
Long-Term Fairness with Unknown Dynamics

A PREPRINT

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June 9, 2023

ABSTRACT

While machine learning can myopically reinforce social inequalities, it may also be used to dynamically seek equitable outcomes. In this paper, we formalize long-term fairness in the context of online reinforcement learning. This formulation can accommodate dynamical control objectives, such as driving equity inherent in the *state* of a population, that cannot be incorporated into static formulations of fairness. We demonstrate that this framing allows an algorithm to adapt to unknown dynamics by sacrificing short-term incentives to drive a classifier-population system towards more desirable equilibria. For the proposed setting, we develop an algorithm that adapts recent work in online learning. We prove that this algorithm achieves simultaneous probabilistic bounds on cumulative loss and cumulative violations of fairness (as statistical regularities between demographic groups). We compare our proposed algorithm to the repeated retraining of myopic classifiers, as a baseline, and to a deep reinforcement learning algorithm that lacks safety guarantees. Our experiments model human populations according to evolutionary game theory and integrate real-world datasets.

1 Introduction

As machine learning (ML) algorithms are deployed for tasks with real-world social consequences (e.g., school admissions, loan approval, medical interventions, etc.), the possibility exists for runaway social inequalities (Crawford and Calo, 2016; Chaney et al., 2018; Fuster et al., 2018; Ensign et al., 2018). While “fairness” has become a salient ethical concern in contemporary research, the closed-loop dynamics of real-world systems comprising ML policies and populations that mutually adapt to each other (Fig. 5 in the supplementary material) remain poorly understood.

In this paper, our primary contribution is to consider the problem of *long-term fairness*, or algorithmic fairness in the context of a dynamically responsive population, as a reinforcement learning (RL) problem subject to constraint. In our formulation, the central learning task is to develop a policy that minimizes cumulative loss (e.g., financial risk, negative educational outcomes, misdiagnoses, etc.) incurred by an ML agent interacting with a human population up to a finite time horizon, subject to constraints on cumulative “violations of fairness”, which we refer to in a single time step as *disparity* and cumulatively as *distortion*.

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Our central hypothesis is that an RL formulation of long-term fairness may allow an agent to learn to **sacrifice short-term utility in order to drive the system towards more desirable equilibria**. The core practical difficulties posed by our general problem formulation, however, are the potentially unknown dynamics of the system under control, which must be determined by the RL agent online (i.e., during actual deployment), and the general non-convexity of the losses or constraints considered. Additionally, we address continuous state and action spaces, in general, which preclude familiar methods with performance guarantees in discrete settings.

Our secondary contributions are 1) to show that long-term fairness can be solved within asymptotic, probabilistic bounds under certain dynamical assumptions and 2) to demonstrate that the problem of long-term fairness can also be addressed more flexibly. For theoretical guarantees, we develop L-UCBFair, an online RL method, and prove sublinear bounds on regret (suboptimality of cumulative loss) and distortion (suboptimality of cumulative disparity) with high probability (Section 3.1). To demonstrate practical solutions without limiting assumptions, we apply R-TD3, an adaptation of a well-known deep reinforcement learning method (viz., TD3) to a Lagrangian relaxation of the central problem with time-dependent regularization. We compare L-UCBFair and R-TD3 to a baseline, myopic policy in interaction with simulated populations initialized with synthetic or real-world data and updated according to evolutionary game theory (Section 4).

This paper considers fairness in terms of statistical regularities across (ideally) socioculturally meaningful *groups*. Acknowledging that internal conflict exists between different statistical measures of fairness, we show that an RL approach to long-term fairness can mitigate trade-offs between fairness defined on the statistics of immediate policy decision *outcomes* (Chen et al., 2022), (e.g., acceptance rate disparities (Dwork et al., 2012; Zemel et al., 2013; Feldman et al., 2015)) and underlying distributional parameters (e.g., qualification rate (Raab and Liu, 2021; Zhang et al., 2020)).

1.1 Related Work

Our effort to formalize long-term fairness as a reinforcement learning problem bridges recent work on “fairness in machine learning”, which has developed in response to the proliferation of data-driven methods in society, and “safe reinforcement learning”, which seeks theoretical safety guarantees in the control of dynamical systems.

Dynamics of Fairness in Machine Learning We distinguish long-term fairness from the dynamics of fair allocation problems Joseph et al. (2016); Jabbari et al. (2017); Tang et al. (2021); Liu et al. (2017) and emphasize side-effects of algorithmic decisions affecting future decision problems. By formalizing long-term fairness in terms of cumulative losses and disparities, we iterate on a developing research trend that accounts for the dynamical response of a human population to deployed algorithmic prediction: both as a singular reaction Liu et al. (2018); Hu et al. (2019); Perdomo et al. (2020) or as a sequence of mutual updates to the population and the algorithm Coate and Loury (1993); D’Amour et al. (2020); Zhang et al. (2020); Heidari et al. (2019); Wen et al. (2019); Liu et al. (2020); Hu and Chen (2018); Mouzannar et al. (2019); Williams and Kolter (2019); Raab and Liu (2021). In particular, Perdomo et al. (2020) introduces the concept of “performative prediction”, analyzing the fixed-points of interactions between a population and an algorithmic classifier, but with state treated as a pure function of a classifier’s actions. For more realistic dynamics, Mouzannar et al. (2019) and Raab and Liu (2021) model updates to qualification rates that depend on both previous state and the classifier’s actions, but only treat myopic classifiers that optimize immediate utility (subject to fairness constraints) rather than learning to anticipate dynamical population responses. Wen et al. (2019) adopted reinforcement learning to achieve long-term fairness. However, they explore a setting that is restricted to discrete state and action spaces. In particular, this recommends a tabular explore-then-commit approach to reinforcement learning that does not generalize to continuous spaces and thus cannot be directly used in our setting. In addition, we provide separate and tighter bounds for both utility and disparity.

Safe Reinforcement Learning L-UCBFair furthers recent efforts in safe RL. While “model-based” approaches, in which the algorithm learns an explicit dynamical model of the environment, constitute one thread of prior work Efroni et al. (2020); Singh et al. (2020); Brantley et al. (2020); Zheng and Ratliff (2020); Kalagarla et al. (2021); Liu et al. (2021); Ding et al. (2021), such algorithms are typified by significant time and space complexity. Among “model-free” algorithms, the unknown dynamics of our setting preclude the use of a simulator that can generate arbitrary state-action pairs Xu et al. (2021); Ding et al. (2020); Bai et al. (2022). While Wei et al. (2022) introduce a model-free and simulator-free algorithm, the tabular setting considered is only applicable to discrete state and action spaces. To tackle continuous state space, Ding et al. (2021); Ghosh et al. (2022) consider linear dynamics: Ding et al. (2021) develop a primal-dual algorithm with safe exploration, and Ghosh et al. (2022) use a softmax policy design. Both algorithms

are based on the work of Jin et al. (2020), which proposed a least squares value iteration method, using an Upper Confidence Bound (UCB) Auer et al. (2002) to estimate a state-action “ Q ” function. To our knowledge, L-UCBFair is the first model-free, simulator-free RL algorithm that provides theoretical safety guarantees for both discrete and **continuous** state and action spaces. Moreover, L-UCBFair achieves bounds on regret and distortion as tight as any algorithm thus far with discrete action space Ghosh et al. (2022).

2 Problem Formulation

A binary classification task is an initial motivating example for our formulation of long-term fairness, though the formal problem we propose is more widely applicable. To this initial task, we introduce “fairness” constraints, then population dynamics, and then cumulative loss and “disparity”, before formalizing the problem of optimizing cumulative loss subject to constraints on cumulative disparity.

We provide notation for the binary classification task: a random individual, sampled i.i.d. from a population, has features $X \in \mathbf{R}^d$, a label $Y \in \{-1, 1\}$, and a demographic group $G \in \mathcal{G}$ (where $\mathcal{G} = [n]$ for $n \geq 2$). Denote the joint distribution of these variables in the population as $s := \Pr(X, Y, G)$. The task is to predict Y (as $\hat{Y}$) from X and G . Specifically, the task is to choose a classifier a , such that $\hat{Y} \sim a(X, G)$, that minimizes some bounded loss $\mathcal{L} \in [0, 1]$ over s . This **basic classification task** is

$$\min_a \mathcal{L}(s, a)$$

Typically, $\mathcal{L}$ corresponds to the expectation value of a loss function L (such as zero-one-loss) defined for individuals drawn from s . We consider this example, but, in general, allow arbitrary, (unit-interval) bounded loss functions $\mathcal{L}$:

$$\mathcal{L}(s, a) \stackrel{\text{e.g.}}{=} \mathbb{E}_{\substack{X, Y, G \sim s \\ \hat{Y} \sim a(X, G)}} [L(Y, \hat{Y})]$$

The standard “**fair**” **classification task** (without a dynamically responsive population) is to constrain classifier a such that the *disparity* $\mathcal{D} \in [0, 1]$ induced on distribution s by a is bounded by some value $c \in [0, 1]$:

$$\min_a \mathcal{L}(s, a) \quad \text{subject to} \quad \mathcal{D}(s, a) \leq c$$

A standard example of disparity is the expected divergence of group acceptance rates β . e.g., when $\mathcal{G} = \{g_1, g_2\}$,

$$\mathcal{D}(s, a) \stackrel{\text{e.g.}}{=} |\beta_{s,a}(g_1) - \beta_{s,a}(g_2)|^2$$

where $\beta_{s,a}(g) := \mathbb{Pr}_{\substack{X, Y, G \sim s \\ \hat{Y} \sim a(X, G)}} (\hat{Y}=1 \mid G=g)$

In this example, $\mathcal{D}$ measures the violation of “demographic parity”: the condition under which a classifier achieves group-independent positive classification rates $\forall g, \Pr(\hat{Y}=1 \mid G=g) = \Pr(\hat{Y}=1)$ Dwork et al. (2012). In this paper, we also consider measures of fairness based on inherent population statistics (e.g., parity of group qualification rates $\Pr(Y=1 \mid G=g)$), which must be driven dynamically Raab and Liu (2021); Zhang et al. (2020). Such notions of disparity are well-suited to an RL formulation of long-term fairness.

State, Action, and Policy Considering the semantics of iterated classification tasks, we identify the distribution $s \in \mathcal{S}$ of individuals in the population as a *state* and the classifier $a \in \mathcal{A}$ as an *action*. While state space $\mathcal{S}$ is arbitrary, we assume that action space $\mathcal{A}$ admits a Euclidean metric, under which it is closed (i.e., $\mathcal{A}$ is isomorphic to $[0, 1]^m, m \in \mathbf{Z}_{>0}$). At a given time τ , a_τ is sampled stochastically according to the current *policy* π_τ : $a_\tau \sim \pi_\tau(s_\tau)$. We assume s_τ is fully observable at time τ . In practice, s_τ must be approximated from finitely many empirical samples, though this caveat introduces well-understood errors that vanish in the limit of infinitely many samples.

Dynamics In contrast to a “one-shot” fair classification task, we assume that a population may react to classification, inducing the distribution s to change. Importantly, such “distribution shift” is a well-known, real-world phenomenon that can increase realized loss and disparity when deployed classification policies are fixed Chen et al. (2022). For classification policies that free to change in response to a mutating distribution s , subsequent classification tasks depend on the (stochastic) predictions made in previous tasks. In our formulation, we assume the existence of dynamical kernel $\mathbf{P}$ that maps a state s and action a at time τ to a *distribution over* possible states at time $\tau + 1$:

$$s_{\tau+1} \sim \mathbf{P}(s_{\tau}, a_{\tau}) \quad (1)$$

We stipulate that $\mathbf{P}$ may be initially unknown, but it does not explicitly depend on time and may be reasonably approximated “online”. While real-world dynamics may depend on information other than the current distribution $\Pr(X, Y, G)$ (e.g., exogenous parameters, history, or additional variables of state), we identify s with the current distribution for simplicity in our treatment of long-term fairness.

Reward and Utility Because standard RL literature motivates *maximizing reward* rather than *minimizing loss*, let us define the instantaneous reward $r \in [0, 1]$ and a separate, instantaneous “utility” $g \in [0, 1]$ for an RL agent as

$$r(s_{\tau}, a_{\tau}) := 1 - \mathcal{L}(s_{\tau}, a_{\tau}) \quad (2)$$

$$g(s_{\tau}, a_{\tau}) := 1 - \mathcal{D}(s_{\tau}, a_{\tau}) \quad (3)$$

where r and g do not explicitly depend on time τ .

Value and Quality Functions Learnable dynamics inspire us to optimize anticipated *cumulative* reward, given constraints on anticipated *cumulative* utility. Let j represent either reward r or utility g . We use the letter V (for “value”) to denote the future expected accumulation of j over steps $[h, \dots, H]$ (without time-discounting) starting from state s , using policy π . Likewise, we denote the “quality” of an action a in state s with the letter Q . For $j \in \{r, g\}$,

$$V_{j,h}^{\pi}(s) = \mathbb{E} \left[\sum_{\tau=h}^H j(s_{\tau}, a_{\tau}) \mid s_h = s \right] \quad (4)$$

$$Q_{j,h}^{\pi}(s, a) = \mathbb{E} \left[\sum_{\tau=h}^H j(s_{\tau}, a_{\tau}) \mid s_h = s, a_h = a \right] \quad (5)$$

By the boundedness of $r, g \in [0, 1]$, V and Q belong to the interval $[0, H - h + 1]$.

“Long-term fairness” via reinforcement learning The central problem explored in this paper is

$$\max_{\pi} V_{r,1}^{\pi}(s) \quad \text{subject to} \quad V_{g,1}^{\pi}(s) \geq \tilde{c} \quad (6)$$

We emphasize that this construction of long-term fairness considers a finite time horizon of H steps and denote the optimal value of π as π^* .

Assumption 2.1 (Slater’s Condition). i.e., “strict primal feasibility”: $\exists \gamma > 0, \bar{\pi}$, such that $V_{g,1}^{\bar{\pi}}(s) \geq \tilde{c} + \gamma$.

Slater’s condition is also adopted in [Efroni et al. \(2020\)](#); [Ding et al. \(2021\)](#); [Ghosh et al. \(2022\)](#).

The Online Setting Given initially unknown dynamics, we formulate long-term fairness for the “online” setting, in which learning is only possible through actual “live” deployment of policy, rather than through simulation. As it is not possible to unconditionally guarantee constraint satisfaction in [Eq. \(6\)](#) over a finite number of episodes, we instead measure two types of *regret*: one that measures the suboptimality of a policy with respect to cumulative incurred loss, which we will continue to call “regret”, and one that measures the suboptimality of a policy with respect to cumulative induced disparity, which we will call “distortion”. Note that we define regret and distortion in [Eq. \(7\)](#) by marginalizing over the stochasticity of state transitions and the sampling of actions:

$$\begin{aligned} \text{Regret}(\pi, s_1) &:= V_{r,1}^{\pi^*}(s_1) - V_{r,1}^{\pi}(s_1) \\ \text{Distortion}(\pi, s_1) &:= \max [0, \tilde{c} - V_{g,1}^{\pi}(s_1)] \end{aligned} \quad (7)$$

3 Algorithms and Analysis

We show that it is possible to provide guarantees for long-term fairness in the online setting. Specifically, we develop an algorithm, L-UCBFair, and prove that it provides probabilistic, sublinear bounds for regret and distortion under a linear MDP assumption ([Assumption 3.2](#)) and properly chosen parameters ([Appendix A.1](#), in the supplementary material). L-UCBFair is the first model-free algorithm to provide such guarantees in the online setting with continuous state and action spaces.

3.1 L-UCBFair

Episodic MDP L-UCBFair inherits from a family of algorithms that treat an episodic Markov decision process (MDP) Jin et al. (2020). Therefore, we first map the long-term fairness problem to the episodic MDP setting, which we denote as $\text{MDP}(\mathcal{S}, \mathcal{A}, H, \mathbf{P}, \mathcal{L}, \mathcal{D})$. The algorithm runs for K episodes, each consisting of H time steps. At the beginning of each episode, which we index with k , the agent commits to a sequence of policies $\pi^k = (\pi_1^k, \pi_2^k, \dots, \pi_H^k)$ for the next H steps. At each step h within an episode, an action $a_h^k \in \mathcal{A}$ is sampled according to policy π_h^k , then the state s_{h+1}^k is sampled according to the transition kernel $\mathbf{P}(s_{h+1}^k, a_h^k)$. $\mathcal{L}$ and $\mathcal{D}$ are the loss and disparity functions. s_1^k is sampled arbitrarily with each episode.

Episodic Regret Because L-UCBFair predetermines its policy for an entire episode, we amend our definition of regret and distortion for the all HK time steps by breaking it into a sum over K episodes of length H .

$$\begin{aligned} \text{Regret}(K) &= \sum_{k=1}^K \left(V_{r,1}^{\pi^*}(s_1^k) - V_{r,1}^{\pi^k}(s_1^k) \right) \\ \text{Distortion}(K) &= \max \left[0, \sum_{k=1}^K \left(\tilde{c} - V_{g,1}^{\pi^k}(s_1^k) \right) \right] \end{aligned} \quad (8)$$

The Lagrangian Consider the Lagrangian function $\mathcal{L}$ associated with Eq. (6), with dual variable $\nu \geq 0$:

$$\mathcal{L}(\pi, \nu) := V_{r,1}^{\pi}(s) + \nu (V_{g,1}^{\pi}(s) - \tilde{c}) \quad (9)$$

L-UCBFair approximately solves the primal problem $\max_{\pi} \min_{\nu} \mathcal{L}(\pi, \nu)$, which is non-trivial, since the objective function is seldom concave in practical parameterizations of π . Let ν^* to denote the optimal value of ν .

Lemma 3.1 (Boundedness of ν^*). For $\bar{\pi}$ and $\gamma > 0$ satisfying Slater's Condition (Assumption 2.1),

$$\nu^* \leq \frac{V_{r,1}^{\bar{\pi}}(s_1) - V_{r,1}^{\bar{\pi}}(s_1)}{\gamma} \leq \frac{H}{\gamma} := \mathcal{V}$$

Lemma 3.1, defines $H/\gamma = \mathcal{V}$ as an upper bound for the optimal dual variable ν^* . $\mathcal{V}$ is an input to L-UCBFair.

Assumption 3.2 (Linear MDP). $\text{MDP}(\mathcal{S}, \mathcal{A}, H, \mathbf{P}, \mathcal{L}, \mathcal{D})$ is a linear MDP with feature map $\phi : \mathcal{S} \times \mathcal{A} \rightarrow \mathbb{R}^d$. For any h , there exist d signed measures $\mu_h = \{\mu_h^1, \dots, \mu_h^d\}$ over $\mathcal{S}$, such that, for any $(s, a, s') \in \mathcal{S} \times \mathcal{A} \times \mathcal{S}$,

$$\mathbb{P}_h(s' | s, a) = \langle \phi(s, a), \mu_h(s') \rangle,$$

In addition, there exist vectors $\theta_{r,h}, \theta_{g,h} \in \mathbb{R}^d$, such that, for any $(s, a) \in \mathcal{S} \times \mathcal{A}$,

$$r(s, a) = \langle \phi(s, a), \theta_{r,h} \rangle; \quad g(s, a) = \langle \phi(s, a), \theta_{g,h} \rangle$$

Assumption 3.2 addresses the curse of dimensionality when state space $\mathcal{S}$ is the space of distributions over X, Y, G . This assumption is also used in Jin et al. (2020); Ghosh et al. (2022), with a similar assumption made in Ding et al. (2021).

3.1.1 Explicit Construction

L-UCBFair, or “LSVI-UCB for Fairness” (Algorithm 1) is based on an optimistic modification of a Least-Squares Value Iteration, where optimism is realized by an Upper-Confidence Bound, as in LSVI-UCB Jin et al. (2020). For each H -step episode k , L-UCBFair maintains estimates for Q_r^k, Q_g^k and a time-indexed policy π^k . In each episode k , L-UCBFair updates Q_r^k, Q_g^k , interacts with the environment, and updates the dual variable ν_k (constant in k).

LSVI-UCB Jin et al. (2020) The estimation of Q is challenging. Consider the Bellman equation: $\forall (s, a) \in \mathcal{S} \times \mathcal{A}$,

$$Q_h^*(s, a) \leftarrow \left[r_h + \mathbb{E}_{s' \sim \mathbf{P}_h(\cdot | s, a)} \max_{a' \in \mathcal{A}} Q_{h+1}^*(s', a') \right] (s, a)$$

It is impossible to iterate over all s, a pairs, since $\mathcal{S}$ and $\mathcal{A}$ are continuous. Instead, LSVI parameterizes $Q_h^*(s, a)$ by the linear form $\mathbf{w}_h^\top \phi(s, a)$, which is supported by Jin et al. (2020). As an additional complication, $\mathbf{P}$ is unknown is estimated by LSVI from empirical samples. Combining these techniques, the Bellman equation can be replaced by a least squares problem.

$$\mathbf{w}_h \leftarrow \underset{\mathbf{w} \in \mathbb{R}^d}{\operatorname{argmin}} \sum_{\tau=1}^{k-1} \left[r_h(s_h^\tau, a_h^\tau) + \max_{a \in \mathcal{A}} Q_{h+1}(s_{h+1}^\tau, a) - \mathbf{w}^\top \phi(s_h^\tau, a_h^\tau) \right]^2 + \varsigma \|\mathbf{w}\|^2$$

In addition, a “bonus term” $\beta (\phi^\top \Lambda_h^{-1} \phi)^{1/2}$ is added to the estimate of Q to encourage exploration.

Adaptive Search Policy Compared to the works of Ding et al. (2021) and Ghosh et al. (2022), the major challenge we face for long-term fairness is a continuous action space $\mathcal{A}$, which renders the independent computation of Q_r^k, Q_g^k for each action impossible. To handle this issue, we propose an adaptive search policy: Instead of drawing an action directly from a distribution over continuous values, π represents a distribution over finitely many, finite regions of $\mathcal{A}$. After sampling a region from a Softmax scheme, the agent draws action a uniformly at random from it.

Definition 3.3. Given a set of distinct actions $I = \{I_0, \dots, I_M\} \subset \mathcal{A}$, where $\mathcal{A}$ is a closed set in Euclidean space, define $\mathcal{I}_i = \{a: \|a - I_i\|_2 \leq \|a - I_j\|_2, \forall i < j\}$ as the subset of actions closer to I_i than to I_j , i.e., the Voronoi region corresponding to locus I_i , with tie-breaking imposed by the order of indices i . Also define the locus function $I(a) = \min_i \operatorname{argmin}_{I_i} \|a - I_i\|_2$.

Lemma 3.4. The Voronoi partitioning described above satisfies $\mathcal{I}_i \cap \mathcal{I}_j = \emptyset, \forall i \neq j$ and $\cup_{i=1}^M \mathcal{I}_i = \mathcal{A}$.

Theorem 3.5. If the number M of distinct loci or regions partitioning $\mathcal{A}$ is sufficiently large, there exists a set of loci I such that $\forall a \in \mathcal{I}_i, i \in M, \|a - I_i\|_2 \leq \epsilon_I$.

In addition to defining a Voronoi partitioning of the action space for the adaptive search policy of L-UCBFair, we make the following assumption:

Assumption 3.6. There exists $\rho > 0$, such that $\|\phi(s, a) - \phi(s, a')\|_2 \leq \rho \|a - a'\|_2$.

This assumption is used to bound the distance of estimated action-value function and value function, thus bound adaptive search policy and the optimal policy.

Dual Update For L-UCBFair, the update method for the dual variable ν in Eq. (9) is also essential. Since $V_{r,1}^\pi(s)$ and $V_{g,1}^\pi(s)$ are unknown, we use $V_{r,1}^k(s)$ and $V_{g,1}^k(s)$ to estimate them. ν is iteratively updated by minimizing Eq. (9) with step-size η , and $\mathcal{V}$ is an upper bound for ν . A similar method is also used in Ding et al. (2021); Ghosh et al. (2022).

3.1.2 Analysis

We next bound the regret and distortion, defined in Eq. (8), realizable by L-UCBFair. We then compare L-UCBFair with existing algorithms for discrete action spaces and discuss the importance of the number of regions M and the distance ϵ_I .

Theorem 3.7 (Boundedness). With probability $1 - p$, there exists a constant b such that L-UCBFair achieves

$$\text{Regret}(K) \leq (b\zeta H^2 \sqrt{d^3} + (\mathcal{V} + 1)H) \sqrt{K} \quad (10)$$

$$\text{Distortion}(K) \leq \frac{b\zeta(1 + \mathcal{V})H^2 \sqrt{d^3}}{\mathcal{V}} \sqrt{K} \quad (11)$$

when selecting parameter values $\varsigma=1, \epsilon_I \leq \frac{1}{2\rho(1+\mathcal{V})KH\sqrt{d}}, \zeta = \log(\log(M)4dHK/p)$, and $\beta = \mathcal{O}(dH\sqrt{\varsigma})$.

For a detailed proof of Theorem 3.7, refer to Appendix A.1. Theorem 3.7 indicates that L-UCBFair reaches $\tilde{\mathcal{O}}(H^2 \sqrt{d^3 K})$ bounds for both regret and distortion with high probability. Compared to the algorithms introduced by Ding et al. (2021); Ghosh et al. (2022), which work with discrete action space, L-UCBFair guarantees the same asymptotic bounds on regret and distortion.

Algorithm 1 L-UCBFair

Input: A set of points $\{I_0, I_1, \dots, I_M\}$ satisfy [Definition 3.3](#). $\epsilon_I = \frac{1}{2\rho(1+\chi)KH}$.
 $\nu_1=0$. $w_{r,h}=w_{g,h}=0$. $\alpha=\frac{\log(M)K}{2(1+\chi+H)}$. $\eta=\chi/\sqrt{KH^2}$. $\beta=C_1dH\sqrt{\log(4\log MdT/p)}$, $\varsigma = 1$.
for episode $k = 1, 2, \dots, K$ **do**
 Receive the initial state s_1^k .
 for step $h = H, H-1, \dots, 1$ **do**
 $\Lambda_h^k \leftarrow \sum_{\tau=1}^{k-1} \phi(s_h^\tau, a_h^\tau) \phi(s_h^\tau, a_h^\tau)^T + \varsigma \mathbf{I}$
 for $j \in \{r, g\}$ **do**
 $w_{j,h}^k \leftarrow (\Lambda_h^k)^{-1} \left[\sum_{\tau=1}^{k-1} \phi(s_h^\tau, a_h^\tau) (j(s_h^\tau, a_h^\tau) + V_{j,h+1}^k(s_{h+1}^\tau)) \right]$
 end for
 for iteration $i = 1, \dots, M$ and index $j \in \{r, g\}$ **do**
 $\xi_{i,j} \leftarrow \left(\phi(\cdot, I_i)^T (\Lambda_h^k)^{-1} \phi(\cdot, I_i) \right)^{1/2}$, $Q_{j,h}^k(\cdot, I_i) \leftarrow \min \left[\langle w_{j,h}^k, \phi(\cdot, I_i) \rangle + \beta \xi_{i,j}, H \right]$
 end for
 $\text{SM}_{h,k}(I_i | \cdot) = \frac{\exp(\alpha(Q_{r,h}^k(\cdot, I_i) + \nu_k Q_{g,h}^k(\cdot, I_i)))}{\sum_j \exp(\alpha(Q_{r,h}^k(\cdot, I_j) + \nu_k Q_{g,h}^k(\cdot, I_j)))}$
 $\pi_h^k(a | \cdot) \leftarrow \frac{1}{\int_{b \in \mathcal{I}(a)} db} \text{SM}_{h,k}(I(a) | \cdot)$
 $V_{r,h}^k(\cdot) \leftarrow \int_{a \in \mathcal{A}} \pi_h^k(a | \cdot) Q_{r,h}^k(\cdot, a) da$, $V_{g,h}^k(\cdot) \leftarrow \int_{a \in \mathcal{A}} \pi_h^k(a | \cdot) Q_{g,h}^k(\cdot, a) da$
 end for
 for step $h = 1, \dots, H$ **do**
 Compute $Q_{r,h}^k(s_h^k, I_i)$, $Q_{g,h}^k(s_h^k, I_i)$, $\pi(I_i | s_h^k)$.
 Take action $a_h^k \sim \pi_h^k(\cdot | s_h^k)$ and observe s_{h+1}^k .
 end for
 $\nu_{k+1} = \max \{ \min \{ \nu_k + \eta (\tilde{c} - V_{g,1}^k(s_1)) \}, \mathcal{V} \}, 0 \}$
end for

3.2 R-TD3

Because assumptions made in [Section 3.1](#) (e.g., the linear MDP assumption) are often violated in the real world, we also consider more flexible and arguably powerful deep reinforcement learning methods. Concretely, we utilize ‘‘Twin-Delayed Deep Deterministic Policy Gradient’’ (TD3) [Fujimoto et al. \(2018\)](#) with the implementation and default parameters provided by the open-source package ‘‘Stable Baselines 3’’ [Raffin et al. \(2021\)](#) on a (Lagrangian) relaxation of the long-term fairness problem. We term this specific algorithm for long-term fairness R-TD3.

In general, methods such as TD3 lack provable safety guarantees, however they may still confirm our hypothesis that agents trained via RL can learn to sacrifice short-term utility in order to drive dynamics towards preferable long-term states and explicitly incorporate dynamical control objectives provided as functions of state.

To treat the long-term fairness problem ([Eq. \(6\)](#)) using unconstrained optimization techniques (i.e., methods like TD3), we consider a time-dependent Lagrangian relaxation of [Eq. \(6\)](#). We train R-TD3 to optimize

$$\min_{\pi} \mathbb{E}_{a_\tau \sim \pi(s_\tau)} \left[\sum_{\tau=1}^H [\kappa_\tau \mathcal{L}(s_\tau, a_\tau) + \lambda_\tau \mathcal{D}(s_\tau, a_\tau)] \right] \quad (12)$$

where $s_{\tau+1} \sim \mathbf{P}(s_\tau, a_\tau)$, $\lambda_\tau = \tau/H$, and $\kappa_\tau = 1 - \lambda_\tau$.

Strictly applied, myopic fairness constraints can lead to undesirable dynamics and equilibria [Raab and Liu \(2021\)](#). Relaxing these constraints (hard $\rightarrow$ soft) for the near future while emphasizing them long-term, we hope to develop classifiers that learn to transition to more favorable equilibria.

3.3 Greedy Baseline

In our experiments, we compare L-UCBFair and R-TD3 to a ‘‘Greedy Baseline’’ agent as a proxy for a myopic status quo in which policy is repeatedly determined by optimizing for immediate utility, without regard for the population

dynamics induced by algorithmic actions. Our chosen algorithm for the greedy baseline is simply gradient descent in f , defined as loss regularized by disparity, performed anew with each time step with fixed parameter λ .

$$f_\tau(\pi) = \mathbb{E}_{a_\tau \sim \pi} [(1 - \lambda)\mathcal{L}(s_\tau, a_\tau) + \lambda\mathcal{D}(s_\tau, a_\tau)] \quad (13)$$

While such an algorithm does not guarantee constraint satisfaction, it is nonetheless “constraint aware” in precisely the same way as a firm that (probabilistically) incurs penalties for violating constraints.

4 Simulated Environments

We describe our experiments with the algorithms we have detailed for long-term fairness as an RL problem: We consider a series of binary ($Y \in \{-1, 1\}$) classification tasks on a population of two groups $\mathcal{G} = \{g_1, g_2\}$ modeled according to evolutionary game theory (using replicator dynamics). We consider two families of distributions of real-valued features for the population: One that is purely synthetic, for which $X \sim \mathcal{N}(Y, 1)$, independent of group G , and one that is based on a logistic regression to real-world data. Both families of distributions are parameterized by the joint distribution $\Pr(Y, G)$. RL agents are trained on episodes of length H initialized with randomly sampled states.

4.1 Setting

The following assumptions simplify our hypothesis space for classifiers in order to better handle continuous state space. These assumptions appeared in [Raab and Liu \(2021\)](#).

Assumption 4.1 (Well-behaved feature). For purely synthetic data, we require X to be a “well-behaved” real-valued feature or “score” within each group. That is,

$$\forall g, \quad \Pr(Y=1 \mid G=g, X=x) \text{ strictly increases in } x$$

As an intuitive example of [Assumption 4.1](#), if Y represents qualification for a fixed loan and X represents credit-score, we require higher credit scores to strongly imply higher likelihood that an individual is qualified for the loan.

Theorem 4.2 (Threshold Bayes-optimality). *For each group g , when [Assumption 4.1](#) is satisfied, the Bayes-optimal, deterministic binary classifier is a threshold policy*

$$\hat{Y} = 1 \text{ if } x \geq A_g \text{ and } -1 \text{ otherwise}$$

where A_g is the feature threshold for group g .

As a result of [Theorem 4.2](#), we consider our action space to be the space of group-specific thresholds, and denote an individual action as the vector $\mathbf{A} := (A_1, A_2, \dots, A_n)$.

4.2 Replicator Dynamics

Our use of replicator dynamics closely mirrors that of [Raab and Liu \(2021\)](#) as an “equitable” model of a population, in which individuals may be modeled identically, independently of group membership, yet persistent outcome disparities may nonetheless emerge from disparate initial conditions between groups. In particular, we parameterize the evolving distribution $\Pr(X, Y \mid G)$, assuming constant group sizes, in terms of “qualification rates” $q_g := \Pr(Y=1 \mid G=g)$ and update these qualification rates according to the discrete-time replicator dynamics:

$$q_g[t+1] = q_g[t] \frac{W_1^g[t]}{\overline{W}^g[t]}; \quad \overline{W}^g[t] := W_1 q_g + (1 - q_g) W_{-1}$$

In this model, the *fitness* $W_y^g > 0$ of label $Y=y$ in group $G=g$ may be interpreted as the “average utility to the individual” in group g of possessing label y , and thus relative *replication rate* of label y in group g , as agents update their labels by mimicking the successful strategies of in-group peers. Also following [Raab and Liu \(2021\)](#), we model W_y^g in terms of acceptance and rejection rates with a group-independent utility matrix U :

$$W_y^g = \sum_{\hat{y} \in \{-1, 1\}} U_{y, \hat{y}} \Pr(\hat{Y}=\hat{y} \mid Y=y, G=g)$$

We choose the matrix U to eliminate dominant strategies (i.e., agents prefer one label over another, independent of classification), assert that agents always prefer acceptance over rejection, and to imply that the costs of qualification are greater than the costs of non-qualification among accepted individuals. While other parameterizations of U are valid, this choice of parameters guarantees internal equilibrium of the replicator dynamics for a Bayes-optimal classifier and “well-behaved” scalar-valued feature X , such that $\Pr(Y=1 \mid X=x)$ is monotonically increasing in x [Raab and Liu \(2021\)](#).

4.3 Data Synthesis and Processing

In addition to a synthetic distribution, for which we assume $X \sim \mathcal{N}(Y, 1)$, independent of G , for all time, we also consider real-world distributions in simulating and comparing algorithms for “long-term fairness”. In both cases, as mentioned above, we wish to parameterize distributions in terms of qualification rates q_g . As we perform binary classification on discrete groups and scalar-valued features, in addition to parameterizing a distribution in terms of q_g , we desire a scalar-valued feature for each example, rather than the multi-dimensional features common to real-world data. Our solution to parameterize a distribution of groups and scalar features is to use an additional learning step for “preprocessing”: Given a static dataset $\mathcal{D}$ from which (X', Y, G) is drawn i.i.d., (e.g., the “Adult Data Set” [Dua and Graff \(2017\)](#)), at each time-step, we train a stochastic binary classifier $\tilde{a}$, such that $\hat{Y}' \sim \tilde{a}(X', G)$ with a loss that re-weights examples by label value, in order to simulate the desired q_g : $\min_{\tilde{a}} \mathbb{E}_{\tilde{a}, \mathcal{D}}[w(X', Y, G)L(Y, \hat{Y}')]$, where $w(X', Y, G) = [(1 - Y)/2 + Yq_g]/\mathbb{E}_{\mathcal{D}}[Y|G]$, L is zero-one loss, and, in our experiments, we choose $\tilde{a}$ according to logistic regression. We interpret $\Pr(\hat{Y}=1)$ as a new, scalar feature value $X \in \mathbf{R}$ mapped from higher-dimensional features X' as the output of a learned “preprocessing” function $\tilde{a}$, [Assumption 4.1](#) is as hard to satisfy in general as solving the Bayes-optimal binary classification task over higher-dimensional features. Nonetheless, we expect [Assumption 4.1](#) to be approximately satisfied by such a “preprocessing” pipeline.

4.4 Linearity of Dynamics

L-UCBFair relies on [Assumption 3.2](#), which asserts the existence of some Hilbert space in which the state dynamics $\mathbf{P}$ are linear. Such linearity for real-world (continuous time) dynamics holds only in infinite-dimensional Hilbert space [Brunton et al. \(2021\)](#) and is not computationally tractable. In addition, the “feature map” ϕ that maps state-action pairs to the aforementioned Hilbert space must be learned by the policy maker. In experiment, we use a neural network to estimate a feature map $\hat{\phi}$ which approximately satisfies the linear MDP assumption. We defer details to [Appendix C.1](#).

5 Experimental Results

Do RL agents learn to seek favorable equilibria against short-term utility? Is a Lagrangian relaxation of long-term fairness sufficient to encourage this behavior? We give positive demonstrations for both questions.

5.1 Losses and Disparities Considered

Our experiments consider losses $\mathcal{L}$ which combine true-positive and true-negative rates, where, for $\alpha, \beta \in [0, 1]$,

$$\mathcal{L}(s, a) = 1 - \alpha \text{tp}(s, a) + \beta \text{tn}(s, a), \quad (14)$$

where $\text{tp}(s, a) = \Pr_{s,a}(\hat{Y}=1, Y=1)$ and $\text{tn}(s, a) = \Pr_{s,a}(\hat{Y}=-1, Y=-1)$. For disparity $\mathcal{D}$, we consider demographic parity (DP) [Dwork et al. \(2012\)](#), equal opportunity (EOp) [Hardt et al. \(2016\)](#), and qualification rate (QR):

Func.	Form	$\xi_{s,a}^y(g) = \Pr_{s,a}(\cdot)$
DP		$\hat{Y}=1 \mid G=g$
QR	$ \xi_{s,a}(g_1) - \xi_{s,a}(g_2) ^2/2$	$\hat{Y}=1 \mid G=g$
EOp		$\hat{Y}=1 \mid G=g$
EO	$\sum_y \xi_{s,a}^y(g_1) - \xi_{s,a}^y(g_2) ^2/2$	$\hat{Y}=\hat{y} \mid Y=y, G=g$

QR does not matter to myopic fair classification, which does not consider mutable population state.

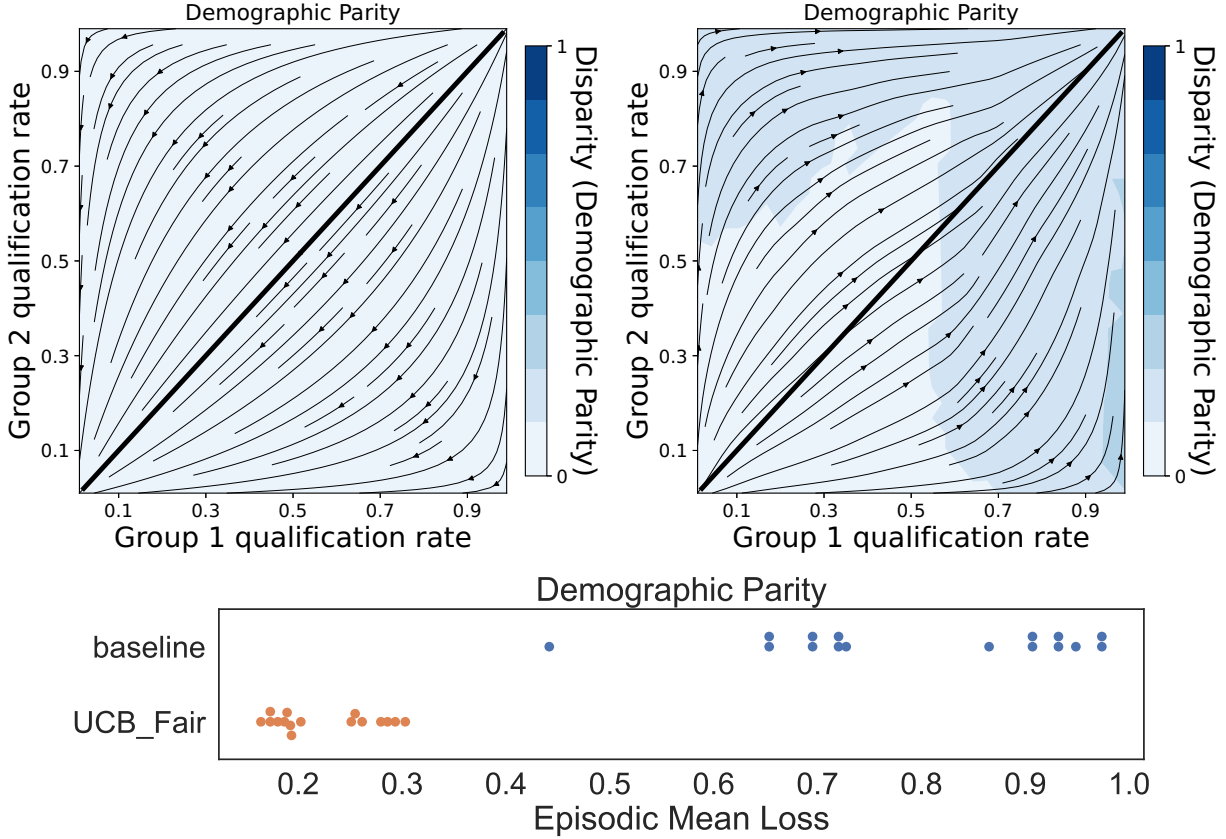


Figure 1: The greedy baseline algorithm (left) and L-UCBFair (right) are tasked to maximize the fraction of true-positive classifications ($\mathcal{L} = 1 - \text{tp}$, Eq. (14)), subject to demographic parity ($\mathcal{D}=\text{DP}$, Section 5.1). The greedy algorithm uses $\lambda=0.5$ in Eq. (13), while L-UCBFair is trained for 2,000 steps on episodes of length 100 prior to generating this “phase portrait”. We depict the expected dynamics (averaged over 20 policy iterations for each state) of the classifier-population system, parameterized by the time-evolving qualification rate in each group (1 on the horizontal, 2 on the vertical). Each group is of equal size and identically modeled by the standard normal $X \sim \mathcal{N}(Y, 1)$. Note that states in the left plot attract to universal non-qualification $\Pr(Y=1)=0$, while the right plot converges to universal qualification. The lower plot shows average loss over pairs of randomly sampled episodes.

5.2 Results

Our experiments show that algorithms trained with an RL formulation of long-term fairness can drive a reactive population toward states with higher utility and fairness, even when short-term utility is *misaligned* with desirable dynamics. Our central hypothesis, that long-term fairness via RL may induce an algorithm to sacrifice short-term utility for better long-term outcomes, is concretely demonstrated by Fig. 1, in which a greedy classifier and L-UCBFair, maximizing true positive rate tp (Section 5.1) subject to demographic parity DP (Section 5.1), drive a population to universal non-qualification ($\Pr(Y=1) \rightarrow 0$) and universal qualification ($\Pr(Y=1) \rightarrow 1$), respectively. Each phase plot shows the dynamics of qualification rates $q_g = \Pr(Y=1 \mid G=g)$, which parameterize the population state s and define the axes, with streamlines; color depicts averaged disparity $\mathcal{D}$ incurred by a in state s .

While both algorithms achieve no or little violation of demographic parity, the myopic algorithm eventually precludes future true-positive classifications (arrows in Fig. 1 approach a low qualification state), while L-UCBFair maintains stochastic thresholds at equilibrium (mean [0.49, 0.38], by group) with a non-trivial fraction of true-positives. The episodic mean loss and disparity training curves for L-UCBFair are depicted in Fig. 3.

We show that RL algorithms that are not limited by the same restrictive assumptions as L-UCBFair are applicable to long-term fairness. In Fig. 2, R-TD3 achieves similar qualitative behavior (i.e., driving near-universal qualification at the expense of short-term utility) when optimizing a loss subject to scheduled disparity regularization. This figure also

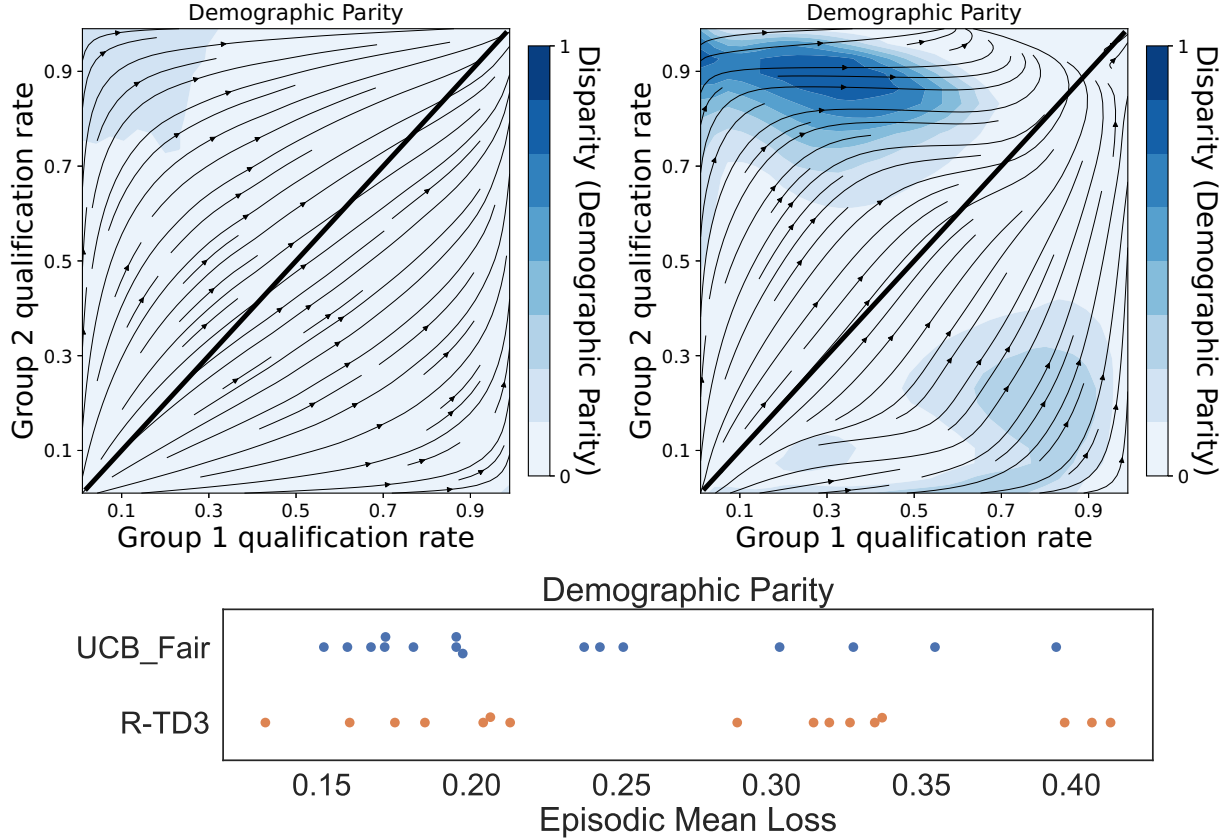


Figure 2: Using a modelled population initialized with the “Adult” dataset, reweighted for equal group representation (Section 4.3), L-UCBFair (left) and R-TD3 (right) are tasked, as in Fig. 1, to maximize the fraction of true-positive classifications ($\mathcal{L} = 1 - \text{tp}$, Eq. (14)), subject to demographic parity ($\mathcal{D}=\text{DP}$, Section 5.1). L-UCBFair performs almost indistinguishably from the experiment on the synthetic dataset (Fig. 1), while R-TD3 learns qualitatively similar behavior with more aggressive short-term violations of the fairness constraint.

highlights the lack of guarantees of R-TD3 in incurring prominent violations of the fairness constraint and failing to convincingly asymptote to the global optimum.

Finally, we demonstrate the capability of RL to utilize notions of fairness that are impossible to treat in the myopic setting, viz. qualification rate parity, in Fig. 4. In this example, while both RL agents achieve qualification rate parity, we note that L-UCBFair fails to realize the optimal equilibrium discovered by R-TD3.

Concluding remarks We have demonstrated that desirable social outcomes that are in conflict with myopic optimization may be realized as solutions to long-term fairness formalized through the lens of reinforcement learning. Moreover, we have shown that this problem both admits solutions with theoretical guarantees and can be relaxed to accommodate a wider class of recent advances in RL. We hope our contributions spur interest in long-term mechanisms and incentive structures for machine learning to be a driver of positive social change.

Acknowledgement This work is partially supported by the National Science Foundation (NSF) under grants IIS-2143895 and IIS-2040800, and CCF-2023495.

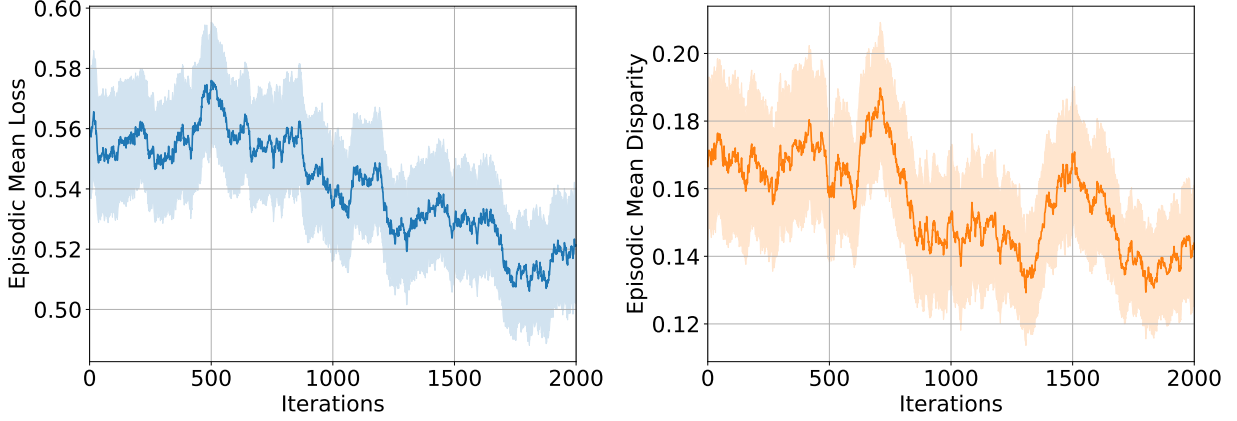


Figure 3: L-UCBFair 20-step sliding mean & std training loss (left) and disparity (right) for the Fig. 1 setting.

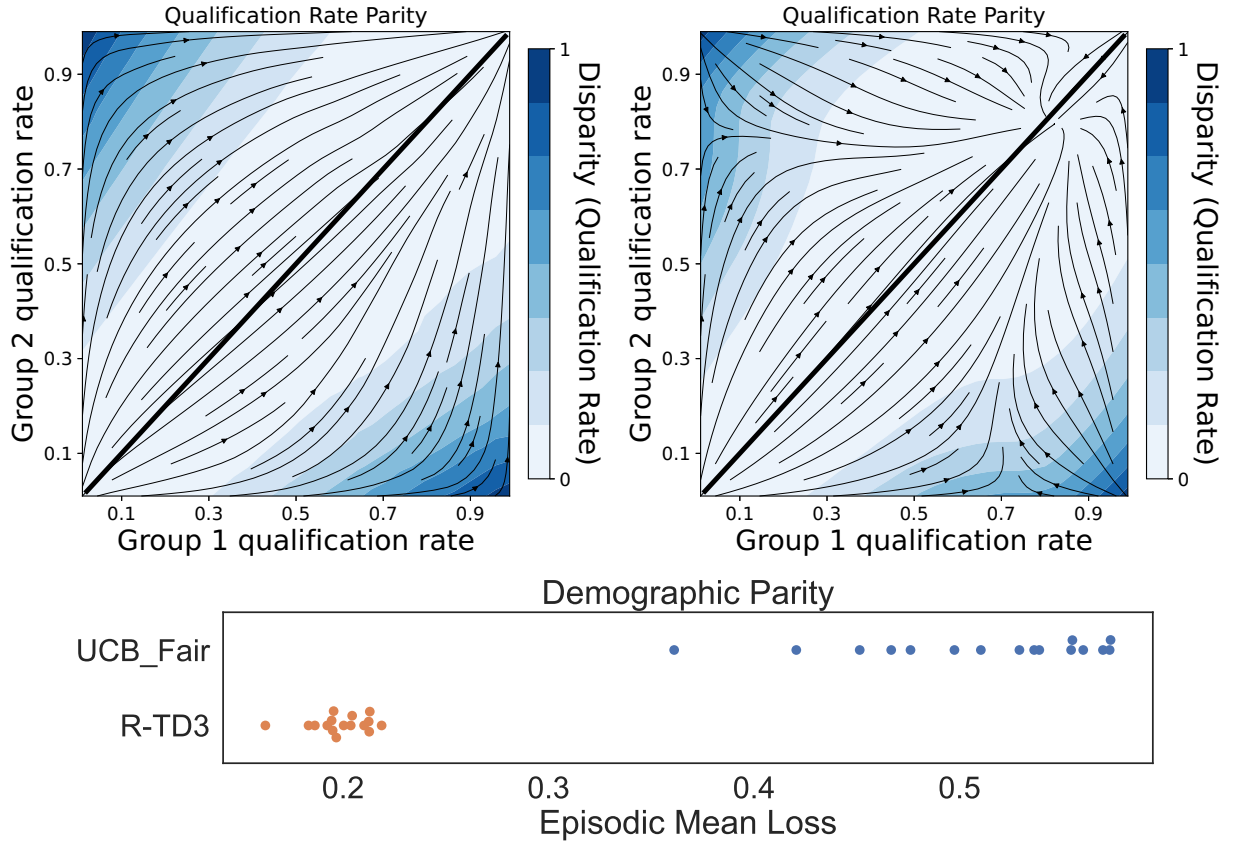


Figure 4: Phase portraits for L-UCBFair (left), and R-TD3 (right) interacting on the synthetic distribution $X \sim \mathcal{N}(Y, 1)$ with groups of equal size. Both algorithms use $\mathcal{L} = 1 - \text{tp} - \text{tn}$ (i.e., zero-one loss) and $\mathcal{D} = \text{QR}$. Shading: qualification rate disparity for the *next* time-step.

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A Proof

Without loss of generality, we assume $\|\phi(s, a)\| \leq 1$ for all $(s, a) \in \mathcal{S} \times \mathcal{A}$, and $\max \{\|\mu_h(\mathcal{S})\|, \|\theta_h\|\} \leq \sqrt{d}$ for all $h \in [H]$.

A.1 Proof of Theorem 3.7

Theorem 3.7 (Boundedness). *With probability $1 - p$, there exists a constant b such that L -UCB*Fair* achieves*

$$\text{Regret}(K) \leq (b\zeta H^2 \sqrt{d^3} + (\mathcal{V} + 1)H) \sqrt{K} \quad (10)$$

$$\text{Distortion}(K) \leq \frac{b\zeta(1 + \mathcal{V})H^2 \sqrt{d^3}}{\mathcal{V}} \sqrt{K} \quad (11)$$

when selecting parameter values $\varsigma = 1$, $\epsilon_I \leq \frac{1}{2\rho(1+\mathcal{V})KH\sqrt{d}}$, $\zeta = \log(\log(M)4dHK/p)$, and $\beta = \mathcal{O}(dH\sqrt{\zeta})$.

Outline The outline of this proof simulates the proof in Ghosh et al. (2022). For brevity, denote $\mathbb{P}_h V_{j,h+1}^\pi(s, a) = \mathbb{E}_{s' \sim \mathbb{P}_h(\cdot | s, a)} V_{j,h+1}^\pi(s')$ for $j = r, g$. Then

$$Q_{j,h}^\pi(s, a) = (r_h + \mathbb{P}_h V_{j,h+1}^\pi)(s, a) \quad (15)$$

$$V_{j,h}^\pi(s) = \langle \pi_h(\cdot | s), Q_{j,h}^\pi(s, \cdot) \rangle_{\mathcal{A}} \quad (16)$$

$$\langle \pi_h(\cdot | s), Q_{j,h}^\pi(s, \cdot) \rangle_{\mathcal{A}} = \sum_{a \in \mathcal{A}} \pi_h(a | s) Q_{j,h}^\pi(s, a) \quad (17)$$

Similar to Efroni et al. (2020), we establish

$$\begin{aligned} & \text{Regret}(K) + \nu \text{Distortion}(K) \\ &= \sum_{k=1}^K \left(V_{r,1}^{\pi^*}(s_1) - V_{r,1}^{\pi_k}(s_1) \right) + \nu \sum_{k=1}^K (b - V_{g,1}^{\pi_k}(s_1)) \\ &\leq \underbrace{\sum_{k=1}^K \left(V_{r,1}^{\pi^*}(s_1) + \nu_k V_{g,1}^{\pi^*}(s_1) \right) - (V_{r,1}^k(s_1) + \nu_k V_{g,1}^k(s_1))}_{\mathcal{T}_1} \\ &\quad + \underbrace{\sum_{k=1}^K (V_{r,1}^k(s_1) - V_{r,1}^{\pi_k}(s_1)) + \nu \sum_{k=1}^K (V_{g,1}^k(s_1) - V_{g,1}^{\pi_k}(s_1))}_{\mathcal{T}_2} \\ &\quad + \underbrace{\frac{1}{2\eta}\nu^2 + \frac{\eta}{2}H^2K}_{\mathcal{T}_3} \end{aligned} \quad (18)$$

$\mathcal{T}_3$ is easily bounded if η . The major task remains bound $\mathcal{T}_1$ and $\mathcal{T}_2$.

Bound $\mathcal{T}_1$ and $\mathcal{T}_2$. We have following two lemmas.

Lemma A.1 (Boundedness of $\mathcal{T}_1$). *With probability $1 - p/2$, we have $\mathcal{T}_1 \leq KH \left(\frac{\log(M)}{\alpha} + 2(1 + \mathcal{V})H\rho\epsilon_I \sqrt{\frac{dK}{\varsigma}} \right)$. Specifically, if $\alpha = \frac{\log(M)K}{2(1+\mathcal{V}+H)}$ and $\varsigma = 1$, we have $\mathcal{T}_1 \leq 2H(1 + \mathcal{V} + H) + 2KH^2(1 + \mathcal{V})\rho\epsilon_I \sqrt{dK}$ with probability $1 - p/2$.*

Lemma A.2 (Boundedness of $\mathcal{T}_2$). *Ghosh et al. (2022) With probability $1 - p/2$, $\mathcal{T}_2 \leq \mathcal{O}((\nu + 1)H^2\zeta\sqrt{d^3K})$, where $\zeta = \log[\log(M)4dHK/p]$.*

Lemma A.2 follows the same logic in Ghosh et al. (2022), and we delay the proof of Lemma A.1 to Section ???. Now we are ready to proof Theorem 3.7.

Proof. For any $\nu \in [0, \mathcal{V}]$, with prob. $1 - p$,

$$\begin{aligned} & \text{Regret}(K) + \nu \text{Distortion}(K) \\ & \leq \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3 \\ & \leq \frac{1}{2\eta} \nu^2 + \frac{\eta}{2} H^2 K + \frac{HK \log M}{\alpha} + 2KH^2(1 + \mathcal{V})\rho\epsilon_I \sqrt{dK} + \mathcal{O}\left((\nu + 1)H^2 \zeta \sqrt{d^3 K}\right) \end{aligned} \quad (19)$$

Taking $\nu = 0, \eta = \frac{\mathcal{V}}{\sqrt{KH^2}}, \alpha = \frac{K \log M}{2(1+\mathcal{V}+H)}, \epsilon_I = \frac{1}{2\rho(1+\mathcal{V})KH\sqrt{d}}$, there exist constant b ,

$$\begin{aligned} \text{Regret}(K) & \leq \frac{\mathcal{V}H}{2} \sqrt{K} + 2H(1 + \mathcal{V} + H) + 2H^2 K(1 + \mathcal{V})\rho\epsilon_I \sqrt{dK} + \mathcal{O}\left(H^2 \zeta \sqrt{d^3 K}\right) \\ & \leq (b\zeta H^2 \sqrt{d^3} + (\mathcal{V} + 1)H) \sqrt{K} = \tilde{\mathcal{O}}(H^2 \sqrt{d^3 K}). \end{aligned}$$

Taking $\nu = \mathcal{V}, \eta = \frac{\mathcal{V}}{\sqrt{KH^2}}, \alpha = \frac{K \log M}{2(1+\mathcal{V}+H)}, \epsilon_I = \frac{1}{2\rho(1+\mathcal{V})KH\sqrt{d}}$,

$$\text{Regret}(K) + \mathcal{V} \text{Distortion}(K) \leq (\mathcal{V} + 1)H \sqrt{K} + (1 + \mathcal{V}) \mathcal{O}\left(H^2 \zeta \sqrt{d^3 K}\right)$$

Following the idea of [Efroni et al. \(2020\)](#), there exists a policy π' such that $V_{r,1}^{\pi'} = \frac{1}{K} \sum_{k=1}^K V_{r,1}^{\pi_k}, V_{g,1}^{\pi'} = \frac{1}{K} \sum_{k=1}^K V_{g,1}^{\pi_k}$. By the occupancy measure, $V_{r,1}^{\pi}$ and $V_{g,1}^{\pi}$ are linear in occupancy measure induced by π . Thus, the average of K occupancy measure also produces an occupancy measure which induces policy π' and $V_{r,1}^{\pi'}$, and $V_{g,1}^{\pi'}$. We take $\nu = 0$ when $\sum_{k=1}^K (b - V_{g,1}^{\pi_k}(s_1^k)) < 0$, otherwise $\nu = \mathcal{V}$. Hence, we have

$$\begin{aligned} & V_{r,1}^{\pi^*}(s_1) - \frac{1}{K} \sum_{k=1}^K V_{r,1}^{\pi_k}(s_1) + \mathcal{V} \max\left((c - \frac{1}{K} \sum_{k=1}^K V_{g,1}^{\pi_k}(s_1), 0)\right) \\ & = V_{r,1}^{\pi^*}(s_1) - V_{r,1}^{\pi'}(s_1) + \mathcal{V} \max(c - V_{g,1}^{\pi'}(s_1), 0) \\ & \leq \frac{\mathcal{V} + 1}{K} H \sqrt{K} + \frac{\mathcal{V} + 1}{K} \mathcal{O}\left(H^2 \zeta \sqrt{d^3 K}\right) \end{aligned} \quad (20)$$

Since $\mathcal{V} = 2H/\gamma$, and using the result of [Lemma A.11](#), we have

$$\max\left(c - \frac{1}{K} \sum_{k=1}^K V_{g,1}^{\pi_k}(s_1^k), 0\right) \leq \frac{\mathcal{V} + 1}{K\mathcal{V}} \mathcal{O}\left(H^2 \zeta \sqrt{d^3 K}\right)$$

In this section we proof [Lemma A.1](#) and [Lemma A.2](#).

A.2 Prepare for [Lemma A.1](#)

In order to bound $\mathcal{T}_1$ and $\mathcal{T}_2$, we introduce the following lemma.

Lemma A.3. *There exists a constant B_2 such that for any fixed $p \in (0, 1)$, with probability at least $1 - p/2$, the following event holds*

$$\left\| \sum_{\tau=1}^{k-1} \phi_{j,h}^{\tau} \left[V_{j,h+1}^k(s_{h+1}^{\tau}) - \mathbb{P}_h V_{j,h+1}^k(s_h^{\tau}, a_h^{\tau}) \right] \right\|_{(\zeta_h^k)^{-1}} \leq B_2 d H q$$

for $j \in \{r, g\}$, where $q = \sqrt{\log[4(B_1 + 1) \log(M) d T / p]}$ for some constant B_1 .

We delay the proof of [Lemma A.3](#) to [Appendix A.4](#).

[Lemma A.3](#) shows the bound of estimated value function $V_{j,h}^k$ and value function $V_{j,h}^{\pi}$ corresponding in a given policy at k . We now introduce the following lemma appeared in [Ghosh et al. \(2022\)](#). This lemma bounds the difference between the value function without bonus in L-UCBFair and the true value function of any policy π . This is bounded using their expected difference at next step, plus a error term.

Lemma A.4. *Ghosh et al. (2022)* There exists an absolute constant $\beta = C_1 dH \sqrt{\zeta}$, $\zeta = \log(\log(M)4dT/p)$, and for any fixed policy π , for the event defined in [Lemma A.3](#), we have

$$\langle \phi(s, a), w_{j,h}^k \rangle - Q_{j,h}^\pi(s, a) = \mathbb{P}_h(V_{j,h+1}^k - V_{j,h+1}^\pi)(s, a) + \Delta_h^k(s, a)$$

for some $\Delta_h^k(s, a)$ that satisfies $|\Delta_h^k(s, a)| \leq \beta \sqrt{\phi(s, a)^T (\Lambda_h^k)^{-1} \phi(s, a)}$.

Lemma A.5. *Ghosh et al. (2022)* With probability at least $1 - p/2$, (for the event defined in [Lemma A.3](#))

$$Q_{r,h}^\pi(s, a) + \nu_k Q_{g,h}^\pi(s, a) \leq Q_{r,h}^k(s, a) + \nu_k Q_{g,h}^k(s, a) - \mathbb{P}_h(V_{h+1}^k - V_{h+1}^{\pi, \nu_k})(s, a)$$

We also introduce the following lemma. This lemma bound the value function by taking L-UCBFair policy and greedy policy.

Lemma A.6. Define $\bar{V}_h^k(\cdot) = \max_a [Q_{r,h}^k(\cdot, a) + \nu_k Q_{g,h}^k(\cdot, a)]$ the value function corresponding to greedy policy, we have

$$\bar{V}_h^k(s) - V_h^k(s) \leq \frac{\log M}{\alpha} + 2(1 + \mathcal{V})H\rho\epsilon_I \sqrt{\frac{dk}{\varsigma}}. \quad (21)$$

Proof. Define a_g the solution of greedy policy,

$$V_h^k(s) - V_h^k(s) = [Q_{r,h}^k(s, a_g) + \nu_k Q_{g,h}^k(s, a_g)] \quad (22)$$

$$- \int_a \pi_{h,k}(a | s) [Q_{r,h}^k(s, a) + \nu_k Q_{g,h}^k(s, a)] da \quad (23)$$

$$\leq [Q_{r,h}^k(s, a_g) + \nu_k Q_{g,h}^k(s, a_g)] \quad (24)$$

$$- \sum_i \text{SM}_\alpha(I_i | x) [Q_{r,h}^k(x, I_i) + \nu_k Q_{g,h}^k(x, I_i)] + 2(1 + \mathcal{V})H\rho\epsilon_I \sqrt{\frac{dk}{\varsigma}} \quad (25)$$

$$\leq \left(\frac{\log \left(\sum_a \exp \left(\alpha \left(Q_{r,h}^k(s, I_i) + \nu_k Q_{g,h}^k(s, I_i) \right) \right) \right)}{\alpha} \right) \quad (26)$$

$$- \sum_i \text{SM}_\alpha(I_i | s) [Q_{r,h}^k(s, I_i) + \nu_k Q_{g,h}^k(s, I_i)] + 2(1 + \mathcal{V})H\rho\epsilon_I \sqrt{\frac{dk}{\varsigma}} \quad (27)$$

$$\leq \frac{\log(M)}{\alpha} + 2(1 + \mathcal{V})H\rho\epsilon_I \sqrt{\frac{dk}{\varsigma}}. \quad (28)$$

The first inequality follows from [Lemma A.10](#) and the second inequality holds because of Proposition 1 in [Pan et al. \(2019\)](#).

A.3 Proof of [Lemma A.1](#)

Now we're ready to proof [Lemma A.1](#).

Proof. This proof simulates Lemma 3 in [Ghosh et al. \(2022\)](#).

We use induction to proof this lemma. At step H , we have $Q_{j,H+1}^k = 0 = Q_{j,H+1}^\pi$ by definition. Under the event in [Lemma A.9](#) and using [Lemma A.4](#), we have for $j = r, g$,

$$|\langle \phi(s, a), w_{j,H}^k(s, a) \rangle - Q_{j,H}^\pi(s, a)| \leq \beta \sqrt{\phi(s, a)^T (\Lambda_H^k)^{-1} \phi(s, a)}$$

$$\text{Thus } Q_{j,H}^\pi(s, a) \leq \min \left\{ \langle \phi(s, a), w_{j,H}^k(s, a) \rangle + \beta \sqrt{\phi(s, a)^T (\Lambda_H^k)^{-1} \phi(s, a)}, H \right\} = Q_{j,H}^k(s, a).$$

From the definition of $\bar{V}_h^k$,

$$\bar{V}_H^k(s) = \max_a [Q_{r,H}^k(s, a) + \nu_k Q_{g,h}^k(s, a)] \geq \sum_a \pi(a | x) [Q_{r,H}^\pi(s, a) + \nu_k Q_{g,H}^\pi(s, a)] = V_H^{\pi, \nu_k}(s)$$

for any policy π . Thus, it also holds for π^* , the optimal policy. Using [Lemma A.6](#) we can get

$$V_H^{\pi^*, \nu_k}(s) - V_H^k(s) \leq \frac{\log M}{\alpha} + 2(1 + \mathcal{V})H\rho\epsilon_I \sqrt{\frac{dk}{\varsigma}}$$

Now, suppose that it is true till the step $h + 1$ and consider the step h . Since, it is true till step $h + 1$, thus, for any policy π ,

$$\mathbb{P}_h(V_{h+1}^{\pi, \nu_k} - V_{h+1}^k)(s, a) \leq (H - h) \left(\frac{\log M}{\alpha} + 2(1 + \mathcal{V})H\rho\epsilon_I \sqrt{\frac{dk}{\varsigma}} \right)$$

From (27) in [Lemma 10](#) and the above result, we have for any (s, a)

$$Q_{r,h}^\pi(s, a) + \nu_k Q_{g,h}^\pi(s, a) \leq Q_{r,h}^k(s, a) + \nu_k Q_{g,h}^k(s, a) + (H - h) \left(\frac{\log M}{\alpha} + 2(1 + \mathcal{V})H\rho\epsilon_I \sqrt{\frac{dk}{\varsigma}} \right)$$

Hence,

$$V_h^{\pi, \nu_k}(s) \leq \bar{V}_h^k(s) + (H - h) \left(\frac{\log M}{\alpha} + 2(1 + \mathcal{V})H\rho\epsilon_I \sqrt{\frac{dk}{\varsigma}} \right)$$

Now, again from [Lemma 11](#), we have $\bar{V}_h^k(s) - V_h^k(s) \leq \frac{\log(|\mathcal{A}|)}{\alpha}$. Thus,

$$V_h^{\pi, \nu_k}(s) - V_h^k(s) \leq (H - h + 1) \left(\frac{\log M}{\alpha} + 2(1 + \mathcal{V})H\rho\epsilon_I \sqrt{\frac{dk}{\varsigma}} \right)$$

Now, since it is true for any policy π , it will be true for π^* . From the definition of V^{π, ν_k} , we have

$$\left(V_{r,h}^{\pi^*}(s) + \nu_k V_{g,h}^{\pi^*}(s) \right) - \left(V_{r,h}^k(s) + \nu_k V_{g,h}^k(s) \right) \leq (H - h + 1) \left(\frac{\log M}{\alpha} + 2(1 + \mathcal{V})H\rho\epsilon_I \sqrt{\frac{dk}{\varsigma}} \right)$$

Hence, the result follows by summing over K and considering $h = 1$.

A.4 Proof of [Lemma A.3](#)

We first define some useful sets. Let $\mathcal{Q}_j = \left\{ Q \mid Q(\cdot, a) = \min \left\{ w_j^T \phi(\cdot, a) + \beta \sqrt{\phi^T(\cdot, a)^T \Lambda^{-1} \phi(\cdot, a)}, H \right\}, a \in \mathcal{A} \right\}$ be the set of Q functions, where $j \in \{r, g\}$. Since the minimum eigen value of Λ is no smaller than one so the Frobenius norm of Λ^{-1} is bounded.

Let $\mathcal{V}_j = \left\{ V_j \mid V_j(\cdot) = \int_a \pi(a | \cdot) Q_j(\cdot, a) da; Q_r \in \mathcal{Q}_r, Q_g \in \mathcal{Q}_g, \nu \in [0, \mathcal{V}] \right\}$ be the set of Q functions, where $j \in \{r, g\}$. Define

$$\Pi = \left\{ \pi \mid \forall a \in \mathcal{A}, \pi(a | \cdot) = \frac{1}{\int_{b \in \mathcal{I}(a)} db} \text{SM}_\alpha(Q_r(\cdot, I(a)) + \nu Q_g(\cdot, I(a))), Q_r \in \mathcal{Q}_r, Q_g \in \mathcal{Q}_g, \nu \in [0, \mathcal{V}] \right\}$$

the set of policies.

It's easy to verify $V_j^k \in \mathcal{V}_j$.

Then we introduce the proof of [Lemma A.3](#). To proof [Lemma A.3](#), we need the ϵ -covering number for the set of value functions([Lemma A.9](#)[Ghosh et al. \(2022\)](#)). To achieve this, we need to show if two Q functions and the dual variable ν are close, then the bound of policy and value function can be derived([Lemma A.7](#), [Lemma A.8](#)). Though the proof of [Lemma A.7](#) and [Lemma A.8](#) are different from [Ghosh et al. \(2022\)](#), we show the results remain the same, thus [Lemma A.9](#) still holds. We'll only introduce [Lemma A.9](#) and omit the proof.

We now proof [Lemma A.7](#).

Lemma A.7. Let π be the policy of *L-UCBFAir* corresponding to $Q_r^k + \nu_k Q_g^k$, i.e.,

$$\pi(a \mid \cdot) = \frac{1}{\int_{b \in \mathcal{I}(a)} db} SM_\alpha(Q_r(\cdot, I(a)) + \nu Q_g(\cdot, I(a))) \quad (29)$$

and

$$\tilde{\pi}(a \mid \cdot) = \frac{1}{\int_{b \in \mathcal{I}(a)} db} SM_\alpha(\tilde{Q}_r(\cdot, I(a)) + \tilde{\nu} \tilde{Q}_g(\cdot, I(a))), \quad (30)$$

if $|Q_j - \tilde{Q}_j| \leq \epsilon'$ and $|\nu - \tilde{\nu}| \leq \epsilon'$, then $|\int_a (\pi(a \mid x) - \tilde{\pi}(a \mid x)) da| \leq 2\alpha\epsilon'(1 + \mathcal{V} + H)$.

Proof.

$$\begin{aligned} & \left| \int_a (\pi(a \mid x) - \tilde{\pi}(a \mid x)) da \right| \quad (31) \\ &= \left| \sum_{i=1}^M \int_{a \in \mathcal{I}_i} (\pi(I(a) \mid x) - \tilde{\pi}(I(a) \mid x)) da \right| \\ &= \left| \sum_{i=1}^M \int_{b \in \mathcal{I}_i} db (\pi(I_i \mid x) - \tilde{\pi}(I_i \mid x)) \right| \\ &\leq \sum_{i=1}^M \left| SM_\alpha(Q_r(s, I_i) + \nu Q_g(s, I_i)) - SM_\alpha(\tilde{Q}_r(s, I_i) + \tilde{\nu} \tilde{Q}_g(s, I_i)) \right| \\ &\leq 2\alpha \left| Q_r(\cdot, I(a)) + \nu Q_g(\cdot, I(a)) - \tilde{Q}_r(\cdot, I(a)) - \tilde{\nu} \tilde{Q}_g(\cdot, I(a)) \right| \quad (32) \end{aligned}$$

The last inequality holds because of Theorem 4.4 in [Epasto et al. \(2020\)](#). Using [Corollary A.13](#), we have

$$\left| \int_a (\pi(a \mid x) - \tilde{\pi}(a \mid x)) da \right| \leq 2\alpha\epsilon'(1 + \mathcal{V} + H) \quad (33)$$

Now since we have [Lemma A.7](#), we can further bound the value functions.

Lemma A.8. If $|\tilde{Q}_j - Q_j^k| \leq \epsilon'$, where $\tilde{Q}_j \in \mathcal{Q}_j$, then there exists $\tilde{V}_j \in \mathcal{V}_j$ such that

$$|V_j^k - \tilde{V}_j| \leq H2\alpha\epsilon'(1 + \mathcal{V} + H) + \epsilon',$$

Proof. For any x ,

$$\begin{aligned} & V_j^k(s) - \tilde{V}_j(s) \\ &= \left| \int_a \pi(a \mid s) Q_j^k(s, a) da - \int_a \tilde{\pi}(a \mid s) \tilde{Q}_j(s, a) da \right| \\ &= \left| \int_a \pi(a \mid s) Q_j^k(s, a) da - \int_a \pi(a \mid s) \tilde{Q}_j(s, a) da + \int_a \pi(a \mid s) \tilde{Q}_j(s, a) da - \int_a \tilde{\pi}(a \mid s) \tilde{Q}_j(s, a) da \right| \\ &\leq \left| \int_a \pi(a \mid s) (Q_j^k(s, a) - \tilde{Q}_j(s, a)) da \right| + \left| \int_a \pi(a \mid s) \tilde{Q}_j(s, a) da - \int_a \tilde{\pi}(a \mid s) \tilde{Q}_j(s, a) da \right| \\ &\leq \epsilon' + H \left| \int_a (\pi(a \mid s) - \tilde{\pi}(a \mid s)) da \right| \\ &\leq \epsilon' + H2\alpha\epsilon'(1 + \mathcal{V} + H) \end{aligned}$$

Using Lemmas above, we can have the same result presented in Lemma 13 of [Ghosh et al. \(2022\)](#) as following.

Lemma A.9. *Ghosh et al. (2022)* There exists a $\tilde{V}_j \in \mathcal{V}_j$ parameterized by $(\tilde{w}_r, \tilde{w}_g, \tilde{\beta}, \Lambda, \tilde{\mathcal{V}})$ such that $\text{dist}(V_j, \tilde{V}_j) \leq \epsilon$ where

$$\left| V_j - \tilde{V}_j \right| = \sup_x \left| V_j(s) - \tilde{V}_r(s) \right|.$$

Let $N_\epsilon^{V_j}$ be the ϵ -covering number for the set $\mathcal{V}_j$, then,

$$\log N_\epsilon^{V_j} \leq d \log \left(1 + 8H \frac{\sqrt{dk}}{\sqrt{\varsigma} \epsilon'} \right) + d^2 \log \left[1 + 8d^{1/2} \beta^2 / \left(\varsigma (\epsilon')^2 \right) \right] + \log \left(1 + \frac{\mathcal{V}}{\epsilon'} \right)$$

where $\epsilon' = \frac{\epsilon}{H2\alpha(1+\mathcal{V}+H)+1}$

Lemma A.10. $|Q_{j,h}^k(s, a) - Q_{j,h}^k(s, I(a))| \leq 2H\rho\epsilon_I \sqrt{\frac{dK}{\varsigma}}.$

Proof.

$$\left| Q_{j,h}^k(s, a) - Q_{j,h}^k(s, I(a)) \right| \tag{34}$$

$$= \left| w_{j,h}^k(s, a)^T (\phi(s, a) - \phi(s, I(a))) \right| \tag{35}$$

$$\leq \|w_{j,h}^k(s, a)\|_2 \|\phi(s, a) - \phi(s, I(a))\|_2 \tag{36}$$

$$\tag{37}$$

From Lemma A.12 and Assumption 3.6 we get the result.

A.5 Preliminary Results

Lemma A.11. *Ding et al. (2021)* Let ν^* be the optimal dual variable, and $C \geq 2\nu^*$, then, if

$$V_{r,1}^{\pi^*}(s_1) - V_{r,1}^{\pi}(s_1) + C [c - V_{g,1}^{\pi}(s_1)]_+ \leq \delta,$$

we have

$$[c - V_{g,1}^{\tilde{\pi}}(x_1)]_+ \leq \frac{2\delta}{C}.$$

Lemma A.12. *Jin et al. (2020)* For any (k, h) , the weight $w_{j,h}^k$ satisfies

$$\|w_{j,h}^k\| \leq 2H\sqrt{dk/\varsigma}$$

Corollary A.13. If $\text{dist}(Q_r, \tilde{Q}_r) \leq \epsilon'$, $\text{dist}(Q_g, \tilde{Q}_g) \leq \epsilon'$, and $|\tilde{\nu}_k - \nu_k| \leq \epsilon'$, then, $\text{dist}(Q_r^k + \nu_k Q_g^k, \tilde{Q}_r + \tilde{\nu}_k \tilde{Q}_g) \leq \epsilon'(1 + \mathcal{V} + H).$

B Additional Figures

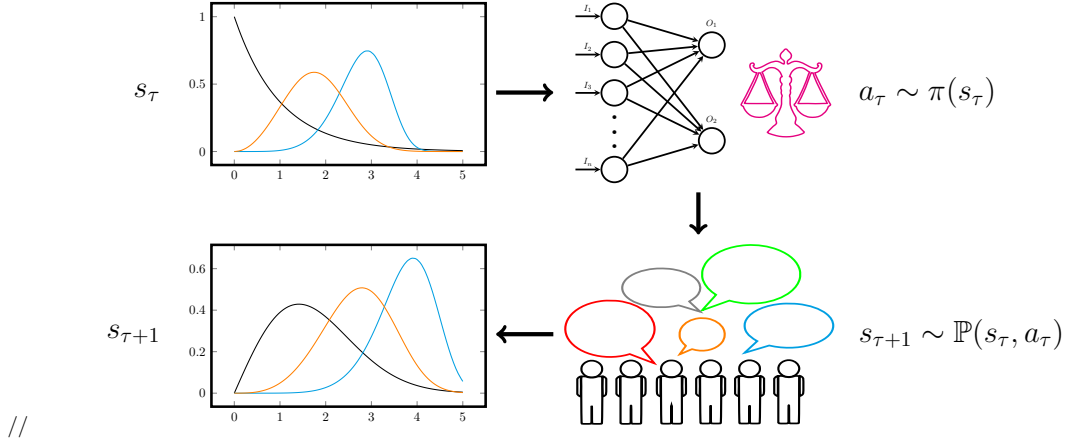


Figure 5: The interaction of an algorithmic classifier and a reactive population. Given state s_τ , the classifier uses policy π to select action a_τ . The population, in state s_τ , reacts to a_τ , transitioning state to $s_{\tau+1}$, then the process repeats.

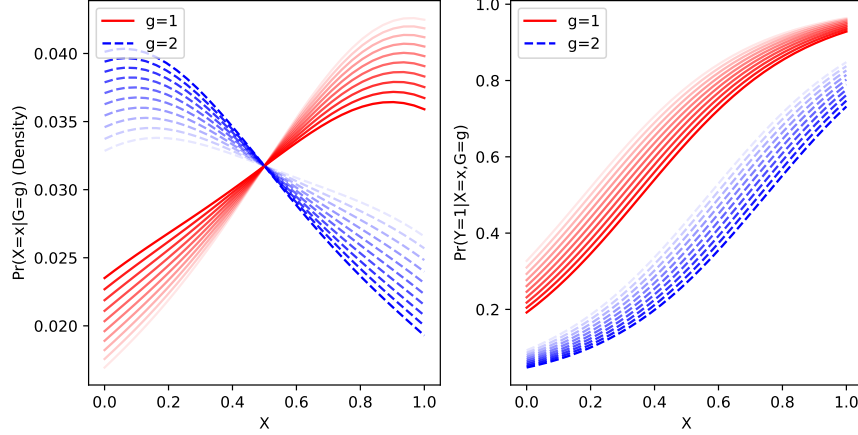


Figure 6: A synthetic distribution is updated according to a dynamical kernel $\mathbf{P}$ based on evolutionary dynamics (Section 4.2), when a classifier repeatedly predicts $\hat{Y}=1$ iff $X \geq 0.5$. We visualize how the distribution of X and conditional qualification rates $\Pr(Y=1 | X)$ change in each group $g \in \{1 \text{ (red, solid), } 2 \text{ (blue, dashed)}\}$, fading the plotted lines over 10 time steps. In this example, the feature values X in each group decrease with time, while the qualification rates of agents at any fixed value of X decrease.

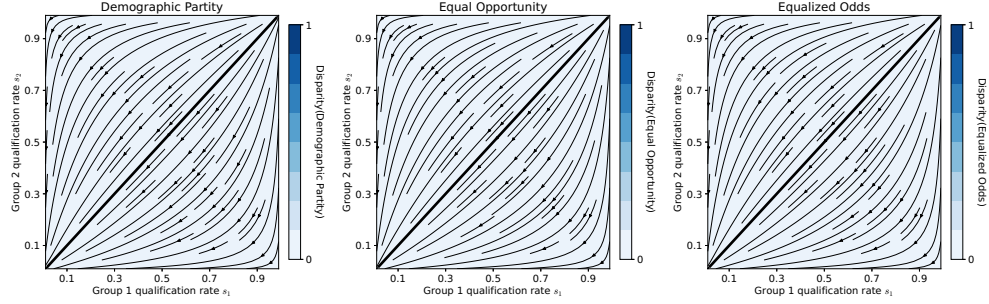
C Experiment

C.1 Experiment Details

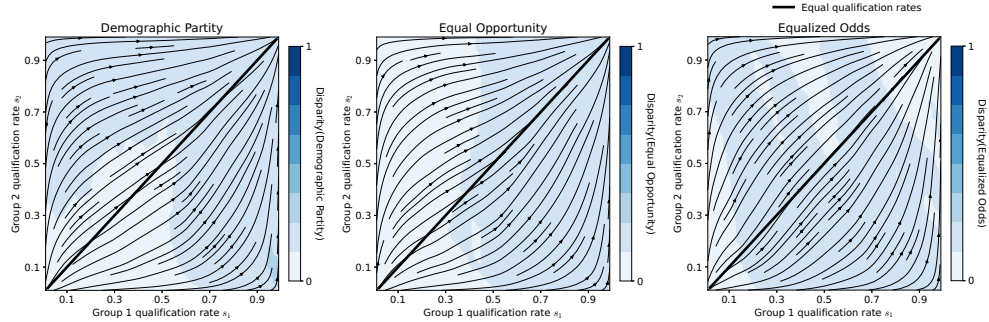
Device and Packages. We run all the experiment on a single 1080Ti GPU. We implement the R-TD3 agent using StableBaseline3 [Raffin et al. \(2021\)](#). The neural network is implemented using Pytorch [Paszke et al. \(2019\)](#).

Neural Network to learn ϕ . We use a multi-layer perceptron to learn ϕ . Specifically, we sample 100000 data points using a random policy, storing s , a , r and g . The inputs of the network are state and action, passing through fully connected (fc) layers with size 256, 128, 64, 64. ReLU is used as activation function between fc layers, while a SoftMax layer is applied after the last fc layer. We treat the outcome of this network as ϕ . To learn ϕ , we apply two separated fc layers (without bias) with size 1 to $\hat{\phi}$ and treat the outputs as predicted r and predicted g . A combination of MSE losses of r and g are adopted. We use Adam as the optimizer. Weight decay is set to 1e-4 and learning rate is set to 1e-3, while batch size is 128.

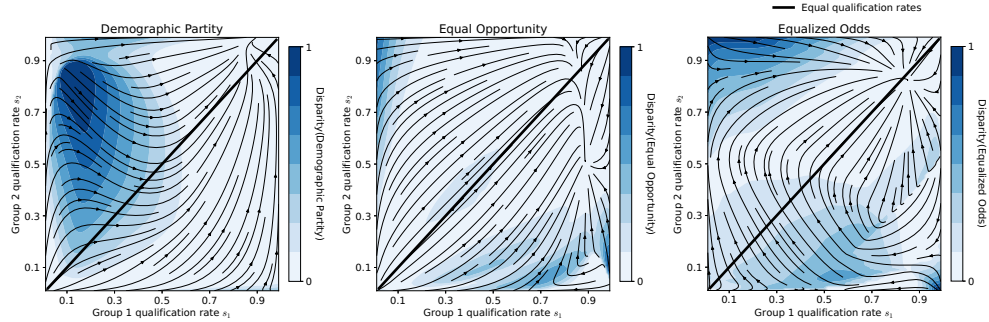
Note that, $\hat{\phi}$ is linear regarding r and g , but the linearity of transition kernel cannot be captured using such a schema. Therefore, equivalently we made an assumption that there always exists measure μ_h such that for given $\hat{\phi}$, the linearity of transition kernel holds. It's a stronger assumption than Assumption 3.2.



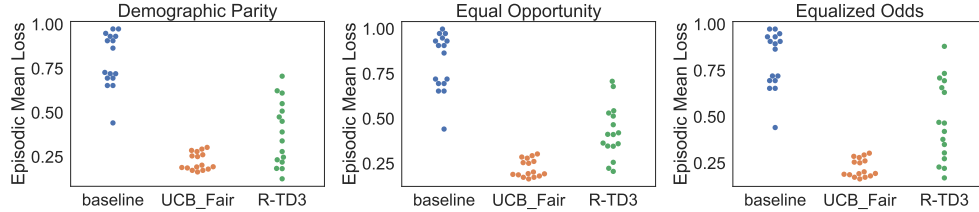
(a) A baseline, greedy classifier performing gradient descent to (locally) maximize true-positives, with static fairness regularization (columns).



(b) L-UCBFair, trained for 2,000 steps on the same, cumulative utility functions.

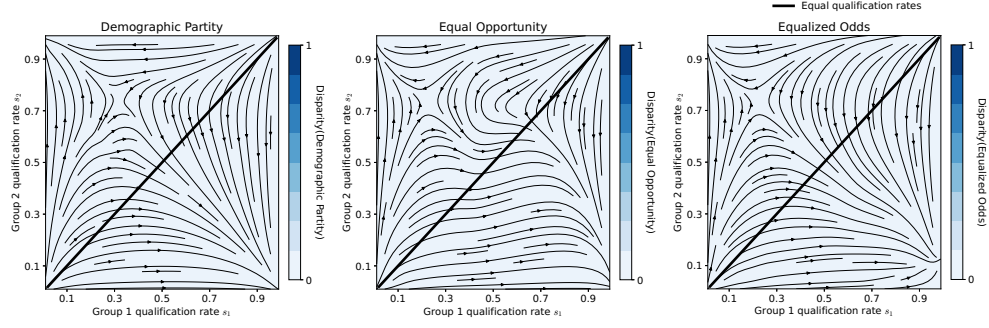


(c) A R-TD3 agent (Section 3.2) trained for 200,000 steps on the same, cumulative utility functions.

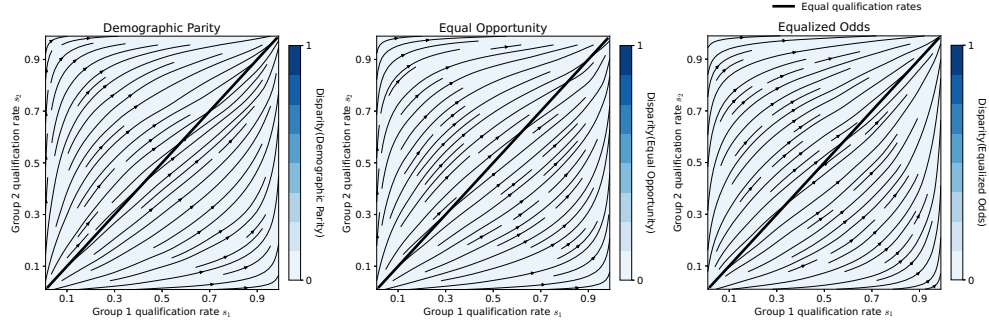


(d)

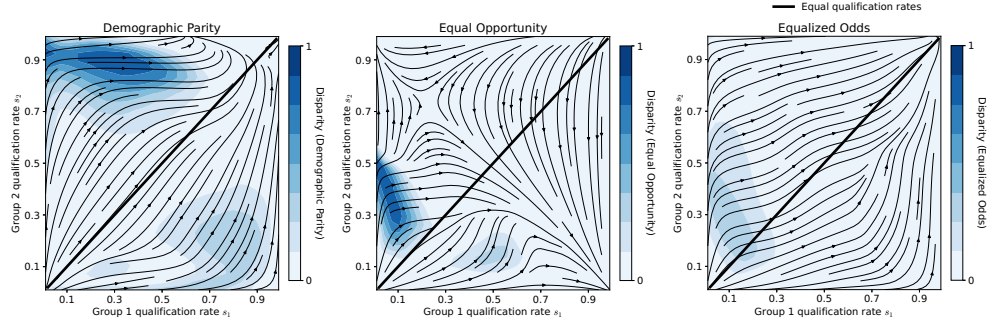
Figure 7: A comparison of learning policies trained to optimize **cumulative true positive fraction** subject to three different regularized fairness constraints (columns) with $\nu = 1$ (L-UCBFair), $\lambda = 0.5$, (greedy agent), and the time-dependent regularization detailed in Section 3.2 (R-TD3). The first policy (top row) is a baseline, myopic policy that greedily seeks to optimize current utility in any state by performing gradient descent. The second policy (bottom row) is trained using deep reinforcement learning (R-TD3) as detailed in Section 3.2 for 200,000 steps before we terminate learning and generate the phase portraits depicted. This is on the synthetic distribution. In all cases, the baseline, greedy policy drives the system to promote unqualified individuals, with low qualification rates in each group, while the R-TD3 agent is able to drive the system to more favorable equilibria characterized by higher qualification rates. The shading in the phase plots depicts the violation of the regularizing fairness constraint within each column, validating the claim that the R-TD3 agent learns to sacrifice short-term utility to drive towards preferable system states.



(a) A baseline, greedy classifier locally maximizing true positive classifications, regularized by fairness (columns).



(b) L-UCBFair, trained for 2,000 steps on the same, cumulative utility functions.



(c) A R-TD3 agent (Section 3.2) trained for 200,000 steps on the same, cumulative utility functions.

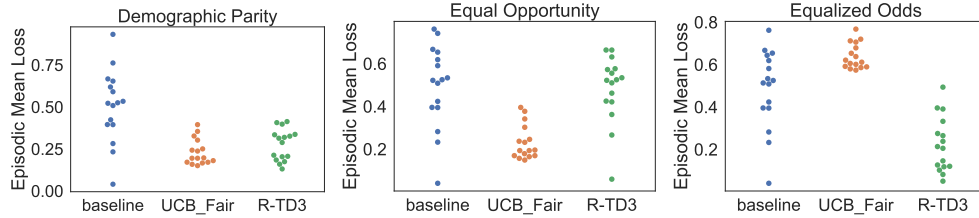
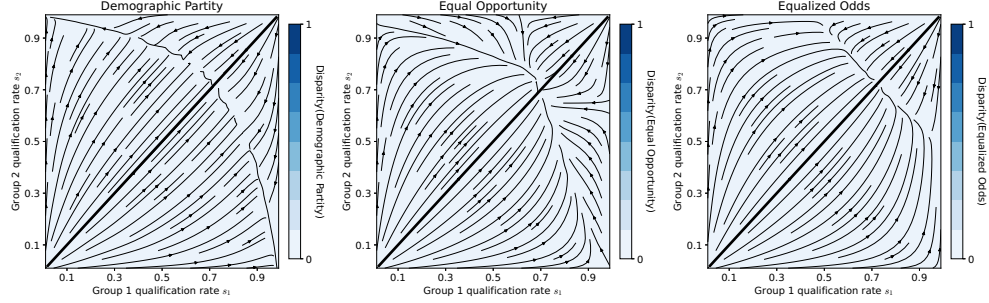
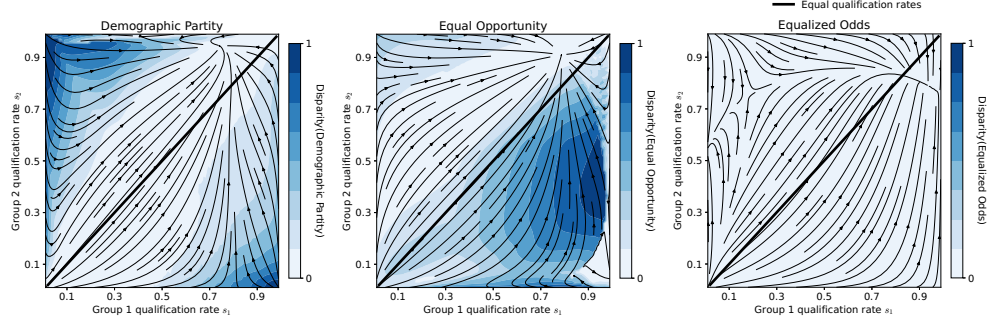


Figure 8: A repetition of the experiment performed in Fig. 7, rewarding **true positive fraction** using data synthesized from the UCI “Adult Data Set”, as detailed in section Section 4.3 with equal group size reweighting. For this experiment, an individual’s sex defined their “group” membership, which is an imbalanced label in the dataset ($\approx 67\%$ male, group 2, vertical axis) that we re-weight for equal representation Section 4. The stark difference between Fig. 7 and this experiment in the qualitative behavior of the greedy agent can be largely explained by the fact that $\Pr(Y=1|X=x)$ is not actually monotonically increasing in x , as stipulated by Assumption 4.1. Indeed, if $\Pr(Y=1|X=x)$ is sufficiently rough, the threshold selected by the baseline agent is liable to appear as if sampled uniformly at random, which is how the initial threshold value is chosen for each of the 20 iterations averaged over for each pair of group qualification rates used to generate the phase portraits above. Despite this failure mode of the baseline agent, however, the R-TD3 agent is still largely able to drive the system towards equilibria with more equal qualification rates in both groups. The line of equal qualification rates in both groups is depicted in black, from the lower-left corner of each phase plot to the upper-right.

C.2 zero one accuracy with more weights on true positive



(a) A baseline, greedy classifier locally maximizing utility, regularized by fairness (columns).



(b) A R-TD3 agent (Section 3.2) trained for 200,000 steps on the same, cumulative utility functions.

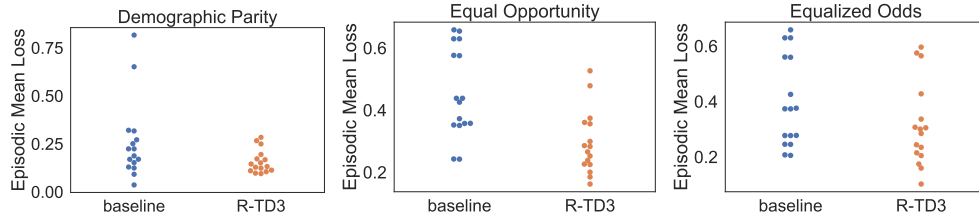
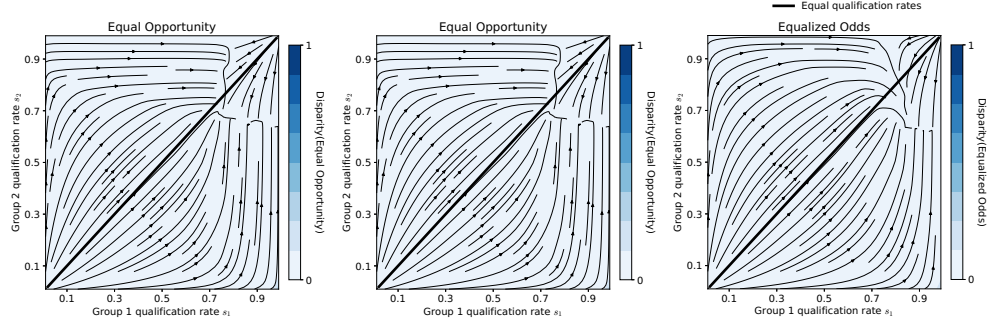
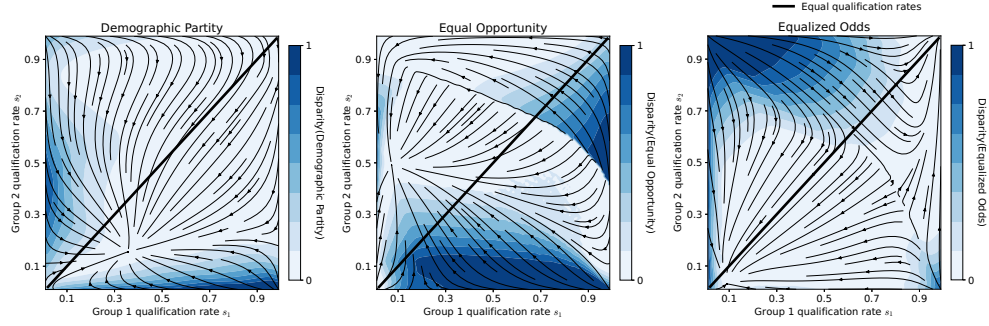


Figure 9: A repetition of the experiment performed in Fig. 7 with a different base utility function (**true positive fraction + 0.8 true negative fraction**), weighting each regularized disparity term with $\gamma = 1$, with the same **synthetic distribution**. While our observations are largely consistent with Fig. 7, we also note that the R-TD3 agent drives a subset of state-space in the third pane to an equilibrium less desired than the one that the myopic agent reaches.



(a) A baseline, greedy classifier locally maximizing utility, regularized by fairness (columns).



(b) A R-TD3 agent (Section 3.2) trained for 200,000 steps on the same, cumulative utility functions.

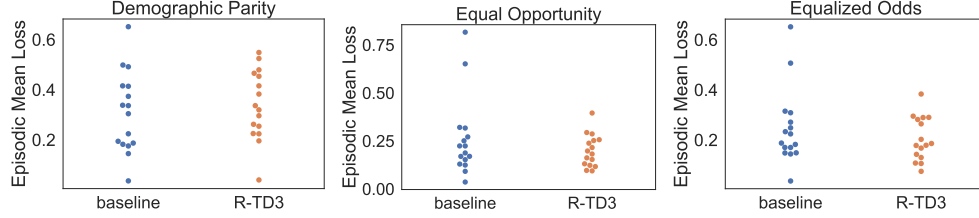


Figure 10: A repetition of the experiment performed in Fig. 9 (i.e., with a base utility function (**true positive fraction + 0.8 true negative fraction**) and $y = 1$ weighted regularized disparity term), on the UCI “Adult Data Set”, as detailed in section Section 4.3 with groups re-weighted for equal representation.

C.3 Training Curves: L-UCBFair

C.3.1

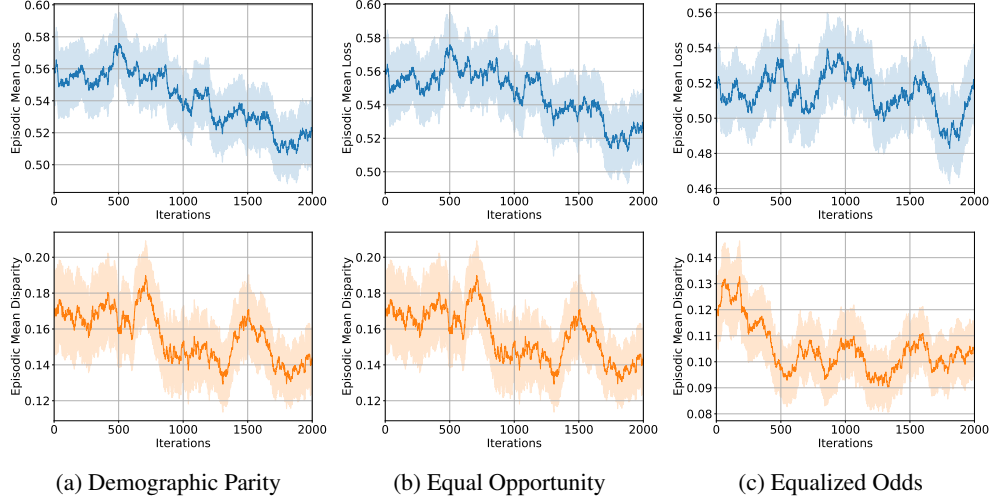


Figure 11: L-UCBFair 20-step sliding mean & std for the setting in Fig. 7.

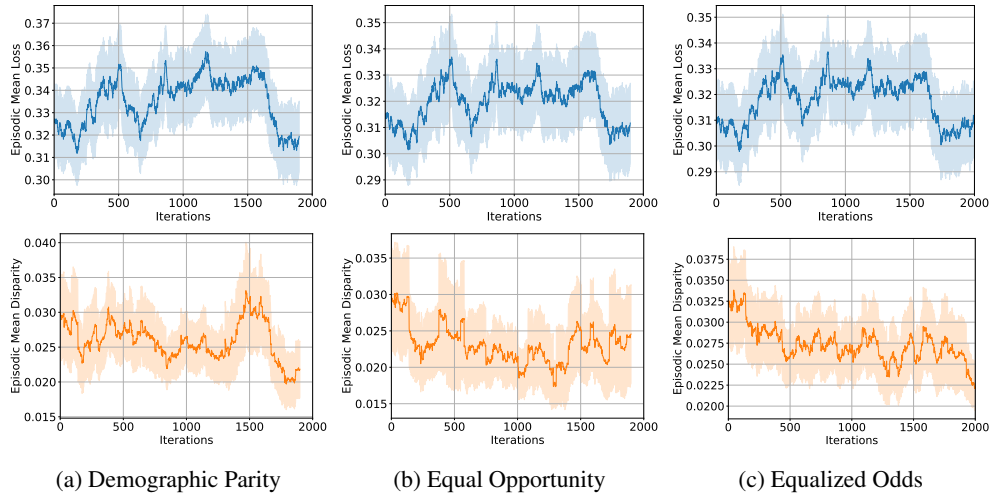


Figure 12: L-UCBFair 20-step sliding mean & std for the setting in Fig. 8.

C.4 Training Curves: R-TD3

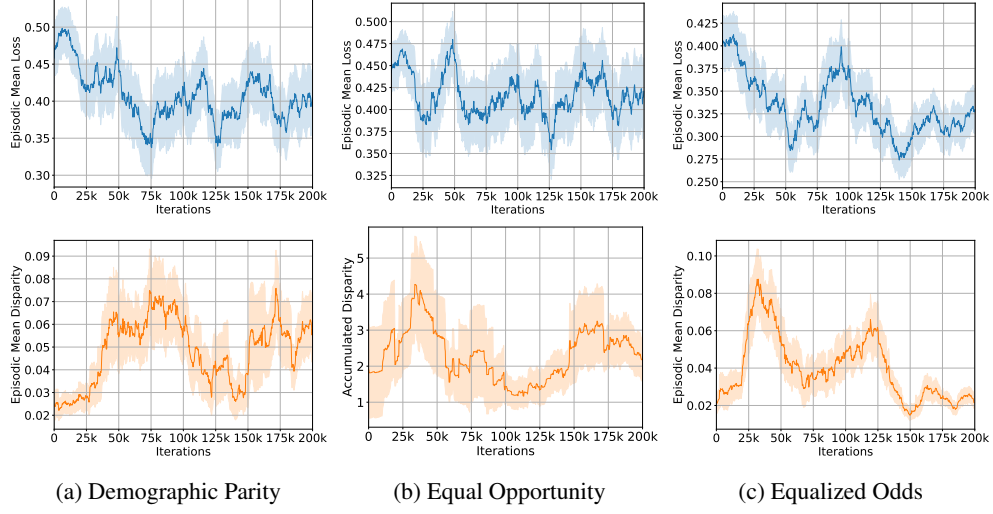


Figure 13: R-TD3 100-step sliding mean & std for the setting in Fig. 8.

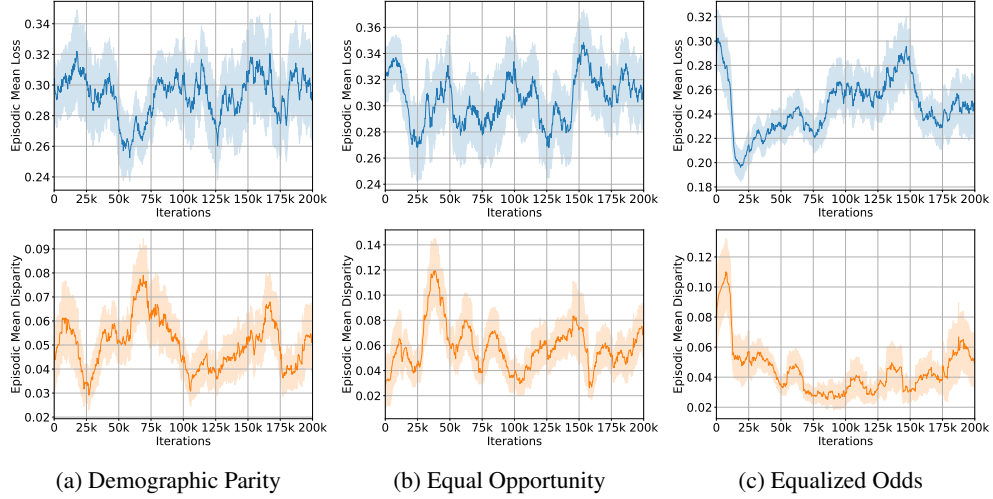


Figure 14: R-TD3 100-step sliding mean & std for the setting in Fig. 10.

C.5 Reduction of Utility

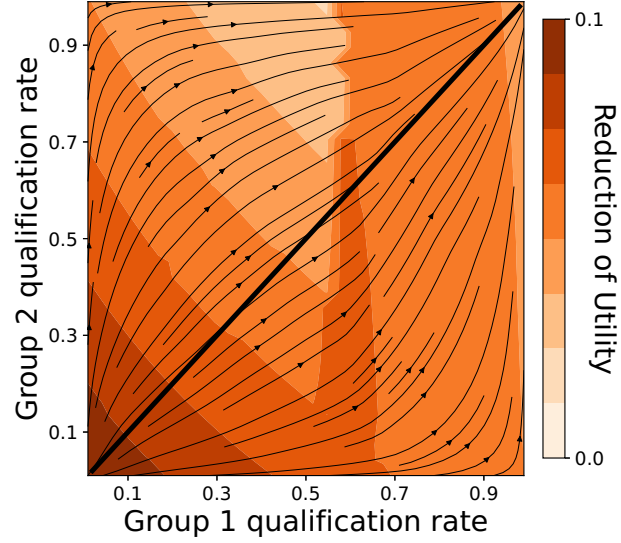


Figure 15: The figure depicts the short-term impact on utility of the UCBFair algorithm compared to a greedy baseline agent that operates without fairness constraints. In this experiment, both algorithms were designed to optimize the fraction of true-positive classifications, but only UCBFair was subject to the additional constraint of demographic parity. As the results indicate, the UCBFair algorithm experiences a reduction in utility compared to the greedy baseline, but it is able to drive the system towards a state that is preferable in the long term.