}_k(Q)$ holds for some $k\geq 0$. Now, for every $0\preceq P'\preceq Q'$ and for every S, we have that $c(P',S)\leq c(Q',S)$ and $\Omega(f(P',S))\leq\Omega(f(Q',S))$. Thus,

$$\begin{aligned}\hat{J}_{k+1}(P')&=\min_S\{c(P',S)+\beta\hat{J}_k(\Omega(f(P',S)))\}\\&\leq\min_S\{c(Q',S)+\beta\hat{J}_k(\Omega(f(Q',S)))\}\\&=\hat{J}_{k+1}(Q').\end{aligned}$$

∎

Proposition 3: For each $k=0,1,2,\dots$ and $P\succeq 0$, we have $J_k(P)\leq\bar{J}_k(P)\leq\hat{J}_k(P)$.

Proof: We again prove the claim by induction. The claim trivially holds for $k=0$ as each $J\in B(\mathbb{S}^n_+(\gamma))$ is an arbitrary constant function. Assume that the claim holds for $k\geq 0$. Then, for each $P'\succeq 0$,

$$\begin{aligned}J_k(P')&\leq J_k(\Theta(P')) &\text{(22a)}\\&\leq\bar{J}_k(\Theta(P')) &\text{(22b)}\\&\leq\hat{J}_k(\Theta(P')) &\text{(22c)}\\&\leq\hat{J}_k(\Omega(P')), &\text{(22d)}\end{aligned}$$

where we first obtain the inequality in (22a) by recalling that $P'\preceq\Theta(P')$ by Lemma 1, which then implies that $J(P')\leq J(\Theta(P'))$ by Lemma 2. The two following inequalities in (22b) and (22c) are each obtained as a result of our induction hypothesis. Finally, we can again invoke the results of Lemmas 1 and 2 to obtain (22d) from (22c). Now, setting $P'=f(P,S)$, we have

$$J_k(f(P,S))\leq\bar{J}_k(\Theta(f(P,S)))\leq\hat{J}_k(\Omega(f(P,S)))$$

for each $P\succeq 0$ and $S\subseteq\{1,\dots,N\}$. Thus,

$$\begin{aligned}&\min_S\{c(P,S)+\beta J_k(f(P,S))\}\\&\quad\leq\min_S\{c(P,S)+\beta\bar{J}_k(\Theta(f(P,S)))\}\\&\quad\leq\min_S\{c(P,S)+\beta\hat{J}_k(\Omega(f(P,S)))\}\end{aligned}$$

which implies that $J_{k+1}(P)\leq\bar{J}_{k+1}(P)\leq\hat{J}_{k+1}(P)$. ∎

C. Upper bound of $\hat{J}_k$

In Lemma 2, we have shown that the function $\hat{J}_k$ is monotonically non-decreasing. Next, we show that the "slope" of $\hat{J}_k$ is bounded. Define a convergent sequence according to

$$L_{k+1}=1+\beta L_k,\quad L_0=0.$$

where $0\leq\beta<1$ is our discount factor. Note that the sequence $L_1,L_2,\dots,L_k,\dots$ is nothing but the sequence of partial sums of a geometric series. Thus, it is straightforward to bound these terms according to $L_k<L:=\frac{1}{1-\beta}$.

Proposition 4: $\hat{J}_k(P+\Delta P)\leq\hat{J}_k(P)+L_k\mathrm{Tr}(\Delta P)$ for every $k=0,1,2,\dots$ and any $P,\Delta P\succeq 0$.

Proof: We prove the claim by induction. The claim trivially holds for $k=0$. Suppose that the claim holds for some $k\geq 0$. For $P'\succeq 0$ and $\Delta P'\succeq 0$, by Lemma 4 (provided in the Appendix), we have that

$$f(P'+\Delta P',S)\preceq f(P',S)+\Delta f \tag{23}$$

for some $\Delta f\succeq 0$ chosen such that $\mathrm{Tr}(\Delta f)\leq\mathrm{Tr}(\Delta P')$. Recalling the monotonicity of $\hat{J}_k$ by Lemma 2, we have

$$\begin{aligned}\hat{J}_k(\Omega(f(P'+\Delta P',S)))&=\hat{J}_k(f(P'+\Delta P',S)+2\epsilon nI)\\&\leq\hat{J}_k(f(P',S)+\Delta f+2\epsilon nI)\\&=\hat{J}_k(\Omega(f(P',S))+\Delta f),\end{aligned}$$

where the first equality is by the definition of $\Omega(\cdot)$, the subsequent inequality is obtained by substituting for the relation in (23), and the final equality is obtained by repackaging the first and third term in $\hat{J}_k(\cdot)$ into the definition of $\Omega(\cdot)$. Using this result,

$$\begin{aligned}&\hat{J}_{k+1}(P'+\Delta P')\\&=\min_S\{c(P',S)+\mathrm{Tr}(\Delta P')\\&\qquad+\beta\hat{J}_k(\Omega(f(P'+\Delta P',S)))\} &\text{(24a)}\\&\leq\min_S\{c(P',S)+\mathrm{Tr}(\Delta P')\\&\qquad+\beta\hat{J}_k(\Omega(f(P',S))+\Delta f)\} &\text{(24b)}\\&\leq\min_S\{c(P',S)+\mathrm{Tr}(\Delta P')\\&\qquad+\beta\hat{J}_k(\Omega(f(P',S)))+\beta L_k\mathrm{Tr}(\Delta f)\} &\text{(24c)}\\&\leq\min_S\{c(P',S)+\beta\hat{J}_k(\Omega(f(P',S)))\\&\qquad+(1+\beta L_k)\mathrm{Tr}(\Delta P')\} &\text{(24d)}\\&=\min_S\{c(P',S)+\beta\hat{J}_k(\Omega(f(P',S)))\}\\&\qquad+L_{k+1}\mathrm{Tr}(\Delta P') &\text{(24e)}\\&=\hat{J}_{k+1}(P')+L_{k+1}\mathrm{Tr}(\Delta P'), &\text{(24f)}\end{aligned}$$

where we have obtained the first equality by noting that $\mathrm{Tr}(P'+\Delta P')=\mathrm{Tr}(P')+\mathrm{Tr}(\Delta P')$ and extracting the latter term from $c(P'+\Delta P',S)$. We then obtain (24b) by applying our previous result. The inequality in (24c) then follows from the induction hypothesis. Recalling that Δf is chosen such that $\mathrm{Tr}(\Delta f)\leq\mathrm{Tr}(\Delta P)$ holds by construction, we substitute for this relation and group the $\mathrm{Tr}(\Delta P)$ terms to obtain (24d).

Noting the definition of L_{k+1} and that $\mathrm{Tr}(\Delta P)$ is independent of our sensor selection S yields (24e). Substituting for the definition of $\hat{J}_k(\cdot)$ then yields (24f). ∎

By Proposition 3, we have $J_k \leq \bar{J}_k \leq \hat{J}_k$. Next, we derive an upper bound of the gap between J_k and $\hat{J}_k$.

Proposition 5: Suppose $\hat{J}_k = \hat{T}^k J_0$ where $J_0 \in B(\mathbb{S}_+^n(\gamma))$ is a constant function. For each $k=0,1,2,\ldots$, we have

$$0 \leq (\hat{T}\hat{J}_k)(P) - (T\hat{J}_k)(P) \leq 2\epsilon n^2 L \quad \forall P \in \mathbb{S}_+^n(\gamma).$$

Proof: The first inequality follows from the monotonicity of $\hat{J}_k$, as proven in Lemmas 1 and 2:

$$\begin{aligned}(T\hat{J}_k)(P) &= \min_S \{c(P,S) + \beta \hat{J}_{k-1}(f(P,S))\} \\ &\leq \min_S \{c(P,S) + \beta \hat{J}_{k-1}(\Omega(f(P,S)))\} \\ &= (\hat{T}\hat{J}_k)(P).\end{aligned}$$

To see the second inequality,

$$\begin{aligned}(\hat{T}\hat{J}_k)(P) &= \min_S \{c(P,S) + \beta \hat{J}_{k-1}(\Omega(f(P,S)))\} && \text{(25a)} \\ &= \min_S \{c(P,S) + \beta \hat{J}_{k-1}(f(P,S) + 2\epsilon n I)\} && \text{(25b)} \\ &\leq \min_S \{c(P,S) + \beta \hat{J}_{k-1}(f(P,S)) \\ &\qquad + L_{k-1}\mathrm{Tr}(2\epsilon n I)\} && \text{(25c)} \\ &\leq \min_S \{c(P,S) + \beta \hat{J}_{k-1}(f(P,S))\} + 2\epsilon n^2 L && \text{(25d)} \\ &= (T\hat{J}_k)(P) + 2\epsilon n^2 L, && \text{(25e)}\end{aligned}$$

where (25b) follows (25a) by the definition of Ω. Then, (25c) follows from the result of Proposition 4. We subsequently obtain (25d) by noting that the final term in (25c) is independent of our sensor selection S and recalling the fact that, for all $k=0,1,2,\ldots$, $L_k \leq L$. Finally, we obtain (25e) by the definition of $(T\hat{J}_k)(P)$. ∎

Proposition 6: Suppose $J_k = T^k J_0$ and $\hat{J}_k = \hat{T}^k J_0$ where $J_0 \in B(\mathbb{S}_+^n(\gamma))$ is a constant function. We have

$$\limsup_{k\to\infty} \|J_k - \hat{J}_k\|_\infty \leq \frac{2\epsilon n^2}{(1-\beta)^2}.$$

Proof: Notice that

$$\begin{aligned}\|J_{k+1} - \hat{J}_{k+1}\|_\infty &= \|TJ_k - \hat{T}\hat{J}_k\|_\infty \\ &= \|TJ_k - T\hat{J}_k + T\hat{J}_k - \hat{T}\hat{J}_k\|_\infty \\ &\leq \|TJ_k - T\hat{J}_k\|_\infty + \|T\hat{J}_k - \hat{T}\hat{J}_k\|_\infty \\ &\leq \beta\|J_k - \hat{J}_k\|_\infty + 2\epsilon n^2 L, && \text{(26)}\end{aligned}$$

where the final inequality is obtained using the fact that T is a contraction mapping, as proven in Proposition 1, to upper bound the first term, and that we can upper bound the second term by the result of Proposition 5. Define a sequence ℓ_k by

$$\ell_{k+1} = \beta \ell_k + 2\epsilon n^2 L \qquad (27)$$

with $\ell_0=0$. We have $\ell_k \leq \frac{2\epsilon n^2 L}{1-\beta} = \frac{2\epsilon n^2}{(1-\beta)^2}$ for $k=0,1,2,\ldots$. Comparing (26) and (27), we have $\|J_k - \hat{J}_k\|_\infty \leq \ell_k$. Thus,

$$\limsup_{k\to\infty} \|J_k - \hat{J}_k\|_\infty \leq \frac{2\epsilon n^2}{(1-\beta)^2},$$

where we have substituted for $L=\frac{1}{1-\beta}$. ∎

D. Main results

We now turn our attention towards the main results of our paper, which are given in the following two theorems.

Theorem 1: Let J^* be the optimal value function characterized by (10). Let $\bar{J}_k = \bar{T}^k J_0$ be the sequence generated by the approximate value iteration with (15) where $J_0 \in B(\mathbb{S}_+^n(\gamma))$ is a constant function. We have

$$\limsup_{k\to\infty} \|J^* - \bar{J}_k\|_\infty \leq \frac{2\epsilon n^2}{(1-\beta)^2}.$$

Proof: Since $0 \leq J_k \leq \bar{J}_k \leq \hat{J}_k$, it follows from Proposition 6 that $\limsup_{k\to\infty} \|J_k - \bar{J}_k\|_\infty \leq \frac{2\epsilon n^2}{(1-\beta)^2}$. Now,

$$\begin{aligned}\limsup_{k\to\infty} \|J^* - \bar{J}_k\|_\infty &\leq \limsup_{k\to\infty} \|J^* - J_k + J_k - \bar{J}_k\|_\infty \\ &\leq \underbrace{\limsup_{k\to\infty} \|J^* - J_k\|}_{=0 \text{ by Proposition 1}} + \limsup_{k\to\infty} \|J_k - \bar{J}_k\|_\infty \\ &= \limsup_{k\to\infty} \|J_k - \bar{J}_k\|_\infty \leq \frac{2\epsilon n^2}{(1-\beta)^2}.\end{aligned}$$

∎

Theorem 2: For all $P \succeq 0$, $J^*(P) \leq J^{\pi'}(P) \leq \bar{J}^*(P)$.

Proof: As in the proof of Theorem 1, set $J_k = T^k J_0$ and $\bar{J}_k = \bar{T}^k J_0$ where $J_0 \in B(\mathbb{S}_+^n(\gamma))$ is a constant function. Define the sequence $J_k^{\pi'}$ by

$$J_{k+1}^{\pi'}(P) = c(P, \pi'(P)) + \beta J_k^{\pi'}(f(P, \pi'(P))).$$

By construction, we have that

$$J_k(P) \leq J_k^{\pi'}(P) \leq \bar{J}_k(P) \;\; \forall P \in \mathbb{S}_+^n(\gamma) \qquad (28)$$

for $k=0$. Now, Suppose (28) holds for some $k \geq 0$. Then,

$$\begin{aligned}J_{k+1}(P) &= \min_S \{c(P,S) + \beta J_k(f(P,S))\} \\ &\leq c(P,\pi'(P)) + \beta J_k(f(P,\pi'(P))) \\ &\leq c(P,\pi'(P)) + \beta J_k^{\pi'}(f(P,\pi'(P))) \\ &= J_{k+1}^{\pi'}(P),\end{aligned}$$

where the second inequality follows from the induction hypothesis. Likewise,

$$\begin{aligned}J_{k+1}^{\pi'}(P) &= c(P,\pi'(P)) + \beta J_k^{\pi'}(f(P,\pi'(P))) \\ &\leq c(P,\pi'(P)) + \beta \bar{J}_k(f(P,\pi'(P))) \\ &= \min_S \{c(P,S) + \beta \bar{J}_k(f(P,S))\} \\ &= \bar{J}_{k+1}(P).\end{aligned}$$

Therefore, (28) holds for every $k=0,1,2,\ldots$. Since J_k, $J_k^{\pi'}$ and $\bar{J}_k$ converge to J^*, $J^{\pi'}$ and $\bar{J}^*$ uniformly, respectively, we have $J^* \leq J^{\pi'} \leq \bar{J}^*$. ∎

Theorems 1 and 2 imply that $\|J^* - J^{\pi'}\|_\infty \leq \frac{2\epsilon n^2}{(1-\beta)^2}$. Therefore, even though the sensor selection policy π' is suboptimal, the performance loss can be made arbitrarily small by selecting a sufficiently small ϵ.

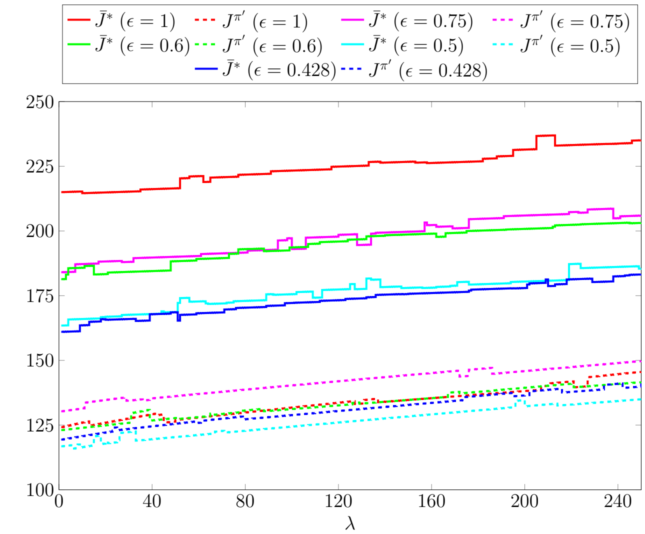

Fig. 1. Value functions $\bar{J}^*(P)$ and $J^{\pi'}(P)$ obtained under $\epsilon = 1, 0.75, 0.6, 0.5$ and $0.428 (= 3/7)$.

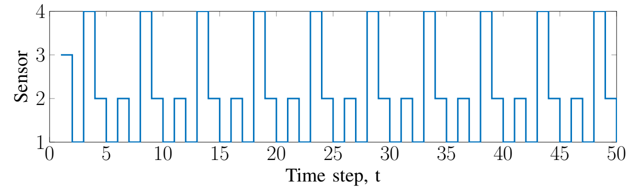

Fig. 2. Sequence of selected sensors when the initial state is $P_0 = 0_{3\times 3}$. Sensor selection policy is computed with $\epsilon = 0.428$.

Method	Sequence	Undiscounted cost
$\epsilon = 0.5$	$\{4, 2, 1\}$	6.4236
$\epsilon = 0.428$	$\{4, 2, 1, 2, 1\}$	6.6941
$\epsilon = 0.6, \epsilon = 1$	$\{2, 2, 1\}$	6.8377
$\epsilon = 0.75$	$\{2, 2, 2, 1\}$	7.3532
[4]	$\{4, 1, 4, 2, 1, 2, 3\}$	6.9404

TABLE II

REPEATING SEQUENCE AND CORRESPONDING UNDISCOUNTED COST.

VI. NUMERICAL EXPERIMENTS

In this section, we perform numerical experiments for the approximate value iteration of the form

$$\bar{J}_{k+1}(P) = (\bar{T}\bar{J}_k)(P) \quad \forall P \in \mathbb{M}(\gamma), k = 0, 1, 2, \ldots \tag{29}$$

where $\bar{T}$ is the Bellman operator defined by (15). Notice that the computation of the right hand side of (29) for each $P \in \mathbb{M}(\gamma)$ is independent of all other $P' \in \mathbb{M}(\gamma)$ and hence is massively parallelizable. We implement (29) on GPUs via the CUDA-enabled MATLAB interface. Through this implementation, the value updates computed according to (29) take under a second per iteration for meshes with cardinalities $|\mathbb{M}(\gamma)|$ of up to several million.

In this example, we revisit the sensor selection problem of a 3D process with four available sensors previously studied in [4]. The system parameters are given by

$$A = \begin{bmatrix} -0.6 & 0.8 & 0.5 \\ -0.1 & 1.5 & -1.1 \\ 1.1 & 0.4 & -0.2 \end{bmatrix}, \; W = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0.75 & -0.2 & -0.65 \\ 0.35 & 0.85 & 0.35 \\ 0.2 & -0.65 & 1.25 \\ 0.7 & 0.5 & 0.5 \end{bmatrix}, \; V = \begin{bmatrix} 0.53 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix}.$$

We adopt the cost function defined in (7), which permits the selection of exactly one active sensor at each time step. We additionally use a discount factor of $\beta{=}0.95$ and set the trace limit to $\gamma{=}15$. To study the impact of the mesh grid size on the resulting value functions, we perform the value iteration (29) with $\epsilon{=}1, 0.75, 0.6, 0.5$ and $0.428 (= 3/7)$. For each value of ϵ, it was observed that 500 iterations of (29) was sufficient to achieve convergence. For each case, we obtained $\bar{J}^*$ and the corresponding policy π' through (18). Fig. 1 displays $\bar{J}^*(P(\lambda))$ evaluated at $P(\lambda){=}0.01\lambda I_3, \lambda{=}1, 2, \ldots, 250$ for each value of ϵ considered. Furthermore, we estimate $J^{\pi'}(P(\lambda))$ by simulating the trajectories P_t, starting with $P_0{=}P(\lambda)$. Fig. 1 confirms that $J^{\pi'}(P){\le}\bar{J}^*(P)$ (Theorem 2) for each value of ϵ considered.

Although the exact value function $J^*(P)$ is not computable, Fig. 1 indicates that a smaller ϵ tends to provide a tighter upper bound $\bar{J}^*(P)$. This relationship holds even if the gap $2\epsilon n^2/(1-\beta)^2$, as obtained in Theorem 1, is too conservative (note that in the case of $\epsilon{=}0.5$, we have that $2\epsilon n^2/(1{-}\beta)^2{=}3600$).

We observe that the sequence of selected sensors under the obtained sensor selection policy always eventually exhibits a periodic behavior. When $\epsilon{=}0.428$, the sequence of selected sensors eventually becomes a repetition of $\{4, 2, 1, 2, 1\}$ regardless of the initial covariance $P(\lambda){=}0.01\lambda I_3, \lambda{=}1, 2, ..., 250$ (Fig. 2). We also observe that different choices of ϵ yield different periodic sequences (Table II). Interestingly, the sequences we obtained are different from the repeating sequence of $\{4, 1, 4, 2, 1, 2, 3\}$ obtained in [4]. Although [4] considered an undiscounted problem (where the performance is evaluated by $\lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} \mathrm{Tr}(P_t)$), it is noteworthy that some of the sequences we obtain outperform the solution obtained in [4] even under an undiscounted setting, as shown in Table II. Our simulation study thus shows that the sensor selection policy obtained by the proposed method performs well in practice, even though the suboptimality guarantee provided by Theorem 1 is conservative.

VII. CONCLUSION AND FUTURE WORK

We considered the problem of synthesizing an optimal sensor selection policy for linear-Gaussian systems. We approached this problem by formulating the sensor selection problem in terms of a value iteration over the continuous space of covariance matrices. To obtain a computationally tractable synthesis procedure, we then developed a point-based value iteration approach over a prespecified, finite set of covariance matrices, and subsequently derived bounds for the suboptimality of the obtained sensor selection policy.

There are several natural extensions of this work. Perhaps the most urgent research direction is to develop methods that efficiently construct the finite set used in the point-based value iteration procedure, as naively using a "mesh grid" yields an impractically large state space for systems with higher-dimensional state spaces. In a similar vein, further work should also consider how to obtain tighter bounds on the suboptimality of the synthesized sensor selection policy. Although the proposed procedure worked comparatively well for the example considered, the obtained bound was exceedingly conservative.

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APPENDIX

A. Supplementary results

Lemma 3: For $M \succ 0$, $N \succ 0$, $X \succeq 0$ and $A \in \mathbb{R}^{n \times n}$, the following inequality holds:

$$\begin{aligned}&((M + AXA^\top)^{-1} + N)^{-1}\\ &\preceq (M^{-1} + N)^{-1}\\ &\qquad + (M^{-1} + N)^{-1} M^{-1} AXA^\top M^{-1} (M^{-1} + N)^{-1}.\end{aligned}$$

Moreover, if A is Schur stable, we have

$$\mathrm{Tr}((M^{-1}+N)^{-1} M^{-1} AXA^\top M^{-1} (M^{-1}+N)^{-1}) \leq \mathrm{Tr}(X).$$

Proof: First, recall the general matrix inversion lemma:

$$(A+BCB^\top)^{-1} = A^{-1} - A^{-1}B(B^\top A^{-1} B + C^{-1})^{-1} B^\top A^{-1} \tag{30}$$

Now, set $F := X^{\frac{1}{2}} A^\top$. Then one can show that

$$\begin{aligned}&((M + AXA^\top)^{-1} + N)^{-1} &(31)\\ &= ((M + F^\top F)^{-1} + N)^{-1} &(32)\\ &= (M^{-1} + N - M^{-1}F^\top (FM^{-1}F^\top + I)^{-1} FM^{-1})^{-1} &(33)\\ &= (M^{-1} + N)^{-1}\\ &\quad + (M^{-1} + N)^{-1} M^{-1} F^\top\\ &\qquad \times (I + FM^{-1}(M - (M^{-1} + N)^{-1}) M^{-1} F^\top)^{-1}\\ &\qquad \times FM^{-1}(M^{-1} + N)^{-1} &(34)\\ &\preceq (M^{-1} + N)^{-1}\\ &\quad + (M^{-1} + N)^{-1} M^{-1} F^\top F M^{-1} (M^{-1} + N)^{-1} &(35)\\ &= (M^{-1} + N)^{-1}\\ &\quad + (M^{-1} + N)^{-1} M^{-1} AXA^\top M^{-1} (M^{-1} + N)^{-1} &(36)\end{aligned}$$

where (32) follows (31) by simply substituting our relation for F. We then obtain (33) by applying the general matrix inversion lemma in (30) to (32) wherein we use $A = M$, $B = F$, and $C = I$ (where I is the identity matrix). To obtain (34), we again apply the general form of the matrix inversion lemma, this time setting $A = (M^{-1} + N)$, $B = FM^{-1}$, and $C = -(FM^{-1}F^\top + I)^{-1}$, and subsequently rearranging terms. The following inequality in (35) is obtained by noting that $M - (M^{-1} + N)^{-1} \succeq 0$, which implies $(I + FM^{-1}(M - (M^{-1} + N)^{-1})M^{-1}F^\top)^{-1} \preceq I^{-1} = I$. Substituting I yields the result. Finally, we obtain (36) by substituting back in our relation for F. To show the second claim, first notice that all the eigenvalues of the matrix

$Z := M^{\frac{1}{2}}(I + M^{\frac{1}{2}}NM^{\frac{1}{2}})^{-1}M^{-\frac{1}{2}}$ satisfy $0 < \lambda_i(Z) < 1$. Thus,

$$\begin{aligned}
&0 \prec M^{-1}(M^{-1}+N)^{-2}M^{-1}\\
&\quad = M^{-\frac{1}{2}}(I + M^{\frac{1}{2}}NM^{\frac{1}{2}})^{-1}M^{\frac{1}{2}}M^{\frac{1}{2}}(I + M^{\frac{1}{2}}NM^{\frac{1}{2}})^{-1}M^{-\frac{1}{2}}\\
&\quad = Z^\top Z \prec I.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\mathrm{Tr}((M^{-1}+N)^{-1}M^{-1}AXA^\top M^{-1}(M^{-1}+N)^{-1})\\
&\leq \mathrm{Tr}(M^{-1}(M^{-1}+N)^{-2}M^{-1}AXA^\top)\\
&\leq \mathrm{Tr}(AXA^\top)\\
&\leq \mathrm{Tr}(X).
\end{aligned}$$

■

Lemma 4: Assume $A \in \mathbb{R}^{n\times n}$ is Schur stable. For every $P \succeq 0$, $\Delta P \succeq 0$ and S, we have $f(P+\Delta P) \preceq f(P,S)+\Delta f$ where $\Delta f \succeq 0$ and $\mathrm{Tr}(\Delta f) \leq \mathrm{Tr}(\Delta P)$.

Proof: This result follows from Lemma 3 by setting $M = APA^\top + W$, $N = C_S^\top V_S^{-1} C_S$, $X = \Delta P$ and $\Delta f = (M^{-1}+N)^{-1}M^{-1}AXA^\top M^{-1}(M^{-1}+N)^{-1}$. ■