

Bruhat interval polytopes, 1-skeleton lattices, and smooth torus orbit closures

Christian Gaetz^{*1}

¹*Department of Mathematics, Cornell University, Ithaca, NY*

Abstract. Introduced by Kodama and Williams, *Bruhat interval polytopes* are generalized permutohedra closely connected to the study of torus orbit closures and total positivity in Schubert varieties. We show that the 1-skeleton posets of these polytopes are lattices and classify when the polytopes are simple, thereby resolving open problems and conjectures of Fraser, of Lee–Masuda, and of Lee–Masuda–Park. In particular, we classify when generic torus orbit closures in Schubert varieties are smooth.

Keywords: Bruhat interval polytope, weak order, Bruhat order, lattice, Schubert variety, torus orbit, smooth

1 Introduction

1.1 Bruhat interval polytopes

For a permutation w in S_n , write $\mathbf{w}$ for the vector $w^{-1}(1), \dots, w^{-1}(n) \in \mathbb{R}^n$. The *Bruhat interval polytope* Q_w is defined as the convex hull:

$$Q_w : \text{Conv} \{ \mathbf{u} \mid \mathbf{u} \preceq \mathbf{w} \} \subset \mathbb{R}^n,$$

where $\preceq$ denotes Bruhat order on S_n (see [Section 2](#)). Bruhat interval polytopes were introduced by Kodama and Williams in [\[18\]](#), where it is shown that they are the images under the *moment map* of the *Schubert variety* X_w in the flag variety, and also of the *totally positive part* $X_w^{\geq 0}$ of the Schubert variety. Therefore, the combinatorics of Q_w encodes information about the actions of the torus and positive torus on X_w and $X_w^{\geq 0}$ respectively.

The combinatorics of Q_w was studied further by Tsukerman and Williams [\[31\]](#), who showed that Q_w is a *generalized permutohedron* in the sense of Postnikov [\[27\]](#) and the matroid polytope of a flag *positroid*. Additional connections to the geometry of matroids were made in [\[6\]](#), and Bruhat interval polytopes have also appeared [\[32\]](#) in the context of BCFW-bridge decompositions [\[1\]](#) from physics, and in the study [\[20, 21, 22, 24\]](#) of *generic torus orbit closures* Y_w in X_w .

^{*}crgaetz@gmail.com. The author is supported by a Klarman Postdoctoral Fellowship at Cornell University.

1.2 The 1-skeleton of Q_w as a lattice

Throughout this work, we study the 1-skeleton poset P_w of Q_w , a partial order on the lower Bruhat interval $[e, w] = \{u \mid u \preceq w\}$.

Definition 1.1. The poset $(P_w, \leq_w)$ has underlying set the Bruhat interval $[e, w]$ and cover relations $u \prec_w v$ whenever Q_w has an edge between vertices u and v and $\ell(v) > \ell(u)$, where ℓ denotes Coxeter length.

When $w = w_0$ is the longest permutation, the polytope Q_w is the *permutohedron*, a fundamental object in algebraic combinatorics, and the poset P_w is the very well-studied *right weak order* (see Section 2). For general w , since edges of Q_w must be Bruhat covers by [31], the order $\leq_w$ is intermediate in strength between right weak order and Bruhat order on $[e, w]$. Since the work of Björner [3] it has been known that the weak order P_{w_0} on S_n is a *lattice*; in our first main theorem, we generalize this to all of the posets P_w .

Theorem A (Proven as Theorem 4.5). *Let $w \in S_n$, then P_w is a lattice.*

As explained below, special cases of this lattice structure confirm a conjecture of Fraser [12], recover several previous results of various authors, imply new properties of Q_w , and suggest interesting directions for future work.

1.2.1 BCFW-bridge decompositions

In the last decade, there has been an explosion of work (see [1]) relating the physical theory of *scattering amplitudes* to the combinatorics and geometry of the *totally nonnegative Grassmannian* $Gr(k, n)_{\geq 0}$ by way of the *amplituhedron*. In this setting, *on-shell diagrams* from physics correspond to *reduced plabic graphs*, which give parametrizations of an important cell decomposition of $Gr(k, n)_{\geq 0}$ [26].

In [1] it is shown that reduced plabic graphs for a given cell may be built up recursively using *BCFW-bridge decompositions*. In [32], Williams showed that these decompositions of plabic graphs correspond to the maximal chains in P_v when v is a Grassmannian permutation, analogous to the fact that reduced words for the longest permutation w_0 correspond to maximal chains in P_{w_0} (weak order). Since weak order is a lattice, Theorem A extends this analogy and implies new structure within the set of BCFW-bridge decompositions. That P_v is a lattice for Grassmannian permutations was conjectured by Fraser [12]. Fraser also conjectured that a larger class of posets, which are not necessarily the 1-skeleton posets of any polytope, are lattices; this problem remains open.

1.2.2 Quotients of weak order

Theorem A is proven by realizing P_w as a quotient of weak order P_{w_0} by an equivalence relation Θ_w which respects the weak order join operation (but does *not* respect the meet

operation!) Thus P_w is a *semilattice quotient* of P_{w_0} but not a *lattice quotient*. The lattice quotients and lattice homomorphisms of weak order have been classified [28, 29]. This work thus suggests that semilattice quotients and homomorphisms of weak order are an intriguing topic for further study.

1.2.3 The parabolic map and the mixed meet

Let $S_n I$ denote the subgroup of S_n generated by a subset I of the simple reflections. Billey, Fan, and Losonczy proved [2] that for any $w \in S_n$ the set $S_n I \cap e, w$ has a unique maximal element $m w, I$ under Bruhat order. Richmond and Slofstra [30] showed that this element $m w, I$ determines whether the projection of the Schubert variety $X_w \subset G/B$ to a partial flag variety G/P is a fiber bundle, and is thus important for understanding the singularities of X_w . We show in [Theorem 4.6](#) that the element $m w, I$ is just the join in P_w of the simple reflections from I , demonstrating the richness of the lattice structure on P_w .

A related operation of *mixed meet* was studied by Bump and Chetard in [9] in relation to certain intertwining operators of representations of reductive groups over nonarchimedean local fields. The mixed meet of $u, v \in S_n$ is the unique Bruhat maximal permutation in $e, u_R \cap e, v$. In the language of [Section 4](#), this element is $\text{bot}_v u$, the unique minimal element under $\leq_R$ in the equivalence class of u under the equivalence relation Θ_v induced on S_n by the normal fan of Q_v . This element is a translate of $\mu_v u$, where μ_v is the *matroid map* obtained by viewing e, v as a Coxeter matroid [7].

1.2.4 The non-revisiting path property

A polytope Q has the *non-revisiting path property* if no shortest path in its 1-skeleton between two vertices returns to a face after having left it. This property has long been of interest in the field of combinatorial optimization. In [14], Hersh proves that any simple polytope whose 1-skeleton poset is a lattice has the non-revisiting path property. Thus, combining [Theorem A](#) and the classification of simple Bruhat interval polytopes in [Theorem B](#) below, we obtain a rich new family of examples to which Hersh's theorem applies. Additionally, in [Section 5](#) we observe that all polytopes Q_w are *directionally simple*. We therefore ask: does Hersh's theorem extend to directionally simple polytopes?

1.3 Bruhat interval polytopes and generic torus orbit closures

1.3.1 Simple Bruhat interval polytopes and smooth torus orbit closures

Let $G = GL_n \mathbb{C}$, let B denote the Borel subgroup of upper triangular matrices, and let T denote the maximal torus of diagonal matrices. The *flag variety* $Fl_n = G/B$ and its Schubert subvarieties $X_w : \overline{BwB}/B$ are of fundamental importance in many areas of

algebraic combinatorics, algebraic geometry, and representation theory. The torus T acts naturally on G/B via left multiplication, and the fixed points $(G/B)^T$ are the points wB for $w \in S_n$, where we identify w with its permutation matrix. The fixed points of the Schubert variety X_w are $\{uB \mid u \preceq w\}$.

Torus orbits in G/B and their closures are a rich family of varieties, studied since Klyachko [17] and Gelfand–Serganova [13] with close connections to matroids and Coxeter matroids [7]. One class of torus orbit closures has received considerable interest [20, 21, 22, 24] of late: *generic* torus orbit closures in Schubert varieties. A torus orbit closure $Y \subset X_w$ is called *generic* if $Y^T = X_w^T$; we write Y_w for a generic torus orbit closure in X_w .

One of the main properties of interest for torus orbits in the flag variety has historically been their singularities [10, 11], and in particular determining when they are smooth. For Schubert varieties themselves, smoothness was famously characterized by Lakshmibai–Sandhya [19] in terms of pattern avoidance. In our next main theorem, we resolve a conjecture of Lee and Masuda [20] by classifying when Y_w is smooth.

Theorem B (Conjectured by Lee–Masuda [20]; Proven below as Corollary 6.3). *Let $w \in S_n$, then Q_w is a simple polytope if and only if it is simple at the vertex w ; equivalently, Y_w is a smooth variety if and only if it is smooth at the point wB .*

Theorem B is proven by showing (see Theorem 6.1) that the degree of a vertex of Q_w is an ordering preserving function of the poset P_w .

By [20], the condition that Y_w is smooth at wB can be checked combinatorially by determining whether a certain graph $\Gamma_w(w)$ is a tree (see Section 3). This tree condition has in turn been characterized combinatorially in terms of pattern avoidance [8], and shown [33] to characterize when X_w is *locally factorial*. By work of Björner–Ekedahl [5] it is also equivalent to the vanishing of the coefficient of q in the associated *Kazhdan–Lusztig polynomial* [15] and thus [16] the vanishing of a certain middle intersection cohomology group of X_w . It would be fascinating to give a purely geometric explanation for the equivalence (by Theorem B) of the smoothness of Y_w with these other geometric conditions on X_w .

1.3.2 Directionally simple polytopes and h -vectors

In Section 5 we show that, even when Q_w is not a simple polytope, it is still *directionally simple* (see Definition 5.1). This fact was also shown in [24] by an involved calculation, but follows directly from our results realizing P_w as a quotient of weak order. This property of Q_w implies that its h -vector has positive entries which count certain permutations according to their number of ascents. In Proposition 5.6 we resolve an open problem of Lee–Masuda–Park [23] by showing that Y_w is smooth if and only if this h -vector is palindromic.

2 Background on the weak and strong Bruhat orders

We refer the reader to [4] for basic definitions and results on Coxeter groups.

We view the symmetric group S_n as a Coxeter group with generators $s_1, \dots, s_{n-1}$, where $s_i : i \ i+1$ is an adjacent transposition. An expression $w = s_{i_1} \cdots s_{i_\ell}$ of minimal length is a *reduced word* for w and in this case the quantity $\ell = \ell(w)$ is the *length* of w . There are two important partial orders on S_n , each graded by length. The *right weak order* $\leq_R$ by definition has cover relations $w \leq_R ws$ whenever s is a simple generator and $\ell(ws) = \ell(w) + 1$; the (strong) *Bruhat order* $\preceq$ has cover relations $w \preceq wt$ whenever $\ell(wt) = \ell(w) + 1$ and t lies in the set T of transpositions ij . We write $[v, w]_R$ and $[v, w]$ for the closed interval between v, w in right weak and Bruhat order respectively.

The *left inversions* of an element $w \in S_n$ are the reflections $T_L w := \{t \in T \mid \ell(tw) < \ell(w)\}$. It is well-known that weak order is characterized by containment of inversions:

Proposition 2.1. *Let $v, w \in S_n$, then $v \leq_R w$ if and only if $T_L v \subseteq T_L w$.*

The symmetric group contains a unique element w_0 of maximum length, and w_0 is the unique maximal element of S_n under both $\leq_R$ and $\preceq$. In fact, weak order is a lattice.

Theorem 2.2 (Björner [3]). *The poset $(S_n, \leq_R)$ is a lattice; we write $\wedge_R$ and $\vee_R$ for the meet and join in right weak order.*

3 The graphs $\tilde{\Gamma}_w$ and Γ_w

Lee and Masuda [20] provided a combinatorial model for determining the edges incident to a vertex u of Q_w .

Definition 3.1 (Lee and Masuda [20]). For $u \preceq w$, the directed graph $\tilde{\Gamma}_w u$ has vertex set $[u, w]$ with directed edges $u \ i, u \ j$ whenever $i < j$, $u \ ij \preceq w$, and $|\ell(u \ ij) - \ell(u)| = 1$. We write $\tilde{E}_w u$ for this set of edges. The directed graph $\Gamma_w u$ is defined to be the transitive reduction of $\tilde{\Gamma}_w u$, with edge set $E_w u$.

Proposition 3.2 (Lee and Masuda [20], Proposition 7.7). *Two vertices u and v of Q_w are connected by an edge of the polytope if and only if $v = u \ ij$ where $u \ i, u \ j \in E_w u$.*

The proofs of the main theorems rely on a detailed development of the properties of directed paths in the graphs $\Gamma_w u$ and $\tilde{\Gamma}_w u$, and how these change as u varies, some of which are sampled below.

3.1 Directed paths and local changes

When the permutation w is understood, we write $a \xrightarrow{u} b$ when $a, b \in \tilde{E}_w u$ and $a \xrightarrow{u} b$ when there is a directed path from a to b in $\tilde{\Gamma}_w u$ (equivalently, in $\Gamma_w u$); we write $a \xRightarrow{u} b$ when $a, b \in E_w u$.

Proposition 3.3. *Let $u, w \in S_n$ with $u \preceq w$, and suppose $ab u \preceq w$, where $u^{-1} a < u^{-1} b$, then $a \xrightarrow{u} b$. In particular, if $ab \in T_L u$, then $a \xrightarrow{u} b$.*

The following proposition is a fundamental ingredient in many of proofs of the subsequent theorems. Its proof is rather involved and must be omitted for the sake of space.

Proposition 3.4. *Suppose that $u, v \in S_n$ satisfy $u \xrightarrow{w} v$ and $ab u$, with $a < b$:*

- (i) *If $c < d$ and $c \xrightarrow{v} d$, then $c \xrightarrow{u} d$;*
- (ii) *If $c > d, c \wedge b, d \wedge a$, and $c \xrightarrow{v} d$, then $c \xrightarrow{u} d$.*

4 The lattice property

4.1 Generalized permutohedra

The normal fan of the permutohedron $\text{Perm}_n = Q_{w_0}$ is the fan determined by the *braid arrangement*, which has defining hyperplanes $x_i - x_j = 0$ for $\forall i \neq j$. The top-dimensional cones $C y$ in this fan are naturally labelled by permutations $y \in S_n$ giving the relative order of the coordinates of a point $x_1, \dots, x_n \in C y$; in particular we have $y \in C y$.

Following Postnikov [27], a polytope whose normal fan coarsens the braid arrangement is called a *generalized permutohedron*. Kodama–Williams [18] showed that Bruhat interval polytopes are generalized permutohedra. This implies that each top-dimensional cone $C_w u$ (where u now runs over the elements of e, w) in the normal fan of Q_w contains some of the $C y$. We write y_w for the equivalence class of y under the equivalence relation Θ_w induced on S_n by the normal fan of Q_w .

We say $y \in S_n$ is a *linear extension* of $\Gamma_w u$ (equivalently, of $\tilde{\Gamma}_w u$) if $y^{-1} i < y^{-1} j$ whenever $i \xrightarrow{u} j$. The following proposition is immediate from the construction of $\Gamma_w u$ in Section 5 of [20] and the discussion of normal fans of generalized permutohedra in Section 3 of [25].

Proposition 4.1 (See [20, 25]). *Let $w \in S_n$ and $u \preceq w$, then u_w is exactly the set of linear extensions of $\Gamma_w u$.*

Somewhat surprisingly, the equivalence classes x_w turn out to be intervals in right weak order. This result was established by other means in [20], but the particulars of our proof will be important later.

Proposition 4.2. *Let $x, w \in S_n$, then x_w contains a unique minimal element $\text{bot}_w x$ and unique maximal element $\text{top}_w x$ under right weak order.*

Proof. Let u be the unique element of $e, w \cap x_w$. By Proposition 4.1, the elements y of x_w are exactly the linear extensions of $\tilde{\Gamma}_w u$. Suppose that $ab \in T_L u$ with $a < b$, then by Proposition 3.3 we have $b \xrightarrow{u} a$, so by Proposition 4.1 we have $ab \in T_L y$ for any $y \in x_w$. Thus by Proposition 2.1 we have $u \leq_R y$, so $\text{bot}_w x = u$.

The reflections occurring as left inversions of some linear extension of $\tilde{\Gamma}_w u$ are exactly those in

$$I : \{ ab \mid a < b \text{ and } a \not\xrightarrow{y} b \}.$$

To see that $\text{top}_w x$ exists, we will demonstrate that the corresponding set of roots, $R = \{e_a - e_b \mid ab \in I\}$, is biclosed, so that $\text{top}_w x$ will be the unique permutation with left inversion set I .

First, note that if $a \xrightarrow{u} b$ and $b \xrightarrow{u} c$, then $a \xrightarrow{u} c$, so R is coclosed.

For closedness, let $a < b < c$ and assume that $a \not\xrightarrow{u} c$, which implies that $u^{-1} a < u^{-1} c$. If $u^{-1} b < u^{-1} a$, then $ab \in T_L u$, so by Proposition 3.3 we have $b \xrightarrow{u} a \xrightarrow{u} c$, so $b \xrightarrow{u} c$. If instead $u^{-1} b > u^{-1} c$, then $bc \in T_L u$, so by Proposition 3.3 we have $a \xrightarrow{u} c \xrightarrow{u} b$, so $a \xrightarrow{u} b$. Otherwise we have $u^{-1} a < u^{-1} b < u^{-1} c$. Consider a path $a \rightarrow a_1 \rightarrow \cdots \rightarrow a_r \rightarrow c_1 \rightarrow \cdots \rightarrow c_s \rightarrow c$, where $u^{-1} a_i \leq u^{-1} b$ and $u^{-1} b < u^{-1} c_j$ for all i, j . If any $a_i > b$, then $a \xrightarrow{u} a_i \xrightarrow{u} b$. If any $c_j < b$, then $b \xrightarrow{u} c_j \xrightarrow{u} c$. Otherwise, since $a_r c_1 u$ covers u in Bruhat order, we must have $a_r = b$, so $a \xrightarrow{u} b \xrightarrow{u} c$. In all cases, we see $a \xrightarrow{u} b$ or $b \xrightarrow{u} c$, so R is closed. $\square$

4.2 The poset structure

Theorem 4.3. *Given $w \in S_n$, the map $\text{top}_w : S_n \rightarrow S_n$ is order preserving with respect to right weak order. That is, if $x \leq_R y$ then $\text{top}_w x \leq_R \text{top}_w y$. Furthermore, the quotient $\text{Weak}_R S_n / \Theta_w$ is isomorphic to P_w via the map $x_w \mapsto \text{bot}_w x$.*

Proof. By the proof of Proposition 4.2, we have for any $z \in S_n$ that

$$T_L \text{top}_w z = \{ cd \mid c < d \text{ and } c \not\xrightarrow{\text{bot}_w z} d \}. \quad (4.1)$$

Thus if $u \leq_w v$ and $ab \in T_L u$ with $a < b$ then by Proposition 3.4(i) and (4.1) we have that

$$T_L \text{top}_w u \subset T_L \text{top}_w v.$$

By Proposition 2.1 we see $\text{top}_w u \leq_R \text{top}_w v$.

Now suppose that $x \leq_R y$ and $xs = tx$ with $x_w \not\leq y_w$, so $C_w u$ and $C_w v$ share a facet along the hyperplane fixed by the reflection t , where $u = \text{bot}_w x$ and $v =$

$\text{bot}_w y$. This implies that there is an edge of Q_w with vertices u and v . Since $u \leq_R x$ by Proposition 4.2, we have that $t \notin T_L u$ by Proposition 2.1 and the fact that $t \notin T_L x$. Since the convex cones $C_w u$ and $C_w v$ share at most one facet, and since $v = ut'$ for some $t' \in T$, we must have in fact that $t = t'$, and thus $\ell v > \ell u$ and $u <_w v$. This establishes that $P_w \sim \text{Weak}_n / \Theta_w$, and that top_w is order preserving, after applying the first paragraph and that fact that $\text{top}_w u = \text{top}_w x$ and $\text{top}_w v = \text{top}_w y$. $\square$

Corollary 4.4. *Let $w \in S_n$, then the map $\text{top}_w : e, w \rightarrow \text{top}_w e, w$ is an isomorphism between the posets P_w and $\text{top}_w e, w, \leq_R$.*

The fact that P_w is a lattice also follows easily from Theorem 4.3.

Theorem 4.5. *For any $w \in S_n$, the poset P_w is a lattice, with join operation given by*

$$u \vee_w v = \text{bot}_w \text{top}_w u \vee_R \text{top}_w v.$$

Proof. Let $z = \text{bot}_w \text{top}_w u \vee_R \text{top}_w v$. Then

$$u \leq_R \text{top}_w u \leq_R \text{top}_w u \vee_R \text{top}_w v,$$

so by Theorem 4.3 we have $u \leq_w z$, and similarly $v \leq_w z$. On the other hand, if $y \geq_w u, v$, then by Theorem 4.3 we have $\text{top}_w y \geq \text{top}_w u, \text{top}_w v$ so $\text{top}_w y \geq \text{top}_w u \vee_R \text{top}_w v$. Thus $y \geq_w z$ and we see that z is the join of u, v in P_w . Since P_w is a finite poset with a join and a unique minimal element, it also has a meet and is thus a lattice. $\square$

Unlike weak order, P_w is not in general a semidistributive lattice.

4.3 The Billey–Fan–Losonczy parabolic map

Let $S_n I$ denote the subgroup of S_n generated by a subset I of the simple reflections. For $w \in S_n$ let $m_{w, I}$ denote the unique maximal element of $S_n I \cap e, w$ under Bruhat order [2] (see Richmond and Slofstra [30] for the importance of these elements in determining the fiber bundle structure of Schubert varieties).

Theorem 4.6. *Let $w \in S_n$, and let I be a set of simple generators, then:*

$$m_{w, I} = \bigvee_w \{s_i \in I \mid s_i \preceq w\}.$$

5 Directionally simple polytopes

Given a polytope $Q \subset \mathbb{R}^d$, say that a cost vector $c \in \mathbb{R}^d$ is *generic* if c is not orthogonal to any edge of Q . A generic cost vector induces an acyclic orientation on the 1-skeleton G_Q by taking edges to be oriented in the direction of greater inner product with c ; we write $G_c Q$ for the resulting acyclic directed graph. It is clear that every face F of Q contains a unique source $\min_c F$ and sink $\max_c F$ with respect to this orientation.

Definition 5.1. We say that a polytope $Q \subset \mathbb{R}^d$ is *directionally simple* with respect to the generic cost vector c if for every vertex v of Q and every set E of edges of $G_c Q$ with source v there exists a face F of Q containing v whose set of edges incident to v is E .

Proposition 5.2. A simple polytope $Q \subset \mathbb{R}^d$ is directionally simple with respect to any generic cost vector.

[Theorem 5.3](#) was proven in [24] by an involved direct computation; here we give a new proof using the results of [Section 4](#).

Theorem 5.3. Let $w \in S_n$, then Q_w is a directionally simple polytope with respect to the cost vector $c = (n, n-1, \dots, 1)$.

[Theorem 5.3](#) shows that Q_w is always directionally simple, in [Section 6](#) we will determine when Q_w is in fact simple.

5.1 h -vectors of directionally simple polytopes

The f -vector of a polytope $Q \subset \mathbb{R}^d$ is the tuple $f(Q) = (f_0, \dots, f_d)$ where f_i is the number of i -dimensional faces of Q . The h -vector $h(Q)$ is defined by the equality of polynomials

$$\sum_{i=0}^d f_i (x-1)^i = \sum_{k=0}^d h_k x^k. \quad (5.1)$$

Proposition 5.4. Let $Q \subset \mathbb{R}^d$ be directionally simple with respect to the generic cost vector c , with h -vector $h(Q) = (h_0, \dots, h_d)$. Then for all $k = 0, \dots, d$ the entry h_k is the number of vertices of Q with out-degree exactly k in $G_c Q$.

Remark. One implication of [Proposition 5.4](#) is that $h_i \geq 0, \forall i$. This by itself is already a very special property of directionally simple polytopes; indeed h -vectors of non-simple polytopes are rarely considered, because they are rarely positive or otherwise interesting.

For $u \in S_n$, write $\text{asc } u$ for the number $n-1-|D_R u|$ of right ascents of u . [Corollary 5.5](#) below is an extension to Q_w of the kind of interpretation for h -vectors of simple generalized permutohedra given by Postnikov–Reiner–Williams [25].

Corollary 5.5. Let $h_0, h_1, \dots$ be the h -vector of Q_w , then for all k we have:

$$h_k = |\{z \in \text{top}_w(e, w) \mid \text{asc } z = k\}|.$$

As explained in [24], the h -vector of Q_w also gives the Poincaré polynomial of the toric variety Y_w , so [Corollary 5.5](#) gives a new formula for that invariant. We can also resolve an open problem raised in [23]:

Proposition 5.6 (Resolves Problem 6.1 of [23]). The variety Y_w is smooth if and only if its Poincaré polynomial is palindromic.

6 Vertex-degree monotonicity

In [Section 4](#) we applied properties of the relation $c \xrightarrow{u} d$ to prove that P_w is a lattice. It is possible to develop and use more refined information about the relation $c \xrightarrow{u} d$ to prove that vertex-degrees of Q_w are monotonic with respect to the partial order $\leq_w$, but we lack the space to do this here. As an application, we resolve a conjecture of Lee–Masuda–Park [\[20\]](#) characterizing smooth generic torus orbit closures in Schubert varieties.

Write $\deg_w u$ for the number of edges of Q_w incident to the vertex u .

Theorem 6.1. *Let $w \in S_n$. If $u \leq_w v$ then $\deg_w u \leq \deg_w v$.*

[Theorem 6.1](#) will follow from the stronger [Theorem 6.4](#) below.

Corollary 6.2. *Let $w \in S_n$, then the polytope Q_w is simple if and only if it is simple at the vertex w .*

Proof. It is clear from [Proposition 3.2](#) and the definition of $E_w e$ that Q_w is always simple at the vertex e . Thus if Q_w is also simple at w , [Theorem 6.1](#) implies that it is simple at every vertex. $\square$

[Corollary 6.2](#) resolves Conjecture 7.17 of Lee–Masuda [\[20\]](#). As described there, [Corollary 6.2](#) has the following geometric interpretation.

Corollary 6.3. *Let Y_w be a generic torus orbit closure in the Schubert variety $X_w : \overline{BwB}/B$, then Y_w is smooth if and only if it is smooth at the torus fixed point wB .*

Write $c \xleftrightarrow{u} d$ if $c \xleftarrow{u} d$ or $c \xrightarrow{u} d$ (note that we never have both $c \xleftarrow{u} d$ and $c \xrightarrow{u} d$).

Theorem 6.4. *Let $w \in S_n$ and suppose $u \leq_w v$ with $c \xleftrightarrow{u} d$, then there is a unique edge of $E_w v$ described by $c \xleftrightarrow{v} d$ or $t \xleftrightarrow{v} t d$. Moreover, the map $\varphi : E_w u \rightarrow E_w v$ sending the edge $c \xleftrightarrow{u} d$ to the corresponding edge of $E_w v$ is an injection.*

[Theorem 6.1](#) follows from [Theorem 6.4](#) since, by [Proposition 3.2](#) we have $\deg_w u = |E_w u|$ for all $u \leq w$.

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The distribution of descents on nonnesting permutations

Sergi Elizalde^{*1}

¹*Department of Mathematics, Dartmouth College, Hanover, NH 03755*

Abstract. Motivated by recent results about descents on Stirling and quasi-Stirling permutations, we consider permutations of the multiset $\{1, 1, 2, 2, \dots, n, n\}$ that avoid the patterns 1221 and 2112. We call these nonnesting permutations, as they can be viewed as nonnesting matchings with labeled arcs. We show that the polynomial describing the distribution of the number of descents is a product of an Eulerian polynomial and a Narayana polynomial. It follows that, rather unexpectedly, this polynomial is palindromic. We provide bijective proofs of these facts by composing various transformations on Dyck paths, including the Lalanne–Kreweras involution.

Keywords: nonnesting, quasi-Stirling, canon permutation, Dyck path, Narayana

1 Introduction

1.1 Permutations of multisets and descents

Given a sequence of positive integers $\pi = \pi_1 \pi_2 \dots \pi_m$, we say that i is a *descent* of π if $\pi_i > \pi_{i+1}$, that it is a *plateau* if $\pi_i = \pi_{i+1}$, and that it is a *weak descent* if $\pi_i \geq \pi_{i+1}$. Denote by $\text{des } \pi$, $\text{plat } \pi$ and $\text{wdes } \pi = \text{des } \pi + \text{plat } \pi$ the number of descents, plateaus and weak descents of π , respectively.

The distribution of descents on the set $\mathcal{S}_n$ of permutations of $n : \{1, 2, \dots, n\}$ is given by the Eulerian polynomials

$$A_n(t) = \sum_{\sigma \in \mathcal{S}_n} t^{\text{des } \sigma},$$

whose generating function is

$$A(t, z) = \sum_{n \geq 0} A_n(t) \frac{z^n}{n!} = \frac{t-1}{t - e^{t-1}z}. \quad (1.1)$$

Note that the Eulerian polynomials are palindromic, in the sense that

$$A_n(t) = t^{n-1} A_n(1/t), \quad (1.2)$$

^{*}sergi.elizalde@dartmouth.edu

since

$$\text{des } \sigma^R = n - 1 - \text{des } \sigma, \quad (1.3)$$

where $\sigma^R = \sigma_n \sigma_{n-1} \dots \sigma_1$ denotes the reversal of σ .

In 1978, Gessel and Stanley [9] introduced *Stirling permutations*, defined as permutations $\pi_1 \pi_2 \dots \pi_{2n}$ of the multiset $n \sqcup n : \{1, 1, 2, 2, \dots, n, n\}$ satisfying that, if $i < j < k$ and $\pi_i = \pi_k$, then $\pi_j > \pi_i$. In pattern avoidance terminology, this condition is equivalent to avoiding the pattern 212. They showed that the distribution of the number of descents on such permutations is related to the Stirling numbers of the second kind. There is an extensive literature on these permutations and their generalizations to other multisets.

In 2019, Archer et al. [1] introduced a variation of Stirling permutations, which they call *quasi-Stirling permutations*. These are permutations $\pi_1 \pi_2 \dots \pi_{2n}$ of $n \sqcup n$ that avoid 1212 and 2121, meaning that there do not exist $i < j < \ell < m$ such that $\pi_i = \pi_\ell$ and $\pi_j = \pi_m$. The number of such permutations is $n! \text{Cat}_n \frac{2n!}{(n-1)!}$, where Cat_n is the n th Catalan number. The generating function for these permutations with respect to the number of descents and plateaus was later found in [6].

Theorem 1.1 ([6]). *Denote by $\overline{\mathcal{Q}}_n$ the set of quasi-Stirling permutations of $n \sqcup n$, and let*

$$\overline{Q}(t, u, z) = \sum_{n \geq 0} \sum_{\pi \in \overline{\mathcal{Q}}_n} t^{\text{des } \pi} u^{\text{plat } \pi} \frac{z^n}{n!}.$$

Then this generating function satisfies the implicit equation

$$\overline{Q}(t, u, z) = A(t, z) \overline{Q}(t, u, z) - t u,$$

where $A(t, z)$ is given by Equation (1.1).

In particular, it is shown that the number of $\pi \in \overline{\mathcal{Q}}_n$ with $n - 1$ descents is equal to $n - 1 \cdot n^{n-1}$. Further generalizations of these results to multisets with an arbitrary number of copies of each element have recently been obtained by Yan et al. [18] and by Fu and Li [8].

1.2 Nonnesting permutations

A permutation π of $n \sqcup n$ can be viewed as a labeled matching of $2n$, by placing an arc with label k between i and j if $\pi_i = \pi_j = k$. The condition that π avoids 1212 and 2121 is equivalent to the fact that this matching is *noncrossing*; see [16, Exer. 60]. With this perspective, it is natural to also consider permutations for which this matching is *nonnesting*; see [16, Exer. 64].



Figure 1: The matchings corresponding to the quasi-Stirling (noncrossing) permutation 4431152253 and the nonnesting permutation 3532521414 $\in \mathcal{C}_5$.

Definition 1.2. A permutation π of the multiset $n \sqcup n$ is called *nonnesting* if it avoids the patterns 1221 and 2112; equivalently, if there do not exist $i < j < \ell < m$ such that $\pi_i = \pi_m$ and $\pi_j = \pi_\ell$. Denote by $\mathcal{C}_n$ the set of nonnesting permutations of $n \sqcup n$.

The above condition on π is equivalent to the requirement that the subsequence of π determined by the first copy of each entry coincides with the subsequence determined by the second copy of each entry. This subsequence, which is a permutation in $\mathcal{S}_n$, will be denoted by $s \pi$. For example, if $\pi = 3532521414 \in \mathcal{C}_5$, then $s \pi = 35214 \in \mathcal{S}_5$.

As in the noncrossing case, the number of nonnesting matchings of $2n$ is again the n th Catalan number [16, Exer. 64]. Since there are $n!$ ways to assign labels to the arcs of a nonnesting matching to form a nonnesting permutation, it follows that

$$|\mathcal{C}_n| = n! \text{Cat}_n = \frac{(2n)!}{n+1}.$$

Motivated by the results on the distribution of the number of descents and plateaus on Stirling and quasi-Stirling permutations, here we describe the distribution of these statistics on nonnesting permutations. We are interested in the polynomial

$$C_n(t, u) = \sum_{\pi \in \mathcal{C}_n} t^{\text{des } \pi} u^{\text{plat } \pi}. \quad (1.4)$$

1.3 Dyck paths and Narayana numbers

Let $\mathcal{D}_n$ be the set of lattice paths from $(0, 0)$ to $(2n, 0)$ with steps $u = (1, 1)$ and $d = (1, -1)$ that do not go below the x -axis. Elements of $\mathcal{D}_n$ are called Dyck paths. A *peak* in a Dyck path is an occurrence of two adjacent steps ud . A peak is called a *low peak* if these steps touch the x -axis, and a *high peak* otherwise. Denote the number of low peaks and the number of high peaks of $D \in \mathcal{D}_n$ by $\text{lpea } D$ and $\text{hpea } D$, respectively. Sometimes it will be convenient to instead draw Dyck paths with steps $e = (1, 0)$ and $n = (0, 1)$ playing the role of u and d , respectively. Consider the *Narayana polynomials*

$$N_n(t, u) = \sum_{D \in \mathcal{D}_n} t^{\text{hpea } D} u^{\text{lpea } D}.$$

From the usual decomposition of Dyck paths by the first return, one can deduce that

$$\sum_{n \geq 0} N_n(t, u) z^n = \frac{2}{1 - \frac{1}{1 - t - 2uz} \sqrt{1 - 2\frac{1}{1 - tz} - 1 - t^2 z^2}}}.$$

A consequence of the above expression is that, for $n \geq 1$,

$$tN_n(t, 1) = N_n(t, t), \quad (1.5)$$

that is, the number of paths in $\mathcal{D}_n$ with $r - 1$ high peaks equals the number of those with r peaks, for all $r \in [n]$. A bijective proof of this fact was given by Deutsch [5]. Another simple bijection $\rho : \mathcal{D}_n \rightarrow \mathcal{D}_n$ with the property that $\text{hpea } D = \text{pea } \rho(D) - 1$ is obtained by applying the inverse of the *rowmotion* map on order ideals of the poset of positive roots in type A; see [17] for details.

An interesting property of the polynomial (1.5) is that it is palindromic, i.e.,

$$N(t, t) = t^{n-1} N(1/t, 1/t); \quad (1.6)$$

equivalently, the number of paths in $\mathcal{D}_n$ with r peaks equals the number of those with $n - 1 - r$ peaks, for all $r \in [n]$. This can be proved bijectively using an involution $\hat{\kappa} : \mathcal{D}_n \rightarrow \mathcal{D}_n$ due to Lanne [13] and Kreweras [11], which has the property that

$$\text{pea } D = n - 1 - \text{pea } \hat{\kappa}(D). \quad (1.7)$$

This extended abstract is structured as follows. In [Section 2](#) we present our main results. The proof of the formula giving the distribution of the number of descents and plateaus on nonnesting permutations can be found in the full version [7]; here we only give an overview of the bijections in the proof. In [Section 3](#) we use these bijections, along with the Lanne-Kreweras involution, to prove some symmetry properties of the distributions of the number of descents and weak descents on nonnesting permutations. Finally, [Section 4](#) discusses some generalizations of our results to permutations of the multiset with k copies of each entry, for any fixed k .

2 Main results

Our main result is the following strikingly simple expression for the polynomial $C_n(t, u)$ from Equation (1.4), as a product of an Eulerian polynomial and a Narayana polynomial.

Theorem 2.1. *For $n \geq 1$,*

$$C_n(t, u) = A_n(t) N_n(t, u).$$

Example 2.2. Here are the polynomials $C_n(t, u)$ for $n \in \{2, 3\}$, along with the factorizations given by [Theorem 2.1](#):

$$\begin{aligned} C_2(t, u) &= u^2 - 1 - u^2 t - t^2 - 1 - t (u^2 - t), \\ C_3(t, u) &= u^3 - 1 - 2u - 4u^3 t - 5 - 8u - u^3 t^2 - 5 - 2u t^3 - t^4 \\ &\quad \left(1 - 4t - t^2\right) \left(u^3 - 1 - 2u t - t^2\right). \end{aligned}$$

As a consequence, we obtain the following two statements about the symmetry of the distribution of descents and weak descents on nonnesting permutations.

Corollary 2.3. *The distribution of the number of weak descents on C_n is symmetric: for all r ,*

$$|\{\pi \in C_n : \text{wdes } \pi = r\}| = |\{\pi \in C_n : \text{wdes } \pi = 2n - r\}|.$$

Proof. This follows from [Theorem 2.1](#) and the fact that both $A_n(t)$ and $N_n(t, t)$ are palindromic. Indeed, Equations (1.2) and (1.6) imply that

$$C_n(t, t) = A_n(t) N_n(t, t) = t^{2n} A(1/t) N_n(1/t, 1/t) = t^{2n} C_n(1/t, 1/t). \quad \square$$

Corollary 2.4. *The distribution of the number of descents on C_n is symmetric: for all r ,*

$$|\{\pi \in C_n : \text{des } \pi = r\}| = |\{\pi \in C_n : \text{des } \pi = 2n - 2 - r\}|.$$

Proof. In addition to Equation (1.2), we now use the fact that $N_n(t, 1) = t^{n-1} N_n(1/t, 1)$, which follows from Equations (1.5) and (1.6). We obtain

$$C_n(t, 1) = A_n(t) N_n(t, 1) = t^{2n-2} A(1/t) N_n(1/t, 1) = t^{2n-2} C_n(1/t, 1). \quad \square$$

Example 2.5. Setting $u = 1$ in [Example 2.2](#), we obtain the following palindromic expressions:

$$\begin{aligned} C_2(t, 1) &= 1 - 2t - t^2 - 1 - t - 1 - t, \\ C_3(t, 1) &= 1 - 7t - 14t^2 - 7t^3 - t^4 - 1 - 4t - t^2 - 1 - 3t - t^2. \end{aligned}$$

It is interesting to note that, in contrast to the symmetry of the Eulerian polynomials, there seems to be no obvious explanation for the symmetries described by [Corollaries 2.3](#) and [2.4](#). We remark that the analogous statements for quasi-Stirling permutations do not hold [6]. Bijective proofs of [Corollaries 2.3](#) and [2.4](#) will be provided in [Section 3](#).

To establish [Theorem 2.1](#), we prove a slightly stronger statement. For $\sigma \in \mathcal{S}_n$, define

$$C_n^\sigma = \{\pi \in C_n : s(\pi) = \sigma\},$$

which can be used to partition the set of nonnesting permutations as $C_n = \bigsqcup_{\sigma \in \mathcal{S}_n} C_n^\sigma$. Letting

$$C_n^\sigma(t, u) = \sum_{\pi \in C_n^\sigma} t^{\text{des } \pi} u^{\text{plat } \pi}, \quad (2.1)$$

we have the following refinement.

Theorem 2.6. For all $\sigma \in \mathcal{S}_n$,

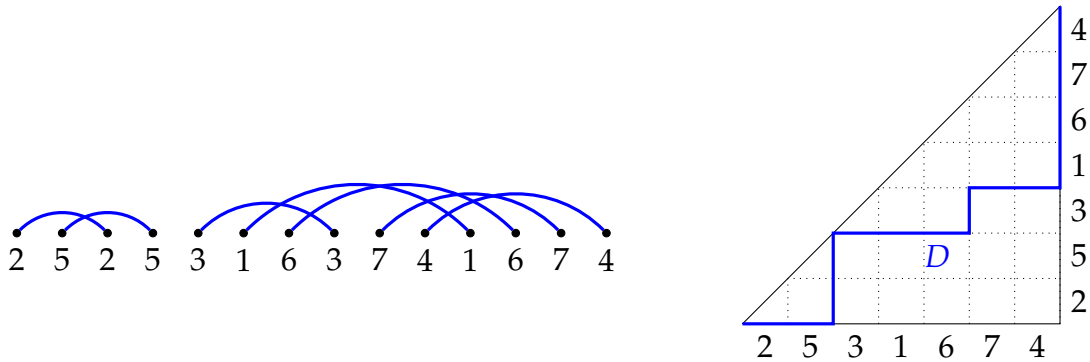
$$C_n^\sigma(t, u) = t^{\text{des } \sigma} N_n(t, u).$$

Note that Theorem 2.1 follows immediately from Theorem 2.6, since

$$C_n(t, u) = \sum_{\sigma \in \mathcal{S}_n} C_n^\sigma(t, u) = \sum_{\sigma \in \mathcal{S}_n} t^{\text{des } \sigma} N_n(t, u) = A_n(t) N_n(t, u).$$

The proof of Theorem 2.6 is omitted in this extended abstract but can be found in the full version [7]. Here we give an overview. A nonnesting permutation $\pi \in \mathcal{C}_n$ is uniquely determined by the underlying nonnesting matching $\text{mat } \pi$ and the subsequence of first (equivalently, second) copies $s \pi \in \mathcal{S}_n$. We denote by $d \pi$ the Dyck path that corresponds to $\text{mat } \pi$ via the standard bijection between nonnesting matchings and Dyck paths. This path is obtained by reading π from left to right and taking an e step for each first copy of an element, and an n step for each second copy; see Figure 2. Under this representation, plateaus of π correspond to low peaks of D , namely, occurrences of adjacent steps en that touch the diagonal, and so

$$\text{plat } \pi = \text{lpea } D. \quad (2.2)$$



in this case, and it follows that

$$C_n^l(t, u) = \sum_{\pi \in \mathcal{C}_n^l} t^{\text{des } \pi} u^{\text{plat } \pi} = \sum_{D \in \mathcal{D}_n} t^{\text{hpea } D} u^{\text{lpea } D} = N_n(t, u).$$

The computation of $C_n^\sigma(t, u)$ for arbitrary σ is done in two stages. We first prove that, for fixed n , the polynomial $C_n^\sigma(t, u)$ depends only on the descent set of σ , by showing that swapping any two non-adjacent entries of σ with consecutive values does not change this polynomial. For any $S \subseteq [n-1]$, this allows us to define $C_n^S(t, u) = C_n^\sigma(t, u)$, where σ is any permutation in $\mathcal{S}_n$ with descent set S , and to reduce the problem to the case where σ has a very specific form. We then prove bijectively that if S' is obtained from S by removing its largest element, then $C_n^S(t, u) = t C_n^{S'}(t, u)$, so one can repeatedly apply this fact to remove all the elements from the descent set of σ , reducing the problem to the case of the identity permutation. Our construction produces a bijection

$$\phi_\sigma : \mathcal{C}_n^\sigma \rightarrow \mathcal{C}_n^l \quad (2.4)$$

satisfying that

$$\text{des } \phi_\sigma(\pi) = \text{des } \pi - \text{des } \sigma \quad \text{and} \quad \text{plat } \phi_\sigma(\pi) = \text{plat } \pi \quad (2.5)$$

for all $\pi \in \mathcal{C}_n^\sigma$.

For example, if

$$\sigma = 281375496 \quad \text{and} \quad \pi = 228183175437954696 \in \mathcal{C}_9^\sigma, \quad (2.6)$$

then $\phi_\sigma(\pi) = 112324354657896789$, where $\text{des } \sigma = 4$, $\text{des } \pi = 9$, and $\text{des } \phi_\sigma(\pi) = 5$. The details of the construction can be found in [7].

We conclude this section with two statements about the symmetry of the distributions of descents and weak descents on the set of nonnesting permutations having a fixed underlying σ . These are simple consequences of [Theorem 2.6](#) and the palindromicity of $N_n(t, t)$ and $N_n(t, 1)$, respectively.

Corollary 2.7. *For each $\sigma \in \mathcal{S}_n$, the distribution of the number of weak descents on $\mathcal{C}_n^\sigma$ is symmetric: for all r ,*

$$|\{\pi \in \mathcal{C}_n^\sigma : \text{wdes } \pi - \text{des } \sigma = r\}| = |\{\pi \in \mathcal{C}_n^\sigma : \text{wdes } \pi - \text{des } \sigma = n - 1 - r\}|.$$

Corollary 2.8. *For each $\sigma \in \mathcal{S}_n$, the distribution of the number of descents on $\mathcal{C}_n^\sigma$ is symmetric: for all r ,*

$$|\{\pi \in \mathcal{C}_n^\sigma : \text{des } \pi - \text{des } \sigma = r\}| = |\{\pi \in \mathcal{C}_n^\sigma : \text{des } \pi - \text{des } \sigma = n - 1 - r\}|.$$

3 Bijective proofs of symmetry

Combining the bijection ϕ_σ from Equation (2.4) with the involution $\hat{\kappa}$ from Section 1.3, one can give bijective proofs of Corollaries 2.3, 2.4, 2.7 and 2.8.

The Lalanne–Kreweras involution $\hat{\kappa}$ can be interpreted as an involution on permutations $\kappa : \mathcal{C}_n^l \rightarrow \mathcal{C}_n^l$, by identifying each $\pi \in \mathcal{C}_n^l$ with its associated Dyck path $d(\pi) \in \mathcal{D}_n$. Specifically, given $\pi \in \mathcal{C}_n^l$ with associated Dyck path $D = d(\pi)$, define $\kappa(\pi) \in \mathcal{C}_n^l$ to be the permutation whose associated Dyck path is $\hat{\kappa}(D)$.

If $\pi \in \mathcal{C}_n^l$, Equations (2.2) and (2.3) imply that

$$\text{wdes } \pi = \text{des } \pi - \text{plat } \pi = \text{hpea } D - \text{lpea } D = \text{pea } D.$$

Thus, the behavior of $\hat{\kappa}$ on the number of peaks described in Equation (1.7) translates to the fact that, for any $\pi \in \mathcal{C}_n^l$,

$$\text{wdes } \pi = n - 1 - \text{wdes } \kappa(\pi). \quad (3.1)$$

Bijjective proof of Corollary 2.7. Define an involution $\Phi_\sigma = \phi_\sigma^{-1} \circ \kappa \circ \phi_\sigma$ from $\mathcal{C}_n^\sigma$ to itself. Adding the two equalities in Equation (2.5),

$$\text{wdes } \phi_\sigma(\pi) = \text{wdes } \pi - \text{des } \sigma \quad (3.2)$$

for every $\pi \in \mathcal{C}_n^\sigma$. Using this property twice and applying Equation (3.1), it follows that

$$\begin{aligned} \text{wdes } \pi - \text{des } \sigma &= \text{wdes } \phi_\sigma(\pi) = n - 1 - \text{wdes } \kappa(\phi_\sigma(\pi)) \\ &= n - 1 - \text{wdes } \Phi_\sigma(\pi) = \text{des } \sigma \end{aligned} \quad (3.3)$$

for every $\pi \in \mathcal{C}_n^\sigma$. Thus, the map Φ_σ provides the desired bijection. $\square$

For example, for σ and π as in Equation (2.6), we have $\Phi_\sigma(\pi) = 281372581347549966$. Note that $\text{wdes } \pi = 10$ and $\text{wdes } \Phi_\sigma(\pi) = 8$, satisfying Equation (3.3).

Bijjective proof of Corollary 2.3. Consider now the composition $\Psi_\sigma = \phi_{\sigma^R}^{-1} \circ \kappa \circ \phi_\sigma$, which is a bijection from $\mathcal{C}_n^\sigma$ to $\mathcal{C}_n^{\sigma^R}$. By definition, the inverse of Ψ_σ is simply Ψ_{σ^R} . Equation (3.2) for σ and σ^R , together with Equation (3.1), imply that

$$\text{wdes } \pi - \text{des } \sigma = n - 1 - \text{wdes } \Psi_\sigma(\pi) = \text{des } \sigma^R.$$

Using Equation (1.3), it follows that

$$\text{wdes } \pi = 2n - \text{wdes } \Psi_\sigma(\pi). \quad (3.4)$$

Thus, the involution $\Psi : \mathcal{C}_n \rightarrow \mathcal{C}_n$ defined by $\Psi(\pi) = \Psi_\sigma(\pi)$ whenever $\pi \in \mathcal{C}_n^\sigma$ provides the desired bijection. $\square$

Bijjective proof of Corollary 2.8. Denote by R the reversal map that takes π to π^R , and define the composition $\bar{\Phi}_\sigma = R \circ \Phi_{\sigma^R} \circ R$, which is an involution from $\mathcal{C}_n^\sigma$ to itself.

Let $\pi \in \mathcal{C}_n^\sigma$. Using the fact that

$$\text{des } \pi^R = 2n - 1 - \text{wdes } \pi, \quad (3.5)$$

together with Equation (1.3), we have

$$\text{des } \pi - \text{des } \sigma = n - \text{wdes } \pi^R - \text{des } \sigma^R. \quad (3.6)$$

Using Equation (3.3) with π^R and σ^R playing the roles of π and σ , respectively, together with Equation (3.6) applied twice, we get

$$\text{des } \pi - \text{des } \sigma = n - 1 - \text{des } \bar{\Phi}_\sigma \pi - \text{des } \sigma,$$

and so $\bar{\Phi}_\sigma$ gives the desired bijection. $\square$

Bijjective proof of Corollary 2.4. Consider the composition $\bar{\Psi}_\sigma = R \circ \Psi_{\sigma^R} \circ R$, which is a bijection from $\mathcal{C}_n^\sigma$ to $\mathcal{C}_n^{\sigma^R}$. The inverse of $\bar{\Psi}_\sigma$ is $\bar{\Psi}_{\sigma^R}$. Using Equation (3.4) with π^R and σ^R playing the roles of π and σ , together with Equation (3.5) applied twice, we get

$$\text{des } \pi = 2n - 1 - \text{wdes } \pi^R = -1 - \text{wdes } \Psi_{\sigma^R} \pi^R = 2n - 2 - \text{des } \bar{\Psi}_\sigma \pi.$$

Thus, the involution $\bar{\Psi} : \mathcal{C}_n \rightarrow \mathcal{C}_n$ defined by $\bar{\Psi} \pi = \bar{\Psi}_\sigma \pi$ whenever $\pi \in \mathcal{C}_n^\sigma$ provides the desired bijection. $\square$

It might be possible to find simpler bijections proving our results about symmetry of the distributions.

Problem 3.1. Give direct bijections proving Corollaries 2.3, 2.4, 2.7 and 2.8 that do not require passing through the case where σ is the identity permutation.

4 Generalizations

Denote by $\sqcup^k n$ the multiset consisting of k copies of each number in $[n]$, so that $\sqcup^2 n = n \sqcup n$. Generalized Stirling permutations, often called k -Stirling permutations, are permutations of $\sqcup^k n$ that avoid the pattern 212. This generalization, originally proposed by Gessel and Stanley [9], has been studied for example in [14, 10, 12].

Similarly, k -quasi-Stirling permutations were defined by the author [6] as those permutations of $\sqcup^k n$ that avoid the patterns 1212 and 2121. Permutations of $\sqcup^k n$ can be viewed as ordered set partitions on kn into n blocks of size k , where block b consists of those $i \in [kn]$ such that $\pi_i = b$, for each $b \in [n]$. With this interpretation, a permutation avoids 1212 and 2121 if the underlying set partition is *noncrossing*, meaning that there

are no $i < j < \ell < m$ so that i, ℓ are in one block and j, m are in another block; see [16, Exer. 159]. The distribution of the number of descents and plateaus in k -quasi-Stirling permutations is given in [6].

There are several ways to generalize nonnesting permutations of $\sqcup^n n$, as given by Definition 1.2, to permutations of $\sqcup^k n$. Next we describe three different generalizations, which arise from distinct ways to view elements of $\mathcal{C}_n$: as pattern-avoiding multipermutations, as labeled nonnesting matchings, and as multipermutations where the subsequences of first copies and of second copies of each entry coincide.

4.1 Permutations avoiding 1221 and 2112

First we consider permutations π of $\sqcup^k n$ that avoid the patterns 1221 and 2112, that is, there do not exist $i < j < \ell < m$ such that $\pi_i = \pi_m / \pi_j = \pi_\ell$. Let $\mathcal{A}_n^k$ denote this set of permutations. For example, $\mathcal{A}_2^3 = \{111222, 112122, 221211, 222111\}$. When $k \geq 3$, this definition is quite restrictive. The distribution of descents and plateaus is relatively simple in this case, see [7] for the proof of the next result.

Theorem 4.1. *For $k \geq 3$ and $n \geq 1$, we have*

$$\sum_{\pi \in \mathcal{A}_n^k} t^{\text{des } \pi} u^{\text{plat } \pi} = u^{k-3} n^2 u^2 t^{n-1} A_n t.$$

Setting $u = 1$ and $u = t$ in Theorem 4.1, respectively, it follows easily that the distribution of the number of descents (respectively, weak descents) on $\mathcal{A}_n^k$ is symmetric. Our proof provides a bijective explanation of these symmetries.

4.2 Nonnesting permutations

A second way to generalize Definition 1.2 to permutations of $\sqcup^k n$ is to require that the underlying set partition is *nonnesting*; see [3, 2, 4] and [16, Exer. 164]. In this case, a block $\{i_1, i_2, \dots, i_k\}_<$ of the partition is represented by $k - 1$ arcs $i_1, i_2, i_2, i_3, \dots, i_{k-1}, i_k$, and the partition is nonnesting if this representation does not contain any pair of nested arcs, i.e., two arcs i, m and j, ℓ such that $i < j < \ell < m$. In terms of the permutation π of $\sqcup^k n$, an arc is placed between i and j if $\pi_i = \pi_j$ and there is no other copy of this value between positions i and j of π . Following Athanasiadis [2], who first considered such permutations in a geometric context, we call these *nonnesting permutations*. Let $\mathcal{B}_n^k$ denote the set of nonnesting permutations of $\sqcup^k n$. Clearly, $\mathcal{A}_n^k \subseteq \mathcal{B}_n^k$, but the converse is not true in general. For example, $\mathcal{B}_2^3 = \mathcal{A}_2^3 \sqcup \{121212, 212121\}$.

A bijection between noncrossing and nonnesting partitions that preserves the sizes of the blocks was given by Athanasiadis [3]. It follows that the number of nonnesting permutations of $\sqcup^k n$ equals the number of those where the underlying partition is

noncrossing, namely, k -quasi-Stirling permutations. This number is given in [6], and so we get

$$|\mathcal{B}_n^k| = \frac{(kn)!}{(k-1)n!1!} = n! \text{Cat}_{n,k},$$

where

$$\text{Cat}_{n,k} = \frac{1}{(k-1)n+1} \binom{kn}{n}$$

is called a k -Catalan number [15, pp. 168–173]. Given the simple formula in Theorem 2.1 for the $k = 2$ case and the results in [6] on k -quasi-Stirling permutations, it is natural to ask for the distribution of the number of descents and the number of plateaus on $\mathcal{B}_n^k$ for $k \geq 3$. Interestingly, unlike in the case $k = 2$, the distribution of the number of descents on $\mathcal{B}_n^k$ fails to be symmetric for $k = 3$ and $n = 4$.

4.3 Canon permutations

A third possible generalization arises when thinking of $\mathcal{C}_n^\sigma$ as the set of permutations of $n \sqcup n$ where both the subsequence of first copies of each entry and the subsequence of second copies of each entry equal σ . For $\sigma \in \mathcal{S}_n$ and $k \geq 1$, define $\mathcal{C}_n^{k,\sigma}$ to be the set of permutations π of $\sqcup^k n$ such that, for each $j \in [k]$, the subsequence of π formed by the j th copy (from the left) of each entry is σ . Let $\mathcal{C}_n^k = \sqcup_{\sigma \in \mathcal{S}_n} \mathcal{C}_n^{k,\sigma}$. We call elements of $\mathcal{C}_n^k$ k -*canon permutations*. By construction, $\mathcal{A}_n^1 = \mathcal{B}_n^1 = \mathcal{C}_n^1 = \mathcal{S}_n$ and $\mathcal{A}_n^2 = \mathcal{B}_n^2 = \mathcal{C}_n^2 = \mathcal{C}_n$, so the three definitions generalize the $k = 1$ and $k = 2$ cases.

It can be shown that $\mathcal{B}_n^k \subseteq \mathcal{C}_n^k$ in general, but the converse does not hold. For example, $\mathcal{C}_2^3 = \mathcal{B}_2^3 \sqcup \{112212, 121122, 212211, 221121\}$.

In forthcoming work, we will show that this definition provides the right setting to generalize Theorems 2.1 and 2.6. We will show that, if we define

$$C_n^k(t, u) = \sum_{\pi \in \mathcal{C}_n^k} t^{\text{des } \pi} u^{\text{plat } \pi} \quad \text{and} \quad C_n^{k,\sigma}(t, u) = \sum_{\pi \in \mathcal{C}_n^{k,\sigma}} t^{\text{des } \pi} u^{\text{plat } \pi}$$

in analogy to Equations (1.4) and (2.1), then for all $\sigma \in \mathcal{S}_n$ and all $k \geq 1$ we have

$$C_n^{k,\sigma}(t, u) = t^{\text{des } \sigma} C_n^{k,\iota}(t, u). \quad (4.1)$$

Thus, summing over all $\sigma \in \mathcal{S}_n$,

$$C_n^k(t, u) = A_n(t) C_n^{k,\iota}(t, u).$$

The proof extends some of the ideas in our proofs of Theorems 2.1 and 2.6 from Dyck paths to the more general setting of standard Young tableaux of rectangular shape. In particular, it yields certain generalizations of the notion of descents on such tableaux, as well as bijective proofs of some equidistribution results for these new statistics.

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Pop, Crackle, Snap (and Pow): Some Facets of Shards

Colin Defant^{*1} and Nathan Williams^{†2}

¹*Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA*

²*Department of Mathematical Sciences, University of Texas at Dallas, Dallas, TX*

Abstract. Reading cut the hyperplanes in a real central arrangement $\mathcal{H}$ into pieces called *shards*, which reflect order-theoretic properties of the arrangement. We show that shards have a natural interpretation as certain generators of the fundamental group of the complement of the complexification of $\mathcal{H}$. Taking only positive expressions in these generators yields a new poset that we call the *pure shard monoid*.

When $\mathcal{H}$ is simplicial, its poset of regions is a lattice, so it comes equipped with a pop-stack sorting operator Pop. In this case, we use Pop to define an embedding Crackle of Reading's shard intersection order into the pure shard monoid. When $\mathcal{H}$ is the reflection arrangement of a finite Coxeter group, we also define a poset embedding Snap of the shard intersection order into the positive braid monoid; in this case, our three maps are related by $\text{Snap} = \text{Crackle} \cdot \text{Pop}$.

Résumé. Reading a coupé les hyperplans dans un arrangement central réel $\mathcal{H}$ en morceaux appelés *shards*, qui reflètent les propriétés ordinales de l'arrangement. Nous montrons que les shards ont une interprétation naturelle en tant que certains générateurs du groupe fondamental du complément de la complexification de $\mathcal{H}$. En ne prenant que des expressions positives dans ces générateurs, on obtient un nouvel ensemble ordonné que nous appelons le *monoïde shard pur*.

Lorsque $\mathcal{H}$ est simplicial, son ensemble de régions est un treillis, et il est équipé d'un pop-stack opérateur de tri Pop. Dans ce cas, nous utilisons Pop pour définir une injection Crackle de l'ordre d'intersection des shards de Reading dans le monoïde shard pur. Lorsque $\mathcal{H}$ est l'arrangement de réflexion d'un groupe de Coxeter fini, nous définissons également une injection d'ensemble ordonné Snap de l'ordre d'intersection des shards dans le monoïde de tresses positives; dans ce cas, nos trois fonctions sont reliées par $\text{Snap} = \text{Crackle} \cdot \text{Pop}$.

Keywords: hyperplane arrangement, fundamental group, shard, shard intersection order, Salvetti complex, pure shard monoid

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[†]nathan.williams1@utdallas.edu. Supported by a Simons Foundation Collaboration Grant.

1 Introduction

Throughout this extended abstract of the article [9], we let $\mathcal{H}$ be a finite central irreducible real hyperplane arrangement in $\mathbb{R}^n$. Salvetti introduced a certain CW complex associated to $\mathcal{H}$ and used it to provide a presentation of the fundamental group of the complement of the complexification of $\mathcal{H}$. We prove that Salvetti's generating set is parameterized by shards, which Reading introduced and used to define his shard intersection order in the case when $\mathcal{H}$ is simplicial. We introduce the *pure shard monoid*, which is the monoid generated by Salvetti's generators; it comes equipped with a natural partial order that we believe deserves further attention. We prove that the interval from the identity element to the *full twist* in the pure shard monoid is self-dual. In the case when $\mathcal{H}$ is an arrangement of rank 2 with m hyperplanes, we prove that this interval is a planar lattice with rank generating function $1 - \left(\sum_{k=1}^{m-1} \binom{m}{k} - 2 \right) q^k$ and with $m2^{m-2}$ maximal chains.

We then assume $\mathcal{H}$ is simplicial. In this case, there is a known characterization of the shard intersection order involving the pop-stack sorting operator Pop on the poset of regions of $\mathcal{H}$. We introduce a new map Crackle , and we prove that it is a poset embedding of the shard intersection order into the pure shard monoid.

Next, we specialize further to the case when $\mathcal{H}$ is the reflection arrangement of a finite Coxeter group W . We introduce another map Snap , and we prove that it is a poset embedding of the shard intersection order on W into the weak order on the positive braid monoid. The restriction of Snap to the set $\text{Sort } W, c$ of c -sortable elements of W originally arose in connection with Deodhar decompositions of noncrossing Catalan varieties. In this setting, we obtain as a corollary that Snap restricts to a poset embedding of the shard intersection order on $\text{Sort } W, c$ —which is isomorphic to the noncrossing partition lattice of W —into the weak order on the positive braid monoid.

Finally, we turn back to arbitrary finite central irreducible real arrangements and define a fourth map Pow . We prove that Pow is a poset embedding of the poset of regions of $\mathcal{H}$ into the pure shard monoid.

2 The Salvetti Complex

We write $\mathcal{R}$ for the set of regions (connected components of the complement) of $\mathcal{H}$, and we fix a base region $B \in \mathcal{R}$ and a point $x_B \in B$. Let $\mathcal{H}_{\mathbb{C}}$ be the union of the complexifications of the hyperplanes in $\mathcal{H}$. The *complexified hyperplane complement* is $\mathbb{C}^n \setminus \mathcal{H}_{\mathbb{C}}$; write $\pi_1 \mathbb{C}^n \setminus \mathcal{H}_{\mathbb{C}}, x_B$ for its fundamental group with base point x_B . For $C, D \in \mathcal{R}$, we write $D \leq C$ if every hyperplane in $\mathcal{H}$ that separates D from B also separates C from B . The resulting poset $\text{Weak } \mathcal{H}, B : \mathcal{R}, \leq$ is the *poset of regions* of $\mathcal{H}$ with respect to B .

Following [10, 15], we construct a CW complex $\text{Sal } \mathcal{H}$ by gluing together oriented dual zonotopes for $\mathcal{H}$ along compatible faces—one zonotope for each choice of base region B , oriented from B to $-B$. The resulting CW complex $\text{Sal } \mathcal{H}$ has the same fundamental group as the complexified hyperplane complement:

$$\pi_1 \text{Sal } \mathcal{H}, B \cong \pi_1 \mathbb{C}^n \setminus \mathcal{H}_{\mathbb{C}, x_B}.$$

The 1-skeleton of $\text{Sal } \mathcal{H}$ is given by orienting all edges of $\text{Weak } \mathcal{H}, B$ away from B , and then for each edge e , adding in a reversed edge e^* . An illustration is given in Figure 1.

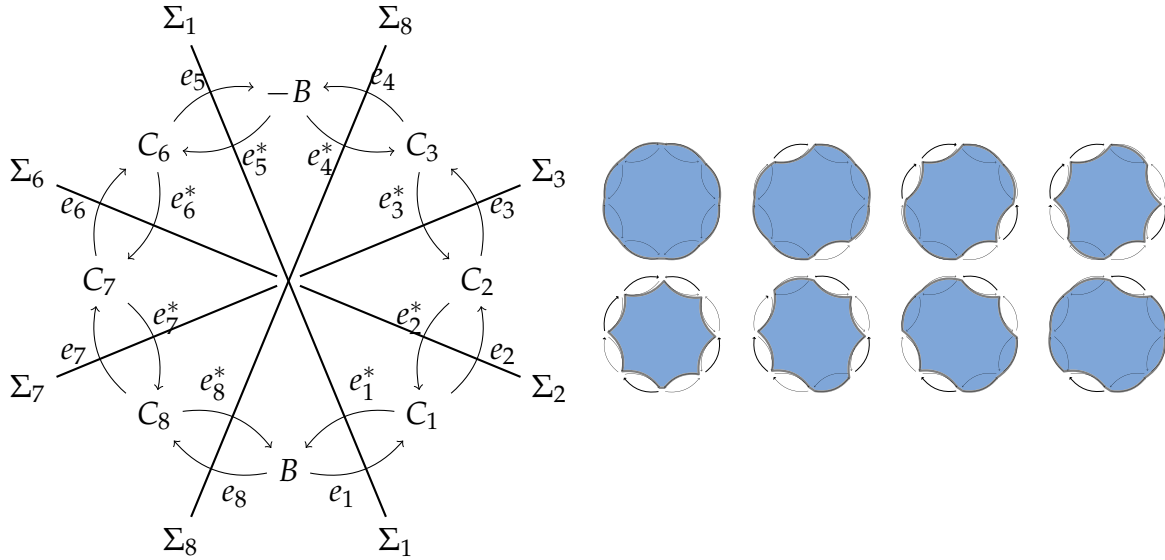


Figure 1: *Left:* the hyperplane arrangement for the dihedral group $I_2(4)$ has eight regions, and its four hyperplanes are cut into six shards. *Right:* the eight 2-cells of $\text{Sal } \mathcal{H}$, indicated in blue. One 2-cell is attached for each of the eight homotopies $e_1 e_2 e_3 e_4 \sim e_8 e_7 e_6 e_5, e_2 e_3 e_4 e_5^* \sim e_1^* e_8 e_7 e_6, \dots, e_8^* e_1 e_2 e_3 \sim e_7 e_6 e_5 e_4^*$.

Given arbitrary regions $C, C' \in \mathcal{R}$, we define a *gallery* from C' to C to be a sequence of edges (of the form e, e^*, e^{-1} , and e^{*-1}) that starts at C' and ends at C . A gallery from C' to C is *positive* if it only uses edges of the form e and e^* (not e^{-1} or e^{*-1}). The gallery is *minimal* if its length is equal to the number of hyperplanes separating C' from C , and it is called a *loop* if $C = C'$. Let us fix a positive minimal gallery $\text{gal } C', C$ from C' to C in $\text{Sal } \mathcal{H}$; any two such galleries from C' to C are homotopic. If e is the edge $C' \xrightarrow{e} C$, we define the corresponding loop $\ell_e \in \pi_1 \text{Sal } \mathcal{H}, B$ by

$$\ell_e: \text{gal } B, C' \cdot e e^* \cdot \text{gal } B, C'^{-1} \in \pi_1 \text{Sal } \mathcal{H}, B. \quad (2.1)$$

Because of homotopies, this definition of the loop does not depend on the choice of the gallery. Write $\mathcal{L}_{\text{edge}} = \mathcal{L}_{\text{edge}} \mathcal{H}, B$ for the set of all such loops ℓ_e . The group $\pi_1 \text{Sal } \mathcal{H}, B$ is generated by $\mathcal{L}_{\text{edge}}$.

3 Shards

We recall some constructions and results from [13, 14]. A subarrangement $\mathcal{A}$ of $\mathcal{H}$ is called *full* if it consists of all hyperplanes of $\mathcal{H}$ containing a particular subset of $\mathbb{R}^n$. For a full rank-2 subarrangement $\mathcal{A}$ of $\mathcal{H}$, let $B_{\mathcal{A}}$ be the region of $\mathcal{A}$ containing B . We say a hyperplane $H \in \mathcal{A}$ is *basic* if its intersection with the boundary of $B_{\mathcal{A}}$ has dimension $n - 1$. For $H, H' \in \mathcal{A}$, we say H *cuts* H' if H is basic in $\mathcal{A}$ but H' is not. Each hyperplane $H \in \mathcal{H}$ is broken into a number of connected pieces if we remove all points in H contained in the hyperplanes of $\mathcal{H}$ that cut H —a *shard* of H is then the closure of one of these connected pieces. We write H_{Σ} for the unique hyperplane containing a shard Σ . Let $\text{III } \mathcal{H}, B$ denote the set of shards.

Each cover relation $C' \prec C$ in $\text{Weak } \mathcal{H}, B$ can be labeled by a shard $\Sigma \subset C' \prec C$, which is the unique shard separating the region C' from the region C ; in this case, we call $\Sigma \subset C' \prec C$ a *lower shard* of C . Let $\text{cov}_{\text{III } \mathcal{H}, B} C$ be the set of lower shards of C .

Now assume $\mathcal{H}$ is simplicial. Then $\text{Weak } \mathcal{H}, B$ is a semidistributive lattice [4], and the set of shards forms an elegant geometric realization of the set of join-irreducible elements. Furthermore, shard intersections encode the canonical join representations of $\text{Weak } \mathcal{H}, B$. The map $C \mapsto \bigcap \text{cov}_{\text{III } \mathcal{H}, B} C$ defines a bijection from $\mathcal{R}$ to the set of arbitrary intersections of shards. In [14], Reading introduced another poset $\text{Shard } \mathcal{H}, B : \mathcal{R}, \preceq$ called the *shard intersection order*, which is defined by

$$C \preceq D \text{ if and only if } \bigcap \text{cov}_{\text{III } \mathcal{H}, B} C \supseteq \bigcap \text{cov}_{\text{III } \mathcal{H}, B} D.$$

As with $\text{Weak } \mathcal{H}, B$, the poset $\text{Shard } \mathcal{H}, B$ is a lattice—but while $\text{Weak } \mathcal{H}, B$ is “tall and slender” (with height equal to the number of hyperplanes and with the number of atoms equal to the dimension), $\text{Shard } \mathcal{H}, B$ is “short and wide” (with height equal to dimension and with the number of atoms equal to the number of shards). When $\mathcal{H}$ is the reflection arrangement of a finite Coxeter group W , the relationship between noncrossing partitions and sortable elements allowed Reading to embed the W -noncrossing partition lattice into the shard intersection order, thereby giving a uniform proof that the noncrossing partition lattice is indeed a lattice. An example is illustrated in Figure 2.

4 Shards and the Salvetti Complex

Whenever we have an edge $C' \xrightarrow{e} C$ in $\text{Sal } \mathcal{H}$, we will write $\Sigma \subset e$ for the associated shard $\Sigma \subset C' \prec C$. Although the generators in $\mathcal{L}_{\text{edge}}$ are a priori indexed by cover relations in $\text{Weak } \mathcal{H}, B$, our first theorem says that they are really indexed by the much smaller set of shards.

Theorem 1 ([9]). *Let $\mathcal{H}$ be a central real hyperplane arrangement. Given edges $C' \xrightarrow{e} C$ and $D' \xrightarrow{f} D$ in $\text{Sal } \mathcal{H}$, the loops ℓ_e and ℓ_f are homotopic if and only if $\Sigma \subset e \subseteq \Sigma \subset f$.*

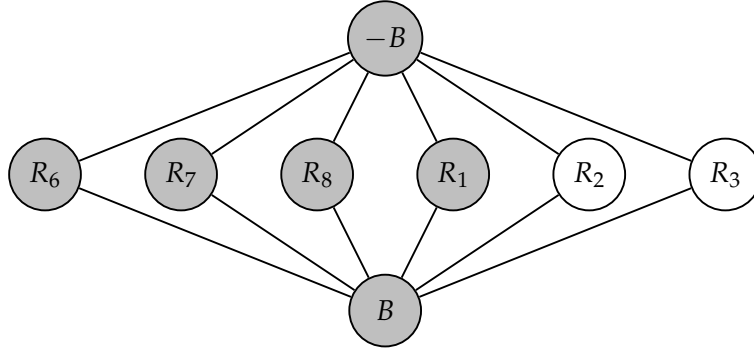


Figure 2: The poset $\text{Shard } \mathcal{H}, B$ for the arrangement of Figure 1, which is the reflection arrangement of the dihedral group $I_2 4$. Gray indicates elements in the image of Reading’s embedding of the $I_2 4$ -noncrossing partition lattice (with respect to a certain Coxeter element); see [9, Section 7.5].

By Theorem 1, it makes sense write $\mathcal{L}_{\text{III}} = \mathcal{L}_{\text{edge}}$, indexing the loops in $\mathcal{L}_{\text{edge}}$ by shards. Thus, for any shard Σ , we define the *shard loop* $\ell_\Sigma = \ell_e$, where e is any edge such that $\Sigma \in e$.

Let us define the *pure shard monoid*, denoted $\mathbf{P}(\mathcal{H}, B)$, to be the submonoid of $\pi_1 \text{Sal } \mathcal{H}, B$ generated by $\mathcal{L}_{\text{III}}$. That is, an element of $\pi_1 \text{Sal } \mathcal{H}, B$ is in $\mathbf{P}(\mathcal{H}, B)$ if and only if it can be represented by a word in the alphabet $\mathcal{L}_{\text{III}}$. This allows us to endow $\mathbf{P}(\mathcal{H}, B)$ with a partial order $\leq$ by declaring that $p \leq p'$ if there is a word over $\mathcal{L}_{\text{III}}$ representing p' that contains a prefix representing p .

The *full twist* is the element Δ^2 of $\pi_1 \text{Sal } \mathcal{H}, B$ defined by

$$\Delta^2 : \text{gal } B, -B \cdot \text{gal } -B, B.$$

When $\mathcal{H}$ is a finite irreducible simplicial arrangement, it is known that the center of $\pi_1 \text{Sal } \mathcal{H}, B$ is an infinite cyclic group generated by Δ^2 [6]. We are especially interested in $[\mathbb{1}, \Delta^2]_{\mathbf{P}}$, the interval between the identity element $\mathbb{1}$ and the full twist Δ^2 in $\mathbf{P}(\mathcal{H}, B)$. This interval is “tall and wide,” but it is *not* a lattice in general, preventing the use of Garside theory to study $\pi_1 \text{Sal } \mathcal{H}, B$. Figure 3 shows this interval when $\mathcal{H}$ is the arrangement from Figure 1.

The combinatorics of the pure shard monoid can be quite involved. However, we can prove the following general result (which is stated as a conjecture in [9]).

Theorem 2. *If $\mathcal{H}$ is a central arrangement, then the interval $[\mathbb{1}, \Delta^2]_{\mathbf{P}}$ is self-dual.*

For a rank-2 arrangement with m hyperplanes, we have the following precise theorem, which tells us that there are exactly $m2^{m-2}$ words over the alphabet $\mathcal{L}_{\text{III}}$ representing Δ^2 .

Theorem 3 ([9]). Let $\mathcal{H}$ be hyperplane arrangement of rank 2 with m hyperplanes. The interval $\mathbb{1}, \Delta^2_{\mathbf{P}}$ is a planar lattice with rank generating function $1 - \left(\sum_{k=1}^{m-1} (2^{\binom{m}{k}} - 2) q^k \right) = q^m$ and with $m2^{m-2}$ maximal chains.

5 Pop

In the next three sections, we discuss three incarnations of the shard intersection order given by three maps Pop, Crackle, and Snap. The first one, defined using the map Pop, is not new, but it inspired our terminology for the other two. For these three sections, we assume that $\mathcal{H}$ is simplicial so that $\text{Weak } \mathcal{H}, B$ is a semidistributive lattice.

Let L be a locally finite meet-semilattice with meet operation denoted by $\wedge$. Motivated by work on *pop-stacks* from enumerative combinatorics and theoretical computer science [1, 5, 17], the first author defined the *pop-stack sorting operator* $\text{Pop}: L \rightarrow L$ in [7] (see also [8]) by

$$\text{Pop } x : x \wedge \bigwedge \{y \in L : y \leq x\}. \quad (5.1)$$

In the case when $L = \text{Weak } \mathcal{H}, B$, we can use Pop to characterize $\text{Shard } \mathcal{H}, B$. For $C \in \mathcal{R}$, let $\Sigma_{\text{Pop } C, C}$ be the set of shards that label the cover relations in the interval $\text{Pop } C, C$; that is,

$$\Sigma_{\text{Pop } C, C} : \{ \Sigma D' : D : \text{Pop } C \leq D' \leq D \leq C \}.$$

By [14, Proposition 5.7], we have

$$C' \preceq C \text{ if and only if } \Sigma_{\text{Pop } C', C'} \subseteq \Sigma_{\text{Pop } C, C}. \quad (5.2)$$

6 Crackle

In this section, we continue to assume $\mathcal{H}$ is simplicial. Let us define the *crackle map* $\text{Crackle}: \mathcal{R} \rightarrow \pi_1 \text{Sal } \mathcal{H}, B$ by

$$\text{Crackle } C : \text{gal } B, \text{Pop } C \cdot \text{gal } \text{Pop } C, C \cdot \text{gal } C, \text{Pop } C \cdot \text{gal } B, \text{Pop } C^{-1}. \quad (6.1)$$

This map generalizes the shard loops ℓ_{Σ} of Equation (2.1) and Theorem 1: if J is a join-irreducible region of $\text{Weak } \mathcal{H}, B$, then $\text{Pop } J$ is the unique region covered by J , so $\text{Crackle } J = \ell_{\Sigma_{\text{Pop } J}, J}$.

Just as Equation (5.2) characterized $\text{Shard } \mathcal{H}, B$ using Pop, we can give a characterization of $\text{Shard } \mathcal{H}, B$ using Crackle. Recall that if P and Q are posets, then a map $\psi: P \rightarrow Q$ is called a *poset embedding* if it is a poset isomorphism from P to its image $\psi(P) \subseteq Q$.

Theorem 4. The map Crackle is a poset embedding from $\text{Shard } \mathcal{H}, B$ into the interval $\mathbb{1}, \Delta^2_{\mathbf{P}}$.

Theorem 4 is illustrated in Figure 3.

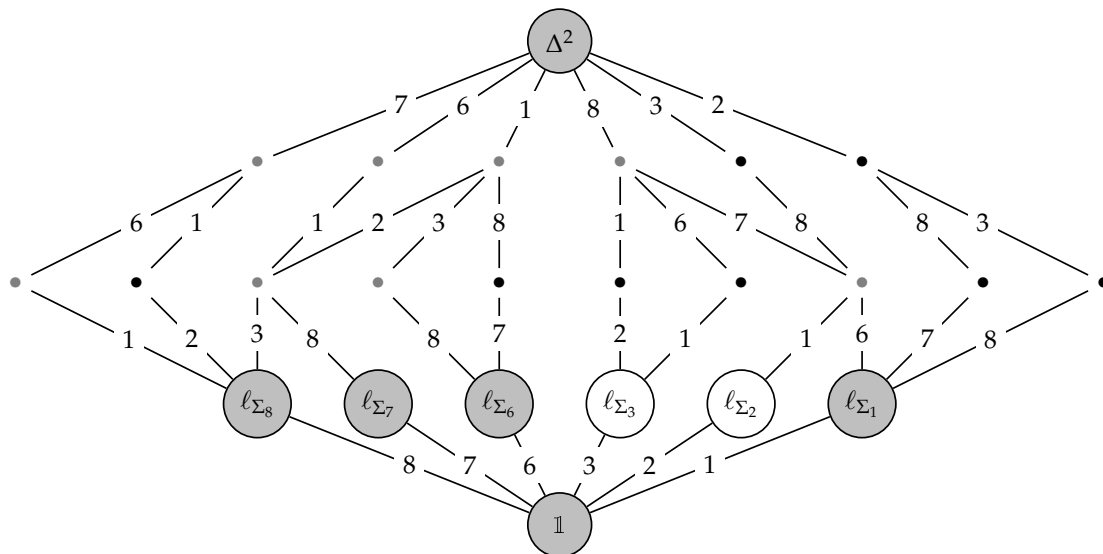


Figure 3: The interval $[1, \Delta^2]_{\mathbf{P}(\mathcal{H}, B)}$ in the pure shard monoid $\mathbf{P}(\mathcal{H}, B)$ between the identity element and the full twist Δ^2 , where $\mathcal{H}, B$ is as in Figure 1. An edge $p \rightarrow p'$ is labeled i when $p' = p \cdot \ell_{\Sigma_i}$. Circled elements are in the image of Crackle. An element is colored gray if it appears as a prefix of a word for Δ^2 using only generators corresponding to noncrossing shards; see [9, Section 7.5].

7 Snap

We now specialize to the case when $\mathcal{H}$ is the reflection arrangement of a finite Coxeter group W . We identify the base region B with the identity element of W ; the free transitive action of W on $\mathcal{R}$ then allows us to identify regions of $\mathcal{H}$ with elements of W . We write $\text{Shard } W := \text{Shard } \mathcal{H}, B$, $\text{Weak } W := \text{Weak } \mathcal{H}, B$, $\mathbf{P} W := \mathbf{P}(\mathcal{H}, B)$, etc.

In this setting, the group $\mathbf{P} W := \pi_1(\mathbb{C}^n \setminus \mathcal{H}_{\mathbb{C}}, x_B)$ is called the *pure braid group* of W , while the group $\mathbf{B} W := \pi_1(\mathbb{C}^n \setminus \mathcal{H}_{\mathbb{C}}/W, x_B)$ is called the *braid group* of W . The Coxeter group W fits into the following well-known exact sequence with its braid and pure braid groups:

$$1 \rightarrow \mathbf{P} W \rightarrow \mathbf{B} W \xrightarrow{\varphi} W \rightarrow 1.$$

Let $\mathbf{S}$ be the set of simple generators of $\mathbf{B} W$ obtained by lifting the set S of simple reflections of W . The generators in $\mathbf{S}$ satisfy the same braid relations as the corresponding simple reflections of W ; the difference is that W also includes the relations stating that the simple reflections are involutions. Thus, the projection $\varphi: \mathbf{B} W \rightarrow W$ is the quotient map that sends each generator $\mathbf{s} \in \mathbf{S}$ to the corresponding $s \in S$ and imposes these additional relations. The submonoid of $\mathbf{B} W$ generated by $\mathbf{S}$ is called the *positive braid monoid* of W and is denoted by $\mathbf{B}^+ W$. The weak order $W, \leq$ is defined by saying

$u \leq v$ if and only if any reduced word for u appears as a prefix of some reduced word for v . Analogously, the weak order $\text{Weak } \mathbf{B} \setminus W : \mathbf{B} \setminus W, \leq$ is defined by saying $\mathbf{u} \leq \mathbf{v}$ if and only if any word over $\mathbf{S}$ representing $\mathbf{u}$ appears as a prefix of some word over $\mathbf{S}$ representing $\mathbf{v}$. In this setting, the full twist Δ^2 is equal to the lift $\mathbf{w}_\circ^2$ of the long element $w_\circ$ of W ; we will also write $\Delta = \mathbf{w}_\circ$.

We can rephrase [Theorem 1](#) when $\mathcal{H}$ is the reflection arrangement of W as follows.

Corollary 1. *Suppose $u, v \in W$ and $s, t \in S$ are such that $u = us$ and $v = vt$. Let $\mathbf{u}, \mathbf{v}, \mathbf{s}, \mathbf{t}$ be the lifts of u, v, s, t , respectively, to $\mathbf{B} \setminus W$. We have $\mathbf{usu}^{-1} = \mathbf{vtv}^{-1}$ if and only if $\Sigma w = ws$ $\Sigma u = ut$.*

For $u, v \in W$, we let $u^v = v^{-1}uv$. Write $\mathbf{c} = \mathbf{s}_1, \dots, \mathbf{s}_n$ for an ordering of the elements of $\mathbf{S}$. For $\mathbf{s} \in \mathbf{S}$ and a positive braid $\mathbf{w} = \mathbf{s}_{i_1} \cdots \mathbf{s}_{i_k} \in \mathbf{B} \setminus W$ with projection $w = \varphi(\mathbf{w}) \in W$, let $\mathbf{s}^{\mathbf{w}} = t, j$, where $t = s^w$ and j counts the number of times t appears in the sequence $s^{s_{i_k}}, s^{s_{i_k}s_{i_{k-1}}}, \dots, s^{s_{i_k}s_{i_{k-1}} \cdots s_1}$.

In [11], a new set of noncrossing W -Catalan objects was introduced as the set of subwords of $\mathbf{c}^{h-1}$ that represent the full twist $\Delta^2 = \mathbf{w}_\circ^2$ and satisfy an additional *Deodhar condition*. When interpreted in the positive braid monoid $\mathbf{B} \setminus W$, this Deodhar condition is equivalent to restricting to those c -sortable elements $\mathbf{w}$ in the interval $[\mathbb{1}, \mathbf{w}_\circ^2]_{\mathbf{B}}$ with the property that for each descent $\mathbf{s}$ of $\mathbf{w}$, we have $\mathbf{s}^{\mathbf{w}} = t, j$ with j even—this is a nonstandard *Deodhar embedding* of the c -sortable elements into the interval $[\mathbb{1}, \mathbf{w}_\circ^2]_{\mathbf{B}}$ (different from the usual lift of W into $\mathbf{B} \setminus W$). The second author speculated that restricting the 2nd c -Fuss–Cambrian lattice to the image of this Deodhar embedding would recover the noncrossing partition lattice $\text{NC}(W, c)$. As the 2nd c -Fuss–Cambrian lattice is a subposet of $\text{Weak } \mathbf{B} \setminus W$, it makes sense to generalize this Deodhar embedding of c -sortable elements to all elements of W .

The *snap map* $\text{Snap}: W \rightarrow \mathbf{B} \setminus W$ is our generalization of the Deodhar embedding. We write Pop for the pop-stack sorting operators on the lattice $\text{Weak } W$ and the meet-semilattice $\text{Weak } \mathbf{B} \setminus W$, relying on the argument of the operator to indicate the context. For $w \in W$, let $\text{des } w$ denote the right descent set of w , let $w_\circ \text{des } w$ be the longest element of the parabolic subgroup of W generated by $\text{des } w$, and write $\mathbf{w}$ and $\mathbf{w}_\circ \text{des } w$ for the usual lifts of w and $w_\circ \text{des } w$ to $\mathbf{B} \setminus W$. Define

$$\text{Snap } w : \text{Pop } \mathbf{w} \cdot \mathbf{w}_\circ \text{des } w^{-2}. \quad (7.1)$$

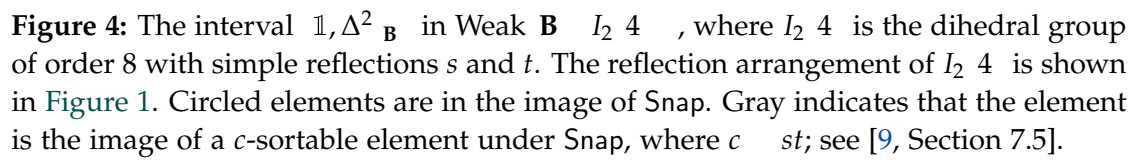
Since $\text{Crackle } w = \text{Pop } \mathbf{w} \cdot \mathbf{w}_\circ \text{des } w^{-2} \cdot \text{Pop } \mathbf{w}^{-1}$ in $\mathbf{P} \setminus W \subseteq \mathbf{B} \setminus W$, it follows that

$$\text{Snap } w = \text{Crackle } w \text{Pop } \mathbf{w}.$$

Just as the shard intersection order was characterized via Pop in [Equation \(5.2\)](#) and via Crackle in [Theorem 4](#), it is also characterized via Snap .

Theorem 5. *The map Snap is a poset embedding from $\text{Shard } W$ into $[\mathbb{1}, \Delta^2]_{\mathbf{B}}$.*

[Theorem 5](#) is illustrated in [Figure 4](#).



8 Pow

In her history “The Untold Tale of Pow!, the Fourth Rice Krispies Elf: A look into the era when the cereal mascots were more than just Snap!, Crackle! and Pop!” [16], Smith writes:

“Lost in the shuffle, however, was a fourth Rice Krispies elf named Pow!
His short life is a time-capsule of an era when everyone was dreaming big.”

Inspired by this fourth mascot, we introduce a map $\text{Pow} : \mathcal{R} \rightarrow \mathbf{P}(\mathcal{H}, B)$ in the general setting when $\mathcal{H}$ is a central irreducible real hyperplane arrangement. Given a region $C \in \mathcal{R}$ and a positive minimal gallery

$$B = C_0 \xrightarrow{e_1} C_1 \xrightarrow{e_2} \cdots \xrightarrow{e_{k-1}} C_{k-1} \xrightarrow{e_k} C_k = C,$$

we let

$$\text{Pow } C : \ell_{\Sigma, e_k} \ell_{\Sigma, e_{k-1}} \cdots \ell_{\Sigma, e_1}.$$

(We prove in [9] that Pow is well-defined.) Just as Crackle embeds the “short and wide” poset $\text{Shard } \mathcal{H}, B$ into the “tall and wide” interval $[\mathbb{1}, \Delta^2_{\mathbf{P}}]$, the map Pow embeds the “tall and slender” poset $\text{Weak } \mathcal{H}, B$ into $[\mathbb{1}, \Delta^2_{\mathbf{P}}]$.

Theorem 6. *The map Pow is a poset embedding from $\text{Weak } \mathcal{H}, B$ into $[\mathbb{1}, \Delta^2_{\mathbf{P}}]$.*

9 Future Work

9.1 Noncrossing Pure Braid Presentations

In future work, we will combine [Theorem 1](#) with Salvetti’s presentation of $\pi_1 \text{Sal } \mathcal{H}, B$, Coxeter–Catalan combinatorics, Cambrian lattices, and noncrossing shards to write explicit presentations of the pure braid groups of finite Coxeter groups. In the special case of the symmetric group and the Tamari lattice, our method will recover Artin’s original presentation of the pure braid group [2, 3].

9.2 The Pure Shard Monoid

The pure shard monoid is an interesting algebraic and order-theoretic structure that deserves further study—in particular, we would like to better understand the elements and the maximal chains in the interval $[\mathbb{1}, \Delta^2_{\mathbf{P}}]$. These maximal chains correspond to $\mathcal{L}_{\text{III}}$ -words representing the full twist Δ^2 . When $\mathcal{H}$ is an arrangement of rank 2, we characterized these words in [9, Proposition 4.7]. It would already be interesting to better understand $[\mathbb{1}, \Delta^2_{\mathbf{P}}]$ for special cases, such as rank-3 arrangements or reflection arrangements of type- A Coxeter groups.

9.3 Infinite Arrangements

We have assumed throughout that $\mathcal{H}$ is finite. It is natural to ask what aspects of the above theory generalize to arrangements with infinitely many hyperplanes.

9.4 Bubbles, Blossom, Buttercup (and Bliss)

We view the bubble sort operator Bubbles as the 0-Hecke action of any reduced word for the long element w_0 in the symmetric group. The *higher Bruhat order* is a partial order defined on these reduced words [12]. We wonder if there are similar “higher Bruhat orders” built from the $\mathcal{L}_{\text{III}}$ -words for Δ^2 . One might expect to find relevant maps Blossom and Buttercup in this theory. We recommend this subsection’s title as the logical name for this proposed work.

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Studying triangulations of even-dimensional cyclic polytopes via directed graphs

Nicholas J. Williams^{*1}

¹*Department of Mathematics and Statistics, Fylde College, Lancaster University, Lancaster, LA1 4YF, United Kingdom*

Abstract. We prove two results on triangulations of even-dimensional cyclic polytopes using the directed graphs (digraphs) implicitly associated to them by Oppermann and Thomas. First, we show that the digraph of a $2d$ -dimensional triangulation encodes whether or not the triangulation has an interior $d-1$ -simplex, and we describe the digraphs of such triangulations. Secondly, we show that the digraph of the triangulation allows one to identify the internal d -simplices which admit bistellar flips. This also allows one to easily compute the outcome of a bistellar flip.

Keywords: Cyclic polytopes, triangulations, bistellar flips

1 Introduction

Cyclic polytopes are a very special family of polytopes. Perhaps their most famous property is that they attain equality in the Upper Bound Theorem [17]: they have the largest possible number of faces of every single dimension out of all polytopes with a given dimension and number of vertices. This result was generalised by Stanley [19], which gave birth to what is now called ‘Stanley–Reisner theory’. Another remarkable property of cyclic polytopes is that for every n and for every sufficiently large collection of points in $\mathbb{R}^m$ in general position, there exists a subset of n points which form the vertices of a polytope combinatorially equivalent to a cyclic polytope [6]. In fact, cyclic polytopes are precisely the polytopes which have this property [3, Proposition 9.4.7]. This is a higher-dimensional version of the Erdős–Szekeres Theorem, and is related to Ramsey theory.

Triangulations of cyclic polytopes are particularly interesting, and for several reasons. For one thing, they are relatively well-understood amongst triangulations of high-dimensional polytopes [7, Section 6.1], admitting elegant combinatorial descriptions [18, 22]. However, there are yet many unresolved problems, such as finding a formula for the number of triangulations of a cyclic polytope. This has been solved in certain cases [2],

^{*}nicholas.williams@lancaster.ac.uk. Nicholas J. Williams was supported by EPSRC grant EP/V050524/1.

but is even still an open problem for three-dimensional cyclic polytopes. But the principal reason triangulations of cyclic polytopes are interesting is that their combinatorial structure describes many phenomena in other areas of mathematics, including representation theory [18, 22], higher category theory [16, 10], integrable systems [9, 23], and scattering amplitudes [1]. In many of these cases, the poset structure on triangulations of cyclic polytopes appears. Two such poset structures were introduced in [16] and [11], but these were proven to be the same in [21].

In this paper, which is an extended abstract of [20], we study triangulations of even-dimensional cyclic polytopes. Even- and odd-dimensional cyclic polytopes behave quite differently, and it is the even-dimensional cyclic polytopes which behave analogously to convex polygons. In particular, we study triangulations of even-dimensional cyclic polytopes by associating digraphs to them, in a way that was done implicitly in [18]. Results aside, this provides a visual tool for studying complex high-dimensional objects which are otherwise impossible to visualise. The digraph $Q(\mathcal{T})$ of a triangulation $\mathcal{T}$ of a $2d$ -dimensional cyclic polytope has the internal d -simplices of $\mathcal{T}$ as its vertices. The arrows of the digraph connect simplices which are adjacent to each other in the triangulation, in a precise sense.

Our first result shows that the digraph $Q(\mathcal{T})$ of a triangulation $\mathcal{T}$ of the cyclic polytope $C(n-2d-1, 2d)$ tells whether or not the triangulation contains interior $d-1$ -simplices, that is, $d-1$ -simplices whose facets do not lie in the boundary of the polytope.

Theorem 1 ([20]). *A triangulation $\mathcal{T}$ of $C(n-2d-1, 2d)$ contains no interior $d-1$ -simplices if and only if $Q(\mathcal{T})$ is a cut digraph and this is the case if and only if $Q(\mathcal{T})$ does not contain an oriented cycle.*

We define cut digraphs in Section 2.2. They are the higher-dimensional analogues of orientations of the type A Dynkin diagram.

As an application of this theorem, we prove that the triangulations of $C(n-2d-1, 2d)$ without interior $d-1$ -simplices form a connected subgraph of the flip graph of triangulations of $C(n-2d-1, 2d)$. The flip graph of a polytope is the graph whose vertices are triangulations of the polytope, with edges between triangulations related by a bistellar flip. Here bistellar flips are the higher-dimensional analogues of flipping a diagonal inside a quadrilateral.

Our second result studies bistellar flips of triangulations of even-dimensional cyclic polytopes. It is obvious that every internal edge of a polygon triangulation can be flipped, but the corresponding statement does not hold in higher dimensions. Indeed, Oppermann and Thomas show that bistellar flips of triangulations of $2d$ -dimensional cyclic polytopes correspond to replacing one internal d -simplex in a triangulation with another. However, not every internal d -simplex in a triangulation of a $2d$ -dimensional cyclic polytope is replaceable in this way. We call an internal d -simplex *mutable* if it can

be replaced in a bistellar flip.

We show that the digraph of a triangulation encodes the mutability of internal d -simplices. We show how the digraph can be decomposed into certain paths, which we call ‘retrograde paths’, such that the following result holds.

Theorem 2 ([20]). *An internal d -simplex is mutable if and only if it does not lie in the middle of a retrograde path in the digraph of the triangulation.*

One can then, in fact, read off the outcome of the bistellar flip from the digraph.

This paper is structured as follows. We start in [Section 2](#) by giving background, primarily on cyclic polytopes and their triangulations. In [Section 3](#), we define the digraph of a triangulation and describe the digraphs of $2d$ -dimensional triangulations without interior $d-1$ -simplices. In [Section 4](#), we present the second main result of the paper, which shows how the digraphs of triangulations can be used to identify mutable internal d -simplices.

2 Background

2.1 Cyclic polytopes

An introduction to cyclic polytopes, as well as historical background, can be found in [14, 4.7]. In this paper, we are only interested in even-dimensional cyclic polytopes, which can be realised in the following way. Let $n \geq 0$ and $d \geq 1$ and consider the curve defined by $q: [0, 2\pi] \rightarrow \mathbb{R}^{2d}$,

$$q(t) = (\cos t, \sin t, \cos 2t, \sin 2t, \dots, \cos dt, \sin dt)$$

which is known as the *Carathéodory curve*, after [4, 5]. The (regular) cyclic polytope $C_{n-2d-1, 2d}$ is the convex hull

$$\text{conv} \left\{ q\left(\frac{1}{n-2d-1}2\pi\right), q\left(\frac{2}{n-2d-1}2\pi\right), \dots, q(2\pi) \right\}.$$

The vertices of $C_{n-2d-1, 2d}$ are labelled by $n-2d-1 : \{1, 2, \dots, n-2d-1\}$, such that $q(2\pi i / (n-2d-1))$ is vertex i .

A *triangulation* of a cyclic polytope $C_{n-2d-1, 2d}$ is a set of $2d$ -simplices whose vertices are vertices of $C_{n-2d-1, 2d}$, whose interiors are pairwise disjoint and whose union is $C_{n-2d-1, 2d}$.

Convention 1. In this paper we specify k -simplices by their vertex tuples, for instance, $A = (a_0, a_1, \dots, a_k) \in [n-2d-1]^k$, where $a_0 < a_1 < \dots < a_k$. We always follow the convention that simplices are denoted by upper case letters, with their vertices denoted by the corresponding lower case letters with subscripts.

Furthermore, we allow vertex tuples to be reordered using cyclically shifted orderings. The *cyclically shifted order* $<_i$ is the order

$$i <_i i-1 <_i \dots <_i n-2d-1 <_i 1 <_i 2 <_i \dots <_i i-1.$$

Hence $<_1$ is the usual order. For example, by changing the order on $n-2d-1$, we allow the simplex $1, 3, 5$ to be represented by $5, 1, 3$ or $3, 5, 1$ as well.

We say that $a_0, a_1, \dots, a_k$ are *cyclically ordered* if $a_0 <_i a_1 <_i \dots <_i a_k$ for some i .

Triangulations of even-dimensional cyclic polytopes were given an elegant combinatorial description in [18], which we now explain. A d -simplex of a triangulation of $C_{n-2d-1, 2d}$ is *internal* if it does not lie within a facet of $C_{n-2d-1, 2d}$, recalling that a *facet* is a face of codimension one. It is clear that a triangulation of a convex polygon is determined by the internal edges of the triangulation; similarly, a triangulation of any $2d$ -dimensional convex polytope is determined by the internal d -simplices of the triangulation, by a theorem of Dey [8]. We write $d\text{-simp } \mathcal{T}$ for the set of internal d -simplices of $\mathcal{T}$. We say that a $d-1$ -simplex of $\mathcal{T}$ is *interior* if all of its facets are internal d -simplices.

There exist combinatorial criteria for when d -simplices in $C_{n-2d-1, 2d}$ are internal d -simplices and for when internal d -simplices intersect in their interiors. Indeed, a d -simplex $A \in n-2d-1 \choose d-1$ is an internal d -simplex in $C_{n-2d-1, 2d}$ if and only if A contains no consecutive elements in the cyclic ordering of $n-2d-1$. A d -simplex A and a d -simplex B are *intertwining* if

$$a_0 < b_0 < a_1 < b_1 < \dots < a_d < b_d$$

is a cyclic ordering, in which case we write $A \bowtie B$. Two internal d -simplices intersect in their interiors if and only if they intertwine, as can easily be seen in the $d-1$ case. A collection of d -simplices is called *non-intertwining* if no pair of its elements are intertwining.

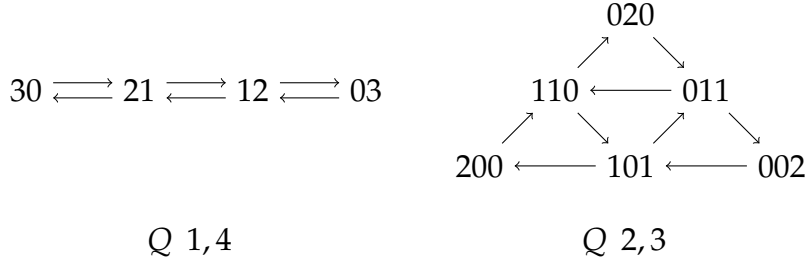
Theorem 3 ([18, Theorem 2.3, Theorem 2.4]). *There is a bijection between triangulations of $C_{n-2d-1, 2d}$ and non-intertwining sets of $n \choose d-1$ d -simplices in $\text{int}_d n-2d-1$, via sending*

$$\mathcal{T} \mapsto d\text{-simp } \mathcal{T}.$$

The corresponding description for triangulations of odd-dimensional cyclic polytopes was found in [22, Section 4].

Triangulations of cyclic polytopes can be *mutated* by operations known as *bistellar flips* [7, Section 2.4]. The following theorem can be taken as a definition of this operation in the case of triangulations of $C_{n-2d-1, 2d}$.

Theorem 4 ([18, Theorem 4.1]). *Triangulations $\mathcal{T}$ and $\mathcal{T}'$ of $C_{n-2d-1, 2d}$ are bistellar flips of each other if and only if $d\text{-simp } \mathcal{T}$ and $d\text{-simp } \mathcal{T}'$ have all but one element in common.*

Figure 1: Examples of the digraphs $Q_{d,n}$


Therefore, given a triangulation $\mathcal{T}$ of $C_{n-2d-1, 2d}$, we call an internal d -simplex $A \in d\text{-simp } \mathcal{T}$ *mutable* if there exists a triangulation $\mathcal{T}'$ such that

$$d\text{-simp } \mathcal{T}' = d\text{-simp } \mathcal{T} \setminus \{A\} \cup \{B\},$$

where $A \neq B$.

2.2 Cuts

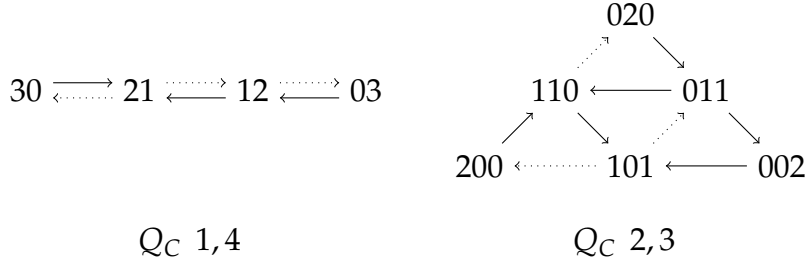
We will associate digraphs to triangulations of even-dimensional cyclic polytopes and show what properties of the triangulation are encoded in the digraph. Indeed, we now define the digraphs which are higher analogues of orientations of the A_n Dynkin diagram, following [15]. The main result of the first part of this paper is that these digraphs tell that a triangulation of $C_{n-2d-1, 2d}$ has no interior $d-1$ -simplices.

Let $Q_{d,n}$ be the digraph with vertices

$$Q_{0,d,n} : \left\{ a_0, \dots, a_d \in \mathbb{Z}_{\geq 0}^{d+1} \mid \sum_{i=0}^d a_i = n-1 \right\}$$

and arrows $Q_{1,d,n} : \{A \rightarrow A - F_i \mid A, A - F_i \in Q_{0,d,n}\}$, where $F_i = \dots, 0, -1, 1, \dots, 0, \dots$, with $F_d = 1, 0, \dots, 0, -1$. See Figure 1 for pictures of these digraphs.

A subset $C \subseteq Q_{1,d,n}$ is called *cut* if it contains exactly one arrow from each $d-1$ -cycle in $Q_{d,n}$. Given a cut C , we write Q_C for the digraph with arrows $Q_{1,d,n} \setminus C$ and refer to this as the *cut digraph*. Examples of cut digraphs can be seen in Figure 2. Note that the cut digraphs of $Q_{1,n}$ are precisely the orientations of the A_n Dynkin diagram.

Figure 2: Cuts of the digraphs $Q_{d,n}$ 

3 Triangulations without interior $d-1$ -simplices

In this section, we introduce the digraph of a triangulation of a $2d$ -dimensional cyclic polytope, following [18]. We give the first main result, which states that the digraph detects when the triangulation has no interior $d-1$ -simplices and describes such digraphs. We outline how this result is proved.

The digraph of a polygon triangulation was implicitly introduced in [12]. The vertices of the digraph are given by internal edges of the triangulation, with an arrow from one internal edge to another if the one precedes the other in the clockwise ordering of the edges of one of the triangles. The key tool used to study triangulations in this paper is the following generalisation of this notion to triangulations of arbitrary even-dimensional cyclic polytopes.

Definition 1. Given a triangulation $\mathcal{T}$ of $C_{n-2d-1, 2d}$, we define the *digraph* $Q(\mathcal{T})$ of $\mathcal{T}$ to be the following directed graph. The vertices are

$$Q_0(\mathcal{T}) = \{d\text{-simp } \tau \in \mathcal{T}\}.$$

The arrows of $Q(\mathcal{T})$ are

$$Q_1(\mathcal{T}) = \left\{ A \rightarrow B \mid \begin{array}{l} A, B \text{ are } d\text{-simplices} \\ \exists s \in [r-1] \text{ such that } A \text{ and } B \text{ share the } s\text{-th } (d-1)\text{-face} \end{array} \right\},$$

where

$$E_i = \{ \dots, 0, 1, 0, \dots \}.$$

This was implicitly introduced in [18], but some work is required to show that the digraph which arises in [18] coincides with the one from the definition above. The details can be found in [20]. Note that in [20] and other related literature, digraphs are known as ‘quivers’.

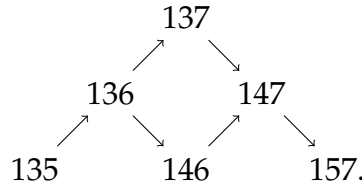
Example 1. Consider the triangulation $\mathcal{T}$ of $C_{8,4}$ with the set of 4-simplices

$$\{12345, 12356, 12367, 12378, 13456, 13467, 13478, 14567, 14578, 15678\}.$$

Here we write 12345 for the 4-simplex $\langle 1, 2, 3, 4, 5 \rangle$, and so on. One can compute the internal d -simplices of $\mathcal{T}$ by finding the internal d -simplices which are faces of these 4-simplices, obtaining

$$d\text{-simp } \mathcal{T} = \{135, 136, 137, 146, 147, 157\}.$$

These form the vertices of $Q(\mathcal{T})$. Using the definition of the arrows of $Q(\mathcal{T})$, one obtains that $Q(\mathcal{T})$ is the digraph



This is a more informative picture than simply considering the set of 4-simplices.

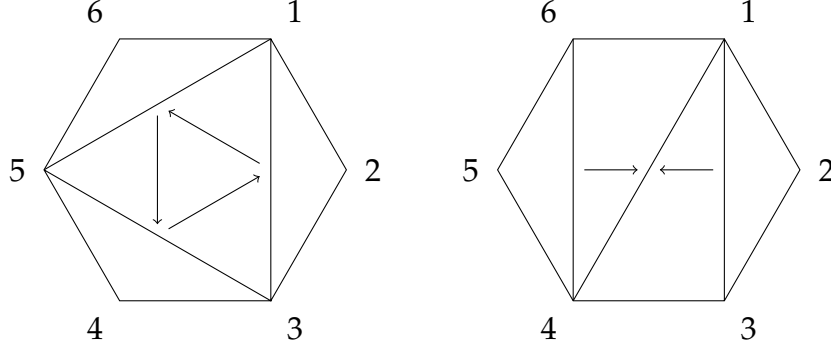
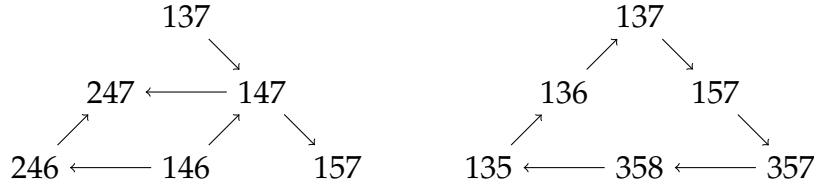
Recall that a $d-1$ -simplex of a triangulation $\mathcal{T}$ of $C_{n-2d-1, 2d}$ is *interior* if all of its facets are internal d -simplices. The first main theorem of the paper describes digraphs of $2d$ -dimensional triangulations with no $d-1$ -simplices.

Theorem 5. *A triangulation $\mathcal{T}$ of $C_{n-2d-1, 2d}$ has no interior $d-1$ -simplices if and only if its digraph $Q(\mathcal{T})$ is a cut of $Q(d, n)$, and this is the case if and only if its digraph has no cycle.*

It was implicit in the relation between the two papers [15] and [18] that cuts of $Q(d, n)$ corresponded to a particular class of triangulations of $C_{n-2d-1, 2d}$, but these papers left open what class of triangulations this was. The embedding of $Q(\mathcal{T})$ into $Q(d, n)$ is not unique, even up to symmetries of $Q(d, n)$. However, there are certain canonical embeddings which are unique up to said symmetries.

As an application of Theorem 5, we prove that the set of triangulations of $C_{n-2d-1, 2d}$ with no interior $d-1$ -simplices forms a connected subgraph of the flip graph of $C_{n-2d-1, 2d}$. Here, the *flip graph* of $C_{n-2d-1, 2d}$ is the graph whose vertices are triangulations of $C_{n-2d-1, 2d}$, with edges connecting triangulations related by a bistellar flip.

Example 2. It is straightforward to illustrate the two-dimensional version of Theorem 5. In Figure 3, the triangulation on the left contains the internal triangle 135, and so the digraph has a cycle, whereas the digraph on the right has no internal triangles, and so its digraph is an orientation of the type A Dynkin diagram.

Figure 3: Triangulations of polygons with and without interior triangles**Figure 4:** Triangulations of $C_{8,4}$ with and without interior 3-simplices

In four dimensions, one cannot draw the triangulations directly, but one can still draw their digraphs. The triangulation on the left-hand side of Figure 4 has set of 4-simplices

$$\{12347, 12356, 12467, 12378, 13478, 14567, 14578, 15678, 23456, 23467\},$$

and has no internal 3-simplices. The triangulation on the right-hand side of Figure 4 has set of 4-simplices

$$\{12345, 12356, 12367, 12378, 13458, 13567, 13578, 34578, 15678, 34567\},$$

with 1357 forming an internal 3-simplex. Note that the cycle this internal 3-simplex gives in the digraph does not exclusively have faces of 1357 as its vertices: 136 and 358 are also present in the cycle, but are not faces of this internal 3-simplex.

4 Identifying mutable d -simplices

In this section, we outline our second main result, which allows one to identify the mutable internal d -simplices of a triangulation from the digraph of the triangulation. It is clear that every internal edge within a polygon triangulation is contained within a

unique quadrilateral, and can therefore be flipped for the other diagonal of this quadrilateral to produce a new triangulation. As given in [Theorem 4](#), the analogue of flipping an internal edge for a triangulation of a $2d$ -dimensional cyclic polytope is exchanging one internal d -simplex for another. Recall from [Section 2.1](#) that if an internal d -simplex can be exchanged for another internal d -simplex then it is called *mutable*. Not every internal d -simplex is mutable, as is shown by the following example.

Example 3. Consider the triangulation $\mathcal{T}$ of $C_{7,4}$ with set of 4-simplices

$$\mathcal{T} = \{12345, 12356, 12367, 13456, 13467, 14567\}.$$

We then have

$$d\text{-simp } \mathcal{T} : \{135, 136, 146\}.$$

The internal 2-simplex 135 can be mutated in this triangulation by replacing it with 246 to obtain the triangulation $\mathcal{T}'$ with

$$d\text{-simp } \mathcal{T}' = \{246, 136, 146\}$$

and set of 4-simplices

$$\mathcal{T}' = \{12346, 12456, 12367, 13467, 14567, 23456\}.$$

Similarly, the internal 2-simplex 146 can be flipped in $\mathcal{T}$ to obtain the triangulation $\mathcal{T}''$ with

$$d\text{-simp } \mathcal{T}'' = \{135, 136, 357\}$$

and set of 4-simplices

$$\mathcal{T}'' = \{12345, 12356, 12367, 13457, 13567, 34567\}.$$

However, one cannot perform a bistellar flip at the internal 2-simplex 136. There is no triangulation of $C_{7,4}$ which has both the internal 2-simplices 135 and 146 apart from $\mathcal{T}$.

The criterion in terms of the digraph for an internal d -simplex to be mutable uses the following notion.

Definition 2. Given a triangulation $\mathcal{T}$ of $C_{n-2d-1, 2d}$, a path of length two

$$\dots, a_{i-1}, a_i, a_{i+1}, \dots \rightarrow \dots, a_{i-1}, b_i, a_{i+1}, \dots \rightarrow \dots, a_{i-2}, b_{i-1}, b_i, a_{i+1}, \dots$$

in $Q(\mathcal{T})$ is *retrograde at* $a_0, \dots, a_{i-1}, b_i, a_{i+1}, \dots, a_d$ if $b_{i-1} < a_i$.

A path of length ≥ 2

$$A_0 \rightarrow A_1 \rightarrow \dots \rightarrow A_l \rightarrow A_{l+1}$$

is *retrograde* if $A_{i-1} \rightarrow A_i \rightarrow A_{i+1}$ is retrograde at A_i for all $i \in [l]$. For $i \in [l]$, we say that A_i is *in the middle* of this retrograde path. We consider paths of length one to be trivially retrograde.

It is straightforward to show that every arrow in the digraph is contained in a unique maximal retrograde path. We then obtain the main theorem of the second half of this paper.

Theorem 6. *Let $\mathcal{T}$ be a triangulation of $C_{n-2d-1, 2d}$. Then an internal d -simplex of $\mathcal{T}$ is mutable if and only if it is not in the middle of a retrograde path in $Q(\mathcal{T})$.*

In fact, one can read off which internal d -simplex should result from the bistellar flip by comparing the adjacent internal d -simplices in the digraph. We do not state this procedure explicitly, but illustrate it in [Example 4](#). This then allows one to determine whether the bistellar flip is increasing or decreasing in the poset structure on the triangulations of $C_{n-2d-1, 2d}$ mentioned in the introduction.

An interesting related question is how the digraph transforms under a bistellar flip. This should give a higher-dimensional version of the quiver mutation of [\[13\]](#). Presently, we are only able to understand limited cases [\[20, 15\]](#).

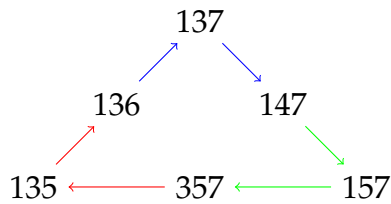
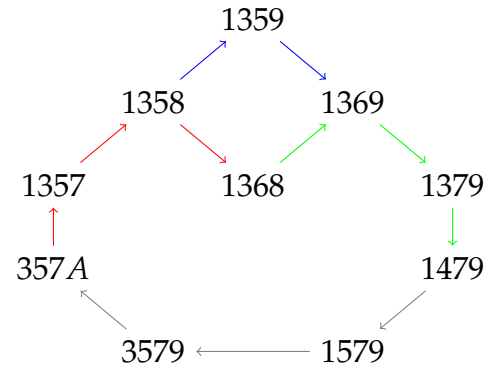
Example 4. We provide examples of how one may use this criterion to identify the mutable internal d -simplices of a triangulation. We also illustrate how to compute the internal d -simplex obtained from the bistellar flip. We represent maximal retrograde paths using consecutive arrows of the same colour.

We first consider the triangulation of $C_{8,4}$ given in [Figure 5a](#). The three retrograde paths in this digraph are $136 \rightarrow 137 \rightarrow 147$, $147 \rightarrow 157 \rightarrow 357$, and $357 \rightarrow 135 \rightarrow 136$. Here, the last path is seen to be retrograde by cyclically reordering using $<_3$ to $357 \rightarrow 351 \rightarrow 361$. The path $135 \rightarrow 136 \rightarrow 137$ is not retrograde since one has increased the same entry twice, instead of increasing the i -th entry and then the $i-1$ -th entry. Cyclically reordering using $<_4$, the path $571 \rightarrow 573 \rightarrow 513$ is not retrograde, since $1 \not< 1$. The mutable internal 2-simplices of the triangulation are then 136, 147, and 357, since 137, 157, and 135 are the ones lying in the middle of retrograde paths.

The bistellar flip at the internal 2-simplex 136 can be computed as follows. We seek an internal 2-simplex abc which is intertwining with 136. Then we must have $a = 2$, since this is the only number between 1 and 3. Moreover $b = 5$ since 135 is adjacent to 136 and $c = 7$ since 137 is adjacent to 136. Hence flipping 136 yields 257.

Note that maximal retrograde paths are not always of length d for a triangulation of $C_{n-2d-1, 2d}$. This is shown by the triangulation of $C_{10,6}$ given in [Figure 5b](#), where we use 'A' to denote 10. We have three retrograde paths of length three, and then the retrograde path $1358 \rightarrow 1359 \rightarrow 1369$ of length two. The mutable internal d -simplices of this triangulation are 357A, 1368, 1479.

We finally illustrate the procedure for computing the outcome of a bistellar flip again, now using 1368. We seek an internal 3-simplex $abcd$ which intertwines with 1368. Then $a = 2$ and $c = 7$ are forced. Looking at the adjacent simplices then gives us $b = 5$ and $d = 9$, yielding 2579 as the outcome of the bistellar flip.

Figure 5: Triangulations for Example 4(a) Triangulation of $C_{8,4}$ (b) Triangulation of $C_{10,6}$ 

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Bijections between planar maps and planar linear normal λ -terms with connectivity condition

Wenjie Fang^{*1}

¹Univ Gustave Eiffel, CNRS, LIGM, F-77454 Marne-la-Vallée, France

Abstract. The enumeration of linear λ -terms has attracted quite some attention recently, partly due to their link to combinatorial maps. In 2019, Zeilberger and Reed conjectured that 3-connected planar linear normal λ -terms have the same counting formula as bipartite planar maps. In this article, we settle this conjecture by giving a direct bijection. We also explore its enumerative consequences. With a similar approach, we give a new and direct bijection between planar linear normal λ -terms and planar maps, whose restriction to 2-connected λ -terms leads to loopless planar maps.

Résumé. L'énumération des λ -termes linéaires a récemment attiré un intérêt grâce à leur lien avec les cartes combinatoires. En 2019, Zeilberger et Reed ont conjecturé que les λ -termes linéaires planaires normaux 3-connexes sont comptés par la même formule que les cartes planaires bipartites. Dans cet article, nous prouvons cette conjecture grâce à une bijection directe. Nous explorons aussi ses conséquences énumératives. Avec la même approche, nous présentons une nouvelle bijection entre les λ -termes linéaires planaires normaux et les cartes planaires, qui est directe et dont la restriction aux λ -termes 2-connexes donne les cartes planaires sans boucle.

Keywords: Bijection, λ -term, combinatorial map, edge-connectivity, enumeration.

As a well-known model of computation, λ -calculus has been well studied in logic and related fields. However, its enumerative aspect only attracted attention relatively recently. One of the most well-studied family is that of *linear λ -terms*, partly due to its connections with combinatorial maps. Such connections were pioneered by Bodini, Gardy and Jacquot in [2], where a simple bijection from linear λ -terms to cubic maps was given. This bijection has led to the study of asymptotic properties and statistics distribution of related λ -terms (see, e.g., [1, 2, 3]), and connections to other objects [7].

Independently, Zeilberger and Giorgetti found in [15] a bijection between *planar linear normal λ -terms* and planar maps, which is not a simple restriction of the one in [2]. Moreover, when restricted to λ -terms that are also *unitless*, i.e., without closed sub-term, this bijection leads to bridgeless planar maps. The unitless condition here is equivalent to the 2-connectedness of the syntactic diagram of the λ -term. Such connections lead naturally to the consideration of higher connectivity conditions on λ -terms. In a talk at

^{*}wenjie.fang@univ-eiffel.fr. Wenjie Fang is partially supported by ANR LambdaComb (ANR-21-CE48-0017).

CLA 2019 [16], Zeilberger and Reed considered a possible interpretation of connectivity in logic. Based on experiments, they proposed the following conjecture.

Conjecture 0.1 ([16]). *Planar 3-connected linear normal λ -terms with $n - 2$ variables are counted by the same formula as bipartite planar maps with n edges given in [13]:*

$$\frac{2^n}{n-1} \binom{2n-1}{n-2}.$$

In a talk at CLA 2020 [11], Grygiel and Yu tried to relate these 3-connected λ -terms to $\beta(0,1)$ -trees, which are in bijection with bicubic planar maps [6] and thus with bipartite planar maps [13], and succeeded in a few special cases. The key of their work is a characterization of skeletons of such λ -terms (Proposition 2.3).

In this article, based on the characterization given by Grygiel and Yu in [11], we prove Conjecture 0.1 by giving a bijection between 3-connected planar linear normal λ -terms and bipartite planar maps via the so-called *degree trees* defined in [8].

Theorem 0.2. *For $n \geq 2$, there is a bijection between 3-connected planar linear normal λ -terms with n variables and bipartite planar maps with $n - 2$ edges.*

Using this new bijection, we study the refined enumeration of such λ -terms under several statistics, and also their asymptotic distribution, which are all done by translating known results on bipartite planar maps.

With a similar approach, we also give a new and direct bijection between planar linear normal λ -terms and planar maps, whose restriction to 2-connected λ -terms leads to loopless planar maps.

Theorem 0.3. *There is a direct bijection between planar linear normal λ -terms with n variables and planar maps with $n - 1$ edges. Furthermore, it sends those λ -terms that are also 2-connected to loopless planar maps.*

For the second bijection, we define a family of node-labeled trees called *v-trees*, which can be seen as description trees (see [6]) of planar maps under a seemingly new recursive decomposition. We then give direct bijections from v-trees to both planar maps and planar linear normal λ -terms, thus linking the two families.

The rest of the article is organized as follows. In Section 1, we define the objects that we study. Then we characterize in Section 2 the skeletons of planar linear normal λ -terms with different connectivity conditions, which is then used in Section 3 to give a bijection from 3-connected planar linear normal λ -terms to degree trees, which are in bijection with bipartite maps. In Section 4, we define v-trees and relate them bijectively to planar linear normal λ -terms, and a direct bijection to planar maps is given in Section 5.

Due to space limit, some results and proofs are omitted in this extended abstract. Readers are referred to the full version [10] for all details.

1 Preliminaries

In mathematical logic, λ -terms can be defined recursively using three constructions:

- Atoms, which are variables $x, y, \dots$;
- Application $t u$ with two λ -terms t and u ;
- Abstraction $\lambda x.t$ with t a λ -term and x a variable. We say that every atom with the variable x in t is *bound* by the abstraction, if not yet bound.

In the following, we consider λ -terms up to α -renaming, i.e., changing the name of a variable in an abstraction and all atoms it binds. A λ -term t is *closed* if all atoms are bound, and it is *linear* if it is closed and each abstraction binds exactly one atom. Computation on λ -terms is done using β -reduction: we replace a sub-term of the form $\lambda x.u v$ in a λ -term by $u x \leftarrow v$, where the sub-term u has all its atoms of x replaced by a copy of v . A λ -term is *β -normal* (or simply *normal*) if no β -reduction is possible. Given a λ -term t , its *skeleton* $\text{Sk } t$ is a plane unary-binary tree defined recursively as follows.

- If t is an atom, then $\text{Sk } t$ is a leaf;
- If $t = u v$, then $\text{Sk } t$ is rooted at a binary node, with $\text{Sk } u$ its left sub-tree, and $\text{Sk } v$ its right sub-tree;
- If $t = \lambda x.u$, then $\text{Sk } t$ is rooted at a unary node, with $\text{Sk } u$ its only sub-tree.

We identify atoms and abstractions in t with their corresponding leaves and unary nodes in $\text{Sk } t$. Many properties of t can be read off from $\text{Sk } t$. For instance, t is normal if no binary node of $\text{Sk } t$ has a unary node as its left child. The *size* of a skeleton is its number of leaves, which is also the number of atoms in the original λ -term.

The skeleton $\text{Sk } t$ of a λ -term t still misses the binding relations between atoms and abstractions. The *syntactic diagram* (or simply *diagram*) of a λ -term t , denoted by $\text{Diag } t$, is obtained from its skeleton $\text{Sk } t$ by replacing each leaf by an edge from its parent to the unary node of abstraction that binds its atom. It is clear that two different λ -terms never share the same diagram. Given a leaf of $\text{Sk } t$, we draw the corresponding edge starting from the parent of the leaf, traveling counter-clockwise and entering the unary node from its right. If such a drawing can be done without intersection, then the λ -term t is *planar*. Figure 1 gives an example of a planar λ -term, with its skeleton and syntactic diagram. We note that this λ -term is also normal.

As diagrams are graphs, we may transfer notions in graph theory to λ -terms. If the diagram of a λ -term t is *2-edge-connected* (or simply *2-connected*), i.e., the removal of any edge does not disconnect the diagram, then we say that t is *2-connected*. Similarly, a λ -term t is *3-edge-connected* (or simply *3-connected*) if the removal of any two edges in

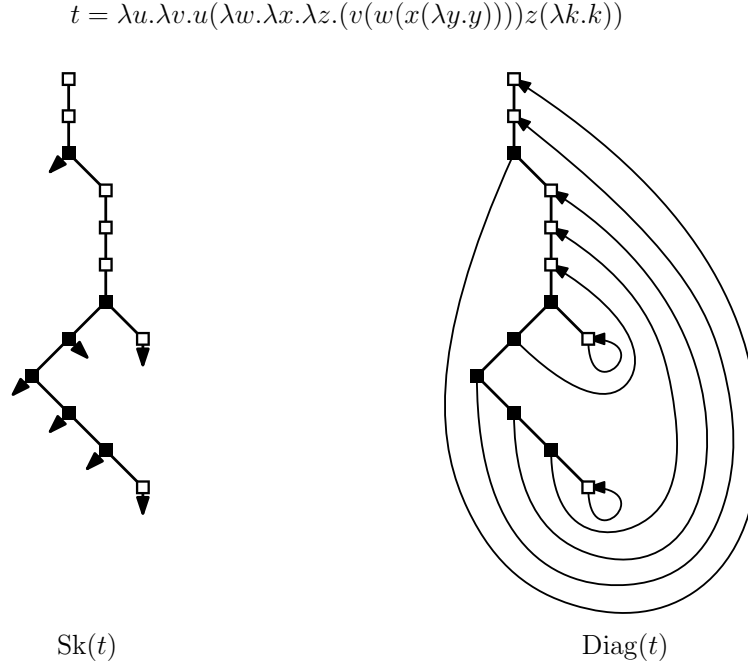


Figure 1: Example of a normal planar λ -term, its skeleton and syntactic diagram.

Diag t does not disconnect Diag t , except possibly the two edges adjacent to the root of Sk t when t starts with an abstraction binding one atom.

In this article, we consider planar linear λ -terms. For planarity, given the skeleton of such a λ -term, we take a clockwise contour walk starting from the root, in which we read off the unary nodes and leaves on first visit. By planarity, we must obtain a well-parenthesized word with each abstraction paired up with the atom it binds. As the binding of atoms is given by planarity, we may identify planar linear λ -terms with their skeletons. By abuse of notation, given a skeleton S of some planar linear λ -term t , we denote by Diag S the diagram Diag t of t .

In the following, we relate previously defined λ -terms to planar maps. A *planar map* is an embedding of a graph on the plane defined up to orientation-preserving continuous deformations, such that its edges only intersect at vertices. The connected components of the complement of the embedding are called the *faces*. The face that extends to infinity is called the *outer face*. Furthermore, we only consider *rooted* planar maps here, which means that we mark a special corner on the outer face, called the *root corner*. The edge next to the root corner in the clockwise order is called the *root edge*, and the vertex of the corner is called the *root vertex*. We also consider the *empty map*, which consists of a single vertex. A planar map is *bipartite* if we can color its vertices with black and white such that each edge links a black and a white vertex. The root vertex is colored black by convention.

2 Characterization of planar linear normal λ -terms with connectivity conditions

We may classify planar linear λ -terms by the connectivity of their diagrams. We denote by $\mathcal{C}_n^1$ the set of planar linear normal λ -terms of size n . Similarly, we denote by $\mathcal{C}_n^2$ (resp. $\mathcal{C}_n^3$) the terms that are 2-connected (resp. 3-connected) in $\mathcal{C}_n^1$. We take $\mathcal{C}^1 = \bigcup_{n \geq 1} \mathcal{C}_n^1$, and we define $\mathcal{C}^2$ and $\mathcal{C}^3$ similarly. It is clear that $\mathcal{C}^3 \subset \mathcal{C}^2 \subset \mathcal{C}^1$.

Due to the correspondence between planar linear λ -terms and their skeletons, in the following we will focus on skeletons instead of λ -terms. We denote by $\mathcal{S}_n^1$ (resp. $\mathcal{S}_n^2$ and $\mathcal{S}_n^3$) the set of skeletons of terms in $\mathcal{C}_n^1$ (resp. $\mathcal{C}_n^2$ and $\mathcal{C}_n^3$). We also take $\mathcal{S}^1 = \bigcup_{n \geq 1} \mathcal{S}_n^1$, and we define $\mathcal{S}^2$ and $\mathcal{S}^3$ similarly. It is clear that $\mathcal{S}^3 \subset \mathcal{S}^2 \subset \mathcal{S}^1$.

We start by characterizing skeletons of planar linear λ -terms. Given a unary-binary tree S and a node u in S , we denote by S_u the sub-tree induced by u , by $\text{leaf } S$ the number of leaves in S , and by $\text{unary } S$ the number of unary nodes in S .

Proposition 2.1. *A unary-binary tree S is in $\mathcal{S}^1$ if and only if*

- **(Linearity)** $\text{leaf } S = \text{unary } S$;
- **(Normality)** No binary node in S has a unary node as its left child;
- **(Connectedness)** For every binary node and leaf u in S , we have $\text{leaf } S_u - \text{unary } S_u$ greater than or equal to the number of consecutive unary nodes above u .

Furthermore, a unary-binary tree $S \in \mathcal{S}^1$ is also in $\mathcal{S}^2$ if and only if

- **(2-connectedness)** For every binary node and leaf u in S , we have $\text{leaf } S_u - \text{unary } S_u$ strictly larger than the number of consecutive unary nodes above u .

For skeletons of λ -terms in $\mathcal{C}^3$, we start with an observation from [11].

Proposition 2.2 (See [11]). *Let $S \in \mathcal{S}^3$ and u its first binary node. The left child of u is a leaf.*

Proof. Let v be the left child of u . If v is not a leaf, then by planarity, the leaves of S_v are connected to an initial segment of the unary nodes above u . Cutting $\{u, v\}$ and the edge below the said initial segment disconnects $\text{Diag } S$, contradicting 3-connectedness. $\square$

We define the *reduced skeleton* of a skeleton S in $\mathcal{S}^3$ to be the unary-binary tree obtained from S by removing the first binary node, its left child (which is a leaf by Proposition 2.2), and the nodes above it. By **(Linearity)** on S , we can reconstruct S from the reduced skeleton of t . We denote by $\mathcal{RS}_n$ the set of reduced skeletons of skeletons in $\mathcal{S}_n^3$, and $\mathcal{RS} = \bigcup_{n \geq 3} \mathcal{RS}_n$. The following characterization of reduced skeletons was first discovered and stated without proof in [11]. Readers are referred to the full version [10] for its proof.

Proposition 2.3. *A unary-binary tree S is in $\mathcal{RS}$ if and only if*

- *(Normality) No binary node in S has a unary node as its left child;*
- *(3-connectedness) For every binary node u in S , let v be its right child, then $\text{leaf } S_v - \text{unary } S_v$ is strictly larger than the number of consecutive unary nodes above u .*

3 Bijection from λ -terms in $\mathcal{C}^3$ to degree trees

A *degree tree*, as defined in [8], is a pair (T, ℓ) where T is a plane tree and ℓ a labeling on the nodes of T such that

- If u is a leaf, then $\ell(u) = 0$;
- For u with children $v_1, v_2, \dots, v_k$ from left to right, let $s(u) = k - \sum_{i=1}^k \ell(v_i)$, then $s(u) - \ell(v_1) \leq \ell(u) \leq s(u)$.

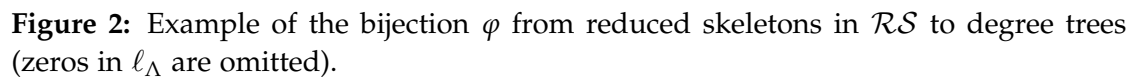
The *size* of a degree tree (T, ℓ) is the number of edges in T . We denote by $\mathcal{T}_n$ the set of degree trees of size n . Given a degree tree (T, ℓ) , we define its *edge labeling* ℓ_Λ as follows: for each internal node u , we put a label $s(u) - \ell(u)$ on its leftmost descending edge, and 0 on all other edges. It is clear that we can recover (T, ℓ) from T and ℓ_Λ . Later, we will use both ℓ and ℓ_Λ . We now define a bijection φ from reduced skeletons to degree trees.

Definition 3.1. Given $S \in \mathcal{RS}$, we perform the following:

1. Remove all leaves from S , and add a new unary root above the old root. Then smooth out all unary nodes and put the number of smoothed-out nodes at each edge. We obtain a binary tree S' with an edge labeling ℓ'_Λ .
2. Perform the classical bijection (*switching left and right*) from binary tree S' to plane tree T' : given a node u , its *right child* in S' is turned into its *leftmost child* in T' , and its *left child* in S' into its sibling immediately to its right. The edge labels of ℓ'_Λ , which are all on rightmost descending edges in S' , are transformed into an edge labeling ℓ_Λ with only non-zero values on the leftmost descending edges of nodes.

We define $\varphi(S)$ to be (T', ℓ) , where ℓ is the node labeling corresponding to ℓ_Λ constructed above. See Figure 2 for an example of the bijection φ .

Proposition 3.2. *A unary-binary tree S is in $\mathcal{RS}_n$ if and only if $\varphi(S)$ is in $\mathcal{T}_{n-2}$.*



We first define several statistics on reduced skeletons. Given $S \in \mathcal{RS}_n$, we define its *excess* $\text{ex } S$ as the number of leaf nodes of S minus the number of unary nodes of S , and we denote by $\text{Appl}_v S$ the number of binary nodes of S whose right child is a leaf, $\text{Appl}_a S$ the number of binary nodes of S whose right child is also a binary node, and $\text{uc}_k S$ the number of maximal chains of unary nodes of length k . In the language of λ -terms, $\text{ex } S$ is the number of abstractions at the beginning of the original term minus one (from the leaf in [Proposition 2.2](#) deleted in the reduced skeleton), $\text{Appl}_v S$ the number of applications of terms to a variable, $\text{Appl}_a S$ the number of applications of terms to another application, and $\text{uc}_k S$ the number of maximal consecutive abstractions of length k , except the one at the root.

Now we introduce some statistics on degree trees. Given T, ℓ a degree tree, we denote by $\text{lnode } T, \ell$ the number of leaves in T , by $\text{znode } T, \ell$ the number of internal nodes whose leftmost descending edge has label 0, by $\text{rlabel } T, \ell$ the label of the root in T, ℓ , and by $\text{edge}_k T, \ell$ the number of edges with label $k \geq 1$. The following transfer of statistics is clear from [Definition 3.1](#).

Proposition 3.3. *Given $S \in \mathcal{RS}_n$, let $T, \ell = \varphi(S)$. Then we have*

$$\begin{aligned} \text{Appl}_v S &= \text{inode } T, \ell, & \text{Appl}_a S &= \text{znode } T, \ell, \\ \text{ex } S &= \text{rlabel } T, \ell, & \text{uc}_k S &= \text{edge}_k T, \ell. \end{aligned}$$

The statistics mentioned in [Proposition 3.3](#) are also transferred to bipartite planar maps. More precisely, let $\mathcal{B}_n$ be the set of bipartite planar maps with n edges. For $M \in \mathcal{B}_n$, we denote by $\text{white } M$ (resp. $\text{black } M$) the number of white (resp. black) vertices in M , and $\text{outdeg } M$ the half-degree of the outer face. For $k \geq 1$, we denote by $\text{face}_k M$ the number of inner faces of degree $2k$. We have the following result from [\[8\]](#).

Proposition 3.4 (See [Proposition 3.3](#) and [3.6](#), and [Remark 1](#) in [\[8\]](#)). *There is a bijection μ from $\mathcal{T}_n$ to $\mathcal{B}_n$. Moreover, for $T, \ell \in \mathcal{T}_n$, let $M = \mu(T, \ell)$, and we have*

$$\begin{aligned} \text{inode } T, \ell &= \text{white } M, & \text{znode } T, \ell &= \text{black } M, \\ \text{rlabel } T, \ell &= \text{outdeg } M, & \text{edge}_k T, \ell &= \text{face}_k M. \end{aligned}$$

We obtain [Theorem 0.2](#) using the bijections from $\mathcal{RS}_n$ to $\mathcal{T}_{n-2}$ and from $\mathcal{T}_{n-2}$ to $\mathcal{B}_{n-2}$.

Proof of Theorem 0.2. Consider $\mu \circ \varphi$ with φ from [Definition 3.1](#) and μ from [Proposition 3.4](#). It is a bijection from $\mathcal{RS}_n$ to $\mathcal{B}_{n-2}$ according to [Propositions 3.2](#) and [3.4](#). $\square$

Combining [Propositions 3.3](#) and [3.4](#), we have the following transfer of statistics from reduced skeletons of λ -terms in $\mathcal{C}^3$ to planar bipartite maps.

Theorem 3.5. *Let $S \in \mathcal{RS}_n$ and $M = \mu(\varphi(S)) \in \mathcal{B}_{n-2}$. Then we have*

$$\begin{aligned} \text{Appl}_v S &= \text{white } M, & \text{Appl}_a S &= \text{black } M, \\ \text{ex } S &= \text{outdeg } M, & \text{uc}_k S &= \text{face}_k M. \end{aligned}$$

A consequence of [Theorem 3.5](#) is that the generating function of $\mathcal{RS}$ refined by all related statistics can be written using that of bipartite maps refined by corresponding statistics. For instance, the enumeration of $\mathcal{RS}$ refined by ex and uc_k for all k is given by the generating function of bipartite maps refined by face degrees, which was found implicitly in [\[4\]](#), then written explicitly in [\[5\]](#).

Furthermore, we can also use [Theorem 3.5](#) to transfer known results on the distribution of various statistics on bipartite planar maps to λ -terms in $\mathcal{S}^3$. As an example, in [\[12\]](#), Liskovets studied the asymptotic vertex degree distribution of many families of planar maps, including Eulerian planar maps, which are duals of bipartite planar maps. Transferring the results in [\[12\]](#) to $\mathcal{RS}$ using [Theorem 3.5](#), we have the following corollary.

Corollary 3.6. *Let X_n be the number of abstractions at the beginning of a λ -term chosen uniformly randomly from $\mathcal{C}_n^3$. Then, for $k \geq 2$, when $n \rightarrow \infty$, we have*

$$\mathbb{P}[X_n = k] \rightarrow \frac{k-1}{3} \binom{2k-2}{k-1} \left(\frac{3}{16}\right)^{k-1}.$$

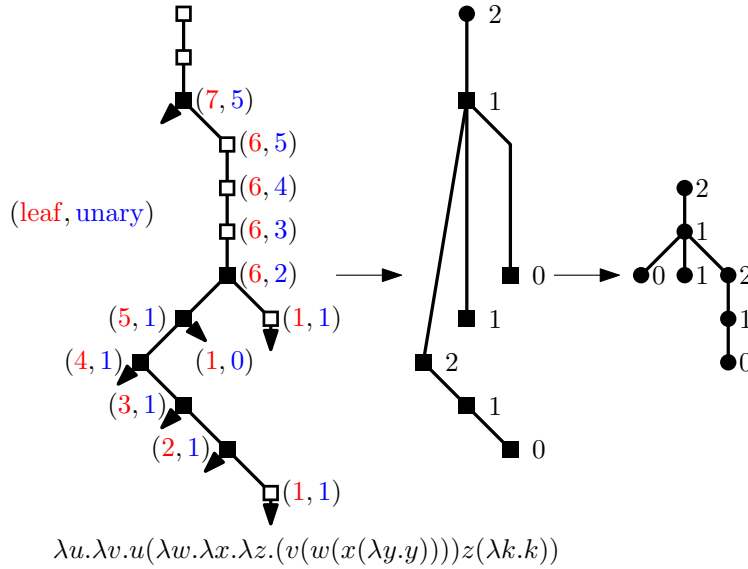


Figure 3: Example of the bijection ψ from skeletons in $\mathcal{S}^1$ to v-trees.

4 Bijection from λ -terms in $\mathcal{C}^1$ and $\mathcal{C}^2$ to v-trees

In the following, we consider skeletons in $\mathcal{S}^1$ and $\mathcal{S}^2$. We start by defining a transformation ψ from these skeletons to some trees with labels on their nodes.

Definition 4.1. Given $S \in \mathcal{S}^1$, We define ψS to be T, ℓ obtained as follows:

1. We label each binary node u by $\text{leaf } S_v - \text{unary } S_v$, where v is the right child of u , and remove all leaves and unary nodes. We now obtain a binary tree S' with node labeling ℓ' .
2. We perform the classical bijection (*switching left and right*) from binary tree S' to plane tree T as in the second step of the definition of φ in [Definition 3.1](#). We keep the labels of ℓ' on S' in ℓ on T .

For the reverse direction, we simply reverse all the operations above, then insert unary nodes *from bottom to top* according to labels, but *only to the right branch* of binary nodes, in accordance with (**Normality**).

We now characterize the labeled plane trees obtained by ψ . A node-labeled tree T, ℓ is a *v-tree* if the following conditions hold:

- Leaves of T are labeled by 0 or 1;
- Every non-root node u with children $v_1, \dots, v_k$ satisfies $0 \leq \ell u \leq 1 - \sum_{i=1}^k \ell v_i$;

- For the root r with children $v_1, \dots, v_k$, we have $\ell(r) = 1 + \sum_{i=1}^k \ell(v_i)$.

The *size* of a v-tree is the number of its edges. We denote by $\mathcal{V}_n$ the set of v-trees of size n , and we take $\mathcal{V} = \bigcup_{n \geq 1} \mathcal{V}_n$. Moreover, we denote by $\mathcal{V}_n$ the set of v-trees *without label 0* of size n , and we take $\mathcal{V} = \bigcup_{n \geq 1} \mathcal{V}_n$. We have the following characterization of $\psi: \mathcal{S}_n^1 \rightarrow \mathcal{V}_n$.

Proposition 4.2. *A unary-binary tree S is in $\mathcal{S}_n^1$ if and only if $T = \psi(S)$ is a v-tree of size $n - 1$. In other words, ψ is a bijection from $\mathcal{S}_n^1$ to $\mathcal{V}_{n-1}$.*

The bijection ψ specializes naturally to skeletons in $\mathcal{S}^2$.

Proposition 4.3. *A unary-binary tree S is in $\mathcal{S}_n^2$ if and only if $T = \psi(S)$ is a v-tree of size $n - 1$ without label 0. In other words, ψ is a bijection from $\mathcal{S}_n^2$ to $\mathcal{V}_{n-1}$.*

Proof. Suppose that u is a non-root non-unary node in S , with k consecutive unary nodes above it leading to a binary node v . By (**Normality**), u is in the right sub-tree of v in S , meaning that $\ell(v) = \text{leaf}(S_u) - \text{unary}(S_u) - k$. We thus conclude the proof by seeing that u satisfies (**2-connectedness**) if and only if $\ell(u) > 0$. $\square$

Remark 1. We note that, unlike degree trees, the definition of v-tree does not distinguish left and right. Therefore, in the definition of ψ , there is no need to switch left and right in the second step. However, we keep the current definition for consistency with φ .

5 Bijection from planar maps to v-trees

We denote by $\mathcal{M}_n$ the set of planar maps with n edges, and by $\mathcal{M}$ the set of all planar maps. For M a planar map, we denote by $\text{outv}(M)$ the number of vertices of M adjacent to the outer face. We now define a direct bijection ρ using an exploration procedure of planar maps, in the vein of [8, 9].

Given a planar map M , we put the label $\text{outv}(M)$ on its root vertex, and then start a contour walk in the clockwise direction from the root corner. Suppose that we are at a corner c and is about to walk along a new edge e starting from v to w . We find the corner c_1 of v on the other side of e and the next corner c_2 in clockwise direction of v that is adjacent to the outer face. We put the label $\text{outv}(U) - 1$ on w , where U is the part of M between c and c_2 . There are two cases, as illustrated on the left part of [Figure 4](#):

1. If $c_1 = c_2$, then e is a bridge, and we do nothing.
2. If $c_1 \neq c_2$, then e separates the outer face with an inner face. In this case, we detach the part of M between c_1 and c_2 while duplicating the vertex v .

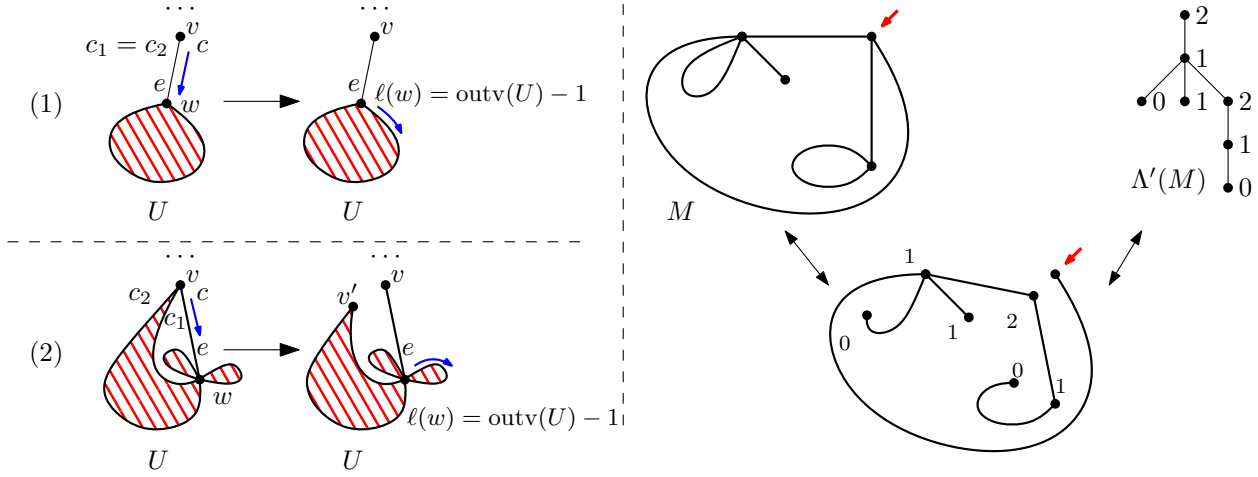


Figure 4: Definition and example of the direct bijection ρ' .

After the whole walk, we obtain a planar map with labels on its vertices, denoted by ρM . The right part of Figure 4 gives an example of the construction of ρM . It is in fact always a tree rooted at the original root corner of M , as every inner face will be eventually “opened up” in the second case above.

Theorem 5.1. *For all $n \geq 0$, the transformation ρ is a bijection from the set $\mathcal{M}_n$ of planar maps with n edges to the set $\mathcal{V}_n$ of v -trees with n edges. Furthermore, for $M \in \mathcal{M}_n$, the label of the root of ρM is $\text{outv } M$.*

Proof of Theorem 0.3. As we may identify λ -terms in $\mathcal{C}^1$ with their skeletons in $\mathcal{S}^1$, the bijection is given by $\psi \circ \rho$. The first part is a consequence of Proposition 4.2 and Theorem 5.1. For the second part, we notice that a label 0 is only produced when opening up a loop in the construction of ρM . We thus conclude by Proposition 4.3. $\square$

Remark 2. Zeilberger and Giorgetti has given a bijection between planar maps and planar linear normal λ -terms in [15], using the Tutte decomposition [13]. Furthermore, Zeilberger also showed in [14] that the same bijection sends terms that are also unitless to bridgeless planar maps, which are the duals of loopless planar maps. We have checked that our bijection is different from the one in [15], even after taking the dual.

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An Algebraic Perspective on Ramsey Numbers

Jesús A. De Loera^{*1} and William J. Wesley^{†1}

¹*Department of Mathematics, University of California, Davis*

Abstract. We study the Ramsey numbers $R(r, s)$ through Hilbert’s Nullstellensatz. We give a polynomial encoding whose solutions correspond to Ramsey graphs, those that do not contain a copy of K_r or $\bar{K}_s$. The Ramsey number is reached the first time the system has no solution. We construct Nullstellensatz certificates of this fact whose degrees are equal to the restricted online Ramsey numbers. These results generalize to other numbers in Ramsey theory, such as Rado, van der Waerden, and Hales–Jewett numbers.

Keywords: Ramsey theory, Hilbert’s Nullstellensatz, polynomial ideals.

1 Introduction

Ramsey numbers are some of the most interesting and mysterious combinatorial numbers [19]. In its simplest, most popular form, the *Ramsey number* $R(r, s)$ is the smallest positive integer n such that every 2-coloring of the edges of K_n contains a copy of K_r in the first color or K_s in the second color. Ramsey numbers can be generalized by allowing more than two colors and graphs other than K_r and K_s . The number $R(G_1, G_2, \dots, G_k)$ is the smallest positive integer n such that every k -coloring of the edges of K_n contains a copy of G_i in color i for some i . If $G_i = K_{r_i}$ for all i , we simply write $R(r_1, r_2, \dots, r_k)$. All of these numbers are finite by Ramsey’s theorem [38]. The goal of this article is to uncover further structure and complexity of Ramsey numbers through algebraic-geometric means.

Our paper introduces algebraic interpretations of $R(r, s)$ and related numbers and graphs using polynomial ideals, varieties, and Nullstellensatz identities (see [9] for an introduction). We tie the values and complexity of $R(r, s)$ to systems of *polynomial equations*. We show that these encodings give interesting information in the computational complexity of Ramsey numbers and Ramsey graphs. The precise models appear in [Section 2](#). Before we state our main results, let us recall some relevant prior context and results:

Computing exact values for Ramsey numbers is a challenge. In fact there are only nine values of $R(r, s)$ with $3 \leq r \leq s$ whose exact values are known, and the only known

^{*}deloera@math.ucdavis.edu.

[†]wjwesley@math.ucdavis.edu

nontrivial Ramsey numbers with more than two colors are $R(3,3,3) = 17$ and $R(3,3,4) = 30$ [21, 7]. Ramsey numbers as small as $R(5,5)$ remain unknown, and the best known bounds are $43 \leq R(5,5) \leq 48$. See [17, 1]. The numbers $R(G_1, G_2)$ are known for some families of graphs, but many cases remain open (see, for example, [37] for a survey of small Ramsey numbers and their best known bounds). The best known asymptotic lower and upper bounds for diagonal Ramsey numbers $R(s, s)$ are given in [41] and [39], respectively.

While we know in practice computing Ramsey numbers is extremely difficult (and considered harder than fighting a war with an alien civilization), it is not clear what is the appropriate computational complexity class to show hardness of computing Ramsey numbers $R(r, s)$. For example, the closely related *arrowing decision problem* asks whether given three graphs F, G, H there is a red-blue edge-coloring of F that contains neither a red G or a blue H ? This decision problem was shown to be in co-NP for fixed choices of G, H [4]. Later Schaefer [40] showed that in general it is in the polynomial hierarchy to answer this queries, but it is not clear what to do with this complexity question when F, G, H are complete graphs K_N, K_r, K_s because there is only one value $R(r, s)$ for each input N, r, s , hence it is not clear how it can be hard for any of the usual classes like NP. See details in [40, 4, 22].

In recent years, Pak and collaborators [36, 25, 35] have proposed another way to measure complexity is by looking at counting sequences. We propose that their point of view could be another way to assert hardness of $R(r, s)$ by counting of Ramsey graphs: *Ramsey (r, s) -graphs* are graphs with no red clique of size r , and no independent set of size s . Clearly, the number of vertices of a Ramsey (r, s) -graph is less than the Ramsey number $R(r, s)$. We are interested in the number of Ramsey (r, s) -graphs on n vertices denoted by $RG(n, r, s)$. What is the complexity of counting the sequence of numbers $\{RG(n, r, s)\}_{n=1}^{\infty}$? From Ramsey's theorem this sequence consists of $R(r, s) - 1$ positive numbers and then an infinite tail of zeroes. We give some examples of $RG(n, r, s)$ in the columns of the table below, which are computed using the #SAT solver RELSAT [2], following the work of successful computations in Ramsey theory using SAT solvers in, for example, [23] and [6] (see also the database at [33]). The hardness of $R(r, s)$ can then be rephrased as the question of whether the counting function $RG(n, r, s)$ is in #P.

n	$RG(n, 3, 3)$	$RG(n, 3, 4)$	$RG(n, 3, 5)$	$RG(n, 3, 6)$	$RG(n, 3, 7)$	$RG(n, 4, 4)$
2	2	2	2	2	2	2
3	6	7	7	7	7	8
4	18	40	41	41	41	62
5	12	322	387	388	388	892
6	0	2812	5617	5788	5789	22484
7	0	13842	113949	133080	133500	923012
8	0	17640	2728617	4569085	4681281	55881692
9	0	0	55650276	220280031	245743539	4319387624

Our Contributions:

In [Section 2](#), we reintroduce the sequence $\{RG(n, r, s)\}_{n=1}^{\infty}$ as the number of solutions of certain zero-dimensional ideals over the polynomial ring $\mathbb{F}_2[x_1, \dots, x_n]$. The solutions are indicator vectors that yield all Ramsey graphs (note, here they are not counted up to symmetry or automorphism classes). Some simple properties of $RG(n, r, s)$, such as the fact that $RG(n, r, s) \leq RG(n, r-1, s)$, follow immediately from [Theorem 1](#).

Theorem 1. *The Ramsey number $R(G_1, \dots, G_k)$ is at most n if and only if there is no solution to the following system over $\mathbb{F}_2$, where $K_n = (V, E)$ is the complete graph on n vertices. Moreover, when the system has solutions, the number of solutions to this system is equal to the number of graphs of order n that avoid copies of G_i in color i . In particular, when $k = 2$, $G_1 = K_r$, and $G_2 = K_s$, this is the number of Ramsey graphs $RG(n, r, s)$.*

$$p_{H,i} : \prod_{e \in E(H)} x_{i,e} = 0 \quad \forall i, 1 \leq i \leq k, \quad \forall H \subseteq K_n, H \sim G_i, \quad (1.1)$$

$$q_e : 1 - \sum_{i=1}^k x_{i,e} = 0 \quad \forall e \in E, \quad (1.2)$$

$$u_{i,j,e} : x_{i,e}x_{j,e} = 0 \quad \forall e \in E, \forall i, j, i \neq j. \quad (1.3)$$

When $k = 2$, $G_1 = K_r$, and $G_2 = K_s$, the Ramsey ideal $RI(n, r, s)$ is ideal of the polynomial ring $\mathbb{F}_2[x_{1,e}, x_{2,e} \mid e \in E(K_n)]$ generated by the polynomials p_H , q_e and $u_{i,j,e}$. Then we have

$$RI(n, r, s) \supseteq RI(n, r-1, s) \supseteq \dots \supseteq RI(n, n, s) \supseteq RI(n, n-1, s) \supseteq \dots$$

and

$$RI(n, r, s) \supseteq RI(n, r, s-1) \supseteq \dots \supseteq RI(n, r, n) \supseteq RI(n, r, n-1) \supseteq \dots$$

A key interesting consequence of [Theorem 1](#) is that the very first value of n for which the system of equations has no solution is equal to the Ramsey number. One important result for our purposes is the famous *Hilbert's Nullstellensatz*, which states that a system of polynomial equations $f_1 = \dots = f_m = 0$ over an algebraically closed field K has no solution if and only if there exist coefficient polynomials $\beta_1, \dots, \beta_m$ such that

$$\sum_{i=1}^m \beta_i f_i = 1. \quad (1.4)$$

We call such an identity (1.4) a *Nullstellensatz certificate*. The *degree* of a certificate is the largest degree of the polynomials β_i . Note that in our case the existence of a Nullstellensatz certificate is equivalent to an upper bound on the Ramsey number. The strong connection between combinatorial problems and the Hilbert Nullstellensatz has

been investigated in, for example, [5, 32, 12, 10, 16, 15, 29, 11]. Here we show a surprising connection of Ramsey numbers to Nullstellensatz certificates.

In [Theorem 2](#) we give a general construction for Nullstellensatz certificates of Ramsey number upper bounds using the polynomial encoding from [Theorem 1](#). In other words, assuming the number of vertices of a Ramsey ideal is n and $n \geq R_{G_1, \dots, G_k}$, we give upper bounds for the degrees of the Nullstellensatz certificates. Surprisingly our construction of Nullstellensatz certificates for the Ramsey ideals yields certificate degrees given by the *restricted online Ramsey numbers* $\tilde{R}_{G_1, \dots, G_k; n}$, first introduced by Conlon et al. in [8]. These numbers are defined in terms of the following *Builder-Painter game*. Each turn Builder selects one edge e of K_n , and Painter selects a color in k and colors e with this color. Builder wins the game once a monochromatic G_i in color i is constructed for some i . The number $\tilde{R}_{G_1, \dots, G_k; n}$ is the minimum number of edges Builder needs to guarantee a victory. We also use the simplified notation $\tilde{R}_{r_1, \dots, r_k; n}$ for $\tilde{R}_{K_{r_1}, \dots, K_{r_k}; n}$. It is trivial that $\tilde{R}_{G_1, \dots, G_k} \leq \binom{R_{G_1, \dots, G_k}}{2}$, but it was shown in [18] that in the case of 2-color classical Ramsey numbers that $\tilde{R}_{r, r; n} \leq \frac{n}{2} - \Omega(n \log n)$ for $n \geq R_{r, r}$. While this degree bound is linear in the number of variables, $k \frac{n}{2}$, it is an improvement over the upper bounds given in, for instance, [26, 28].

Theorem 2. *If $n \geq R_{G_1, \dots, G_k}$, then there is an explicit Nullstellensatz certificate of degree $\tilde{R}_{G_1, \dots, G_k; n} - 1$ that the statement $R_{G_1, \dots, G_k} > n$ is false using the encoding in [Theorem 1](#). In particular, in the case of 2-color classical Ramsey numbers, this implies that if $n \geq R_{r, s}$, then there exists a Nullstellensatz certificate of degree $\tilde{R}_{r, s; n} - 1$ that the statement $R_{r, s} > n$ is false.*

The proof of [Theorem 2](#) does not rely on the graph-theoretic properties of Ramsey numbers, and in fact it applies to the whole of Ramsey theory. In particular, we can modify the encoding in [Theorem 1](#) to suit several well-known problems in Ramsey theory (see, for example, [19, 27]). We express these problems using the general framework below.

Definition 3. *Let k be a positive integer, and let $\{S_n\}$ be a sequence of sets. For c in k , the set of colors, let $\mathcal{P}_n^c$ be a subset of S_n . A triple $A : \{S_n\}, \{\mathcal{P}_n^c\}; k$ is a Ramsey-type problem if the following hold:*

- (i) $S_i \subseteq S_{i+1}$ for $i \geq 1$,
- (ii) $\mathcal{P}_i^c \subseteq \mathcal{P}_{i+1}^c$ for $i \geq 1, 1 \leq c \leq k$,
- (iii) *There exists an integer N such that for all $i \geq N$ and every k -coloring of S_i there is a color c and some element $X \in \mathcal{P}_i^c$ where each element of X is assigned color c .*

The smallest such N is called the Ramsey-type number for A , and is denoted R_A .

We see that in the problem of computing classical Ramsey numbers $R_{r,s}$, we have $S_n = E(K_n) = \{i, j : 1 \leq i < j \leq n\}$. The families $\mathcal{P}_n^1$ and $\mathcal{P}_n^2$ consist of all the sets of edges of induced subgraphs of K_n containing r and s vertices, respectively. As another example, the problem of computing *Schur numbers* asks for the smallest n such that every k -coloring of n contains a monochromatic solution to the equation $x + y = z$. In this case we have $S_n = n$, and for all c we have $\mathcal{P}_n^c = \{\{x, y, z\} : \{x, y, z\} \subseteq n, x + y = z\}$.

As we see in [Section 2](#), the encoding in [Theorem 1](#) can be modified to give bounds for many other Ramsey-type numbers, including Schur, Rado, van der Waerden, and Hales–Jewett numbers [\[19, 27\]](#). We can define numbers analogous to the restricted online Ramsey numbers for Ramsey-type problems in terms of another Builder-Painter game. We define this game for a fixed n as follows.

For each turn, Builder selects one object from S_n and Painter assigns it a color in $[k]$. Builder wins once there is a color c and an element $X \in \mathcal{P}_n^c$ where every element of X is assigned color c . Define the number $\tilde{R}_k(\mathcal{P}_n^1, \dots, \mathcal{P}_n^k; S_n)$ to be the smallest number of turns for which Builder is guaranteed a victory. In this notation, the restricted online Ramsey number $\tilde{R}_{r,s;n}$ is equal to $\tilde{R}_2(\mathcal{P}_n^1, \mathcal{P}_n^2; S_n)$ with $\mathcal{P}_n$ and S_n defined as above for the Ramsey number $R_{r,s}$. [Theorem 4](#) generalizes [Theorems 1 and 2](#).

Theorem 4. *Let $A = \{S_n\}, \{\mathcal{P}_n^c\}; k$ be a Ramsey-type problem. Then for each n , the Ramsey-type number for A is strictly greater than n if and only if the following system of equations has no solution over $\overline{\mathbb{F}_2}$.*

$$\begin{aligned} p_{X,c} : \prod_{s \in X} x_{c,s} &= 0 & \forall X \in \mathcal{P}_n^c, 1 \leq c \leq k \\ q_s : 1 - \sum_{i=1}^k x_{i,s} &= 0 & \forall s \in S_n, \\ u_{i,j,s} : x_{i,s} x_{j,s} &= 0 & \forall s \in S_n, \forall i, j, 1 \leq i < j \leq k. \end{aligned}$$

If $n \geq R_A$, then the minimal degree of a Nullstellensatz certificate for this system is at most $\tilde{R}_k(\mathcal{P}_n^1, \dots, \mathcal{P}_n^k; S_n) - 1$. Moreover, the number of solutions to this system is equal to the number of k -colorings of S_n such that for every color c , each set $X \in \mathcal{P}_n^c$ contains an object that is not assigned color c .

For example, in the case of Schur numbers, the number of solutions to this system is exactly the number of k -colorings of n that do not contain any monochromatic solutions to $x + y = z$. In [Section 2](#) we give some examples of values of $\tilde{R}_{r,s;n}$ and $\tilde{R}(\mathcal{P}_n^1, \dots, \mathcal{P}_n^k; S_n)$ and discuss the Nullstellensatz certificates for the associated polynomial systems.

2 Ramsey and Hilbert's Nullstellensatz

In this section we give several encodings of the problem of finding an upper bound for $R_{r,s}$ in terms of the feasibility of a system of polynomial equations. In the simplest version of the encoding, the variables correspond to edges in the graph K_n , and the solutions of the system correspond to graph colorings that avoid monochromatic copies of K_r and K_s . If the system is infeasible for some n , then no such coloring exists, hence $R_{r,s} \leq n$.

Many combinatorial problems can be encoded as a system of polynomial equations, including colorings, independence sets, partitions, etc. (see, e.g., [5, 3, 14, 12, 16, 24, 30, 32]). A Nullstellensatz certificate for such a combinatorial polynomial system is therefore a proof that a combinatorial theorem is true. We are interested on bounding the Nullstellensatz degree for our Ramsey systems.

There are known general “algebraic geometers” upper bounds for the degree of a Nullstellensatz certificate, so the above procedure terminates, even when these bounds are exponential and sharp in general [26]. However, the exponential bounds should not be bad news for combinatorialists. First, it has been shown [28] that for “combinatorial ideals”, the bounds are much better, linear in the number of variables. Over finite fields there are degree bounds that are independent of the number of variables [20], and a recent paper [34] gives substantial improvements to these bounds. The bounds we give in Theorems 2 and 4 for our systems of equations are better than the above bounds. Moreover, it has been documented that in practice the degrees of Nullstellensatz certificates of NP-hard problems (e.g., non-3-colorability), tend to be small “in practice” (see, for example, [31, 29, 15] and the references therein), especially when the polynomial encodings are over finite fields. Note also that when we know the degree of the Nullstellensatz certificate, one can compute explicit coefficients of the Nullstellensatz certificate using a linear algebra system derived by equating the monomials of the identity. This has been exploited in practical computation with great success, see [10, 15, 29].

We now prove Theorem 1 of our encoding for Ramsey numbers over $\mathbb{F}_2$ below.

Proof of Theorem 1. Suppose there is a solution $\mathbf{x}$ to the system over $\mathbb{F}_2$. For each edge e of K_n and each color i , the system has a variable $x_{i,e}$. The polynomials $u_{i,j,e}$ guarantee that for a given e , at most one variable $x_{i,e}$ is nonzero. From the polynomials q_e , we then see that exactly one index i such that $x_{i,e} = 1$, and let $\phi(\mathbf{x})$ be the coloring χ where $\chi(e)$ is this index. Color each edge e of K_n with the color $\chi(e)$. In the equations involving the polynomials p_H , for each subgraph H of K_n with $H \sim G_i$, there is at least one edge e in H with $x_{i,e} = 0$. Therefore $\chi(e) \neq i$, so there is no monochromatic copy of G_i in color i .

Conversely, if we have a coloring χ of the edges of K_n with no monochromatic G_i in color i , then let $\psi(\chi)$ be the solution $\mathbf{x}$ where $x_{i,e}$ is 1 if $\chi(e) = i$ and 0 otherwise. One can check easily that $\mathbf{x}$ satisfies the system of equations. The maps ϕ and ψ are inverses

of each other, and so the number of solutions to the system is equal to the number of colorings of K_n with no monochromatic G_i in color i .

For the first chain of ideals, observe that for a fixed i , the polynomial $\prod_{e \in E(H)} x_{i,e}$ divides $\prod_{e \in E(H')} x_{i,e}$ if and only if H is a subgraph of H' . Since every copy of K_{r-1} in K_n contains a copy of K_r as a subgraph, in the ideal $RI(n, r-1, s)$, every polynomial of the form $\prod_{e \in E(H')} x_{i,e}$ with $H' \sim K_{r-1}$ is divisible by a generator $\prod_{e \in E(H)} x_{i,e}$ of $RI(n, r, s)$ with $H \sim K_r$. The ideals in the chain are equal for $r > n$ since in this case K_r is not a subgraph of K_n . The proof for the second chain of ideals is similar. $\square$

Before we prove [Theorem 2](#), we show a special case as a warm-up example. There is a simple certificate of the fact that $R(r, 2) \leq r$.

Example 5. For all r , there exists a Nullstellensatz certificate of degree $\frac{r}{2} - 1$ of the statement $R(r, 2) \leq r$.

Proof. Label the edges of K_r from 1 to $n = \frac{r}{2}$. The following identity is a certificate that $R(r, 2) \leq r$. Polynomials in parentheses are part of the system of equations in [Theorem 1](#).

$$\begin{aligned} & 1 - \prod_{i=1}^n (1 - x_{1,i} - x_{2,i} - \dots - x_{r,i}) \\ & \quad - \prod_{i=1}^n (1 - x_{1,i} - x_{2,i} - \dots - x_{r,i}) \\ & \quad - \prod_{i=1}^n (1 - x_{1,i} - x_{2,i} - \dots - x_{r,i}) \\ & \quad - \prod_{i=1}^n (1 - x_{1,i} - x_{2,i} - \dots - x_{r,i}) \end{aligned}$$

$\square$

In the proof of [Theorem 2](#), we show how to translate a strategy for Builder into a Nullstellensatz certificate. This method can be used to construct a certificate for all (known) upper bounds for $R(G_1, \dots, G_s)$. Notably, better strategies for Builder yield lower degree certificates. In [Example 5](#), this is not a concern since the order in which Builder selects edges does not matter, and in fact $\tilde{R}(r, 2; r) = \frac{r}{2}$. Painter simply colors every edge Builder selects the first color, and Builder wins when all $\frac{r}{2}$ edges are selected.

The proofs of [Theorems 2](#) and [4](#) are similar, and in fact [Theorem 2](#) follows from [Theorem 4](#), but for the sake of concreteness we begin with [Theorem 2](#).

Proof of Theorem 2. Number the edges of K_n from 1 to $\frac{n}{2}$. A t -turn game state g is a set $\{e_{i_1}, c_1, e_{i_2}, c_2, \dots, e_{i_t}, c_t\}$ of pairs of edges $e_{i_j} \in E$ chosen by Builder and colors $c_j \in k$ chosen by Painter. A game is complete if there is some color $c \in k$ where Painter has colored a monochromatic G_c in color c . Let $d : \tilde{R}(G_1, \dots, G_k; n)$. If Builder follows an optimal strategy for choosing edges, then the game lasts at most d turns, that is $t \leq d$.

For a t -turn game state g , define the monomial $\pi(g)$ to be $\pi(g) : \prod_{j=1}^t x_{c_j, e_{i_j}}$. Similarly, for any monomial $f = \prod_{j=1}^t x_{c_j, e_{i_j}}$ with distinct e_{i_j} , let $\sigma(f)$ denote the game state

$\{ e_{i_1}, c_1, e_{i_2}, c_2, \dots, e_{i_t}, c_t \}$. We will describe an algorithm to construct a Nullstellensatz certificate of the form

$$\sum_{i=1}^k \sum_{H \sim G_i} \beta_{H,i} p_{H,i} - \sum_{e \in E} \gamma_e q_e = 1. \quad (2.1)$$

Denote the left-hand side of Equation (2.1) by L . For each $i \in [k]$, initialize $\beta_{H,i}$ to 0 for all $H \sim G_i$. Initialize γ_e to 0 for all edges e except e_1 , and set $\gamma_{e_1} = 1$. Then repeat the following:

1. Expand and simplify L so that L is a sum of monomials. If $L = 1$, then we are done. Otherwise, at least one term in L is a nonconstant monomial f .
2. If σf is a completed game state, then $p_{H,i}$ divides f for some color i and $H \sim G_i$. Then set $\beta_{H,i} \leftarrow \beta_{H,i} - \frac{f}{p_{H,i}}$. This results in $L \leftarrow L - f$, which cancels the original f in the certificate since it is an expression over $\overline{\mathbb{F}_2}$, which has characteristic 2.
3. If σf is not a completed game state, then let e be an edge that Builder should choose in an optimal strategy from the game state σf . Set $\gamma_e \leftarrow \gamma_e + f$. Since $f q_e = f \sum_{i=1}^k f x_{i,e}$, we obtain $L \leftarrow L - f \sum_{i=1}^k f x_{i,e}$. This results in the cancellation of f in L , but adds k additional terms (one for each of Painter's k choices for coloring e) to L . Note that if σf is a t -turn game state, then $\sigma f x_{i,e}$ is a $t+1$ -turn game state for all i .

By the symmetry of K_n , it is arbitrary which edge Builder selects first. Therefore each nonconstant term that appears in L corresponds to a game state where Builder (but not necessarily Painter) has followed an optimal strategy. Since terms that correspond to completed games are cancelled out in step 3, this procedure terminates, resulting in a Nullstellensatz certificate. Because Builder follows an optimal strategy, the maximal degree of any term in any γ_e is $d-1$, so the degree of the certificate is $d-1$. $\square$

To illustrate the importance of Builder's strategy in this method, observe that one can construct a degree 7 certificate for the statement $R_{3,3} \leq 6$ using the following strategy: For the first five turns, Builder selects each edge incident to some vertex v . No matter how Painter colors these edges, three must be colored the same color. Call these edges vw_1, vw_2, vw_3 . For the next three turns, Builder selects the edges w_1w_2, w_1w_3 , and w_2w_3 , and Painter must construct a monochromatic triangle. However, if Builder plays poorly and selects, for example, the edges $1,2, 2,3, 3,4, 4,5, 5,6, 1,6, 1,4, 2,5$, and $3,6$, then no matter what Painter does there are *no* monochromatic triangles, and this leads to a higher degree certificate.

The proof of Theorem 2 shows that the polynomials can "simulate" a tree of Builder-Painter games. However, in general the degrees of certificates can be strictly smaller

than the bounds given in [Theorem 2](#). For example, by a computational search, it can be shown that $\tilde{R}(3, 3; 6) = 8$. However, there exists a Nullstellensatz certificate of degree 5 using the encoding in [Theorem 1](#), which is better than the bound given in [Theorem 2](#). Due to space constraints we skip the proof of [Theorem 4](#) but it can be found in [\[13\]](#).

As an application of [Theorem 4](#), let $\mathcal{E}$ be a linear equation, and let $R_k \mathcal{E}$ denote the k -color Rado number for $\mathcal{E}$, the smallest n such that every k -coloring of $[n]$ contains a monochromatic solution to $\mathcal{E}$. Let $X_{n, \mathcal{E}}$ be the set of all solutions over $[n]$ to $\mathcal{E}$. Let $\mathcal{P}_n^c : X_{n, \mathcal{E}}$ for all c . If $R_k \mathcal{E}$ exists, then $\{[n]\}, \{\mathcal{P}_n^c\}; k$ is a Ramsey-type problem, and we have the following corollary.

Corollary 6. *Let $\mathcal{E}$ be the linear equation $\sum_{j=1}^t a_j y_j = a_0$ with a finite Rado number $R_k \mathcal{E}$. Let $X_{n, \mathcal{E}} = \{m_1, \dots, m_t : \sum_{j=1}^t a_j m_j = a_0, 1 \leq m_j \leq n\}$ be the set of solutions over $[n]$ to $\mathcal{E}$. Then for every n , the following system has no solution over $\mathbb{F}_2$ if and only if $n \geq R_k \mathcal{E}$.*

$$\begin{aligned} \prod_{j=1}^t x_{i, m_j} &= 0 & \forall m_1, \dots, m_t \in X_{n, \mathcal{E}}, 1 \leq i \leq k, \\ 1 - \sum_{i=1}^k x_{i, m} &= 0 & 1 \leq m \leq n, \\ x_{i, m} x_{j, m} &= 0 & 1 \leq m \leq n, 1 \leq i < j \leq k. \end{aligned}$$

The degree of a minimal Nullstellensatz certificate for this system has degree at most $\tilde{R}_k(X_{n, \mathcal{E}}, \dots, X_{n, \mathcal{E}}; n) - 1$.

As an example, let $\mathcal{E}$ denote the equation $x + 3y + 3z$, and let $X_{9, \mathcal{E}}$ be the solutions to $\mathcal{E}$ over $[9]$ as above. It is known that $R_2 \mathcal{E} = 9$ [\[27\]](#). However, Builder can select, in order, the integers 4, 6, 9, 3, and 7 to win the Builder-Painter game in at most 5 turns: since $(6, 4, 6)$ is a solution, 4 and 6 must be different colors, and then since $(9, 6, 9)$ and $(3, 3, 4)$ are solutions, 4 and 9 must be one color and 3 and 6 are the other color. But then $(3, 6, 7)$ and $(9, 4, 7)$ are solutions, so there is a monochromatic solution no matter which color Painter selects for 7. Therefore $\tilde{R}_2(X_{9, \mathcal{E}}, X_{9, \mathcal{E}}; 9) \leq 5$, and the minimal degree of a Nullstellensatz certificate for the system of equations in [Corollary 6](#) is at most 4. In fact, some computations show the minimal degree is 2.

Similarly, the encoding in [Theorem 1](#) for the Schur number $S(2) = R_2(x + y = z)$ also gives an example of Nullstellensatz certificates that are smaller than the ones given by games. It is well-known that $S(2) = 5$, and from the encoding in [Theorem 4](#), we have $S(2) \leq 5$ if and only if the following system of equations has no solutions over $\mathbb{F}_2$.

$$\begin{aligned} 1 - x_{1, i} - x_{2, i} &= 0, & 1 \leq i \leq 5, \\ x_{i, 1} x_{i, 2} &= 0, & x_{i, 2} x_{i, 4} &= 0, \\ x_{i, 1} x_{i, 3} x_{i, 4} &= 0, & x_{i, 1} x_{i, 4} x_{i, 5} &= 0, & x_{i, 2} x_{i, 3} x_{i, 5} &= 0. \end{aligned} \quad \left. \vphantom{\begin{aligned} 1 - x_{1, i} - x_{2, i} &= 0, \\ x_{i, 1} x_{i, 2} &= 0, \\ x_{i, 1} x_{i, 3} x_{i, 4} &= 0, \end{aligned}} \right\} i = 1, 2$$

A computer search shows that the number $\tilde{R}_2(X_{5,x,y,z}, X_{5,x,y,z}; 5) = 5$, where $X_{5,x,y,z}$ is the set of positive integer solutions to $x + y + z \leq 5$. We computed a degree 3 Nullstellensatz certificate for the above system of equations, which is an improvement on the bound in Theorem 4. We omit the certificate here for space considerations, but it can be found in [13]. Moreover, by Theorem 4 there are similar consequences for van der Waerden numbers, Hales–Jewett numbers, and essentially any other Ramsey-type quantity. We are unable to include them here due to space, but we invite readers to see more in [13].

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Faces of Directed Edge Polytopes

Yasuhide Numata^{*1}, Yusuke Takahashi², and Dai Tamaki⁺³

¹Department of Mathematics, Hokkaido University, Sapporo, Japan

²Graduate School of Science and Technology, Shinshu University, Matsumoto, Japan

³Department of Mathematics, Shinshu University, Matsumoto, Japan

Abstract. Given a finite quiver (directed graph) without loops and multiedges, the convex hull of the column vector of the incidence matrix is called the directed edge polytope and is an interesting example of lattice polytopes. In this article, we give a complete characterization of facets of the directed edge polytope of an arbitrary finite quiver without loops and multiedges in terms of the connectivity and the existence of a rank function. Our result can be regarded as an extension of the result of Higashitani et al. [6] on facets of symmetric edge polytopes to directed edge polytopes. When the quiver in question has a rank function, we obtain a characterization of faces of arbitrary dimensions. This article is an extended abstract of [10].

Résumé. Pour un graphe orienté fini simple, l'enveloppe convexe du vecteur colonne de la matrice d'incidence est appelée le polytope d'arête orienté. C'est un intéressant exemple de polytopes intégraux. Dans cet article, nous donnons une caractérisation complète des facettes de polytope d'arête orienté en termes de connectivité et d'existence d'une fonction de rang. Notre résultat peut être considéré comme une extension du résultat de Higashitani et al. [6] sur les facettes de polytope d'arête symétrique. Pour un graphe orienté avec une fonction de rang, on obtient une caractérisation de faces de dimensions arbitraires. Cet article est un résumé étendu de [10].

Keywords: Kantorovich–Rubinstein polytopes; fundamental polytopes; symmetric edge polytopes; directed edge polytopes; characterization of faces; f -vectors

1 Introduction

For a motivation of optimal transportation problems, Vershik proposed to study the fundamental polytope constructed from a finite metric space in [11]. For a metric space (n, d) , the *fundamental polytope* $\text{KR}(n, d)$ is defined to be the convex hull of

$$\left\{ \epsilon_{i,j}^d - \frac{e_i - e_j}{d(i,j)} \mid i, j \in n, i \neq j \right\},$$

^{*}nu@math.sci.hokudai.ac.jp. The first author was partially supported by JSPS KAKENHI Grant Number JP18K03206.

⁺rivulus@shinshu-u.ac.jp. The third author was partially supported by JSPS KAKENHI Grant Number JP20K03579.

where $\{e_1, \dots, e_n\}$ is the standard orthonormal basis of $\mathbb{R}^n$. The polytope $\text{KR}(n, d)$ is also called the *Kantorovich–Rubinstein polytope* [7, 8]. In the case of a tree-like metric space, Delucchi and Hoessley proved a nice formula of the f -vector by using the relation between tree-like metric spaces and hyperplane arrangements in [3]. The starting point of this work was the second author's attempt to extend their work to graphs with cycles.

For a finite simple graph G whose vertex set is $[n]$, we can define a metric d_G on $[n]$ by the minimum length of paths. The Kantorovich–Rubinstein polytope $\text{KR}(n, d_G)$ has already been studied as the *symmetric edge polytope* $\text{SE}(G)$, introduced by Matsui et al. [9]. If G is the complete graph K_n , then it is also called the *root polytope* of the root system A_{n-1} and Cho completely determined its faces in [1].

We can also generalize the definition of symmetric edge polytope to a finite quiver (directed graph). In this article, a quiver $Q = (Q_0, Q_1)$ means a quiver without loops and multiedges. Since Q has no loop and no multiedges, Q_1 is realized as a subset of $Q_0 \times Q_0 \setminus \{(i, i) \mid i \in Q_0\}$. Let Q be a quiver with $Q_0 = [n]$. We define $\varepsilon_{i,j} = e_i - e_j$ for $(i, j) \in Q_1$. Let $\varepsilon(Q)$ be the set of $\varepsilon_{i,j}$ for all $(i, j) \in Q_1$. We define the *directed edge polytope* $\text{DE}(Q)$ to be the convex hull of $\varepsilon(Q)$. In other words, $\text{DE}(Q)$ is the convex hull of column vectors of the incidence matrix of Q . The directed edge polytope is the convex polytope whose set of vertices is $\varepsilon(Q)$. See also [5]. The symmetric edge polytope $\text{SE}(G)$ is nothing but $\text{DE}(D(G))$, where $D(G)$ is the double of G , i.e. the quiver whose edge set is the collection of direct edges v, w and w, v for adjacent vertices v, w in G .

The aim of this article is to give an explicit combinatorial description of all facets of $\text{DE}(Q)$ for a quiver Q and combinatorial descriptions of facets of $\text{SE}(G) = \text{KR}(G_0, d_{\text{graph}})$ for a finite simple graph.

In general, the directed edge polytope $\text{DE}(Q)$ with $Q_0 = [n]$ is a convex polytope in the vector space $\mathbb{R}^n$. Since the set of vertices of $\text{DE}(Q)$ is given by $\varepsilon(Q)$, we can describe any face of $\text{DE}(Q)$ in the form $\text{DE}(R)$ for a subquiver R with $R_0 \subseteq Q_0$. Let us call such a subquiver a *luf subquiver*.

Our problem is to determine the luf subquiver R of Q such that $\text{DE}(R)$ is a facet of $\text{DE}(Q)$, i.e. $\text{DE}(R)$ is a face of $\text{DE}(Q)$ and $\dim \text{DE}(R) = \dim \text{DE}(Q) - 1$. The existence of a rank function plays a key role in both problems. We call $\rho: Q_0 \rightarrow \mathbb{R}$ a *rank function* of Q if $\rho(v) - 1 \leq \rho(w)$ for each edge $(v, w) \in Q_1$. As in Remark 3.3, such a function makes the vertex set Q_0 into a graded poset. To consider the dimension, we also need to consider a connected component C of a quiver Q , i.e. a maximal subquiver such that any two distinct vertices in C are connected by an undirected walk in C , where an undirected walk is a sequence $v_0, v_1, \dots, v_l$ of vertices in Q such that $(v_t, v_{t+1}) \in Q_1$ or $(v_{t+1}, v_t) \in Q_1$ for all t . We define the *coconnectivity* $c(Q)$ of Q by $c(Q) = |Q_0| - |\pi_0(Q)|$, where $\pi_0(Q)$ stands for the set of connected components of Q . We can describe the dimension of $\text{DE}(Q)$ by the coconnectivity as follows:

Theorem 1.1. *For a finite quiver Q without loops and multiedges, we have*

$$\dim \text{DE } Q = \begin{cases} c(Q) - 1 & \text{if } Q \text{ has a rank function} \\ c(Q) & \text{otherwise} \end{cases}.$$

To determine faces, the contraction is also a key notion. We say that a luff subquiver R of Q is *component-wise full* if $C_1 = \{v, w \in Q_1 \mid v, w \in C_0\}$ for each connected component C of R . For a component-wise full subquiver R of a quiver Q , we define an equivalence relation $\sim$ on Q_0 by

$$v \sim w \iff v \text{ and } w \text{ are in the same connected component of } R,$$

and the equivalence class of $v \in Q_0$ is denoted by $\bar{v}$. We define the contraction Q/R to be the quiver such that $Q/R_0 = Q_0/\sim$ and $Q/R_1 = \{v, w \mid v, w \in Q_1 \setminus R_1\}$. Roughly speaking, Q/R is the quiver obtained from Q by collapsing each connected component of R to a point. We can characterize facets as follows:

Theorem 1.2. *Let Q be a finite quiver without loops and multiedges. For a lluf subquiver R of Q with $\dim \text{DE } R = \dim \text{DE } Q - 1$, $\text{DE } R$ is a facet of $\text{DE } Q$ if and only if one of the following conditions holds:*

1. $c(R) = c(Q) - 1$, R is a component-wise full subquiver of Q , and the contraction Q/R is acyclic.
2. $c(R) = c(Q)$ and there exists a rank function ρ of R such that

$$\rho(v) - \rho(w) = 1 \quad \rho(v') - \rho(w') = 1 > 0$$

for any $v, w, v', w' \in Q_1 \setminus R_1$.

By iterating the process of taking facets, we can obtain lower dimensional faces. Thus we obtain a characterization of faces of $\text{DE } Q$ for a quiver Q with a rank function.

Theorem 1.3. *Suppose Q has a rank function. For a proper subquiver R of Q , the polytope $\text{DE } R$ is a face of $\text{DE } Q$, if and only if R is a component-wise full subquiver of Q and Q/R is acyclic.*

In the case of $D(G)$ for a simple graph G , it does not have a rank function and the condition (2) in Theorem 1.2 applies. Hence we have the following:

Corollary 1.4. *Let G be a finite simple graph. For a lluf subquiver R of $D(G)$ with $\dim \text{DE } R = \dim \text{SE } G - 1$, the following are equivalent:*

1. $\text{DE } R$ is a facet of $\text{SE } G$.

2. $c \in R \iff c \in D \setminus G$ and there exists a rank function ρ of R such that

$$\rho(v) - \rho(w) = 1 \iff \rho(v') - \rho(w') = 1 > 0$$

for any $v, w, v', w' \in D \setminus G \setminus R_1$.

3. $c \in R \iff c \in D \setminus G$ and there exists a map ρ from the set of vertices of G to $\mathbb{R}$ such that

$$\rho(v) - \rho(w) = \begin{cases} 1 & v, w \in R_1 \\ -1 & w, v \in R_1 \\ 0 & \text{otherwise} \end{cases}$$

for $v, w \in D \setminus G \setminus R_1$.

Remark 1.5. This is essentially equivalent to a characterization of facets of symmetric edge polytopes in Higashitani–Jochemko–Michalek [6] for a connected graph G . See [10, Remark 5.5] for the details.

Remark 1.6. In the case of the Kantorovich–Rubinstein polytope $\text{KR}(n, d)$ for a general metric d , Gordon and Petrov [4] defined a quiver $Q(F)$ with vertex set n from a face F of $\text{KR}(n, d)$ by

$$Q(F) = \left\{ v, w \mid \epsilon_{v,w}^d \in F \right\}.$$

They proved as Theorem 3 that E is a subset of $Q(F)$ for some facet F of $\text{KR}(n, d)$ if and only if there exists a function $\rho: E \rightarrow \mathbb{R}$ such that $\rho(v) - \rho(w) = d(v, w)$ for $v, w \in E$.

When $d = d_G$ for a graph G with the vertex set n , $\text{KR}(n, d_G) = \text{SE}(G)$, the third condition in Corollary 1.4 is closely related to their condition. In fact, if F is a facet of $\text{SE}(G)$ represented by a luff subquiver R of $D \setminus G$ by $F = \text{DE}(R)$, then $R_1 \subset Q(F)$ and Theorem 3 of Gordon-Petrov says that there exists a function $\rho: n \rightarrow \mathbb{R}$ such that $\rho(v) - \rho(w) = 1$ for $v, w \in R_1$.

On the other hand, Corollary 1.4 gives a complete characterization of lluf subquivers R of $D \setminus G$ such that $\text{DE}(R)$ is a facet of $\text{KR}(n, d_G)$, while Gordon-Petrov's Theorem 3 does not.

This article is organized as follows: After fixing notation and terminology in Section 2, we see sketches of proofs for Theorem 1.1 in Section 3.1, and for Theorem 1.2 in Section 3.2. We end this paper with sample computations in Section 4. This article is an extended abstract of the preprint [10]. We will omit details. See [10] for the details.

2 Notation and terminology

Here we fix notation and terminology for quivers. A quiver Q is said to be

1. *acyclic* if there do not exist $v_0, \dots, v_l \in Q_0$ such that $l > 1$, $v_l = v_0$, and $v_t, v_{t-1} \in Q_1$ for $t = 0, \dots, l-1$,
2. *asymmetric* if $w, v \notin Q_1$ for any $v, w \in Q_1$, and
3. *symmetric* if $v, w \in Q_1$ for all $w, v \in Q_1$.

Note that a quiver may not be symmetric nor asymmetric.

The sequence $v_0, v_1, \dots, v_l$ is called a *directed cycle* if $l > 1$, $v_l = v_0$, $v_t, v_{t-1} \in Q_1$ for $t = 0, \dots, l-1$, and $v_i \neq v_j$ for any pair i, j with $0 \leq i < j < l$. By definition, a quiver is acyclic if and only if it does not contain a directed cycle.

We define the *underlying graph* of a quiver Q to be the (undirected) graph obtained from Q by using all vertices of Q and by replacing all directed edges of Q with undirected edges. Underlying graphs may have multiple edges. The underlying graph of Q is simple if and only if Q is asymmetric.

In order to describe faces of directed edge polytopes, we need subquivers. A quiver R is called a *subquiver* of Q if $R_0 \subset Q_0$ and $R_1 \subset Q_1$. We say that a subquiver R of Q is

1. *proper* if R_1 is a proper subset of Q_1 ,
2. *full* if $R_1 = \{ v, w \in Q_1 \mid v, w \in R_0 \}$, and
3. *luff* if $R_0 = Q_0$.

We make use of (undirected) walks to define connectivity of quivers. Let Q be a quiver. An *undirected walk* from v_0 to v_l in Q is a sequence $v_0, v_1, \dots, v_l$ of vertices in Q such that $v_t, v_{t-1} \in Q_1$ or $v_{t-1}, v_t \in Q_1$ for all t . An undirected walk $v_0, v_1, \dots, v_l$ is called

1. *closed* if $v_0 = v_l$, and
2. an *undirected cycle* if it is closed and $v_i \neq v_j$ for any pair i, j with $0 \leq i < j < l$.

We say a quiver Q is *connected* if, for any pair v, w of vertices of Q , there exists an undirected walk from v to w . A connected maximal subquiver of Q is called a *connected component* of Q . The set of all connected components of Q is denoted by $\pi_0 Q$. The number $|Q_0| - |\pi_0 Q|$ is denoted by $c Q$ and is called the *coconnectivity* of Q . Note that a quiver is connected if and only if the underlying graph is connected. We say that a luff subquiver R of Q is *component-wise full* if each connected component C of R is a full subquiver of Q .

3 Sketch of proof

Here we give a sketch of proofs of [Theorems 1.1](#) and [1.2](#). We omit the details. See [\[10\]](#) for the details.

3.1 Dimension

Here we give a sketch of proofs of [Theorem 1.1](#). Note that even if Q is connected, a subquiver R representing a face of DE Q may not be connected. It turns out that the number of connected components is closely related to the dimension of DE R . In fact, an upper bound is given by the coconnectivity c_R .

Lemma 3.1. *Define a vector subspace V_Q of $\mathbb{R}^{Q_0}$ by*

$$V_Q = \bigcap_{R \in \pi_0 Q} \kappa_{R_0}^\perp,$$

where $\kappa_{R_0}^\perp$ is the hyperplane orthogonal to $\kappa_{R_0} = \sum_{v \in R_0} e_v$ in $\mathbb{R}^n$. Then $\text{DE } Q \subset V_Q$ and we have $\dim \text{DE } Q \leq |Q_0| - |\pi_0 Q| - c_Q$.

It turns out that $\dim \text{DE } Q$ varies depending on the existence of a rank function, since such a function defines another hyperplane that contains $\text{DE } Q$.

Definition 3.2. For a quiver Q , a map $\rho: Q_0 \rightarrow \mathbb{R}$ is called a *rank function* of Q if it satisfies $\rho v - 1 \leq \rho w$ for each edge $v, w \in Q_1$.

Remark 3.3. The following are equivalent for a quiver Q :

1. Q has a rank function ρ .
2. Q is asymmetric and satisfies

$$|\{t \mid v_t, v_{t-1} \in Q_1\}| = |\{t \mid v_{t-1}, v_t \in Q_1\}|$$

for each undirected closed walk $v_0, v_1, \dots, v_n$ in Q .

3. Q is asymmetric and satisfies

$$|\{t \mid v_t, v_{t-1} \in Q_1\}| = |\{t \mid v_{t-1}, v_t \in Q_1\}|$$

for each undirected cycle $v_0, v_1, \dots, v_n$ in Q .

4. Q is the Hasse diagram of a graded poset $(Q_0, \leq)$ with the rank function ρ .

Let Q have a rank function ρ . Since $Q_0 = n$, we regard a rank function as a vector in $\mathbb{R}^n$. The affine hyperplane $\rho^\perp = \{ \delta \in \mathbb{R}^n \mid \delta \in \rho^\perp \}$ is independent of choice of an edge $v, w \in Q_1$. We define H_ρ to be $\rho^\perp$ for some edge $v, w \in Q_1$. The hyperplane H_ρ contains $\text{DE } Q$, and is transversal to the hyperplanes defined by connected components of Q . We have the following upper bound of $\dim \text{DE } Q$.

Lemma 3.4. *If Q has a rank function, then $\dim \text{DE } Q \leq c(Q) - 1$.*

In order to obtain lower bounds of $\dim \text{DE } Q$, we consider the case of a quiver whose underlying graph is acyclic. Then the dimension of the vector space spanned by $\{ \varepsilon_{v,w} \mid v, w \in Q_1 \}$ is given by $c(Q)$. Thus we obtain

$$\dim \text{DE } Q = \dim \text{aff} \left(\varepsilon_{v,w} \mid v, w \in Q_1 \right) = c(Q) - 1,$$

where aff denotes the affine hull. Hence we have the following.

Lemma 3.5. *For any quiver Q , we have $\dim \text{DE } Q \geq c(Q) - 1$.*

For those quivers that do not have rank functions, we have the following lower bound.

Lemma 3.6. *If Q does not have a rank function, then $\dim \text{DE } Q \geq c(Q)$.*

Now [Theorem 1.1](#) follows from [Lemmas 3.1 and 3.4 to 3.6](#).

3.2 Facets

Let R be a lluf subquiver of Q so that both $\text{DE } R$ and $\text{DE } Q$ are contained in $\mathbb{R}^{Q_0}$. In order to prove [Theorem 1.2](#), we would like to know when $\text{DE } R$ is a face of $\text{DE } Q$ and $\dim \text{DE } R = \dim \text{DE } Q - 1$.

We first obtain the following relation between the coconnectivities of Q and R by the dimension condition.

By [Theorem 1.1](#), we have the following:

Lemma 3.7. *Let R be a lluf subquiver of Q . If $\text{DE } R$ is a facet of $\text{DE } Q$, then $c(R) = c(Q)$ or $c(R) = c(Q) - 1$.*

If a facet $\text{DE } R$ of $\text{DE } Q$ satisfies $c(R) = c(Q)$, then Q has no rank function but R has a rank function. In this case, we obtain [Lemma 3.8](#), which implies necessary condition of [Theorem 1.2](#) for the case where $c(R) = c(Q)$:

Lemma 3.8. *Let R be a lluf subquiver of Q with $c(R) = c(Q)$. If $\text{DE } R$ is a facet of $\text{DE } Q$, then the subquiver R has a rank function $\rho \in \mathbb{R}^{Q_0}$ such that*

$$\rho(v) - \rho(w) = 1, \quad \rho(v') - \rho(w') = 1 > 0.$$

for $v, w, v', w' \in Q_1 \setminus R_1$ and Q does not have a rank function.

If a facet $\text{DE } R$ of $\text{DE } Q$ satisfies $c_R > c_Q$, then both Q and R have a rank function or both has no rank function. In this case, we have the following:

Lemma 3.9. *Let R be a subquiver of Q with $c_R = c_Q - 1$. If $\text{DE } R$ is a facet of $\text{DE } Q$, then R is a component-wise full subquiver of Q .*

If Q/R has a directed cycle, then we have a pair R^1, R^2 of connented components of R with an edge of Q connecting R^1 to R^2 and an edge of Q connecting R^2 to R^1 . The hyperplanes corrsponding to those edges separates the connected components R^1, R^2 . Hence we have the following:

Lemma 3.10. *Let R be a subquiver of Q with $c_R = c_Q - 1$. If $\text{DE } R$ is a facet of $\text{DE } Q$, then Q/R is acyclic.*

Lemmas 3.9 and 3.10 imply nessesary condition of Theorem 1.2 for the case where $c_R = 1 = c_Q$

By the calculation of the supporting hyperplanes, we have Lemmas 3.11 and 3.12.

Lemma 3.11. *Let R be a component-wise full subquiver of Q . If Q/R is acyclic, then $\text{DE } R$ is a face of $\text{DE } Q$.*

Lemma 3.12. *Let R be a lluf proper subquiver of a quiver Q . If R has a rank function $\rho \in \mathbb{R}^{Q_0}$ such that*

$$\rho(v) - \rho(w) = 1, \quad \rho(v') - \rho(w') = 1 > 0$$

for any $v, w, v', w' \in Q_1 \setminus R_1$, then $\text{DE } R$ is a face of $\text{DE } Q$.

4 Examples

Here we apply our main theorem to some special quivers, and consider the f -vector, i.e., the sequence of the number f_d of the faces of dimension d .

4.1 Quivers related to forests

First we consider the case of an asymmetric quiver Q with no undirected closed walk, i.e., the case where the underlying graph of Q is a forest. In this case, we have

$$|\text{vert DE } Q| = |Q_1| = c_Q.$$

On the other hand, Q has a rank function and we have $\dim \text{DE } Q = c_Q - 1$ by Theorem 1.1. Hence $\text{DE } Q$ is a simplex of dimension $|Q_1| - 1 = c_Q - 1$. We also have $f_d = \binom{|Q_1|}{d-1}$, and $\text{DE } R$ is a face of $\text{DE } Q$ for any lluf subquiver R of Q .

Next we consider a forest $G = G_0, G_1$ and $\text{SE } G$. Since a quiver whose underlying graph is G has a rank function, it is a facet of $\text{SE } G$. Hence $f_d = \binom{|G_1|}{d-1} 2^{d-1}$.

4.2 Quivers related to cycles

Here we consider a quiver related to the cycle with m edges and m vertices. In this case, we regard the vertex set Q_0 as $\mathbb{Z}/m\mathbb{Z} = \{\bar{1}, \dots, \bar{m} = \bar{0}\}$. Moreover, we define O^+ to be the set $\{\bar{i} - \bar{1}, \bar{i} \mid \bar{i} \in Q_0\}$ of ‘clock-wise’ edges, and O^- to be the set $\{\bar{i} - \bar{1}, \bar{i} \mid \bar{i} \in Q_0\}$ of ‘anticlock-wise’ edges.

First we consider the quiver whose underlying graph is a cycle.

Example 4.1. Let Q be an asymmetric quiver whose underlying graph is the cycle with $2n - 1$ edges. In this case, Q has no rank function. Hence $\text{DE } Q$ is a $2n$ dimensional polytope with $2n - 1$ vertices. Thus $\text{DE } Q$ is a simplex of dimension $2n$.

Next consider the case of even cycle.

Example 4.2. We consider an asymmetric quiver Q whose underlying graph is an even cycle. Let $m = 2n$. We define $Q_1 = Q_1 \cap O^+$, $Q_1^- = Q_1 \cap O^-$. If $|Q_1| = |Q_1^-| = n$, then Q has a rank function. It follows from [Theorem 1.1](#) that $\dim \text{DE } Q = 2n - 2$. Since $|\text{vert DE } Q| = |Q_1| = 2n$, $\text{DE } Q$ is not a simplex.

Let R be a lluf subquiver of Q whose directed edge polytope $\text{DE } R$ is a facet of $\text{DE } Q$. Since R also has a rank function, we have

$$2n - 3 = \dim \text{DE } R = |Q_0| - |\pi_0(R)| - 1 = 2n - 1 - |\pi_0(R)|,$$

which implies that $|Q_1 \setminus R_1| = 2$. Let $Q_1 \setminus R_1 = \{e', e''\}$. The acyclicity of Q/R following from [Theorem 1.3](#) allows us to assume that $e' \in Q_1$ and $e'' \in Q_1^-$. This is a characterization of facets of $\text{DE } Q$.

For such a subquiver R , $\text{DE } R$ is a simplex of dimension $2n - 3$. Since faces of a simplex are in one-to-one correspondence to subsets of the vertex set, for a lluf subquiver S of Q , $\text{DE } S$ is a proper face of $\text{DE } Q$ of dimension d if and only if $|S_1 \cap Q_1| < n$, $|S_1 \cap Q_1^-| < n$, and $|S_1| = d + 1$. Hence

$$f_d = \binom{2n}{d+1} - 2 \binom{n}{d+1-n},$$

where the binomial coefficient $\binom{m}{k}$ equals 0 if $m < k$ or $k < 0$.

Next we consider a cycle G and $\text{SE } G$.

Example 4.3. Let C_{2n} be the cycle with $2n$ edges and $Q = D C_{2n}$. In this case, we have $Q_1 = O^+ \cup O^-$. The symmetric edge polytope $\text{SE } C_{2n} = \text{DE } Q$ is a $2n - 1$ dimensional polytope by [Theorem 1.1](#). The faces of $\text{SE } C_{2n}$ can be determined as follows.

By (3) of [Corollary 1.4](#), for a lluf subquiver R of Q with

$$\dim \text{DE } R = \dim \text{SE } C_{2n} - 1 = 2n - 2,$$

DE R is a facet of SE C_{2n} if and only if $c \in R \iff c \in Q$ and $2n - 1$ and there exists a function $\rho : \mathbb{Z}/2n\mathbb{Z} \rightarrow \mathbb{R}$ such that

$$\rho(\overline{i-1}) - \rho(\bar{i}) = \begin{cases} 1 & \overline{i-1}, \bar{i} \in R_1 \\ -1 & \bar{i}, \overline{i-1} \in R_1 \\ 0 & \text{otherwise} \end{cases},$$

which implies that only one of $\overline{i-1}, \bar{i}$ or $\bar{i}, \overline{i-1}$ belongs to R_1 for each i . Let $R_1 \cap O = R_1^+$ and $R_1^- = R_1 \cap O^-$. Then we have $|R_1^+| = |R_1^-|$.

Since DE R is of dimension $2n - 2$,

$$|R_1| + |\text{vert DE } R| \geq 2n - 1.$$

By the condition on ρ , we see that the underlying graph of R must be the whole C_{2n} . Hence facets of SE C_{2n} are in bijective correspondence to subsets of cardinality n in O . Hence we have

$$f_{2n-2} = \binom{2n}{n}.$$

Note that facets of SE C_{2n} are polytopes in [Example 4.2](#). In particular, faces of codimension 2 in SE C_{2n} are simplices of dimension $2n - 3$, which means that all faces of SE C_{2n} except for facets are simplices. In other words, for $d < 2n - 2$ and a lluf subquiver R of Q , DE R is a face of dimension d in SE C_{2n} if and only if $|R_1| = d + 1$, $|R_1 \cap O| < n$, $|R_1 \cap O^-| < n$, and $R_1 \cap -R_1^- = \emptyset$. Hence it follows from direct calculation that

$$f_d = \binom{2n}{d+1} 2^{d-1}.$$

We remark that D'Ali, Delucchi, and Michałek [2] also performed the same computation based on the characterization of facets by Higashitani et al. [6].

Example 4.4. Consider the case of an odd cycle C_{2n-1} . Let $Q = D C_{2n-1}$. The symmetric edge polytope SE $C_{2n-1} = \text{DE } Q$ is a $2n$ -dimensional polytope by [Theorem 1.1](#).

For a lluf subquiver R of Q , suppose that $\dim \text{DE } R = 2n - 1$. By the same argument as in [Example 4.3](#), DE R is a facet of SE C_{2n-1} if and only if $c \in R \iff c \in Q$, $|R_1| = |R_1^-| = n$, and $R_1 \cap -R_1^- = \emptyset$. Hence we have

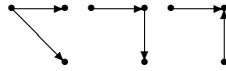
$$f_{2n-1} = n - 1 \binom{2n-1}{n} = \frac{(2n-1)!}{n!n!} = 2n - 1 \binom{2n}{n}.$$

Thus, for $d < 2n - 2$ and a lluf subquiver R of Q , DE R is a face of SE C_{2n-1} of dimension d if and only if $|R_1| = d + 1$, $|R_1^+| < n$, $|R_1^-| < n$, and $R_1 \cap -R_1^- = \emptyset$. Hence it follows from direct calculation that

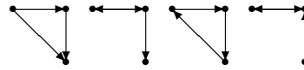
$$f_d = \binom{2n-1}{d+1} 2^{d-1}.$$

4.3 Quivers of small size

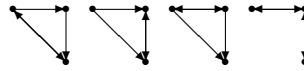
As a final remark, we give examples of f -vectors of directed edge polytopes for connected quivers with three vertices. The directed edge polytopes for the following quivers are 1-dimensional simplices:



The directed edge polytopes for the following quivers are 2-dimensional simplices:



Next we give examples of the case where directed edge polytopes are not simplices. The f -vectors for the directed edge polytopes for the following quivers are $1, 4, 4, 1$:



The f -vector for the directed edge polytope for the following quiver is $1, 5, 5, 1$:



The f -vector for the directed edge polytope for the following quiver is $1, 6, 6, 1$:



In the case of the quivers with three vertices, we have only three kinds of f -vectors for directed edge polytopes which are not simplices. In the case of the quivers with four vertices, we, however, have more kinds of f -vectors as follows:

$1, 5, 8, 5, 1$,	$1, 6, 10, 6, 1$,	$1, 7, 13, 8, 1$,	$1, 8, 16, 10, 1$,	$1, 9, 19, 12, 1$,
$1, 9, 17, 10, 1$,	$1, 10, 20, 12, 1$,	$1, 6, 11, 7, 1$,	$1, 7, 14, 9, 1$,	$1, 8, 17, 11, 1$,
$1, 8, 14, 8, 1$,	$1, 9, 15, 8, 1$,	$1, 7, 12, 7, 1$,	$1, 5, 9, 6, 1$,	$1, 7, 11, 6, 1$,
$1, 9, 18, 11, 1$,	$1, 10, 21, 13, 1$,	$1, 6, 12, 8, 1$,	$1, 7, 15, 10, 1$,	$1, 6, 9, 5, 1$,
$1, 10, 19, 11, 1$,	$1, 11, 22, 13, 1$,	$1, 12, 24, 14, 1$,	$1, 8, 13, 7, 1$,	$1, 4, 4, 1$,
$1, 8, 18, 12, 1$,	$1, 8, 15, 9, 1$,	$1, 8, 12, 6, 1$,	$1, 9, 16, 9, 1$.	

Acknowledgements

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On the Topology of Cut Complexes of Graphs

Margaret Bayer¹, Mark Denker², Marija Jelić Milutinović³,
Rowan Rowlands⁴, Sheila Sundaram^{*5}, and Lei Xue⁶

^{1,2}Department of Mathematics, University of Kansas, Lawrence, KS, USA

³Department of Mathematics, University of Belgrade, Belgrade, Serbia

⁴Department of Mathematics, University of Washington, Seattle, WA, USA

⁵Pierrepont School, Westport, CT, USA

⁶Department of Mathematics, University of Michigan, Ann Arbor, MI, USA

Abstract. For a positive integer k and a finite graph G , we define the k -cut complex $\Delta_k G$ to be the pure simplicial complex in which the complement of each face contains a set of k vertices inducing a disconnected subgraph of G . This generalises a simplicial complex considered by John Eagon and Victor Reiner (1998), who use $\Delta_2 G$ to reformulate and extend a famous theorem of Ralf Fröberg (1990) relating certain Stanley-Reisner ideals to chordal graphs. In particular their combined results imply that $\Delta_2 G$ is shellable if and only if G is a chordal graph.

We investigate $\Delta_k G$ with this inspiration, using techniques from algebraic and combinatorial topology. We describe the effect of various graph operations on the cut complex, consider its shellability, and determine the homotopy type and Betti numbers of $\Delta_k G$ for various families of graphs. When the homotopy type is a wedge of spheres, we also determine the group representation on the rational homology, notably in the case of complete multipartite graphs.

Keywords: Graph complex, chordal graph, shellability, homotopy, homology representation

1 Introduction

In recent years, there has been much interest in the topology of simplicial complexes associated with graphs. A comprehensive reference is Jonsson's book [9].

A *graph complex* is a simplicial complex associated to a finite graph G . In this paper we introduce a new family of graph complexes which we call *cut complexes*. We consider only simple graphs. Our work is motivated by a famous theorem of Ralf Fröberg (1990) connecting commutative algebra and graph theory through topology. We investigate our new complexes in the spirit of Fröberg's theorem, relating topological properties of the cut complex to structural properties of the graph.

^{*}shsund@comcast.net

For a field $\mathbb{K}$ and a finite simplicial complex Δ with vertex set $n = \{1, 2, \dots, n\}$, the Stanley-Reisner ideal of Δ is the ideal I_Δ of the polynomial ring $\mathbb{K}[x_1, \dots, x_n]$ generated by monomials $x_{i_1} \cdots x_{i_k}$, where $\{i_1, \dots, i_k\}$ runs over the inclusion-minimal subsets of n which are NOT faces of Δ . The Stanley-Reisner ring $\mathbb{K}[\Delta]$ is the quotient of the polynomial ring $\mathbb{K}[x_1, \dots, x_n]$ by the ideal I_Δ .

For a graph G , the *clique complex* $\Delta(G)$ is the simplicial complex whose simplices are subsets of vertices of G , in which every pair of vertices is connected by an edge of G . Fröberg [7] characterised ideals generated by monomials which have a *linear resolution*, by first reducing to the case of square-free monomial ideals. The ideal I_Δ is generated by quadratic square-free monomials precisely when the simplicial complex Δ is $\Delta(G)$ for some graph G (see [6, Proposition 8]). Hence Fröberg's theorem can be stated as follows:

Theorem 1. (Fröberg) [[7], [6, p. 274]] *A Stanley-Reisner ideal I_Δ generated by quadratic square-free monomials has a linear resolution if and only if Δ is the clique complex $\Delta(G)$ of a chordal graph G .*

Define the combinatorial *Alexander dual* of a simplicial complex Δ [5, p.188] on n vertices to be

$$\Delta^\vee : \{F \subset n : n \setminus F \notin \Delta\}.$$

The i th homology of Δ and the $n - i - 3$ th cohomology of $\Delta^\vee$ are isomorphic by Alexander duality in the sphere S^{n-2} .

For a graph G , write $\Delta_2(G)$ for the Alexander dual $\Delta(G)^\vee$ of the clique complex $\Delta(G)$. The facets of $\Delta_2(G)$ are the complements of independent sets of size 2 in G . Eagon and Reiner's [6, Proposition 8] reformulation of Fröberg's theorem includes the following equivalences.

Theorem 2. *The graph G is chordal $\iff \Delta_2(G)$ is shellable $\iff \Delta_2(G)$ is vertex decomposable.*

Inspired by this theorem, we introduce the following generalisation of the simplicial complex $\Delta_2(G)$. Let $k \geq 1$. Define a complex whose facets are complements of sets F of size k in G such that the induced subgraph of G on the vertex set F is disconnected; we call this the k -cut complex of G , and denote it by $\Delta_k(G)$. A different generalisation, the total k -cut complex $\Delta_k^t(G)$, is treated in [1]. The two notions coincide for $k = 2$.

We examine the topology of the cut complex $\Delta_k(G)$ and consider how it is affected by properties of the graph G (Section 3). For example, in analogy with Fröberg's theorem, Proposition 21 asserts that graph chordality implies shellability of the 3-cut complex Δ_3 . We show that for many common families of graphs (Section 5), the homotopy type of $\Delta_k(G)$ is a wedge of spheres in a single dimension. Often there is a simplicial group action on the cut complex $\Delta_k(G)$, which in turn acts on the rational homology. We determine this homology representation, notably in the case of complete multipartite graphs [2].

2 Definitions

General references for simplicial complexes, shellability and topology are [3], [8] and [15], and [17] for graph theory. All graphs in this paper are simple (no loops and no multiple edges).

Definition 3. A simplicial complex Δ on a set A is a collection of subsets of A

$$\sigma \in \Delta \text{ and } \tau \subseteq \sigma \Rightarrow \tau \in \Delta.$$

The elements of Δ are called its *faces* or *simplices*. If the collection of subsets is empty, i.e. Δ has no faces, we call Δ the void complex. Otherwise Δ always contains the empty set as a face.

The *dimension* of a face σ , $\dim \sigma$, is one less than its cardinality; thus the dimension of the empty face is -1 , and the 0-dimensional faces are the *vertices* of Δ . A *d-face* or *d-simplex* is a face of dimension d . The maximal faces of Δ are called its *facets*, and the maximum dimension of a facet is the *dimension* $\dim \Delta$ of the nonvoid simplicial complex Δ . We write $\Delta \langle \mathcal{F} \rangle$ for the simplicial complex Δ whose set of facets is $\mathcal{F}$.

In this paper all simplicial complexes will be *finite*, that is, the vertex set is finite.

A (nonvoid) simplicial complex is *pure* if all its facets have the same dimension, which is then the dimension of the complex.

The *join* of two simplicial complexes Δ_1 and Δ_2 with disjoint vertex sets is the complex

$$\Delta_1 * \Delta_2 = \{ \sigma \cup \tau : \sigma \in \Delta_1, \tau \in \Delta_2 \}.$$

Thus the join $\Delta_1 * \Delta_2$ contains Δ_1 and Δ_2 as subcomplexes.

The *cone* over Δ and the *suspension* of Δ are the complexes

$$\text{cone } \Delta = \Delta * \Gamma_1, \text{ susp } \Delta = \Delta * \Gamma_2 = \Delta * \{u\} \cup \Delta * \{v\},$$

where Γ_1 is the 0-dimensional simplicial complex with one vertex, and Γ_2 is the 0-dimensional complex with two vertices u, v .

Definition 4 ([3, Sec. 11.2]). An ordering $F_1, F_2, \dots, F_t$ of the facets of a simplicial complex Δ is a *shelling* if, for every j with $1 < j \leq t$, $\left(\bigcup_{i=1}^{j-1} \langle F_i \rangle \right) \cap \langle F_j \rangle$ is a simplicial complex whose facets all have cardinality $|F_j| - 1$, where $\langle F_i \rangle$ is the simplex generated by the face F_i . The simplicial complex Δ is *shellable* if it admits a shelling order of its facets.

Remark 5. By convention, the void complex is shellable. The complex whose only face is the empty set is vacuously shellable. The complex with a unique nonempty facet is (also vacuously) shellable, and contractible.

See Figures 1(b) and 2(b) for examples of shellable and nonshellable complexes.

In combinatorial topology, shellability is an important tool for determining the homotopy type of simplicial complexes, thanks to the following theorem of Björner.

Theorem 6 ([3, (9.19) and Sec. 11]). *A pure shellable simplicial complex of dimension d has the homotopy type of a wedge of spheres, all of dimension d (it may be contractible).*

Definition 7. Let $G = (V, E)$ be a graph on $|V| = n$ vertices. If S is a subset of the vertex set V , write $G[S]$ to denote the *induced subgraph* of G whose vertex set is S .

Definition 8. Let $G = (V, E)$ be a graph on $|V| = n$ vertices, and let $k \geq 2$. Define the k -cut complex of the graph G to be the $n - k - 1$ -dimensional simplicial complex

$$\Delta_k G := \langle F \subseteq V, |F| = n - k \mid G[V \setminus F] \text{ is disconnected} \rangle.$$

The facets of the cut complex $\Delta_k G$ are the vertex subsets of G of size $n - k$ whose removal disconnects the graph G . Thus σ is a face of the cut complex $\Delta_k G$ if and only if its complement $V \setminus \sigma$ contains a subset S of size k such that the induced subgraph $G[S]$ is disconnected. Note the inclusion $\Delta_{k-1} G \subseteq \Delta_k G$ for $k \geq 2$, and the fact that the vertices of $\Delta_k G$ are a subset of the vertices of the graph G . See Figures 1 and 2.

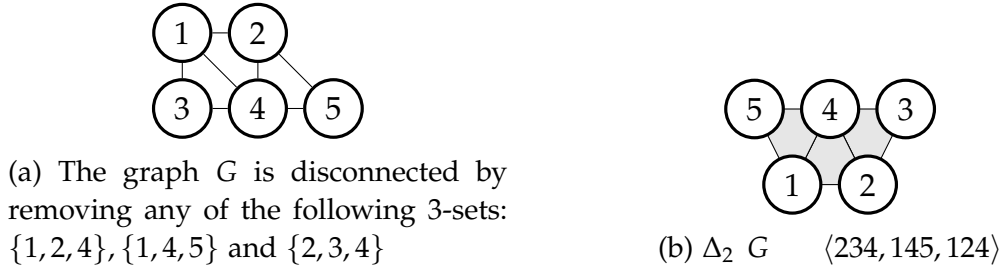


Figure 1: (Shellable) 2-cut complex of graph G

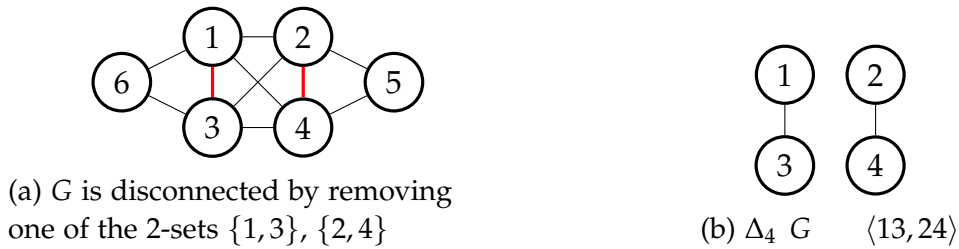


Figure 2: (Nonshellable) 4-cut complex of graph G

Example 9. Let G be a graph on n vertices. We record some easy facts about cut complexes.

1. $\Delta_k G$ is void if $k = 1$ or $k > n$.

2. $\Delta_n G$ is $\begin{cases} \text{the void complex,} & \text{if } G \text{ is connected,} \\ \text{the } -1 \text{-dimensional complex } \{\emptyset\}, & \text{otherwise.} \end{cases}$
3. $\Delta_k G$ is void for $n - k \leq r - 1$ if G is r -connected, since at least r vertices must be removed to disconnect the graph.
4. If G is the complete graph K_n , then $\Delta_k G$ is void for all $k \geq 1$.
5. If $G = E_n$ is the edgeless graph on n vertices, then for $2 \leq k \leq n - 1$, $\Delta_k G$ is the $n - k - 1$ -skeleton of an $n - 1$ -dimensional simplex, hence shellable [4].

In view of Item (1.) above, we will assume $n \geq k \geq 2$ for the cut complex $\Delta_k G$.

In order to determine the homotopy type of a simplicial complex Δ , and especially a group representation on the rational homology, it is often useful to work with the face lattice $\mathcal{L} \Delta$ of Δ . In many cases the automorphism group of the graph G acts simplicially on the cut complex $\Delta_k G$.

Clearly the symmetric group $\mathfrak{S}_n$ acts on the cut complex $\Delta_k E_n$ of the edgeless graph E_n . The face lattice $\mathcal{L} \Delta_k E_n$ is a rank-selected subposet of the Boolean lattice B_n of subsets of an n -element set, consisting of subsets of size at most $n - k$. Using the fact that this is shellable (see e.g. [15]) in conjunction with a well-known result of Solomon [12], Observation (5.) above gives us our first nontrivial result for the homotopy type and equivariant homology of a cut complex:

Proposition 10. *Let E_n be the edgeless graph on n vertices. If $k \geq n$, the cut complex $\Delta_k E_n$ is void. If $2 \leq k \leq n - 1$, the cut complex is shellable and homotopy equivalent to a wedge of $\binom{n-1}{k-1}$ spheres in dimension $n - k - 1$:*

$$\Delta_k E_n \simeq \bigvee_{\substack{n-1 \\ k-1}} \mathbb{S}^{n-k-1}.$$

The $\mathfrak{S}_n$ -representation on the unique nonvanishing homology of $\Delta_k E_n$ is the irreducible module indexed by the partition $(k, 1^{n-k})$ of n .

Definition 11 ([17]). A graph is *chordal* if it has no induced cycle of size greater than 3.

Figure 1 shows a chordal graph and its shellable 2-cut complex, whereas Figure 3 is an example of a nonchordal graph, with nonshellable 2-cut complex. These examples also illustrate Fröberg's Theorem 2.

3 Constructive Theorems

In this section we consider the effect of some common graph operations on the cut complex. We begin with what may be viewed as a universal property of cut complexes.

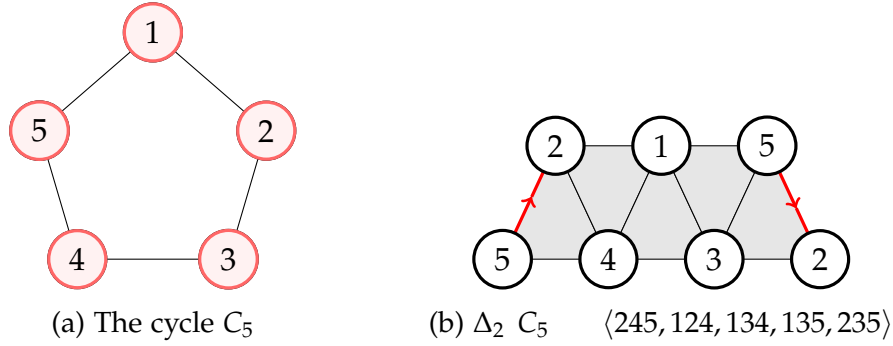


Figure 3: The 2-cut complex for C_5 is a Möbius strip

Theorem 12. *Let Δ be any pure simplicial complex. There exists some k and some chordal graph G such that Δ is equal to the cut complex $\Delta_k G$.*

The construction starts with a clique whose vertices correspond to the vertex set of the complex Δ . For each facet of Δ , add a vertex which is connected to every vertex of that facet. If Δ has n vertices, t facets and dimension d , the resulting graph G has $n + t$ vertices, and $\Delta_{n+t-d-1} G = \Delta$. Figure 4 illustrates the procedure.

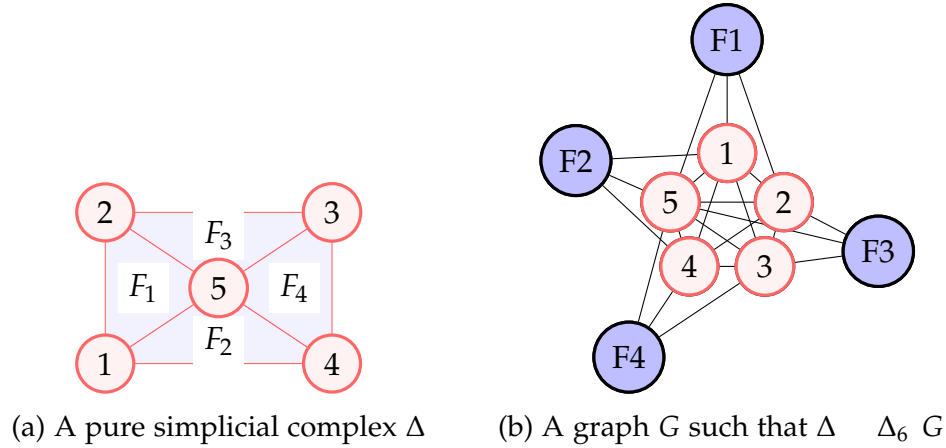


Figure 4: The construction of Theorem 12

Let $G = (V, E)$ be a graph. Recall that we assume $k \geq 2$ for the cut complex $\Delta_k G$. The link [3, Sec. 9.9] of a face σ of a simplicial complex Δ is defined to be the simplicial complex

$$\text{lk}_\Delta \sigma = \{ \tau \in \Delta \mid \sigma \cap \tau = \emptyset, \text{ and } \sigma \cup \tau \in \Delta \}.$$

Lemma 1. *Let $v \in V$. Then $\Delta_k G \setminus v = \text{lk}_{\Delta_k G} \{v\}$, if v is a vertex of $\Delta_k G$. Otherwise $\Delta_k G \setminus v$ is the void complex.*

By applying this lemma repeatedly and using the fact that links preserve shellability [3, Sec. 11.2], we obtain the following important corollary.

Corollary 13. *Let $W \subseteq V$. If the k -cut complex $\Delta_k G$ is shellable, so is $\Delta_k G \setminus W$. Equivalently, if $\Delta_k G$ is shellable for a graph G , then $\Delta_k H$ is shellable for every induced subgraph H of G .*

Definition 14. If G_1, G_2 are graphs, their disjoint union is the graph $G_1 \sqcup G_2$ having vertex set equal to the union of the vertex sets of G_1 and G_2 , and edge set equal to the union of the edge sets of G_1 and G_2 .

Theorem 15. $\Delta_k G_1 \sqcup G_2$ is shellable if and only if $\Delta_k G_1$ and $\Delta_k G_2$ are shellable.

For instance, from [Example 9](#), this theorem tells us that the k -cut complex of the disjoint union of two complete graphs is always shellable.

Definition 16. Given graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ on disjoint vertex sets, their join $G_1 * G_2$ is their disjoint union with the set of all edges between V_1 and V_2 included.

Theorem 17. $\Delta_k G_1 * G_2$ is shellable if and only if between $\Delta_k G_1$ and $\Delta_k G_2$, one is shellable and the other is the void complex.

The homotopy type of the cut complex of the join of two graphs bears a precise relationship to the homotopy types of the cut complexes of the individual graphs. Passing to the face lattices gives an alternative proof of the homotopy equivalence, and also shows that it is group-equivariant, a fact we will need in [Section 5](#), where we will apply the full force of this theorem to complete bipartite graphs, which are joins of edgeless graphs.

Theorem 18. Let $k \geq 1$, let G_i be a graph with vertex set $V_i, i = 1, 2$, where $V_1 \cap V_2 = \emptyset$. Assume $\Delta_k G_i$ is not void for at least one $i = 1, 2$.

If only one cut complex is void, then $\Delta_k G_1 * G_2$ is contractible.

Now assume both cut complexes $\Delta_k G_i, i = 1, 2$, are nonvoid. Then there is a homotopy equivalence

$$\text{susp } \Delta_k G_1 * \Delta_k G_2 \simeq \Delta_k G_1 * G_2. \quad (3.1)$$

Moreover, there is a group-equivariant poset map from the product $\mathcal{L} \Delta_k G_1 \times \mathcal{L} \Delta_k G_2$ of face lattices, to the face lattice of the simplicial complex $\Delta_k G_1 * G_2$ which induces a group-equivariant homotopy equivalence of the respective order complexes. This in turn gives a group-equivariant $H_1 \times H_2$ -homotopy equivalence $\text{susp } \Delta_k G_1 * \Delta_k G_2 \simeq_{H_1 \times H_2} \Delta_k G_1 * G_2$, where H_i is a group acting simplicially on the cut complex $\Delta_k G_i$.

Hence we have the following isomorphism of $H_1 \times H_2$ -modules in rational homology:

$$\tilde{H}_d \Delta_k G_1 * G_2 \sim \bigoplus_{p+q=d-2} \tilde{H}_p \Delta_k G_1 \otimes \tilde{H}_q \Delta_k G_2.$$

The proof of Equation (3.1) uses gluing homeomorphisms and homotopy equivalences from algebraic topology [8, Ex. 0.14], [1, Prop. 2.7]. To establish the group-equivariant homotopy equivalence of face lattices, we use an equivariant Quillen fibre lemma [11], [16]. The Künneth theorem then gives the isomorphism in homology.

Definition 19. Given graphs G_1 and G_2 on disjoint vertex sets, a *wedge product* of G_1 and G_2 , $G_1 \vee G_2$, is formed by taking one vertex from G_1 and one vertex from G_2 , and identifying them. Note that a wedge product of two graphs is not unique in general.

For example, any tree on n vertices can be obtained as a wedge product of a smaller tree on $n - 1$ vertices with the complete graph K_2 .

Theorem 20. Let $k \geq 2$, and let G_1, G_2 be graphs. Then $\Delta_k G_1 \vee G_2$ is shellable if and only if $\Delta_k G_1$ and $\Delta_k G_2$ are shellable.

The cycle graph gives a counterexample (see Theorem 27) to the converse of the next result, offered in the spirit of Fröberg's Theorem 2.

Proposition 21. If G is chordal, then $\Delta_3 G$ is shellable.

An example of a chordal graph whose 4-cut complex is not shellable appears in Figure 2.

Corollary 13 says that if $\Delta_k H$ is not shellable for some induced subgraph H of G , then $\Delta_k G$ is not shellable. It is thus natural to seek a description of the *minimal* graphs whose k -cut complex fails to be shellable, for fixed k . We will call such a graph a *minimal forbidden subgraph* for k -cut complex shellability.

Proposition 22. Every minimal forbidden subgraph for 3-cut complex shellability is at least 3-connected.

4 Reduced Euler characteristics and Betti numbers

Our proofs of the preceding results determine the homotopy type, but do not give Betti numbers. In this section we present a method for determining the Betti numbers of a cut complex, under certain favourable conditions. Recall [13] that the reduced Euler characteristic of a simplicial complex Δ is the Möbius number $\mu \mathcal{L} \Delta$ of its face lattice $\mathcal{L} \Delta$. The idea is to describe the face poset of the cut complex as a subposet of the truncated Boolean lattice, and then use poset topology techniques as in [13, Lemma 3.16.4], [14, Proposition 1.3] to determine the Möbius number.

We say a subset A of the vertex set $V G$ of a graph G is *connected* (resp. *disconnected*) if the induced subgraph $G A$ is connected (resp. disconnected).

Let G be a graph with vertex set $V G$ of size n . Let $P_{n,k}$ be the truncated Boolean lattice $B_n^{\leq n-k}$ of subsets of n of size at most $n - k$. Clearly the face poset of $\Delta_k G$

is a subposet of $P_{n,k}$. Let A^c denote the complement of a subset A of n , and define $\mathcal{Z}_k G$ to be the subset $\{A^c \in P_{n,k} : |A| = k, A \text{ induces a connected subgraph of } G\}$. Thus $\mathcal{Z}_k G$ is the set of maximal elements of $P_{n,k}$ which are not facets of the cut complex $\Delta_k G$. Also the number of facets of $\Delta_k G$ is $\binom{n}{k} - |\mathcal{Z}_k G|$.

Theorem 23. *We have the following.*

1. *The face poset of $\Delta_k G$ is a subposet of $P_{n,k} \setminus \mathcal{Z}_k G$.*
2. *The following are equivalent:*
 - (i) *The face poset of $\Delta_k G$ coincides with $P_{n,k} \setminus \mathcal{Z}_k G$.*
 - (ii) *Let $A^c \in \mathcal{Z}_k G$, $X \subset A^c$, $|X| = n - k - 1$. Then X^c contains a disconnected set of size k .*
 - (iii) *Let $A \subset V G$ be connected of size k . For every $x \notin A$ there is a $y \in A$ such that $A \setminus \{y\} \cup \{x\}$ is disconnected.*
3. *If any condition in Part (2.) holds, the reduced Euler characteristic $\mu_{\Delta_k G}$ of $\Delta_k G$ is given by*

$$-1^{n-k-1} \mu_{\Delta_k G} = \binom{n-1}{k-1} - |\mathcal{Z}_k G| = |\{F : F \text{ is a facet of } \Delta_k G\}| - \binom{n-1}{k}.$$

Furthermore, in this case the nonzero homology of $\Delta_k G$ is torsion-free and occurs in at most two dimensions, $n - k - 1$ and $n - k - 2$.

4. *If any condition in Part (2.) holds and $\Delta_k G$ is shellable, then it is homotopy equivalent to a wedge of $|\mu_{\Delta_k G}|$ spheres in dimension $n - k - 1$. It is contractible if $\mu_{\Delta_k G} = 0$.*

The technical conditions in Part (2.) above, as well as variations, hold for several common families of graphs, including trees, cycles and grid graphs.

5 Families of Graphs

We now examine specific classes of graphs. [Theorem 23](#) can be applied to compute Betti numbers for cut complexes of forests and cycles, which satisfy the conditions in Part (2.) of [Theorem 23](#).

Trees

Recall that for every k , the k -cut complex of the complete graph K_2 is the void complex ([Example 9](#)). Since trees may be recursively constructed by taking a wedge with K_2 , and forests are disjoint unions of trees, [Theorems 15](#) and [17](#) give:

Proposition 24. *If G is a forest, $\Delta_k G$ is shellable for all $k \geq 2$.*

For any (finite) graph G , let $\tau_k G$ denote the number of subtrees of G with k vertices. We have $\tau_2 \mathcal{F}_n = n - c$, where c is the number of connected components of the forest $\mathcal{F}_n$. If the tree is a path P_n , then $\tau_k P_n = n - k + 1$. From [Theorem 23](#), we obtain:

Theorem 25. *Let $\mathcal{F}_n$ be a forest on n vertices. Then the shellable cut complex $\Delta_k \mathcal{F}_n$ is homotopy equivalent to a wedge of $\binom{n-1}{k-1} - \tau_k \mathcal{F}_n$ spheres in dimension $n - k - 1$ if this number is nonzero, and contractible otherwise.*

In particular, for a tree T_n on n vertices, $\Delta_k T_n$ is contractible if $k = 2$, and has the homotopy type of a wedge of $\binom{n-1}{k-1} - \tau_k T_n$ spheres of dimension $n - k - 1$ if $k \geq 3$.

Cycles

For the cycle graph C_n , we may assume $n \geq 4$ and $2 \leq k \leq n - 2$. If $n = 4$, the one-dimensional cut complex $\Delta_2 C_4$ has two facets $\{1, 3\}$ and $\{2, 4\}$, and is homotopy equivalent to the 0-sphere S^0 ; it is thus not shellable.

Recall from [Figure 3](#) that the 2-dimensional cut complex $\Delta_2 C_5$ is in fact a Möbius strip, and hence it is homotopy equivalent to the one-sphere S^1 . Using a discrete Morse matching [\[9\]](#), it was shown more generally in [\[1\]](#) that:

Proposition 26 ([\[1, Theorem 3.9\]](#)). *Let C_n be a cycle graph, $n \in \mathbb{N}$, and $n \geq 5$. Then the $n - 3$ -dimensional cut complex $\Delta_2 C_n$ is homotopy equivalent to the sphere S^{n-4} .*

We show that for $k \geq 3$, the cut complex of the cycle graph is shellable. The Betti number follows by verifying the conditions in [Theorem 23](#).

Theorem 27. *For all $n > k > 2$, the $n - k - 1$ -dimensional cut complex $\Delta_k C_n$ is shellable. It has the homotopy type of a wedge of $\binom{n-1}{k-1} - n$ spheres of dimension $n - k - 1$.*

Bipartite graphs

Let E_n denote the edgeless graph on n vertices. The complete bipartite graph $K_{m,n}$ is the join of two edgeless graphs E_m and E_n . We use the structural result of [Theorem 18](#), completely determining both the homotopy type and homology representation for the cut complexes of all complete multipartite graphs. For lack of space we state our results only for the complete bipartite case.

Let V_λ denote the irreducible module of the symmetric group $\mathfrak{S}_n$ indexed by the integer partition λ of n . The group $\mathfrak{S}_m \times \mathfrak{S}_n$ (resp. the wreath product $\mathfrak{S}_2 \wr \mathfrak{S}_n$) induces a representation on the rational homology of $\Delta_k K_{m,n}$, $m \leq n$, (resp. $m > n$).

From [Theorem 17](#), [Theorem 18](#) and [Proposition 10](#), we have, since $K_{m,n} = E_m * E_n$:

Theorem 28. *Let $1 \leq m \leq n$ and $2 \leq k$. Then $\Delta_k K_{m,n}$ is shellable if and only if $m < k$. Furthermore, if $m < k \leq n$, then $\Delta_k K_{m,n}$ is contractible, and if $k > n$, the cut complex is void and hence shellable.*

If $k \leq m \leq n$, the $m - n - k - 1$ -dimensional complex $\Delta_k K_{m,n}$ is homotopy equivalent to a wedge of $\binom{m-1}{k-1} - \binom{n-1}{k-1}$ spheres of dimension $m - n - 2k$.

The representation of $\mathfrak{S}_m \times \mathfrak{S}_n$ on the rational homology is the irreducible $V_{k,1^{m-k}} \otimes V_{k,1^{n-k}}$ indexed by the pair of integer partitions $(k, 1^{m-k}), (k, 1^{n-k})$.

If $m = n$, the representation of $\mathfrak{S}_2 \times \mathfrak{S}_n$ is $\Phi_{\mathfrak{S}_2} V_{k,1^{n-k}}$, where $\Phi_{\mathfrak{S}_2}$ equals sgn^{n-k-1} , and sgn is the sign representation of $\mathfrak{S}_2$.

The last statement uses [14, Proposition 2.3] to compute the homology representation for a product of posets.

The homotopy type and homology representation determined in Theorem 28 bear a striking resemblance to that of a complex studied by Linusson, Shareshian and Welker [10, Theorem 1.4, Conjecture 1.15], a connection we propose to explore further.

Grid Graphs

The m by n grid graph $G_{m,n}$ is the Cartesian product of two paths, $P_m \times P_n$. Here we describe our most significant results on grid graphs. The first result is proved by a discrete Morse matching [9].

Theorem 29. [1, Theorem 3.10] For $n, m \geq 2$, the $mn - 3$ -dimensional cut complex $\Delta_2 G_{m,n}$ has the homotopy type of a wedge of $(m-1)(n-1)$ spheres of dimension $mn - 4$.

Complexes associated to grid graphs appear to be difficult to analyse. We obtain shellability of $\Delta_3 G_{m,n}$ below as an unexpected corollary of Proposition 22. The Betti number is the result of a generalisation of Theorem 23, exploiting the fact that grid graphs are triangle-free.

Theorem 30. Let $m, n \geq 2$. The 3-cut complex $\Delta_3 G_{m,n}$ is shellable and has the homotopy type of a wedge of $\binom{mn-1}{2} - 5mn + 5m + n - 3$ spheres in dimension $mn - 4$.

For $k = 4, 6$, we use poset topology tools to show that the homology of $\Delta_k G_{m,n}$ is torsion-free and concentrated in the top two dimensions, and give formulas for the reduced Euler characteristic.

We conjecture that $\Delta_k G_{m,n}$ is shellable for $mn - 3 \geq k \geq 4$.

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An SL_4 -web basis from hourglass plabic graphs

Christian Gaetz^{*1}, Oliver Pechenik^{†2}, Stephan Pfannerer^{‡3},
Jessica Striker^{§4}, and Joshua P. Swanson^{¶5}

¹*Dept. of Mathematics, Cornell University, Ithaca, NY, USA*

²*Dept. of Combinatorics & Optimization, University of Waterloo, Waterloo, ON, Canada*

³*Institute of Discrete Mathematics and Geometry, Technische Universität Wien, Austria*

⁴*Dept. of Mathematics, North Dakota State University, Fargo, ND, USA*

⁵*Dept. of Mathematics, University of Southern California, Los Angeles, CA, USA*

Abstract. We introduce *hourglass plabic graphs* and prove that certain of these graphs index a rotation-invariant SL_4 -web basis, a structure that has been sought since Kuperberg’s introduction of the SL_3 -web basis in 1996. These graphs exhibit connections to the combinatorics of standard Young tableaux, crystals, alternating sign matrices, six-vertex configurations, and plane partitions.

Keywords: web, promotion, plabic graph, alternating sign matrix, six-vertex model

1 Introduction

The irreducible representations of the symmetric group S_n are the *Specht modules* S^λ indexed by integer partitions λ . For the case of 3-row rectangles, Kuperberg [17] famously introduced a diagrammatic “web” basis of the Specht module $S^{3 \times b}$ (and more generally for other spaces of invariant tensors). Kuperberg’s web basis has many important applications to quantum link invariants, cluster algebras, and algebraic geometry. From a combinatorial perspective, key properties of the web basis are that the long cycle $c = 12 \dots n$ and the long element $w_0 = n \ n - 1 \ \dots 1$ both act in this basis as scalar multiples of permutation matrices. Indeed, (up to scalars) c acts diagrammatically as a rotation [21] and w_0 as a reflection [19], yielding an easily accessible pictorial understanding of these actions. These facts are particularly powerful in dynamical algebraic

^{*}crgaetz@gmail.com. Christian Gaetz was partially supported by a Klarman Postdoctoral Fellowship at Cornell University and by an NSF Postdoctoral Research Fellowship (DMS-2103121).

[†]oliver.pechenik@uwaterloo.ca. Oliver Pechenik was partially supported by a Discovery Grant (RGPIN-2021-02391) and Launch Supplement (DGEGR-2021-00010) from the Natural Sciences and Engineering Research Council of Canada.

[‡]stephan.pfannerer@tuwien.ac.at. Stephan Pfannerer was partially supported by the Austrian Science Fund (FWF) P29275 and is a recipient of a DOC Fellowship of the Austrian Academy of Sciences.

[§]jessica.striker@ndsu.edu. Jessica Striker was partially supported by a Simons Foundation/SFARI grant (527204, JS). The team’s visit to NDSU was partially supported by NSF grant (DMS-2000592).

[¶]swansonj@usc.edu.

combinatorics, where for example Petersen–Pylyavskyy–Rhoades [21] used them to obtain a much more direct proof of the 3-row case of Rhoades’ [24] cyclic sieving result for rectangular Young tableaux under the action of tableau promotion.

The 2-row analogue of Kuperberg’s web basis can be combinatorially identified with noncrossing matchings and is algebraically closely related to the Temperley–Lieb algebra (see [16]). Kuperberg also exhibits analogues in Lie types B_2 and G_2 . Kuperberg [17] writes that “the main open problem [...] is how to generalize [these web bases] to higher rank.” Despite progress on identifying generators and relations [4, 8, 14, 18], as well as constructions of non-rotating bases [5, 6, 9, 29], no rotation-invariant web basis for rectangular shapes of more than 3 rows has yet been obtained. However, see [20] for a putative web basis in a non-rectangular setting and [7] for the 2-column case.

Our main result is a rotation-invariant web basis for the 4-row rectangular Specht module $S^{4 \times b}$ (and also for more general spaces of tensor invariants). Our diagrams are a generalization of Postnikov’s plabic graphs (originally introduced for the study of totally positive parts of flag varieties), where our main combinatorial innovation is the introduction of *hourglass edges* that locally break planarity in a mild way. Our new hourglass webs have surprising connections to alternating sign matrices, the 6-vertex model of statistical physics, and plane partitions. A novel ingredient of our proof is the use of Kashiwara’s crystal bases to identify and establish appropriate local growth rules for webs.

This paper is an extended abstract only; the full version will be available on the arXiv.

2 Background

2.1 Tableaux, promotion, and evacuation

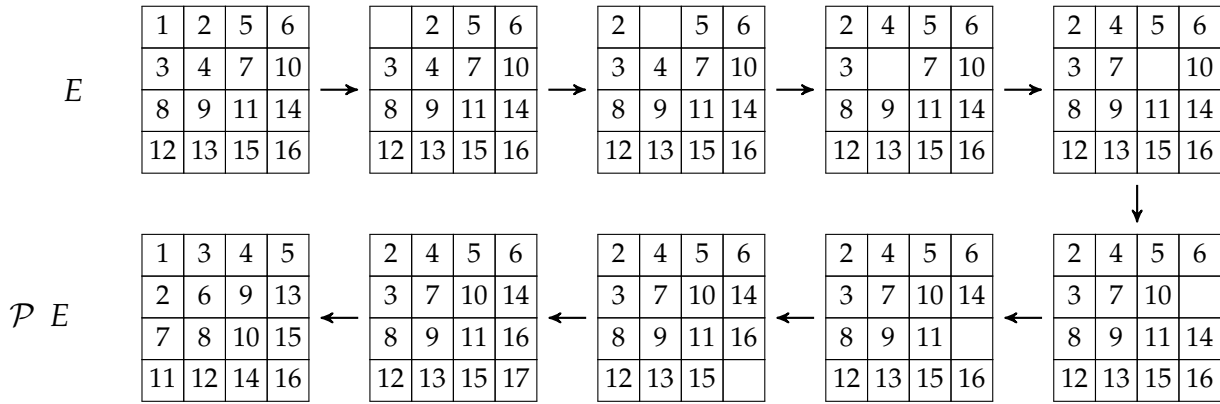
A *standard Young tableau* of *rectangular shape* $a \times b$ is a bijective filling of an $a \times b$ grid of boxes with a rows and b columns using the values $1, 2, \dots, ab$ such that each row is increasing left-to-right and each column is increasing top-to-bottom. Our running example will be

$$E \quad \begin{array}{|c|c|c|c|} \hline 1 & 2 & 5 & 6 \\ \hline 3 & 4 & 7 & 10 \\ \hline 8 & 9 & 11 & 14 \\ \hline 12 & 13 & 15 & 16 \\ \hline \end{array} \quad (2.1)$$

which is a 4×4 standard Young tableau. We write $\text{SYT } a \times b$ for the set of all standard Young tableaux of shape $a \times b$. In this paper, our focus is the case $a = 4$.

Promotion (Schützenberger [25]) is a bijection $\mathcal{P} : \text{SYT } a \times b \rightarrow \text{SYT } a \times b$ defined as follows. Given $T \in \text{SYT } a \times b$, first delete the entry 1 from the box b . Fill the (now) empty box b by sliding into b the smaller of the two values appearing immediately to the

right of b and immediately below b . This slide yields a new empty box b' . Repeat this sliding process until the empty box appears in the bottom right corner of the tableau. Finally, fill the bottom right corner with $ab - 1$ and then subtract 1 from all entries. The result is the promotion $\mathcal{P} T \in \text{SYT } a \times b$. It is known [10] that for $\text{SYT } a \times b$, we have $\mathcal{P}^{ab} = \text{id}$. *Classical webs* (cf. Section 2.3) provide a pictorial proof of this fact for $a \leq 3$; our main result yields such a proof for $a = 4$. For the tableau E , this process is as follows:



For $T \in \text{SYT } a \times b$, the *evacuation* is the tableau $\mathcal{E} T \in \text{SYT } a \times b$ obtained by rotating T by 180° and then replacing each entry i by $ab - 1 - i$. Note that evacuation is an involution. Together $\mathcal{P}$ and $\mathcal{E}$ generate a dihedral group action on $\text{SYT } a \times b$, as $\mathcal{P}^{-1} \circ \mathcal{E} = \mathcal{E} \circ \mathcal{P}$.

2.2 Plabic graphs

Definition 2.1 (Postnikov [22]). A *plabic graph* is a planar undirected graph, embedded in a disk, whose boundary vertices have degree 1 and are labelled $b_1, b_2, \dots$ in clockwise order and whose vertices are each colored black or white.

Plabic graphs were introduced by Postnikov [22] in his study of the totally positive Grassmannian, and have since proven [23] to be fundamental objects in the theories of cluster algebras [26] and KP solitons [15] and in the physics of scattering amplitudes [1].

For each boundary vertex b_i of a plabic graph G , one may obtain another boundary vertex $b_{\pi(i)}$ by beginning a walk on the unique edge incident to b_i and turning left at each white vertex and right at each black vertex, until the boundary is reached. The function π is in fact a permutation, denoted $\text{trip } G$. If this set of walks from the boundary vertices avoids certain *bad double crossings*, then G is called *reduced*. Two reduced plabic graphs share the same trip permutation if and only if they are connected by a sequence of certain graphical transformations called *moves* (see [22]).

2.3 Classical webs

Consider an $a \times n$ matrix of distinct variables. Multiplication by the special linear group SL_a induces an SL_a -module structure on the space of polynomials in these variables. An important task in representation theory is to describe the submodule of invariant polynomials. This submodule is the homogeneous coordinate ring of the *Grassmannian* of a -planes in $\mathbb{C}^n$; moreover, various multihomogeneous components yield concrete constructions of the *Schur modules* (the irreducible polynomial representations of SL_n) and of the *Specht modules* (the irreducible representations of the symmetric group S_n). In this abstract, we restrict attention to Specht modules, although our results are more general.

Our Specht modules $S^{a \times b}$ are associated to rectangular shapes $a \times b$. To compute with Specht modules, it is useful to have explicit vector space bases of $S^{a \times b}$. Various useful choices of basis are known, but we are especially interested in *web bases* [17]. Web bases have a variety of useful features. For combinatorial purposes, the most important such features are that web bases are diagrammatic (meaning that the basis elements are indexed by planar diagrams) and that the actions of the long cycle c and the long element w_0 are by (scalar multiples of) permutation matrices that are pictorially apparent (up to straightforward signs, c acts by rotation of diagrams [21] and w_0 by reflection [19]). For example, in [21], these facts for web bases were used to give substantially more direct proofs of cyclic sieving phenomena for the action of promotion on 2- and 3-row rectangular tableaux (following the original Kazhdan–Lusztig theory proof of Rhoades [24] for the general rectangular case). Our main result may allow an analogous proof in the 4-row case.

Unfortunately, rotation-invariant web bases are only known in a small number of cases. For the Specht modules $S^{2 \times b}$, the web diagrams are the noncrossing matchings of $2b$ points on the boundary of a disk [16]. For $S^{3 \times b}$, Kuperberg [17] introduced SL_3 -webs, which are the bipartite plane graphs embedded in a disk such that all boundary vertices have degree 1, all internal vertices have degree 3, all internal faces have at least 6 sides, and all boundary vertices share a color. It has long been desired to obtain web bases for other shapes, especially 4-row rectangles. For example, [4, 8, 14, 18] describe families of diagrams that are too large to be bases, together with the relations among the corresponding invariant polynomials, whereas [5, 6, 29] provide algorithms to construct non-rotating bases. A rotating basis for certain nonrectangular shapes was recently given in [20] and a web basis for 2-column rectangles in [7]. Our main result is an explicit description of a rotation-invariant web basis for $S^{4 \times b}$, the first extension to higher-row rectangular shapes since Kuperberg’s 1996 paper [17].

3 Main results

In this section, we present our main results and new constructs. Sections 3.1 and 3.2 contain new definitions and theorems needed for our main theorems in Sections 3.3 and 3.4. Theorem 3.6 gives our main bijection between standard tableaux and equivalence classes of hourglass plabic graphs, and the corresponding web basis is given in Theorem 3.7.

3.1 Hourglass plabic graphs

Definition 3.1. An *hourglass plabic graph* is a bipartite plabic graph with black boundary vertices, with hourglass edges $\circ \infty \bullet$ allowed between internal vertices, whose internal vertices are all 4-valent.

Hourglass plabic graphs admit moves of two kinds: *square moves* and *benzene moves* (see Figure 1). Hourglass plabic graphs G and G' are *equivalent* if one may be obtained from the other by a sequence of square and benzene moves. We write G for the equivalence class of G . Equivalence of classical plabic graphs and webs under square moves has appeared before, while benzene equivalence is a new and crucial feature of hourglass plabic graphs.

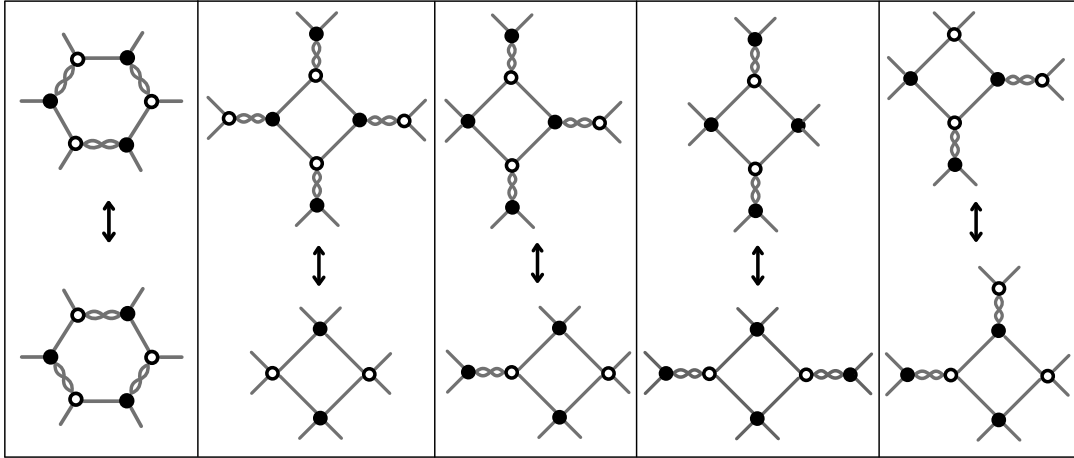


Figure 1: All moves that can be performed on an hourglass plabic graph, up to rotation and reversal of vertex colors. The leftmost move is a *benzene move* and the others are *square moves*.

Definition 3.2. An hourglass plabic graph G is *fully reduced* if no $G' \in G$ contains any of the following substructures:

1. An interior vertex incident to fewer than three other vertices,

2. A 2-cycle (treating an hourglass edge as a single edge), or
3. A 4-cycle containing an hourglass edge.

A key feature of hourglass plabic graphs is that they have *three* trip permutations $\text{trip}_1 G$, $\text{trip}_2 G$, and $\text{trip}_3 G$. These are defined analogously to the trip permutation of a plabic graph, but we take the i -th left at white vertices and i -th right at black vertices when computing $\text{trip}_i G$ (see Figure 2).

Theorem 3.3. *Two fully reduced hourglass plabic graphs G and G' are equivalent if and only if $\text{trip}_i G = \text{trip}_i G'$ for $i = 1, 2, 3$.*

3.2 Promotion permutations

Motivated by trip permutations of hourglass plabic graphs, we want to associate tuples of permutations to rectangular standard Young tableaux. For $T \in \text{SYT } a \times b$, $1 \leq i \leq a - 1$ and $1 \leq j \leq ab$, denote with $p^{i,j}$ the unique entry of $\mathcal{P}^{j-1} T$ that moves from row $i - 1$ to row i in the sliding process when applying promotion to $\mathcal{P}^{j-1} T$.

Definition-Theorem 3.4. *For $1 \leq i \leq a - 1$, the map $\text{prom}_i T : j \mapsto p^{i,j} - j - 1 \pmod{ab}$ is a fixed-point-free permutation on the set $\{1, \dots, ab\}$. We call $\text{prom}_i T$ the i -th **promotion permutation** of T .*

The **anti-exceedance set** of a permutation π is $\text{Aexc } \pi : \{i \mid \pi^{-1} i > i\}$. We denote with $\text{rot } \pi = c^{-1} \circ \pi \circ c$ the **rotation** of π and we denote with $\text{refl } \pi = w_0 \pi w_0$ its **reverse-complement**. The promotion permutations of T satisfy the following properties.

Theorem 3.5. *Let $T \in \text{SYT } a \times b$, then for $1 \leq i \leq a - 1$*

1. $\text{rot } \text{prom}_i T = \text{prom}_i \mathcal{P} T$,
2. $\text{refl } \text{prom}_i T = \text{prom}_i \mathcal{E} T$,
3. $\text{prom}_i T = \text{prom}_{a-i} T^{-1}$, and
4. $\text{Aexc } \text{prom}_i T = \{e \mid e \text{ is in the first } i \text{ rows of } T\}$.

The promotion permutations in one-line notation of our running example (2.1) are

$$\begin{aligned} \text{prom}_1 E &= 4 \ 3 \ 14 \ 10 \ 9 \ 7 \ 8 \ 16 \ 13 \ 11 \ 12 \ \underline{6} \ \underline{5} \ 15 \ \underline{2} \ \underline{1}, \\ \text{prom}_2 E &= 14 \ 9 \ 16 \ 15 \ 11 \ 8 \ 13 \ \underline{6} \ \underline{2} \ 12 \ \underline{5} \ \underline{10} \ \underline{7} \ \underline{1} \ \underline{4} \ \underline{3}, \text{ and} \\ \text{prom}_3 E &= 16 \ 15 \ \underline{2} \ \underline{1} \ 13 \ 12 \ \underline{6} \ \underline{7} \ \underline{5} \ \underline{4} \ \underline{10} \ \underline{11} \ \underline{9} \ \underline{3} \ \underline{14} \ \underline{8}. \end{aligned} \tag{3.1}$$

(The anti-exceedances of each permutation are underscored.) Note that $\text{prom}_1 E$ and $\text{prom}_3 E$ are inverses of each other and that $\text{prom}_2 E$ is an involution.

3.3 The main bijection

Given an hourglass plabic graph G , let $\text{rot } G$ denote the hourglass plabic graph obtained from G by cyclically rotating the graph with respect to the boundary labels, and let $\text{refl } G$ denote the hourglass plabic graph obtained by reflecting the graph (but not the boundary labels) with respect to a diameter of the disk intersecting the boundary between b_{4b} and b_1 .

Theorem 3.6. *For an equivalence class $\mathcal{G}$ of fully reduced hourglass plabic graphs with $4b$ boundary vertices, there is a unique $4 \times b$ standard Young tableau $\mathcal{T} \in \mathcal{G}$ whose first i rows contain the entries $\text{Aexc } \text{trip}_i \mathcal{G}$ for $i = 1, 2, 3$. Furthermore, the map $\mathcal{T}$ is a bijection satisfying:*

1. $\text{trip}_i \mathcal{G} = \text{prom}_i \mathcal{T} \mathcal{G}$ for $i = 1, 2, 3$,
2. $\mathcal{T}(\text{rot } G) = \mathcal{P}(\mathcal{T}(G))$, and
3. $\mathcal{T}(\text{refl } G) = \mathcal{E}(\mathcal{T}(G))$.

See Figure 2 for an example.

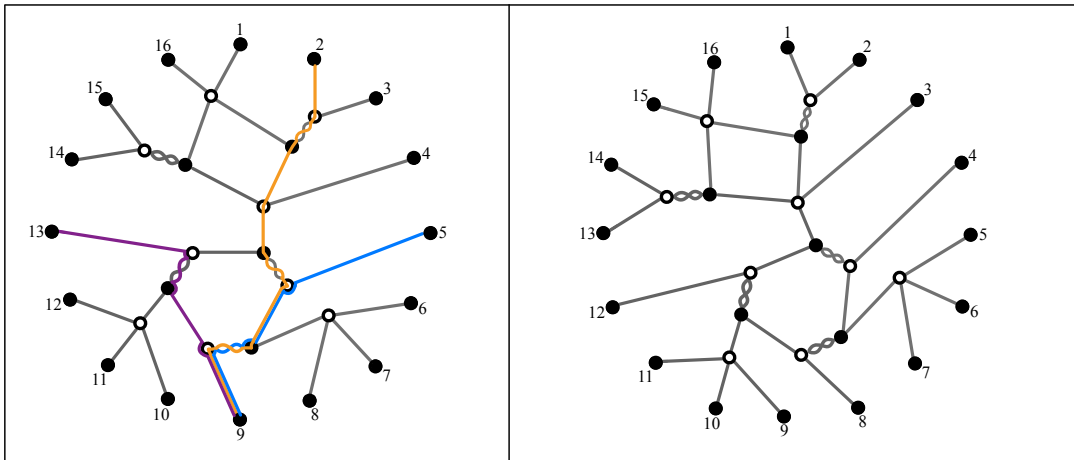


Figure 2: An hourglass plabic graph G representing $\mathcal{G} \in E$ for our running example tableau E (left) and its rotation $\text{rot } G$ (right), corresponding to $\mathcal{P}(E)$. The paths computing $\text{trip}_1 \mathcal{G} = 9 \rightarrow 13$, $\text{trip}_2 \mathcal{G} = 9 \rightarrow 2$, and $\text{trip}_3 \mathcal{G} = 9 \rightarrow 5$ are drawn in purple, orange, and blue respectively, matching the values of $\text{prom}_i \mathcal{E} = 9$ computed in (3.1).

The inverse $\mathcal{G}$ of the map $\mathcal{T}$ in Theorem 3.6 involves *growth rules*, which operate on the *lattice word* of a standard Young tableau T . In the lattice word $L = \ell_1 \ell_2 \dots$ of T , ℓ_k is the row index of the box labeled k in T . The lattice condition requires that, for any prefix of L and for all $i \leq j$, the number of i 's is not less than the number of j 's. For example, the lattice word associated to (2.1) is $L = 1122112332344344$.

For $T \in \text{SYT } 2 \times b$, we may construct a non-crossing circular matching as follows. First place $2b$ vertices in a line, each with a dangling edge labeled ℓ_i from the lattice word L of T . Now iteratively “cap off” adjacent pairs of dangling strands labeled 1 on the left, 2 on the right. This is analogous to matching parentheses. After all dangling strands have been capped off, wrap the resulting figure around a circle.

For $T \in \text{SYT } 3 \times b$, Khovanov–Kuperberg [13] extended this construction to produce Kuperberg’s trivalent non-elliptic webs. The algorithm starts with $3b$ black vertices and dangling strands again labeled by the lattice word of T . Strands are combined iteratively using a collection of 14 growth rules (see [21, Figure 4] under $1 \leftrightarrow 1, 0 \leftrightarrow 2, \bar{1} \leftrightarrow 3$). For instance, adjacent strands labeled 1, 2 dangling from black vertices may be combined in a new trivalent white vertex with a new dangling strand labeled 3. Khovanov–Kuperberg showed that the algorithm is independent of all choices and is bijective.

We extend these constructions to $T \in \text{SYT } 4 \times b$. More precisely, we give a bijection $\mathcal{G} : \text{SYT } 4 \times b \rightarrow \{G \mid G \text{ a fully reduced hourglass plabic graph}\}$ inverse to the bijection $\mathcal{T}$ from Theorem 3.6. The algorithm involves approximately 100 growth rules; see Figure 3 for some examples. An intriguing feature of the 4-row growth rules is the presence of “witnesses,” which are dangling strands which are not themselves combined. Moreover, the result is *not* independent of choices, though the result is well-defined as an equivalence class of fully reduced hourglass plabic graphs. Our key technique for generalizing the Khovanov–Kuperberg growth rules is to observe that each rule corresponds to a pair of corresponding vertices in isomorphic Kashiwara crystal graphs.

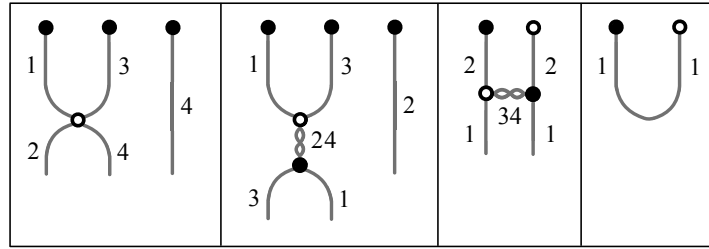


Figure 3: Some examples of 4-row growth rules.

3.4 The SL_4 -web basis

Kuperberg [17] introduced SL_3 -webs as a combinatorial model for the category of $\text{SL}_3 \mathbb{C}$ -modules. In particular, bipartite trivalent webs with $3b$ black boundary vertices correspond to *invariant polynomials*. These are elements of

$$\text{Inv } \mathbb{C}^3 \otimes^b \text{Hom}_{\text{SL}_3 \mathbb{C}} \mathbb{C}^3 \otimes^b, \mathbb{C} \subset \mathbb{C} x_{i1}, x_{i2}, x_{i3} : 1 \leq i \leq b,$$

where $\mathbb{C}^3$ is the defining representation. For example, 3 black boundary vertices connected to a single white vertex corresponds to the 3×3 determinant polynomial in

9 variables x_{ij} for $1 \leq i, j \leq 3$. Kuperberg showed that the invariants associated to SL_3 -webs form a basis for $\text{Inv } \mathbb{C}^3 \otimes^b$. A group based at LACIM [2] showed more directly that Kuperberg–Khovanov’s growth rules applied to $T \in \text{SYT } 3 \times b$ produce webs whose invariants have grevlex-leading terms that correspond naturally to T itself. See [2, Thm. 5.2] for details.

Using Fraser–Lam–Le’s monomial expansion [8, Lemma 5.4], we extend the result of [2] to hourglass plabic graphs.

Each equivalence class G of fully reduced reduced hourglass plabic graphs contains a **top** hourglass plabic graph G^{top} , whose corresponding *symmetrized six-vertex configuration* (see Section 4) has all triangles oriented counterclockwise. For example the graphs in Figure 2 are top fully reduced hourglass plabic graphs. The top element G^{top} is unique up to square moves, and thus the web invariant polynomial $\llbracket G^{\text{top}} \rrbracket$ is well-defined.

Theorem 3.7. *The web invariant polynomials $\llbracket G^{\text{top}} \rrbracket$ of top fully reduced hourglass plabic graphs with $4b$ boundary vertices are a rotation-invariant basis for the invariant space $\text{Inv } \mathbb{C}^4 \otimes^b$.*

4 The six-vertex model and combinatorial connections

Fully reduced hourglass plabic graphs may be transformed to directed graphs in which all boundary vertices are degree 1 and oriented inward and all interior vertices are degree 4 and are sources, sinks, or any rotation of  (**transmitting vertices**). The bijection proceeds by orienting all non-hourglass edges from black to white vertex, removing the vertex coloring, and collapsing each hourglass edge and incident vertices to a single vertex, producing a transmitting vertex configuration. This is invertible, as each source corresponds to a black vertex, each sink corresponds to a white vertex, and each transmitting vertex uniquely expands to an hourglass edge with one black and one white vertex. Condition 3 of Definition 3.2 corresponds to each three-cycle being oriented (in the entire equivalence class). See Figure 4. A similar construction appears in [9].

These graphs can be seen as configurations of a symmetrized version of the *six-vertex model with domain wall boundary conditions*. Configurations of this model on a square grid involve boundary edges oriented inward along the left and right boundary and outward along the top and bottom, while interior edge configurations include transmitting vertices as well as  and . Such a configuration may be transformed to our convention by reversing the direction of all vertical edges, permuting transmitting vertices and producing a source or sink at each non-transmitting vertex.

On a square grid, these graphs are well-known to correspond to **alternating sign matrices (ASM)**, matrices whose entries are in $\{0, \pm 1\}$, whose rows and columns sum to 1, and whose nonzero entries alternate in sign along each row and column. In our convention, sinks correspond to 1, sources to -1 , and transmitting vertices to 0. In fact,

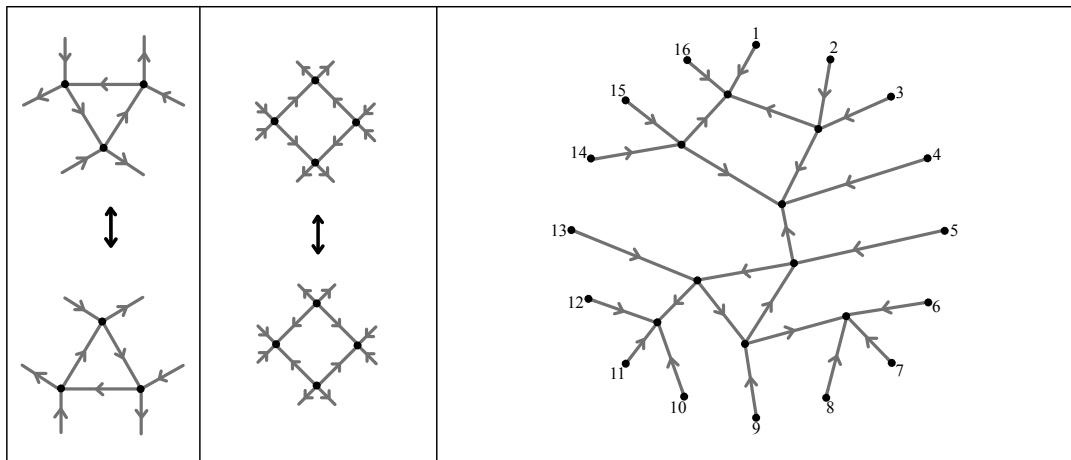


Figure 4: The six-vertex configuration of our running example (2.1) (right), together with six-vertex versions of the first two moves in Figure 1 (left).

this connection yields an intriguing enumeration; the $4 \times n$ standard Young tableau in which the numbers appear sequentially, left-to-right then top-to-bottom has $\prod_{j=0}^{n-1} \frac{3j+1}{n-j}!$ elements in its equivalence class, corresponding to all $n \times n$ alternating sign matrices.

The moves of Figure 1 transform to studied moves on six-vertex configurations and alternating sign matrices: the benzene move corresponds to the *star-triangle relation* associated to the *Yang-Baxter equation*, while the square move may be seen as a *fiber toggle* in the alternating sign matrix tetrahedral poset [28] or as travelling along an edge of the *ASM polytope* [27]. Composing these toggles at even squares followed by odd squares produces the well-studied *gyration* action of [30] on *fully-packed loops*, objects in bijection with alternating sign matrices. Fully-packed loops on generalized domains (such as ours) were considered by Cantini–Sportiello in their proof of the *Razumov–Stroganov conjecture* [3] and relate to *chained alternating sign matrices* [11].

The benzene move (in the plabic formulation of Figure 1) can be seen as a transformation on a *dimer cover* (perfect matching) of the hexagonal lattice, where the hourglass edges correspond to the included edges of the dimer cover. The transformation inverts a cycle of alternate hourglass and non-hourglass edges. This is well-known to correspond to adding and removing boxes from an associated *plane partition*.

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The associative-commutative spectrum of a binary operation

Jia Huang^{*1} and Erkki Lehtonen^{+2,3}

¹Department of Mathematics and Statistics, University of Nebraska, Kearney, NE 68849, USA

²Department of Mathematics, Khalifa University, P.O. Box 127788, Abu Dhabi, United Arab Emirates

³Centro de Matemática e Aplicações, Faculdade de Ciências e Tecnologia, Universidade Nova de Lisboa, Quinta da Torre, 2829-516 Caparica, Portugal

Abstract. We initiate the study of a quantitative measure for the failure of a binary operation to be commutative and associative. We call this measure the associative-commutative spectrum as it extends the associative spectrum (also known as the sub-associativity type), which measures the nonassociativity of a binary operation. In fact, the associative-commutative spectrum (resp., associative spectrum) is the cardinality of the symmetric (resp., nonsymmetric) operad obtained naturally from a groupoid (a set with a binary operation). In this paper we provide some general results on the associative-commutative spectrum, precisely determine this measure for certain binary operations, and propose some problems for future study.

Keywords: Associative-commutative spectrum; associative spectrum; binary operation; tree

1 Introduction

Associativity and commutativity are important properties for binary operations. Although many familiar operations satisfy both properties, some only satisfy one or neither of them. Moreover, nonassociativity and noncommutativity arise in many algebraic structures, such as Lie algebras, Poisson algebras, and so on. The operad theory models both nonassociativity and noncommutativity using binary trees. It was developed by Boardman, May, Vogt, and others, with applications recently found in many branches of mathematics (see, e.g., Loday and Vallette [15]). We recall some basic definitions below.

A *nonsymmetric operad* is an indexed family $\mathcal{P} = \{\mathcal{P}_n\}_{n \geq 1}$ of sets with an identity element $1 \in \mathcal{P}_1$ and, for all positive integers $n, m_1, \dots, m_n$, a *composition map* $\circ: \mathcal{P}_n \times$

^{*}huangj2@unk.edu.

[†]e.lehtonen@fct.unl.pt. Erkki Lehtonen was partially supported by National Funds through the FCT—Fundação para a Ciência e a Tecnologia, I.P., under the scope of the project UIDB/00297/2020 (Center for Mathematics and Applications) and the project PTDC/MAT-PUR/31174/2017.

$\mathcal{P}_{m_1} \times \cdots \times \mathcal{P}_{m_n} \rightarrow \mathcal{P}_{m_1 \cdots m_n}$, $P, P_1, \dots, P_n \mapsto P \circ P_1, \dots, P_n$ such that $P \circ 1, \dots, 1 = P$, $1 \circ P = P$ and $P \circ P_1 \circ P_{1,1}, \dots, P_{1,m_1}, \dots, P_n \circ P_{n,1}, \dots, P_{n,m_n} = P \circ P_1, \dots, P_n \circ P_{1,1}, \dots, P_{1,m_1}, \dots, P_{n,1}, \dots, P_{n,m_n}$ for any $P \in \mathcal{P}_n$, $P_i \in \mathcal{P}_{n_i}$, $P_{i,j} \in \mathcal{P}_{n_{i,j}}$ ($1 \leq i \leq n$, $1 \leq j \leq n_i$). A nonsymmetric operad can thus be seen as a many-sorted algebra with a nullary operation 1 and operations $\circ_{n,m_1,\dots,m_n}$ for all positive integers $n, m_1, \dots, m_n$, but for notational simplicity, the same symbol $\circ$ is used to denote all of the latter. The elements of $\mathcal{P}_n$ are called n -ary operations. The Hilbert series of a nonsymmetric operad $\mathcal{P}$ is $\sum_{n=1}^{\infty} |\mathcal{P}_n| t^n$.

On the other hand, a *symmetric operad* is a nonsymmetric operad $\mathcal{P} = \{\mathcal{P}_n\}_{n \geq 1}$ with a right action of the symmetric group $\mathfrak{S}_n$ on $\mathcal{P}_n$ for each $n \geq 1$ satisfying the equivariance conditions $P \cdot w \circ P_{w^{-1}1}, \dots, P_{w^{-1}n} = P \circ P_1, \dots, P_n \cdot w$ and $P \circ P_1 \cdot w_1, \dots, P_n \cdot w_n = P \circ P_1, \dots, P_n \cdot w_1, \dots, w_n$ for any $P \in \mathcal{P}_n$, $w \in \mathfrak{S}_n$, $P_i \in \mathcal{P}_{m_i}$, and $w_i \in \mathfrak{S}_{m_i}$. Here by abuse of notation, the permutation w on the right side of the first equation is a permutation of the set $\{1, \dots, m_1 \cdots m_n\}$ that breaks the set into n blocks of sizes $m_1, \dots, m_n$ and then permutes the n blocks by w . The Hilbert series of a symmetric operad $\mathcal{P} = \{\mathcal{P}_n\}_{n \geq 1}$ is $\sum_{n=1}^{\infty} \frac{|\mathcal{P}_n|}{n!} t^n$.

Now recall that a *groupoid* is a set G with a single binary operation $*$.¹ A *bracketing* of n variables is a groupoid term over the set $X_n : \{x_1, \dots, x_n\}$ of variables that is obtained by inserting parentheses in $x_1 * x_2 * \cdots * x_n$ in a valid way. With $\mathcal{P}_* n$ denoting the set of all n -ary term operations on $G, *$ induced by the bracketings of n variables, we obtain the non-symmetric operad $\mathcal{P}_* : \{\mathcal{P}_* n\}_{n \geq 1}$. The cardinality $|\mathcal{P}_* n|$ measures to some extent the failure of $*$ to be associative. In general, we have $1 \leq |\mathcal{P}_* n| \leq C_{n-1}$ where $C_n : \frac{1}{n+1} \binom{2n}{n}$ is the ubiquitous *Catalan number*.

Csákány and Waldhauser [2] called the sequence $s_n^a * : |\mathcal{P}_* n|_{n \in \mathbb{N}}$ the *associative spectrum* of the binary operation $*$, while Brait and Silberger [1] named it the *subassociativity type* of the groupoid $G, *$. Independently, Hein and the first author [5] proposed the study of $s_n^a * = |\mathcal{P}_* n|$ for a binary operation $*$ and provided an explicit formula when $*$ satisfies *k-associativity* (a generalization of associativity). The associative spectra of many other binary operations have been determined [6, 8, 12, 13, 14].

For each $n \geq 1$, let $\overline{\mathcal{P}}_* n$ be the set of all n -ary term operations induced on $G, *$ by *full linear terms* of n variables, i.e., groupoid terms over X_n in which each variable $x_1, \dots, x_n$ occurs exactly once (but in an arbitrary order, as opposed to bracketings). This gives a symmetric operad $\overline{\mathcal{P}}_* : \{\overline{\mathcal{P}}_* n\}_{n \geq 1}$ with Hilbert series $\sum_{n=1}^{\infty} \frac{|\overline{\mathcal{P}}_* n|}{n!} t^n$. We call the sequence $s_n^{ac} *_{n \in \mathbb{N}}$, where $s_n^{ac} * : |\overline{\mathcal{P}}_* n|$, the *associative-commutative spectrum* (in brief, *ac-spectrum*) of the binary operation $*$, which measures both the nonassociativity and the noncommutativity of $*$. We will determine the ac-spectra for some binary operations and exhibit connections to other interesting combinatorial objects and results.

It is clear that $s_n^{ac} * \geq 1$ and the equality holds for all $n \in \mathbb{N}$ if and only if $*$ is both

¹Note that the term *groupoid* has a different meaning in category theory.

commutative and associative. On the other hand, we have the following upper bounds.

- (i) Since full linear terms over X_n are in bijection with (ordered) binary trees with n labeled leaves, we have $s_n^{\text{ac}} * \leq n!C_{n-1}$ for an arbitrary binary operation $*$.
- (ii) Since the equivalence classes of full linear terms over X_n modulo the equational theory of associative groupoids (semigroups) are in bijection with the set of permutations of $\{1, \dots, n\}$, we have $s_n^{\text{ac}} * \leq n!$ if $*$ is an associative binary operation.
- (iii) Since the equivalence classes of full linear terms over X_n modulo the equational theory of commutative groupoids are in bijection with unordered binary trees with n labeled leaves, we have $s_n^{\text{ac}} * \leq 2n - 2 ! / 2^{n-1} n - 1 !$ [16, A001147] if $*$ is commutative. This upper bound is the solution to Schröder's third problem [17, p. 178].

In Section 3 we show that the above upper bounds can be achieved by the free groupoid on one generator, the free associative groupoid (i.e., the free semigroup) on two generators, and the free commutative groupoid on one generator, respectively.

In Section 4 we focus on associative or commutative binary operations. For an associative noncommutative binary operation $*$, we show that its ac-spectrum $s_n^{\text{ac}} *$ attains the upper bound $n!$ if it has a neutral element (i.e., identity element), and give some other examples for which $s_n^{\text{ac}} * < n!$. We provide some concrete examples of commutative groupoids whose ac-spectra reach the upper bound $2n - 2 ! / 2^{n-1} n - 1 !$, but for the arithmetic, geometric, and harmonic means, we show that their ac-spectra coincide with an interesting sequence that counts ways to express 1 as an ordered sum of powers of 2 [16, A007178]. The last example shows that the ac-spectrum of a commutative operation may not achieve the upper bound $2n - 2 ! / 2^{n-1} n - 1 !$ even if it is *totally nonassociative*, i.e., its associative spectrum equals the upper bound C_{n-1} . However, we show that the converse does hold: a commutative groupoid is totally nonassociative if its ac-spectrum reaches that upper bound.

In Section 5 we show that the ac-spectrum of some anticommutative algebras over a field, including the cross product and certain Lie brackets, is exactly two times the upper bound $2n - 2 ! / 2^{n-1} n - 1 !$ for the ac-spectrum of a commutative operation.

In Section 6, we determine the ac-spectra of some more examples of totally nonassociative operations, including the exponentiation, the (converse) implication, and the negated disjunction (NOR). The exponentiation and the converse implication satisfy the identity $x yz \approx x zy$ and their ac-spectrum reaches the upper bound n^{n-1} for the ac-spectrum of any binary operation satisfying the above identity. Here n^{n-1} shows up because it is the number of unordered rooted trees with n labeled vertices. The negated disjunction is commutative and its ac-spectrum reaches the upper bound $2n - 2 ! / 2^{n-1} n - 1 !$ for commutative operations. Together with Example 4.2, this completely describes the ac-spectra of all two-element groupoids.

In Section 7 we obtain a formula involving the Stirling numbers of the second kind for the ac-spectrum of a binary operation $*$ satisfying the property that any two full linear terms agree on $*$ if and only if the right depth sequences of their corresponding binary

trees are congruent modulo k (this is equivalent to the k -associativity mentioned earlier). An example is given by $a * b : a + e^{2\pi i/k} b$, which becomes addition and subtraction when $k = 1$ and $k = 2$. Related to this example is the operation $a * b : e^{2\pi i/k} a + b$. When $k = 2$ this becomes the *double minus operation* $a \ominus b : -a - b$ whose associative spectrum is $|\mathcal{P}_* n| = \lfloor 2^n/3 \rfloor$ [16, A000975], as shown by Csákány and Waldhauser [2, § 5.3] and independently by the first author, Mickey, and Xu [8]. We show that $s_n^{\text{ac}} \ominus 2^n - \lfloor 2^n/3 \rfloor$, which is the well-known *Jacobsthal sequence* [16, A001045]. The more general operation $a * b : e^{2\pi i/k} a + b$ satisfies the k -depth-equivalence studied recently by the second author. Computations show that neither the associative spectrum nor the ac-spectrum of this operation is in OEIS [16] when $k > 2$.

In Section 8, we give remarks and indicate possible directions for further research.

2 Preliminaries

In this section we briefly recall some fundamental concepts that are necessary for our work and introduce the (*fine*) *associative-commutative spectrum* of a groupoid.

Let $*$ be a binary operation on a set G . Then $\mathbf{G} = (G, *)$ is called a *groupoid*. Define $\mathbb{N} : \{1, 2, 3, \dots\}$ and $[n] : \{1, \dots, n\}$. Let X be a set of variables; some common choices are $X_\omega : \{x_1, x_2, \dots\}$ and $X_n : \{x_1, \dots, x_n\}$ for $n \in \mathbb{N}$. A *term* over X is of the form $x_{i_1} * \dots * x_{i_\ell}$ with $\ell - 1$ pairs of parentheses inserted in a valid way, where $x_{i_1}, \dots, x_{i_\ell} \in X$. We may omit $*$ from a term if there is no confusion to do so.

Let $T(X)$ denote the set of all terms over X . A term $t \in T(X)$ is *linear* if no variable occurs more than once in t , or *full* if every variable $x \in X$ occurs in t . A full linear term over X_n can be obtained by inserting $n - 1$ pairs of parentheses in a valid way into $x_{\sigma(1)} * \dots * x_{\sigma(n)}$ for some permutation $\sigma \in \mathfrak{S}_n$. If σ happens to be the identity permutation then we get a *bracketing* over X_n . Let F_n and B_n denote the set of all full linear terms over X_n and the set of all bracketings over X_n , respectively. It is well known that the number of bracketings over X_n equals the $(n - 1)$ -st Catalan number $C_{n-1} = \frac{1}{n} \binom{2n-2}{n-1}$, i.e., $|B_n| = C_{n-1}$. Consequently, $|F_n| = n! C_{n-1}$.

Each $t \in F_n$ corresponds to an n -ary operation $t^{\mathbf{G}}$ on G . Given two terms $s, t \in F_n$, we have an *identity* $s \approx t$ satisfied by $\mathbf{G}$ if $s^{\mathbf{G}} = t^{\mathbf{G}}$. The *fine associative-commutative spectrum* (in brief, *fine ac-spectrum*) of $\mathbf{G}$ is the sequence $\sigma_n^{\text{ac}} \mathbf{G} : n \in \mathbb{N}$, where $\sigma_n^{\text{ac}} \mathbf{G}$ is the set of all $s, t \in F_n$ with $s^{\mathbf{G}} = t^{\mathbf{G}}$. It is clear that $\sigma_n^{\text{ac}} \mathbf{G}$ is an equivalence relation on F_n . The *associative-commutative spectrum* (in brief, *ac-spectrum*) of $\mathbf{G}$ is the sequence $s_n^{\text{ac}} \mathbf{G} : n \in \mathbb{N}$, where $s_n^{\text{ac}} \mathbf{G} : = |F_n / \sigma_n^{\text{ac}} \mathbf{G}|$, i.e., the number of equivalence classes of $\sigma_n^{\text{ac}} \mathbf{G}$. Equivalently, $s_n^{\text{ac}} \mathbf{G}$ is the number of distinct term operations on $\mathbf{G}$ induced by the full linear terms over X_n , in symbols, $s_n^{\text{ac}} \mathbf{G} = |F_n^{\mathbf{G}}| = |\{t^{\mathbf{G}} \mid t \in F_n\}|$. The *fine associative spectrum* $\sigma_n^{\text{a}} \mathbf{G} : n \in \mathbb{N}$ and the *associative spectrum* $s_n^{\text{a}} \mathbf{G} : n \in \mathbb{N}$ of $\mathbf{G}$ were defined analogously by Liebscher and Waldhauser [14] by taking bracketings instead of

full linear terms, i.e., by replacing F_n with B_n in the above definitions. These numbers satisfy $1 \leq s_n^a \mathbf{G} \leq |B_n| = C_{n-1}$ and $1 \leq s_n^{ac} \mathbf{G} \leq |F_n| = n!C_{n-1}$. We say $\mathbf{G}$ is *totally nonassociative* if $s_n^a \mathbf{G} = C_{n-1}$ for all $n \geq 1$. For notational simplicity, we may write t^* for $t^{\mathbf{G}}$ or $s_n^{ac} *$ for $s_n^{ac} \mathbf{G}$.

It is easy to see that isomorphic or antiisomorphic groupoids have the same associative spectrum and the same ac-spectrum. The following facts follow immediately from the fact that varieties (classes of groupoids axiomatized by identities) are closed under homomorphic images, subgroupoids, and direct products.

- (i) If $\mathbf{A}$ is a homomorphic image of $\mathbf{B}$, then $\sigma_n^{ac} \mathbf{A} \supseteq \sigma_n^{ac} \mathbf{B}$ and $s_n^{ac} \mathbf{A} \leq s_n^{ac} \mathbf{B}$.
- (ii) If $\mathbf{A}$ is a subgroupoid of $\mathbf{B}$, then $\sigma_n^{ac} \mathbf{A} \supseteq \sigma_n^{ac} \mathbf{B}$ and $s_n^{ac} \mathbf{A} \leq s_n^{ac} \mathbf{B}$.
- (iii) If $\mathbf{C} = \mathbf{A} \times \mathbf{B}$, then $\sigma_n^{ac} \mathbf{C} = \sigma_n^{ac} \mathbf{A} \cap \sigma_n^{ac} \mathbf{B}$ and $s_n^{ac} \mathbf{C} \geq \max\{s_n^{ac} \mathbf{A}, s_n^{ac} \mathbf{B}\}$.

Now let T be a *rooted tree*, i.e., a tree with a distinguished vertex called the *root* and with edges oriented away from the root (usually downward). A vertex in T is a *leaf* if it has no children, or an *internal vertex* otherwise. A rooted tree T is *ordered*² if the children of each internal vertex are linearly ordered, or *unordered* otherwise. Given an ordered tree T , the unordered tree obtained from T by simply ignoring the order of children of each internal vertex is called the *underlying unordered tree* of T and denoted by T^u . A rooted tree is *labeled* if all of its vertices are labeled. Given a vertex v in a rooted tree T , the *subtree of T rooted at v* consists of all the vertices and edges weakly below v . A rooted tree T is a *binary tree* if each internal vertex has exactly two children. A binary tree is *leaf-labeled* if its leaves are labeled. The *left subtree* T_L and *right subtree* T_R of an ordered binary tree T are the subtrees rooted at the left child and at the right child of the root of T , respectively. If S and T are two ordered binary trees, then $S \wedge T$ is the ordered binary tree whose left and right subtrees are S and T , respectively. One can naturally extend these definitions to unordered binary trees by not distinguishing left and right.

Let T be a binary tree with n leaves labeled $1, \dots, n$ in some order. The *left depth* $\delta_T i$, *right depth* $\rho_T i$, and *depth* $d_T i$ of a leaf i in an ordered binary tree T is the number of left, right, and all steps in the path from the root to the leaf labeled i . This leads to the *left depth sequence* $\delta_T : \delta_T 1, \dots, \delta_T n$, the *right depth sequence* $\rho_T : \rho_T 1, \dots, \rho_T n$ and the *depth sequence* $d_T : d_T 1, \dots, d_T n$ of T . If the leaves of T are labeled $1, \dots, n$ from left to right, then each sequence above determines the ordered binary tree T uniquely [2, 5, 8]. The depth sequence can also be defined for unordered leaf-labeled binary trees.

There is a bijection between the set $T(X)$ of all terms over X and the set of ordered binary trees with leaves labeled by the variables in X (if $X = X_\omega$ then we may identify a label x_i with i), defined recursively as follows: each variable $x \in X$ corresponds to a tree with just one vertex labeled with x , and if the terms t_1 and t_2 correspond to trees T_1 and

²Ordered trees are often called *plane trees* since a plane embedding of a tree induces a cyclic ordering of the neighbours of each vertex; moreover, if the root is drawn at the top – following our drawing convention – then the embedding specifies a linear ordering for the children of each internal vertex.

T_2 , respectively, then the term $t_1 t_2$ corresponds to the tree $T_1 \wedge T_2$. We write T_t for the binary tree corresponding to the term t via this bijection, and we write t_T for the term corresponding to the binary tree T via its inverse map.

3 Free groupoids

In this section we show that the various upper bounds for the associative-commutative spectra mentioned in Section 1 can be achieved by certain free groupoids.

Recall that a *groupoid* $\mathbf{G} = (G, *)$ is a set G with a single binary operation $*$. A groupoid is *commutative* if it satisfies the identity $x * y \approx y * x$ (*commutative law*), and it is *associative* if it satisfies the identity $x * (y * z) \approx (x * y) * z$ (*associative law*). An associative groupoid is called a *semigroup*. The *free semigroup* over X is isomorphic to the semigroup $\mathbf{X} = (X^+, \cdot)$ of nonempty words over X endowed with the operation $\cdot$ of concatenation.

We first show that the upper bound $s_n^{\text{ac}} \mathbf{G} \leq n! C_{n-1}$ for the ac-spectrum $s_n^{\text{ac}} \mathbf{G}$ of an arbitrary groupoid $\mathbf{G}$ becomes an equality when $\mathbf{G}$ is the free groupoid on one generator.

Proposition 3.1. *If $\mathbf{G}$ is a free groupoid with one generator then $s_n^{\text{ac}} \mathbf{G} = n! C_{n-1}$ for all $n \geq 1$.*

If $\mathbf{G}$ is a semigroup then $s_n^{\text{ac}} \mathbf{G} \leq n!$ for all $n \in \mathbb{N}$, since all bracketings over X_n induce the same term operation on $\mathbf{G}$ and it is only the order of variables in a full linear term that matters. We show that the equality in this upper bound holds when $\mathbf{G}$ is the free semigroup with two generators. Note that the ac-spectrum of the free semigroup with one generator is the constant 1 sequence.

Proposition 3.2. *If $\mathbf{G}$ is a free semigroup with two generators then $s_n^{\text{ac}} \mathbf{G} = n!$ for all $n \geq 1$.*

If $\mathbf{G}$ is a commutative groupoid then $s_n^{\text{ac}} \mathbf{G}$ is bounded above by the number $(2n - 2)! / 2^{n-1} (n - 1)!$ of unordered binary trees with n labeled leaves [16, A001147], i.e., the solution to Schröder's third problem; see, e.g., Stanley [17, p. 178].

Proposition 3.3. *If $\mathbf{G}$ is a free commutative groupoid with one generator then its ac-spectrum achieves the above upper bound: $s_n^{\text{ac}} \mathbf{G} = (2n - 2)! / 2^{n-1} (n - 1)!$ for all $n \geq 1$.*

4 Associative or commutative groupoids

In this section we study the ac-spectra of some associative or commutative groupoids.

First assume that $*$ is an associative binary operation on a set G , i.e., $\mathbf{G} = (G, *)$ is a semigroup. We know that the upper bound $s_n^{\text{ac}} \mathbf{G} \leq n!$ is reached by the free semigroup with two generators (Proposition 3.2). Now we provide another example of a family of groupoids for which this upper bound is achieved.

Recall that a *monoid* is a semigroup $\mathbf{G} = (G, *)$ with a *neutral element* (or *identity element*), i.e., an element $e \in G$ such that $e * x = x = x * e$ for all $x \in G$.

Proposition 4.1. *If $\mathbf{G} = (G, *)$ is a noncommutative monoid, then $s_n^{\text{ac}} \mathbf{G} = n!$ for all $n \geq 1$.*

The above proposition would no longer be true if we omitted the assumption that the groupoid has a neutral element, as shown by the following example.

Example 4.2. Csákány and Waldhauser [2, Section 4] determined the associative spectrum of every two-element groupoid. Such a groupoid is isomorphic or anti-isomorphic to $\mathbf{G} = \{0, 1\}, *$, where $x * y$ is defined as one of the following: (1) 1, (2) x , (3) $\min\{x, y\}$, (4) $x \downarrow y \bmod 2$, (5) $x \uparrow y \bmod 2$, (6) $x \downarrow y$ (negated disjunction, NOR) or (7) $x \rightarrow y$ (implication). We have $s_n^{\text{ac}} \mathbf{G} = 1$ for all $n \in \mathbb{N}$ if $*$ defined by (1), (3), or (4) since $*$ is both associative and commutative in these three cases. The operation $*$ defined by (2) is associative but not commutative, and we have $s_n^{\text{ac}} \mathbf{G} = n$ for all $n \in \mathbb{N}$. For the operation $*$ defined by (5), we can show that $s_1^{\text{ac}} \mathbf{G} = 1$, $s_2^{\text{ac}} \mathbf{G} = 2$, and $s_n^{\text{ac}} \mathbf{G} = 2n$ for all $n \geq 3$. The groupoids given by (6) and (7) are totally nonassociative and their ac-spectra will be determined in Section 6.

Assume that $\mathbf{G} = (G, *)$ is a commutative groupoid. Recall that we have an upper bound $s_n^{\text{ac}} \mathbf{G} \leq 2n - 2 \cdot 2^{n-1} / (n - 1)!$ which is attained by a free commutative groupoid with one generator (Proposition 3.3). The following lemma shows that any commutative groupoid $\mathbf{G}$ reaching this upper bound must be totally nonassociative.

Lemma 4.3. *Let $\mathbf{G} = (G, *)$ be a commutative groupoid. If $s_n^{\text{ac}} \mathbf{G} = 2n - 2 \cdot 2^{n-1} / (n - 1)!$ for $n \in \mathbb{N}$, then $\mathbf{G}$ is totally nonassociative, i.e., $s_n^{\text{a}} \mathbf{G} = C_{n-1}$ for all $n \in \mathbb{N}$.*

The converse of Lemma 4.3 does not hold. If $*$ is the arithmetic, geometric, or harmonic mean, then $s_n^{\text{a}} \mathbf{G} = C_{n-1}$ for all $n \geq 1$ (see Csákány and Waldhauser [2]). However, as we are going to show next, its ac-spectrum agrees with an interesting sequence in OEIS [16, A007178], which enumerates different ways to write 1 as an ordered sum of n powers of 2 (i.e., compositions of 1 into powers of 2) and is also related to the prefix codes or Huffman codes (see, e.g., Even and Lempel [3], Giorgilli and Molteni [4], Knuth [9], Krenn and Wagner [10] and Lehr, Shallit and Tromp [11]).

Proposition 4.4. *If $*$ is the arithmetic mean on $\mathbb{R}$ or the geometric/harmonic mean on $\mathbb{R}^+$ then $s_n^{\text{ac}} \mathbf{G} =$ equals the number of ways to write 1 as an ordered sum of n powers of 2 for all $n \geq 1$.*

Proposition 4.5. *For the rock-paper-scissors operation $*$ defined on $\{\text{rock}, \text{paper}, \text{scissors}\}$ by $x * y = y * x$: x if x beats y or $x = y$ (rock beats scissors, scissors beat paper, and paper beats rock) we have $s_n^{\text{ac}} \mathbf{G} = 2n - 2 \cdot 2^{n-1} / (n - 1)!$ and $s_n^{\text{a}} \mathbf{G} = C_{n-1}$ for all $n \geq 1$.*

We next study nonassociative commutative groupoids with neutral elements. An example is the Jordan algebras of $n \times n$ self-adjoint matrices over $\mathbb{R}, \mathbb{C}$, or $\mathbb{H}$ (the algebra of quaternions) with a product defined by $x \circ y = (xy + yx) / 2$. The identity matrix I_n is the neutral element for this commutative algebra.

Theorem 4.6. *Let $\mathbf{G} = (G, *)$ be a commutative groupoid with neutral element e . Then either*

- (i) $\mathbf{G}$ is associative, in which case $s_n^{\text{a}} \mathbf{G} = s_n^{\text{ac}} \mathbf{G} = 1$ for all $n \geq 1$, or
- (ii) $s_n^{\text{a}} \mathbf{G} = C_{n-1}$ and $s_n^{\text{ac}} \mathbf{G} = 2n - 2 \cdot 2^{n-1} / (n - 1)!$ for all $n \geq 1$.

5 Anticommutative algebras

We now turn our attention to ac-spectra of bilinear products in algebras over a field. An algebra over a field $\mathbb{F}$ of characteristic not 2 is *anticommutative* if it satisfies the identity $xy \approx -yx$, which implies the identity $xx \approx 0$ since $xx \approx -xx$.

Given an anticommutative algebra over a field, we can turn the product $*$ into a commutative bilinear product $\otimes$ as follows. Let A be the universe of the algebra. Let g be any choice function on the collection $C : \{\{a, -a\} \mid a \in A\}$ and let $f : A \rightarrow C$, $f a : \{a, -a\}$. (Recall that a *choice function* on a collection C of subsets of some base set X is a mapping $g : C \rightarrow X$ such that $g S \in S$ for every $S \in C$. Note that any map f arising in this way is *even*, i.e., it satisfies $f a = f -a$ for all $a \in A$.) Now we can fix a basis B of the vector space A , and for all basis vectors $a, b \in B$, we define $a \otimes b : g f a * b$. This partial operation extends to a commutative bilinear product on A . (It is well known that any partial operation on A with domain B extends in a unique way to a bilinear product on A , and a bilinear product is commutative if and only if its restriction to the basis is commutative.) Such a product $\otimes$ will be referred to as a *commutative version* of $*$.

Theorem 5.1. *Let $*$ be the bilinear product of an anticommutative algebra over a field with a commutative version $\otimes$ satisfying $s^{\otimes} / \pm t^{\otimes}$ for any terms $s, t \in F_n$ with T_s^u / T_t^u . Then*

- (i) $s_n^{\text{ac}} \otimes = 2n - 2 ! / 2^{n-1} n - 1 !$ and $s_n^a \otimes = C_{n-1}$ for all $n \geq 1$;
- (ii) $s_n^{\text{ac}} * = 2s_n^{\text{ac}} \otimes$ for all $n \geq 2$ and $s_n^a * = C_{n-1}$ for all $n \geq 1$.

Since the cross product is anticommutative, we can determine its ac-spectrum by using a commutative operation associated with it. Define $\bowtie$ on a three-dimensional real vector space V with a basis $\{u, v, w\}$ by letting $x \bowtie x : 0$ for all $x \in \{u, v, w\}$ and $x \bowtie y : z$ for all distinct $x, y \in \{u, v, w\}$, where $z \in \{u, v, w\} \setminus \{x, y\}$, and extending this bilinearly from $\{u, v, w\}$ to V . This operation occurs in recent studies of the Norton algebras of certain distance regular graphs and it is commutative and totally nonassociative [7, Example 3.11, Remark 5.10].

Corollary 5.2. *For the cross product $\times$ on $\mathbb{R}^3$ and its commutative version $\bowtie$, we have*

- (i) $s_n^{\text{ac}} \bowtie = 2n - 2 ! / 2^{n-1} n - 1 !$ for all $n \geq 1$,
- (ii) $s_n^{\text{ac}} \times = 2s_n^{\text{ac}} \bowtie = 2n - 2 ! / 2^{n-2} n - 1 !$ for all $n \geq 2$, and
- (iii) $s_n^a \times = s_n^a \bowtie = C_{n-1}$ for all $n \geq 1$.

Now we study Lie algebras. A triple e, f, h of nonzero elements of a Lie algebra is called an $\mathfrak{sl}_2$ -triple if $e, f = h, h, e = 2e$, and $h, f = -2f$. It is well known that $\mathfrak{sl}_2$ -triples exist in every semisimple Lie algebra over a field of characteristic zero.

Corollary 5.3. *Let $\mathbf{L}$ be a Lie algebra over a field of characteristic distinct from 2 with an $\mathfrak{sl}_2$ -triple. For the Lie bracket $-, -$ of $\mathbf{L}$, it holds that $s_n^{\text{ac}} -, - = 2n - 2 ! / 2^{n-2} n - 1 !$ for all $n \geq 2$ and $s_n^a -, - = C_{n-1}$ for all $n \geq 1$.*

6 Totally nonassociative operations

In this section we focus on the ac-spectra of some totally nonassociative operations that are not commutative or anticommutative. Recall that a binary operation $*$ is said to be *totally nonassociative* if $s_n^a * \leq C_{n-1}$ for all $n \geq 1$. The arithmetic, geometric, and harmonic means, the cross product on $\mathbb{R}^3$, and the Lie brackets of Lie algebras over fields of characteristic distinct from 2 with an $\mathfrak{sl}_2$ -triple are all totally nonassociative and their ac-spectra have been determined in earlier sections. There are many other examples of totally nonassociative operations [2, 7]. We will study the exponentiation, the implication, and the negated disjunction (NOR) in this section.

Recall that any binary operation $*$ satisfies $s_n^{ac} * \leq n!C_{n-1}$ for all $n \geq 1$, and the equality is achieved by the free groupoid on one generator (Proposition 3.1). We show that if $*$ is the exponentiation then $s_n^{ac} *$ is strictly less than this upper bound.

Proposition 6.1. *Let $\mathbf{G} = (G, *)$ be a groupoid satisfying the identity $xy \cdot z \approx xz \cdot y$. If $s, t \in T X_\omega$ are linear terms such that the corresponding ordered labeled trees P_s and P_t have equal underlying unordered trees, i.e., $P_s^u = P_t^u$, then $s^{\mathbf{G}} = t^{\mathbf{G}}$. (See Section 2 for definitions.) Consequently, $s_n^{ac} \mathbf{G} \leq n^{n-1}$. Moreover, if the equality holds, then $s_n^a \mathbf{G} = C_{n-1}$.*

Proposition 6.2. *For $\mathbf{G} = (\mathbb{R}_{\geq 0}, *)$, where $*$ is the exponentiation operation defined by $a * b : a^b$ for all $a, b \in \mathbb{R}_{\geq 0}$, we have $s_n^{ac} \mathbf{G} = n^{n-1}$.*

Next we study the implication $\rightarrow$ defined on $\{0, 1\}$ by $x \rightarrow y : 0$ if $x, y = 1, 0$ or $x \rightarrow y : 1$ otherwise.

Proposition 6.3. *For $\mathbf{G} = (\{0, 1\}, \rightarrow)$, we have $s_n^{ac} \mathbf{G} = n^{n-1}$.*

Now we study the ac-spectrum of the groupoid $\mathbf{G} = (\{0, 1\}, \downarrow)$, where $\downarrow$ is the *negated disjunction* (NOR), defined by the rule $x \downarrow y = 1$ if and only if $x = y = 0$.

Proposition 6.4. *For $\mathbf{G} = (\{0, 1\}, \downarrow)$, we have $s_n^{ac} \mathbf{G} = (2n - 2)! / 2^{n-1} (n - 1)!$.*

7 Depth equivalence relations

In this section we study binary operations $*$ satisfying the property that two full linear terms agree on $*$ if and only if their corresponding binary trees are equivalent with respect to certain attributes related to the depths of the leaves.

A groupoid $\mathbf{G} = (G, *)$ and the corresponding binary operation $*$ are said to be *right k -associative* if $\mathbf{G}$ satisfies the identity $x_1 x_2 \cdots x_{k-1} \cdot_R x_k \cdot_2 \approx x_1 x_2 \cdots x_{k-2} \cdot_R (x_{k-1} \cdot_2 x_k)$, where $\cdots \cdot_R$ is a shorthand for the rightmost bracketing of the variables occurring between the square brackets, e.g., $x_1 x_2 \cdots x_{k-1} \cdot_R = x_1 x_2 \cdots x_{k-1} \cdot_2 \cdots$. Typical examples of k -associative operations are the ones defined by $a * b : a \cdot_\omega b$ for all $a, b \in \mathbb{C}$, where

$\omega = e^{2\pi i/k}$ is a k -th primitive root of unity. This reduces to addition and subtraction when $k = 1, 2$, respectively. One can also define the *left k -associativity* similarly. The left or right k -associativity becomes the usual associativity when $k = 1$.

Previous work [5] showed that the equivalence relation on binary trees induced by the left k -associativity is the same as the congruence relation on the left depth sequences of binary trees modulo k ; this is called the *k -left-depth-equivalence relation* by the second author. The number of equivalence classes is called the *k -modular Catalan number*, which counts many restricted families of Catalan objects and has interesting closed formulas [5]. Of course, the right k -associativity corresponds to the *k -right-depth-equivalence relation*, whose equivalence classes are also counted by the k -modular Catalan number.

Now we consider a stronger form of the right k -associativity. The *k -right-depth-equivalence relation* extends immediately from binary trees with unlabeled leaves to ones with labeled leaves. Let T and T' be binary trees with n leaves labeled by $x_1, \dots, x_n$ (in an arbitrary order). We say that T and T' are *k -right-depth-equivalent* if $\rho_T(x_i) \equiv \rho_{T'}(x_i) \pmod k$ for all $i \in [n]$, i.e., the right-depth sequences ρ_T and $\rho_{T'}$ are componentwise congruent modulo k . Suppose that a binary operation $*$ satisfies the property that any two full linear terms agree on $*$ if and only if T_s and T_t are k -right-depth-equivalent, i.e.,

$$\forall s, t \in F_n, s^* = t^* \iff \rho_{T_s}(x_i) \equiv \rho_{T_t}(x_i) \pmod k, \quad i = 1, 2, \dots, n. \quad (7.1)$$

It is clear that such a binary operation $*$ must be k -right-associative. The above example $a * b : a = e^{2\pi i/k} b$ satisfies property (7.1) and another example is given by $f * g : xf = yg$ for all $x, y \in \mathbb{C} \mid x, y \neq y^k - 1$ [6]. The associative spectrum of these examples is given by the k -modular Catalan numbers mentioned above. We determine the ac-spectrum $s_n^{\text{ac}} *$ of a binary operation $*$ satisfying property (7.1). If $k = 1$ then we clearly have $s_n^{\text{ac}} = 1$ for all $n \geq 1$. Thus we assume $k \geq 2$ in the remainder of this section.

For $k = 2$ we are looking at subtraction. We have $s_n^{\text{ac}} = 2^n - 2$ for $n \geq 2$ since the term operations in F_n^- are precisely the operations of the form $a_1, \dots, a_n \mapsto \pm a_1 \pm a_2 \cdots \pm a_n$ with at least one plus sign and at least one minus sign. For $k \geq 2$ we need to use the *Stirling number of the second kind* $S(n, k)$, which counts partitions of the set $[n] = \{1, 2, \dots, n\}$ into k (unordered) blocks.

Theorem 7.1. *Let $*$ be a binary operation satisfying property (7.1) with $k \geq 2$. Then*

$$s_n^{\text{ac}} = k! S(n, k) = n \sum_{0 \leq i \leq k-2} i! S(n-1, i), \quad \forall n \geq 1.$$

Remark 7.2. When $k = 3$ we have $n \sum_{1 \leq i \leq k-2} i! S(n-1, i) = n$ for all $n \geq 2$. When $k = 4$, the sequence $n \sum_{1 \leq i \leq k-2} i! S(n-1, i)$ is recorded in OEIS [16, A058877] and has simple closed formulas: $n2^{n-1} - n \sum_{1 \leq j \leq n} (n-2-j) 2^{n-j-1} = \sum_{1 \leq j \leq n-1} \binom{n}{j} (n-j)$.

Recently, Hein and the first author [6] generalized the k -associativity to the *k, ℓ -associativity*, based on the example $f * g : xf = yg$ for all $x, y \in \mathbb{C} \mid x^k = 1, y^\ell = 1$,

which satisfies the following for $i = 1, 2, \dots, n$.

$$\forall s, t \in F_n, s^* = t^* \iff \delta_{T_s} x_i \equiv \delta_{T_t} x_i \pmod{k}, \quad \rho_{T_s} x_i \equiv \rho_{T_t} x_i \pmod{\ell} \quad (7.2)$$

For $*$ satisfying the above property (7.2), computations show that $s_n^{\text{ac}} *$ is not in OEIS.

Next, define $a * b : e^{2\pi i/\ell} a e^{2\pi i/k} b$ for all $a, b \in \mathbb{C}$, which generalizes the example $a * b : a e^{2\pi i/k} b$ mentioned earlier. When $k = \ell$, one sees that two full linear terms agree on $*$ if and only if their corresponding binary trees are k -depth-equivalent, i.e.,

$$\forall s, t \in F_n, s^* = t^* \iff d_{T_s} x_i \equiv d_{T_t} x_i \pmod{k}. \quad (7.3)$$

Further generalizations of depth equivalence were studied recently by the second author.

For $k = 2$, the resulting operation is the *double minus operation* $a \ominus b : -a - b$. The first author, Mickey, and Xu [8] showed that $s_n^a \ominus$ coincides with an interesting sequence in OEIS [16, A000975]. We show that $s_n^{\text{ac}} \ominus$ (or the ac-spectrum of any binary operation satisfying property (7.3) with $k = 2$) agrees with the well-known *Jacobsthal sequence* [16, A001045]. For $k \geq 3$, computations show that neither $s_n^a *$ nor $s_n^{\text{ac}} *$ occurs in OEIS.

Theorem 7.3. *For $n \geq 1$ we have $s_n^{\text{ac}} \ominus = 2^n - (-1)^n / 3$.*

8 Remarks and questions

Csákány and Waldhauser [2, Section 4] examined the associative spectra of all two-element groupoids (up to isomorphism). The ac-spectra of two-element groupoids are given in Example 4.2, Proposition 6.3 and Proposition 6.4. As possible directions for further research, one could study the associative spectra and the ac-spectra of groupoids with three elements or groupoids satisfying some properties weaker than associativity, such as alternative, flexible, or power associative groupoids. We already studied the Jordan algebra in Section 4, which is commutative (hence flexible), power associative, but not associative. Another example is the Okubo algebra, which is flexible and power associative but not associative nor alternative.

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Automorphisms of undirected Bruhat graphs

Christian Gaetz^{*1}, and Yibo Gao^{†2}

¹Department of Mathematics, Cornell University, Ithaca, NY

²Department of Mathematics, University of Michigan, Ann Arbor, MI

Abstract. The *undirected Bruhat graph* $\Gamma_{u,v}$ has the elements of the Bruhat interval u, v as vertices, with edges given by multiplication by a reflection. Famously, $\Gamma_{e,v}$ is regular if and only if the Schubert variety X_v is smooth, and this condition on v is characterized by pattern avoidance. In this work, we classify when $\Gamma_{e,v}$ is *vertex-transitive*; surprisingly this class of permutations is also characterized by pattern avoidance and sits nicely between the classes of smooth permutations and self-dual permutations. This leads us to a general investigation of automorphisms of $\Gamma_{u,v}$ in the course of which we show that *special matchings*, which originally appeared in the theory of Kazhdan–Lusztig polynomials, can be characterized, for the symmetric and right-angled groups, as certain $\Gamma_{u,v}$ -automorphisms which are conjecturally sufficient to generate the orbit of e under $\text{Aut } \Gamma_{e,v}$.

Keywords: Bruhat order, Bruhat graph, vertex transitive, Schubert variety, smooth

1 Introduction

The (*directed*) *Bruhat graph* $\hat{\Gamma}$ of a Coxeter group W is the directed graph with vertex set W and directed edges $w \rightarrow wt$ whenever $\ell(wt) > \ell(w)$ and t is a reflection. Write $\hat{\Gamma}_{u,v}$ for its restriction to a Bruhat interval $u, v \subset W$, and simply $\hat{\Gamma}_v$ for its restriction to e, v . These graphs appear ubiquitously in the combinatorics of Coxeter groups and Bruhat order [14], the topology of flag, Schubert, and Richardson varieties as the GKM-graph for the natural torus action [17, 18], and in the geometry of these varieties and related algebra, for example in the context of Kazhdan–Lusztig polynomials [4, 5, 12, 13].

In all of these contexts, the directions of the edges, and sometimes additional edge labels, are centrally important. In this work, however, we study the associated *undirected* graphs $\Gamma_{u,v}$ and $\Gamma_v : \Gamma_{e,v}$. In particular, from the perspective of the undirected graph, it is very natural to study graph automorphisms (in contrast, the directed Bruhat graph $\hat{\Gamma}$ has very few automorphisms [23]), and these automorphisms end up having close connections to previous work on smooth Schubert varieties [20, 9], self-dual Bruhat intervals [15], Billey–Postnikov decompositions [2, 22], and special matchings [8].

^{*}crgaetz@gmail.com. C.G. was supported by a Klarman Postdoctoral Fellowship at Cornell University and by a National Science Foundation Postdoctoral Research Fellowship (DMS-2103121).

[†]gaoyibo@umich.edu.

1.1 Regular, vertex-transitive, and self-dual Bruhat graphs

The following well-known theorem, combining results of Lakshmibai–Sandhya [20] and Carrell–Peterson [9], helped establish the fundamental nature of both the Bruhat graph and pattern avoidance conditions in the combinatorial and geometric study of Schubert varieties.

Theorem 1 (Lakshmibai–Sandhya [20], Carrell–Peterson [9]). *The following are equivalent for a permutation w in the symmetric group $\mathfrak{S}_n$:*

- (S1) *the undirected Bruhat graph Γw is a regular graph,*
- (S2) *the permutation w avoids the patterns 3412 and 4231,*
- (S3) *the poset e, w is rank-symmetric, and*
- (S4) *the Schubert variety X_w is smooth.*

In light of (S3), it is natural to ask whether e, w is in fact self-dual as a poset when X_w is smooth. This turns out to not always be the case, but the smaller class of self-dual intervals also admits a nice characterization by pattern avoidance:

Theorem 2 (G.–G. [15]). *The following are equivalent for a permutation $w \in \mathfrak{S}_n$:*

- (SD1) *the Bruhat interval e, w is self-dual as a poset, and*
- (SD2) *the permutation w avoids the patterns 3412 and 4231 as well as 34521, 54123, 45321, and 54312.*

In our first main theorem here, we characterize by pattern avoidance those permutations w such that Γw is *vertex-transitive*; this characterization implies that this class of permutations sits nicely between the classes of self-dual permutations (Theorem 2) and smooth permutations (Theorem 1).

Theorem 3. *The following are equivalent for a permutation $w \in \mathfrak{S}_n$:*

- (VT1) *the undirected Bruhat graph Γw is a vertex-transitive graph,*
- (VT2) *the permutation w avoids the patterns 3412 and 4231 as well as 34521 and 54123.*

The proof of Theorem 3 is quite technical and is omitted in this extended abstract.

Since vertex-transitive graphs are necessarily regular, it is clear that the permutations from Theorem 3 are a subset of those from Theorem 1, and this is borne out by comparing conditions (S2) and (VT2). It is not at all conceptually clear, however, why the self-dual permutations of Theorem 2 should in turn be a subset of those from Theorem 3, even though this fact is easily seen by comparing conditions (VT2) and (SD2). A conceptual bridge between these two classes of permutations is provided by Conjecture 1.

Conjecture 1. Let $w \in \mathfrak{S}_n$ and let $\mathcal{O} = \{\varphi \in \Gamma_w \mid \varphi \in \text{Aut } \Gamma_w\}$ be the orbit of the identity under graph automorphisms of Γ_w , then $\mathcal{O} = \Gamma_{e,v}$ for some $v \leq w$.

Indeed, if $\Gamma_{e,w}$ is self-dual, then $w \in \mathcal{O}$, and so if [Conjecture 1](#) holds we must have $\mathcal{O} = \Gamma_{e,w}$. That is, Γ_w must be vertex-transitive.

In the course of the proof of [Theorem 3](#) and the refinement (see [Section 3](#)) of [Conjecture 1](#), we are led to consider certain automorphisms of $\Gamma_{u,v}$ arising from perfect matchings on the Hasse diagram of u,v . That these automorphisms are the same thing as the previously well-studied *special matchings* on u,v is the subject of our second main theorem.

1.2 Special matchings and Bruhat automorphisms

Special matchings (see the definition in [Section 2.3](#)) on Bruhat intervals were introduced [[6](#), [8](#)] because they can be used to define a recurrence for *Kazhdan–Lusztig R-polynomials* [[19](#)] which allows for the resolution of the *Combinatorial Invariance Conjecture* in the case of lower intervals e,w . These matchings are intended to generalize many of the combinatorial properties of the matching on W induced by multiplication by a simple reflection s . Special matchings on Bruhat intervals and related posets have since found several other combinatorial and topological applications and been generalized in several ways [[1](#), [7](#), [21](#)], and are completely classified on lower Bruhat intervals. [[11](#)].

In [Theorem 4](#) and [Conjecture 2](#) below we give a new characterization of special matchings of Bruhat intervals u,v in terms of automorphisms of $\Gamma_{u,v}$. This characterization is notable because it expresses the special matching condition, originally formulated as a condition only on Bruhat covers, as a condition on the global structure of the undirected Bruhat graph.

A Coxeter group W is called *right-angled* if every pair of simple generators either commutes or generates an infinite dihedral group.

Theorem 4. Let W be a right-angled Coxeter group or the symmetric group and let $u \leq v$ be elements of W . Then a perfect matching of the Hasse diagram of u,v is a special matching if and only if it is an automorphism of $\Gamma_{u,v}$.

Conjecture 2. [Theorem 4](#) holds for arbitrary Coxeter groups W .

1.3 Outline

In [Section 2](#), we cover background and definitions relating to Coxeter groups, Bruhat order and Bruhat graphs, Billey–Postnikov decompositions, and special matchings. In [Section 3](#) we give a more precise version of [Conjecture 1](#) in terms of *almost reducible decompositions* and some partial results towards resolving the conjecture. [Section 4](#) outlines

the proof of [Theorem 4](#). The proof of [Theorem 4](#) relies on a structural property of Bruhat order, the existence of upper bounds of *butterflies*, which may be of independent interest. Most of the proofs are omitted for this extended abstract, notably including those of [Theorem 3](#), [Theorem 7](#), [Proposition 4](#), [Lemma 1](#) and [Lemma 3](#). Readers may refer to [\[16\]](#) for additional details.

2 Background and definitions

2.1 Bruhat graphs and Bruhat order

The *directed Bruhat graph* $\widehat{\Gamma}$ of a Coxeter group W is the directed graph with vertex set W and directed edges $w \rightarrow wt$ whenever t is a reflection with $\ell(wt) > \ell(w)$. Note that, since T is closed under conjugation, the “left” and “right” versions of $\widehat{\Gamma}$ in fact coincide. The (*undirected*) *Bruhat graph* Γ is the associated simple undirected graph. The directed graph $\widehat{\Gamma}$ is much more commonly considered in the literature, and often called “the Bruhat graph” but, since our focus in this work is on the undirected graph Γ , when directedness is not specified we mean the undirected graph.

The (*strong*) *Bruhat order* $W, \leq$ is the partial order on W obtained by taking the transitive closure of the relation determined by $\widehat{\Gamma}$. We write $[u, v]$ for the interval $\{w \in W \mid u \leq w \leq v\}$ in Bruhat order. For $u \leq v$, we write $\widehat{\Gamma}_{[u, v]}$ and $\Gamma_{[u, v]}$ for the restrictions of $\widehat{\Gamma}, \Gamma$ to the vertex set $[u, v]$; when u is the identity element e , we sometimes write simply $\widehat{\Gamma}_v$ and Γ_v .

Fundamental properties of Bruhat order can be found in [\[3\]](#).

For $w \in W$, we write $\text{Supp } w$ for the *support* of w : the set of simple reflections appearing in some (equivalently, every) reduced word for w . We say the element $w \in W$ has a *disjoint support decomposition* if it may be expressed as a nontrivial product $w = w'w''$ with $\text{Supp } w' \cap \text{Supp } w'' = \emptyset$ (note that, in this case, we have $w' = w_J$ and $w'' = w_{J'} w_J$ with $J' \cap \text{Supp } w'' = \emptyset$).

Proposition 1. *Let $w = w'w''$ be a disjoint support decomposition, then:*

$$\begin{aligned} \widehat{\Gamma}_w &\sim \widehat{\Gamma}_{w'} \times \widehat{\Gamma}_{w''}, \\ \Gamma_w &\sim \Gamma_{w'} \times \Gamma_{w''}, \\ e, w &\sim e, w' \times e, w''. \end{aligned}$$

In each case, the isomorphism is given by group multiplication.

[Proposition 1](#) implies that when $w = w'w''$ is a disjoint support decomposition, $\Gamma_w, \widehat{\Gamma}_w$, and e, w have automorphisms induced by automorphisms for w', w'' .

2.2 Billey–Postnikov decompositions

Definition 1 (Billey–Postnikov [2], Richmond–Slofstra [22]). *Let W be a Coxeter group and $J \subseteq S$, the parabolic decomposition $w = w^J w_J$ of w is a Billey–Postnikov decomposition or BP-decomposition if*

$$\text{Supp } w^J \cap J \subseteq D_L w_J .$$

BP-decompositions were introduced by Billey and Postnikov in [2] in the course of their study of pattern avoidance criteria for smoothness of Schubert varieties in all finite types. The following characterizations of BP-decompositions will be useful to us.

Proposition 2 (Richmond–Slofstra [22]). *For $w \in W$ and $J \subseteq S$, the following are equivalent:*

1. $w = w^J w_J$ is a BP-decomposition,
2. the multiplication map $(e, w^J \cap W^J) \times (e, w_J) \rightarrow (e, w)$ is a bijection,
3. w_J is the maximal element of $W_J \cap (e, w)$.

2.3 Special matchings

The *Hasse diagram*, denoted $H(P)$, of a poset P is the undirected graph with vertex set P and edges $\{x, y\}$ whenever $x <_P y$ is a cover relation in P . Note that the Hasse diagram $H(W)$ of Bruhat order on W is a (non-induced) subgraph of Γ . A *perfect matching* of a graph G is a fixed-point-free involution $M : G \rightarrow G$ such that $\{x, M(x)\}$ is an edge of G for all $x \in G$.

Definition 2 (Brenti [6], Brenti–Caselli–Marietti [8]). *A perfect matching M on the Hasse diagram of a poset P is a special matching if, for every cover relation $x <_P y$, either $M(x) = y$ or $M(x) <_P M(y)$.*

For $u \leq v \in W$ it is not hard to check, using the Lifting Property, that for $s \in D_L v \setminus D_L u$ (resp. $s \in D_R v \setminus D_R u$) left (resp. right) multiplication by s determines a special matching of the Hasse diagram $H(u, v)$. In fact, this motivated the definition of special matching [6]. Proposition 3 below, a special case of the classification by Caselli and Marietti [10, 11] of special matchings of lower Bruhat interval, observes that *middle multiplication* is also a special matching for lower intervals.

Proposition 3 (Special case of Theorem 5.1 in [10]). *Suppose $w = w^J w_J$ is a BP-decomposition of w and in addition we have $\text{Supp } w^J \cap \text{Supp } w_J = \{s\}$, then the middle multiplication map*

$$\phi : x \mapsto x^J s x_J,$$

is a special matching of (e, w) .

2.4 From special matchings to Bruhat automorphisms

The following result of Waterhouse shows that $\widehat{\Gamma}$ has no nontrivial automorphisms as a directed graph.

Theorem 5 (Waterhouse [23]). *Let W be an irreducible Coxeter group which is not dihedral, then $\text{Aut } W, \leq$ (equivalently, $\text{Aut } \widehat{\Gamma}$) is generated by the graph automorphisms of the Dynkin diagram of W and the group inversion map on W .*

In this paper we study the much richer sets of automorphisms of Γ and particularly of its subgraphs $\Gamma_{u,v}$. Although it is stated only for lower intervals e, v , the proof of Theorem 10.3 in [8] also applies to general intervals u, v and yields Theorem 6 below:

Theorem 6 (Theorem 10.3 of [8]). *Let $u \leq v$ be elements of a Coxeter group W . Any special matching M of the Hasse diagram $H_{u,v}$ is an automorphism of $\Gamma_{u,v}$.*

Corollary 1. *Let $w \in W$. If $s \in D_L w$ (resp. $D_R w$) then left (resp. right) multiplication by s is an automorphism of Γ_w . If $w = w^J w_J$ is a BP-decomposition of w with $\text{Supp } w^J \cap \text{Supp } w_J = \{s\}$, then middle multiplication by s is an automorphism of Γ_w .*

Theorem 6 provides one direction of Theorem 4 and Conjecture 2 for arbitrary Coxeter groups. This implies that special matchings on Bruhat intervals, although defined by a local condition (that is, a condition on cover relations), respect the global structure of Bruhat graphs. The reverse direction is the subject of Section 4.

3 Identity orbits in Bruhat graphs

We describe a more precise version of Conjecture 1, taking into account the automorphisms described in Section 2.4 and Theorem 3. In light of Proposition 1, it is sufficient to consider permutations $w \in \mathfrak{S}_n$ which have full support and do not admit a disjoint support decomposition; we call such permutations *Bruhat irreducible*.

Definition 3. *A Bruhat irreducible permutation $w \in \mathfrak{S}_n$ is almost reducible at J, i if $w^J w_J$ is a BP-decomposition with $\text{Supp } w^J \cap J = \{s_i\}$ and $s_i \notin D_L w, D_R w$.*

Definition-Proposition 1. *If a Bruhat irreducible $w \in \mathfrak{S}_n$ is almost reducible at J, i , then $J = \{s_1, \dots, s_i\}$ or $\{s_i, \dots, s_{n-1}\}$. We say it is left-almost-reducible at i if $J = \{s_1, \dots, s_i\}$, and right-almost-reducible at i if $J = \{s_i, \dots, s_{n-1}\}$.*

Definition 4. *For a Bruhat irreducible permutation $w \in \mathfrak{S}_n$, let*

$$\{i_1 < \dots < i_k\} = \{i \mid w \text{ is right-almost-reducible at } i\}$$

and define $A_R w := s_{i_1} \dots s_{i_k}$. Similarly, let $\{j_1 < \dots < j_t\}$ be the set of j at which w is left-almost-reducible and define $A_L w := s_{j_t} \dots s_{j_1}$.

Theorem 7. *Let w be Bruhat irreducible. Then the following three elements commute pairwise:*

$$A_R w, A_L w, w_0 D_L w \cap D_R w.$$

The following is a strengthened version of [Conjecture 1](#).

Conjecture 3. *Let $w \in \mathfrak{S}_n$ be Bruhat irreducible and let $\mathcal{O}$ denote the orbit of e under graph automorphisms of Γw . Define*

$$v w := w_0 D_L w \cdot A_R w \cdot w_0 D_L w \cap D_R w \cdot A_L w \cdot w_0 D_R w,$$

then $\mathcal{O} = e, v w$.

Proposition 4. *Let $w \in \mathfrak{S}_n$ be Bruhat irreducible and such that Γw is vertex-transitive, then $v w = w$, so [Conjecture 3](#) holds in this case.*

The following proposition shows that the element $v w$ is indeed in the identity orbit of Γw . An automorphism of Γw sending e to $v w$ may be obtained by composing various left, right, and middle multiplication automorphisms (see [Section 2.4](#)).

Proposition 5. *Let $w \in \mathfrak{S}_n$ be Bruhat irreducible and let $\mathcal{O}$ be the orbit of e under graph automorphisms of Γw , then $v w \in \mathcal{O}$.*

4 From Bruhat automorphisms to special matchings via butterflies

In [Theorem 8](#) below we give a converse to [Theorem 6](#) for certain Coxeter groups. [Theorem 6](#) and [Theorem 8](#) together imply [Theorem 4](#).

Theorem 8. *Let $u \leq v$ be elements of a Coxeter group W which is right-angled or a symmetric group, then any perfect matching of $H(u, v)$ which is an automorphism of $\Gamma(u, v)$ is a special matching.*

The proof of [Theorem 8](#) relies on [Lemma 1](#), a structural property of Bruhat order involving butterflies.

Definition 5. *We say that elements x_1, x_2, y_1, y_2 of a Coxeter group W form a butterfly if $x_1 \leq y_1, y_2$ and $x_2 \leq y_1, y_2$.*

The butterfly structures are essential to the analysis of Bruhat automorphisms and special matchings, and are of interest on their own.

Lemma 1. *Let W be a Coxeter group which is right-angled or the symmetric group, and suppose that $x_1, x_2, y_1, y_2 \in u, v$ form a butterfly. Then there is an element $z \in u, v$ with $y_1, y_2 \leq z$.*

Proof of Theorem 8 given Lemma 1. Let $u \leq v$ be elements of a Coxeter group W which is right-angled or the symmetric group, and let M be a perfect matching of $H_{u,v}$ which is an automorphism of $\Gamma_{u,v}$. Suppose that M is not a special matching; since M is a $\Gamma_{u,v}$ -automorphism, the violation of the special matching property must consist of elements $x \leq y$ with $M y \leq M x$. Choose x, y so that y has maximal length among all such violations in u, v .

Now, note that $x, M y, y, M x$ form a butterfly, so by Lemma 1 there exists an element $z \in u, v$ with $y, M x \leq z$. We must have $M z \geq z$, for otherwise each of $y, M x$, and $M z$ would each cover both x and $M y$, but this substructure cannot occur in Bruhat order of a Coxeter group (see Theorem 3.2 of [8]). Since height-two intervals in Bruhat order are diamonds (see Chapter 2 of [3]), there exists an element $w \leq z$ with $y \leq w \leq M z$.

Suppose that $M w \leq w$, then since M is an automorphism of the Bruhat graph we must have $M w \leq z$ and $M y \leq M w$. Now, since $y \leq z$, we know $M y \rightarrow M z$ in $\hat{\Gamma}_{u,v}$, but the height-three interval $M y, M z$ contains at least three elements— $y, M w$, and $M x$ at height one, contradicting Proposition 3.3 of [14].

We conclude that $w \leq M w$. However this too is a contradiction, since $w \leq M z$ is a violation of the special matching condition with $\ell M z \geq \ell y$. Thus M must be a special matching. $\square$

We conjecture that a slight weakening of Lemma 1 holds for arbitrary Coxeter groups.

Conjecture 4. *Let W be any Coxeter group, and suppose that the elements $x_1, x_2, y_1, y_2 \in u, v$ form a butterfly. Then there is an element $z \in u, v$ with $y_1, y_2 \leq z$ or with $z \leq x_1, x_2$.*

Remark 1. *The weakening of Lemma 1 conjectured for general Coxeter groups in Conjecture 4 is necessary even for finite Coxeter groups. For example, the finite Coxeter group of type F_4 has a butterfly:*

$$\begin{aligned} x_1 &= s_2 s_3 s_4 s_2 s_3 s_1 s_2 s_3 s_4 s_3 s_2 s_3 s_1 s_2 s_3 s_1 s_2, \\ x_2 &= s_3 s_2 s_3 s_4 s_3 s_2 s_3 s_1 s_2 s_3 s_4 s_2 s_3 s_1 s_2 s_3 s_2, \\ y_1 &= s_2 s_3 s_1 s_2 s_3 s_4 s_3 s_2 s_3 s_1 s_2 s_3 s_4 s_2 s_3 s_1 s_2 s_3, \\ y_2 &= s_3 s_2 s_3 s_4 s_3 s_2 s_3 s_1 s_2 s_3 s_4 s_2 s_3 s_1 s_2 s_3 s_1 s_2, \end{aligned}$$

which has a lower bound $z = s_2 s_3 s_4 s_2 s_3 s_1 s_2 s_3 s_4 s_3 s_2 s_3 s_1 s_2 s_3 s_2 \leq x_1, x_2$ but no upper bound $y_1, y_2 \leq z'$. See Figure 1 for a Bruhat interval that contains $z \leq x_1, x_2 \leq y_1, y_2$.

We end with a few more structural results concerning butterflies, in the setting of finite Weyl groups.

Lemma 2. *In a finite Weyl group, $w \leq ws_\alpha$ if and only if $\alpha \notin \text{Inv}_R w$ and there does not exist $\beta_1, \beta_2 \in \text{Inv}_R w$ such that $\beta_2 = -s_\alpha \beta_1$. Moreover, if $w \leq ws_\alpha$ and $\beta \in \Phi$ satisfies $s_\alpha \beta \in \Phi^-$, then $\beta \in \text{Inv}_R w$ if and only if $\beta \in \text{Inv}_R ws_\alpha$.*

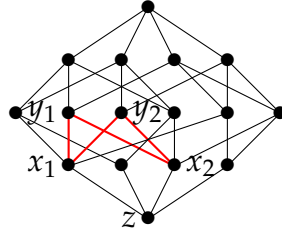


Figure 1: A butterfly in F_4 which does not admit an upper bound.

Lemma 3. *Let W be a finite Weyl group of simply-laced type, and let $x_1, x_2 \leq y_1, y_2$ form a butterfly. Then there exists $u \leq x_1, x_2$ and $z \geq y_1, y_2$ in W .*

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Extremal tensor products of Demazure crystals are direct sums of Demazure crystals

Sami Assaf^{*1}, Anne Dranowski^{†1}, and Nicolle González^{‡2}

¹Department of Mathematics, University of Southern California, Los Angeles, USA

²Department of Mathematics, University of California, Berkeley, USA

Abstract. We give a new necessary and sufficient condition for when tensor products of Demazure crystals decompose as direct sums of Demazure crystals. Our local criterion depends on the string property which Demazure crystals, and more generally, extremal crystals, exhibit. Our characterization implies that tensor products of Demazure crystals are direct sums of Demazure crystals if and only if they are extremal.

Keywords: Demazure crystals, extremal crystals, excellent filtrations, tensor products.

1 Introduction

In his study of the representations of quantum groups $U_q(\mathfrak{g})$ for $\mathfrak{g}$ a complex semisimple Lie algebra, Kashiwara [9], based on work of Lusztig [15], introduced *crystal bases* upon which, in the $q \rightarrow 0$ limit, the action of the Chevalley operators could be easily described. The crystal bases form the vertices of a *crystal graph*, a directed, colored graph with edges given by deformed Chevalley operators. The combinatorial structure of the crystal encodes the highest weight theory of the corresponding $U_q(\mathfrak{g})$ -modules. Thus to any irreducible highest weight representation $V(\lambda)$, we associate the highest weight crystal $\mathcal{B}(\lambda)$ whose character agrees with the Weyl character of the module.

Given the monoidal structure of the category of $U_q(\mathfrak{g})$ -modules, Kashiwara defined a crystal structure on the set $\mathcal{B}_1 \otimes \mathcal{B}_2$ which aligns with the tensor product of the corresponding modules. In particular, the fact that $V(\lambda) \otimes V(\mu)$ admits a *good filtration*, i.e. a filtration by Weyl modules, is reflected in the fact that $\mathcal{B}(\lambda) \otimes \mathcal{B}(\mu)$ decomposes as a direct sum of highest weight crystals.

Demazure [4] considered a family of submodules generated by *extremal weight elements* under the Borel subalgebra, known eponymously as *Demazure modules*. The associated *Demazure crystals*, introduced by Littelmann [14] and generalized by Kashiwara [10], arise as truncations of the crystals for $U_q(\mathfrak{g})$ -modules. As in the classical case, Demazure crystals encode the combinatorial structure of the corresponding Demazure

^{*}shassaf@usc.edu S.A. supported by Simons Award 953878.

[†]dranowsk@usc.edu

[‡]nicolle@math.berkeley.edu

modules. Hence, each Demazure module $V_w(\lambda)$ has an associated crystal $B_w(\lambda)$, indexed by a highest weight λ and an element w of the Weyl group W of $\mathfrak{g}$.

Filtrations by Demazure modules are known as *excellent filtrations*. Unlike with tensor products of Weyl modules, tensor products of Demazure modules do not always admit excellent filtrations [8]. Thus a natural question to consider is when can $V_w(\lambda) \otimes V_u(\nu)$ be filtered by Demazure modules.

In this paper we answer this question from a crystal theoretic perspective by considering a larger family of subcrystals, which we call *extremal subsets*. Extremal subsets are characterized by the string property which states that every i -string of the crystal which intersects the subset is either entirely contained in the subset or intersects in only the top element. Kashiwara [10] showed every Demazure crystal is extremal, though the converse does not hold. We show that tensor products of Demazure crystals $\mathcal{B}_w(\lambda) \otimes \mathcal{B}_u(\nu)$ decompose as sums of Demazure crystals if and only if $\mathcal{B}_w(\lambda) \otimes \mathcal{B}_u(\nu)$ is extremal. By studying tensor products of extremal subcrystals, we give a local criterion for when tensor products of Demazure crystals are extremal, thus giving a local characterization of precisely when $\mathcal{B}_w(\lambda) \otimes \mathcal{B}_u(\nu)$ decomposes as a sum of Demazure crystals.

Our results generalize work of Lakshimbai, Littelmann, and Magyar [13] and Joseph [6] in which they prove $\{u_\lambda\} \otimes \mathcal{B}_u(\nu)$ decomposes as a direct sum of Demazure crystals. Our local criterion also provides an alternative characterization to Kouno's global condition [12] for when $\mathcal{B}_w(\lambda) \otimes \mathcal{B}_u(\nu)$ remains Demazure. For full details, see [1].

2 Crystal graphs

Let $\mathfrak{g}$ be a complex semisimple Lie algebra. In this section, we review normal $\mathfrak{g}$ -crystals. For a thorough treatment of crystals, see [11].

2.1 Highest weight crystals

Let P be the weight lattice of $\mathfrak{g}$ and let I be the vertex set of the Dynkin diagram. For every $i \in I$ we have a simple root $\alpha_i \in P$ and a simple coroot $\alpha_i^\vee \in P^\vee = \text{Hom}_{\mathbb{Z}}(P, \mathbb{Z})$. Given $\lambda \in P$ and $\mu^\vee \in P^\vee$ we write $\langle \mu^\vee, \lambda \rangle$ for the integer obtained by the natural symmetric pairing on weights and coweights. Write W for the Weyl group generated by the set of simple reflections s_i associated to $\alpha_i^\vee \in P^\vee$ and P^+ for the set of dominant weights $\{\lambda \in P : \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0} \text{ for all } \alpha_i^\vee \in P^\vee\}$.

Definition 1. A (finite) normal $\mathfrak{g}$ -crystal is a nonempty set $\mathcal{B}$, together with *crystal operators* $e_i, f_i : \mathcal{B} \rightarrow \mathcal{B} \sqcup \{0\}$, a *weight map* $\text{wt} : \mathcal{B} \rightarrow P$, and *string operators* $\varepsilon_i(b) := \max \{k \in \mathbb{Z}_{\geq 0} \mid e_i^k(b) \in \mathcal{B}\}$ and $\varphi_i(b) := \max \{k \in \mathbb{Z}_{\geq 0} \mid f_i^k(b) \in \mathcal{B}\}$, such that for every $i \in I$ and for every $b, b' \in \mathcal{B}$:

- (1) $b' = e_i(b)$ if and only if $b = f_i(b')$ in which case $\text{wt}(b') = \text{wt}(b) + \alpha_i$;

$$(2) \quad \varphi_i(b) - \varepsilon_i(b) = \langle \alpha_i^\vee, \text{wt}(b) \rangle.$$

The finite-dimensional, irreducible, integrable representations of $U_q(\mathfrak{g})$ are naturally indexed by the integral dominant weights. For each $\lambda \in P^+$, let $\mathcal{B}(\lambda)$ denote the crystal for the irreducible highest weight representation $V(\lambda)$.

Given a highest weight crystal $\mathcal{B}$, the associated *crystal graph* is the directed, I -colored graph with vertex set $\mathcal{B}$ and with an i -edge from b to $f_i(b)$ provided the latter is nonzero.

A crystal is *connected* if its underlying (undirected) graph is connected. Henceforth, we refer to (elements of) crystals and (vertices of) their graphs interchangeably.

Example A. The *standard crystal* $\mathcal{B}(1, 0^{n-1})$ for $\mathfrak{sl}_n(\mathbb{C})$ has basis $\{i \mid i = 1, \dots, n\}$, weight map $\text{wt}(i) = (0^{i-1}, 1, 0^{n-i-1})$, and lowering operators $f_j(i) = i + 1$ if $j = i + 1$ and $f_j(i) = 0$ otherwise. We draw the crystal graph for $\mathcal{B}(1, 0^{n-1})$ as shown in Figure 1.

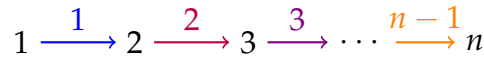


Figure 1: The $\mathfrak{sl}_n(\mathbb{C})$ -crystal $\mathcal{B}(1, 0^{n-1})$.

For any $i \in I$ and $X \subseteq \mathcal{B}$, let $\mathcal{F}_i(X) = \{f_i^m(x) \mid x \in X \text{ and } m \in \mathbb{Z}_{\geq 0}\} \setminus \{0\}$. For $s_{i_1} \cdots s_i$ a reduced expression for $w \in W$, let $\mathcal{F}_w(X) = \mathcal{F}_{i_1} \cdots \mathcal{F}_i(X)$. When $w = w_0$ is the longest element we write omit the subscript and write $\mathcal{F}(X)$.

Joseph [7] proves that the set $\mathcal{F}_w(X)$ is independent of the choice of reduced expression for w and so is well-defined. The sets $\mathcal{E}_i(X)$, $\mathcal{E}_w(X)$ and $\mathcal{E}(X)$ are similarly defined using raising operators.

An element $b \in \mathcal{B}$ is a *highest weight element* if $\mathcal{E}_i(\{b\}) = \{b\}$ for all i . Let b_λ denote the highest weight element of the irreducible highest weight crystal $\mathcal{B}(\lambda)$.

2.2 Demazure crystals

The Weyl group W is equipped with a partial order $\prec$ called *Bruhat order* defined on any $u, v \in W$ by $u \prec v$ if and only if there exists a reduced word for v which contain a reduced word for u as a subword. See [3] for a reference on Bruhat order.

Demazure crystals are subsets $\mathcal{B}_w(\lambda) \subseteq \mathcal{B}(\lambda)$ depending on a choice of $w \in W$. They were introduced by Littelmann who showed for classical $\mathfrak{g}$ that their characters are the characters of Demazure modules $V_w(\lambda)$ [5, 6].

Definition 2 ([10]). For $\lambda \in P^+$ and $w \in W$, the *Demazure crystal* $\mathcal{B}_w(\lambda)$ is

$$\mathcal{B}_w(\lambda) = \mathcal{F}_w(\{b_\lambda\}). \quad (2.1)$$

Kashiwara [10] generalized Littelmann construction to arbitrary $\mathfrak{g}$ and showed $\mathcal{B}_w(\lambda)$ satisfies the following properties.

- (1) $\mathcal{E}(\mathcal{B}_w(\lambda)) \subset \mathcal{B}_w(\lambda)$;
- (2) if $s_i w \prec w$, then $\mathcal{B}_w(\lambda) = \{f_i^m(b) \mid m \geq 0, b \in \mathcal{B}_{s_i w}(\lambda), e_i(b) = 0\} \setminus \{0\}$;
- (3) for any i -string S , $S \cap \mathcal{B}_w(\lambda)$ is either $\emptyset$ or S or $\{b\}$, where $b \in S$ and $e_i(b) = 0$.

For any $i \in I$, an i -string is any connected subset of a crystal closed under both $\mathcal{E}_i$ and $\mathcal{F}_i$. Equivalently, an i -string is a subset of the form $\mathcal{F}_i(\{b\})$ where $e_i(b) = 0$.

Demazure crystals are nested according to Bruhat order [10], i.e. $\mathcal{B}_v(\lambda) \subseteq \mathcal{B}_w(\lambda)$ whenever $v \prec w$. We tighten this result as follows.

Given $\lambda \in P^+$, let W_λ be the stabilizer subgroup of λ in W . The minimal (resp. maximal) length coset representatives of wW_λ are denoted by $\lfloor w \rfloor^\lambda$ (resp. $\lceil w \rceil^\lambda$).

Proposition 3. *Let $\lambda \in P^+$ and $v, w \in W$. Then $v \preceq \lceil w \rceil^\lambda$ if and only if $\mathcal{B}_v(\lambda) \subseteq \mathcal{B}_w(\lambda)$. Moreover, $\mathcal{B}_v(\lambda) = \mathcal{B}_w(\lambda)$ only when $v \in wW_\lambda$.*

Example B. Consider $\mathcal{B}_{s_2}(2, 2, 0) \subset \mathcal{B}(2, 2, 0)$ in Figure 2. Here $w = s_1$ and $\lambda = (2, 2, 0)$. Since $s_1 \in W_\lambda$, we have $\lceil w \rceil^\lambda = s_2 s_1 \succ s_1$, and so $\mathcal{B}_{s_2 s_1}(2, 2, 0) = \mathcal{B}_{s_2}(2, 2, 0)$. Likewise, since $\lceil w \rceil^\lambda \prec s_1 s_2 s_1$, we have $\mathcal{B}_{s_2 s_1}(2, 2, 0) \subsetneq \mathcal{B}_{s_1 s_2 s_1}(2, 2, 0) = \mathcal{B}_{s_1 s_2}(2, 2, 0) = \mathcal{B}(2, 2, 0)$.

2.3 Extremal crystals

Following work of the extremal authors [2], we consider subsets satisfying property (3).

Definition 4. A subset $X \subseteq \mathcal{B}(\lambda)$ is *extremal* if X is nonempty and for any i -string S of $\mathcal{B}(\lambda)$, $S \cap X$ is either $\emptyset$ or S or $\{b\}$ where $b \in S$ and $e_i(b) = 0$.

Notice any subset of $\mathcal{B}(\lambda)$ satisfying Kashiwara's property (3) necessarily satisfies property (1) as well. In particular, if $X \subset \mathcal{B}(\lambda)$ is extremal, then $\mathcal{E}X \subset X$, and so $b_\lambda \in X$.

As Kashiwara proves [10], all Demazure crystals are extremal subsets. The converse, however, is false. Not all extremal subsets are Demazure crystals.

Example C. Let $\mathfrak{g} = \mathfrak{sl}_3$ and $\lambda = (2, 2, 0)$. Then $X = \{b_\lambda, f_2(b_\lambda), f_2^2(b_\lambda), f_1 f_2(b_\lambda)\}$ (seen in the middle of Figure 2) is extremal, but not Demazure. In particular, $\mathcal{B}_{s_2}(2, 2, 0) \subsetneq X \subsetneq \mathcal{B}(2, 2, 0)$. Similarly, $Y = \{b_\lambda, f_2(b_\lambda), f_2^2(b_\lambda), f_1 f_2^2(b_\lambda), f_1^2 f_2^2(b_\lambda)\}$ is also an extremal subset of $\mathcal{B}(2, 2, 0)$ containing $\mathcal{B}_{s_1}(2, 2, 0)$ that is not Demazure.

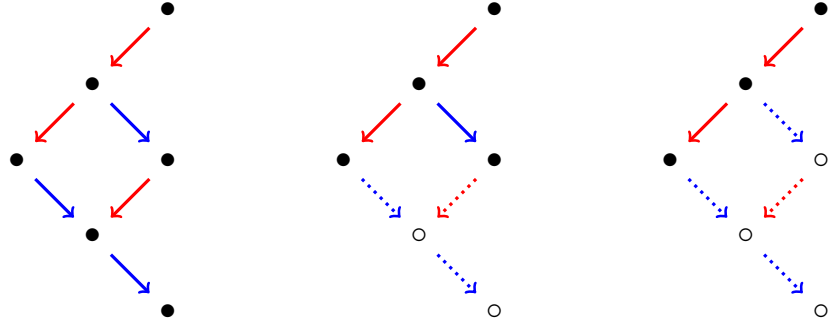


Figure 2: The $\mathfrak{sl}_3$ -crystal $B(2,2,0)$ (left), an extremal subset (middle), and the Demazure crystal $B_{s_2}(2,2,0)$ (right) with f_1 and f_2 depicted by blue and red arrows, respectively.

3 Tensor products of crystals

3.1 Kashiwara's tensor product rule

Given $\mathfrak{g}$ -crystals $\mathcal{B}_1$ and $\mathcal{B}_2$, the *direct sum* $\mathcal{B}_1 \oplus \mathcal{B}_2$ is their disjoint union with corresponding operators. Since any graph decomposes into the disjoint union of its connected components, every $\mathfrak{g}$ -crystal decomposes as a direct sum of highest weight crystals.

Definition 5. The *tensor product* $\mathcal{B}_1 \otimes \mathcal{B}_2$ has vertex set $\{b_1 \otimes b_2 \mid b_1 \in \mathcal{B}_1 \text{ and } b_2 \in \mathcal{B}_2\}$, crystal operator f_i defined by

$$f_i(b_1 \otimes b_2) = \begin{cases} f_i(b_1) \otimes b_2 & \text{if } \varepsilon_i(b_2) < \varphi_i(b_1), \\ b_1 \otimes f_i(b_2) & \text{if } \varepsilon_i(b_2) \geq \varphi_i(b_1), \end{cases}$$

e_i defined analogously, $\text{wt}_i(b) = \langle \alpha_i^\vee, \text{wt}(b) \rangle$, $\text{wt}(b_1 \otimes b_2) = \text{wt}(b_1) + \text{wt}(b_2)$, and $\varepsilon_i(b_1 \otimes b_2) = \max(\varepsilon_i(b_1), \varepsilon_i(b_2) - \text{wt}_i(b_1))$ and $\varphi_i(b_1 \otimes b_2) = \max(\varphi_i(b_2), \varphi_i(b_1) + \text{wt}_i(b_2))$.

Kashiwara [9] proves this tensor product is associative and noncommutative and proves $\mathcal{B}(\lambda) \otimes \mathcal{B}(\mu)$ is a crystal for $V(\lambda) \otimes V(\mu)$.

Example D. Consider the tensor product $\mathcal{B}(1,1,0) \otimes \mathcal{B}(1,0,0)$, where

$$\mathcal{B}(1,1,0) = a_1 \xrightarrow{\text{red}} a_2 \xrightarrow{\text{blue}} a_3 \quad \text{and} \quad \mathcal{B}(1,0,0) = b_1 \xrightarrow{\text{blue}} b_2 \xrightarrow{\text{red}} b_3.$$

Then, $\varphi_2(a_1) = \varphi_1(a_2) = \varepsilon_1(b_2) = \varepsilon_2(b_3) = 1$ and $\varphi_1(a_1) = \varphi_2(a_2) = \varepsilon_2(b_2) = \varepsilon_1(b_3) = 0$. Thus, as seen in Figure 3, $\mathcal{B}(1,1,0) \otimes \mathcal{B}(1,0,0)$ will decompose into two connected components with highest weights $(2,1,0)$ and $(1,1,1)$, respectively. Thus $\mathcal{B}(1,1,0) \otimes \mathcal{B}(1,0,0) \cong \mathcal{B}(2,1,0) \oplus \mathcal{B}(1,1,1)$, as expected from the decomposition of the tensor product of the corresponding modules.

3.2 Tensor products of Demazure crystals

The tensor product $\mathcal{B}_w(\lambda) \otimes \mathcal{B}_u(\mu)$ is not always a direct sum of Demazure crystals.

Example E. Consider the $\mathfrak{sl}_3$ -crystals $\mathcal{B}_{s_2}(1,1,0)$ and $\mathcal{B}_{s_1}(1,1,0)$. Their tensor product, show in the middle diagram of Figure 3, is not a direct sum of Demazure crystals. In fact, it is not even extremal.

Kouno [12] characterized w, u, λ, μ such that $\mathcal{B}_w(\lambda) \otimes \mathcal{B}_u(\mu)$ is a direct sum of Demazure crystals.

Recall that for any $\lambda \in P^+$, we denote by W_λ be the stabilizer subgroup of λ in W and by $\lfloor w \rfloor^\lambda$ and $\lceil w \rceil^\lambda$ the minimal and maximal length coset representatives of wW_λ , respectively. For any $\sigma \in W$, let $W_\sigma \subseteq W$ denote the parabolic subgroup

$$W_\sigma = \langle s_i \in W \mid s_i \sigma \prec \sigma \rangle.$$

Theorem 6 (Kouno [12]). *Let $\lambda, \mu \in P^+$ and $u, w \in W$. Then $\mathcal{B}_w(\lambda) \otimes \mathcal{B}_u(\mu)$ is a direct sum of Demazure crystals if and only if $\lfloor w \rfloor^\lambda \in W_{\lceil u \rceil^\mu}$.*

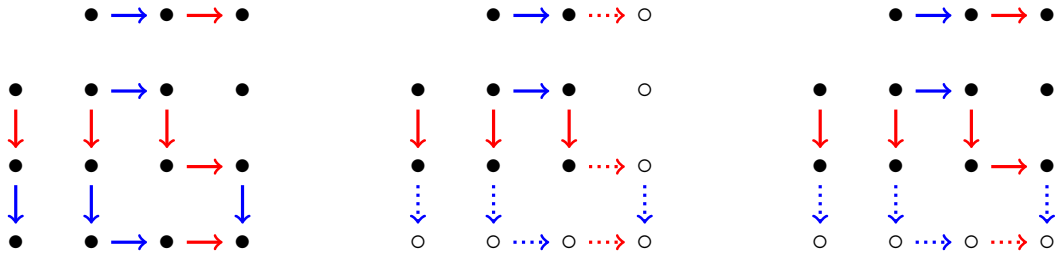


Figure 3: The tensor products $B(1,1,0) \otimes B(1,0,0)$ (left), $B_{s_2}(1,1,0) \otimes B_{s_1}(1,0,0)$ (middle), and $B_{s_2}(1,1,0) \otimes B(1,0,0) = B_{s_2 s_1}(2,1,0) \oplus B_e(1,1,1)$ (right) with f_1 and f_2 depicted by blue and red arrows, respectively.

Example F. Consider $\lambda, \mu \in P^+$, and suppose W has a longest element, which we denote by w_0 . Then $\lfloor w \rfloor^\lambda \in W = W_{w_0}$ for any $w \in W$, and so $\mathcal{B}_w(\lambda) \otimes \mathcal{B}(\mu)$ always decomposes into Demazure crystals; see Figure 3.

Example G. Let $\mathfrak{g} = \mathfrak{sl}_3$, and consider $B_{s_2}(1,1,0) \otimes B_{s_1}(1,0,0)$. Then $W_{(1,0,0)} = \{s_2, e\}$, thus $\lceil s_1 \rceil^{(1,0,0)} = s_1 s_2$ and so $W_{s_1 s_2} = \{s_1\}$. However, $W_{(1,1,0)} = \{s_1, e\}$ so that $\lfloor s_2 \rfloor^{(1,1,0)} = s_2 \notin W_{s_1 s_2}$. As seen in Figure 3, $B_{s_2}(1,1,0) \otimes B_{s_1}(1,1,0)$ is indeed not Demazure.

Recall the tensor product of crystals is not commutative, though Kashiwara [10] showed $\mathcal{B}(\lambda) \otimes \mathcal{B}(\mu)$ is isomorphic to $\mathcal{B}(\mu) \otimes \mathcal{B}(\lambda)$. We remark this does *not* hold for Demazure crystals; that is, $\mathcal{B}_w(\lambda) \otimes \mathcal{B}_u(\mu)$ is *not* isomorphic to $\mathcal{B}_u(\mu) \otimes \mathcal{B}_w(\lambda)$ in general.

Indeed, by Kouno's characterization $B_w(\lambda) \otimes \{u_\mu\}$ is a direct sum of Demazure crystals only when $w \in W_\mu$. However, Joseph [7] proved $\{u_\mu\} \otimes B_w(\lambda)$ always decomposes as a direct sum of Demazure crystals.

Example H. Take $\mathfrak{g} = \mathfrak{sl}_3$, then $B_e(1, 1, 0) \otimes_{B_{s_2}}(1, 1, 0) \cong B_e(2, 2, 0) \oplus B_e(2, 1, 1)$, as seen in Figure 5 (middle), is a direct sum of Demazure crystals. However, $B_{s_2}(1, 1, 0) \otimes B_e(1, 1, 0)$ in Figure 5 (right) is not even extremal, let alone Demazure.

3.3 Tensor products of extremal crystals

Just as tensor products of Demazure crystals are not always Demazure, tensor products of extremal subsets are not always extremal. For instance, in the rightmost diagram of Figure 5, we see that $B_{s_1}(1, 1, 0) \otimes B_e(1, 1, 0)$ is not extremal even though both factors are.

Example I. Consider $X = \{b_\lambda, f_2(b_\lambda), f_2^2(b_\lambda), f_1 f_2(b_\lambda)\} \subset B_{(2,2,0)}$, an extremal though not Demazure subset. As seen in Figure 4, $X \otimes X$ decomposes into connected components $Y_1 \oplus Y_2 \oplus Y_3 \subset B_{(4,4,0)} \oplus B_{(4,3,1)} \oplus B_{(4,2,2)}$ where neither Y_1 nor Y_2 are extremal subsets.

However, if the resulting tensor product of two subsets of crystals is itself extremal, this imposes some structure on the underlying subsets themselves.

Proposition 7. *If $X \otimes Y \subset \mathcal{B}(\lambda) \otimes \mathcal{B}(\mu)$ is an extremal subset, then $\mathcal{E}(X) \subset X \sqcup \{0\}$. Furthermore, if $\mathcal{E}(Y) \subset Y \sqcup \{0\}$, then $X \subset \mathcal{B}(\lambda)$ is an extremal subset.*

Example J. Let $\mathfrak{g} = \mathfrak{sl}_3$, $\lambda = (4, 4, 0)$, and consider $X = \{b_\lambda\}$ and $Y = \{b_\lambda, f_2(b_\lambda), f_2^2(b_\lambda)\}$ subsets of $B(4, 4, 0)$. Then $X \otimes Y \cong B_e(8, 8, 0) \oplus B_e(8, 7, 1) \oplus B_e(8, 6, 2)$ is a sum of Demazure subsets, though Y is not extremal (but it is closed under $\mathcal{E}$).

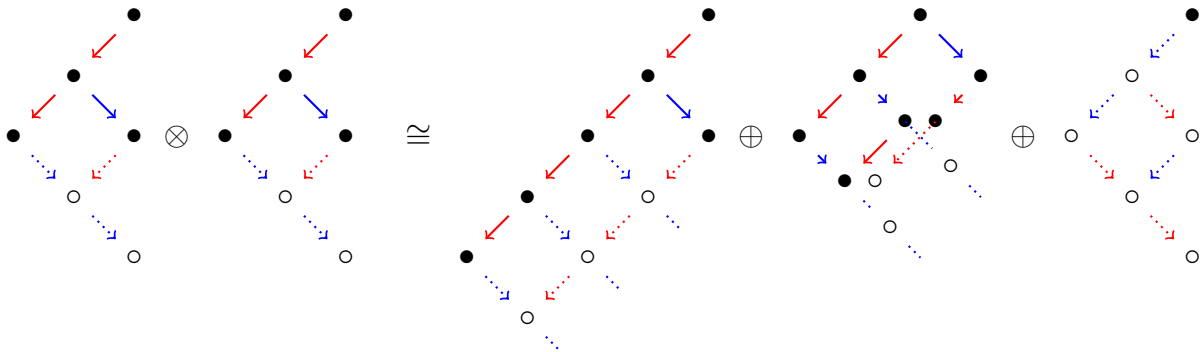


Figure 4: The summands of the extremal but not Demazure subgraph of $B_{(2,2,0)} \otimes B_{(2,2,0)} \cong B_{(4,4,0)} \oplus B_{(4,3,1)} \oplus B_{(4,2,2)}$ with f_1 and f_2 depicted by blue and red arrows.

4 Characterization of extremal tensor products

Determining when the tensor product of extremal subsets remains extremal depends solely on the following elements.

Definition 8. For $i \in I$, an element $x \otimes y \in \mathcal{B}(\lambda) \otimes \mathcal{B}(\mu)$ is called an i -hinge if $e_i(x \otimes y)$ and $f_i(x \otimes y)$ are both nonzero with $e_i(x \otimes y) = e_i(x) \otimes y$ and $f_i(x \otimes y) = x \otimes f_i(y)$.

We say $x \otimes y \in X \otimes Y \subset \mathcal{B}(\lambda) \otimes \mathcal{B}(\mu)$ is a *broken i -hinge* if $f_i(y) \notin Y$.

Example K. Consider $B(1, 1, 0) = a_1 \xrightarrow{\text{red}} a_2 \xrightarrow{\text{blue}} a_3$. Then the element $a_2 \otimes a_1 \in B(1, 1, 0) \otimes B(1, 1, 0)$ (seen in the leftmost diagram of Figure 5) is a 2-hinge since $\varepsilon_2(a_2) = 1$ and $\varphi_2(a_2) = 0$ but $\varepsilon_2(a_1) = 0$ with $\varphi_2(a_1) = 1$. In particular, the subset $B_{s_2}(1, 1, 0) \otimes B_e(1, 1, 0)$ (rightmost in Figure 5) contains a broken 2-hinge since $f_2(a_1) \notin B_e(1, 1, 0)$.

Theorem 9. Let $X \subset \mathcal{B}(\lambda)$ and $Y \subset \mathcal{B}(\mu)$ be extremal subsets. Then $X \otimes Y$ is an extremal subset of $\mathcal{B}(\lambda) \otimes \mathcal{B}(\mu)$ if and only if $X \otimes Y$ contains no broken i -hinge for any $i \in I$.

In particular, if $X = \{b_\lambda\} \subset \mathcal{B}(\lambda)$ has only the highest weight element or if $Y = \mathcal{B}(\mu)$ contains all possible elements, then $X \otimes Y$ contains no i -hinges for any i . Thus both $\{b_\lambda\} \otimes \mathcal{B}_u(\mu)$ and $\mathcal{B}_w(\lambda) \otimes \mathcal{B}(\mu)$ are extremal subsets of $\mathcal{B}(\lambda) \otimes \mathcal{B}(\mu)$.

Recall every Demazure subset is extremal, though the converse is false.

Any subset $\mathcal{B}_w(\lambda) \otimes \mathcal{B}_u(\mu) \subset \mathcal{B}(\lambda) \otimes \mathcal{B}(\mu)$ which is a direct sum of Demazure crystals is also an extremal subset. Amazingly, the converse of this statement is also true.

Theorem 10. For $\lambda, \mu \in P^+$ and $w, u \in W$, we have $\mathcal{B}_w(\lambda) \otimes \mathcal{B}_u(\mu)$ is an extremal subset of $\mathcal{B}(\lambda) \otimes \mathcal{B}(\mu)$ if and only if $\lfloor w \rfloor^\lambda \in W_{\lceil u \rceil^\mu}$.

Combining this with Theorem 6, we derive the following result.

Corollary 11. For $\lambda, \mu \in P^+$ and $w, u \in W$, the tensor product $\mathcal{B}_v(\lambda) \otimes \mathcal{B}_w(\mu)$ is a direct sum of extremal subsets if and only if $\mathcal{B}_w(\lambda) \otimes \mathcal{B}_u(\mu)$ is a direct sum of Demazure crystals.

Thus, $\mathcal{B}_v(\lambda) \otimes \mathcal{B}_w(\mu)$ is a sum of Demazure crystals precisely when it doesn't contain a broken i -hinge for any $i \in I$. Hence Corollary 11 gives a local characterization of tensor products of Demazure crystals that does not rely on the values of λ, μ, w, u .

Example L. Take $\lambda = (1, 1, 0)$ and $w = e$ and $v = s_2$ as in Figure 5. Then $\lfloor w \rfloor^\lambda = e$ and $\lceil v \rceil^\lambda = s_2 s_1$, thus $\lfloor w \rfloor^\lambda \in W_{\lceil v \rceil^\lambda}$ so $B_e(1, 1, 0) \otimes B_{s_2}(1, 1, 0)$ is extremal. Conversely, $\lfloor v \rfloor^\lambda = s_2$ and $\lceil w \rceil^\lambda = s_1$ so $\lfloor v \rfloor^\lambda \notin W_{\lceil w \rceil^\lambda}$ and thus $B_{s_2}(1, 1, 0) \otimes B_e(1, 1, 0)$ is not extremal.

It is important to note that Corollary 11 is false if we replace $\mathcal{B}_v(\lambda)$ and $\mathcal{B}_w(\mu)$ with arbitrary extremal subsets. This can be seen in Figure 4, where $X = Y$ is extremal and non-Demazure but the tensor product is neither.

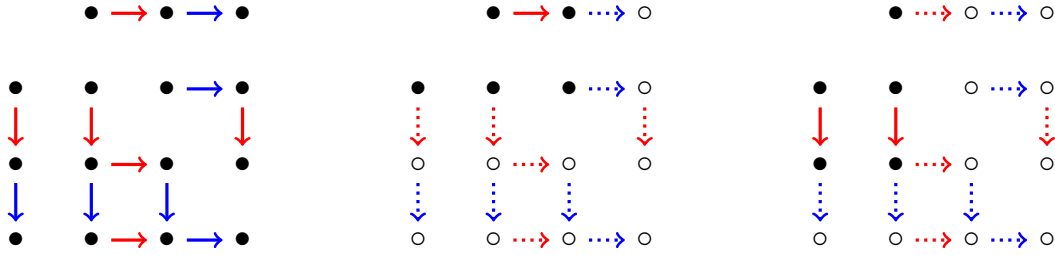


Figure 5: The tensor products $B(1,1,0) \otimes B(1,1,0)$ (left), $B_e(1,1,0) \otimes B_{s_2}(1,1,0)$ (middle), and $B_{s_2}(1,1,0) \otimes B_e(1,1,0)$ (right) with f_1 and f_2 depicted by blue and red arrows.

5 Application to tensor squares

Even when $\mathcal{B}_w(\lambda) \otimes \mathcal{B}_u(\mu)$ is not a direct sum of Demazure crystals, some connected components of it may be. For instance, in [Example M](#) and [Figure 6](#), $B_{s_2s_1}(2,1,0)^{\otimes 2}$ decomposes into four connected components, two of which are Demazure and two of which are not even extremal. In particular, the component of weight $(4,2,0)$ is a Demazure crystal. Using [Corollary 11](#), we show that the highest weight component is always Demazure.

Theorem 12. *For $\lambda \in P^+$ and $w \in W$, the m -fold tensor product*

$$\mathcal{F}(\{b_\lambda \otimes \cdots \otimes b_\lambda\}) \cap \mathcal{B}_w(\lambda) \otimes \cdots \otimes \mathcal{B}_w(\lambda) \subset \mathcal{B}(\lambda) \otimes \cdots \otimes \mathcal{B}(\lambda)$$

is isomorphic to $\mathcal{B}_w(m\lambda)$. In particular, it is a Demazure crystal.

Example M. Let $\mathfrak{g} = \mathfrak{sl}_3$ and consider $B(2,1,0) \otimes B(2,1,0) \cong B(4,2,0) \oplus B(3,3,0) \oplus B(4,1,1) \oplus B(3,2,1)^{\oplus 2} \oplus B(2,2,2)$. The subset $B_{s_2s_1}(2,1,0) \otimes B_{s_2s_1}(2,1,0)$ decomposes into four connected components; see [Figure 6](#). Only the components with highest weights $(4,2,0)$ and $(3,3,0)$ are Demazure. The components with highest weights $(3,2,1)$ and $(4,1,1)$ are not even extremal. The remaining two highest weights do not appear.

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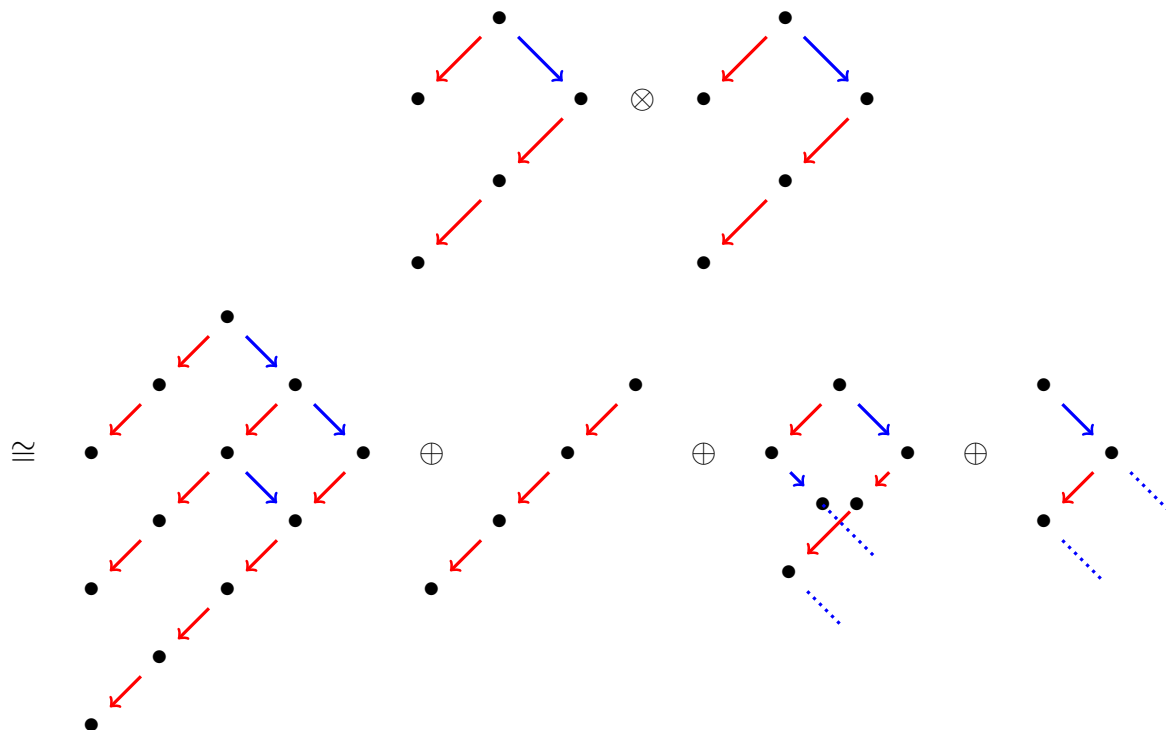


Figure 6: The tensor product of $B_{S_2S_1}(2, 1, 0) \otimes B_{S_2S_1}(2, 1, 0)$ decomposed into connected components, some of which are Demazure and some of which are not extremal.

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Effective algebraicity for solutions of systems of functional equations with one catalytic variable

Hadrien Notarantonio^{1,2} and Sergey Yurkevich^{1,3}

¹Inria Saclay, France

²Sorbonne Université, France

³University of Vienna, Austria

Abstract. We study systems of $n \geq 1$ discrete differential equations of order $k \geq 1$ in one catalytic variable and provide a constructive and elementary proof of algebraicity of their solutions. This yields effective bounds and a systematic method for computing the minimal polynomials. Our approach is a generalization of the pioneering work by Bousquet-Mélou and Jehanne (2006).

Keywords: algebraic functional equations, combinatorics, algebraic algorithms, algebraicity bounds, catalytic variables

1 Introduction

Numerous combinatorial enumeration problems reduce to the study of functional equations which can be solved by a uniform method introduced by Bousquet-Mélou and Jehanne in the seminal work [5]. These functional equations, usually called discrete differential equations (DDEs) with one catalytic variable, involve a bivariate generating function $F \in \mathbb{Q}[u, t]$ associated to the enumeration problem, and are of the form

$$F(t, u) = f(u) + t \cdot Q(F(t, u), \Delta_a F(t, u), \dots, \Delta_a^k F(t, u), t, u), \quad (1.1)$$

where $k \in \mathbb{N}$ (called the order of the DDE), f and Q are polynomials, and (for some $a \in \mathbb{Q}$, usually 0 or 1) Δ_a^ℓ is the ℓ th iteration of the operator $\Delta_a : \mathbb{Q}[u, t] \rightarrow \mathbb{Q}[u, t]$ defined by

$$\Delta_a F(t, u) := \frac{F(t, u) - F(t, a)}{u - a}.$$

In their paper, Bousquet-Mélou and Jehanne designed a “non-linear kernel method” which allows one to prove that the unique solution of (1.1) is always an algebraic function over $\mathbb{Q}[t, u]$. Significantly in practice, this approach yields an algorithm for finding the minimal polynomial of the specialization $F(t, a)$ and of the bivariate series $F(t, u)$.

The main contribution of the present paper is a generalization of this method to the case of *systems of discrete differential equations*. More precisely, we shall prove the following theorem. Here and in the following, we let $\mathbb{K}$ be a field of characteristic 0.

Theorem 1. Let $n, k \geq 1$ be integers and $f_i \in \mathbb{K}[u]$, $Q_i \in \mathbb{K}[y_1, \dots, y_{n-k-1}, t, u]$ be polynomials for $i = 1, \dots, n$. Set $\nabla^k F := F, \Delta_a F, \dots, \Delta_a^k F$. Then the system of equations

$$\begin{cases} (\mathbf{E}_{F_1}): F_1 - f_1(u) - t \cdot Q_1 \nabla^k F_1, \dots, \nabla^k F_n, t, u, \\ \vdots \\ (\mathbf{E}_{F_n}): F_n - f_n(u) - t \cdot Q_n \nabla^k F_1, \dots, \nabla^k F_n, t, u \end{cases} \quad (1.2)$$

admits a unique vector of solutions $F_1, \dots, F_n \in \mathbb{K}[u][t^{-n}]$, and all its components are algebraic functions over $\mathbb{K}[t, u]$.

The key idea, analogous to the one in [5], for proving this theorem is to define a deformation of (1.2) that ensures the applicability of a multi-dimensional analog of the “non-linear kernel method”. Stated explicitly, we show in Lemma 3 that after deforming the equations as in (3.1), the polynomial in u defined by the determinant of the Jacobian matrix associated to the numerator equations in (1.2) (considered with respect to the F_i) has exactly nk solutions in an extension of the ring $\bigcup_{d \geq 1} \overline{\mathbb{K}}[t^{1/d}]$. Then, after a process of “duplication of variables”, we construct a zero-dimensional and radical polynomial ideal, a non-trivial element of which must be the desired annihilating polynomial. The most difficult step consists in proving the invertibility of a certain Jacobian matrix (Lemma 5 and Lemma 6) in order to justify the zero-dimensionality. We remark that an alternative, and possibly more practical, strategy is to reduce the initial system to a single functional equation. Our Proposition 7 ensures that such a reduction preserves the roots guaranteed in the deformation step, however, as we will show in Section 4, this method is not guaranteed to produce a zero-dimensional polynomial ideal in the end.

Similarly to the work by Bousquet-Mélou and Jehanne, our proof is effective, in the sense that it produces an algorithm for finding the minimal polynomials of the power series of interest. Moreover, we can deduce a bound on the algebraicity degree of each F_i . When the field $\mathbb{K}$ is effective (e.g. $\mathbb{K} = \mathbb{Q}$), we can also bound the arithmetic complexity of our algorithm, that is the number of operations $+, -, \times, \div$ performed in $\mathbb{K}$. Denoting by $\text{totdeg } P$ the total degree of a multivariate polynomial P , we obtain:

Theorem 2. In the setting of Theorem 1, let $F_1, \dots, F_n \in \mathbb{K}[u][t^{-n}]$ be the vector of solutions and $\delta := \max(\deg f_1, \dots, \deg f_n, \text{totdeg } Q_1, \dots, \text{totdeg } Q_n)$. The algebraicity degree of each $F_i(t, u)$ over $\mathbb{K}[t, u]$ is in $O(nk\delta^{4n^2k^2} n^{-2} / nk!^{nk})$. Also, when $\mathbb{K}$ is effective, there exists an algorithm computing the minimal polynomial of any $F_i(t, u)$ in $2nk\delta^{O(n^2k)} \text{ ops. in } \mathbb{K}$.

Discrete differential equations are ubiquitous in enumerative combinatorics [6, 7, 12]. Systems of DDEs also appear in a variety of different contexts throughout combinatorics, for instance for hard particles on planar maps [5, §5.4], inhomogeneous lattice paths [8], or certain orientations with n edges [3, §5]. The usual strategy for solving these systems of equations is to reduce a given system to a scalar equation and then apply the method

of Bousquet-Mélou and Jehanne. This approach is usually ad-hoc. Also, since the reduced equation is in general not of the form (1.1), the theory of [5] may not be applicable.

In the literature there exist two methods to overcome these theoretical issues. First, a deep theorem in commutative algebra by Popescu [10], so-called “nested Artin approximation”, guarantees that equations of the form (1.2) always admit an algebraic solution (see also [3, Thm. 16] for a statement of this theorem). Note that the nested condition is automatically satisfied in this case and that the uniqueness of the solution is obvious. Yet, a drawback of using Popescu’s theorem is that its proof seems non-constructive and can thus only be applied as a “black box”, whereas in practice one is often interested in the explicit minimal polynomials annihilating the solutions. Secondly, the frequent case when all polynomials $Q_1, \dots, Q_n$ in (1.2) are linear functions was effectively solved in the recent FPSAC article [8] by Buchacher and Kauers and also studied by Asinowski, Bacher, Banderier and Gittenberger [1]. From this viewpoint, it is safe to say that our contribution is a common generalization of central results by Bousquet-Mélou and Jehanne [5] and the “vectorial kernel method” of [1, 8], and at the same time an effective and elementary proof of a special case of Popescu’s theorem [10].

We shall highlight the following two examples of systems of DDEs in more detail.

Example 1. The following system of DDEs for the generating function of certain planar orientations was considered in [3, Eq. 27] and solved in the same work:

$$\begin{cases} (\mathbf{E}_{F_1}): F_1(t, u) = 1 + t \cdot (u - 2uF_1(t, u) - 2uF_2(t, 1) - u \frac{F_1(t, u) - uF_1(t, 1)}{u-1}), \\ (\mathbf{E}_{F_2}): F_2(t, u) = t \cdot (2uF_1(t, u) - F_2(t, u) - uF_1(t, u) - uF_2(t, 1) - u \frac{F_2(t, u) - uF_2(t, 1)}{u-1}). \end{cases} \quad (1.3)$$

From our perspective, (1.3) has the advantage that it does not require any deformation and, as we will show in Section 2, it can be solved fast by a direct application of our method. It is thus a good illustration of the simplest non-trivial case of our approach.

Example 2. This example of a system of DDEs modelling a particular case of hard particles on planar maps was introduced and solved in [5, Section 11]:

$$\begin{cases} (\mathbf{E}_{F_1}): F_1(t, u) = F_2(t, u) + tu^2F_1(t, u) + tu \frac{uF_1(t, u) - F_1(t, 1)}{u-1}, \\ (\mathbf{E}_{F_2}): F_2(t, u) = 1 + tsuF_1(t, u) - F_2(t, u) + tsu \frac{F_2(t, u) - F_2(t, 1)}{u-1}. \end{cases} \quad (1.4)$$

To apply our method directly, a deformation step (3.1) is necessary (see Section 4).

The structure of the paper is as follows: In Section 2 we explain our method in the case of two equations of order one under the genericity assumption that no deformation is necessary. We summarize the method in an algorithm and showcase it explicitly on Example 1. Section 3 is devoted to the proofs of Theorem 1 and Theorem 2. In the last Section 4 we briefly explore an improvement to our approach which in theory has a better algorithmic complexity but which requires a new genericity assumption. We also discuss possible future works.

2 The case of two generic equations of first order

Before proving our main theorem in [Section 3](#), we introduce our method in the situation of two equations of order 1 and under a genericity assumption on the input system.

Starting with (1.2), we first multiply (E_{F_1}) and (E_{F_2}) by $u - a^{m_1}$ and $u - a^{m_2}$ respectively (for $m_1, m_2 \in \mathbb{N}$) in order to obtain a system with polynomial coefficients in u . By a slight abuse of notation, we shall still write (E_{F_1}) and (E_{F_2}) for those equations. Note that this system induces polynomials E_1, E_2 in $\mathbb{K}[x_1, x_2, z_0, z_1, t, u]$ whose specializations to $x_1 = F_1(t, u), x_2 = F_2(t, u), z_0 = F_1(t, a), z_1 = F_2(t, a)$ are zero.

Example 1 (cont.). Multiplying (E_{F_1}) and (E_{F_2}) in [Example 1](#) by $u - 1$ gives

$$\begin{cases} E_1 & 1 - x_1 \cdot u - 1 & t \cdot 2u^2x_1^2 - u^2z_0 & 2u^2z_1 - 2ux_1^2 & u^2 & ux_1 - 2uz_1 - u \\ E_2 & x_2 \cdot 1 - u & t \cdot 2u^2x_1x_2 & u^2x_1 - 2ux_1x_2 - ux_1 & ux_2 - uz_1 \end{cases}$$

In the spirit of [5], we now take the derivative of both equations with respect to u :

$$\begin{pmatrix} \partial_{x_1}E_1 & \partial_{x_2}E_1 \\ \partial_{x_1}E_2 & \partial_{x_2}E_2 \end{pmatrix} \cdot \begin{pmatrix} \partial_u F_1 \\ \partial_u F_2 \end{pmatrix} - \begin{pmatrix} \partial_u E_1 \\ \partial_u E_2 \end{pmatrix} = 0. \quad (2.1)$$

Define $\text{Det} : \partial_{x_1}E_1 \cdot \partial_{x_2}E_2 - \partial_{x_1}E_2 \cdot \partial_{x_2}E_1 \in \mathbb{K}[x_1, x_2, z_0, z_1, t, u]$ to be the determinant of the square matrix above. One can show that $\text{Det}(F_1(t, u), F_2(t, u), F_1(t, a), F_2(t, a), t, u) \in \mathbb{K}[t, u]$ admits either 0, 1 or 2 distinct non-zero solutions $u \in U(t) \in \bigcup_{d \geq 1} \overline{\mathbb{K}}[t]^{1/d} : \overline{\mathbb{K}}[t]^{1/d} \rightarrow \overline{\mathbb{K}}[t]^{1/d}$. We assume now that there exist 2 such solutions $U_1, U_2 \in \overline{\mathbb{K}}[t]^{1/d}$; we prove in [Section 3](#) that it is always the case up to the deformation (3.1).

Exploiting the idea of [8], we now define $v : \partial_{x_1}E_2, -\partial_{x_1}E_1 \in \mathbb{K}[x_1, x_2, z_0, z_1, t, u]^2$ and plug U_1 for u into v and (2.1). Note that v is an element of the left-kernel of the square matrix in (2.1) mod $\text{Det}(x_1, x_2, z_0, z_1, t, u)$. After multiplication of (2.1) by v on the left, we find a new polynomial relation between $F_1(t, U_i), F_2(t, U_i), F_1(t, a), F_2(t, a), t$ and U_i , namely $\partial_{x_1}E_1 \cdot \partial_u E_2 - \partial_{x_1}E_2 \cdot \partial_u E_1 = 0$ when evaluated at $x_1 = F_1(t, U_i), x_2 = F_2(t, U_i), z_0 = F_1(t, a), z_1 = F_2(t, a), u = U_i$. Denote $P \in \mathbb{K}[x_1, x_2, z_0, z_1, t, u]$ this polynomial. Define the polynomial system $\mathcal{S} : E_1, E_2, \text{Det}, P \in \mathbb{K}[t, x_1, x_2, z_0, z_1, u]^4$. It admits the solutions $F_1(t, U_i), F_2(t, U_i), F_1(t, a), F_2(t, a), U_i \in \overline{\mathbb{K}}[t]^{1/d}$, for $i \in \{1, 2\}$.

Example 1 (cont.). Continuing [Example 1](#), we find

$$\begin{cases} \text{Det} & 4tu^2x_1 - 4tux_1 & tu - u & 1 & 2tu^2x_1 - 2tux_1 & tu - u & 1 \\ P & -2tx_1x_2 - tx_1 & tx_2 - tz_1 - x_2 & P_1 \cdot u & P_2 \cdot u^2 & P_3 \cdot u^3, \end{cases}$$

where P_1, P_2, P_3 are explicit (but relatively big) polynomials in $\mathbb{Q}[x_1, x_2, z_0, z_1, t]$.

Now, generalizing naturally the steps of [5], we define for $i \in \{0, 1\}$ the polynomial systems $\mathcal{S}_i : \mathcal{S}(x_{2i-1}, x_{2i-2}, z_0, z_1, t, u_{i-1})$ by “duplicating” variables. We call the situation “generic” if the ideal $\langle \mathcal{S}_0, \mathcal{S}_1, m \cdot u_1 - u_2 - 1 \rangle$ has dimension 0 over $\mathbb{K}[t]$. In

this case, in order to find an annihilating polynomial of $F_1(t, a)$, it is enough to compute a non-zero element of $\langle S_0, S_1, m \cdot u_1 - u_2 - 1 \rangle \cap \mathbb{K}[z_0, t]$.

Example 1 (cont.). Continuing Example 1, we compute¹ a generator of the polynomial ideal $\langle S_0, S_1, m \cdot u_1 - u_2 - 1 \rangle \cap \mathbb{Q}[z_0, t]$. It has degree 13 in z_0 and 14 in t . In particular, it contains in its factors the minimal polynomial of $F_1(t, 1)$ given by $64t^3z_0^3 - 48t^3 - 72t^2 - 2t z_0^2 - 15t^3 - 9t^2 - 19t - 1 z_0 - t^3 - 27t^2 - 19t - 1$.

We summarize the presented algorithm in a more compact form:

Algorithm 1: Solving systems of two discrete differential equations of order 1.

Input: A “generic” system of two DDEs $(E_{F_1}), (E_{F_2})$ of order 1.

Output: A non-zero $R \in \mathbb{K}[z_0, t]$ annihilating $F_1(t, a)$.

- 1 Replace (E_{F_1}) and (E_{F_2}) by their respective numerators and denote by E_1 and E_2 the associated polynomials in $\mathbb{K}[x_1, x_2, z_0, z_1, t, u]$.
 - 2 Compute $\text{Det} : \partial_{x_1} E_1 \cdot \partial_{x_2} E_2 - \partial_{x_1} E_2 \cdot \partial_{x_2} E_1$ and $P : \partial_{x_1} E_1 \cdot \partial_u E_2 - \partial_{x_1} E_2 \cdot \partial_u E_1$.
 - 3 Set $S : E_1, E_2, \text{Det}, P \in \mathbb{K}[x_1, x_2, z_0, z_1, t, u]$.
 - 4 For $0 \leq i \leq 1$, define $S_i : S[x_{2i-1}, x_{2i-2}, z_0, z_1, t, u_{i-1}]$.
 - 5 **Return** a non-zero element of $\langle S_0, S_1, m \cdot u_1 - u_2 - 1 \rangle \cap \mathbb{K}[z_0, t]$.
-

As already stated, Section 3 ensures that for a non-generic input, after a deformation, we can use an algorithm in spirit of Algorithm 1.

We remark that if the strategy above is applied in the case of a single equation of first order $F_1 = f(u) \cdot t \cdot Q_1(F_1, \Delta_a F_1, t, u)$, the presented method simplifies to the classical algorithm in [5] relying on studying the ideal $\langle E_1, \partial_{x_1} E_1, \partial_u E_1 \rangle$. Stated explicitly, $\partial_{x_1} E_1$ plays the role of Det and $\partial_u E_1$ plays the role of P (as we can take here $v = 1$).

3 Proofs of Theorem 1 and Theorem 2

We start by proving Theorem 1. As explained before, the statement and proof can be seen as a generalization of [5, Theorem 3] and [8, Theorem 2], so several steps are done analogously. Without loss of generality we assume that $a = 0$ and set $\Delta : \Delta_0$.

Denote by $m_1, \dots, m_n$ the least positive integers greater than or equal to k such that multiplying (E_{F_i}) in (1.2) by u^{m_i} removes the apparent denominator in u . Set $\beta : \lfloor 2M/k \rfloor$ and $\alpha : 3n^2k \cdot \beta - 1 - 3nM$, where $M : m_1 + \dots + m_n$. Let ϵ be a new variable, $\mathbb{L} : \mathbb{K}[\epsilon]$, and let $\gamma_{i,j} \ 1 \leq i, j \leq n$ be defined by $\gamma_{i,i} = i^k$ and $\gamma_{i,j} = t^\beta$ for $i \neq j$. Then consider the following system which is a deformation of (1.2):

$$\begin{cases} (\mathbf{E}_{G_1}): G_1 = f_1(u) \cdot t^\alpha \cdot Q_1(\nabla^k G_1, \nabla^k G_2, \dots, \nabla^k G_n, t^\alpha, u) - t \cdot \epsilon^k \cdot \sum_{i=1}^n \gamma_{1,i} \cdot \Delta^k G_i, \\ \vdots \\ (\mathbf{E}_{G_n}): G_n = f_n(u) \cdot t^\alpha \cdot Q_n(\nabla^k G_1, \nabla^k G_2, \dots, \nabla^k G_n, t^\alpha, u) - t \cdot \epsilon^k \cdot \sum_{i=1}^n \gamma_{n,i} \cdot \Delta^k G_i. \end{cases} \quad (3.1)$$

¹All computations in this paper have been performed in Maple using `msolve` [2].

The fixed point nature of these equations still implies that there exists a unique solution $G_1, \dots, G_n \in \mathbb{L}[u, t]^n$. Remark that the equalities $F_i(t^\alpha, u) = G_i(t, u, 0)$ relate the formal power series solutions of (1.2) and of (3.1). Hence, showing that each G_i is algebraic over $\mathbb{L}[t, u]$ is enough to prove [Theorem 1](#). Moreover, as we will see later, the algebraicity of each G_i follows from the algebraicity of $G_1(0), \dots, \partial_u^{k-1} G_1(0), \dots, G_n(0), \dots, \partial_u^{k-1} G_n(0)$. Here, and in what follows, we shall use the short notation $G_i(u) \equiv G_i(t, u, \epsilon)$, $\partial_0 G_i(u) \equiv G_i(u, G_i(0), \partial_u G_i(0), \dots, \partial_u^{k-1} G_i(0))$ and $A(u) \equiv A(\partial_0 G_1, \dots, \partial_0 G_n, t, u)$ for any polynomial $A \in \mathbb{L}[X_1, \dots, X_n, t, u]$ with $X_j : x_j, z_{k-j-1}, \dots, z_{k-j-1}$. In the case $n = 1$, this notation implies that for any $0 \leq i \leq k-1$, the variable z_i stands for $\partial_u^i F_1(t, a)$.

Let us define $Y_{i,0} : x_i$ and $Y_{i,j} : x_i - z_{k-i-1} - \dots - \frac{u^{j-1}}{(j-1)!} z_{k-i-1-j-1} / u^j$ for $j \geq 1$. With these definitions, (3.1) is equivalent to the system of polynomial equations

$$\begin{cases} E_1 : u^{m_1} f_1 - x_1 - t^\alpha \cdot Q_1(Y_{1,0}, \dots, Y_{1,k}, Y_{2,0}, \dots, Y_{n,k}, t^\alpha, u) - \epsilon^k \cdot t \sum_{i=1}^n \gamma_{1,i} \cdot Y_{i,k} = 0, \\ \vdots \\ E_n : u^{m_n} f_n - x_n - t^\alpha \cdot Q_n(Y_{1,0}, \dots, Y_{1,k}, Y_{2,0}, \dots, Y_{n,k}, t^\alpha, u) - \epsilon^k \cdot t \sum_{i=1}^n \gamma_{n,i} \cdot Y_{i,k} = 0. \end{cases} \quad (3.2)$$

Like in (2.1), we take the derivative with respect to u of these equations and find

$$\begin{pmatrix} \partial_{x_1} E_1 & \dots & \partial_{x_n} E_1 \\ \vdots & \ddots & \vdots \\ \partial_{x_1} E_n & \dots & \partial_{x_n} E_n \end{pmatrix} \cdot \begin{pmatrix} \partial_u G_1 \\ \vdots \\ \partial_u G_n \end{pmatrix} - \begin{pmatrix} \partial_u E_1 \\ \vdots \\ \partial_u E_n \end{pmatrix} = 0. \quad (3.3)$$

Let $\text{Det} \in \mathbb{L}[X_1, \dots, X_n, t, u]$ be the determinant of the square matrix $\partial_{x_j} E_i$ $1 \leq i, j \leq n$ above. The following lemma on the number of distinct solutions to $\text{Det}(u) = 0$ is the first main step in our proof.

Lemma 3. *$\text{Det}(u) = 0$ admits exactly nk distinct non-zero solutions $U_1, \dots, U_{nk} \in \overline{\mathbb{L}}[t^{\frac{1}{k}}]$.*

Proof. Note that we have

$$\text{Det}(u) = \det \begin{pmatrix} -u^{m_1} & t\epsilon^k \gamma_{1,1} u^{m_1-k} & \dots & t\epsilon^k \gamma_{1,n} u^{m_1-k} \\ \vdots & \ddots & \ddots & \vdots \\ t\epsilon^k \gamma_{n,1} u^{m_n-k} & \dots & -u^{m_n} & t\epsilon^k \gamma_{n,n} u^{m_n-k} \end{pmatrix} = O(t^\alpha u^{M-nk}).$$

For every i we first divide the i^{th} row by u^{m_i-k} . Then, using the definition of $\gamma_{i,j}$ and $\alpha, \beta \geq n$, we see that the matrix above becomes diagonal mod t^{n-1} and its determinant mod t^{n-1} simplifies to $u^{M-nk} \cdot \prod_{j=1}^n -u^k - t\epsilon^k j^k \pmod{t^{n-1}}$. Hence, computing the first terms of a solution in u by using Newton polygons, we find nk distinct nonzero solutions $u = U_1(t), \dots, U_{nk}(t)$ whose first terms are given by $\zeta^\ell \cdot t^{\frac{1}{k}} \cdot \epsilon = O(t^{\frac{2}{k}}), \dots, \zeta^\ell \cdot n \cdot t^{\frac{1}{k}} \cdot \epsilon = O(t^{\frac{2}{k}}) \in \overline{\mathbb{L}}[t^{\frac{1}{k}}]$, for ζ a k -primitive root of unity and for all $1 \leq \ell \leq k$. Finally, note that the constant coefficient in t of $\prod_{j=1}^n -u^k - t\epsilon^k j^k$ has degree nk so by [5, Theorem 2] there cannot be more than nk nonzero solutions to $\text{Det}(u) = 0$ in $\overline{\mathbb{L}}[t^{\frac{1}{k}}]$. $\square$

Now, let P be the determinant of the square matrix $\partial_{x_j} E_i$ $1 \leq i, j \leq n$ where the last column $\partial_{x_n} E_1, \dots, \partial_{x_n} E_n$ is replaced by $\partial_u E_1, \dots, \partial_u E_n$. It is easy to see with standard linear algebra arguments that if $\text{Det} = 0$ then (3.3) implies that $P = 0$. Hence, we define the polynomial system $\mathcal{S} : E_1, \dots, E_n, \text{Det}, P$ in $\mathbb{L}[t, X_1, \dots, X_n, u]$. Note that $\mathcal{S}$ is a system with exactly $n + 2$ equations in the $nk + n + 1$ variables $z_0, \dots, z_{nk-1}, x_1, \dots, x_n, u$ (here t and ϵ are parameters). We wish to construct a zero-dimensional ideal, so we consider the *duplicated system* $\mathcal{S}_{\text{dup}} : \mathcal{S}_1, \dots, \mathcal{S}_{nk} \subset \mathbb{L}[t, x_1, \dots, x_{n2k}, z_0, \dots, z_{nk-1}, u_1, \dots, u_{nk}]$. This system is built from $nk + n + 2$ equations and $nk + n + 2$ variables.

The following lemma is proven in [4, Lemma 2.10] as a consequence of Hilbert's Nullstellensatz and [9, Theorem 16.19]:

Lemma 4. Assume that the Jacobian matrix $\text{Jac}_{\mathcal{S}_{\text{dup}}}$ of $\mathcal{S}_{\text{dup}}$, considered with respect to the variables $x_1, \dots, x_n, u_1, \dots, x_{n2k-n}, \dots, x_{n2k}, u_{nk}, z_0, \dots, z_{nk-1}$, is invertible at the point

$$\mathcal{P} = (G_1, U_1, \dots, G_n, U_1, \dots, G_1, U_{nk}, \dots, G_n, U_{nk}, U_{nk}, G_1, 0, \dots, \partial_u^{k-1} G_1, 0, \dots, G_n, 0, \dots, \partial_u^{k-1} G_n, 0) \in \overline{\mathbb{L}}[t^{\frac{1}{*}}]^{nk+n+1} \times \mathbb{L}[t]^{nk}.$$

Then the saturated ideal $\langle \mathcal{S}_{\text{dup}} \rangle : \det \text{Jac}_{\mathcal{S}_{\text{dup}}}^\infty$ is zero-dimensional and radical over $\mathbb{L}[t]$. Moreover, $\mathcal{P}$ lies in the zero set of $\langle \mathcal{S}_{\text{dup}} \rangle : \det \text{Jac}_{\mathcal{S}_{\text{dup}}}^\infty$.

Therefore, in order to conclude the algebraicity of $G_i = 0, \dots, \partial_u^{k-1} G_i = 0$ over $\mathbb{L}[t]$ for all $1 \leq i \leq n$, it is enough to justify that $\text{Jac}_{\mathcal{S}_{\text{dup}}}$ is invertible at $\mathcal{P}$. Then, by Lemma 4, it will follow that it is possible to apply effective techniques from polynomial elimination theory and find annihilating polynomials for the power series of interest.

The idea for proving that $\det \text{Jac}_{\mathcal{S}_{\text{dup}}}(\mathcal{P}) \neq 0$, analogous to the proof in [5], is to show first that $\text{Jac}_{\mathcal{S}_{\text{dup}}}(\mathcal{P})$ can be rewritten as a block triangular matrix. We will then show that each such block is invertible by carefully analyzing its lowest valuation in t .

If $A \in \mathbb{L}[t, X_1, \dots, X_n, u]$, we shall define its " i^{th} duplicated polynomial" as $A^i : A(X_{n-i-1}, \dots, X_{n-i-1}, u_i)$. Then the Jacobian matrix $\text{Jac}_{\mathcal{S}_{\text{dup}}}(\mathcal{P})$ has the shape

$$\text{Jac}_{\mathcal{S}_{\text{dup}}}(\mathcal{P}) = \begin{pmatrix} A_1 & 0 & B_1 \\ & \ddots & \vdots \\ 0 & A_{nk} & B_{nk} \end{pmatrix} \in \overline{\mathbb{L}}[t^{\frac{1}{*}}]^{(nk+n+2) \times (nk+n+2)},$$

where the matrices $A_i \in \overline{\mathbb{L}}[t^{\frac{1}{*}}]^{(n+2) \times (n+1)}$ and $B_i \in \overline{\mathbb{L}}[t^{\frac{1}{*}}]^{(n+2) \times nk}$ are given by:

$$A_i : \begin{pmatrix} \partial_{x_1} E_1^i U_i & \dots & \partial_{x_n} E_1^i U_i & \partial_u E_1^i U_i \\ \vdots & \ddots & \vdots & \vdots \\ \partial_{x_1} E_n^i U_i & \dots & \partial_{x_n} E_n^i U_i & \partial_u E_n^i U_i \\ \partial_{x_1} \text{Det}^i U_i & \dots & \partial_{x_n} \text{Det}^i U_i & \partial_u \text{Det}^i U_i \\ \partial_{x_1} P^i U_i & \dots & \partial_{x_n} P^i U_i & \partial_u P^i U_i \end{pmatrix}, B_i : \begin{pmatrix} \partial_{z_0} E_1^i U_i & \dots & \partial_{z_{nk-1}} E_1^i U_i \\ \vdots & \ddots & \vdots \\ \partial_{z_0} E_n^i U_i & \dots & \partial_{z_{nk-1}} E_n^i U_i \\ \partial_{z_0} \text{Det}^i U_i & \dots & \partial_{z_{nk-1}} \text{Det}^i U_i \\ \partial_{z_0} P^i U_i & \dots & \partial_{z_{nk-1}} P^i U_i \end{pmatrix}.$$

Using $\det U_i = 0$ and (3.3), we see that the first $n \times n - 1$ submatrix of each A_i has rank at most $n - 1$. Hence, after performing operations on the first n rows, we can transform the n^{th} row of A_i into the zero vector. It follows that after the suitable transformation and a permutation of rows, $\text{Jac}_{\mathcal{S}_{\text{dup}}} \mathcal{P}$ can be rewritten as a block triangular matrix. To give the precise form of the determinant of $\text{Jac}_{\mathcal{S}_{\text{dup}}} \mathcal{P}$, we first define

$$R: \det \begin{pmatrix} \partial_{x_1} E_1^i U_i & \dots & \partial_{x_{n-1}} E_1^i U_i & y_1 \\ \vdots & \ddots & \vdots & \vdots \\ \partial_{x_1} E_n^i U_i & \dots & \partial_{x_{n-1}} E_n^i U_i & y_n \end{pmatrix} \in \mathbb{K} \{ \partial_{x_j} E_\ell^i U_i \}_{1 \leq \ell \leq n, 1 \leq j \leq n-1} y_1, \dots, y_n. \quad (3.4)$$

Then it follows that $\det \text{Jac}_{\mathcal{S}_{\text{dup}}} \mathcal{P} = \pm \left(\prod_{i=1}^{nk} \det \text{Jac}_i U_i \right) \cdot \det \Lambda$, where

$$\text{Jac}_i u: \begin{pmatrix} \partial_{x_1} E_1^i u & \dots & \partial_{x_n} E_1^i u & \partial_{u_i} E_1^i u \\ \vdots & \ddots & \vdots & \vdots \\ \partial_{x_1} E_{n-1}^i u & \dots & \partial_{x_n} E_{n-1}^i u & \partial_{u_i} E_{n-1}^i u \\ \partial_{x_1} \text{Det}^i u & \dots & \partial_{x_n} \text{Det}^i u & \partial_{u_i} \text{Det}^i u \\ \partial_{x_1} P^i u & \dots & \partial_{x_n} P^i u & \partial_{u_i} P^i u \end{pmatrix} \in \mathbb{L} u \quad t^{-n-1 \times n-1}, \text{ and}$$

$$\Lambda: (R \partial_{z_j} E_1^i U_i, \dots, \partial_{z_j} E_n^i U_i)_{1 \leq i, j \leq nk} \in \overline{\mathbb{L}} \quad t^{\frac{1}{k} \cdot nk \times nk}. \quad (3.5)$$

The proof that this product is non-zero is the content of [Lemma 5](#) and [Lemma 6](#).

Lemma 5. *For each $i = 1, \dots, nk$, the determinant of $\text{Jac}_i U_i$ is non-zero.*

Proof sketch. To prove that $\det \text{Jac}_i U_i \neq 0$ we will show that $\text{val}_t \det \text{Jac}_i U_i < \infty$, where val_t denotes the valuation in t . The main idea here is to expand $\det \text{Jac}_i U_i$ with respect to the last column and show that the least valuation comes from the product of $\partial_{u_i} \text{Det}^i U_i$ by the determinant of its associated submatrix, which we denote by $\mathcal{M}$.

Since $\partial_{x_j} \text{Det}^i U_i = O(t^\alpha)$, it is clear that for $1 \leq j \leq n$ the minors associated to $\partial_{u_i} E_{j-1}^i U_i$ and $\partial_{u_i} P^i U_i$ are in $O(t^\alpha)$. It remains to show that the product of $\partial_{u_i} \text{Det}^i U_i$ by $\det \mathcal{M}$ is of valuation in t strictly lower than α . Assuming² that $U_i^k = n^k t \epsilon^k = O(t^{k-1/k})$, one computes for $j = 1, \dots, n-1$ and $\ell = 1, \dots, n$ that $\text{val}_t \mathcal{M}_{j,\ell} = \text{val}_t \partial_{x_j} E_\ell^i U_i = \beta - \frac{m_j}{k}$ if $j \neq \ell$ and $\frac{m_j}{k}$ if $j = \ell$. Moreover, it follows from the definition of P and expansion along the last row of the matrix which defines P that the term with lowest t -valuation in $\partial_{x_n} P^i$ is given by the product of $\partial_{x_n, u_i} E_n^i$ by the determinant of the associated sub-matrix of $\partial_{u_i} E_n^i$. Computing this valuation while using that α, β are chosen sufficiently large, we find that $\text{val}_t \partial_{x_n} P^i = \sum_{i=1}^n \frac{m_i}{k} - \frac{1}{k} = M - 1/k$. It follows that the only monomial in the determinant of $\mathcal{M}$ that has no dependency on β comes from the product of diagonal elements of $\mathcal{M}$. Using the definition $\alpha = 3n^2 k \cdot \lfloor 2M/k \rfloor$ and $\beta = \lfloor 2M/k \rfloor$, we conclude that $\text{val}_t \partial_{u_i} \text{Det}^i U_i \cdot \det \mathcal{M} < \alpha$. $\square$

²The other case (i.e. $U_i^k = m^k t \epsilon^k = O(t^{k-1/k})$, $m < n$) works similarly up to a few technical adaptations.

Lemma 6. *The determinant of Λ is non-zero.*

Proof sketch. Proving $\det \Lambda \neq 0$ is again done by analyzing the first terms of $\det \Lambda$. We prove that mod t^α the determinant factors as a product of U_i , the Vandermonde determinant $\prod_{i < j} U_i - U_j$, and a non-zero polynomial $H(t)$. The actual computation is somewhat technical, since $\det \Lambda$ is defined as the determinant of the $nk \times nk$ matrix $R(\partial_{z_j} E_1^i U_i, \dots, \partial_{z_j} E_n^i U_i)_{i,j}$ whose entries are themselves determinants of the $n \times n$ matrices (3.4). We shall give an exposition of the proof, omitting technical details.

We denote $R_j(u) : R(\partial_{z_j} E_1^i u, \dots, \partial_{z_j} E_n^i u)$ and compute $R_j(u_i) \bmod t^\alpha$ for variables $u_1, \dots, u_{nk}$. Note that the latter is a non-zero polynomial in $\mathbb{L}[u_1, \dots, u_{nk}, t]$ which is independent of the polynomials $Q_1, \dots, Q_n$. Let $\tilde{\Lambda} = (R_j(u_i))_{1 \leq i, j \leq nk}$ be the matrix Λ with the U_i replaced by the variables u_i . With tedious but explicit computations it is possible to show that each element in the i^{th} row of $\tilde{\Lambda} \bmod t^\alpha$ is a polynomial in u_i of degree $\leq M - 1$ and valuation $\geq M - nk$. Moreover, all entries of $\tilde{\Lambda} \bmod t^\alpha$ have degree in t bounded by $t^{\beta - 1}$. The choice for α and β ensures that $\det \tilde{\Lambda} \bmod t^\alpha = \det \tilde{\Lambda} \bmod t^\alpha$.

As we have $M \geq nk$, it is possible to factor out u_i^{M-nk} from the i^{th} row of $\tilde{\Lambda} \bmod t^\alpha$ when computing its determinant. This yields polynomials of degree at most $nk - 1$ in u_i on the i^{th} row. Moreover, it is obvious that if $u_i = u_j$ for some $i \neq j$, the determinant of $\tilde{\Lambda}$ vanishes. Hence, we can also factor out the Vandermonde determinant $\prod_{i < j} u_i - u_j$. As this latter product is of degree $nk - 1$ in u_i , we conclude that

$$\det \tilde{\Lambda} \equiv \prod_{i=1}^{nk} u_i^{M-nk} \cdot \prod_{i < j} u_i - u_j \cdot H(t) \bmod t^\alpha, \quad (3.6)$$

for some non-zero polynomial $H \in \mathbb{L}[t]$ whose degree only depends on β . Recall that $\tilde{\Lambda}(U_1, \dots, U_{nk}) = \Lambda$, and all U_i are distinct with valuation in t of $1/k$ by Lemma 3. Using this, equation (3.6) and $\alpha > M - nk - n - n^2k - \beta - 1$, we obtain $\det \Lambda \neq 0$. $\square$

Having now proved that $\det \text{Jac}_{S_{\text{dup}}} \neq 0$ at $\mathcal{P}$, we can apply Lemma 4 and obtain that the specialized series $G_i(0), \dots, \partial_u^{k-1} G_i(0)$ are all algebraic over $\mathbb{K}[t, \epsilon]$. The algebraicity of the complete formal power series $G_1, \dots, G_n$ over $\mathbb{K}[t, u, \epsilon]$ then follows again by [4, Lemma 2.10] from the invertibility of the Jacobian matrix of $E_1, \dots, E_n$ considered with respect to the variables $x_1, \dots, x_n$ (with $t, u, z_0, \dots, z_{nk-1}$ viewed as parameters). The equalities $F_i(t^\alpha, u) = G_i(t, u, 0)$ finally imply that $F_1, \dots, F_n$ are also algebraic over $\mathbb{K}[t, u]$.

As already mentioned, a strength of the presented method is that it is effective. Recall that Theorem 2 summarizes a bound on the algebraicity degree of all $F_i(t, u)$ and estimates the arithmetic complexity of the algorithm which computes $F_i(t, a)$.

Proof sketch of Theorem 2. Using the definition of α and β in the proof of Theorem 1, the result is proven along the same lines as the results in [4, Section 3]. The algebraicity

bound of the shape $n^{2n^2k^2} k^{n-1} n^{2k^2} n^{n-2} n \delta^{n^2k^2} n^{n-2} n / nk !^{nk}$ is a consequence of Bézout's theorem applied to the ideal defined in Lemma 4, while the announced complexity is a consequence of [11, Theorem 2]. Note that the factor $nk !^{nk}$ comes from exploiting the action of the symmetric group $\mathfrak{S}_{nk}$ on the zero set of the duplicated system. $\square$

4 Summary and future work

We can summarize the strategy presented in Section 3 as follows:

1. Set up the deformed system (3.1) and the polynomials $\text{Det}, P \in \mathbb{L}[X_1, \dots, X_n, t, u]$:

$$\text{Det} : \det \begin{pmatrix} \partial_{x_1} E_1 & \dots & \partial_{x_n} E_1 \\ \vdots & \ddots & \vdots \\ \partial_{x_1} E_n & \dots & \partial_{x_n} E_n \end{pmatrix} \quad \text{and} \quad P : \det \begin{pmatrix} \partial_{x_1} E_1 & \dots & \partial_{x_{n-1}} E_1 & \partial_u E_1 \\ \vdots & \ddots & \vdots & \vdots \\ \partial_{x_1} E_n & \dots & \partial_{x_{n-1}} E_n & \partial_u E_n \end{pmatrix}.$$

2. Set up the duplicated polynomial system $\mathcal{S}_{\text{dup}}$, consisting of the duplications of the polynomials E_i, Det, P . It has $nk(n-2)$ variables and equations.
3. Compute a non-trivial element of $\langle \mathcal{S}_{\text{dup}} \rangle : \det \text{Jac}_{\mathcal{S}_{\text{dup}}}^\infty \cap \mathbb{K}[t, z_0, \epsilon]$ and set ϵ to 0.

As illustrated in Example 1, the deformation step is not always needed. In fact, it is clear that for a generic system the equation $\text{Det } u = 0$ will have nk distinct non-zero solutions in $\overline{\mathbb{K}}[t^{\frac{1}{\star}}]$. Moreover, generically, a non-trivial element of both $\langle \mathcal{S}_{\text{dup}} \rangle : \det \text{Jac}_{\mathcal{S}_{\text{dup}}}^\infty$ and $\langle \mathcal{S}_{\text{dup}} \rangle : \prod_{i/j} u_i - u_j^\infty$ contains the sought annihilating polynomial. In general, however, the deformation is important, as the following example shows:

Example 2 (cont.). For s randomly chosen in $\mathbb{Q}$ in (1.4), one cannot apply Algorithm 1 because the ideal $\langle \mathcal{S}_0, \mathcal{S}_1, m \cdot u_1 - u_2 - 1 \rangle$ is not 0-dimensional, despite the fact $\text{Det } u = 0$ has 2 distinct solutions. However, after the deformation (3.1), the system becomes zero-dimensional and can be solved systematically via rather big computations.

Our strategy produces a polynomial system with $nk(n-2)+1$ variables and equations (the additional variable and equation come from the saturation). Since, already for small values of n, k , solving such systems are often out of reach, we wish to briefly introduce an approach that has a better algorithmic complexity. The idea is to reduce (by eliminating $F_2, \dots, F_n$) the initial system to a single functional equation $R = 0$, and then to use Bousquet-Mélou and Jehanne's method [5]. This reduces to solving a polynomial system with just $3nk$ variables and equations. In order to make this approach work, there are two necessary conditions: the equation $\partial_{x_1} R = 0$ should contain enough (that is nk) roots in $\overline{\mathbb{K}}[t^{\frac{1}{\star}}]$ and the corresponding ideal should be zero-dimensional. Note that R is not a DDE anymore in general, so these conditions are not guaranteed. The following proposition ensures that our deformation takes care of the first part, and the example right after shows that the second condition can still fail in practice.

Proposition 7. Let $(\mathbf{E}_{F_1}), \dots, (\mathbf{E}_{F_n})$ be as in [Theorem 1](#) and let $E_1, \dots, E_n$ be the polynomials obtained after deforming $(\mathbf{E}_{F_1}), \dots, (\mathbf{E}_{F_n})$ as in (3.2). Let $U_1, \dots, U_{nk} \in \overline{\mathbb{K}} \epsilon^{-t^{\frac{1}{*}}}$ be the distinct non-zero series solutions in u of the equation $\text{Det } u = 0$ and let $R \in \langle E_1, \dots, E_n \rangle : \text{Det}^\infty \cap \mathbb{L}[x_1, z_0, \dots, z_{nk-1}, t, u]$. Then $U_1, \dots, U_{nk}$ are also solutions of $\partial_{x_1} R = 0$.

Proof. Since $R \in \langle E_1, \dots, E_n \rangle$, there exist $V_1, \dots, V_n \in \mathbb{L}[x_1, \dots, x_n, z_0, \dots, z_{nk-1}, t, u]$ such that $R = U_\ell \sum_{i=1}^n E_i U_\ell V_i$ for any $\ell = 1, \dots, nk$. Differentiating with respect to x_j for $j = 1, \dots, n$ and using $E_i U_\ell = 0$ and that R does not depend on x_j for $j \geq 2$, yields

$$\begin{pmatrix} \partial_{x_1} R U_\ell \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \partial_{x_1} E_1 U_\ell & \dots & \partial_{x_1} E_n U_\ell \\ \vdots & \ddots & \vdots \\ \partial_{x_n} E_1 U_\ell & \dots & \partial_{x_n} E_n U_\ell \end{pmatrix} \begin{pmatrix} V_1 U_\ell \\ \vdots \\ V_n U_\ell \end{pmatrix}. \quad (4.1)$$

By definition of U_ℓ , the matrix $\partial_{x_j} E_i U_\ell$ is singular and [Lemma 3](#) implies that each of its $(n-1) \times (n-1)$ minors is non-zero. It follows that we can express the first row of the matrix as a linear combination of the other rows, then (4.1) implies that $\partial_{x_1} R U_\ell = 0$. $\square$

Example 2 (cont.). For s randomly chosen in $\mathbb{Q}$ in (1.4), reducing to a single equation R (by taking the resultant with respect to x_2), we indeed find that $\partial_{x_1} R = 0$ has two distinct roots in $\overline{\mathbb{K}} \epsilon^{-t^{\frac{1}{*}}}$. However, the computation of a Gröbner basis reveals that the corresponding ideal has positive dimension.

Future work. The present work proves constructively and elementarily the algebraicity of solutions of systems of DDEs with one catalytic variable. Practical experiments (which are also based on further algorithmic tools under development) make us believe that our method has good potential for practical unresolved combinatorial examples as well. Moreover, there are three most natural directions for further work. They will deal with complexity improvements for practical computations and theoretical generalizations:

1. Exploit the strategy *hybrid guess-and-prove*, which was used in [4, Section 2.2.2] to tackle first order scalar DDEs efficiently, and which turns out to be useful when dealing with huge polynomial systems.
2. [Proposition 7](#) ensures that the deformation (3.1) guarantees nk distinct roots of $\partial_{x_1} R = 0$, however, as demonstrated above, the corresponding ideal might still have positive dimension. Investigate whether it is possible to overcome this issue.
3. Extend the results to a higher number of “nested” catalytic variables, where the algebraicity is still guaranteed by Popescu’s theorem, but with no effective version.

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Volume rigidity and algebraic shifting

Denys Bulavka^{*1}, Eran Nevo⁺², and Yuval Peled^{‡2}

¹Department of Applied Mathematics, Faculty of Mathematics and Physics, Charles University, Prague, Czech Republic

²Einstein Institute of Mathematics, Hebrew University, Jerusalem 91904, Israel

Abstract. We study the generic volume rigidity of $d - 1$ -dimensional simplicial complexes in $\mathbb{R}^{d-1}$, and show that the volume rigidity of a complex can be identified in terms of its exterior shifting. In addition, we establish the volume rigidity of triangulations of several 2-dimensional surfaces and prove that, in all dimensions > 1 , volume rigidity is *not* characterized by a corresponding hypergraph sparsity property.

Keywords: rigidity, algebraic shifting, exterior algebra

1 Introduction

Let K be an n -vertex $d - 1$ -dimensional simplicial complex and $\mathbf{p} : V K \rightarrow \mathbb{R}^{d-1}$ be a generic mapping of its vertices, in the sense that its $d - 1$ n coordinates are algebraically independent over $\mathbb{Q}$. This paper deals with the infinitesimal version of the following problem: is there a *non-trivial* continuous motion of the vertices starting at $\mathbf{p}$ that preserves the volumes of all the $d - 1$ -simplices in K ? By “non-trivial” we mean that, for some $d - 1$ -simplex on $V K$ that is not in K , its volume would change along the motion. It is easy to show that the continuous and infinitesimal versions coincide for generic embeddings, as is the case for graph rigidity [2].

Volume Rigidity. The signed volume of a $d - 1$ -face $\sigma = \{v_1, \dots, v_d\} \in K$ with respect to $\mathbf{p}$ is given by the determinant of the $d \times d$ matrix

$$M_{\mathbf{p},\sigma} = \begin{pmatrix} \mathbf{p}_{v_1} & \dots & \mathbf{p}_{v_d} \\ 1 & \dots & 1 \end{pmatrix}.$$

Observe that, for every $1 \leq i \leq d - 1$, $1 \leq j \leq d$, the derivative of the signed volume $\det M_{\mathbf{p},\sigma}$ with respect to the i -th coordinate of $\mathbf{p}_{v_j}$ is given by the cofactor $C_{i,j} M_{\mathbf{p},\sigma}$

^{*}dbulavka@kam.mff.cuni.cz. Denys Bulavka was supported by the project “Grant Schemes at CU” (reg. no. CZ.02.2.69/0.0/0.0/19_073/0016935), by the grant no. 21-32817S of the Czech Science Foundation (GAČR) and by Charles University project PRIMUS/21/SCI/014.

⁺nevo@math.huji.ac.il. Eran Nevo is partially supported by the Israel Science Foundation grant ISF-2480/20.

[‡]yuval.peled@mail.huji.ac.il.

— that is, the determinant of the submatrix obtained by removing the i -th row and j -th column multiplied by -1^{i+j} .

The *volume-rigidity* matrix $\mathfrak{V}(K, \mathbf{p})$ of the pair $(K, \mathbf{p})$ is a $(d-1)n \times f_{d-1}(K)$ matrix, where the columns are indexed by the $(d-1)$ -faces of K , and every vertex is associated with a block of $(d-1)$ rows. The column vector $\mathbf{v}_\sigma$ corresponding to a $(d-1)$ -face $\sigma = \{v_1, \dots, v_d\} \in K$ is defined by

$$\mathbf{v}_\sigma(v_{i,j}) = C_{i,j} M_{\mathbf{p},\sigma}, \quad i \in [d], j \in [d-1],$$

and 0 elsewhere. Here $\mathbf{v}_\sigma(v_{i,j})$ denotes the j -th coordinate of $\mathbf{v}_\sigma$ in the block of v_i . In words, $\mathfrak{V}$ is the Jacobian of the function $\mathbf{p} \mapsto \det M_{\mathbf{p},\sigma}$ for $\sigma \in K$, viewing $\mathbf{p}$ as a $(d-1)n$ -dimensional vector.

This matrix was introduced in [16, Appendix A] along with the description of a *trivial* $(d^2 - d - 1)$ -subspace of the left kernel of $\mathfrak{V}(K, \mathbf{p})$, arising from the volume-preserving transformations of $\mathbb{R}^{d-1}$. Concretely, the trivial subspace consists of all $(d-1)n$ -dimensional vectors z obtained by choosing a $(d-1) \times (d-1)$ matrix A whose trace is zero and a vector $u \in \mathbb{R}^{d-1}$, and letting $z_v = A \cdot \mathbf{p}_v = u$ for every vertex v . The following definition suggests itself.

Definition 1.1. An n -vertex $(d-1)$ -dimensional simplicial complex K is called *volume-rigid* if

$$\text{rank } \mathfrak{V}(K, \mathbf{p}) = (d-1)n - (d^2 - d - 1),$$

for a generic $\mathbf{p} : V(K) \rightarrow \mathbb{R}^{d-1}$.

Exterior shifting. Algebraic shifting was introduced by Kalai (see e.g. [11] and the survey [12]) and has been studied extensively in algebraic combinatorics. Here we present a variant of exterior shifting. The standard basis e_i $i \in [n]$ of $\mathbb{R}^n$ induces the basis e_σ $\sigma \subseteq [n]$ of its exterior algebra $\wedge \mathbb{R}^n$. Consider a generic basis $f_1, \dots, f_n$ of $\mathbb{R}^n$, where without loss of generality we assume that $f_1 = \mathbf{1} \in \mathbb{R}^n$, namely, the other $n^2 - n$ coordinates in this basis are algebraically independent over $\mathbb{Q}$. Consider the *exterior face ring* $\wedge K = \wedge \mathbb{R}^n / \langle e_\sigma : \sigma \notin K \rangle$, and let q denote the natural quotient map. Given a partial order $<$ on the power set of $[n]$, define

$$\Delta^{<} K = \{\sigma : q(f_\sigma) \notin \text{span}_{\mathbb{R}}\{q(f_\tau) : \tau < \sigma, |\tau| = |\sigma|\}\}. \quad (1.1)$$

Of special importance in our case is the partial order $<_p$ defined by $\sigma = \{\sigma_1 < \dots < \sigma_m\} \leq_p \tau = \{\tau_1 < \dots < \tau_{m'}\}$ if $m = m'$ and $\sigma_i \leq \tau_i$, $\forall i \in [m]$. Corollary 2.3 asserts that $\Delta^p K : \Delta^{<_p} K$ is a shifted simplicial complex independent of the generic choice of f . (Note that $\Delta^p K$ may have more faces than K .)

1.1 Main results.

Our main result is a characterization of volume rigidity in the setting of Kalai's exterior shifting.

Theorem 1.2. *Fix $d \geq 3$. An n -vertex $d - 1$ -dimensional simplicial complex K is volume-rigid if and only if $\{1, 3, 4, \dots, d, n\} \in \Delta^p K$.*

In the 2-dimensional case we are able to derive the volume rigidity of triangulations of the following surfaces.

Corollary 1.3. *Every triangulation of the 2-sphere, the torus, the projective plane or the Klein bottle is volume rigid. In addition, every triangulation of the 2-sphere and the torus minus a single triangle is also volume-rigid. In particular, every simplicial disc with a 3-vertex boundary is minimally volume-rigid.*

In the case of the 2-sphere we give a complete mathematical proof. For the other surfaces, we reduce — via edge contractions á la Whiteley [23] — to irreducible triangulations, whose volume-rigidity we verify numerically.

Hypergraph sparsity was introduced by Streinu and Theran [20], generalizing results on graph sparsity, prominently by White and Whiteley [24] who studied it from a matroid perspective. We say that a $d - 1$ -complex is $d - 1, d^2 - d - 1$ -sparse (resp. tight) if every subset A of its vertices of cardinality at least d spans at most $d - 1 |A| - d^2 - d - 1$ simplices of dimensions $d - 1$ (resp. and equality holds when A equals the entire vertex set).

Clearly, a vertex subset A spanning more $d - 1$ -simplices induces a non trivial linear dependence between the columns of $\mathfrak{V}(K, \mathbf{p})$, and it is natural to ask whether this characterizes all the linear dependencies in the volume rigidity matrix. Using Theorem 1.2, we show that the answer is negative, hence a Laman-type condition for volume-rigidity does not hold true¹.

Corollary 1.4. *For every $d \geq 3$, there exists a $d - 1, d^2 - d - 1$ -tight $d - 1$ -complex that is not volume-rigid.*

1.2 Relation to previous works.

The maximal independent sets of columns of $\mathfrak{V} \binom{n}{d}, \mathbf{p}$, for all generic embeddings $\mathbf{p}$, form the bases of the same matroid. For $d = 2$ they correspond to spanning trees, namely the bases in the graphic matroid on $\binom{n}{2}$. Kalai [10] introduced for every integer $k \geq 1$ the k -hyperconnectivity matroid on $\binom{n}{2}$, where $k = 1$ corresponds to the graphic

¹Corollary 1.4 shows that Prop.1 in the preprint [19] from 2007 is a misstatement.

²The k -hyperconnectivity matroid is derived from an embedding of the vertex set into $\mathbb{R}^k$. Studying higher hyperconnectivity translates to increasing the dimension of the embedding space.

matroid, and identified its bases in terms of exterior shifting (w.r.t. the lexicographic order): G is a basis if and only if the edges of $\Delta^{\text{lex}} G$ form the initial segment that ends with $\{k, n\}$, w.r.t. the lex-order.

Here, in [Theorem 1.2](#), rather than increasing the dimension of the embedding space and staying with graphs, we increase also the dimension of the pure complex, by the same number, and characterize the bases of the resulted d -volume-rigidity matroid in terms of exterior shifting w.r.t. the partial order $<_p$.

The fact that $d-1, d^2-d-1$ -sparse complexes form the independent sets of a matroid on $\binom{n}{d}$ was asserted in [\[15, 24\]](#). Additional matroidal and algorithmic properties of sparsity matroids were studied by Streinu and Theran in [\[20, 21\]](#). By [Corollary 1.4](#), the $d-1, d^2-d-1$ -sparsity matroid strictly contains the $d-1$ -volume-rigidity matroid for all $d \geq 3$. It would be interesting to find further combinatorial conditions that once imposed on the bases of the sparsity-matroid would give the bases of the volume-rigidity matroid.

The remainder of the paper is organized as follows. In [Section 2](#) we establish the connection between volume rigidity and exterior shifting, and prove [Theorem 1.2](#). Afterwards, in [Section 3](#) we investigate the effect of local moves on volume rigidity and prove [Corollary 1.3](#). In the following [Section 4](#) we prove [Corollary 1.4](#), and we conclude in [Section 5](#) with some related open problems.

2 Volume rigidity and Δ^p .

This section is devoted to studying the basic properties of the shifted complex $\Delta^p K$, and to establishing the connection between $\Delta^p K$ and K 's volume rigidity.

Basic properties of Δ^p .

We start by briefly exploring some useful properties of the complex $\Delta^p K$ that appears in [Theorem 1.2](#). Given a partially ordered set (poset) $(P, <)$ and an element $x \in P$ we denote by $P_{<,x}$ the prefix $\{y \in P : y \leq x\}$.

Claim 2.1. *Let $(P, <)$ be a poset and $x \in P$, then there exists a linear extension $<_l$ of $<$ such that $P_{<_l,x} = P_{<,x}$.*

Proof. View the sets $A = P_{<,x}$ and $B = P \setminus P_{<,x}$ as posets with the partial order induced by $<$. Extend each of these posets linearly, and concatenate the extensions such that the elements in A are smaller than those in B . $\square$

We will mainly work with the partial order $<_p$ on the power set of $[n]$ and denote the set of its linear extensions by $\mathcal{L}$. We usually denote an element in $\mathcal{L}$ by $<_l$ and the corresponding shifted complex by $\Delta^l K : \Delta^{<_l} K$.

Claim 2.2. $\Delta^p K = \bigcup_{<_l \in \mathcal{L}} \Delta^l K$.

Proof. On the one hand, if $\sigma \in \Delta^l K$ for some $<_l \in \mathcal{L}$ then $q f_\sigma$ is not spanned by $B_{l,\sigma} : \{q f_\tau : \tau <_l \sigma\}$, which contains the vector set $B_{p,\sigma}$. Therefore, by the definition of $\Delta^<$ in (1.1), we find that $\Delta^p K \supseteq \Delta^l K$. On the other hand, for every $\sigma \in \Delta^p K$, there exists by the previous claim a linear extension $<_l \in \mathcal{L}$ satisfying $B_{p,\sigma} \subseteq B_{l,\sigma}$ hence $\sigma \in \Delta^l K$. $\square$

Corollary 2.3. For every simplicial complex K there holds that $\Delta^p K$ is a shifted simplicial complex independent of the choice of the generic basis f . In addition, $\Delta^p K = K$ if K is shifted. $\square$

That $\Delta^p K$ is downwards closed follows exactly as in the proof for $\Delta^{\text{lex}} K$. The rest of Corollary 2.3 follows immediately from the above decomposition of $\Delta^p K$ and the fact that the basic properties of algebraic shifting in [11] – being shifted, and independence the the generic f chosen – hold in every linear extensions of $<_p$, as remarked in [11, p.58].

Volume rigidity and Δ^p .

We are now ready to prove Theorem 1.2. We denote $\sigma_0 = \{1, 3, \dots, d, n\}$ and observe that the prefix $B : \{\tau \leq_p \sigma_0 : |\tau| = d\}$ consists of the subsets d and $d \setminus \{i\} \cup \{v\}$ for $2 \leq i \leq d$ and $d - 1 \leq v \leq n$. We define a linear transformation $\psi : \bigoplus_{i=2}^d \wedge^1 \mathbb{R}^n \rightarrow \wedge^d \mathbb{R}^n$ given by

$$\psi(m_2, \dots, m_d) = \sum_{i=2}^d f_{d \setminus \{i\}} \wedge m_i.$$

Lemma 2.4. The image of ψ is spanned by $\{f_\tau : \tau \in B\}$, and its kernel is $d^2 - d - 1$ - dimensional.

Proof. The fact that $f_\tau \in \text{im } \psi$ for every $\tau \in B$ can be shown directly. Indeed, we have $\psi(0, \dots, 0, f_d) = f_d$ and by taking $m_i = f_v$, $m_{i'} = 0 \forall i' \neq i$ we have that $\psi(0, \dots, f_v, \dots, 0) = f_{d \setminus \{i\} \cup \{v\}}$ for $2 \leq i \leq d$ and $d - 1 \leq v \leq n$. To show that these $1 + n - d - d - 1$ linearly independent vectors span the image of ψ , we will construct $d^2 - d - 1$ linearly independent vectors in $\ker \psi$ which actually completes the proof by the rank-nullity theorem since $1 + n - d - d - 1 = d^2 - d - 1 = n - d - 1$.

First, for every $2 \leq i \leq d$ and $j \in d \setminus \{i\}$ consider the vector defined by setting $m_i = f_j$ and $m_{i'} = 0$ for every $i' \neq i$. Then,

$$\psi(m_2, \dots, m_d) = f_{d \setminus \{i\}} \wedge f_j = 0$$

since $j \in d \setminus \{i\}$. This amounts to $d - 1$ vectors in $\ker \psi$, and the remaining $d - 2$ are given by vectors of the form $m_i = a_i f_i$, $2 \leq i \leq d$, where the scalars $a_2, \dots, a_d$ satisfy

$\sum_{i=2}^d -1^i a_i = 0$. Indeed,

$$\psi_{m_2, \dots, m_d} = \sum_{i=2}^d a_i f_{d \setminus \{i\}} \wedge f_i = \sum_{i=2}^d a_i (-1)^{d-i} f_d = 0.$$

The linear independence of these $d^2 - d - 1$ vectors follows directly from the linear independence of $f_1, \dots, f_d$. $\square$

Proof of Theorem 1.2. Identify the vertices of K with the set $[n]$. Without loss of generality, assume that $f_{d-1} \in K \times [d-1, n]^{d^2-d-1}$, as otherwise K is not volume-rigid and $\{1, 3, 4, \dots, d, n\} \notin \Delta^p K$. In addition, suppose that the generic embedding $\mathbf{p} : V(K) \rightarrow \mathbb{R}^{d-1}$ is obtained from the vectors $f_2, \dots, f_d$ in the generic basis of $\mathbb{R}^n$ by taking $f_i \cdot v$ $\mathbf{p}(v)_{i-1}$ for every $2 \leq i \leq d$ and $v \in [n]$.

Consider the (i, v) -unit vector $e_{i,v} = m_{2,\dots,m_d}$ in the domain of ψ , for $2 \leq i \leq d$ and $v \in [n]$, defined by $m_i = e_v$ and $m_{i'} = 0, \forall i' \neq i$. Then,

$$\begin{aligned} \psi_{e_{i,v}} &= f_{d \setminus \{i\}} \wedge e_v \\ &= (-1)^{d-i} f_1 \wedge \dots \wedge f_{i-1} \wedge e_v \wedge f_{i+1} \wedge \dots \wedge f_d \\ &= (-1)^{d-i} (-1)^{d-1} f_2 \wedge \dots \wedge f_{i-1} \wedge e_v \wedge f_{i+1} \wedge \dots \wedge f_d \wedge f_1. \end{aligned}$$

Let $\sigma = \{v_1, \dots, v_d\} \subset [n]$. Clearly, for the inner product on $\wedge \mathbb{R}^n$ with orthonormal basis $\{e_\sigma : \sigma \subset [n]\}$, we have $\langle e_\sigma, \psi_{e_{i,v}} \rangle = 0$ if $v \notin \sigma$. Otherwise, by the identification of $\mathbf{p}$ with $f_2, \dots, f_d$ above and $f_1 = \mathbf{1}$, if $v = v_j$ then $\langle e_\sigma, \psi_{e_{i,v}} \rangle$ is equal to $(-1)^{i-1}$ times the determinant of the matrix that is obtained from $M_{\mathbf{p},\sigma}$ by replacing its $(i-1)$ -th row with the j -th d -dimensional all-ones row vector. Consequently,

$$\langle e_\sigma, \psi_{e_{i,v}} \rangle = (-1)^{i-1} C_{i-1,j} M_{\mathbf{p},\sigma}.$$

Thus, by letting $q : \wedge \mathbb{R}^n \rightarrow \wedge K$ be the natural quotient map, and by choosing the basis $\{e_\sigma : \sigma \in K\}$ for $\wedge K$, we find that the $f_{d-1} \in K \times [d-1, n]$ matrix representation of $q \circ \psi$ is equal — up to multiplying some of its columns by -1 and reordering them — to the transpose of the volume-rigidity matrix $\mathfrak{V}(K, \mathbf{p})$. Therefore, K is volume-rigid if and only if $\dim \ker q \circ \psi = d^2 - d - 1$. In other words, K is *not* volume-rigid if and only if there exists a non-zero $f \in \text{im } \psi$ such that $q(f) = 0$. By the characterization of ψ 's image in Lemma 2.4, f can be written as a non-trivial linear combination $f = \sum_{\tau \in B} \lambda_\tau f_\tau$.

To conclude the proof we claim that $q(f) = 0$ for some $f \in \text{span}\{f_\tau : \tau \in B\}$ if and only if $\sigma_0 \notin \Delta^p K$. Indeed, on one direction, $q(\sum_{\tau \in B} \lambda_\tau f_\tau) = 0$ implies that for some $\tau \in B$, $q(f_\tau)$ is a linear combination of its predecessors in $\prec_p$. By (1.1), $\tau \notin \Delta^p K$ and since $\Delta^p K$ is shifted then $\sigma_0 \notin \Delta^p K$. On the other hand, by manipulating the linear combination which asserts that $\sigma_0 \notin \Delta^p K$, we obtain a non-zero vector $f = \sum_{\tau \in B} \lambda_\tau f_\tau$ satisfying $q(f) = 0$. $\square$

3 Volume rigidity, local moves and homology

We turn to study the effect of local combinatorial moves on volume rigidity. We start by proving a volume-rigidity analog of Whiteley's vertex splitting [23], by which he showed that every triangulation of the 2-sphere has a 3-rigid 1-skeleton.

Lemma 3.1 (Edge contraction). *Let K be a pure $d - 1$ -dimensional simplicial complex, $e = \{u, w\} \in K$ such that at least $d - 1$ facets in K contain e . Let K' to be the simplicial complex obtained from K by contracting the edge e , i.e. by identifying the vertex u with w , and removing duplicates. If K' is volume rigid then so is K .*

Sketch of proof. Without loss of generality assume that $u < w$ are the first among the n vertices of K , as the vertex labels do not affect volume-rigidity. We will construct an auxiliary $(d - 1) \times n$ matrix A such that

$$\text{rank } \mathfrak{V}(K, \mathbf{p}) \geq \text{rank } A \quad d - 1 \leq n \leq d^2 - d - 1.$$

First, we replace the position of the vertex w , i.e. $\mathbf{p}_w$, by the position of the vertex u , i.e. $\mathbf{p}_u$. Formally we define a new (non-generic) placement of vertices $\mathbf{p}'$ that coincides with $\mathbf{p}$ on all vertices except w on which we set it to equal to $\mathbf{p}_u$. Clearly, since $\mathbf{p}$ is generic, there holds $\text{rank } \mathfrak{V}(K, \mathbf{p}) \geq \text{rank } \mathfrak{V}(K, \mathbf{p}')$. To obtain A , we add the rows in $\mathfrak{V}(K, \mathbf{p}')$ corresponding to the vertex u to the rows corresponding to the vertex w , an operation that does not change the rank. We first claim that the submatrix of A which corresponds to the columns of the facets L containing $e = \{u, w\}$ is supported on the rows corresponding to u . Second, we claim that the submatrix $A_{u,L}$ of A corresponding to the $d - 1$ rows of u and the columns of L has a full rank of $d - 1$. Third, we consider the complement submatrix $A' : A_{\{u\}^c, L^c}$ whose rows correspond to all the vertices except u , and columns to all the facets that are not in L . We observe that A' contains as a submatrix the generic volume rigidity matrix $\mathfrak{V}(K', \mathbf{p})$ — where $\mathbf{p}$ is viewed here as a generic embedding of $V(K') = V(K) \setminus \{u\}$ into $\mathbb{R}^{d-1}$. In the end the matrix A takes the form

$$A = \begin{matrix} & \begin{matrix} L & L^c \end{matrix} \\ \begin{matrix} u \\ \{u\}^c \end{matrix} & \begin{pmatrix} A_{u,L} & * \\ 0 & A' \end{pmatrix} \end{matrix}.$$

Combining the assumption that K' is volume-rigid and the claim that $A_{u,L}$ has full rank we obtain

$$\text{rank } \mathfrak{V}(K, \mathbf{p}) \geq \text{rank } A \geq \text{rank } A_{u,L} + \text{rank } A' \geq (d - 1) + (d^2 - d - 1),$$

as claimed. □

The next two lemmas are direct analogs of basic results in graph rigidity [3] asserting that gluing preserves volume-rigidity.

Lemma 3.2. *Let K be $(d-1)$ -volume-rigid, $v \notin V(K)$ and $S \subseteq V(K)$ such that $|S| \geq d$, then $K \cup v *_{d-1}^S$ is $(d-1)$ -volume-rigid.*

Lemma 3.3 (Union of volume-rigid complexes). *Let K and L be $(d-1)$ -volume-rigid complexes such that $|V(K) \cap V(L)| \geq d$. Then $K \cup L$ is $(d-1)$ -volume-rigid.*

3.1 Proof of Corollary 1.3

Barnette and Edelson [5, 6] proved that every compact surface without boundary admits only finitely many *irreducible* triangulations, namely, triangulations where every edge contraction would result in a simplicial complex not homeomorphic to the given surface. Thus, by Lemma 3.1, in order to conclude that for a given surface S every simplicial complex that triangulates it is volume-rigid, it is enough to verify if for the irreducible triangulations of S . Those are known for the surfaces indicated in Corollary 1.3: one for the 2-sphere (namely the boundary of a tetrahedron), two for the projective plane [4], 21 for the torus [13] and 29 for the Klein bottle [14, 22]. Clearly the boundary of the tetrahedron is volume-rigid, and we verified by computer that the irreducible triangulations K of the other surfaces mentioned above are volume-rigid – for this it was enough to find some embedding $\mathbf{p}_K : V(K) \rightarrow \mathbb{R}^2$ such that $\text{rank } \mathfrak{B}(K, \mathbf{p}_K) = 2|V(K)| - 5$.

Remark 3.4. The fact that every triangulation K of the 2-sphere is volume-rigid follows also from combining the 3-hyperconnectivity of its graph with the Cohen-Macaulay property. Indeed, the first property says that $\{3, |V(K)|\} \in \Delta^{\text{lex}}(K)$, and as K is Cohen-Macaulay then $\Delta^{\text{lex}}(K)$ is pure and hence $\{1, 3, |V(K)|\} \in \Delta^{\text{lex}}(K)$ as $\Delta^{\text{lex}}(K)$ is shifted, which implies, by Claim 2.2, that $\{1, 3, |V(K)|\} \in \Delta^p(K)$, and we are done by Theorem 1.2.

To prove the second part of Corollary 1.3 we are left to show that removing one triangle from a triangulated 2-sphere or torus preserves volume-rigidity, done next. A pure simplicial complex is a *minimal cycle* (over some coefficients commutative ring $\mathbb{F}$) if there exists an $\mathbb{F}$ -linear combination of its facets whose boundary vanishes, and no proper nonempty subset of its facets has this property. For example, every triangulation of a compact connected surface (resp. and orientable) is a minimal cycle over $\mathbb{Z}_2$ (resp. $\mathbb{Z}$).

Lemma 3.5. *If K is a $(d-1)$ -dimensional volume rigid minimal cycle over $\mathbb{Z}$, then $K \setminus \sigma$ is volume rigid for every $\sigma \in K$.*

We first give short proof for the special case $d = 3$ and conclude the proof of Corollary 1.3. The proof for the general case can be found in the full version of this article [7].

Proof of Lemma 3.5 (d = 3). As K is a minimal cycle over $\mathbb{Z}$, its 2-dimensional homology with $\mathbb{R}$ -coefficients is one dimensional, and for each facet σ of K , for $K \setminus \sigma$ this homology vanishes. By the translation of homology in terms of algebraic shifting, $\Delta^{\text{lex}} K \ni \{2, 3, 4\} \notin \Delta^{\text{lex}} K \setminus \sigma$, and as shifting preserves containment we conclude

$$\Delta^{\text{lex}} K \setminus \sigma \subseteq \Delta^{\text{lex}} K \setminus \{\{2, 3, 4\}\}.$$

Note that in this dimension³ $\tau <_p \{1, 3, n\}$ iff $\tau <_{\text{lex}} \{1, 3, n\}$, and thus: $\{1, 3, n\} \in \Delta^p K$ (by Theorem 1.2), hence $\{1, 3, n\} \in \Delta^{\text{lex}} K$, and by the displayed equality above also $\{1, 3, n\} \in \Delta^{\text{lex}} K \setminus \sigma$, so finally $\{1, 3, n\} \in \Delta^p K \setminus \sigma$, equivalently, $K \setminus \sigma$ is volume-rigid. $\square$

Proof of Corollary 1.3. This is immediate from Lemma 3.1, Lemma 3.5 for the case $d = 3$, and the discussion in the beginning of Section 3.1. $\square$

4 Volume rigidity and sparsity

Proof of Corollary 1.4. Let $d \geq 3$ and K be the $d - 1$ -dimensional simplicial complex obtained from the graph $K_{3,3}$ by iterating the cone operation $d - 2$ times. Then K is $d - 1, d^2 - d - 1$ -sparse. (Indeed, $K_{3,3}$ is $2, 3$ -sparse, and if a pure $k - 1$ -dimensional simplicial complex is $k - 1, k^2 - k - 1$ -sparse then its cone is $k, k^2 - k - 1 - 1$ -sparse.) Thus, by completing it to a basis in the $d - 1, d^2 - d - 1$ -sparsity matroid, we find a basis K' containing K , so K' is $d - 1, d^2 - d - 1$ -tight.

In order to show that K' is not volume-rigid, by Theorem 1.2 it is enough to show that

$$\{1, 2, \dots, d - 2, d - 1, d - 2\} \in \Delta^p K',$$

as $\{1, 2, \dots, d - 2, d - 1, d - 2\} \not\leq_p \{1, 3, 4, \dots, d, n\}$ and using tightness.

The displayed equation above follows from basic properties of this shifting operator, proved in the same way as for exterior shifting w.r.t. the lex-order:

- If K is a subcomplex of K' then $\Delta^p K \subseteq \Delta^p K'$.
- Cone and Δ^p commute, namely, if $K = v * L$ for a simplicial complex L then $\Delta^p K = 1 * \Delta^p L = 1$.

Here for a family F of subsets of m , $F = 1 : \{B = 1 : B \in F\}$, and $B = 1 : \{i = 1 : i \in B\}$ (so $\emptyset = 1 - \emptyset$). To conclude the proof it is left to note that $\{3, 4\} \in \Delta^{\text{lex}} K_{3,3}$ and hence, by Claim 2.2, also $\{3, 4\} \in \Delta^p K_{3,3}$. $\square$

³For $d > 3$, $\{\tau : \tau <_p \{1, 3, 4, \dots, d, n\}\}$ is smaller than $\{\tau : \tau <_{\text{lex}} \{1, 3, 4, \dots, d, n\}\}$.

5 Concluding remarks

We end up with some related open problems. An obvious one is to extend [Corollary 1.3](#) to include all surface triangulations.

Conjecture 5.1. *Every triangulation of a compact connected surface without boundary, minus a single triangle, is volume-rigid.*

The problem we face in applying Fogelsanger’s decomposition [9] (see also [8, Sec. 3.3]) to volume rigidity of surfaces is that the pieces in the decomposition include triangle faces not existing in the original triangulation, and thus the gluing lemmas we could prove, e.g. [Lemma 3.3](#), are not strong enough to settle [Conjecture 5.1](#).

Higher dimensions: It is known that for every triangulation K of the 2-sphere on n vertices, minus a single triangle, its exterior shifting $\Delta^{\text{lex}} K \setminus \Delta^p K$ consists exactly of the triangle $13n$ and all the triangles that are smaller than it in the lex-order, and their subsets. This is a sufficient condition for volume rigidity by [Theorem 1.2](#). The following conjecture deals with a higher-dimensional counterpart of this fact.

Conjecture 5.2. *For every $d \geq 3$, every triangulation K of the $d - 1$ -sphere minus a single $d - 1$ -simplex is volume rigid.*

It is also natural to ask whether the stronger property of $\{1, 3, 4, \dots, d, n\} \in \Delta^{\text{lex}} K$ holds true. This is known, and tight, for stacked spheres [17] (also [18, Example 2.1.8]). Let us remark that the conclusion $\{1, 3, 4, \dots, d, n\} \in \Delta^s K$ for Kalai’s symmetric shifting operator $\Delta^s \cdot$ is equivalent to the hard-Lefschetz isomorphism from degree 1 to degree $d - 1$ in a generic Artinian reduction of the Stanley-Reisner ring of K over the field of reals; the later isomorphism was proved recently by Adiprasito [1].

Back to general complexes,

Problem 5.3. *For every dimension, find a combinatorial characterization of the corresponding volume-rigidity matroid.*

The combinatorial characterization problem is important for the d -rigidity matroid (and is open for $d \geq 3$). The d -rigidity of a graph G on n vertices is equivalent to $\{d, n\} \in \Delta^s G$. In view of this fact, we ask:

Problem 5.4. *Define a version of symmetric shifting $\Delta^{sp} \cdot$ and find a matroid on $\binom{n}{d}$ such that its bases K are exactly those satisfying $\Delta^{sp} K = \{\tau : \tau \leq_p \{1, 3, 4, \dots, d, n\}\}$.*

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Invariant theory for the free left-regular band and a q -analogue

Sarah Brauner^{*1}, Patricia Commins^{†1}, and Victor Reiner^{‡1}

¹*School of Mathematics, University of Minnesota, Minneapolis, MN*

Abstract. We examine from an invariant theory viewpoint the monoid algebras for two monoids having large symmetry groups. The first monoid is the free left-regular band on n letters, defined on the set of all injective words, that is, the words with at most one occurrence of each letter. This monoid carries the action of the symmetric group. The second monoid is one of its q -analogues, considered by K. Brown, carrying an action of the finite general linear group. In both cases, we show that the invariant subalgebras are semisimple commutative algebras, and characterize them using *Stirling* and *q -Stirling numbers*.

We then use results from the theory of random walks and random-to-top shuffling to decompose the entire monoid algebra into irreducibles, simultaneously as a module over the invariant ring and as a group representation. Our irreducible decompositions are described in terms of *derangement symmetric functions* introduced by Désarménien and Wachs.

Keywords: left-regular band, random-to-top, Stirling number, symmetric group, general linear group, unipotent character.

To the memory of Georgia Benkart.

1 Introduction

Motivated by results on mixing times for shuffling algorithms on permutations, Bidigare [2] and Bidigare, Hanlon and Rockmore [1] developed a complete spectral analysis for a class of random walks on the chambers of a hyperplane arrangement. Their work relied heavily on the *Tits semigroup* structure on the cones of the arrangement. Later, Brown [4] generalized their analysis to random walks coming from semigroups $\mathcal{L}$ which form a *left-regular band (LRB)*, meaning that $x^2 = x$ and $xyx = xy$ for all x, y in $\mathcal{L}$. Left-regular bands have since been studied by many others; see Margolis, Saliola, and Steinberg [10] for extensive work along with a historical discussion in their Chapter 1.

^{*}braun622@umn.edu. Supported by NSF Graduate Research Fellowship

[†]commi010@umn.edu. Supported by NSF Graduate Research Fellowship

[‡]reiner@umn.edu. Supported by NSF grant DMS-2053288

Some left-regular bands come equipped with natural symmetry groups. In his PhD thesis [2], Bidigare studied the Tits semigroup algebra of a reflection arrangement under the action of its corresponding reflection group. He discovered that the invariant subalgebra is anti-isomorphic to a well-studied algebra: Solomon's descent algebra. The shuffling operators contained in the invariant subalgebra are random walks on the reflection group (see [4, Theorem 8]) and include interesting examples such as random-to-top shuffling and inverse riffle shuffling. The close relationship between the Tits semigroup algebra and the descent algebra has proved to be useful beyond shuffling; for example, Saliola used this viewpoint of the descent algebra in his computation of its quiver [16].

This abstract is based on [3], where we study two examples of left-regular bands M , related to those discussed by Brown, with large groups of monoid automorphisms G :

- the *free LRB on n letters* [4, §1.3], denoted $\mathcal{F}_n$, with G the symmetric group $\mathfrak{S}_n$, and
- a q -analogue $\mathcal{F}_n^q$ related to monoids in [4], and G the general linear group $GL_n \mathbb{F}_q$.

Inspired by Bidigare's work, we study these left-regular bands under the action of their symmetry groups. In particular, for both monoids $M = \mathcal{F}_n, \mathcal{F}_n^q$, we examine the *monoid algebra* $R = \mathbf{k}M$ with coefficients in a commutative ring $\mathbf{k}$ with 1, and answer the two *main questions of invariant theory* for G acting on R :

Question 1.1. *What is the structure of the invariant subalgebra R^G ?*

Question 1.2. *What is the structure of R , simultaneously as an R^G -module and a G -representation?*

Notably, both questions are answered for the two monoids $M = \mathcal{F}_n, \mathcal{F}_n^q$ in parallel.

We answer [Question 1.1](#) by showing that (when $q \in \mathbf{k}^\times$) the invariant subalgebra R^G for both monoids is semisimple, commutative, and generated by a single element. This generator acts semisimply on R with eigenvalues in $0, 1, \dots, n$ for $M = \mathcal{F}_n$ and $0_q, 1_q, \dots, n_q$ for $M = \mathcal{F}_n^q$. Our analysis uses the combinatorics of Stirling and q -Stirling numbers.

Our answer to [Question 1.2](#) involves decomposing the eigenspaces of the generator of R^G on R as G -representations (for $\mathbf{k}$ a field in which $|G|$ is invertible). We do so by (i) introducing and studying filtrations on R and (ii) inductively constructing eigenvectors. We describe these eigenspaces in terms of *derangement symmetric functions* first introduced by Désarménien and Wachs [5]. Derangement symmetric functions have connections to many well-studied objects in combinatorics such as the complex of injective words [13], random-to-top and random-to-random shuffling [21], higher Lie characters [21], and configuration spaces [9]; see [3, §3.3] for historical details. We add to this list by showing they form crucial building blocks for the invariant theory of $\mathbf{k}\mathcal{F}_n$ and $\mathbf{k}\mathcal{F}_n^q$.

1.1 Outline

In [Section 2](#) we define the two monoids of interest and their respective symmetry groups. In [Section 3](#) we determine the structure of the invariant subalgebras for both monoid algebras, answering [Question 1.1](#) with [Theorem 3.4](#). In order to present our answer to [Question 1.2](#) for both monoids in parallel, we use the language of symmetric functions. Accordingly, [Section 4](#) reviews the famous Frobenius characteristic map and its lesser-known q -analogue. We also give background on the *derangement symmetric function* here. Finally, in [Section 5](#) we answer [Question 1.2](#) with [Theorem 5.1](#).

2 The left-regular bands and their symmetry groups

Definition 2.1. The *free left-regular band* (or LRB) on n letters $\mathcal{F}_n$ (see [[4](#), §1.3], [[18](#), §14.3.1]) consists, as a set, of all words $\mathbf{a} = a_1, a_2, \dots, a_\ell$ with letters a_i from $\{1, 2, \dots, n\}$ and no repeated letters, that is, $a_i \neq a_j$ for $1 \leq i < j \leq \ell$. The set $\mathcal{F}_n$ becomes a semigroup under the following operation: if $\mathbf{b} = b_1, \dots, b_m$ is another word in $\mathcal{F}_n$, then their product is

$$\mathbf{a} \cdot \mathbf{b} : \quad a_1, \dots, a_\ell, b_1, \dots, b_m^\wedge,$$

where we adopt the notation from Brown [[4](#)] that for a sequence $\mathbf{c} = c_1, \dots, c_p$, the subsequence $\mathbf{c}^\wedge = c_1, \dots, c_p^\wedge$ is obtained by removing any letter c_i that appears already in the prefix $c_1, c_2, \dots, c_{i-1}$.

Example 2.2. In $\mathcal{F}_5$, one has $2, 5, 1 \cdot 4, 2, 3, 1 = 2, 5, 1, 4, 3$.

One can check that the empty word ϵ is an identity element for this operation, and hence $\mathcal{F}_n$ is not only a semigroup, but a *monoid*. For any word $\mathbf{a} \in \mathcal{F}_n$, the *length* $\ell(\mathbf{a}) = \ell$ lies anywhere in the range $0 \leq \ell \leq n$. There is a left action on $\mathcal{F}_n$ by $\mathfrak{S}_n$, the symmetric group on n letters, where $w \in \mathfrak{S}_n$ acts by $w \cdot a_1, \dots, a_\ell = w(a_1), \dots, w(a_\ell)$.

The q -analogue of $\mathcal{F}_n$ that we will consider will be denoted $\mathcal{F}_n^q$, defined as follows.

Definition 2.3. As a set, $\mathcal{F}_n^q$ consists of all partial flags of subspaces $\mathbf{A} = A_1, \dots, A_\ell$ where A_i is an i -dimensional $\mathbb{F}_q$ -linear subspace of $\mathbb{F}_q^n$, and $A_1 \subset A_2 \subset \dots \subset A_\ell$. Again the *length* $\ell(\mathbf{A}) = \ell$ lies in the range $0 \leq \ell \leq n$. We multiply $\mathbf{A}$ with another flag $\mathbf{B} = B_1, \dots, B_m$ in $\mathcal{F}_n^q$ by:

$$\mathbf{A} \cdot \mathbf{B} : \quad A_1, \dots, A_\ell, A_\ell \cap B_1, A_\ell \cap B_2, \dots, A_\ell \cap B_m^\wedge.$$

As before, for a sequence $\mathbf{C} = C_1, \dots, C_p$ of nested subspaces $C_1 \subseteq C_2 \subseteq \dots \subseteq C_p$, the subsequence $\mathbf{C}^\wedge$ is obtained by removing any C_i appearing in the prefix $C_1, \dots, C_{i-1}$.

As in the $q = 1$ case, $\mathcal{F}_n^q$ is not only a semigroup but a *monoid* since the empty flag is an identity element¹. Our symmetry group in this case is $GL_n : GL_n \mathbb{F}_q$, and $g \in GL_n$ acts on the left by $g \cdot A_1, \dots, A_\ell \mapsto g \cdot A_1, \dots, g \cdot A_\ell$.

Let $\mathbf{k}$ be a commutative ring with 1. For any finite monoid M (such as $M = \mathcal{F}_n, \mathcal{F}_n^q$), the *monoid algebra* $R = \mathbf{k}M$ is the free $\mathbf{k}$ -module with basis elements given by the elements $\mathbf{a}$ of M , and multiplication extended $\mathbf{k}$ -linearly. A group G of monoid automorphisms of M (such as $G = \mathfrak{S}_n, GL_n$) acts as ring automorphisms on $\mathbf{k}M$.

Following Brown [4], we use *chambers* to refer to words and flags of maximum length in $\mathcal{F}_n$ and $\mathcal{F}_n^q$. In $\mathbf{k}\mathcal{F}_n$, the chamber subspace is $\mathbf{k}\mathfrak{S}_n$. In $\mathbf{k}\mathcal{F}_n^q$, the chamber subspace is spanned by complete flags of $\mathbb{F}_q^n$, equivalent to the coset space $\mathbf{k} GL_n / B$.

3 Answer to Question 1.1

Our first goal is to answer Question 1.1 by describing the invariant rings R^G . Importantly, since the groups G permute the monoid elements M , the monoid algebra $R = \mathbf{k}M$ becomes a permutation representation of G . Therefore the invariant subalgebra R^G has as a $\mathbf{k}$ -basis the orbit sums $\{\sum_{\mathbf{a} \in \mathcal{O}} \mathbf{a}\}$ as one runs through all G -orbits $\mathcal{O}$ on M .

For both monoids $M = \mathcal{F}_n, \mathcal{F}_n^q$, one can easily identify the G -orbits as elements of a fixed length. In particular, the G -invariant subalgebras R^G have $\mathbf{k}$ -bases $\{x_\ell\}_{\ell=0,1,\dots,n}$ and $\{x_\ell^q\}_{\ell=0,1,\dots,n}$, defined by

$$x_\ell : \sum_{\substack{\mathbf{a} \in \mathcal{F}_n \\ \ell(\mathbf{a}) = \ell}} \mathbf{a}, \quad x_\ell^q : \sum_{\substack{\mathbf{A} \in \mathcal{F}_n^q \\ \ell(\mathbf{A}) = \ell}} \mathbf{A}.$$

Example 3.1. Let $q = 2, n = 3, \ell = 1$, and let e_1, e_2, e_3 be standard basis vectors for $V = \mathbb{F}_2^3$. Using the notation $\langle v_1, v_2, \dots, v_m \rangle$ for the $\mathbb{F}_q$ -span of the vectors $\{v_1, v_2, \dots, v_m\}$ in V , one has

$$x_1^2 = \langle e_1 \rangle + \langle e_2 \rangle + \langle e_3 \rangle + \langle e_1, e_2 \rangle + \langle e_1, e_3 \rangle + \langle e_2, e_3 \rangle + \langle e_1, e_2, e_3 \rangle.$$

We will show that both invariant subalgebras R^G have another natural basis consisting of powers of x_1, x_1^q when $q \in \mathbf{k}^\times$. Henceforth, set $x := x_1$ and $x^q := x_1^q$.

Our proof that the powers of x and x_1^q form a basis of R^G involves Stirling numbers and one of their q -analogues². Recall the standard q -analogue of $n \in \mathbb{Z}_{\geq 0}$:

$$n_q := 1 + q + q^2 + \dots + q^{n-1}.$$

¹The authors thank an anonymous referee for pointing out that both $\mathcal{F}_n, \mathcal{F}_n^q$ are instances of the (right-) *free Rhodes expansion* of the lattice semigroups for the Boolean algebra and finite vector space lattices, using the rank one elements as generators; see [12, §4].

²See [15, §1.1, 1.2] for a history of two standard q -analogues, one being the $S_q(n, k)$ in Definition 3.2.

Definition 3.2. Define the classical Stirling numbers $S(n, k)$ and a q -analogue $S_q(n, k)$ recursively as follows. When $n = k = 0$, let $S(n, k) = S_q(n, k) = 1$. When $n = k \geq 1$ with $n = 0$ or $k = 0$, set $S(n, k) = S_q(n, k) = 0$. For n and k both at least 1, define

$$\begin{aligned} S(n, k) &= S(n-1, k-1) + k \cdot S(n-1, k), \\ S_q(n, k) &= q^{k-1} \cdot S_q(n-1, k-1) + k \cdot S_q(n-1, k). \end{aligned} \quad (3.1)$$

The recurrences in Equation (3.1) lead to a change of basis expression in R^G .

Lemma 3.3. Let $\mathbf{k}$ be a commutative ring with 1, and let $R = \mathbf{k}M$ with $M = \mathcal{F}_n$ or $\mathcal{F}_n^q$. Then the (q) -Stirling numbers $S(m, k)$, $S_q(m, k)$ are the expansion coefficients for the powers $\{x^m\}_{m=0,1,\dots,n}$ and $\{x^{q^{-m}}\}_{m=0,1,\dots,n}$ in the $\mathbf{k}$ -bases $\{x_k\}_{k=0,1,\dots,n}$, $\{x_k^q\}_{k=0,1,\dots,n}$ of R^G :

$$\begin{aligned} x^m &= \sum_k S(m, k) x_k, \\ x^{q^{-m}} &= \sum_k S_q(m, k) x_k^q. \end{aligned}$$

Thus unitriangularity of $\{S(m, k)\}$ shows $\{x_k\}_{k=0,1,\dots,n}$ always gives a $\mathbf{k}$ -basis for R^G , while triangularity of $\{S_q(m, k)\}$ shows $\{x_k^q\}_{k=0,1,\dots,n}$ are a $\mathbf{k}$ -basis for R^G if and only if $q \in \mathbf{k}^\times$.

Proof Idea. The lemma follows from induction on the Stirling recurrences (3.1) after proving the following multiplication rules in R^G : for $\ell = 0, 1, \dots, n$,

$$\begin{aligned} x \cdot x_\ell &= \ell x_\ell + x_{\ell+1}, \\ x^{q^{-\ell}} \cdot x_\ell^q &= \ell \cdot q x_\ell^q + q^\ell x_{\ell+1}^q. \end{aligned}$$

Note that these rules allow one to solve for each x_ℓ or x_ℓ^q in terms of powers of x or $x^{q^{-1}}$ inductively. The Stirling numbers just happen to be the change-of-basis coefficients. $\square$

We use Lemma 3.3, as well as properties of minimal polynomials, to describe the structure of the invariant subalgebras, answering Question 1.1.

Theorem 3.4. Let $\mathbf{k}$ be any commutative ring with 1, and $R = \mathbf{k}M$ for either of the monoids $M = \mathcal{F}_n, \mathcal{F}_n^q$, with symmetry groups $G = \mathfrak{S}_n, GL_n$. If $M = \mathcal{F}_n^q$, assume that q is in $\mathbf{k}^\times$.

(i) The unique $\mathbf{k}$ -algebra map $\mathbf{k}[X] \rightarrow R$ defined by

$$X \mapsto \begin{cases} x & \text{if } M = \mathcal{F}_n, \\ x^{q^{-1}} & \text{if } M = \mathcal{F}_n^q, \end{cases}$$

induces an algebra isomorphism $\mathbf{k}[X] / f(X) \cong R^G$ where

$$f(X) = \begin{cases} X(X-1)(X-2) \cdots (X-n) & \text{if } M = \mathcal{F}_n, \\ X(X-1)_q(X-2)_q \cdots (X-n)_q & \text{if } M = \mathcal{F}_n^q. \end{cases}$$

Hence R^G is commutative and generated by x or x^q .

(ii) If $\mathbf{k}$ is a field where $|G|$ is invertible, then x or x^q acts semisimply on any finite-dimensional R^G -module, with eigenvalues contained in the lists

$$\begin{cases} 0, 1, 2, \dots, n & \text{if } M \in \mathcal{F}_n, \\ 0_q, 1_q, 2_q, \dots, n_q & \text{if } M \in \mathcal{F}_n^q. \end{cases}$$

An old, but interesting, observation is that multiplication by x acts on the chamber subspace of $\mathbf{k}\mathcal{F}_n$ as a (rescaled) version of the random-to-top operator, see for instance B. Steinberg [18, Prop. 14.5]. This observation will be used in the proof of [Theorem 5.1](#).

Example 3.5. If $n = 4$ and $w = 3, 1, 4, 2$, then

$$x \cdot w = \begin{matrix} 1 & 2 & 3 & 4 \\ 1, 3, 4, 2 & 2, 3, 1, 4 & 3, 1, 4, 2 & 4, 3, 1, 2 \end{matrix}$$

which (after scaling by $\frac{1}{4}$) is the result of random-to-top shuffling on w as an element of $\mathbf{k}\mathfrak{S}_4$.

Remark 3.6. In unpublished notes, Garsia [7] (see also Tian [20]), studies the *top-to-random* shuffling operator, which is adjoint or transpose to the random-to-top operator. There he sketches a proof that its minimal polynomial is $X(X-1)(X-2)\cdots(X-n)$. In light of the fact that an operator and its transpose have the same minimal polynomial, Garsia's sketch is closely related to the part of our proof of [Theorem 3.4](#) dealing with $M \in \mathbf{k}\mathcal{F}_n$.

4 Symmetric function background

4.1 Symmetric functions, $\mathfrak{S}_n$ - and unipotent GL_n -representations

We review here the relation between the ring of symmetric functions Λ and representations of $\mathfrak{S}_n$; see Sagan [14], Stanley [17] as references, and for undefined terminology. We then review the parallel story for R. Steinberg's unipotent representations of GL_n ; see [8, §4.2, 4.6, 4.7] as a reference.

The ring of symmetric functions Λ (of bounded degree, in infinitely many variables) may be viewed as a polynomial algebra $\mathbb{Z}[h_1, h_2, \dots, e_1, e_2, \dots]$ where h_n, e_n are the complete homogeneous and elementary symmetric functions of degree n . One may view Λ as a graded $\mathbb{Z}$ -algebra $\Lambda = \bigoplus_{n=0}^{\infty} \Lambda^n$, which we wish to relate to the direct sum

$$\mathcal{C} \otimes : \bigoplus_{n=0}^{\infty} \mathcal{C} \otimes \mathfrak{S}_n,$$

where $\mathcal{C} \mathfrak{S}_n$ denotes the $\mathbb{Z}$ -module of *virtual characters* of $\mathfrak{S}_n$. That is, $\mathcal{C} \mathfrak{S}_n$ is the free $\mathbb{Z}$ -module on the basis of irreducible characters $\{\chi^\lambda\}$ indexed by the partitions λ of n , or alternatively, the $\mathbb{Z}$ -submodule of class functions on $\mathfrak{S}_n$ of the form $\chi - \chi'$ for genuine characters χ, χ' . One makes $\mathcal{C} \mathfrak{S}$ a graded algebra via the *induction product* defined by

$$\begin{aligned} \mathcal{C} \mathfrak{S}_{n_1} \times \mathcal{C} \mathfrak{S}_{n_2} &\longrightarrow \mathcal{C} \mathfrak{S}_{n_1 + n_2} \\ f_1, f_2 &\longmapsto f_1 * f_2 : \quad f_1 \otimes f_2 \uparrow_{\mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2}}^{\mathfrak{S}_{n_1 + n_2}} \end{aligned}$$

where $- \uparrow_H^G$ is the usual *induction* of class functions on a subgroup H to class functions on G . One then has the *Frobenius characteristic isomorphism* of $\mathbb{Z}$ -algebras $\mathcal{C} \mathfrak{S} \xrightarrow{\text{ch}} \Lambda$, mapping

$$\begin{aligned} \mathcal{C} \mathfrak{S} &\xrightarrow{\text{ch}} \Lambda \\ \mathbf{1}_{\mathfrak{S}_n} &\longmapsto h_n, \\ \chi^\lambda &\longmapsto s_\lambda. \end{aligned}$$

Here s_λ is the *Schur function*. For a composition $\alpha = \alpha_1, \alpha_2, \dots, \alpha_\ell$, we use the standard shorthand $h_\alpha : h_{\alpha_1} h_{\alpha_2} \cdots h_{\alpha_\ell}$.

There is a parallel story for a certain subset of GL_n -representations. Specifically, there is a collection of irreducible GL_n -representations $\{\chi_q^\lambda\}$, indexed by partitions λ of n , which are the irreducible constituents occurring within the GL_n -permutation action on the set GL_n/B of *complete flags of subspaces* in $V = \mathbb{F}_q^n$. They were studied by R. Steinberg [19], and are now called the *unipotent characters* of GL_n . Denoting by $\mathcal{C} GL_n$ the free $\mathbb{Z}$ -submodule of the class functions on GL_n with unipotent characters $\{\chi_q^\lambda\}$ as a basis, one can define the *parabolic* or *Harish-Chandra induction* product on the direct sum $\mathcal{C} GL : \bigoplus_{n=0}^\infty \mathcal{C} GL_n$ as follows:

$$\begin{aligned} \mathcal{C} GL_{n_1} \times \mathcal{C} GL_{n_2} &\longrightarrow \mathcal{C} GL_{n_1 + n_2} \\ f_1, f_2 &\longmapsto f_1 * f_2 : \quad \left(f_1 \otimes f_2 \uparrow_{GL_{n_1} \times GL_{n_2}}^{P_{n_1, n_2}} \right) \uparrow_{P_{n_1, n_2}}^{GL_{n_1 + n_2}} \end{aligned}$$

Here P_{n_1, n_2} is the maximal parabolic subgroup of $GL_{n_1 + n_2}$ setwise stabilizing the $\mathbb{F}_q$ -span of the first n_1 standard basis vectors, and $- \uparrow_{GL_{n_1} \times GL_{n_2}}^{P_{n_1, n_2}}$ is the *inflation* operation that creates a $GL_{n_1} \times GL_{n_2}$ -representation from a P_{n_1, n_2} -representation, by precomposing with the surjective homomorphism $P_{n_1, n_2} \twoheadrightarrow GL_{n_1} \times GL_{n_2}$ sending $\begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \mapsto \begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix}$.

This parabolic induction turns out to make $\mathcal{C} GL$ into an associative, commutative $\mathbb{Z}$ -algebra. One then has a q -analogue of the Frobenius isomorphism $\mathcal{C} GL \xrightarrow{\text{ch}_q} \Lambda$ that sends $\mathbf{1}_{GL_n} \mapsto h_n$ and $\chi_q^\lambda \mapsto s_\lambda$. For example, the chamber subspaces $\mathbf{k} \mathfrak{S}_n$ and $\mathbf{k} GL_n/B$ of $\mathbf{k} \mathcal{F}_n$ and $\mathbf{k} \mathcal{F}_n^q$ carry the same (q)-Frobenius characteristic image h_{1^n} .

4.2 (q -)derangement numbers and representations

A central role in this story is played by the classical *derangement numbers* d_n , and the q -*derangement numbers* $d_n(q)$ of Wachs [22]:

$$\begin{aligned} d_n &: n! \sum_{k=0}^n \frac{(-1)^k}{k!} = n! \left(\frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots + \frac{(-1)^n}{n!} \right), \\ d_n(q) &: n! \sum_{k=0}^n \frac{(-1)^k}{k!_q}. \end{aligned}$$

There are two well-known combinatorial models for d_n counting permutations in $\mathfrak{S}_n$:

- *derangements*, which are the fixed-point free permutations, or
- *desarrangements*, which are permutations $w = w_1, w_2, \dots, w_n$ whose first ascent position i with $w_i < w_{i-1}$ (using $w_{n-1} = n-1$) occurs for an even position i .

Wachs [22] and later Désarménien and Wachs [6] gave various interpretations for $d_n(q)$. In particular, $d_n(q)$ is still closely related to derangements and desarrangements. Letting D_n, E_n denote the derangements and desarrangements in S_n , and defining the *major index* statistic of a permutation $w = w_1, \dots, w_n$ as $\text{maj } \sigma = \sum_{i: w_i > w_{i-1}} i$, one has

$$d_n(q) = \sum_{\sigma \in D_n} q^{\text{maj } \sigma} = \sum_{\sigma \in E_n} q^{\text{maj } \sigma^{-1}}.$$

These $d_n, d_n(q)$ are the dimensions for a pair of representations of $\mathfrak{S}_n$ and GL_n , which we call the *derangement representation* $\mathcal{D}_n$ and its (unipotent) q -analogue $\mathcal{D}_n^q$. Both have the same symmetric function image $\mathfrak{d}_n$ under the Frobenius maps ch and ch_q . The symmetric function $\mathfrak{d}_n$ was originally introduced by Désarménien and Wachs [5] and has many equivalent descriptions (see [3, Proposition 3.1]). Here, we define $\mathfrak{d}_n$ by its decomposition into Schur functions; this description is due to Reiner and Webb [13]. To do so, we must define desarrangement tableaux.

A standard Young tableau Q with n cells written in English notation, has *descent set*

$$\text{Des } w : \{i \in \{1, 2, \dots, n-1\} : i+1 \text{ appears south and weakly west of } i \text{ in } Q\}.$$

For example, $Q = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 6 \\ \hline 4 & \\ \hline 5 & \\ \hline \end{array}$ has $\text{Des } Q = \{1, 3, 4\}$. A *desarrangement tableau* is a standard

Young tableau Q with n cells for which the smallest element of $\{1, 2, \dots, n\} \setminus \text{Des } Q$ is even. Thus the example tableau Q given above is a desarrangement tableau.

Definition 4.1. $\mathfrak{d}_n = \sum_Q s_{\lambda(Q)}$ where Q runs through the desarrangement tableaux of size n .

Example 4.2. We compute $\mathfrak{d}_n$ for $0 \leq n \leq 4$.

n	desarrangement tableaux Q	$\mathfrak{d}_n$
0	$\emptyset$	1
1	(none)	0
2	$\begin{array}{ c } \hline 1 \\ \hline 2 \\ \hline \end{array}$	$s_{1,1}$
3	$\begin{array}{ c c } \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array}$	$s_{2,1}$
4	$\begin{array}{ c } \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline \end{array}$ $\begin{array}{ c c } \hline 1 & 3 \\ \hline 2 & \\ \hline 4 & \\ \hline \end{array}$ $\begin{array}{ c c } \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}$ $\begin{array}{ c c c } \hline 1 & 3 & 4 \\ \hline 2 & & \\ \hline \end{array}$	$s_{1,1,1,1}$ $s_{2,1,1}$ $s_{2,2}$ $s_{3,1}$

5 Answer to Question 1.2

We are now prepared to answer [Question 1.2](#) by examining the monoid algebras $R = \mathbf{k}M$ as modules over both the symmetry groups $\mathbf{k}G$ and their invariant subalgebras R^G . Since R carries commuting actions of R^G and of $\mathbf{k}G$, we will describe R simultaneously as a module over both. Henceforth, assume that $\mathbf{k}$ is a field in which $|G|$ is invertible.

We will utilize two important features of our setting. First, since $\mathbf{k}M$ is finite-dimensional over $\mathbf{k}$, it is semisimple both as an R^G -module from [Theorem 3.4\(ii\)](#), and as a $\mathbf{k}G$ -module by Maschke's Theorem. Second, by [Theorem 3.4\(ii\)](#), we have that R^G is generated by the single element x or x^q , which acts diagonalizably with certain eigenvalues λ all lying in $\mathbf{k}$. It follows that in order to understand the simultaneous R^G - and $\mathbf{k}G$ -module structure of any module V , it suffices to decompose the eigenspaces $(\ker(x - \lambda)|_V \text{ or } \ker(x^q - \lambda)|_V)$ as G -representations.

Hence, we are able to answer [Question 1.2](#) for $j = 0, 1, \dots, n$

- when $M = \mathcal{F}_n$ by describing $\ker(x - j)|_{\mathbf{k}\mathcal{F}_n}$, the j -eigenspace of x acting on $\mathbf{k}\mathcal{F}_n$, as an $\mathfrak{S}_n$ -representation, and
- when $M = \mathcal{F}_n^q$ by describing $\ker(x^q - j^q)|_{\mathbf{k}\mathcal{F}_n^q}$, the j^q -eigenspace of x^q acting on $\mathbf{k}\mathcal{F}_n^q$, as a GL_n -representation.

Theorem 5.1. Let $\mathbf{k}$ be a field in which $|G|$ is invertible. Then x, x^q act diagonalizably on $\mathbf{k}\mathcal{F}_n, \mathbf{k}\mathcal{F}_n^q$, and for each $j = 0, 1, 2, \dots, n$, their eigenspaces carry G -representations with the same Frobenius map images

$$\text{ch } \ker(x - j)|_{\mathbf{k}\mathcal{F}_n} = \sum_{\ell=j}^n h_{n-\ell, j} \cdot \mathfrak{d}_{\ell-j} = \text{ch}_q \ker(x^q - j^q)|_{\mathbf{k}\mathcal{F}_n^q}.$$

In other words, one has G -representation isomorphisms

$$\begin{aligned} \ker(x - j |_{\mathbf{k}\mathcal{F}_n}) &\sim \bigoplus_{\ell=0}^n \mathbf{1}_{\mathfrak{S}_{n-\ell}} * \mathbf{1}_{\mathfrak{S}_j} * \mathcal{D}_{\ell-j}, \\ \ker(x^q - j^q |_{\mathbf{k}\mathcal{F}_n^q}) &\sim \bigoplus_{\ell=0}^n \mathbf{1}_{GL_{n-\ell}} * \mathbf{1}_{GL_j} * \mathcal{D}_{\ell-j}^q. \end{aligned}$$

Proof Idea. The crucial idea is to introduce a filtration on $\mathbf{k}M$ by length,

$$0 = \mathbf{k}M_{\geq n+1} \subset \mathbf{k}M_{\geq n} \subset \mathbf{k}M_{\geq n-1} \subset \cdots \subset \mathbf{k}M_{\geq 1} \subset \mathbf{k}M_{\geq 0} = \mathbf{k}M,$$

where $M_{\geq \ell}$ is the $\mathbf{k}$ -span of words of length at least ℓ for $M = \mathcal{F}_n$ and flags of length at least ℓ for $M = \mathcal{F}_n^q$. By semisimplicity, there is an isomorphism of both R^G -modules and G -representations:

$$\mathbf{k}M \sim \bigoplus_{\ell=0}^n \mathbf{k}M_{\geq \ell} / \mathbf{k}M_{\geq \ell+1}. \quad (5.1)$$

Our approach is to study the eigenspaces of x, x^q on each summand in Equation (5.1).

The bottom of the filtration (when $\ell = n$) is the chamber subspace of $M = \mathbf{k}\mathcal{F}_n, \mathbf{k}\mathcal{F}_n^q$, which we write as $\mathfrak{C}_n, \mathfrak{C}_n^q$, respectively. We prove that

$$\text{ch } \ker(x - j |_{\mathfrak{C}_n}) = h_j \mathfrak{d}_{n-j} = \text{ch}_q \ker(x^q - j^q |_{\mathfrak{C}_n^q}). \quad (5.2)$$

The key ingredients in proving Equation (5.2) are (i) inductive constructions of explicit j -, j^q -eigenvectors³ of $x|_{\mathfrak{C}_n}, x^q|_{\mathfrak{C}_n^q}$ from nullvectors of $x|_{\mathfrak{C}_{n-j}}, x^q|_{\mathfrak{C}_{n-j}^q}$, (ii) the dimensions of $\ker(x|_{\mathfrak{C}_n}), \ker(x^q|_{\mathfrak{C}_n^q})$ following from work of Phatarfod [11], Brown [4], and (iii) a recursive description of $\mathfrak{d}_n$ from [5]: $h_{1^n} = \sum_{j=0}^n \mathfrak{d}_j h_{n-j}$.

Finally, we address the remaining summands of Equation (5.1) by reinterpreting the action of x, x^q on $\mathbf{k}M_{\geq \ell} / \mathbf{k}M_{\geq \ell+1}$ in terms of the eigenspaces of $\mathfrak{C}_\ell, \mathfrak{C}_\ell^q$ and using properties of induced representations. $\square$

Example 5.2. We illustrate Theorem 5.1 computing the Frobenius map image for each j -eigenspace of x on $\mathbf{k}\mathcal{F}_3$, or equivalently the q -Frobenius map image for each j^q -eigenspace of x^q on $\mathbf{k}\mathcal{F}_3^q$. The table below shows these symmetric functions in the j^{th} row, decomposed into columns labeled by ℓ , which index the filtration factors from Equation (5.1) that contribute a term.

³The third author is grateful to Michelle Wachs for explaining to him the constructions in the case that $M = \mathbf{k}\mathcal{F}_n$ in 2002, in the context of random-to-top shuffling.

Frobenius map images for eigenspaces of x, x^q on $\mathbf{k}\mathcal{F}_3, \mathbf{k}\mathcal{F}_3^q$:

	$\ell = 0$	$\ell = 1$	$\ell = 2$	$\ell = 3$
$j = 0$	$h_3 \cdot \mathfrak{d}_0$ $h_3 \cdot s$ s_3	$h_2 \cdot \mathfrak{d}_1$ $h_2 \cdot 0$ 0	$h_1 \cdot \mathfrak{d}_2$ $h_1 \cdot s_{1,1}$ $s_{2,1} \quad s_{1,1,1}$	$h_0 \cdot \mathfrak{d}_3$ $h_0 \cdot s_{2,1}$ $s_{2,1}$
$j = 1$		$h_{2,1} \cdot \mathfrak{d}_0$ $h_{2,1} \cdot s$ $s_3 \quad s_{2,1}$	$h_{1,1} \cdot \mathfrak{d}_1$ $h_{1,1} \cdot 0$ 0	$h_1 \cdot \mathfrak{d}_2$ $h_1 \cdot s_{1,1}$ $s_{2,1} \quad s_{1,1,1}$
$j = 2$			$h_{2,1} \cdot \mathfrak{d}_0$ $h_{2,1} \cdot s$ $s_3 \quad s_{2,1}$	$h_2 \cdot \mathfrak{d}_1$ $h_2 \cdot 0$ 0
$j = 3$				$h_3 \cdot \mathfrak{d}_0$ $h_3 \cdot s$ s_3

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Rowmotion on 321-avoiding permutations

Ben Adenbaum^{*1} and Sergi Elizalde^{†1}

¹Department of Mathematics, Dartmouth College, Hanover, NH 03755, USA

Abstract. We give a natural definition of rowmotion for 321-avoiding permutations, by translating, through bijections involving Dyck paths and the Lanne–Kreweras involution, the analogous notion for antichains of the positive root poset of type A . We prove that some permutation statistics, such as the number of fixed points, are homomesic under rowmotion, meaning that they have a constant average over its orbits. Finally, we show that the Armstrong–Stump–Thomas equivariant bijection between antichains in types A and B and non-crossing matchings can be described more naturally in terms of the Robinson–Schensted–Knuth correspondence on permutations.

Keywords: rowmotion, homomesy, pattern avoidance

1 Introduction

Let $\mathcal{S}_n$ denote the set of permutations of $\{1, 2, \dots, n\}$. We say that $\pi \in \mathcal{S}_n$ is 321-avoiding if there do not exist $i < j < k$ such that $\pi(i) > \pi(j) > \pi(k)$. Let $\mathcal{S}_n^{321}$ denote the set of 321-avoiding permutations in $\mathcal{S}_n$. We can represent $\pi \in \mathcal{S}_n^{321}$ as an $n \times n$ array with crosses in squares $(i, \pi(i))$ for $1 \leq i \leq n$; we call this the *array* of π . Rows and columns are indexed using cartesian coordinates, so that (i, j) denotes the cell in the i th column from the left and j th row from the bottom. We say that $(i, \pi(i))$ is a *fixed point* (respectively *excedance*, *weak excedance*, *deficiency*, *weak deficiency*) if $\pi(i) = i$ (respectively $\pi(i) > i$, $\pi(i) \geq i$, $\pi(i) < i$, $\pi(i) \leq i$).

Let P be a finite poset, and let $\mathcal{A}(P)$ denote the set of antichains of P . *Antichain rowmotion* is the map $\rho_{\mathcal{A}} : \mathcal{A}(P) \rightarrow \mathcal{A}(P)$ defined as follows: for $A \in \mathcal{A}(P)$, let $\rho_{\mathcal{A}}(A)$ be the minimal elements of the complement of the order ideal generated by A . See [Section 2.3](#) for more details and definitions.

Historically, rowmotion was first described for general posets by Brouwer and Schrijver [2], and then again by Cameron and Fon-der-Flaas [3] as a composition of certain involutions called toggles. The name of rowmotion comes from the work of Striker and Williams [17] where, for certain posets, rowmotion is described as a composition of toggles along the rows, and related to another operation called promotion.

Restricting our attention to the poset of positive roots for the type A root system, we will show that antichains of this poset are in bijection with Dyck paths and with

^{*}benjamin.m.adenbaum.gr@dartmouth.edu

[†]sergi.elizalde@dartmouth.edu

321-avoiding permutations. This will allow us to define a natural rowmotion operation on $\mathcal{S}_n^{321}$.

When studying rowmotion, it is common to look for statistics that exhibit a property called *homomesy* [12]. Given a set S and a bijection $\tau : S \rightarrow S$ so that each orbit of the action of τ on S has finite order, we say that a statistic on S is *homomesic* under this action if its average on each orbit is constant. More specifically, the statistic is said to be *c-mesic* if its average over each orbit is c . We prove that several statistics on 321-avoiding permutations, including the number of fixed points, are homomesic under rowmotion.

Furthermore, we use the viewpoint of 321-avoiding permutations to shed new light into a celebrated bijection of Armstrong, Stump and Thomas [1] between antichains in root posets of finite Weyl groups (also known as nonnesting partitions) and noncrossing partitions. We will show that, in the case of types A and B , the Armstrong–Stump–Thomas (AST) bijection has a simple interpretation in terms of the Robinson–Schensted–Knuth (RSK) correspondence applied to 321-avoiding permutations.

2 Background

In this section we review some notions about Dyck paths, noncrossing matchings, and the RSK correspondence, in particular as it applies to 321-avoiding permutations. We also provide a basic overview of rowmotion.

2.1 Dyck paths

Let $\mathcal{D}_n$ be the set of words over $\{u, d\}$ consisting of n u s and n d s, and satisfying that every prefix contains at least as many u s as d s. Elements of $\mathcal{D}_n$ are called Dyck paths, and they will be drawn in three different ways as lattice paths in $\mathbb{Z}^2$ starting at the origin. Replacing u and d with $(0, 1)$ and $(1, 0)$ (respectively, $(1, 0)$ and $(0, 1)$), we obtain paths that stay weakly above (respectively, below) the diagonal $y = x$. We denote these by $\mathcal{D}_n^{\geq}$ (respectively, $\mathcal{D}_n^{\leq}$). The sets $\mathcal{D}_n^{\geq}$ and $\mathcal{D}_n^{\leq}$ are in bijection with each other, by simply reflecting along the diagonal. The third way to draw Dyck paths that we will use is when u and d are replaced with $(1, 1)$ and $(1, -1)$, respectively. In all cases, a pair of consecutive steps ud is called a *peak*, and a pair du is called a *valley*. Interpreting u and d steps of $D \in \mathcal{D}_n$ as opening and closing parentheses, respectively, and matching them in the usual way, a pair of matched steps will be called a *tunnel*, following [5].

Several bijections between 321-avoiding permutations and Dyck paths are known. For $\pi \in \mathcal{S}_n^{321}$, let $E_p \pi \in \mathcal{D}_n^{\geq}$ be the path whose peaks occur at the weak excedances of π , let $E_v \pi \in \mathcal{D}_n^{\geq}$ be the path whose valleys occur at the excedances of π , and let $D_v \pi \in \mathcal{D}_n^{\leq}$ be the path whose valleys occur at the weak deficiencies of π . The bijection that maps $E_p \pi$ to $D_v \pi$ is known as the *Lalanne–Kreweras involution* on Dyck paths [8,

9], which we denote by $LK = D_v \circ E_p^{-1}$; see the examples in Figure 1.

Let $\mathcal{N}_n$ denote the set of *noncrossing matchings* of $\{1, 2, \dots, 2n\}$, i.e., perfect matchings with the property that there do not exist $i < j < k < \ell$ such that i is matched with k and j is matched with ℓ . We will draw the points $1, 2, \dots, 2n$ around a circle in clockwise order, with a line segment connecting each pair of matched points. There is a straightforward bijection between Dyck paths and noncrossing matchings.

Definition 2.1. Let $\text{Match} : \mathcal{D}_n \rightarrow \mathcal{N}_n$ be the bijection defined as follows. Given $D \in \mathcal{D}_n$, the points i and j are matched in $\text{Match } D$ if the steps of D in positions i and j form a tunnel.

Define promotion of Dyck paths to be the following map $\text{Pro} : \mathcal{D}_n \rightarrow \mathcal{D}_n$. Given $D \in \mathcal{D}_n$, consider its first-return decomposition $D = uAdB$ where A, B are Dyck paths, and let $\text{Pro } D = AuBd$. The reason for this name is that, as observed by White [14, Sec. 8], applying this operation to a path $D \in \mathcal{D}_n$ is equivalent to applying promotion on standard Young tableaux of shape n, n .

2.2 RSK and 321-avoiding permutations

The RSK correspondence is a bijection between permutations and pairs of standard Young tableaux of the same shape. We refer the reader to [16, Sec. 7.11] for definitions. Given a permutation $\pi \in \mathcal{S}_n$, we denote its image by $\text{RSK } \pi = P, Q$, where P and Q are standard Young tableaux of the same shape. In particular, π is 321-avoiding if and only if P and Q have at most two rows. This property is used to define the following map from $\mathcal{S}_n$ 321 to $\mathcal{D}_n$.

Definition 2.2. Let $\pi \in \mathcal{S}_n$ 321, and suppose that $\text{RSK } \pi = P, Q$. Define a Dyck path $\widehat{\text{RSK}} \pi$ as follows. For $1 \leq i \leq n$, let the i th step be a u if i is in the top row of P , and a d otherwise; let the $(2n - 1 - i)$ th step be a d if i is in the top row of Q , and a u otherwise.

2.3 Rowmotion

Let P be a finite poset. An *antichain* of P is a subset of pairwise incomparable elements. An *order ideal* of P is a subset I with the property that if $x \in I$ and $y \leq x$, then $y \in I$. Similarly, an *order filter* of P is a subset F with the property that if $x \in F$ and $x \leq y$, then $y \in F$. Let $\mathcal{A} P$, $\mathcal{I} P$ and $\mathcal{F} P$ denote the sets of antichains, order ideals and order filters of P , respectively. The complementation map Θ , defined on subsets S of P by $\Theta S = P \setminus S$, restricts to a bijection between $\mathcal{I} P$ and $\mathcal{F} P$.

Following [7], define the up-transfer map $\Delta : \mathcal{I} P \rightarrow \mathcal{A} P$ by letting ΔI be the set of maximal elements of $I \in \mathcal{I} P$, and the down-transfer map $\nabla : \mathcal{F} P \rightarrow \mathcal{A} P$ by

letting ∇F be the set of minimal elements of $F \in \mathcal{F} P$. The inverses of these maps are given by

$$\begin{aligned}\Delta^{-1} A &= \{x \in P : x \leq y \text{ for some } y \in A\}, \\ \nabla^{-1} A &= \{x \in P : x \geq y \text{ for some } y \in A\},\end{aligned}$$

for $A \in \mathcal{A} P$. Antichain rowmotion is the map $\rho_A : \mathcal{A} P \rightarrow \mathcal{A} P$ defined as the composition $\rho_A = \nabla \circ \Theta \circ \Delta^{-1}$.

The poset of positive roots in type A , which we denote by $\mathbf{A}^{n-1}$, can be described as the set of intervals $\{i, j : 1 \leq i \leq j \leq n-1\}$ ordered by inclusion. It is a ranked poset, with rank function given by $\text{rk } i, j = j - i$. The set $\mathcal{A} \mathbf{A}^{n-1}$ is in bijection with $\mathcal{D}_n$. One such bijection consists of mapping each path P to the antichain αP whose elements are at the valleys of P . This bijection allows us to define rowmotion on Dyck paths as $\rho_D = \alpha^{-1} \circ \rho_A \circ \alpha$. A second bijection between $\mathcal{A} \mathbf{A}^{n-1}$ and $\mathcal{D}_n$ consists of mapping the antichain A to the path δA whose peaks are at the elements of the antichain. Note that $\rho_A = \alpha \circ \delta$ and $\rho_D = \delta \circ \alpha$.

Panyushev [10, 11] considered the map on $\mathcal{A} \mathbf{A}^n$ defined by mapping an antichain given by $\{i_1, j_1, \dots, i_k, j_k\}$ to $\{i'_1, j'_1, \dots, i'_{n-k}, j'_{n-k}\}$ with $\{i'_1, \dots, i'_{n-k}\} = \{1, 2, \dots, n\} \setminus \{j_1, \dots, j_k\}$ and $\{j'_1, \dots, j'_{n-k}\} = \{1, 2, \dots, n\} \setminus \{i_1, \dots, i_k\}$. As noted in [7], this map equals the composition $\alpha \circ \text{LK} \circ \alpha^{-1}$, so we will denote it by LK_A . Hopkins and Joseph show in [7, Thm. 3.5] that a certain map known as *antichain rowvacuation*, denoted in general by Rvac_A , coincides with LK_A in the case of the type A root poset.

3 A rowmotion operation on 321-avoiding permutations

To define rowmotion on 321-avoiding permutations, first consider the following bijection to antichains of $\mathbf{A}^{n-1}$.

Definition 3.1. Let $\text{Exc} : \mathcal{S}_n \text{ 321} \rightarrow \mathcal{A} \mathbf{A}^{n-1}$ be the bijection where, for $\pi \in \mathcal{S}_n \text{ 321}$, we define

$$\text{Exc } \pi = \{i, \pi i - 1 : i, \pi i \text{ is an excedance of } \pi\}.$$

See the left column of Figure 1 for an example of $\text{Exc} : \mathcal{S}_9 \text{ 321} \rightarrow \mathcal{A} \mathbf{A}^8$, which maps the permutation 241358967 to the antichain $\{1, 1, 2, 3, 6, 7, 7, 8\}$.

To see that Exc is a bijection, note that it can be written as $\text{Exc} = \delta^{-1} \circ E_p = \alpha \circ E_v$. The map Exc provides the following natural way to translate rowmotion into an operation on $\mathcal{S}_n \text{ 321}$.

Definition 3.2. Rowmotion on 321-avoiding permutations is the map $\rho_S : \mathcal{S}_n \text{ 321} \rightarrow \mathcal{S}_n \text{ 321}$ defined by $\rho_S = \text{Exc}^{-1} \circ \rho_A \circ \text{Exc}$.

An equivalent description of ρ_S can be given in terms of the bijections to Dyck paths, as the composition

$$\rho_S = E_v^{-1} \circ E_p. \quad (3.1)$$

Figure 1 gives examples of ρ_S computed in both ways. Note that if $\pi \in \mathcal{S}_n$ 321 has upper and lower paths given by $P = E_p \pi$ and $Q = D_v \pi$, then $\sigma = \rho_S \pi$ has upper and lower paths given by $\rho_D P = E_p \sigma$ and $\rho_D^{-1} Q = D_v \sigma$. Equivalently, at the level of antichains, we have $\text{Exc } \sigma = \rho_A \text{Exc } \pi$ and $\text{Exc } \sigma^{-1} = \rho_A^{-1} \text{Exc } \pi^{-1}$, noting that $\text{Exc } \pi^{-1}$ is the antichain formed by the deficiencies of π . This follows from the fact that

$$\text{LK}_A \circ \rho_A = \rho_A^{-1} \circ \text{LK}_A, \quad (3.2)$$

which was proved by Panyushev [11, Thm. 3.5].

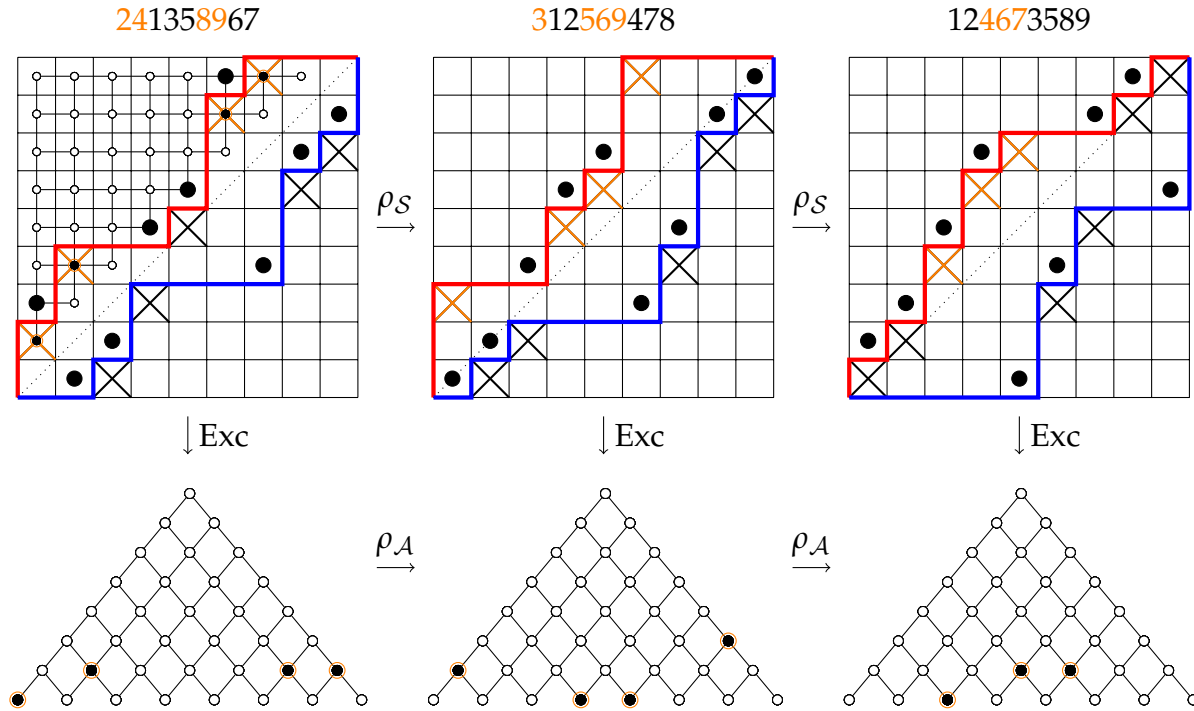


Figure 1: Two applications of rowmotion starting at $\pi = 241358967 \in \mathcal{S}_9$ 321, computed using Definition 3.2, or alternatively the composition (3.1). In the upper left diagram, the crosses represent π , the red path is $E_p \pi$, the blue path is $D_v \pi = \text{LK}(E_p \pi)$, and the dots represent $\rho_S \pi = E_v^{-1} E_p \pi = 312569478$.

A diagram of our bijections for permutations, paths and antichains, as well as their interactions, appears in Figure 2. We can translate the Lanne–Kreweras involution for Dyck paths into an involution on $\mathcal{S}_n$ 321 by defining

$$\text{LK}_S = E_p^{-1} \circ \text{LK} \circ E_p. \quad (3.3)$$

Using that $\text{LK}_{\mathcal{A}} = \alpha \circ \text{LK} \circ \alpha^{-1}$, the maps $\text{LK}_{\mathcal{S}}$ and $\text{LK}_{\mathcal{A}}$ are related by

$$\text{LK}_{\mathcal{S}} = E_p^{-1} \circ \alpha^{-1} \circ \text{LK}_{\mathcal{A}} \circ \alpha \circ E_p. \quad (3.4)$$

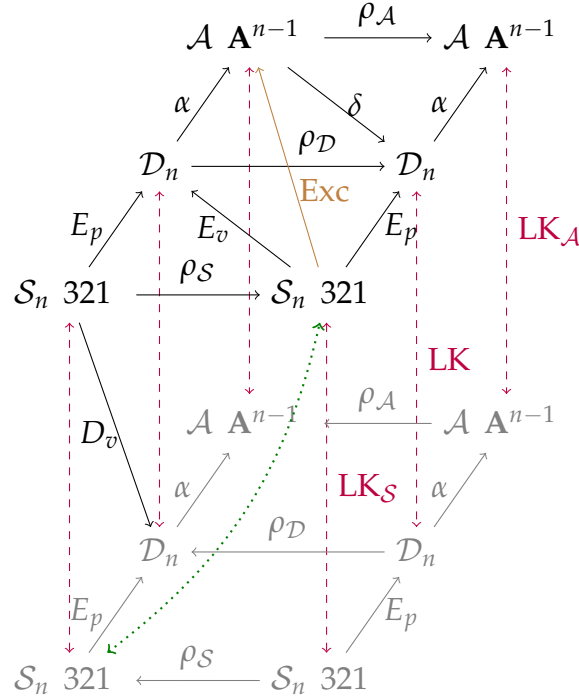


Figure 2: Diagram of the bijections $\rho_{\mathcal{S}}$, $\rho_{\mathcal{D}}$, $\rho_{\mathcal{A}}$, α , δ , E_p , E_v , D_v , $\text{LK}_{\mathcal{S}}$, $\text{LK}_{\mathcal{A}}$, and $\text{LK}_{\mathcal{D}}$. The vertical dashed arrows are various versions of the Lalanne–Kreweras involution, and the dotted curved arrow is the map sending a permutation to its inverse.

The map $\text{LK}_{\mathcal{S}}$ is closely related to the operation that sends each permutation to its inverse, as the next lemma shows.

Lemma 3.3. *If $\pi \in \mathcal{S}_n \text{ 321}$ then $\pi^{-1} = \rho_{\mathcal{S}} \text{LK}_{\mathcal{S}} \pi = \text{LK}_{\mathcal{S}} \rho_{\mathcal{S}}^{-1} \pi$.*

Finally, let us mention that rowmotion on 321-avoiding permutations provides a more direct proof of a result of Hopkins and Joseph [7, Thm. 6.2] that the number of antichains of $\mathbf{A}^{n-1}$ that are fixed by the involution $\text{LK}_{\mathcal{A}} \circ \rho_{\mathcal{A}}$ is $\lfloor \frac{n}{2} \rfloor$, by instead reducing it to a classical result of Simion and Schmidt [15] on the enumeration of 321-avoiding involutions.

4 Statistics and homomesies

In this section we show that certain statistics on 321-avoiding permutations exhibit homomesy under the action of $\rho_{\mathcal{S}}$.

4.1 The number of fixed points and the statistics h_i and ℓ_i

The first statistic that we consider is the number of fixed points of a permutation π , denoted by $\text{fp } \pi = |\{i : \pi(i) = i\}|$. Let $\text{exc } \pi = |\{i : \pi(i) > i\}|$ and $\text{wexc } \pi = |\{i : \pi(i) \geq i\}|$ denote the number of excedances and weak excedances of π , respectively.

Theorem 4.1. *The statistic fp is 1-mesic under the action of ρ_S on $\mathcal{S}_n$ 321.*

Proof. Let $\pi \in \mathcal{S}_n$ 321, and let $\mathcal{O}$ be the orbit of π under ρ_S . Then

$$\text{fp } \pi = \text{wexc } \pi - \text{exc } \pi = n - \text{exc } \pi^{-1} - \text{exc } \pi = n - |\text{Exc } \pi| - |\text{Exc } \pi^{-1}|.$$

Summing over the orbit,

$$\frac{1}{|\mathcal{O}|} \sum_{\pi \in \mathcal{O}} \text{fp } \pi = n - \frac{1}{|\mathcal{O}|} \sum_{\pi \in \mathcal{O}} |\text{Exc } \pi| - \frac{1}{|\mathcal{O}|} \sum_{\pi \in \mathcal{O}} |\text{Exc } \pi^{-1}|.$$

As noted above equation (3.2), applying ρ_S to π corresponds to applying ρ_A to $\text{Exc } \pi$ and ρ_A^{-1} to $\text{Exc } \pi^{-1}$. It then follows by Lemma 3.3 that the sets $\{\text{Exc } \pi : \pi \in \mathcal{O}\}$ and $\{\text{Exc } \pi^{-1} : \pi \in \mathcal{O}\}$ are complete orbits under ρ_A . It is known [1] that the antichain cardinality statistic is $\frac{n-1}{2}$ -mesic under the action of ρ_A on $\mathcal{A} \mathbf{A}^{n-1}$. Thus $\frac{1}{|\mathcal{O}|} \sum_{\pi \in \mathcal{O}} |\text{Exc } \pi| = \frac{1}{|\mathcal{O}|} \sum_{\pi \in \mathcal{O}} |\text{Exc } \pi^{-1}| = \frac{n-1}{2}$, and we conclude that fp is 1-mesic. $\square$

See Figure 3 for a computation of $\text{fp } \pi$ over a rowmotion orbit for $n = 4$. As Theorem 4.1 asserts, the average of this statistic is 1 over each orbit.

Next we consider two families of statistics on 321-avoiding permutations, and show that they are also homomesic under rowmotion. The first family are the statistics h_i introduced by Hopkins and Joseph [7]. For $1 \leq i \leq n-1$, they define h_i on antichains $A \in \mathcal{A} \mathbf{A}^{n-1}$ as

$$h_i(A) = \sum_{j=1}^i \mathbb{1}_{j,i} \in A = \sum_{j=i}^{n-1} \mathbb{1}_{i,j} \in A,$$

where $\mathbb{1}_{i,j} \in A$ is the indicator function that equals 1 if $i, j \in A$ and 0 otherwise. For $\pi \in \mathcal{S}_n$ 321, we now define $h_i \pi$ by $h_i \text{Exc } \pi$. In terms of the array of $\pi \in \mathcal{S}_n$ 321, $h_i \pi$ counts the number of crosses of the form $j, i-1$ with $1 \leq j \leq i$, or i, j with $i-1 \leq j \leq n$.

Hopkins and Joseph prove in [7, Thm. 4.3] that the statistics h_i on antichains are 1-mesic under ρ_A . This result can be translated in terms of 321-avoiding permutations as follows.

Theorem 4.2 ([7]). *For $1 \leq i \leq n-1$, the statistic h_i is 1-mesic under the action of ρ_S on $\mathcal{S}_n$ 321.*

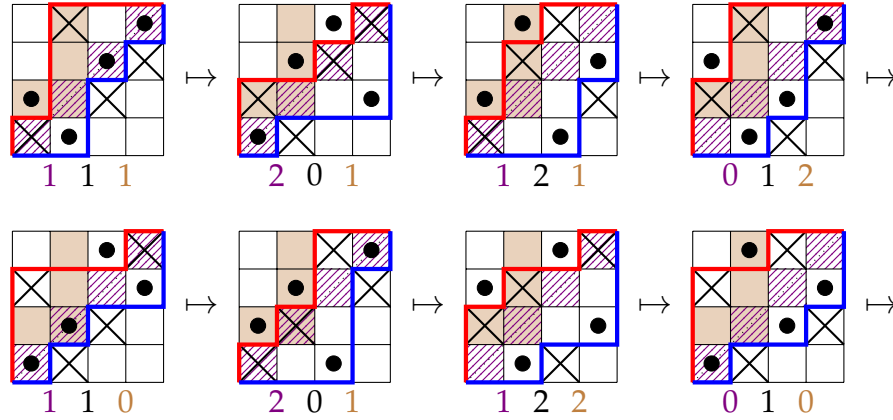


Figure 3: The rowmotion orbit containing 1423. The numbers below each diagram are the values $\text{fp } \pi$, $h_2 \pi$ and $\ell_2 \pi$, from left to right.

Next we define a new family of permutation statistics, that we denote by ℓ_i for $1 \leq i \leq n$. For $\pi \in \mathcal{S}_n^{321}$, let $\ell_i \pi$ be the number of crosses in the array of π of the form j, i with $1 \leq j \leq i$, plus the number of crosses of the form i, j with $i < j \leq n$. It turns out that the statistics ℓ_i are homomesic as well, see Figure 3 for an example.

Theorem 4.3. *For $1 \leq i \leq n$, the statistic ℓ_i is 1-mesic under the action of ρ_S on $\mathcal{S}_n^{321}$.*

4.2 The sign statistic

In this subsection we describe how rowmotion interacts with the sign of a 321-avoiding permutation.

Theorem 4.4. *For all $\pi \in \mathcal{S}_n^{321}$,*

$$\text{sgn } \rho_S \pi = \text{sgn } \text{LK}_S \pi \begin{cases} \text{sgn } \pi & \text{if } n \text{ is odd,} \\ -\text{sgn } \pi & \text{if } n \text{ is even.} \end{cases}$$

It is a classical result of Simion and Schmidt [15, Prop. 2] that, when n is even, the set $\mathcal{S}_n^{321}$ contains the same number of odd and even permutations. A bijective proof of this fact was given by Reifegerste [13]. Theorem 4.4 gives two new bijections, ρ_S and LK_S , between the subsets of odd and even permutations in $\mathcal{S}_n^{321}$. Furthermore, LK_S has the additional property of being a sign-reversing involution on $\mathcal{S}_n^{321}$.

Finally, we note the following two immediate consequences of Theorem 4.4.

Corollary 4.5. *For even n , the statistic sgn on $\mathcal{S}_n^{321}$ is 0-mesic under the action of ρ_S .*

Corollary 4.6. *For even n , the map LK on $\mathcal{D}_n$ has no fixed points.*

Corollary 4.6 is equivalent to [10, Thm. 4.6] for even n .

5 The Armstrong–Stump–Thomas bijection as a map on permutations

In [1], Armstrong, Stump and Thomas constructed a bijection between antichains in root posets of finite Weyl groups and noncrossing matchings, having the property that it translates rowmotion on antichains into rotation of noncrossing matchings. In type A , we can interpret antichains as 321-avoiding permutations via the bijection Exc . It turns out that, with this interpretation, the Armstrong–Stump–Thomas bijection is equivalent to the well-known Robinson–Schensted–Knuth correspondence restricted to 321-avoiding permutations. This fact extends to the root poset in type B .

5.1 AST in type A

The Armstrong–Stump–Thomas bijection is described in [1] in much more generality, but for our purposes we will be restricting exclusively to the cases of type A and B . We follow the description given by Defant and Hopkins [4]. In type A , we denote this bijection by AST .

Definition 5.1 (AST in type A [4]). Let $\text{AST} : \mathcal{A} \mathbf{A}^{n-1} \rightarrow \mathcal{N}_n$ be the bijection where the image of $A \in \mathcal{A} \mathbf{A}^{n-1}$ is obtained as follows. For each $1 \leq i \leq n$, the vertex $2n - 1 - i$ of the matching will be denoted by $\bar{i}$, so that the vertices are $1, 2, \dots, n, \bar{n}, \overline{n-1}, \dots, \bar{1}$ in clockwise direction. For each i from 1 to n , consider two options:

- if $i, j - 1 \in A$ for some j , match the vertex j with the nearest unmatched vertex in counterclockwise direction;
- otherwise, match the vertex $\bar{i}$ with the nearest unmatched vertex in clockwise direction.

We can now state our main result in this section.

Theorem 5.2. *Let Match , $\widehat{\text{RSK}}$, Exc and AST be the bijections from Definitions 2.1, 2.2, 3.1 and 5.1, respectively. Then*

$$\text{AST} = \text{Match} \circ \widehat{\text{RSK}} \circ \text{Exc}^{-1}.$$

See Figure 4 for examples of these maps. It is shown in [1] that the bijection AST is equivariant in the sense that $\text{AST} \circ \rho_A = \text{Rot} \circ \text{AST}$.

5.2 AST in type B

Let $\mathbf{B}^m$ denote the positive root poset of the type B_m root system. The poset $\mathbf{B}^m$ is isomorphic to the quotient of $\mathbf{A}^{2m-1}$ by the relations of $i, j \sim 2m - j, 2m - i$ for all

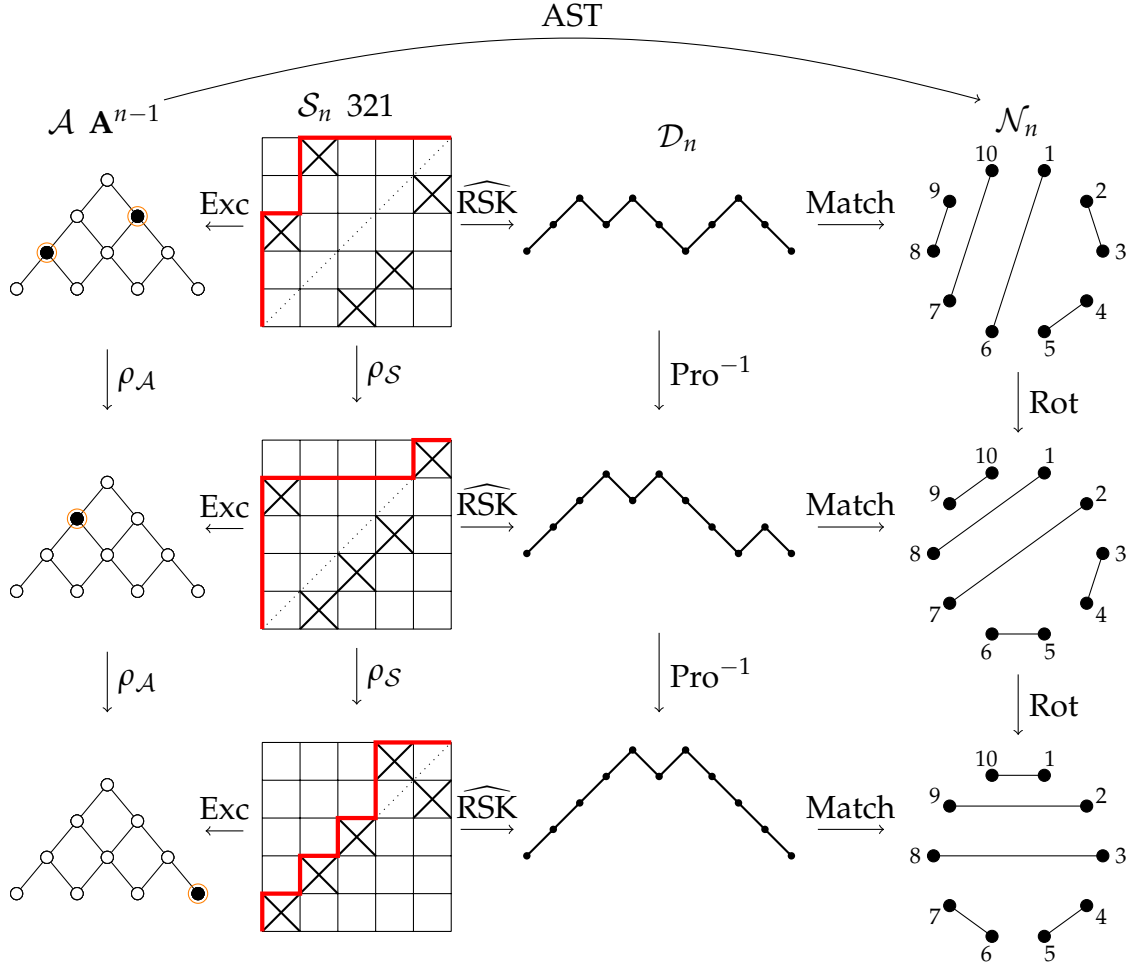


Figure 4: The three terms of the respective orbits of the elements of $\mathcal{A} \mathbf{A}^4$, S_5 321, D_5 , and $\mathcal{N}_5$ associated to and beginning with the permutation 35124 together with the respective actions of ρ_A , ρ_S , Pro^{-1} , and Rot .

$i, j \in \mathbf{A}^{2m-1}$. In type B , the Armstrong–Stump–Thomas map [1] is the map $\text{AST}_B : \mathcal{A} \mathbf{B}^m \rightarrow \mathcal{N}_{2m}$ defined as follows. For $A \in \mathcal{A} \mathbf{B}^m$, consider the antichain in $\mathbf{A}^{2m-1}$ given by

$$\hat{A} = \{i, j : i, j \in A \text{ or } 2m - j, 2m - i \in A\}. \quad (5.1)$$

Now let $\text{AST}_B A = \text{AST} \hat{A}$, where AST is the map from Definition 5.1.

It is shown in [1, Lemma 3.5] that the map ρ_A^n on $\mathcal{A} \mathbf{A}^{n-1}$, obtained by applying rowmotion n times, sends an antichain $\{i_1, j_1, \dots, i_k, j_k\}$ to the antichain $\{n - j_1, n - i_1, n - j_2, n - i_2, \dots, n - j_k, n - i_k\}$. The antichains that are invariant under this map are those that are symmetric under vertical reflection. When $n = 2m$, these are the antichains in the image of the embedding $A \mapsto \hat{A}$ from $\mathcal{A} \mathbf{B}^m$ into $\mathcal{A} \mathbf{A}^{2m-1}$ given

by (5.1). Thus the image of AST_B consists of the matchings in $\mathcal{N}_{2m}^S$ that are invariant under applying Rot^{2m} , i.e., under rotation by 180° . Denoting by $\mathcal{N}_n^S$ the set of centrally symmetric matchings in $\mathcal{N}_n$, one concludes that AST_B is a bijection between $\mathcal{A}(\mathbf{B}^m)$ and $\mathcal{N}_{2m}^S$.

Our alternative description of AST in terms of RSK allows us to give a new proof of this property of AST_B . Unlike the proof in [1], which relies on the properties of rowmotion, ours uses well-known properties of the RSK correspondence.

For $\pi \in \mathcal{S}_n$, its *reverse complement* is the permutation π^{rc} such that $\pi^{rc}(i) = n + 1 - \pi(n + 1 - i)$ for all i . The array of π^{rc} is obtained by rotating the array of π by 180° . Thus, the array of $\pi^{rc^{-1}}$ is obtained by reflecting the array of π along the secondary diagonal, i.e., the one passing through the bottom-left and upper-right corners of the array. The map Exc from Definition 3.1 restricts to a bijection between permutations $\pi \in \mathcal{S}_n$ 321 such that $\pi = \pi^{rc^{-1}}$, and antichains in $\mathcal{A}(\mathbf{A}^{2m-1})$ that are invariant under vertical reflection, which in turn are in bijection with $\mathcal{A}(\mathbf{B}^m)$. The behavior of RSK under these symmetries is well understood.

Theorem 5.3 ([16, Thm. A1.2.10]). *Suppose that $\text{RSK}(\pi) = (P, Q)$. Then $\text{RSK}(\pi^{rc}) = (\text{Evac}(P), \text{Evac}(Q))$, and $\text{RSK}(\pi^{rc^{-1}}) = (\text{Evac}(Q), \text{Evac}(P))$.*

These properties allow us to show the following without use of AST.

Theorem 5.4. *For $\pi \in \mathcal{S}_n$ 321, the matchings $\text{Match}(\widehat{\text{RSK}}(\pi^{rc^{-1}}))$ and $\text{Match}(\widehat{\text{RSK}}(\pi))$ are 180° rotations of each other. Thus, the map $\text{Match} \circ \widehat{\text{RSK}}$ restricts to a bijection between $\{\pi \in \mathcal{S}_n \text{ 321} : \pi = \pi^{rc^{-1}}\}$ and $\mathcal{N}_n^S$.*

Finally, another byproduct of our permutation perspective is that we can enumerate antichains of $\mathbf{B}^n$ which are fixed under the action of $\text{LK}_{\mathcal{A}} \circ \rho_{\mathcal{A}}$, answering a question of Hopkins and Joseph [6, Remark 6.7].

Proposition 5.5.

$$\#\{A \in \mathcal{A}(\mathbf{B}^n) : \text{LK}_{\mathcal{A}} \circ \rho_{\mathcal{A}}(A) = A\} = 2^n.$$

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Sums of Weighted Lattice Points of Polytopes

Jesús A. De Loera^{*1}, Laura Escobar^{†2},
Nathan Kaplan^{‡3}, and Chengyang Wang^{§1}

¹*Department of Mathematics, University of California, Davis*

²*Department of Mathematics and Statistics, Washington University in St. Louis*

³*Department of Mathematics, University of California, Irvine*

Abstract. We study the problem of counting lattice points of a polytope that are weighted by an Ehrhart quasi-polynomial of a family of parametric polytopes. As applications one can compute integrals and maximum values of such quasi-polynomials. We not only recover some known identities in representation theory and semigroup theory but obtain new ones.

Keywords: quasi-polynomials, Ehrhart functions, combinatorial identities, polyhedra.

1 Introduction

We are given a rational convex polytope P in $\mathbb{R}^n$ and $w(x)$ a quasi-polynomial function in n variables. We call w a *weight function* (the precise definition of the quasi-polynomials we use here is below, but for now you can think of w as a polynomial). A computational problem arising throughout the mathematical sciences is to compute, or at least estimate,

$$L_P(w, t) = \sum_{x \in tP \cap \mathbb{Z}^n} w(x). \quad (1.1)$$

Note that, for a fixed w as the polytope P is dilated by an integer factor $t \in \mathbb{N}$ we obtain a function of t , which we call the *weighted Ehrhart quasi-polynomial* for the pair (P, w) . The name is natural as when $w(x) = 1$ then $L_P(1, t)$ yields the classical *Ehrhart quasi-polynomial*. We recommend Chapter 4 of [25] or [8] and the references there for excellent introductions to Ehrhart functions and Ehrhart quasi-polynomials.

One can prove $L_P(w, t)$ is a quasi-polynomial in the sense that it is a function in the variable t which is a sum of monomials up to degree $d + M$, where $M = \deg w$, but whose coefficients E_m are periodic functions of $n \in \mathbb{N}$:

$$L_P(w, t) = \sum_{m=0}^{d+M} E_m t^m.$$

^{*}deloera@math.ucdavis.edu. Supported by NSF Grant DMS-1818969.

[†]laurae@wustl.edu. Supported by NSF Grant DMS-1855598 and NSF CAREER Grant DMS-2142656.

[‡]nckaplan@math.uci.edu. Supported by NSF Grants DMS-1802281 and DMS-2154223.

[§]cyywang@ucdavis.edu. Supported by NSF Grant DMS-1818969.

The evaluation of Equation (1.1) at the dilation tP of P is obviously what we need in order to understand this Ehrhart quasi-polynomial. The leading coefficient of $L_P(w, t)$ is given by the integral of w over the polytope P . These integrals were studied in [6], [7] and more recently in [5].

We will illustrate soon many important examples of such weighted Ehrhart problems. For now note they appear in enumerative combinatorics [2], algebraic combinatorics [4, 10], statistics [16, 11], and in symbolic integration and optimization [5, 14], among others.

Our Contributions:

We now outline the main contributions whose details will appear in the forthcoming full version. The main theorem is a surprisingly simple way to evaluate the function $L_P(w, t)$ where P is a rational polytope and $w(x)$ is a very general weight function. The key idea is that we build a new polytope, the *weight lifting polytope* P^* , for which these functions become simply $L_{P^*}(1, t)$, in other words, just a “standard” lattice point counting function. This way (often) the weighted Ehrhart polynomial P is equivalent to the (usual) Ehrhart polynomial of P^* . Clearly, P^* will depend on both P and w :

Theorem 1.1 (The existence of weight lifting polytopes). *Let P be a rational convex polytope in the form $\{\mathbf{x} \mid \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$, where $\mathbf{A} \in \mathbb{Z}^{s \times n}$, $\mathbf{b} \in \mathbb{Z}^s$. Let $Q(x_1, \dots, x_n)$ be the parametric family of rational convex polytopes parameterized by $x_1, \dots, x_n$, given by*

$$Q(x_1, \dots, x_n) = \left\{ \mathbf{y} \mid \mathbf{C}\mathbf{y} = \sum_{i=1}^n x_i \mathbf{d}_i + \mathbf{e}, \mathbf{y} \geq \mathbf{0} \right\},$$

where $\mathbf{C} \in \mathbb{Z}^{r \times m}$, $\mathbf{d}_i, \mathbf{e} \in \mathbb{Z}^r$. Using Q define $w(\mathbf{x})$ to be the multivariate Ehrhart quasi-polynomial function in n variables that counts the number of lattice points in the parametric polytope $Q(x_1, \dots, x_n)$ when x_i are chosen integers, i.e.,

$$w(x_1, \dots, x_n) = |Q(x_1, \dots, x_n) \cap \mathbb{Z}^m|.$$

1. There is a weight lifting polytope $P^* \subset \mathbb{R}^{n+m}$ defined by

$$P^* = \left\{ \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \mid \mathbf{A}^* \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} \mathbf{b} \\ -\mathbf{e} \end{pmatrix}, \mathbf{x} \geq \mathbf{0}, \mathbf{y} \geq \mathbf{0} \right\}$$

where

$$\mathbf{A}^* = \begin{bmatrix} & \mathbf{A} & & \mathbf{0} \\ \mathbf{d}_1 & \mathbf{d}_2 & \cdots & \mathbf{d}_n & -\mathbf{C} \end{bmatrix},$$

for which the summation of the lattice points of P weighted by w equals the number of lattice points of P^* .

2. Moreover, when $\mathbf{e} = 0$, the construction is parametric in the sense that the weight w is a homogeneous function, $(tP)^* = t(P^*)$, and

$$L_P(w, t) = |(tP)^* \cap \mathbb{Z}^{n+m}| = |t(P^*) \cap \mathbb{Z}^{n+m}| = L_{P^*}(1, t).$$

Remark 1.2. To the best of our knowledge the first version of [Theorem 1.1](#) appeared in print in work by Ardila and Brugallé (see [4, Section 4]), but in [4] the weights $w(x)$ were special polynomials and in that case some of the consequences we show were not possible. In [Section 2](#), we present a direct constructive/algorithmic proof of [Theorem 1.1](#) and describe several interesting special cases depending on the type of Ehrhart quasi-polynomials (in particular, we recover the results of [4]).

Remark 1.3. The second half of [Theorem 1.1](#) uses special weights that are by construction non-negative. But we note that most of the proof of the theorem works even when $w(x)$ takes negative or zero values over P . The function $L_P(w, t)$ still makes sense, but what we obtain is not a traditional Ehrhart polynomial, because, for example, the leading coefficient could be negative, and volumes are never negative.

Remark 1.4. [Theorem 1.1](#) says the weight $w(x)$ can be any Ehrhart quasi-polynomial. In [Section 2](#), we carefully discuss many ways to express polynomials in terms of these quasi-polynomial weights. A key point of our paper is that [Theorem 1.1](#) is more versatile and expressive because it applies to more functions than just polynomial weights. In fact, [Section 2](#) shows w can have many different representations (e.g., polynomials), some more efficient than others. To demonstrate the power in [Section 3](#) we present applications to Combinatorial Representation Theory and Number Theory.

[Corollary 1.5](#) below is a notable new consequence of [Theorem 1.1](#) that can be applied to many problems of interest. For example, these ideas can be applied to integration and maximization of Kostka numbers, Littlewood–Richardson coefficients, and any other combinatorial invariant that is given by an Ehrhart quasi-polynomial.

Corollary 1.5. *Let w be weight obtained from an Ehrhart quasi-polynomial function of a parametric polyhedron Q , whose parameters are defined over the lattice points of a polytope P . Here P, Q, w are just as in [Theorem 1.1](#). Using the weight lifting polytope construction of [Theorem 1.1](#) one can integrate and maximize w over P as follows:*

- One can compute the integral $\int_P w(x)dx$ reformulated as a volume computation of the weight lifting polytope P^* .
- One can solve the maximization problem and determine $\max_{\alpha \in P \cap \mathbb{Z}^n} w(\alpha)$. It reduces to counting the lattice points of a finite sequence of weight lifting polytopes which contain each other and can be read from P^* efficiently.

We sketch the proof of [Corollary 1.5](#) in [Section 2](#). In the forthcoming full version we will include computational experiments with LattE [13].

2 Proofs of Theorem 1.1 and sketch of other results

Here we present proofs of Theorem 1.1 and some variations of it.

Proof of Theorem 1.1. Note that there is a natural projection map $\pi : P^* \rightarrow P$ via $(\mathbf{x}, \mathbf{y}) \mapsto \mathbf{x}$. It suffices to show that for any fixed $\mathbf{x} \in P \cap \mathbb{Z}^n$, $w(\mathbf{x}) = |\pi^{-1}(\mathbf{x}) \cap \mathbb{Z}^{n+m}|$. Recall that $(\mathbf{x}, \mathbf{y}) \in \pi^{-1}(\mathbf{x})$ if and only if $\mathbf{A}\mathbf{x} = \mathbf{b}$ and $\mathbf{C}\mathbf{y} = \sum_{i=1}^n x_i \mathbf{b}_i + \mathbf{e}$ where $\mathbf{x} \geq \mathbf{0}, \mathbf{y} \geq \mathbf{0}$. Given $\mathbf{x} \in P \cap \mathbb{Z}^n$, we see that $(\mathbf{x}, \mathbf{y}) \in \pi^{-1}(\mathbf{x}) \cap \mathbb{Z}^{n+m}$ if and only if $\mathbf{y} \in Q(x_1, \dots, x_n) \cap \mathbb{Z}^m$. Hence, for a fixed $\mathbf{x} \in P \cap \mathbb{Z}^n$, $|\pi^{-1}(\mathbf{x}) \cap \mathbb{Z}^{n+m}| = |Q(\mathbf{x}) \cap \mathbb{Z}^m| = w(\mathbf{x})$.

We now consider second part of Theorem 1.1. We show that P^* is parametric with respect to $\mathbf{b}$ in the following sense. If $P = \{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$, then

$$P^* = \left\{ \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \middle| \mathbf{A}^* \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} \mathbf{b} \\ -\mathbf{e} \end{pmatrix}, \mathbf{x} \geq \mathbf{0}, \mathbf{y} \geq \mathbf{0} \right\}.$$

Therefore, $tP = \{\mathbf{x} : \mathbf{A}\mathbf{x} = t\mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ and

$$(tP)^* = \left\{ \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \middle| \mathbf{A}^* \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} t\mathbf{b} \\ -\mathbf{e} \end{pmatrix}, \mathbf{x} \geq \mathbf{0}, \mathbf{y} \geq \mathbf{0} \right\}.$$

Given $\mathbf{e} = \mathbf{0}$, we can see that $(tP)^* = t(P^*)$. By the first part of the proof we conclude

$$L_P(w, t) = |(tP)^* \cap \mathbb{Z}^{n+m}| = |t(P^*) \cap \mathbb{Z}^{n+m}| = L_{P^*}(1, t). \quad \square$$

Now we outline more results and corollaries of Theorem 1.1. From now on we deal with the most general *quasi-polynomial weighted* case, i.e., $w(x)$ is a non-constant quasi-polynomial as in the statement of Theorem 1.1.

Definition 2.1. Let $Q = \{\mathbf{y} \mid \mathbf{C}\mathbf{y} = \mathbf{d}, \mathbf{y} \geq \mathbf{0}\} \subseteq \mathbb{R}^m$ where $\mathbf{C} \in \mathbb{Z}^{r \times m}, \mathbf{d} \in \mathbb{Z}^r$ be a rational polytope. For every integer t , we define $tQ = \{\mathbf{y} \mid \mathbf{C}\mathbf{y} = t\mathbf{d}, \mathbf{y} \geq \mathbf{0}\}$.

Example 2.2. Consider the $(m-1)$ -dimensional standard simplex

$$\Delta_{m-1} = \{\mathbf{y} \mid y_1 + \dots + y_m = 1, y_i \geq 0\}.$$

Then $-2\Delta_{m-1} = \{\mathbf{y} \mid y_1 + \dots + y_m = -2 \cdot 1, y_i \geq 0\}$.

Definition 2.3. A function $w(t)$ is a **late-dilated** Ehrhart quasi-polynomial if

$$w(t) = |(t - c)Q \cap \mathbb{Z}^m|,$$

where $c \in \mathbb{Z}$ and Q is a rational polytope of the form given in Definition 2.1.

Example 2.4. The function $\binom{t}{m-1}$ is a late-dilated Ehrhart polynomial in the variable t , because $\binom{t}{m-1} = |(t - m + 1)\Delta_{m-1} \cap \mathbb{Z}^m|$.

Corollary 2.5. Let $w_1, w_2, \dots, w_n$ be n late-dilated Ehrhart quasi-polynomials, i.e., $w_i(t) = |(t - c_i)Q_i \cap \mathbb{Z}^{m_i}|$ where $Q_i = \{\mathbf{y}_i \mid \mathbf{C}_i \mathbf{y}_i = \mathbf{d}_i, \mathbf{y}_i \geq \mathbf{0}\}$ and $\mathbf{C}_i \in \mathbb{Z}^{r_i \times m_i}, \mathbf{d}_i \in \mathbb{Z}^{r_i}, c_i \in \mathbb{Z}$. Consider a rational polytope of the form $P = \{\mathbf{x} \mid \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\} \subseteq \mathbb{R}^n$ where $\mathbf{A} \in \mathbb{Z}^{s \times n}, \mathbf{b} \in \mathbb{Z}^s$ and the multivariate function $w(\mathbf{x}) = \prod_{i=1}^n w_i(x_i)$. There exists a weight lifting polytope $P^* \subseteq \mathbb{R}^{n^*}$ of P , where $n^* = n + m_1 + \dots + m_n$, such that

$$\sum_{\mathbf{x} \in P \cap \mathbb{Z}^n} w(\mathbf{x}) = |P^* \cap \mathbb{Z}^{n^*}|.$$

Proof. We need only show that there is a rational polytope $Q(x_1, \dots, x_n)$ of the form given in [Theorem 1.1](#) for which $w(\mathbf{x}) = |Q(x_1, \dots, x_n) \cap \mathbb{Z}^{m_1 + \dots + m_n}|$ and then apply [Theorem 1.1](#). Let $Q(x_1, \dots, x_n) = \prod_{i=1}^n (x_i - c_i)Q_i$. Specifically, $Q(x_1, \dots, x_n)$ has the form

$$\left\{ \begin{pmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_n \end{pmatrix} \left| \begin{pmatrix} \mathbf{C}_1 & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{C}_n \end{pmatrix} \begin{pmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_n \end{pmatrix} = x_1 \begin{pmatrix} \mathbf{d}_1 \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{pmatrix} + \cdots + x_n \begin{pmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \mathbf{d}_n \end{pmatrix} + \mathbf{e}, \mathbf{y} \geq \mathbf{0} \right\}. \quad \square$$

Corollary 2.6. For every monomial $w(\mathbf{x}) = \mathbf{x}^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$, there exists a weight lifting polytope $P^* \subseteq \mathbb{R}^{n^*}$ where $n^* = n + 2|\alpha| = n + 2 \sum_{i=1}^n \alpha_i$ such that

$$\sum_{\mathbf{x} \in P \cap \mathbb{Z}^n} w(\mathbf{x}) = |P^* \cap \mathbb{Z}^{n^*}|.$$

Proof. By [Corollary 2.5](#), we just need to show that $x_i^{\alpha_i}$ is a late-dilated Ehrhart polynomial. It is well known that $(k+1)^{\alpha_i}$ is the Ehrhart polynomial of the α_i -dimensional hypercube of length k . In particular, the hypercube has the form

$$tQ_i = \left\{ \begin{pmatrix} y_1 \\ \vdots \\ y_{\alpha_i} \\ z_1 \\ \vdots \\ z_{\alpha_i} \end{pmatrix} \left| \begin{bmatrix} 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_{\alpha_i} \\ z_1 \\ \vdots \\ z_{\alpha_i} \end{pmatrix} = t \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, y_i \geq 0, z_i \geq 0 \right\}. \quad \square$$

Corollary 2.7. For every polynomial $w(\mathbf{x}) = \sum_{\alpha \in I} c_\alpha \mathbf{x}^\alpha = \sum_{\alpha \in I} c_\alpha x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$, there exist $|I|$ weight lifting polytopes P_α^* indexed by the exponents of monomials such that

$$\sum_{\mathbf{x} \in P \cap \mathbb{Z}^n} w(\mathbf{x}) = \sum_{\alpha \in I} c_\alpha |P_\alpha^* \cap \mathbb{Z}^{n^*}|.$$

Proof. This follows directly from [Corollary 2.6](#) □

Remark 2.8. [Corollary 2.7](#) implies that if $w(x)$ is a polynomial with $|I|$ nonzero monomials, then we can compute the sum of lattice points of P weighted by w by counting integral points in $|I|$ weight lifting polytopes.

We give another two corollaries of [Theorem 1.1](#).

Corollary 2.9. Consider the polynomial $w(\mathbf{x}) = \prod_{i=1}^n \binom{x_i + \alpha_i - 1}{\alpha_i - 1}$. There exists a weight lifting polytope $P^* \subseteq \mathbb{R}^{n^*}$ where $n^* = n + |\alpha|$ such that

$$\sum_{\mathbf{x} \in P \cap \mathbb{Z}^n} w(\mathbf{x}) = |P^* \cap \mathbb{Z}^{n^*}|.$$

Proof. Recall that $\binom{x_i + \alpha_i - 1}{\alpha_i - 1}$ is the Ehrhart polynomial of the standard $(\alpha_i - 1)$ -simplex $1 = y_1 + \dots + y_{\alpha_i}$ with $y_i \geq 0$. In particular, the simplex has the form

$$tQ_i = \left\{ \begin{pmatrix} y_1 \\ \vdots \\ y_{\alpha_i} \end{pmatrix} \left| \begin{bmatrix} 1 & \dots & 1 \end{bmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_{\alpha_i} \end{pmatrix} = t \cdot 1, y_i \geq 0 \right. \right\}.$$

Applying [Corollary 2.5](#) gives the weight lifting polytope from the statement. $\square$

Corollary 2.10. Consider the polynomial $w(\mathbf{x}) = \prod_{i=1}^n \binom{x_i}{\alpha_i - 1}$. There exists a weight lifting polytope P^* of the dimension $n^* = n + |\alpha|$ such that $\sum_{\mathbf{x} \in P \cap \mathbb{Z}^n} w(\mathbf{x}) = |P^* \cap \mathbb{Z}^{n^*}|$.

Proof. The function $\binom{x_i}{\alpha_i - 1}$ is a late-dilated Ehrhart polynomial because $\binom{x_i + \alpha_i - 1}{\alpha_i - 1}$ is the Ehrhart polynomial of the standard $(\alpha_i - 1)$ -simplex. Applying [Corollary 2.5](#) gives the weight lifting polytope from the statement. $\square$

Note that $\{ \binom{x+k-1}{k-1} \mid k = 1, 2, \dots \}$ and $\{ \binom{x}{k-1} \mid k = 1, 2, \dots \}$ are two well-known bases of the vector space of polynomials in x .

Corollary 2.11. For every monomial $w(\mathbf{x}) = \mathbf{x}^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$, there exist at most $\alpha_1 \dots \alpha_n$ weight lifting polytopes P_β^* indexed by the vector β and $P_\beta^* \subset \mathbb{R}^{n^*}$ where $n^* = n + |\beta|$ such that

$$\sum_{\mathbf{x} \in P \cap \mathbb{Z}^n} w(\mathbf{x}) = \sum_{\beta \leq \alpha} c_\beta |P_\beta^* \cap \mathbb{Z}^{n^*}|.$$

Proof. Let $v_k(x)$ be one of the two binomial bases described above. We can transform the monomial basis $\{x^k \mid k = 0, 1, 2, \dots\}$ into the binomial basis,

$$x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n} = \sum_{\beta \leq \alpha} c(\alpha, \beta) \cdot v_{\beta_1}(x_1) v_{\beta_2}(x_2) \dots v_{\beta_n}(x_n).$$

By [Corollaries 2.9](#) and [2.10](#), for each β and each polynomial $v_{\beta_1}(x_1) v_{\beta_2}(x_2) \dots v_{\beta_n}(x_n)$, there exists a corresponding weight lifting polytope $P_\beta^* \subset \mathbb{R}^{n+|\beta|}$. $\square$

Remark 2.12. In [Corollary 2.6](#) we express the weighted sum of lattice points of P using a single $P^* \subset \mathbb{R}^{n+2|\alpha|}$, but in [Corollary 2.11](#) we express this sum using at most $\alpha_1 \alpha_2 \cdots \alpha_n$ polytopes of lower dimension $P_\beta^* \subset \mathbb{R}^{n+|\beta|}$.

Proof of Corollary 1.5. Applying [Theorem 1.1](#) to P gives a weight lifting polytope P^* for which $L_P(w, t) = L_{P^*}(1, t)$. Applying a classical result relating the volume and lead coefficient of the Ehrhart quasi-polynomial of P^* completes the proof. Both $L_P(w, t)$ and $L_{P^*}(1, t)$ are quasi-polynomial functions of t , and concretely, this equality implies that their leading coefficients are the same.

We can then replace integration of $w(x)$ over P with computation of the leading coefficient of $L_{P^*}(1, t)$, which is equivalent to computing the volume of P^* . Note that this transformation can be carried out in a number of steps that is polynomial in the size of the inputs describing P^* .

For the second claim, we start by recalling an elementary fact. Let $S = \{s_1, \dots, s_r\}$ be a set of non-negative real numbers. Then $\max\{s_i \mid s_i \in S\} = \lim_{k \rightarrow \infty} \sqrt[k]{\sum_{j=1}^r s_j^k}$. The arithmetic mean of S is at most its maximum value, which in turn is at most as big as $\sum_i s_i$. We apply these ideas to the set $S = \{w(\alpha) \mid \alpha \in P \cap \mathbb{Z}^n\}$. This gives upper and lower bounds for each positive integer k :

$$L_k = \sqrt[k]{\frac{\sum_{\alpha \in P \cap \mathbb{Z}^n} w(\alpha)^k}{|P \cap \mathbb{Z}^d|}} \leq \max\{w(\alpha) : \alpha \in P \cap \mathbb{Z}^n\} \leq \sqrt[k]{\sum_{\alpha \in P \cap \mathbb{Z}^n} w(\alpha)^k} = U_k.$$

As $k \rightarrow \infty$, L_k and U_k approach this maximum value monotonically (from below and above, respectively). Trivially, if the difference between the (rounded) upper and lower bounds becomes strictly less than 1, we have determined $\max\{w(x) \mid x \in P \cap \mathbb{Z}^n\} = \lceil L_k \rceil$. Thus the process terminates with the correct value. Finally, the key value in the sequences L_k and U_k is the term $L_P(w^k, t) = \sum_{\alpha \in P \cap \mathbb{Z}^n} w(\alpha)^k$. [Corollary 2.5](#) describes how to construct the weight lifting polytope P^* corresponding to the pair P and $w(\alpha)^k$. $\square$

3 Applications

[Theorem 1.1](#) has applications beyond integration and maximization of Ehrhart quasi-polynomials. In this section we discuss how to use it to find new algebraic combinatorial identities by carefully choosing the polytope P and reinterpreting the weight function w in terms of Ehrhart quasi-polynomials of some polytopes Q_i .

3.1 Weighted Ehrhart in number theory

Simultaneous Core Partitions. We first describe an area in which weighted Ehrhart machinery has already been applied to prove a significant result. Let λ be a partition

and $\mathcal{H}(\lambda)$ denote its multiset of hook lengths. The partition λ is called an a -core partition if no element of $\mathcal{H}(\lambda)$ is divisible by a . If λ is both an a -core partition and a b -core partition, then we say that it is an (a, b) -core partition. There is an extensive literature about statistical properties of sizes of simultaneous core partitions [12, 22]. Anderson proved that if a and b are relatively prime positive integers then the number of (a, b) -core partitions is $\frac{1}{a+b} \binom{a+b}{a}$ [1]. Johnson proved a conjecture of Armstrong, showing that the average size of an (a, b) -core partition is $(a + b + 1)(a - 1)(b - 1)/24$ [18]. Johnson's proof fits into the framework of weighted Ehrhart theory.

Suppose that a and b are relatively prime positive integers. It is not hard to show that a -core partitions are in bijection with elements of $\Lambda_a = \{(c_0, \dots, c_{a-1}) \in \mathbb{Z}^a : \sum_i c_i = 0\}$. Let $r_a(x)$ be the remainder when x is divided by a . We use cyclic indexing for elements $\mathbf{c} \in \Lambda_a$, that is, for $k \in \mathbb{Z}$ we set $c_k = c_{r_a(k)}$. Simultaneous (a, b) -core partitions are in bijection with the elements of Λ_a satisfying the inequalities $c_{i+b} - c_i \leq \left\lfloor \frac{b+i}{a} \right\rfloor$ for each $i \in \{0, 1, \dots, a-1\}$ [18, Lemma 23]. In this way, we see that (a, b) -core partitions are in bijection with integer points in a rational polytope $\text{SC}_a(b)$. The size of the a -core partition corresponding to $\mathbf{c} = (c_0, \dots, c_{a-1})$ is $h_a(\mathbf{c}) = \frac{a}{2} \sum_{i=0}^{a-1} (c_i^2 + ic_i)$ [18, Theorem 22]. Therefore, Anderson's theorem is equivalent to computing the number of integer points in $\text{SC}_a(b)$, and Johnson's theorem is equivalent to computing $\sum_{\mathbf{c} \in \text{SC}_a(b)} h_a(\mathbf{c})$.

Johnson computes this weighted sum of lattice points by relating it to a sum over the subset of integer points $(z_0, \dots, z_{a-1})$ of the dilation of the standard simplex $b\Delta_{a-1}$ that satisfy $\sum iz_i \equiv 0 \pmod{a}$. Johnson then shows that the sum he needs to compute is equal to $1/a$ times the sum of a quadratic function w taken over all integer points of $b\Delta_{a-1}$. In order to conclude, he applies a result from Euler–Maclaurin theory, which is a version of the first part of Corollary 1.5, and also applies a version of weighted Ehrhart reciprocity that appears in [4].

By Corollary 2.7, there exists a family of weight lifting polytopes $P_\alpha^* \subset \mathbb{R}^{n^*}$ such that

$$\sum_{x \in b\Delta_{a-1} \cap \mathbb{Z}^a} w(x) = \sum_{\alpha \in I} c_\alpha \left| P_\alpha^* \cap \mathbb{Z}^{n^*} \right|.$$

It seems likely that further study of these kinds of weight lifting polytopes can lead to new techniques in the study of simultaneous core partitions.

Numerical Semigroups. A numerical semigroup S is an additive submonoid of $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ with finite complement. The elements of $\mathbb{N}_0 \setminus S$ are the *gaps* of S , denoted $G(S) = \{h_1, \dots, h_g\}$. The *weight* of S is defined by $w(S) = (h_1 + \dots + h_g) - (1 + 2 + \dots + g)$. The motivation for studying $w(S)$ comes from the theory of Weierstrass semigroups of algebraic curves [3, Chapter 1, Appendix E].

Numerical semigroups containing m are in bijection with integer points $(x_1, \dots, x_{m-1})$ in the *Kunz polyhedron* $P_m \subset \mathbb{R}^{m-1}$, which is defined via bounding inequalities

$$x_i + x_j \geq x_{i+j} \text{ if } i + j < m, \quad x_i + x_j + 1 \geq x_{i+j-m} \text{ if } i + j > m.$$

Let $NS(m, g)$ be the set of numerical semigroups containing m with genus g . These semigroups are in bijection with the integer points of $P_{m,g}$, the polytope we get from P_m by adding the additional constraint $\sum x_i = g$. For a more extensive discussion of the connection between numerical semigroups containing m and integer points in the Kunz polyhedron, see [19, Section 4]. If $(k_1, \dots, k_{m-1})$ is the integer point corresponding to a semigroup S of genus g , then $w(S) = \frac{m}{2} \sum_{i=1}^{m-1} k_i(k_i - 1) + \sum_{i=1}^{m-1} ik_i - g(g+1)/2$. There has been recent interest in statistical properties of weights of families of semigroups, see for example [21, Section 5] and [20].

By Corollary 2.7, there exists a family of weight lifting polytopes $P_\alpha^* \subset \mathbb{R}^{n^*}$ such that

$$\sum_{S \in NS(m, g)} w(S) = \sum_{S \in P_{m, g} \cap \mathbb{Z}^{m-1}} w(S) = \sum_{\alpha \in I} c_\alpha \left| P_\alpha^* \cap \mathbb{Z}^{n^*} \right|.$$

Studying this family of polytopes and applying a version of Corollary 1.5 suggests an approach to the following two questions:

1. What is the maximum of $w(S)$ for $S \in NS(m, g)$?
2. For fixed m , what is the main term in the expression for $\sum_{S \in NS(m, g)} w(S)$ as $g \rightarrow \infty$?

3.2 Weighted Ehrhart in combinatorial representation theory

There is a long tradition of using lattice points of polytopes in representation theory (see [15] and the references there). Here, as an application of Theorem 1.1, we provide new connections.

Maximizing Kostka numbers. Fix a partition $\lambda \vdash n$ and let $SSYT(\lambda)$ denote the set of semi-standard Young tableaux of shape λ . The Schur function s_λ is

$$s_\lambda(x) = \sum_{T \in SSYT(\lambda)} x^T = \sum_{\alpha \in \text{comp}(n)} K_{\lambda\alpha} x^\alpha,$$

where $\text{comp}(n)$ is the set of weak compositions n and $K_{\lambda\alpha}$ is the *Kostka number* that counts the number of tableaux in $SSYT(\lambda)$ with content α . Evaluating s_λ at $x_1 = 1, x_2 = 1, \dots, x_N = 1, x_{N+1} = 0, x_{N+2} = 0, \dots$ yields

$$|SSYT(\lambda, N)| = \sum_{\alpha \in N\text{-comp}(n)} K_{\lambda\alpha},$$

where $SSYT(\lambda, N)$ is the set of semi-standard Young tableaux of shape λ and entries bounded by N and $N\text{-comp}(n)$ is the set of weak composition of n with N parts.

A weak composition of n with N parts is a lattice point in the scaled standard $(N-1)$ -simplex $n\Delta_{N-1}$. The Kostka number $K_{\lambda\alpha}$ equals the number of lattice points in the Gelfand–Tsetlin polytope $GT(\lambda, \alpha)$ (see e.g., [15]), so $w(\alpha) = K_{\lambda\alpha}$ is a weight function. There have been contributions to understanding the behavior of $K_{\lambda\alpha}$ as (λ, α) vary and

an example is [17] in which it is shown that they are log-concave. Applying the method in Corollary 1.5 one can use the weight lifting polytope given by Theorem 1.1 to compute $\max_{\alpha \in N\text{-comp}(n)} K_{\lambda\alpha}$ and we will include generated data in the full version of this paper.

Robinson–Schensted–Knuth (RSK) identity. Fix partitions $\mu, \nu \vdash n$ and recall the famous *RSK identity* (for details see e.g., [23]):

$$\sum_{\lambda \vdash n} K_{\lambda\mu} K_{\lambda\nu} = N_{\mu,\nu}.$$

The left sum is over partitions of n and the summands are products of Kostka numbers. In fact, the left side of the identity is a weighted sum over the lattice points of

$$P = \{\mathbf{x} \in \mathbb{R}^n \mid x_1 + \cdots + x_n = n, x_1 \geq x_2 \geq \cdots \geq x_n \geq 0\}. \quad (3.1)$$

This is because the weight function $w(\lambda) = K_{\lambda\mu} K_{\lambda\nu}$ is the number of lattice points in the Cartesian product $GT(\lambda, \mu) \times GT(\lambda, \nu)$ of two Gelfand–Tsetlin polytopes. The right-hand side of RSK, $N_{\mu,\nu}$, is the number of lattice points in the transportation polytope

$$Mat_{n,n}(\mu, \nu) = \left\{ (z_{ij})_{1 \leq i,j \leq n} \mid \sum_j z_{ij} = \mu_i, \sum_i z_{ij} = \nu_j, z_{ij} \geq 0 \right\}.$$

While RSK provides more information (e.g., a bijection), Theorem 1.1 gives a new polytope whose number of lattice points is the sum $\sum_{\lambda \vdash n} K_{\lambda\mu} K_{\lambda\nu}$.

Corollary 3.1 (A new RSK-like identity). *There exists a weight lifting polytope $P^*(\mu, \nu) \subseteq \mathbb{R}^{n^2+2n}$ which is combinatorially different from $Mat_{n,n}(\mu, \nu)$ such that*

$$\sum_{\lambda \vdash n} K_{\lambda\mu} K_{\lambda\nu} = |P^*(\mu, \nu) \cap \mathbb{Z}^{n^2+2n}|.$$

Littlewood–Richardson Coefficients. Schur functions are central objects in representation theory and combinatorics. The skew Schur function for partitions $\lambda, \mu \vdash n$ is

$$s_{\lambda/\mu}(x) = \sum_{\alpha \in \text{comp}(n)} K_{\lambda/\nu, \alpha} x^\alpha,$$

where the sum is over all compositions of n and $K_{\lambda/\nu, \alpha}$ counts the number of skew semi-standard Young tableaux of shape λ/ν and weight α . The Littlewood–Richardson rule (see e.g., [24]) expresses the skew Schur functions in terms of Schur functions,

$$s_{\lambda/\mu}(x) = \sum_{\nu \vdash n} c_{\mu\nu}^\lambda s_\nu(x).$$

Comparing the expression of the coefficient of the monomial x^α yields

$$K_{\lambda/\nu, \alpha} = \sum_{\nu \vdash n} c_{\mu\nu}^\lambda K_{\nu\alpha}.$$

The Littlewood–Richardson coefficient $c_{\mu\nu}^\lambda$ counts the number of lattice points in the hive polytope $H_{\mu\nu}^\lambda$ (see e.g., [9]). Applying Theorem 1.1 to the simplex in (3.1) and $w(\nu) = c_{\mu\nu}^\lambda K_{\nu\alpha}$, which counts the number of lattice points in $H_{\mu\nu}^\lambda \times GT(\nu, \alpha)$, we obtain the following corollary.

Corollary 3.2. *There exists a weight lifting polytope $P^*(\lambda/\mu, \alpha) \subseteq \mathbb{R}^{n^2+2n}$ such that*

$$\sum_{\lambda \vdash n} c_{\mu\nu}^\lambda K_{\nu\alpha} = |P^*(\lambda/\mu, \alpha) \cap \mathbb{Z}^{n^2+2n}|.$$

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Noncrossing partitions of an annulus

Laura G. Brestensky^{*1} and Nathan Reading⁺¹

¹*Department of Mathematics, NC State University, Raleigh, NC, United States*

Abstract. The noncrossing partition poset associated to a Coxeter group W and Coxeter element c is the interval $[1, c]_T$ in the absolute order on W . We construct a new model of noncrossing partitions for W of classical affine type, using planar diagrams. The model in type $\tilde{A}$ consists of noncrossing partitions of an annulus. In type $\tilde{C}$, the model consists of symmetric noncrossing partitions of an annulus or noncrossing partitions of a disk with two orbifold points. Following the lead of McCammond and Sulway, we complete $[1, c]_T$ to a lattice by factoring the translations in $[1, c]_T$, but the combinatorics of the planar diagrams leads us to make different choices about how to factor.

Résumé. Le poset de partitions noncroisées associé à un groupe de Coxeter W et un élément c de Coxeter est l'intervalle $[1, c]_T$ dans l'ordre absolu sur W . Nous construisons un nouveau modèle de partitions noncroisées dans le type affine, en utilisant des diagrammes planaires. Le modèle de type $\tilde{A}$ se compose des partitions noncroisées d'un anneau. Le modèle de type $\tilde{C}$ se compose des partitions symétriques noncroisées d'un anneau ou des partitions noncroisées d'un disque avec deux points d'orbifold. En suivant l'exemple de McCammond et Sulway, nous complétons $[1, c]_T$ à un treillis en factorisant les translations dans $[1, c]_T$, mais le combinatoire des diagrammes planaires nous conduit à faire des choix différents sur la manière de factoriser.

Keywords: absolute order, affine Coxeter group, annulus, noncrossing partition

1 Introduction

Noncrossing partitions associated to a Coxeter group are algebraic/combinatorial objects that figure prominently in the analysis of the associated Artin group. In the classical finite types A, B, and D, noncrossing partitions are best understood in terms of certain planar diagrams. The full version [10] of this paper and its sequel [11] extend planar diagrams for noncrossing partitions to the classical affine types. Furthermore, in [20], which can be thought of as a “prequel” to these papers, the combinatorics of these planar diagrams is generalized to the setting of marked surfaces (in the sense of cluster algebras). Planar diagrams for noncrossing partitions of types A and $\tilde{A}$ are generalized

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⁺reading@math.ncsu.edu. Nathan Reading was partially supported by the Simons Foundation under award number 581608 and by the National Science Foundation under award number DMS-2054489.

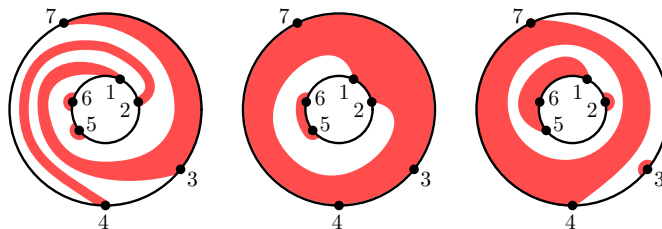


Figure 1: Some noncrossing partitions of an annulus

by noncrossing partitions of marked surfaces (without “punctures”), while in the other classical finite and affine types, the generalization is in terms of symmetric noncrossing partitions of a marked surface with (or without) double points.

Noncrossing partitions of a cycle were introduced by Kreweras [14]. Biane [3] connected noncrossing partitions of a cycle to finite Coxeter groups of type A (the symmetric groups) and showed that the lattice of noncrossing partitions is isomorphic to $[1, c]_T$, the interval between the identity and a Coxeter element c in the absolute order. The analogous interval in a finite Coxeter group of type B is modeled by centrally symmetric noncrossing partitions [1, 21], and there is an analogous planar model [1] for type D. These planar models can be understood uniformly in terms of the Coxeter plane, which was defined and studied in [12, 22]. Specifically, realizing group elements as permutations of an orbit, one projects to the Coxeter plane and interprets the cycle structure of the permutations as blocks in a partition [19].

The interval $[1, c]_T$ in a finite Coxeter group W serves as a Garside structure for the corresponding spherical Artin group $\text{Art}(W)$, leading to a “dual presentation” of $\text{Art}(W)$ [2, 7] and proving desirable properties of $\text{Art}(W)$. The fact that $[1, c]_T$ is a lattice is crucial to the dual presentation.

Outside of finite type, the interval $[1, c]_T$ need not be a lattice. The case of affine type is treated in a series of papers that begins by extending crucial results on rigid motions [6] to affine type [5] and continues with an analysis of the failure of the lattice property in Coxeter groups of affine type [16]. The series culminates in [17], in which McCammond and Sulway extend the affine Coxeter group W to a larger group, thereby extending the interval $[1, c]_T$ to a lattice. The larger lattice serves as a Garside structure for a supergroup of the Euclidean Artin group $\text{Art}(W)$, which inherits desirable (previously conjectured) properties from the supergroup.

In this extended abstract, we describe planar models, in affine types $\tilde{A}$ and $\tilde{C}$, for the intervals $[1, c]_T$ and the larger lattices constructed by McCammond and Sulway, **omitting all proofs**. In type $\tilde{A}$, the resulting model consists of *noncrossing partitions of an annulus*, defined in Section 3 and exemplified in Figure 1. In type $\tilde{C}$, the model consists of *symmetric noncrossing partitions of an annulus*, defined in Section 4. Some examples are shown in the top row of Figure 2. To see the symmetry, think of the pictures as

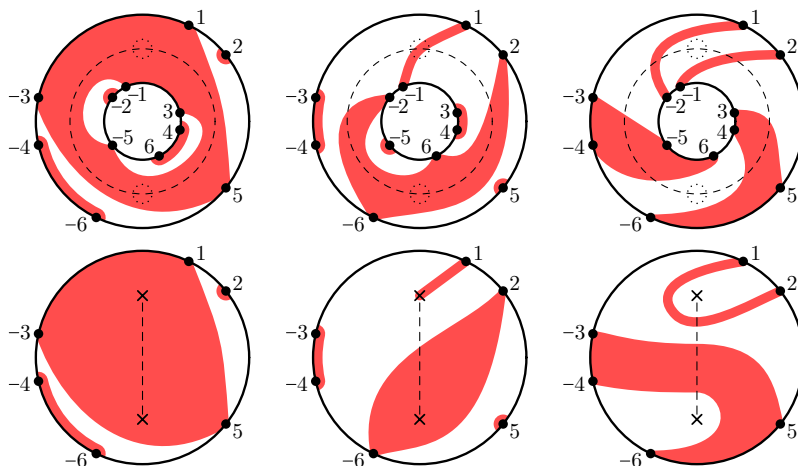


Figure 2: Some symmetric noncrossing partitions of an annulus and corresponding noncrossing partitions of the two-orbifold disk

perspective drawings of a circular cylinder; the symmetry is a rotation of the cylinder along a vertical axis. (The intersections of this axis with the cylinder are indicated in the pictures by small dotted circles). Symmetric noncrossing partitions of an annulus can equivalently be represented as *noncrossing partitions of a two-orbifold disk*, as shown in the bottom row of Figure 2. The fixed points of the symmetry in the annulus become order-2 orbifold points in the disk, marked with an $\times$ in the pictures. The dotted circle cuts the annulus into two pieces (a fundamental domain of the symmetry and its one translate). The image of the circle under the quotient map is shown as a dotted line.

The key idea behind McCammond and Sulway's extension of $[1, c]_T$ to a lattice is the notion of “factoring” translations in an affine Coxeter group W . Certain elements of W act (in the affine representation) as translations, and the translations in $[1, c]_T$ were implicated (in [16]) in the failure of the lattice property. To overcome the failure, each translation in $[1, c]_T$ is factored into two or more translations not already contained in W . The larger group containing W is generated by W and these new *factored translations*.

One of the motivating questions for the current project was how factored translations would fit into planar models. Would projecting the Coxeter plane lead to a useful model of $[1, c]_T$ only, or of the larger interval (incorporating factored translations) only, or of both? In type $\tilde{A}$, the best possible thing happened: The planar diagrams model the larger interval, and the elements of the smaller interval are distinguished by a simple criterion. The elements of $[1, c]_T$ are the noncrossing partitions of the annulus with no *dangling annular blocks*. A dangling annular block is a block that is an annulus and has numbered points on only one component of its boundary. Thus the leftmost and middle picture in Figure 1 represent elements of $[1, c]_T$, while the rightmost picture represents an element in the larger interval, not in $[1, c]_T$.

We pause to mention two aspects of the project that are omitted from this extended abstract: First, we omit the construction of the Coxeter plane and the projection of an orbit to the Coxeter plane, which lead naturally to a model on the annulus. Second, we omit a detailed discussion connecting our constructions to the work of McCammond and Sulway. (We only mention here that the planar model on the annulus suggests a scheme for factoring translations that is different from that in [17].)

Remark 1.1. Most of the results of [10, 11] reported here have appeared in Laura Brestensky's thesis [9]. Some parts of [10, 11] (and thus of this extended abstract) are revisions of material from [9].

Remark 1.2. Some other constructions are superficially or more deeply similar to the constructions in this paper. Details are in [10], but here we mention [13, 18, 15].

2 Background

Given a Coxeter system (W, S) , a *Coxeter element* is an element $c \in W$ that can be written as a product, in some chosen order, of the elements of S , each repeated exactly once in the product. The set of *reflections* of W is $T = \{ws w^{-1} : w \in W, s \in S\}$. The *absolute length* or *reflection length* $\ell_T(w)$ of an element $w \in W$ is the number of letters in a shortest expression for w as a product of elements of T . The *absolute order* $\leq_T$ is the partial order on W defined by $u \leq_T w$ if and only if $\ell_T(w) = \ell_T(u) + \ell_T(u^{-1}w)$.

When W is finite, the interval $[1, c]_T$ is a lattice [8], often called the *W -noncrossing partition lattice*, but when W is infinite, $[1, c]_T$ can fail to be a lattice. Planar models of the W -noncrossing partition lattice are well known in types A, B, and D [1, 3, 14, 21]. In [10] and [11], we construct planar diagrams for $[1, c]_T$ in the classical affine types.

The Coxeter group W of type $\tilde{A}_{n-1}$ can be realized as the group $\tilde{S}_n$ of *affine permutations*. (See, e.g. [4, Section 8.3].) These are permutations π of $\mathbb{Z}$ such that $\pi(i+n) = \pi(i) + n$ for all $i \in \mathbb{Z}$ and $\sum_{i=1}^n \pi(i) = \binom{n+1}{2}$. We also consider the larger group $S_{\mathbb{Z}(\text{mod } n)}$ (sometimes called the extended affine symmetric group) of *periodic permutations*: permutations π of $\mathbb{Z}$ such that $\pi(i+n) = \pi(i) + n$ for all $i \in \mathbb{Z}$.

We will describe permutations $\pi \in S_{\mathbb{Z}(\text{mod } n)}$ by their cycle structure. A finite cycle in π may not contain two entries that are equivalent modulo n , and for every finite cycle $(a_1 \ a_2 \ \cdots \ a_k)$ in π , the cycle $(a_1 + \ell n \ a_2 + \ell n \ \cdots \ a_k + \ell n)$ is also present in π for every $\ell \in \mathbb{Z}$. We write $(a_1 \ a_2 \ \cdots \ a_k)_n$ for the infinite product $\prod_{\ell \in \mathbb{Z}} (a_1 + \ell n \ a_2 + \ell n \ \cdots \ a_k + \ell n)$ of cycles. An infinite cycle in π necessarily has two entries that are equivalent modulo n , and, for any entry a_1 in the cycle, the cycle is completely determined by the sequence of entries from a_1 to the next entry a_{k+1} that is equivalent to a_1 modulo n . Thus we can represent infinite cycles in π by writing $(\cdots a_1 \ a_2 \ \cdots a_{k+1} \ \cdots)$.

A periodic permutation π is determined by the entries $\pi(1), \pi(2), \dots, \pi(n)$, sometimes called the “window” of π . The affine permutations are the periodic permutations

whose window sums to $\binom{n+1}{2}$.

Write $\ell_i = (\cdots i \ i + n \cdots)$ for $i = 1, \dots, n$ and $L = \{\ell_1, \dots, \ell_n\} \cup \{\ell_1^{-1}, \dots, \ell_n^{-1}\}$. One can show that $S_{\mathbb{Z}(\text{mod } n)}$ is generated by $T \cup L$. We define a partial order $\leq_{T \cup L}$ on $S_{\mathbb{Z}(\text{mod } n)}$ in the same way as the absolute order $\leq_T$ on $\tilde{S}_n$. We write $[1, c]_{T \cup L}$ for the interval between 1 and c in this order, where the subscript specifies not only the generating set, but indirectly also the group where the interval lives.

The Coxeter diagram of type $\tilde{A}_{n-1}$ is a cycle on the simple reflections $\{s_1, \dots, s_n\}$, with each s_i adjacent to s_{i-1} and s_{i+1} (taking indices modulo n). We record the choice of a Coxeter element c as follows: The numbers $1, \dots, n$ are placed in clockwise order over one full turn about the center of the annulus. The number i is placed on the outer boundary if and only if s_{i-1} precedes s_i in c or on the inner boundary if and only if s_i precedes s_{i-1} in c . We identify the numbers $1, \dots, n$ with their positions on the boundary of the annulus, and call them *outer points* or *inner points*. This construction creates a bijection from the set of Coxeter elements to the set of all partitions of $\{1, \dots, n\}$ into a nonempty set of outer points and a nonempty set of inner points.

Example 2.1. The pictures in Figure 1 shows the case where $W = \tilde{S}_7$ and the Coxeter element is $c = s_6 s_5 s_2 s_1 s_3 s_4 s_7 = (\cdots 3 \ 4 \ 7 \ 10 \cdots)(\cdots 6 \ 5 \ 2 \ 1 \ -1 \cdots)$.

Lemma 2.2. Let c be a Coxeter element of $\tilde{S}_n$, represented as partition of $\{1, \dots, n\}$ into inner points and outer points. If $a_1, \dots, a_k$ are the outer points in increasing (therefore clockwise) order and $b_1, \dots, b_{n-k}$ are the inner points in decreasing (therefore counterclockwise) order, then c is

$$(\cdots a_1 \ a_2 \ \cdots a_k \ a_1 + n \cdots)(\cdots b_1 \ b_2 \ \cdots b_{n-k} \ b_1 - n \cdots).$$

3 Noncrossing partitions of an annulus

Fix a Coxeter element c in $\tilde{S}_n$, encoded by a choice of inner and outer points on an annulus as described in Section 2. We refer to this annulus as A . The inner and outer points are collectively called *numbered points*. The pair consisting of A and the set of numbered points is an example of a marked surface in the sense of [20] (where the numbered points were called marked points). The results quoted here for the annulus were established in [9] and are quoted in [10] as special cases of results of [20].

A *boundary segment* of A is the portion of the boundary of A between two adjacent numbered points. A boundary segment may have both endpoints at the same numbered point if that is the only numbered point on one component of the boundary of A . Two subsets of A are related by *ambient isotopy* if they are related by a homeomorphism from A to itself, fixing the boundary ∂A pointwise and homotopic to the identity by a homotopy that fixes ∂A pointwise at every step.

An *arc* in A is a non-oriented curve in A , with endpoints at numbered points (possibly coinciding), that does not intersect itself except possibly at its endpoints and, except

for its endpoints, does intersect the boundary of A . We exclude the possibility that an arc bounds a monogon in A and the possibility that an arc combines with a boundary segment to bound a digon in A . An *embedded block* in A is one of the following:

- a *trivial block*, meaning a singleton consisting of a numbered point in A ;
- an arc or boundary segment in A ;
- a *disk block*, meaning a closed disk in A whose boundary is a union of arcs and/or boundary segments of A ;
- a *dangling annular block*, meaning a closed annulus in A with one component of its boundary a union of arcs and/or boundary segments of A and the other component of its boundary a circle in the interior of the A that can't be contracted in A to a point; or
- a *nondangling annular block*, meaning a closed annulus in A with each component of its boundary a union of arcs and/or boundary segments of A .

The first two types of embedded blocks are *degenerate disk blocks*. The last two types of embedded blocks are referred to less specifically as *annular blocks*.

A *noncrossing partition* of A is a collection $\mathcal{P} = \{E_1, \dots, E_k\}$ of disjoint embedded blocks (the *blocks of $\mathcal{P}$*) such that every numbered point is contained in some E_i and at most one E_i is an annular block. Given that the blocks are pairwise disjoint, restricting to at most one annular block serves only to rule out the case of two dangling annular blocks, one containing numbered points on its inner boundary and the other containing numbered points on its outer boundary. Noncrossing partitions are considered up to ambient isotopy. We refer to different, but isotopic, choices of the embedded blocks $E_1, \dots, E_k$ as different *embeddings* of the same noncrossing partition.

Example 3.1. We can now give more context to [Figure 1](#). These are noncrossing partitions of the annulus from [Example 2.1](#). Degenerate blocks are shown with some thickness, to make them visible.

We define a partial order on noncrossing partitions of A , called (in light of [Theorem 3.3](#) below) the *noncrossing partition lattice*: Noncrossing partitions $\mathcal{P}$ and $\mathcal{Q}$ of A have $\mathcal{P} \leq \mathcal{Q}$ if and only if there exist embeddings of $\mathcal{P}$ and $\mathcal{Q}$ such that every block of $\mathcal{P}$ is contained in some block of $\mathcal{Q}$. We write $\widehat{NC}_c^A$ for the set of noncrossing partitions of A with this partial order.

Example 3.2. Continuing [Example 3.1](#), we see that the left noncrossing partition shown in [Figure 1](#) is less than the middle noncrossing partition, but that is the only order relation among the three noncrossing partitions shown.

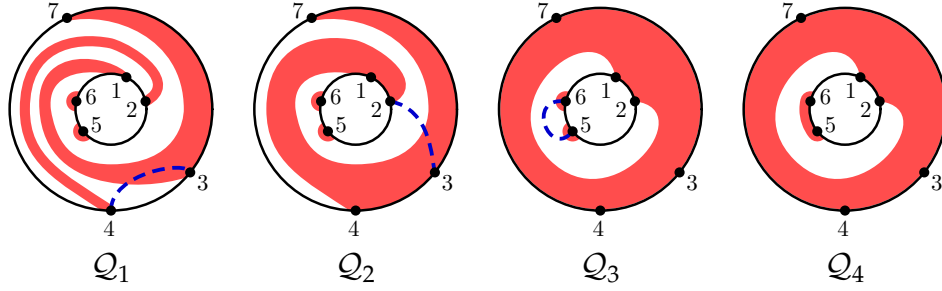


Figure 3: Q_1 Q_2 Q_3 Q_4 , with simple connectors

Theorem 3.3. The poset $\widetilde{NC}_c^A$ of noncrossing partitions of an annulus with marked points on both boundaries, n marked points in all, is a graded lattice, with rank function given by n minus the number of non-annular blocks.

Given $\mathcal{P} \in \widetilde{NC}_c^A$, a **simple connector** for $\mathcal{P}$ is an arc or boundary segment κ in the annulus that starts in some block E of $\mathcal{P}$, leaves E , and is disjoint from all other blocks until it enters some block of E' of $\mathcal{P}$, which it does not leave again. (Possibly $E = E'$.) The **augmentation** of $\mathcal{P}$ along κ , written $\mathcal{P} \cup \kappa$, is the smallest noncrossing partition containing $\mathcal{P}$ and κ . A more detailed definition is given in [10].

Proposition 3.4. If $\mathcal{P}, \mathcal{Q} \in \widetilde{NC}_c^A$, then $\mathcal{P}$ is covered by $\mathcal{Q}$ if and only if there exists a simple connector κ for $\mathcal{P}$ such that $\mathcal{Q} = \mathcal{P} \cup \kappa$.

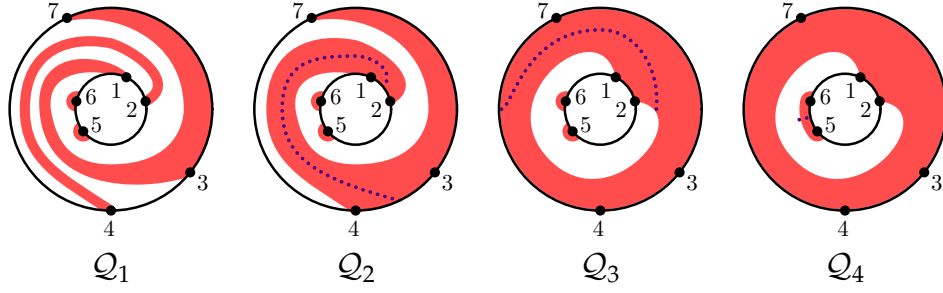
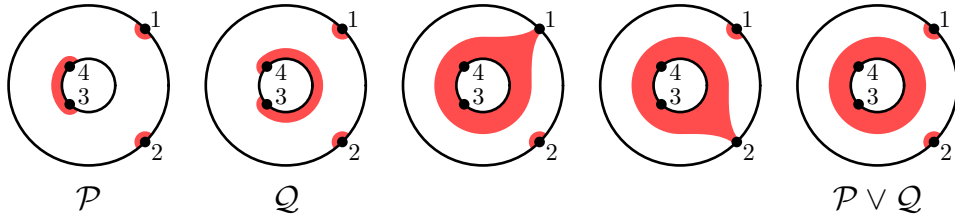
Example 3.5. Figure 3 shows noncrossing partitions Q_1, Q_2, Q_3 , and Q_4 with $Q_1 \prec Q_2 \prec Q_3 \prec Q_4$. A simple connector is shown for each cover relation.

Suppose $\mathcal{P} \in \widetilde{NC}_c^A$ and λ is a non-self-intersecting curve contained in some block E of $\mathcal{P}$, with each endpoint at a non-numbered point on the boundary E . Then λ is a **cutting curve** for $\mathcal{P}$ if it either (when E is annular or non-annular) cuts E into two pieces, each having at least one numbered point, or (when E is annular), cuts E into one non-annular piece. We allow the degenerate case where E is an arc or boundary segment and λ is a non-numbered point in E . Write $\mathcal{P} - \lambda$ for the noncrossing partition obtained from $\mathcal{P}$ by cutting E in this way. A more detailed definition is given in [10].

Proposition 3.6. If $\mathcal{P}, \mathcal{Q} \in \widetilde{NC}_c^A$, then $\mathcal{P} \prec \mathcal{Q}$ if and only if there exists a cutting curve for $\mathcal{Q}$ such that $\mathcal{P} = \mathcal{Q} - e$.

Example 3.7. Figure 4 shows the same noncrossing partitions Q_1, Q_2, Q_3, Q_4 pictured in Figure 3, this time with the relevant cutting curves shown.

Write $\widetilde{NC}_c^{A,o}$ for the subposet of $\widetilde{NC}_c^A$ consisting of noncrossing partitions of A with no dangling annular blocks.

Figure 4: Q_1 Q_2 Q_3 Q_4 , with cutting curvesFigure 5: Joins in $\widetilde{NC}_c^A$ may need dangling annular blocks

Proposition 3.8. Suppose $\mathcal{P}, \mathcal{Q} \in \widetilde{NC}_c^{A, \circ}$. Then $\mathcal{Q}$ covers $\mathcal{P}$ in $\widetilde{NC}_c^{A, \circ}$ if and only if $\mathcal{Q}$ covers $\mathcal{P}$ in $\widetilde{NC}_c^A$.

Example 3.9. The poset $\widetilde{NC}_c^{A, \circ}$ (noncrossing partitions of the annulus with no dangling annular blocks) can fail to be a lattice. Take $n = 4$ and $c = s_4 s_3 s_1 s_2$. The left two pictures in Figure 5 show noncrossing partitions $\mathcal{P}, \mathcal{Q} \in \widetilde{NC}_c^{A, \circ}$. The next two pictures show minimal upper bounds for $\mathcal{P}$ and $\mathcal{Q}$ in $\widetilde{NC}_c^{A, \circ}$. The rightmost picture shows the join of $\mathcal{P}$ and $\mathcal{Q}$ in $\widetilde{NC}_c^A$, which is not in $\widetilde{NC}_c^{A, \circ}$ because it has a dangling annular block.

The *Kreweras complement* is an anti-automorphism $\text{Krew} : \widetilde{NC}_c^A \rightarrow \widetilde{NC}_c^A$. For each numbered point i , choose a point i' on the boundary of the annulus A between i and the next numbered point clockwise of i (for i outer) or counterclockwise of i (for i inner). Given $\mathcal{P} \in \widetilde{NC}_c^A$, first construct a noncrossing partition $\mathcal{P}'$ on the annulus with numbered points $1', \dots, n'$ whose blocks are the maximal embedded blocks that are disjoint from the nontrivial blocks of $\mathcal{P}$, together with trivial blocks $\{i'\}$ for every i' contained in a block of $\mathcal{P}$. Then $\text{Krew}(\mathcal{P})$ is obtained from $\mathcal{P}'$ by applying a homeomorphism of A that rotates each i' back to i while moving interior points of A as little as possible.

Example 3.10. Figure 6 shows the Kreweras complements of the noncrossing partitions shown in Figure 1.

We now define the map $\text{perm} : \widetilde{NC}_c^A \rightarrow S_{\mathbb{Z}(\text{mod } n)}$ that will serve as an isomorphism to $[1, c]_{\text{TUL}}$. Given $\mathcal{P} \in \widetilde{NC}_c^A$, we define $\text{perm}(\mathcal{P})$ by reading the cycle notation of a permutation from the embedding of the partition in the annulus. Each component

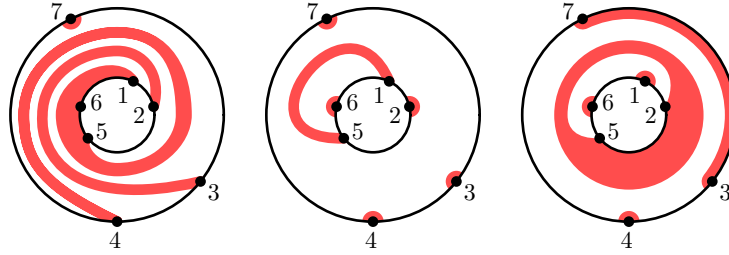


Figure 6: Kreweras complements of the noncrossing partitions from Figure 1

of the boundary of each block is read as a cycle by following the boundary, keeping the interior of the block on the right. (Each degenerate disk block is first thickened to a disk with one or two numbered points on its boundary, and is thus read as a singleton cycle or transposition.) The entries in the cycle are the numbered points encountered along the boundary, but adding a multiple of n to each, with the multiple of n increases or decreases according to how the block wraps around the annulus. Specifically, the multiple of n changes when we cross the *date line*, a radial line segment separating n and 1. When a numbered point is reached, we record its value (between 1 and n) plus wn , where w is the number of times we have crossed the date line clockwise minus the number of times we have crossed counterclockwise, since starting to read the cycle. When we return to the numbered point where we started, if we have added or subtracted a nonzero multiple of n , then we continue around the boundary again, obtaining an infinite cycle. Otherwise, if we have recorded the values $a_1, a_2, \dots, a_k$ around the boundary, we obtain the infinite product $(a_1 \ a_2 \ \dots \ a_k)_n$ of cycles. More specifically, a disk block becomes such an infinite product of finite cycles, a dangling annular block becomes a single infinite cycle (either increasing or decreasing), and a non-dangling annular block becomes a pair of infinite cycles, one increasing and one decreasing.

Example 3.11. Labeling the noncrossing partitions shown in Figure 1 (Example 3.1) as $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ from left to right, we apply perm to obtain the following permutations in $\tilde{S}_7$:

$$\begin{aligned} \text{perm}(\mathcal{P}_1) &= (1 \ -8 \ -4)_7 (2 \ -3)_7 (5)_7 (6)_7 \\ \text{perm}(\mathcal{P}_2) &= (\dots 1 \ -5 \ -6 \ \dots) (\dots 3 \ 4 \ 7 \ 10 \ \dots) (5 \ 6)_7 \\ \text{perm}(\mathcal{P}_3) &= (1 \ -1 \ -2)_7 (2)_7 (3)_7 (\dots 4 \ 7 \ 11 \ \dots) \end{aligned}$$

Theorem 3.12. The map $\text{perm} : \widetilde{NC}_c^A \rightarrow S_{\mathbb{Z}(\text{mod } n)}$ is an isomorphism from $\widetilde{NC}_c^A$ to the interval $[1, c]_{T \cup L}$ in $S_{\mathbb{Z}(\text{mod } n)}$. It restricts to an isomorphism from $\widetilde{NC}_c^{A, \circ}$ to the interval $[1, c]_T$ in $\tilde{S}_n$.

Theorem 3.13. If $\mathcal{P} \in \widetilde{NC}_c^A$, then $\text{perm}(\text{Krew}(\mathcal{P})) = \text{perm}(\mathcal{P})^{-1}c$.

4 Noncrossing partitions of a two-orbifold disk

Using a standard technique of “folding” a Coxeter group of type $\tilde{A}$, we realize the Coxeter group of type $\tilde{C}_{n-1}$ as the group $\tilde{S}_{2n}^s$ of *affine signed permutations* (permutations $\pi : \mathbb{Z} \rightarrow \mathbb{Z}$ with $\pi(i + 2n) = \pi(i) + 2n$ and $\pi(-i) = -\pi(i)$ for all $i \in \mathbb{Z}$). A Coxeter element c in $\tilde{S}_{2n}^s$ is encoded by a decomposition of $\{\pm 1, \dots, \pm(n-1)\}$ into outer and inner points, with i outer if and only if $-i$ is inner.

A corresponding “folding” of planar diagrams realizes the interval $[1, c]_T$ as the set of noncrossing partitions that are symmetric with respect to a certain symmetry ϕ . Viewing pictures of the annulus as perspective drawings of a cylinder, ϕ rotates this cylinder a half-turn about a vertical axis. In the top-row pictures in Figure 2, the two intersections of the axis with the cylinder are marked with small dotted circles. The dashed circle cuts the annulus into two pieces, with the action of ϕ mapping each piece to the other. Thus the quotient of the annulus modulo ϕ is obtained from the annulus by deleting everything inside the dashed circle and identifying each point on the dashed circle with the other point at the same height in the picture. The result is a disk C with two orbifold points, called the *two-orbifold disk*, which appears in the bottom-row pictures in Figure 2.

An *arc* in C is a curve in C that does not intersect itself except possibly at its endpoints, does not intersect the boundary or orbifold points of C , except possibly at its endpoints, and is of one of the following two types: An *ordinary arc* has both endpoints on the boundary of C , while an *orbifold arc* has one endpoint on the boundary and the other at an orbifold point. An arc may not bound a monogon in C unless that monogon contains one or both orbifold points, and it may not combine with a boundary segment to bound a digon unless that digon contains one or both orbifold points. An *embedded block* in C is one of the following:

- a *trivial block*, meaning a singleton consisting of a numbered point in C ;
- an arc (ordinary or orbifold) or boundary segment in C ; or
- a *disk block*, meaning a closed disk in C whose boundary is a union of ordinary arcs and/or boundary segments of C .

The first two types of blocks are *degenerate disk blocks*. An embedded block can contain one or both orbifold points.

A *noncrossing partition* of C is a collection of pairwise disjoint embedded blocks such that every numbered point is contained in one of the embedded blocks. Noncrossing partitions are considered up to ambient isotopy. Write $\overline{NC}_c^C$ for the set of noncrossing partitions of the two-orbifold disk C associated to the Coxeter element c , partially ordered with $\mathcal{P} \leq \mathcal{Q}$ if and only if there exist embeddings of $\mathcal{P}$ and $\mathcal{Q}$ such that each block in $\mathcal{P}$ is a subset of a block in $\mathcal{Q}$.

Example 4.1. The noncrossing partitions of the two-orbifold disk on bottom row of [Figure 2](#) exemplify the case where W is of type $\tilde{C}_6$ and $c = s_6 s_4 s_3 s_0 s_1 s_2 s_5$. The top row shows the corresponding symmetric noncrossing partitions on an annulus.

Theorem 4.2. $\widetilde{NC}_c^C$ is a graded lattice, with rank function given by

$$(n - 1) - (\# \text{ blocks of } \mathcal{P}) + (\# \text{ orbifold points enclosed by blocks of } \mathcal{P}).$$

Folding allows us to define a map $\text{perm}^C : \widetilde{NC}_c^C \rightarrow \tilde{S}_{2n}^s$ such that the following theorem holds.

Theorem 4.3. The map perm^C is an isomorphism from $\widetilde{NC}_c^C$ to $[1, c]_T$.

The map perm^C reads clockwise around blocks of a noncrossing partition to define cycles, using the *toggle line* (shown dotted in the bottom-row pictures of [Figure 2](#)) and the *date line* (a segment down from the bottom orbifold point). We omit the details.

Example 4.4. Labeling the noncrossing partitions shown in [Figure 2](#) as $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ from left to right, we apply perm^C to obtain the following permutations in $\tilde{C}_6$:

$$\begin{aligned} \text{perm}^C(\mathcal{P}_1) &= ((\cdots 1 \ 5 \ 11 \ 15 \ \cdots)) ((2))_{14} ((4 \ 6))_{14} \\ \text{perm}^C(\mathcal{P}_2) &= (1 \ -1)_{14} (2 \ 8 \ 12 \ 6)_{14} ((3 \ 4))_{14} ((5))_{14} \\ \text{perm}^C(\mathcal{P}_3) &= ((1 \ -2))_{14} ((5 \ 8 \ 4 \ 3))_{14} \end{aligned}$$

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Triangular-Grid Billiards and Plabic Graphs

Colin Defant^{*1} and Pakawut Jiradilok⁺¹

¹*Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA*

Abstract. Given a polygon P in the triangular grid, we obtain a permutation π_P via a natural billiards system in which beams of light bounce around inside of P . The different cycles in π_P correspond to the different trajectories of light beams. We prove that

$$\text{area } P \geq 6 \text{ cyc } P - 6 \quad \text{and} \quad \text{perim } P \geq \frac{7}{2} \text{ cyc } P - \frac{3}{2},$$

where $\text{area } P$ and $\text{perim } P$ are the (appropriately normalized) area and perimeter of P , respectively, and $\text{cyc } P$ is the number of cycles in π_P . The inequality concerning $\text{area } P$ is tight, and we characterize the polygons P satisfying $\text{area } P = 6 \text{ cyc } P - 6$. These results can be reformulated in the language of Postnikov's plabic graphs as follows. Let G be a connected reduced plabic graph with essential dimension 2. Suppose G has n marked boundary points and v (internal) vertices, and let c be the number of cycles in the trip permutation of G . Then we have

$$v \geq 6c - 6 \quad \text{and} \quad n \geq \frac{7}{2}c - \frac{3}{2}.$$

We end with a discussion of numerous ideas for future work.

Keywords: triangular grid, billiards, plabic graph, essential dimension, trip permutation, cycle

1 Introduction

In this extended abstract of the article [2], we introduce a natural class of billiards systems that surprisingly appear to be new. While traditional billiards systems exhibit continuous behavior that is best analyzed from a geometric or analytic point of view, our systems are constrained so that they are essentially combinatorial. Indeed, we will reformulate these systems in terms of trip permutations of Postnikov's plabic graphs. Nevertheless, our proof methods require a heavy dose of Euclidean geometry. These new billiards systems lead to several promising open problems and ideas for future work, some of which we will discuss.

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2 Triangular-Grid Billiards

Consider the infinite triangular grid in the plane, scaled so that each equilateral triangular grid cell has side length 1 and oriented so that some of the grid lines are horizontal. We refer to the sides of these grid cells as *panes* because we will imagine that each pane either allows light to pass through it (like a window pane) or reflect off of it (like a mirror pane). Define a *grid polygon* to be a (not necessarily convex) polygon whose boundary is a union of panes. We assume that the boundary of a grid polygon (viewed as a closed curve) does not intersect itself. Suppose P is a grid polygon whose boundary panes are $b_1, \dots, b_n$, listed clockwise. Pick some boundary pane b_i , and emit a colored beam of light from the midpoint of b_i into the interior of P so that the light beam forms a 60° angle with b_i and travels either northeast, southeast, or west (depending on the orientation of b_i). The light beam will travel through the interior of P until reaching the midpoint of a different boundary pane $b_{\pi(i)}$, which it will meet at a 60° angle. This defines a permutation $\pi_P: n \rightarrow n$ (where $n = \{1, \dots, n\}$) called the *billiards permutation* of P . For example, if P is the grid polygon in Figure 1, then the cycle decomposition of π_P is

1 3 32 26 6 30 2 33 25 12 14 9 21 19 29 28 4 31 5 24 13 10 20 27 7 22 23 15 17 8 11 18 16 .

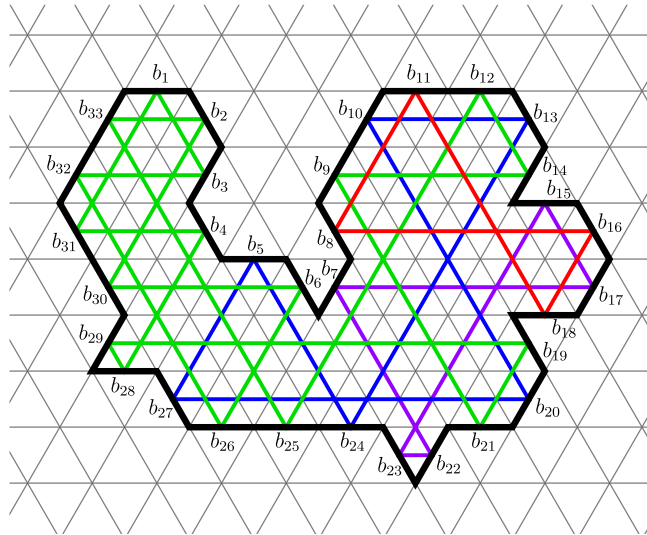


Figure 1: A grid polygon P with 33 boundary panes. The billiards permutation π_P has 4 cycles. We have colored the 4 different trajectories with different colors.

Let us imagine that the boundary panes of P are mirrors (and all other panes are transparent windows). When the light beam emitted from b_i reaches $b_{\pi(i)}$, it will bounce

off in such a way that the reflected beam forms a 60° angle with $b_{\pi^{-1}i}$. This reflected beam will then travel to $b_{\pi^2 i}$, where it will bounce off at a 60° angle and continue on to $b_{\pi^3 i}$, and so on. We will be interested in the cycles of π . Given points p and p' in the plane, let us write $[p, p']$ for the line segment whose endpoints are p and p' . Let $\text{mid } s$ denote the midpoint of a line segment s . If $c = i_1 i_2 \cdots i_r$ is a cycle of π , then we define the *trajectory* of c to be

$$\text{traj } c = \bigcup_{j=1}^r [\text{mid } b_{i_j}, \text{mid } b_{\pi i_j}].$$

The description of π in terms of light beam billiards is convenient because we can imagine that the beams of light corresponding to different cycles have different colors; thus, we will use different colors to draw different trajectories (see Figure 1).

The investigation of billiards in planar regions is a classical and much-beloved topic in both dynamical systems and recreational mathematics. However, the typical questions considered in previous works concern systems where the beams of light can have arbitrary initial positions and arbitrary initial directions. In contrast, our setup imposes a great deal of rigidity by requiring each beam of light to start at the midpoint of a boundary pane and begin its journey in a direction that forms a 60° angle with that boundary pane. Although several traditional dynamically-flavored billiards problems (such as determining the existence of periodic trajectories) become trivial or meaningless under our rigid conditions, our setting affords some fascinating combinatorial/geometric questions.

The major players in our story are the following quantities associated with a grid polygon P . The *perimeter* of P , denoted $\text{perim } P$, is simply the number of boundary panes of P . We define the *area* of P , denoted $\text{area } P$, to be the number of triangular grid cells in P .¹ We write $\text{cyc } P$ for the number of cycles in the associated permutation π_P , which is the same as the number of different light beam trajectories in the associated billiards system. Our main theorems address the following extremal question concerning the possible relationships between these quantities: How big must $\text{area } P$ and $\text{perim } P$ be in comparison with $\text{cyc } P$?

Theorem 1 ([2]). *If P is a grid polygon, then*

$$\text{area } P \geq 6 \text{ cyc } P - 6.$$

Theorem 2 ([2]). *If P is a grid polygon, then*

$$\text{perim } P \geq \frac{7}{2} \text{ cyc } P - \frac{3}{2}.$$

The inequality in Theorem 1 is tight, and we can characterize the grid polygons that achieve equality. Define a *unit hexagon* to be a grid polygon that is a regular hexagon of

¹Thus, our area measure is just the Euclidean area multiplied by the normalization factor $4/\sqrt{3}$.

side length 1. Let us construct a sequence of grid polygons P_k $k \geq 1$ as follows. First, let P_1 be a unit hexagon. For $k \geq 2$, let $P_k = P_{k-1} \cup Q_k$, where Q_k is a unit hexagon such that $P_{k-1} \cap Q_k$ is a single pane. We call a grid polygon P_k obtained in this manner a *tree of unit hexagons*; see Figure 2 for an example with $k = 9$. It is not difficult to prove by induction that $\text{cyc } P_k = k + 1$ for all $k \geq 1$. Thus, $\text{area } P_k = 6k = 6 \text{cyc } P_k - 6$.

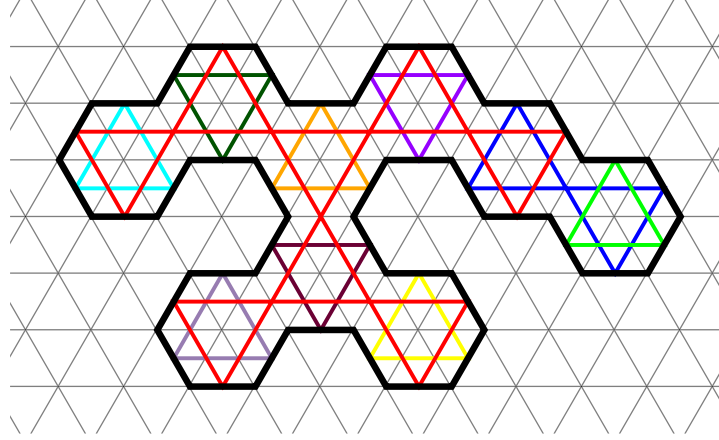


Figure 2: A tree of unit hexagons P_9 with $\text{cyc } P_9 = 10$, $\text{area } P_9 = 54$, and $\text{perim } P_9 = 38$.

Theorem 3 ([2]). *If P is a grid polygon, then $\text{area } P = 6 \text{cyc } P - 6$ if and only if P is a tree of unit hexagons.*

On the other hand, we believe that Theorem 2 is not tight. After drawing several examples of grid polygons, we have arrived at the following conjecture.

Conjecture 1 ([2]). *If P is a grid polygon, then*

$$\text{perim } P \geq 4 \text{cyc } P - 2.$$

If Conjecture 1 is true, then it is tight. Indeed, if P_k is a tree of unit hexagons as described above, then $\text{perim } P_k = 4k + 2 = 4 \text{cyc } P_k - 2$.

Of fundamental importance in our analysis of the billiards system of a grid polygon P are the *triangular trajectories* of P , which are just the trajectories of the 3-cycles in π_P . One of the crucial ingredients in the proofs of Theorems 1–3 is the following result, which we deem to be noteworthy on its own.

Theorem 4 ([2]). *Let P be a grid polygon, and let c be a cycle of size m in π_P . Then the trajectory $\text{traj } c$ intersects at most $m - 2$ triangular trajectories of P (excluding $\text{traj } c$ itself if $m = 3$).*

Suppose we take a single trajectory in the billiards system of a grid polygon P . This trajectory is a union of several line segments. If we delete the points where two of these line segments intersect, then we will cut the trajectory into several open line segments that we call the *fragments* of the trajectory. Fragments end up being crucial in our proofs of the above theorems. After writing the article [2], we proved that the number of fragments of a trajectory is always divisible by 3. The challenge to prove this divisibility result will appear soon in the *Problems and Solutions* section of the American Mathematical Monthly, so we will not spoil the solution here.

3 Plabic Graphs

A *plabic graph* is a planar graph G embedded in a disc such that each vertex is colored either black or white. We assume that the boundary of the disc has n marked points labeled clockwise as $b_1^*, \dots, b_n^*$ so that each b_i^* is connected via an edge to exactly one vertex of G . Following [6], we will also assume that every vertex of G is incident to exactly 3 edges, including edges connected to the boundary of the disc (the study of general plabic graphs can be reduced to this case). In his seminal article [9], Postnikov introduced plabic graphs—along with several other families of combinatorial objects—in order to parameterize cells in the totally nonnegative Grassmannian. These graphs have now found remarkable applications in a variety of fields such as cluster algebras, knot theory, polyhedral geometry, scattering amplitudes, and shallow water waves [1, 3, 4, 5, 6, 7, 8, 10].

Imagine starting at a marked boundary point b_i^* and traveling along the unique edge connected to b_i^* . Each time we reach a vertex, we follow the *rules of the road* by turning right if the vertex is black and turning left if the vertex is white. Eventually, we will reach a marked boundary point $b_{\pi(i)}^*$. The path traveled is called the *trip* starting at b_i^* . Considering the trips starting at all of the different marked boundary points yields a permutation $\pi = \pi_G : n \rightarrow n$ called the *trip permutation* of G . We say G is *reduced* if it has the minimum number of faces among all plabic graphs with the same trip permutation. Figure 3 shows a reduced plabic graph G whose trip permutation is the cycle $\pi_G = 13524$.

Given a grid polygon P , one can obtain a reduced plabic graph $G \in P$ via a planar dual construction. Let us say an equilateral triangle with a horizontal side is *right-side up* (respectively, *upside down*) if its horizontal side is on its bottom (respectively, top). We refer to this property of a triangle (right-side up or upside down) as its *orientation*. Place a black (respectively, white) vertex at the center of each right-side up (respectively, upside down) equilateral triangular grid cell inside of P . Whenever two such grid cells share a side, draw an edge between the corresponding vertices. Finally, encompass P in a disc, draw a marked point b_i^* on the boundary of the disc corresponding to each

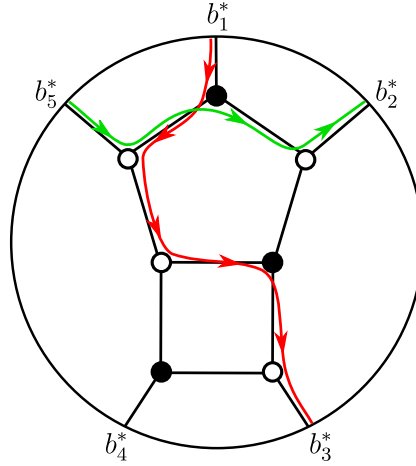


Figure 3: A reduced plabic graph whose trip permutation is the cycle 13524 . The fact that $\pi_G 1 = 3$ is illustrated by the red trip starting at b_1^* and ending at b_3^* . Similarly, $\pi_G 5 = 2$ because the green trip starting at b_5^* ends at b_2^* .

boundary pane b_i of P , and draw an edge from b_i^* to the vertex drawn inside the unique grid cell that has b_i as a side. (See Figure 4.)

It is immediate from the relevant definitions that the trip permutation $\pi_{G \setminus P}$ is equal to the billiards permutation π_P . For example, if P and $G \setminus P$ are as in Figure 4, then $\pi_P = \pi_{G \setminus P}$ is the permutation with cycle decomposition $174359 \ 268$.

In the recent paper [6], Lam and Postnikov introduced *membranes*, which are certain triangulated 2-dimensional surfaces embedded in a Euclidean space. The definition of a membrane relies on a choice of an irreducible root system, and most of the discussion in [6] centers around membranes of type A . They discussed how type A membranes are in a sense dual to plabic graphs, and they further related type A membranes to the theory of cluster algebras. A membrane is *minimal* if it has the minimum possible surface area among all membranes with the same boundary; Lam and Postnikov showed how to associate a reduced plabic graph $G \setminus M$ to each minimal type A membrane M . They then defined the *essential dimension* of a reduced plabic graph G_0 to be the smallest positive integer d such that there exists a minimal membrane M of type A_d with $G \setminus M = G_0$. They proved that if G_0 has n marked boundary points, then the essential dimension of G_0 is at most $n - 1$, with equality holding if and only if there exists $k \in \{1, \dots, n - 1\}$ such that $\pi_{G_0} i = i + k \pmod n$ for all $i \in \{1, \dots, n\}$ (this is equivalent to saying that G_0 corresponds to the top cell in the totally nonnegative Grassmannian $\text{Gr}_{k,n}^{\geq 0}$). Other than this result, there is essentially nothing known about essential dimensions of plabic graphs. Our original motivation for this project was to initiate the investigation of essential dimensions by studying plabic graphs of essential dimension 2 in detail.

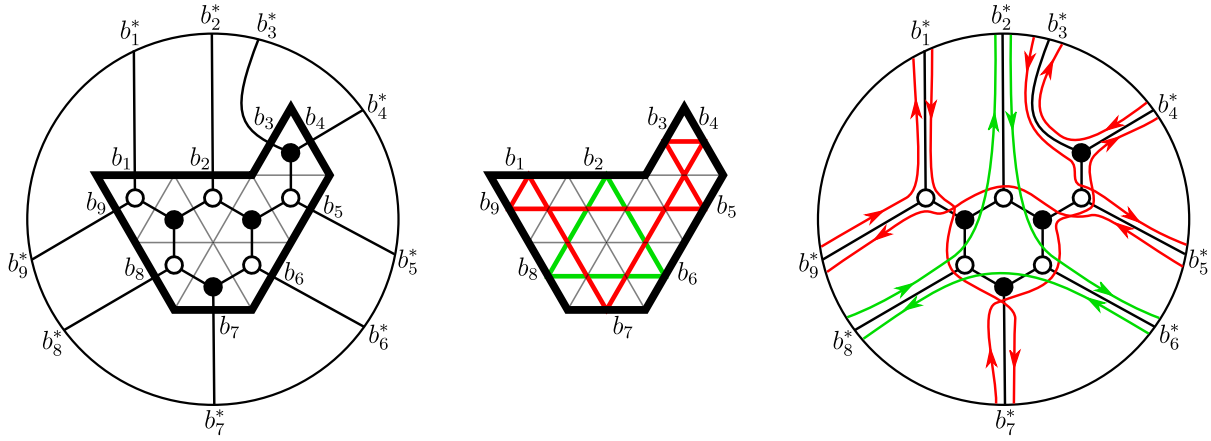


Figure 4: On the left is a grid polygon P overlaid with the corresponding plabic graph G_P . The middle image shows the trajectories in the billiards system of P to illustrate that its billiards permutation is $\pi_P = 174359 \ 268$. The right image shows the trips of G_P to illustrate that its trip permutation is $\pi_{G_P} = 174359 \ 268$.

Consider the class of triangulated surfaces in the triangular grid that can be obtained by iteratively wedging grid polygons. In other words, Q is in this class if there are grid polygons $P_1, \dots, P_k$ such that $P_{i-1} \cap P_1 \cup \dots \cup P_i$ is a single point for all $i \in k-1$ and such that $Q = P_1 \cup \dots \cup P_k$. In this case, we call the grid polygons $P_1, \dots, P_k$ the *components* of Q . See Figure 5. As mentioned in [6], the class we have just described is the same as the class of membranes of type A_2 . Such a membrane M is automatically minimal (since it is determined by its boundary). In order to understand these membranes and their associated reduced plabic graphs, it suffices to understand grid polygons and their associated reduced plabic graphs. Indeed, the reduced plabic graphs associated to the components of M are basically the same as the connected components of the reduced plabic graph associated to M ; thus, restricting our focus to grid polygons is the same as restricting our focus to connected plabic graphs. Furthermore, if $M = P$ is a grid polygon, then the definition that Lam and Postnikov gave for the reduced plabic graph G_M associated to M (viewed as a membrane) is exactly the same as the definition that we gave in Section 3 for the reduced plabic graph G_P associated to P (viewed as a grid polygon). In other words, understanding plabic graphs of essential dimension 2 and their trip permutations is equivalent to understanding grid polygons and their billiards permutations.

As a consequence of the preceding discussion, we can reformulate Theorems 1 and 2 in the language of plabic graphs.

Corollary 1 ([2]). *Let G be a connected reduced plabic graph with essential dimension 2. Suppose G has n marked boundary points and v vertices, and let c be the number of cycles in the trip*

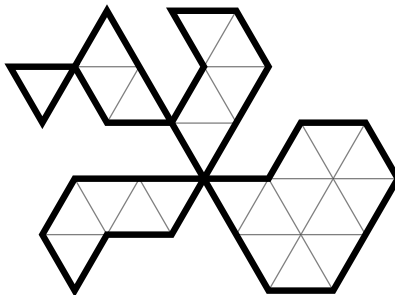


Figure 5: A membrane of type A_2 with 5 components.

permutation π_G . Then

$$v \geq 6c - 6 \quad \text{and} \quad n \geq \frac{7}{2}c - \frac{3}{2}.$$

4 Reflections and Next Directions

We believe our work just scratches the surface of rigid combinatorial billiards systems and their connections with plabic graphs and membranes. In this section, we discuss several variations and potential avenues for future research.

4.1 Perimeter vs. Cycles

Recall [Conjecture 1](#), which says that $\text{perim } P \geq 4 \text{cyc } P - 2$ for every grid polygon P . The grid polygons P satisfying $\text{perim } P = 4 \text{cyc } P - 2$ seem more sporadic and unpredictable than the equality cases of [Theorem 1](#) (see [Figure 6](#)), which are just the trees of unit hexagons by [Theorem 3](#). This gives a heuristic hint as to why [Conjecture 1](#) is more difficult to prove than [Theorem 1](#).

4.2 Other Families of Plabic Graphs

Let G be a connected reduced plabic graph with n marked boundary points and v vertices, and let c be the number of cycles in the trip permutation π_G . [Corollary 1](#) provides inequalities that say how large n and v must be relative to c in the case when G has essential dimension 2. One can ask for similar inequalities when G is taken from some other interesting family of plabic graphs. One natural candidate for such a family is the collection of plabic graphs of essential dimension 3; we refer to [\[6\]](#) for further details concerning the definition. It is also natural to consider plabic graphs that can be obtained from polygons in other planar grids besides the triangular grid; [Figure 7](#) shows

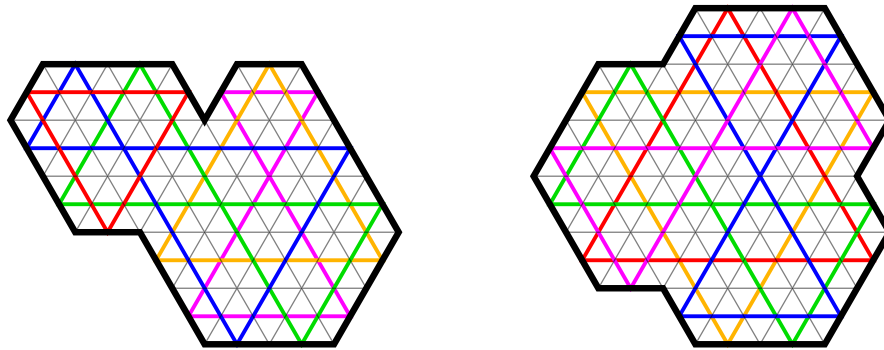


Figure 6: Two grid polygons with perimeter 18, each of which has 5 cycles in its billiards system.

some examples (in these examples, we dismiss our earlier assumption that all vertices in a plabic graph are trivalent).

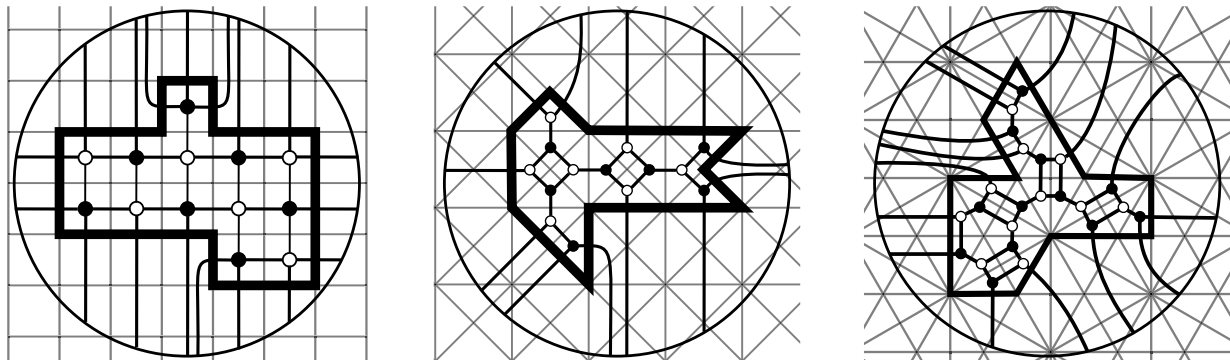


Figure 7: Plabic graphs obtained from polygons in different planar grids.

If G is a plabic graph obtained from the square grid (as on the left of [Figure 7](#)), then it is not too difficult to prove that $v \geq 3c - 2$ and that $n \geq 4c$; moreover, these bounds are tight.

4.3 Regions with Holes

Suppose Q is a region in the triangular grid obtained from a grid polygon by cutting out some number of polygonal holes. We can define the billiards system for Q in the same way that we defined it for a grid polygon. It would be interesting to obtain analogues of [Theorems 1, 2, 3, and 4](#) in this more general setting. The resulting analogues of [Theorems 1 and 2](#) might need to incorporate the genus of Q . Indeed, [Figure 8](#) shows

a region Q with genus 1 for which the inequalities in [Theorems 1 and 2](#) are false as written.

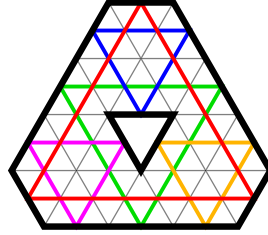


Figure 8: Trajectories in a triangular grid region of genus 1.

4.4 Higher Dimensions

Nathan Williams has suggested the following natural way to use algebra to extend our billiards systems into higher dimensions.

Let $\mathcal{H}'_d$ be the Coxeter arrangement of type $\tilde{A}_d$. This is the collection of hyperplanes $H_{i,j}^k: \{x_1, \dots, x_{d-1} \in \mathbb{R}^{d-1} : x_i - x_j = k\}$ such that $i, j \in d-1$ are distinct and $k \in \mathbb{Z}$. We identify $\mathbb{R}^{d-1}/\text{span}\{1, \dots, 1\}$ with $\mathbb{R}^d$ and let $\mathcal{H}_d$ be the hyperplane arrangement in $\mathbb{R}^d$ obtained by taking the images of the hyperplanes in $\mathcal{H}'_d$ under the natural quotient map $\mathbb{R}^{d-1} \rightarrow \mathbb{R}^{d-1}/\text{span}\{1, \dots, 1\}$. The closures of the connected components of $\mathbb{R}^d \setminus \bigcup_{H \in \mathcal{H}_d} H$ are called *alcoves*. When $d = 2$, alcoves are the same as the triangular grid cells.

For $1 \leq i \leq d$, let $s_i: \mathbb{R}^d \rightarrow \mathbb{R}^d$ be the reflection through (the image in the quotient of) the hyperplane $H_{i,i}^0$. Let s_0 be the reflection through (the image in the quotient of) $H_{1,d}^1$. Then $s_0, s_1, \dots, s_d$ are the simple reflections generating the affine symmetric group $\tilde{S}_{d-1}$. Note that if X is an alcove and $0 \leq i \leq d$, then the region $s_i \cdot X$ obtained by applying the reflection s_i to X is also an alcove. This defines a free transitive action of $\tilde{S}_{d-1}$ on the set of alcoves, so we can identify $\tilde{S}_{d-1}$ with the set of alcoves.

Let us fix a finite set $P \subseteq \tilde{S}_{d-1}$; we can think of P as a finite set of alcoves. For $w \in P$ and $j \in \mathbb{Z}/(d-1)\mathbb{Z}$, let

$$\tau_j w = \begin{cases} ws_j & \text{if } ws_j \in P \\ w & \text{if } ws_j \notin P. \end{cases}$$

Define $\varphi: P \times \mathbb{Z}/(d-1)\mathbb{Z} \rightarrow P \times \mathbb{Z}/(d-1)\mathbb{Z}$ by $\varphi(w, j) = (\tau_j w, j+1)$. We call an orbit of the map φ a *trajectory* of P .

Each pair $(w, j) \in P \times \mathbb{Z}/(d-1)\mathbb{Z}$ encodes the alcove w together with a “direction” given by j . Applying φ moves us one step in this direction unless doing so would take us out of P ; in the latter case, we stay at w and change direction. If $d = 2$ and P is a

region in the triangular grid, then the trajectories defined using φ correspond precisely to the trajectories in the billiards system of P .

It would be interesting to understand the relationship between the number of alcoves in P and the number of trajectories of P when $d > 2$. One might hope to obtain an inequality relating these quantities for arbitrary d . [Theorem 1](#) relates these quantities when $d = 2$ under the assumption that P is simply connected (this assumption makes the problem different but not necessarily easier). In this more general setting, it could be simpler to remove the hypothesis that P is simply connected (and thus consider regions with holes, as in [Section 4.3](#)).

4.5 Random and Infinite Billiards Systems

We would like to formulate a notion of a *random billiards system* so that one can ask about the expected number of trajectories. However, there are several possible random models one could consider, and it is not clear which is the best.

One potential random model of an *infinite* billiards system is as follows. Fix a probability $p \in [0, 1]$. Choose a random set P of grid cells in the triangular grid by adding each grid cell into P with probability p so that all choices are independent. The set P is a region that is almost surely disconnected and of infinite area. We can define trajectories just as we did for polygons, except these trajectories might be infinite. It would be very interesting to understand the distribution of trajectory lengths in this random billiards system. Suppose we start with a specific fixed grid cell Δ . Given that $\Delta \in P$, what is the probability that the trajectory that travels west through Δ is infinite?

4.6 Cyclic Billiards Systems

It would be interesting to obtain estimates for the number of grid polygons P such that $\text{cyc } P = 1$.

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Promotion for fans of Dyck paths

Joseph Pappe^{*1}, Stephan Pfannerer^{†2},
Anne Schilling^{‡1}, and Mary Claire Simone^{§1}

¹*Department of Mathematics, University of California, Davis, CA, USA*

²*Institut für Diskrete Mathematik und Geometrie, TU Wien, Austria*

Abstract. We construct an injection from the set of r -fans of Dyck paths of length n into the set of chord diagrams on n that intertwines promotion and rotation. This is done in two different ways, namely as fillings of promotion matrices and in terms of Fomin growth diagrams. Our analysis uses the fact that r -fans of Dyck paths can be viewed as highest weight elements of weight zero in crystals of type B_r , which in turn can be analyzed using virtual crystals. On the level of Fomin growth diagrams, the virtualization process corresponds to the Roby–Krattenthaler blow up construction. Our construction generalizes to vacillating tableaux as well. We give a cyclic sieving phenomenon on r -fans of Dyck paths using the promotion action.

Keywords: Crystal bases, promotion, Fomin growth and chord diagrams, Dyck paths

1 Introduction

Interest in invariant subspaces goes back to Rumer, Teller and Weyl [22], who studied the quantum mechanical description of molecules. In particular, they devised diagrammatic bases for the invariant spaces. For $SL\ n$, a diagrammatic basis for the invariant space was constructed by Cautis, Kamnitzer and Morrison [2], generalizing Kuperberg’s webs [11] for $SL\ 2$ and $SL\ 3$.

The dimension of the invariant subspace of a tensor product $V^{\otimes N}$ of an irreducible representation V of a Lie algebra $\mathfrak{g}$ is equal to the number of highest weight elements of weight zero in $\mathcal{B}^{\otimes N}$, where $\mathcal{B}$ is the crystal basis associated to V [25, 19]. The symmetric group acts on $V^{\otimes N}$ by permuting tensor positions. By Schur–Weyl duality, this action commutes with the action of the Lie group. In particular, the symmetric group acts on the invariant space of $V^{\otimes N}$. It was shown by Westbury [25] that the action of the long cycle corresponds to the action of promotion on highest weight elements of weight zero in $\mathcal{B}^{\otimes N}$. In this setting promotion is defined using Henriques’ and Kamnitzer’s commutor [5], see [4, 25, 26].

^{*}jhpappe@ucdavis.edu.

[†]stephan.pfannerer@tuwien.ac.at. Recipient of a DOC Fellowship of the Austrian Academy of Sciences.

[‡]anne@math.ucdavis.edu. Partially supported by NSF grants DMS–1760329 and DMS–2053350.

[§]mcsimone@ucdavis.edu.

In general, it is desirable to have a correspondence between highest weight elements of weight zero in $\mathcal{B}^{\otimes N}$ and diagram bases, such as chord diagrams, which intertwine promotion and rotation. For Kuperberg's webs [11], this was achieved by Petersen, Pylyavskyy, Rhoades [18] and Patrias [17] by showing that the growth algorithm of Khovanov and Kuperberg [8] intertwines promotion with rotation. For the vector representation of the symplectic group and the adjoint representation of the general linear group, such a correspondence between highest weight elements of weight zero and chord diagrams which intertwines promotion and rotation was given in [19].

In this paper, we construct an injection from the set of r -fans of Dyck paths of length n into the set of chord diagrams on n that intertwines promotion and rotation. There is a natural correspondence between r -fans of Dyck paths and highest weight elements in the tensor product of the spin crystal of type B_r . We present this injection in two different ways: (1) as fillings of promotion matrices [13] (see Section 3.1); (2) in terms of Fomin growth diagrams [3, 10] (see Section 3.2).

While the first description shows that the map intertwines promotion and rotation, the second description shows injectivity. Our proof strategy uses virtualization of crystals (see for example [1]) and results of [19] for oscillating tableaux of weight zero (or equivalently highest weight words of weight zero for the vector representation type C_r):

1. Find a virtual crystal morphism for the spin crystals of type B_r into the r -th tensor power of the crystal of the vector representation of type C_r (Section 2.1).
2. Use virtualization to map a fan of Dyck paths to an oscillating tableau (Section 2.2).
3. Show that this virtualization commutes with promotion and the filling rules.
4. Show that blowing up the filling of the growth diagram corresponds to the filling of the oscillating tableau. In this sense, the blow up on growth diagrams is the analogue of the virtualization on crystals.

An overview of our strategy is shown in Figure 1. Our analysis also generalizes to vacillating tableaux, details of which appear in the long version of this paper [16].

Having the injective map to chord diagrams gives a first step towards a diagrammatic basis for the invariant subspaces. In addition, Fontaine and Kamnitzer [4] as well as Westbury [25] tied the promotion action on highest weight elements of weight zero to the cyclic sieving phenomenon introduced by Reiner, Stanton and White [20]. In Section 4.2, we make this cyclic sieving phenomenon more concrete by providing the polynomial in terms of the energy function. We also conjecture another polynomial, which is the q -deformation of the number of r -fans of Dyck paths, to give a cyclic sieving phenomenon.

The paper is organized as follows. In Section 2, we give a brief review of crystal bases and virtual crystals and provide the virtual crystals for the spin representation of type B_r into type C_r . We also define promotion on crystals via the crystal commutor.

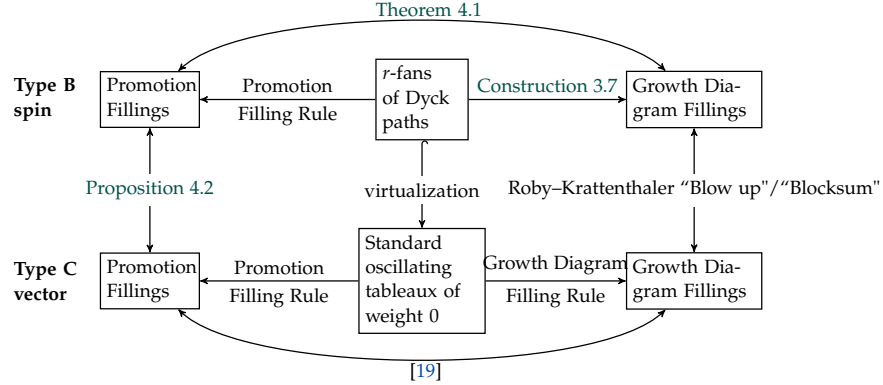


Figure 1: Overview of strategy and results for r -fans of Dyck paths

In Section 3, we give the various filling rules to construct the map to chord diagrams. Section 4 is reserved for the statements of our main results. The long version [16] of this extended abstract contains all proofs and more details on vacillating tableaux.

We thank S. Hopkins, V. Reiner, M. Rubey and B. Westbury for discussions.

2 Crystal bases

Crystal bases form a combinatorial skeleton of representations of quantum groups associated to Lie algebras. They were first introduced by Kashiwara [7] and Lusztig [14]. Axiomatically, for a given root system Φ with index set I and weight lattice Λ , a *crystal* is a nonempty set $\mathcal{B}$ together with maps $e_i, f_i: \mathcal{B} \rightarrow \mathcal{B} \sqcup \{\emptyset\}$, $\varepsilon_i, \varphi_i: \mathcal{B} \rightarrow \mathbb{Z}$, $\text{wt}: \mathcal{B} \rightarrow \Lambda$ for $i \in I$, satisfying certain conditions (see for example [1, Definition 2.13]). The operators e_i and f_i are called *raising* and *lowering operators*. The map wt is the *weight map*. The map ε_i (resp. φ_i) measures how often e_i (resp. f_i) can be applied to the given crystal element. For all crystals considered in this paper, $\varepsilon_i b = \max\{k \geq 0 \mid e_i^k b \neq \emptyset\}$ for $b \in \mathcal{B}$ and similarly for $\varphi_i b$. An element $b \in \mathcal{B}$ is called *highest weight* if $e_i b = \emptyset$ for all $i \in I$.

Here we define certain crystals for the root systems B_r and C_r explicitly. Let $\mathbf{e}_i \in \mathbb{Z}^r$ be the i -th unit vector with 1 in position i and 0 everywhere else.

Definition 2.1. The *spin crystal* of type B_r , denoted by $\mathcal{B}_{\text{spin}}$, consists of all r -tuples $\epsilon = \epsilon_1, \epsilon_2, \dots, \epsilon_r$, where $\epsilon_i \in \{\pm 1\}$. The weight of ϵ is $\text{wt } \epsilon = \frac{1}{2} \sum_{i=1}^r \epsilon_i \mathbf{e}_i$. The crystal operator f_r annihilates ϵ unless $\epsilon_r = 1$. If $\epsilon_r = 1$, f_r acts on ϵ by changing ϵ_r from 1 to -1 and leaving all other entries unchanged. The crystal operator f_i for $1 \leq i < r$ annihilates ϵ unless $\epsilon_i = 1$ and $\epsilon_{i+1} = -1$. In the latter case, f_i acts on ϵ by changing ϵ_i to -1 and ϵ_{i+1} to 1. The crystal operator e_i is defined similarly.

The crystal $\mathcal{B}_{\text{spin}}$ of type B_3 is depicted in Figure 2.

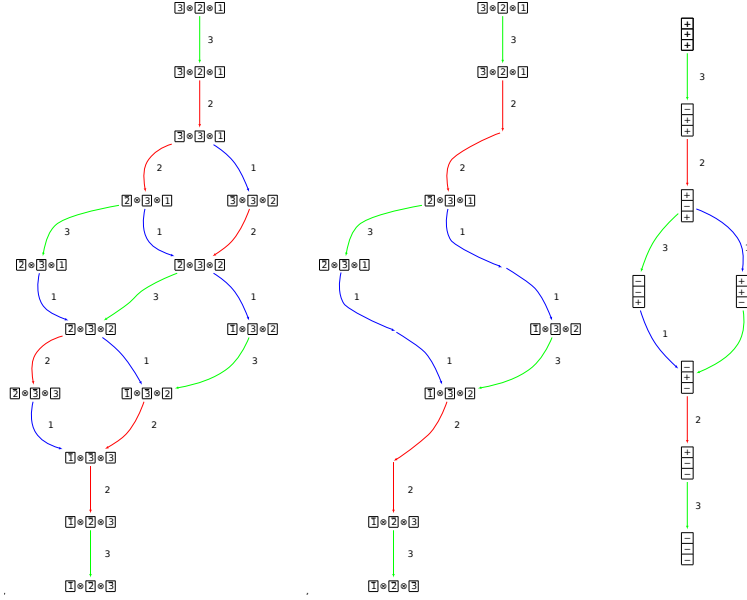


Figure 2: Left: One component of the crystal $\widehat{\mathcal{V}} = \mathcal{C}_{\square}^{\otimes 3}$ of type C_3 . Middle: The virtual crystal $\mathcal{V}$ inside $\widehat{\mathcal{V}}$ of type B_3 . Right: The spin crystal $\mathcal{B}_{\text{spin}}$ of type B_3 .

Definition 2.2. The *crystal for the vector representation* $\mathcal{C}_{\square}$ of type C_r consists of the elements $\{1, 2, \dots, r, \bar{r}, \dots, \bar{2}, \bar{1}\}$. The crystal operator f_i for $1 \leq i < r$ maps i to $i-1$, maps $\overline{i-1}$ to $\bar{i}$ and annihilates all other elements. The crystal operator f_r maps r to $\bar{r}$ and annihilates all other elements. The crystal operators e_i are defined similarly. Furthermore, $\text{wt } i = \mathbf{e}_i$ and $\text{wt } \bar{i} = -\mathbf{e}_i$.

A remarkable property of crystals is that they respect *tensor products*. Given two crystals $\mathcal{B}$ and $\mathcal{C}$ associated to the same root system Φ , the tensor product $\mathcal{B} \otimes \mathcal{C}$ as a set is the Cartesian product $\mathcal{B} \times \mathcal{C}$. The weight of $b \otimes c \in \mathcal{B} \otimes \mathcal{C}$ is the sum of the weights $\text{wt } b \otimes c = \text{wt } b + \text{wt } c$. For more information, see [1, Section 2.3].

2.1 Virtual crystals

Stembridge [23] characterized crystals associated to simply-laced root systems in terms of local rules on the crystal graph. Crystals for non-simply-laced root systems can be constructed using virtual crystals [1, Chapter 5].

In this paper, we utilize virtual crystals to construct Fomin growth diagrams and the promotion operators for type B_r using results for type C_r . Hence let us briefly review the set-up for virtual crystals. Let $X \hookrightarrow Y$ be an embedding of Lie algebras such that the fundamental weights ω_i and simple roots α_i map as $\omega_i^X \mapsto \gamma_i \sum_{j \in \sigma} \omega_j^Y$ and $\alpha_i^X \mapsto \gamma_i \sum_{j \in \sigma} \alpha_j^Y$. Here γ_i is a multiplication factor, $\sigma: I^X \rightarrow I^Y / \text{aut}$ is a bijection and

aut is an automorphism on the Dynkin diagram for Y .

Definition 2.3. If there is an embedding of Lie algebras $X \hookrightarrow Y$, then $\mathcal{V} \subseteq \widehat{\mathcal{V}}$ is a *virtual crystal* for the root system Φ^X if

- V1.** The ambient crystal $\widehat{\mathcal{V}}$ is a crystal associated to a representation for the root system Φ^Y with crystal operators $\widehat{e}_i, \widehat{f}_i, \widehat{\varepsilon}_i, \widehat{\varphi}_i$ for $i \in I^Y$ and weight function wt .
- V2.** If $b \in \mathcal{V}$ and $i \in I^X$, then $\widehat{e}_j b$ has the same value for all $j \in \sigma i$ and that value is a multiple of γ_i . The same is true for $\widehat{f}_j b$.
- V3.** The subset $\mathcal{V} \sqcup \{\emptyset\} \subseteq \widehat{\mathcal{V}} \sqcup \{\emptyset\}$ is closed under the virtual crystal operators $e_i : \prod_{j \in \sigma i} \widehat{e}_j^{\gamma_i}$ and $f_i : \prod_{j \in \sigma i} \widehat{f}_j^{\gamma_i}$ and $\varepsilon_i b = \max\{k \geq 0 \mid e_i^k b \neq \emptyset\}$, $\varphi_i b = \max\{k \geq 0 \mid f_i^k b \neq \emptyset\}$ for $b \in \mathcal{V}$.

The tensor product of two virtual crystals for the same embedding $X \hookrightarrow Y$ is again a virtual crystal (see for example [1, Theorem 5.8]).

We will now apply the theory of virtual crystals to the embedding $B_r \hookrightarrow C_r$. In this setting $I^{C_r} = I^{B_r} = \{1, 2, \dots, r\}$, $\sigma i = \{i\}$, $\gamma_i = 2$ for $1 \leq i < r$ and $\gamma_r = 1$. We consider as the ambient crystal $\widehat{\mathcal{V}} = \mathcal{C}_{\square}^{\otimes r}$. Define an ordering $<$ on the set $r \cup \bar{r}$ as follows $1 < 2 < \dots < r < \bar{r} < \dots < \bar{1}$. Denote by $|\cdot|$ the map from $r \cup \bar{r}$ to r that sends letters to their corresponding unbarred values.

Definition 2.4. Let $\mathcal{V} \subseteq \widehat{\mathcal{V}}$ be given by $\mathcal{V} = \{v_r \otimes v_{r-1} \otimes \dots \otimes v_1 \in \widehat{\mathcal{V}} \mid v_i > v_j \text{ and } |v_i| \neq |v_j| \text{ for all } i > j\}$. Let $f_i = \widehat{f}_i^2, e_i = \widehat{e}_i^2$ for $1 \leq i < r$ and $f_r = \widehat{f}_r, e_r = \widehat{e}_r$.

Definition 2.5. Let $\Psi: \mathcal{B}_{\text{spin}} \rightarrow \mathcal{V}$ be the map $\Psi = \epsilon_1 \epsilon_2 \dots \epsilon_r = v_r \otimes v_{r-1} \otimes \dots \otimes v_1$, where $v_r > v_{r-1} > \dots > v_1$ such that if $\epsilon_i = -$ then v contains an i and if $\epsilon_i = +$ then v contains an $\bar{i}$ for all $1 \leq i \leq r$.

Proposition 2.6. The map Ψ is a bijective map that intertwines the crystal operators on $\mathcal{B}_{\text{spin}}$ and $\mathcal{V}$. Furthermore, $\mathcal{V}$ is a virtual crystal for the embedding of Lie algebras $B_r \hookrightarrow C_r$.

An example of the virtual crystal construction for $\mathcal{B}_{\text{spin}}$ is given in Figure 2.

2.2 Highest weights of weight zero

A weight $\lambda \in \Lambda$ is called *minuscule* if $\langle \lambda, \alpha^\vee \rangle \in \{0, \pm 1\}$ for all coroots $\alpha^\vee$. A crystal $\mathcal{B}$ is called *minuscule* if $\text{wt } b$ is minuscule for all $b \in \mathcal{B}$. Note that $\mathcal{B}_{\text{spin}}$ is a minuscule crystal (see for example [1, Chapter 5.4]). A weight λ is called *dominant* if $\langle \lambda, \alpha_i^\vee \rangle \geq 0$ for all $i \in I$. Except for spin weights, dominant weights can be identified with partitions. A *partition* λ is a sequence $\lambda = \lambda_1, \lambda_2, \dots, \lambda_\ell$ such that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell \geq 0$. We identify partitions that differ by trailing zeroes.

Let $\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_n$ be minuscule crystals. For any highest weight element $u = u_n \otimes \dots \otimes u_1 \in \mathcal{B}_n \otimes \dots \otimes \mathcal{B}_1$, we may bijectively associate a sequence of dominant weights $\emptyset \neq \mu^0, \mu^1, \dots, \mu^n$, where $\mu^q = \sum_{i=1}^q \text{wt } u_i$. If μ^n is zero, u is a *highest weight element of weight zero*. Note that the number of highest weight elements of weight zero in a tensor product of crystals is equal to the dimension of the invariant subspace [25, 19].

Definition 2.7 (Sundaram [24]). An *r -symplectic oscillating tableau* O of length n and shape μ is a sequence of partitions $O = \emptyset \neq \mu^0, \mu^1, \dots, \mu^n = \mu$ such that the Ferrers diagrams of two consecutive partitions differ by exactly one cell, and each partition μ^i has at most r nonzero parts.

Next we relate highest weight elements of weight zero in $\mathcal{B}_{\text{spin}}^{\otimes n}$ of type B_r and r -fans of Dyck paths.

Definition 2.8. An *r -fan of Dyck paths* F of length n is a sequence $F = \emptyset \neq \mu^0, \mu^1, \dots, \mu^n = \emptyset$ of partitions μ^i with at most r parts such that the Ferrers diagram of two consecutive partitions differs by exactly one cell in each part. In other words, μ^i differs from μ^{i-1} by $\pm 1, \pm 1, \dots, \pm 1$ for $0 \leq i < n$.

Since $\mathcal{B}_{\text{spin}}$ of type B_r is minuscule, by the above discussion $\epsilon = \epsilon_n \otimes \dots \otimes \epsilon_1 \in \mathcal{B}_{\text{spin}}^{\otimes n}$ is highest weight if and only if $\sum_{i=1}^q \text{wt } \epsilon_i$ is dominant for all $1 \leq q \leq n$. Hence highest weight elements of weight zero can be identified with an r -fan of Dyck paths of length n : the j -th entry of ϵ_i is ± 1 if and only if the j -th Dyck path has an up-step at position i . In particular, for a highest weight element ϵ of weight zero, the sequence of dominant weights $\mu^q = \sum_{i=1}^q 2\text{wt } \epsilon_i$ for $0 \leq q \leq n$ defines an r -fan of Dyck paths consistent with Definition 2.8.

Example 2.9. For $r = 3$ and $n = 4$, $F = (000, 111, 220, 111, 000)$ is a 3-fan of Dyck paths corresponding to $\epsilon = (-, -, -) \otimes (-, -,) \otimes (, -, -) \otimes (, ,) \in \mathcal{B}_{\text{spin}}^{\otimes 4}$.

Following Definition 2.5, we obtain an embedding from the set of r -fans of Dyck paths into the set of oscillating tableaux.

Definition 2.10. For an r -fan of Dyck paths $F = \emptyset \neq \lambda^0, \dots, \lambda^n = \emptyset$ define the oscillating tableau $\iota_{F \rightarrow O} F = \mu^0, \dots, \mu^n$ as follows. Let $v^t = \Psi(\lambda^t - \lambda^{t-1})$ for $1 \leq t \leq n$ with Ψ as in Definition 2.5. Then $\mu^{tr+s} = \lambda^t + \sum_{i=1}^s \text{wt } v_i^{t-1}$ for $0 \leq t < n, 0 \leq s < r$.

2.3 Promotion via crystal commutator

For finite crystals B_λ of classical type of highest weight λ , Henriques and Kamnitzer [5] introduced the crystal commutator as follows. Let $\eta_{B_\lambda}: B_\lambda \rightarrow B_\lambda$ be the Lusztig involution, which maps the highest weight vector to the lowest weight vector and interchanges the crystal operators f_i with $e_{i'}$, where $w_0 \alpha_i = -\alpha_{i'}$ under the longest element w_0 . This can

be extended to tensor products of such crystals by mapping each connected component to itself using the above. Then the *crystal commutator* is defined as

$$\sigma_{B_\lambda, B_\mu} : B_\lambda \otimes B_\mu \rightarrow B_\mu \otimes B_\lambda \quad \text{with} \quad b \otimes c \mapsto \eta_{B_\mu \otimes B_\lambda} \eta_{B_\mu} c \otimes \eta_{B_\lambda} b.$$

Definition 2.11 ([4, 25, 26]). Let C be a crystal and $u \in C^{\otimes n}$ a highest weight element. Then *promotion* pr on u is defined as $\sigma_{C^{\otimes n-1}, C} u$.

Example 2.12. Promotion is $\sigma_{\mathcal{B}_{\text{spin}}^{\otimes 3}, \mathcal{B}_{\text{spin}}} \epsilon = - , - , - \otimes - , \quad , \quad \otimes \quad , - , - \otimes \quad , \quad ,$ for ϵ in Example 2.9.

2.4 Promotion via local rules

Adapting local rules of van Leeuwen [12], Lenart [13] gave a combinatorial realization of the crystal commutator $\sigma_{A,B}$ by constructing an equivalent bijection between the highest weight elements of $A \otimes B$ and $B \otimes A$ respectively. We define Lenart's construction in the case of promotion. The *local rules* of Lenart [13] can be stated as follows: four

weight vectors $\lambda, \mu, \kappa, \nu \in \Lambda$ depicted in a square diagram $\begin{array}{|c|c|} \hline \lambda & \nu \\ \hline \kappa & \mu \\ \hline \end{array}$ satisfy the local rule, if $\mu - \text{dom}_W \kappa = \nu - \lambda$, where W is the Weyl group of the root system Φ for A and B and $\text{dom}_W \rho$ is the dominant weight in the Weyl orbit of ρ .

Let C be a minuscule crystal and $\emptyset = \mu^0, \mu^1, \dots, \mu^n = \mu$ the sequence of partitions corresponding to the highest weight element $u \in C^{\otimes n}$. Let $\hat{\emptyset} = \hat{\mu}^0, \hat{\mu}^1, \dots, \hat{\mu}^n = \mu$ be the sequence of partitions obtained by constructing a skewed grid as in (2.1) and recursively computing $\hat{\mu}^1, \dots, \hat{\mu}^{n-1}$ using Lenart's local rules.

$$\begin{array}{ccccccc} \mu^0 & \mu^1 & \mu^2 & \dots & \mu^{n-1} & \mu^n \\ \hline & \mu^0 & \mu^1 & \dots & \mu^{n-2} & \mu^{n-1} & \mu^n \\ \hline \end{array} \quad (2.1)$$

It follows from Lenart [13, Theorem 4.4] that $\emptyset = \hat{\mu}^0, \hat{\mu}^1, \dots, \hat{\mu}^n = \mu$ is the unique sequence of partitions corresponding to the highest weight element $\text{pr } u$.

3 Chord diagrams

3.1 Promotion-evacuation diagrams

In this section we summarize the method to obtain a map from highest weight words of weight zero to chord diagrams that intertwines promotion and rotation.

Definition 3.1. A *chord diagram* of size n is a graph with n vertices depicted on a circle labelled $1, \dots, n$ in counter-clockwise orientation. The *rotation* of a chord diagram is obtained by rotating all edges clockwise by $\frac{2\pi}{n}$ around the center of the diagram.

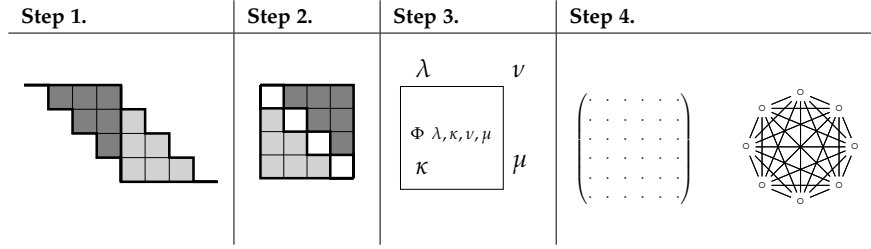


Figure 3: Overview of the steps in our map

In our setting, chord diagrams are undirected graphs with possibly multiple edges between the same two vertices. We can therefore identify a chord diagram with its *adjacency matrix*, which is a symmetric $n \times n$ matrix $M = (m_{ij})_{1 \leq i, j \leq n}$ with non-negative integer entries m_{ij} equal to the number of edges between vertex i and vertex j . We denote by $\text{rot } M$ the adjacency matrix corresponding to the rotation of the chord diagram.

We now outline the idea to construct a rotation and promotion intertwining map for oscillating tableaux and r -fans of Dyck paths. A visual guideline is given in Figure 3.

Construction 3.2. The construction is given as follows:

Step 1 Iteratively calculate promotion of a highest weight word of weight zero and length n using Lenart's schema (2.1) a total of n times.

Step 2 Group the results into a square grid, called the *promotion matrix*.

Step 3 Fill the cells of the square grid with certain non-negative integers according to a filling rule Φ that only depends on the four corners of the cells in the schema (2.1).

Step 4 Regard the filling as the adjacency matrix of a graph, which is the chord diagram.

Definition 3.3. The *filling rule* for oscillating tableaux and fans of Dyck paths, with cells labelled as in Step 3 of Figure 3, is $\Phi(\lambda, \kappa, \nu, \mu)$ = number of negative entries in $\kappa - \nu - \lambda$.

Remark 3.4. For oscillating tableaux at most one negative entry can occur, so that $\Phi(\lambda, \kappa, \nu, \mu)$ is either 0 or 1. Note that the filling rules are new even in the case of oscillating tableaux as the proofs in [19] did not follow this construction.

Definition 3.5. Denote by M_O (resp. M_F) the function that maps an r -symplectic oscillating tableau (resp. r -fan of Dyck paths) of length n to an $n \times n$ adjacency matrix using Construction 3.2 and the filling rule in Definition 3.3.

Proposition 3.6. The map M_X for $X \in \{O, F\}$ intertwines promotion and rotation, that is $M_X \circ \text{pr} = \text{rot} \circ M_X$.

3.2 Fomin growth diagrams

Generally speaking, a *Fomin growth diagram* is a means to bijectively map sequences of partitions satisfying certain constraints to non-negative integer fillings of a Ferrers shape,

drawn in French notation [3, 10]. To map a 0/1 filling of a Ferrers shape to a sequence of partitions, we iteratively label all corners of the cells of the shape with partitions by certain local forward rules, F1-F6 [10, p. 4]. Conversely, given partitions labelling the right and top corners of a cell, the local backwards rules B1-B6 determine the last partition and the 0/1 filling of the cell [10, p. 5].

To map a non-negative integer filling of a Ferrers shape to a sequence of partitions, we produce a “blow up” construction of the original shape for the Burge variant which contains south-east chains of 1’s, as done by [10, Section 4.4]. If a cell is filled with a positive entry m , we replace the cell with an $m \times m$ grid of cells with 1’s along the diagonal. If there exist several nonzero entries in one column or row, we arrange the grids of cells so that the 1’s form a south-east chain in each column and row.

Since the filling of the blow up growth diagram consists of 1’s and 0’s, we now apply the forward local rules F1-F6 to obtain the partition labels of all corners. We then “shrink back” the labelled blow up growth diagram to obtain a labelling of the original Ferrers diagram by only using the partitions labelling the intersections that occurred in the original Ferrers diagram. The labelling from the blow up construction is given explicitly by the local forward rules $F^4_0 - F^4_2$ and local backward rules $B^4_0 - B^4_2$ [10, p. 22].

Construction 3.7. Let $F = \emptyset, \mu^1, \dots, \mu^n = \emptyset$ be an r -fan of Dyck paths. The associated triangular growth diagram is the Ferrers shape $(n-1, n-2, \dots, 2, 1, 0)$. Label the cells according to the following specification:

1. Label the north-east corners of the cells on the main diagonal from the top-left to the bottom-right with the partitions in F .
2. For each $i \in \{0, \dots, n-1\}$ label the corner on the first subdiagonal adjacent to the labels μ^i and μ^{i-1} with the partition $\mu^i \cap \mu^{i-1}$.
3. Use backwards rules Burge $B^4_0 - B^4_2$ to obtain all other labels and cell fillings.

For oscillating tableaux, [Construction 3.7](#) reduces to rules B1-B6 and F1-F6.

We denote by G_F (resp. G_O) the symmetric $n \times n$ matrix one obtains from the filling of the growth diagram by putting zeros in the unfilled cells and along the diagonal and completing this to a symmetric matrix. Starting from a filling of a growth diagram one obtains the r -fan by filling the cells of a growth diagram, setting all vectors on corners on the bottom and left border of the diagram to be the empty partition and applying the forwards growth rules Burge $F^4_0 - F^4_2$ (resp. F1-F6).

4 Main results

4.1 Results for oscillating tableaux and r -fans of Dyck paths

Our main result states that the fillings of the growth diagrams ([Construction 3.7](#)) and the fillings of the promotion matrices ([Definition 3.3](#)) coincide. For oscillating tableaux

this was not stated explicitly in [19], but can be deduced from the proofs.

Theorem 4.1. *For an oscillating tableau of weight zero O and an r -fan of Dyck paths F , we have $G_O O = M_O O$ and $G_F F = M_F F$. In particular, the maps M_O and M_F are injective.*

Proposition 4.2. *For an r -fan of Dyck paths F , $\iota_{F \rightarrow O} \circ \text{pr}_{\mathcal{B}_{\text{spin}}} F = \text{pr}_{\mathcal{C}_{\square}}^r \circ \iota_{F \rightarrow O} F$. In addition, $M_F F = \text{blocksum}_r M_O \iota_{F \rightarrow O} F$, where $\text{blocksum}_r M_O \iota_{F \rightarrow O} F$ is the matrix obtained by replacing $r \times r$ blocks in $M_O \iota_{F \rightarrow O} F$ with the sum of their entries.*

4.2 Cyclic sieving

The cyclic sieving phenomenon was introduced by Reiner, Stanton and White [20].

Definition 4.3. Let X be a finite set and C be a cyclic group generated by c acting on X . Let $\zeta \in \mathbb{C}$ be a $|C|^{th}$ primitive root of unity and $f(q) \in \mathbb{Z}[q]$ be a polynomial in q . Then the triple (X, C, f) exhibits the *cyclic sieving phenomenon* if for all $d \geq 0$ we have that the size of the fixed point set of c^d (denoted X^{c^d}) satisfies $|X^{c^d}| = f(\zeta^d)$.

We state cyclic sieving phenomena for the promotion action on oscillating tableaux and fans of Dyck paths. For this we need the *local energy function* (see for example [15]) $H: \mathcal{B} \otimes \mathcal{B} \rightarrow \mathbb{Z}$. The local energy function is constant on connected components.

For the crystal $\mathcal{C}_{\square}$ of type C_r , using the ordering $1 < 2 < \dots < r < \bar{r} < \dots < \bar{2} < \bar{1}$, we have that $H(a \otimes b) = 0$ if $a \leq b$ and $H(a \otimes b) = 1$ if $a > b$. The classical highest weight elements in $\mathcal{B}_{\text{spin}}^{\text{af}} \otimes \mathcal{B}_{\text{spin}}^{\text{af}}$ are $\epsilon_1, \dots, \epsilon_r \otimes \dots$, with ϵ_i for $1 \leq i \leq k$ and $\epsilon_i = -$ for $k < i \leq r$ for some $0 \leq k \leq r$. Denoting by m the number of $-$ in the ϵ_i , we have $H(\epsilon_1, \dots, \epsilon_r \otimes \dots) = \left\lfloor \frac{m \epsilon_1, \dots, \epsilon_r}{2} \right\rfloor$. The *energy function* $E: \mathcal{B}^{\otimes N} \rightarrow \mathbb{Z}$ is defined for $b_1 \otimes \dots \otimes b_N \in \mathcal{B}^{\otimes N}$ as $E(b_1 \otimes \dots \otimes b_N) = \sum_{i=1}^{N-1} i H(b_i \otimes b_{i+1})$. Define a polynomial in q using the energy function for highest weight elements in $\mathcal{B}^{\otimes n}$ of weight zero

$$f_{n,r}(q) = q^{c_{n,r}} \sum_{\substack{b \in \mathcal{B}^{\otimes n} \\ \text{wt } b = 0, e_i(b) = 0 \text{ for } 1 \leq i \leq r}} q^{E(b)},$$

where r is the rank of the root system and $c_{n,r}$ is a constant depending on the type.

Theorem 4.4. *Let X be the set of highest weight elements in $\mathcal{B}^{\otimes n}$ of weight zero, where $\mathcal{B}$ is minuscule. Then $(X, C_n, f_{n,r}(q))$ exhibits the cyclic sieving phenomenon, where C_n is the cyclic group of order n given by the action of promotion pr on $\mathcal{B}^{\otimes n}$.*

Note that $\mathcal{C}_{\square}$ and $\mathcal{B}_{\text{spin}}$ are minuscule, and hence Theorem 4.4 gives a cyclic sieving phenomenon for oscillating tableaux and fans of Dyck paths. See also [4, 25]. For the type A vector representation, the energy function is the major index and Theorem 4.4 relates to results in [21].

Recall that highest weight elements of weight zero in $\mathcal{B}_{\text{spin}}^{\otimes 2n}$ of type B_r are in bijection with r -fans of Dyck paths of length $2n$. Denote by D_n^r the set of all r -fans of Dyck paths of length $2n$. We have $|D_n^r| = \prod_{1 \leq i \leq j \leq n-1} \frac{i+j-2r}{i-j}$, see [9]. Define the q -analogue as $g_{n,r}(q) = \prod_{1 \leq i \leq j \leq n-1} \frac{i+j-2r}{i-j} q$ where $m_q = 1 + q + q^2 + \dots + q^{m-1}$. The following conjecture is equivalent to [6, Conjecture 5.9] on plane partitions and root posets.

Conjecture 4.5. *The triple $(D_n^r, C_{2n}, g_{n,r}(q))$ exhibits the cyclic sieving phenomenon, where C_{2n} is the cyclic group of order $2n$ that acts on D_n^r by applying promotion.*

Example 4.6. We have $q^{-4}f_{2,2}(q) = g_{2,2}(q) = q^4 + q^2 + 1$ and

$$\begin{aligned} g_{3,2}(q) &= q^{12} + q^{10} + q^9 + 2q^8 + q^7 + 2q^6 + q^5 + 2q^4 + q^3 + q^2 + 1, \\ q^{-6}f_{3,2}(q) &= q^{10} + q^9 + 2q^8 + q^7 + 3q^6 + q^5 + 2q^4 + q^3 + q^2 + 1. \end{aligned}$$

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Deficit and q, t -symmetry in triangular Dyck paths

Loïc Le Mogne^{*1} and Viviane Pons^{†1}

¹CNRS, IRL CRM Montréal, Canada – Université Paris-Saclay, CNRS, Laboratoire Interdisciplinaire des Sciences du Numérique, Orsay, France

Abstract. We study the q, t -enumeration of triangular Dyck paths considered by Bergeron and Mazin. To do so, we introduce the notion of *triangular* and *sim-sym* tableaux and the *deficit* statistic which is a new interpretation of the *dinv*. We use it to obtain new results and proofs on triangular 2-partitions and an interesting conjecture for a certain lattice interval q, t, r -enumeration.

Résumé. Nous étudions la symétrie q, t dans l'énumération des chemins de Dyck triangulaires décrits par Bergeron et Mazin. Nous introduisons les notions de *tableau triangulaire* et *sim-sym* ainsi que la statistique du *deficit* qui ré-interprète le *dinv*. Nous utilisons cela pour obtenir de nouveaux résultats et nouvelles démonstrations pour les 2-partages ainsi qu'une conjecture sur un comptage trivarié q, t, r des intervalles de certains treillis.

Keywords: triangular partitions, q, t -Catalan, Dyck paths

1 Introduction

In [2], Bergeron and Mazin study a certain family of partitions that they call *triangular partitions*, motivated by the results of [4]. A triangular partition is the maximal partition lying under a given line. The sub-partitions of a triangular partition are called the triangular Dyck paths, generalizing classical and rational Dyck paths. The q, t -enumeration of paths, such as the q, t -Catalan numbers [7], has raised interesting combinatorial questions in recent years, related in particular to representation theory. The Negut formula (see for instance [1, 5, 6]) gives a q, t -enumeration of the sub-partitions of any given partition but the signification of each q, t monomial is not clear and the coefficients are not always positive. In a recent work generalizing the *shuffle theorem* [4], the authors found out a combinatorial interpretation when the partition is triangular, using two statistics generalizing the classical *area* and *dinv* of a Dyck path. Indeed, in this case the q, t polynomials appear as the coefficients of some symmetric functions and are

^{*}loic.le-mogne@universite-paris-saclay.fr.

[†]viviane.pons@lisn.upsaclay.fr. Supported by Project PAGCAP ANR-21-CE48-0020.

by nature symmetric themselves. More generally, Conjecture 7.1.1 from [4] states that the Negut formula gives positive coefficients if the partition lies under a certain convex curve. As expressed in [1], Bergeron further conjectures that this condition is not only sufficient but necessary. Moreover, the q, t -enumeration appears to be *Schur positive* in the triangular case [2, Conjecture 1].

In terms of combinatorics, this raises two (very) difficult questions: finding an elementary, combinatorial proof of the q, t -symmetry for the enumeration of triangular Dyck paths and proving that they are Schur positive. This includes the classical and rational Dyck path cases, which are special cases of triangular Dyck paths. This is the problem that motivates this extended abstract. Our main results are stated in [Theorems 2](#) and [5](#) where we prove the conjecture in the (limited) case of 2-partitions and extend the symmetry to a certain interval enumeration. Moreover, we introduce new combinatorial tools such as the *triangular* and *sim-sym* tableaux as well as the *deficit* statistic which explore the combinatorial interpretation of the *dinv* statistic. This led us to state [Conjecture 4](#) in relation to interval enumeration in the generalized Tamari lattices of [10].

The paper is organized as follows. [Section 2](#) recalls the basic notions related to partitions and the definition and main properties of triangular partitions from [2]. In [Section 3](#), we define and explore some new combinatorial tools on triangular partitions, namely the *triangular tableau*, the *deficit* statistic and the *sim-sym* tableaux. The main result of this section is [Proposition 3.4](#) where we prove that the *deficit* statistic actually gives the *dinv*. [Section 4](#) uses these tools to obtain explicit results on 2-partitions: [Theorems 2](#) and [3](#). Finally, the lattice enumerations are succinctly presented in [Section 5](#) in order for us to state [Conjecture 4](#) and [Theorem 5](#).

2 Background

2.1 Partitions and Young tableaux

Definition 2.1. A [partition](#)¹ λ of size n is a tuple $\lambda_1 \geq \lambda_2 \geq \dots \lambda_k > 0$ such that $|\lambda| : \sum_i^k \lambda_i = n$. For convenience, we consider partitions ending with an infinite number of 0-parts, $\lambda_j = 0$ for $j > k$. A [k-partition](#) is a partition $\lambda = \lambda_1, \dots, \lambda_{k'}$ with $k' \leq k$. We say that a partition μ is a [sub-partition](#) of another partition λ if $\mu_i \leq \lambda_i$ for all i .

As usual, we represent partitions by their [Ferrers diagram](#) (French style) such that each value λ_i , from bottom to top, corresponds to a line of i square cells as shown on [Figure 1](#).

Definition 2.2. A [standard Young tableau](#) on a partition λ is an integer filling of the Ferrers diagram of λ (we place a number in each cell) such that the numbers are strictly increasing along

¹This is actually an integer partition. As there is no ambiguity in the paper, we simply call them partitions.

each row and each column. We identify the cells of λ by their coordinates in the Ferrers diagram (the cell $(0,0)$ is the bottom left-most cell). We consider a tableau θ as a function from the cells of λ to $\{1, \dots, |\lambda|\}$ and write $\theta(c)$ the value of the cell c in θ .

See Figure 1 for an example. *Schur functions* form a famous basis of symmetric functions indexed by partitions. Any symmetric polynomial can thus be expressed as a sum of Schur functions. A symmetric polynomial is said to be *Schur positive* if all the coefficients in its expansion are positive integers. For the precise definition and properties of Schur functions, we refer the reader to [8].

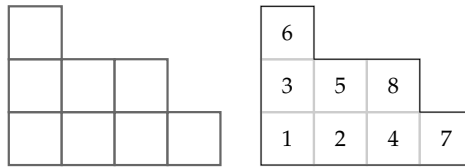


Figure 1: The Ferrers diagram of $(4,3,1)$ and an example of standard Young tableau.

2.2 Triangular partitions

Definition 2.3 (From [2]). A *triangular partition* λ is a partition such that there exist two real numbers r and s with $\lambda_j = \lfloor r - \frac{jr}{s} \rfloor$ for all $j \leq s$, and $\lambda_j = 0$ otherwise.

We say that the line joining $(r,0)$ and $(0,s)$ is an *r - s -line*. Then λ is the greatest partition lying under the r - s -line. We also say that this line *cuts off* λ . See an example on Figure 2.

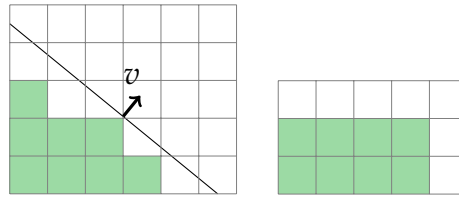


Figure 2: Left: the triangular partition $(4,3,1)$ with one line that cuts it off and its slope vector. Right: a non-triangular partition $(4,4)$.

When there exists an r - s -line cutting off λ with r and s integer values, λ is said to be a *rectangular* partition. It is further said to be *rational* when r and s are coprime. Later on, it will be important to characterize the *slope* of the line. In accordance with [2], this can be done through a single parameter v which characterizes a *slope vector*.

Definition 2.4. The **slope vector** of an r - s -line is the vector orthogonal to the line and of coordinates $v, 1 - v$. For simplicity, we write “the slope v ”. We say that λ admits a slope v if it is cut off by an r - s -line of slope vector $v, 1 - v$.

A triangular partition admits an infinite number of slope vectors which corresponds to certain open interval for the value of v . Bergeron and Mazin [2] give a simple way to compute the slope interval using the hooks of the partition. For each cell c of λ , let a_c be the **arm** of the cell, *i.e.* the number of cells lying to the right of c in λ , and ℓ_c be the **leg** of c , *i.e.* the number of cells lying above c in λ . Then

$$v^-_{c,\lambda} : \frac{\ell_c}{a_c + \ell_c + 1} \quad v_{c,\lambda} : \frac{\ell_c + 1}{a_c + \ell_c + 1}. \quad (2.1)$$

We say that $v^-_{c,\lambda}, v_{c,\lambda}$ is the **admissible interval** of the cell c . The slope interval of λ is the intersection of all admissible intervals $v^-_{c,\lambda}, v_{c,\lambda}$ where $v^-_{\lambda} : \max_{c \in \lambda} v^-_{c,\lambda}$ and $v_{\lambda} : \min_{c \in \lambda} v_{c,\lambda}$. In particular, λ is a triangular partition if and only if $v^-_{\lambda} < v_{\lambda}$ [2, Lemma 1.2].

2.3 Area, sim and q, t -enumeration

In this paper, we are interested in the q, t -enumeration of sub-partitions of a given triangular partition λ . Following the terminology of [2], we call such objects **triangular Dyck paths**.

Definition 2.5. A **triangular Dyck path** is a tuple λ, μ such that μ is a sub-partition of λ and λ is a triangular partition. The **area** of a triangular Dyck path is the number of cells belonging to λ but not to μ , given by $\text{area}_{\lambda, \mu} : \sum_{i \in \lambda} \lambda_i - \mu_i$.

Note that we do not require that μ is also triangular. See Figure 3 for an example. This name is justified in the sense that when λ is the staircase partition, *i.e.* $\lambda = n - 1, n - 2, \dots, 1$, then the triangular Dyck paths λ, μ correspond to actual Dyck paths of length n . Similarly, when λ is a rational partition, then its sub-partitions are rational Dyck paths.

Definition 2.6 (From [2], Section 4.1). Let λ, μ be a triangular Dyck path. A cell c of μ is said to be **similar** if

$$v^-_{c,\mu} \leq \frac{v^-_{\lambda} + v_{\lambda}}{2} < v_{c,\mu}. \quad (2.2)$$

In other words, the **mean slope** of the slope interval of λ belongs to the semi-closed admissible interval of the cell c in μ . Then $\text{sim}_{\lambda, \mu}$ is the number of similar cells in the triangular Dyck path λ, μ .

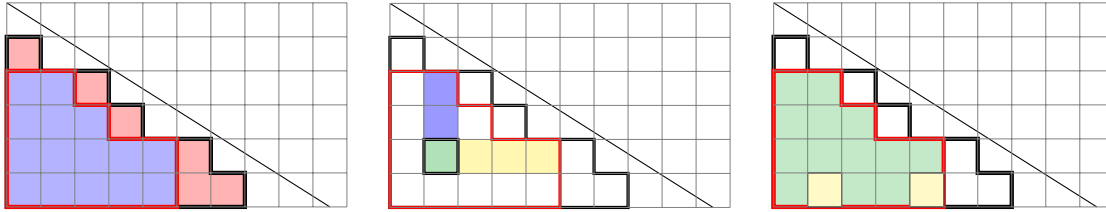


Figure 3: A triangular Dyck path λ, μ . The left picture shows μ in blue and the area in red with $\text{area}_\lambda \mu = 6$. The middle picture illustrates the hook of a given cell in μ . The right picture shows all similar cells in green with $\text{sim}_\lambda \mu = 13$.

For example on Figure 3, the mean slope of λ , $\frac{45}{112}$ belongs to the admissible interval of the selected cell given by $v^-(c, \mu) = \frac{1}{3}$ and $v^+(c, \mu) = \frac{1}{2}$. It is important to understand that the similarity of a cell depends on both partitions of the triangular Dyck path. Indeed, the mean slope is computed on λ but the admissible cell interval is computed on μ . If λ, μ is a triangular Dyck path such that all the cells of μ are similar, then in particular μ is triangular: a slight perturbation of the mean slope of λ will always cut off μ (see [2, Lemma 4.1]). When all cells of μ are similar cells, we say that λ, μ is a *mean-similar* triangular Dyck path.

The *sim* statistic already appears in [4] and is a generalization of the famous *div* statistic that can be found in [7] for the classical q, t -Catalan enumeration. Indeed, when λ is the staircase partition, then the *sim* of λ, μ is actually the *div* of the corresponding Dyck path.

We can now define the q, t polynomials associated to a triangular partition λ as:

$$A_\lambda(q, t) = \sum_{\mu \subseteq \lambda} q^{\text{area}_\lambda \mu} t^{\text{sim}_\lambda \mu}. \quad (2.3)$$

These polynomials have been shown to be symmetric in q and t via an algebraic proof [4]. However, the question of finding a combinatorial proof is still open even for the classical case. For example, for the partition $\lambda = (3, 2)$, we have the following:

$$A_{3,2}(q, t) = q^5 + q^4 t + q^3 t^2 + q^2 t^3 + q t^4 + t^5 + q^3 t + q^2 t^2 + q t^3 \quad (2.4)$$

As the $A_\lambda(q, t)$ polynomials are symmetric, they can be expressed in terms of Schur functions with the following conjecture.

Conjecture 1 (Conjecture 1 of [2]). *The polynomials $A_\lambda(q, t)$ are Schur positive.*

See [2] for many examples of computation, here are $A_{3,2,1}(q, t)$ and $A_{3,2}(q, t)$:

$$A_{3,2,1}(q, t) = s_6(q, t) + s_{4,1}(q, t) + s_{3,1}(q, t), \quad A_{3,2}(q, t) = s_5(q, t) + s_{3,1}(q, t). \quad (2.5)$$

3 Combinatorial interpretation of similar cells

In this paper, we explore some combinatorial tools related to the aforementioned conjecture and obtain a proof for the case of 2-partitions. The first step is to define a certain Young tableau associated to each triangular partition.

3.1 The triangular Young tableau

Proposition 3.1. *Let λ be a triangular partition. Then for all $0 \leq k \leq |\lambda|$, there exists a unique mean-similar triangular Dyck path λ, μ of area k . Besides, if λ, α and λ, μ are two mean-similar triangular Dyck paths with $|\alpha| \leq |\mu|$, then $\alpha \subseteq \mu$.*

In other words, if λ, μ is such that all cells of μ are similar cells, then there is a unique way to remove a cell from μ such that all remaining cells stay similar. This is a consequence of [2, Lemma 4.1]: the mean-slope of λ can always be slightly shifted to obtain an irrational slope. By moving this irrational slope towards the origin, you will “touch” the cells of λ one by one, which gives the removal order on the cells of the sub-partitions in order to stay mean-similar, see Figure 4 for an example. This leads to the following definition.

Definition 3.2. The **triangular Young tableau** of a triangular partition λ with $|\lambda| = n$ is the unique standard Young tableau such that for all $0 \leq k \leq n$, the cells 1 to k form a sub-partition μ with λ, μ a mean-similar triangular Dyck path.

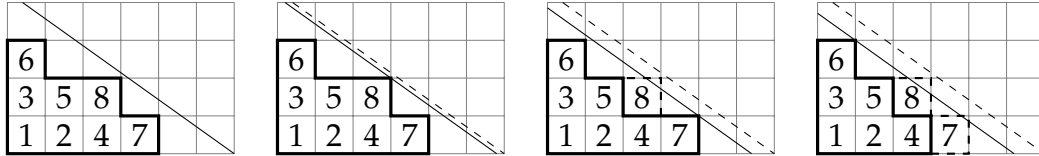


Figure 4: The triangular Young tableaux of $(4, 3, 1)$ with its slope construction.

3.2 Deficit of a triangular Dyck path

Definition 3.3. Let λ, μ be a triangular Dyck path and θ a standard Young tableau of shape λ . We say that there is a **θ -inversion** in λ, μ if there exists a cell $c = (x, y)$ in μ and a cell $c' = (x', y')$ in $\lambda \setminus \mu$ such that $\theta(c) > \theta(c')$. In particular, c cannot be in the same line or column as c' . Then we say that the cell $d = (\min(x, x'), \min(y, y'))$, which is at the hook of c and c' , is a **θ -deficit cell**. We call **deficit** of the triangular Dyck path λ, μ with tableau θ , written $\text{def}_\theta \mu$, the number of θ -deficit cells.

See Figure 5 for an example: 1 is a deficit cell because it is at the hook of 6 and 4, and 2 is a deficit cell because it is at the hook of 5 and 7. Note that it can happen that multiple θ -inversions give the same deficit cell. The following proposition gives a more combinatorial approach to the computation of similar cells.

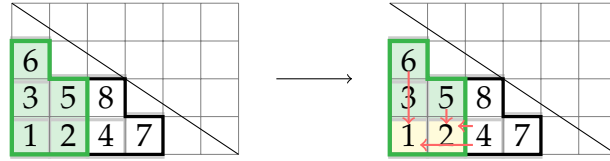


Figure 5: As a sub-partition of $4, 3, 1$, the partition $2, 2, 1$ has two deficit cells 1 & 2.

Proposition 3.4. *Let λ be a triangular partition and θ its triangular Young tableau as in Definition 3.2. Then for any triangular Dyck path λ, μ , the θ -deficit cells of λ, μ are exactly the non-similar cells of μ . In particular, we have $|\lambda| - \text{area}_\lambda \mu = \text{sim}_\lambda \mu = \text{def}_\theta \mu$.*

In the rest of the paper, we write $\text{def}_\lambda \mu : \text{def}_\theta \mu$ of a triangular Dyck path where θ is the triangular Young tableau.

For the staircase partition (which gives classical Dyck paths), the triangular tableau is the *top-down* tableau where we label the corners of the partition in decreasing order from top to bottom as shown in Figure 6. The deficit then gives a new interpretation for the classical *dinv* statistic.

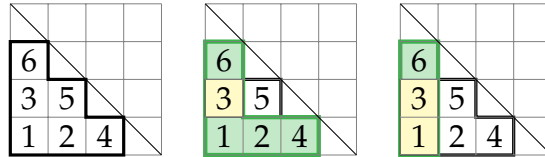


Figure 6: The triangular tableau of the staircase partition $3, 2, 1$ and two examples of sub-partitions (Dyck paths). In the left example, we have $\mu = 3, 1, 1$ and $\text{sim}_\lambda \mu = 4$. In the second example, we have $\mu = 1, 1, 1$ and $\text{sim}_\lambda \mu = 1$.

3.3 Sim-sym tableau

The deficit statistic is defined for any tableau θ and just happens to coincide with the non-similar cells when θ is the triangular Young tableau of λ . It is then natural to extend the definition of the *sim* statistic to any Young tableau.

Definition 3.5. Let θ be a standard Young tableau of triangular shape λ . Then for any triangular Dyck paths λ, μ , we say that a cell of μ is **θ -similar** if it is not a θ -deficit cell. We write $\text{sim}_\theta \mu$ the number of θ -similar cells.

With that definition, again it is natural to extend the definition of (2.3) to this new setting, we define

$$A_\theta(q, t) = \sum_{\mu \subseteq \lambda} q^{\text{area}_\lambda \mu} t^{\text{sim}_\theta \mu} \quad (3.1)$$

where θ is a standard Young tableau of shape λ . These polynomials are not symmetric in general. On the other hand, the triangular Young tableau is not the only tableau giving the symmetry. This led us to introduce the following objects which we believe are of high interest when studying the q, t -symmetry.

Definition 3.6. A **sim-sym** (similar-symmetric) tableau is a Young tableau θ of triangular shape λ such that $A_\theta(q, t) = A_\lambda(q, t)$.

It is important to note that this equality does not necessarily mean that, for all triangular Dyck paths, one has $\text{sim}_\theta \mu = \text{sim}_\lambda \mu$. It only means we have a bijection over the triangular Dyck paths of λ which keeps the *area* constant while sending the θ -*sim* statistic to the classical *sim*. Figure 7 gives three examples of sim-sym tableaux for $4, 2, 1$.

The natural question that follows would be to characterize the sim-sym tableaux. This is not an easy question. One way to find different sim-sym tableaux is to choose a different slope to sweep down the partition. Indeed, the triangular Young tableau is defined using a slight deformation of the mean-slope (to ensure that it is irrational). But any irrational slope would actually give a tableau, and computational explorations suggest that these are always sim-sym tableaux. Nevertheless, they are not the only examples. For example, for the partition $4, 2, 1$, we find only two slope-tableaux which are the first two tableaux of Figure 7 (the first one is the mean-slope triangular tableau), but there are actually seven sim-sym tableaux. In the rest of the paper, we characterize sim-sym tableaux on 2-partitions, and explore some interesting conjectures related to the lattice structure of triangular Dyck-paths and sim-sym tableaux.

6				
3	5			
1	2	4	7	

7				
3	5			
1	2	4	6	

7				
4	6			
1	2	3	5	

Figure 7: Three sim-sym tableaux of $4, 2, 1$: the first two are slope-tableaux but not the last one.

Recall that $s_{k,\ell} q, t = \sum_i \sum_k q^i t^k \ell^{-i}$. In particular, each monomial $q^i t^j$ appears only once in (4.1). The proof then is purely combinatorial, building an explicit bijection between monomials $q^i t^j$ and triangular Dyck paths which *area* equals to i and *deficit* equals to $m - n - i - j$. In the general case, monomials might appear with coefficients greater than 1, which explains why this proof does not generalize easily.

4.3 Sim-sym tableaux on 2-partitions

The key combinatorial ingredient in the proof of [Theorem 2](#) is the triangular tableau and especially the alternation of values between the two rows. This leads to the following theorem.

Theorem 3 (Characterization of sim-sym tableau on 2-partitions). *Let $\lambda = (m, n)$ be a triangular partition with $n \geq 2$. Then there are exactly $m - 2n + 2$ sim-sym tableaux of shape λ . The values on the shortest row of each tableau are given in decreasing order by $k, k-2, k-4, \dots, k-2n+1$ with $3n-2 < k \leq m-n$.*

Furthermore, when $n = 2$, we have the aforementioned tableaux as well as the tableau containing values 2 and 5 on its shortest row.

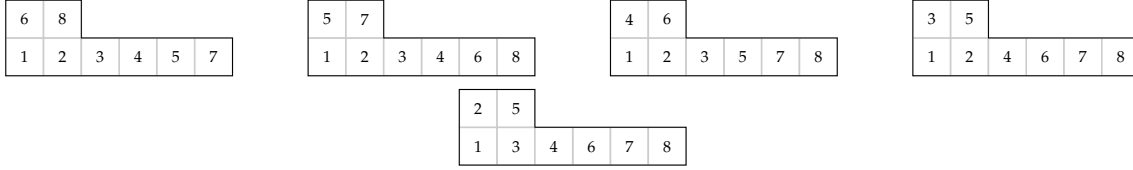


Figure 9: The five different sim-sym tableaux on $(6, 2)$.

5 Lattice and interval enumeration

In some instances, the polynomials $A_\lambda(q, t)$ of (2.3) occur as the expansion in two variables of the characters of a certain GL_2 -action. Hence, they are Schur positive, as the Schur functions are the characters of the irreducibles in this context. For all triangular partitions λ , one can define a symmetric function A_λ corresponding to the Hilbert series of alternating component of multivariate diagonal harmonic polynomials [1]. The trivariate case has been studied in particular in the context of the Tamari and m -Tamari lattices [3] with various conjectures emitted in [9]. For example, we have that

$$A_{3,2,1} = s_6 + s_{4,1} + s_{3,1} + s_{1,1,1}. \quad (5.1)$$

When expanded on two variables, the term $s_{1,1,1}(q, t)$ is 0 and we get (2.5), the (q, t) -enumeration of Dyck paths. For the Tamari and m -Tamari cases, the A_λ expanded in three variables q , t , and r are conjectured to enumerate the intervals of the m -Tamari lattices with q counting the *distance* of the interval (length of the maximal chain) and t counting the *dinv* of the maximal element. The signification of the r statistic as well as a combinatorial proof that the polynomials are symmetric in q , t and r , and of their Schur positivity is still open. A wider question would be to find a combinatorial interpretation for the terms appearing in the Schur expansion of the general polynomials A_λ .

Considering the three variables expansion for the triangular case, the first question would be to find a lattice structure “compatible” with the q, t, r -enumeration. In [10], the authors describe a generalization of the Tamari lattice for all sub-partitions of any given partition λ . But this lattice does not always give the proper enumeration. In particular, the triangular tableau does not always define a maximal chain in the lattice, which seems to be an important property. This led us to define the *top-down tableau* by iteratively removing the corners of the partition from top to bottom (see last tableau of Figure 7) and to state the following conjecture.

Conjecture 4. *If λ is a triangular partition such that its top-down tableau θ is a sim-sym tableau, then*

$$A_\lambda(q, t, 1) = \sum_{\lambda, \mu_1 \leq \lambda, \mu_2} q^{\text{dist}_\lambda(\mu_1, \mu_2)} t^{\text{sim}_\theta(\mu_2)} \quad (5.2)$$

where the sum runs over intervals in the λ -Tamari lattice as defined in [10] and $\text{dist}_\lambda(\mu_1, \mu_2)$ is the length of the maximal chain between the triangular Dyck paths λ, μ_1 and λ, μ_2 .

In the Tamari and m -Tamari cases, the top-down tableau is the triangular tableau and this is the trivariate conjecture found for example in [9]. In the triangular case, there are partitions such as $4, 2, 1$ where the top-down tableau is not the triangular tableau (it is the last one of Figure 7), and actually allows us to find the A_λ enumeration. However, there are cases such as $4, 3, 1$ where the top-down tableau is not sim-sym and the question of finding a compatible lattice structure is still open. Nevertheless, in the case of triangular 2-partitions, the top-down tableau is always sim-sym by Theorem 3. Besides, in this case the λ -Tamari lattice has a very neat structure as shown on Figure 10, which allowed us to perform some recursive enumeration, and prove the conjecture.

Theorem 5. *For λ a triangular 2-partition and θ its top-down tableau, then*

$$A_\lambda(q, t, 1) = \sum_{\lambda, \mu_1 \leq \lambda, \mu_2} q^{\text{dist}_\lambda(\mu_1, \mu_2)} t^{\text{sim}_\theta(\mu_2)} \quad (5.3)$$

summing over intervals of the λ -Tamari lattice.

Note that in the case of 2-partitions, the Schur expression of A_λ is actually the same as for $A_\lambda(q, t)$, and is thus given by Theorem 2.

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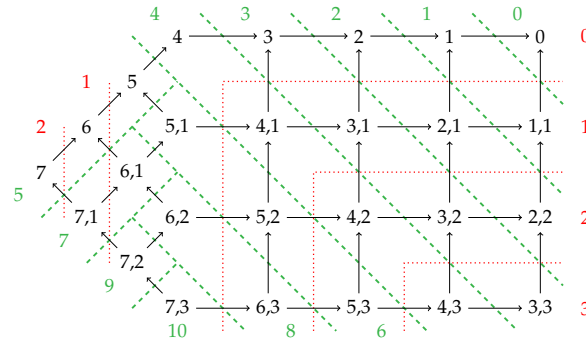


Figure 10: The structure of the λ -lattice on a 2-partition. Elements share the same θ -deficit between two red dotted lines and the same θ -sim between two green dashed lines.

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Combinatorial formulas for shifted dual stable Grothendieck polynomials

Joel Brewster Lewis^{*1} and Eric Marberg⁺²

¹*Department of Mathematics, George Washington University*

²*Department of Mathematics, Hong Kong University of Science and Technology*

Abstract. The K -theoretic Schur P - and Q -functions GP_λ and GQ_λ may be concretely defined as weight generating functions for semistandard shifted set-valued tableaux. These symmetric functions are the shifted analogues of stable Grothendieck polynomials, and were introduced by Ikeda and Naruse for applications in geometry. Nakagawa and Naruse specified families of dual K -theoretic Schur P - and Q -functions gp_λ and gq_λ via a Cauchy identity involving GP_λ and GQ_λ . They conjectured that the dual power series are weight generating functions for certain shifted plane partitions. We prove this conjecture. We also derive a related generating function formula for the images of gp_λ and gq_λ under the ω involution of the ring of symmetric functions. This confirms a conjecture of Chiu and the second author. Using these results, we verify a conjecture of Ikeda and Naruse that the GQ -functions are a basis for a ring.

Keywords: K -theoretic Schur P - and Q -functions, Cauchy identities, shifted tableaux, set-valued tableaux, plane partitions

1 Introduction

The main results of this extended abstract are explicit combinatorial generating functions for certain families of “dual” power series that were originally defined indirectly by Cauchy identities. The formulas that we establish were conjectured in [2, 20]. We start by giving a summary of the power series involved and the generating functions derived. We then explain an application of our formulas to resolve a conjecture of Ikeda and Naruse from [8]. We also provide a comparison with the work of Lam and Pylyavskyy in the unshifted case [9]. For the full version of these results, with proofs, see [13, 16].

2 Shifted set-valued generating functions

Throughout this section, $\lambda = \lambda_1 > \lambda_2 > \cdots > \lambda_k > 0$ denotes a strict integer partition. The *shifted Young diagram* of λ is $SD_\lambda : \{ (i, i - j + 1) \in k \times \mathbb{Z} : 1 \leq j \leq \lambda_i \}$ where

^{*}jblewis@gwu.edu. Partially supported by Simons Foundation grant 634530.

⁺emarberg@ust.hk. Partially supported by Hong Kong RGC grant GRF 16306120.

$k : \{1, 2, \dots, k\}$. Elements of SD_λ are called *positions* or *boxes*. A *shifted set-valued tableau* of shape λ is a filling T of SD_λ by finite, nonempty subsets of $\frac{1}{2}\mathbb{Z}$. Throughout, we let $i' : i - \frac{1}{2}$ for $i \in \mathbb{Z}$ and refer to half-integers as *primed numbers*.

Let T_{ij} denote the entry assigned by T to box $i, j \in \text{SD}_\lambda$, and write $i, j \in T$ when i, j is in the domain of T . The *diagonal positions* of T are the boxes $i, j \in T$ with $i = j$. A shifted set-valued tableau T is *semistandard* if all of the following conditions hold:

- (S1) its entries T_{ij} are nonempty finite subsets of $\{1' < 1 < 2' < 2 < \dots\}$,
- (S2) $\max T_{ij} \leq \min T_{i-1, j}$ and $\max T_{ij} \leq \min T_{i, j-1}$ for all relevant $i, j \in T$,
- (S3) no unprimed number appears in different boxes within the same column, and
- (S4) no primed number appears in different boxes within the same row.

We draw shifted tableaux in French notation; for example, both

$$\begin{array}{c}
 \begin{array}{cccc}
 & & 345 \\
 & 2' & 3' \\
 1 & 2' & 2 & 3'3
 \end{array}
 \quad \text{and} \quad
 \begin{array}{cccc}
 & & 5 \\
 & 3' & 3 \\
 12 & 2 & 23' & 34
 \end{array}
 \end{array}$$

are semistandard shifted set-valued tableaux of shape $4, 2, 1$.

Let $\beta, x_1, x_2, x_3, \dots$ be commuting indeterminates. Define $|T| : \sum_{i, j \in T} |T_{ij}|$ and $\mathbf{x}^T : \prod_i x_i^{a_i - b_i}$ where a_i and b_i are the number of times that i and i' appear in T , respectively. Our examples above both have $|T| = 10$ and $\mathbf{x}^T = x_1 x_2^3 x_3^4 x_4 x_5$.

Definition 2.1. Let $\text{ShSVT}_Q \lambda$ be the set of all semistandard shifted set-valued tableaux of shape λ and let $\text{ShSVT}_P \lambda$ be the subset of such tableaux with no **primed** numbers in any diagonal positions. The *K-theoretic Schur P- and Q-functions* indexed by λ are the formal power series $GP_\lambda : \sum_{T \in \text{ShSVT}_P \lambda} \beta^{|T| - |\lambda|} \mathbf{x}^T$ and $GQ_\lambda : \sum_{T \in \text{ShSVT}_Q \lambda} \beta^{|T| - |\lambda|} \mathbf{x}^T$.

The functions GP_λ and GQ_λ are sometimes defined by these formulas with $\beta = -1$. This loses no generality since we can recover GP_λ from $\sum_{T \in \text{ShSVT}_P \lambda} -1^{|T| - |\lambda|} \mathbf{x}^T$ by substituting $x_i \mapsto -\beta x_i$ then dividing by $\beta^{|\lambda|}$. Similar comments apply to GQ_λ .

Both GP_λ and GQ_λ belong to ring $\mathbb{Z} \llbracket x_1, x_2, \dots \rrbracket$. If $\deg \beta = -1$ then GP_λ and GQ_λ are homogeneous of degree $|\lambda|$, but if $\deg \beta = 0$ then the power series have unbounded degree. Both power series are symmetric in the x_i variables [8, Thm. 9.1]. Setting $\beta = 0$ turns GP_λ and GQ_λ into the classical *Schur P- and Q-functions* P_λ and Q_λ . It follows that as λ ranges over all strict partitions, the sets $\{GP_\lambda\}$ and $\{GQ_\lambda\}$ are linearly independent. While $Q_\lambda = 2^{\ell(\lambda)} P_\lambda$, each GQ_λ is a more complicated (though still finite) $\mathbb{Z} \beta$ -linear combination of GP_μ 's [2, Thm. 1.1].

Ikeda and Naruse introduced GP_λ and GQ_λ in [8] for applications in geometry. Specializations of these symmetric functions represent the structure sheaves of Schubert varieties in the torus-equivariant K -theory of the maximal isotropic Grassmannians of orthogonal and symplectic types [8, Cor. 8.1]; see also [19, 20, 21]. These power series further appear as “stable limits” of K -theory classes of certain orbit closures in [17, 18].

3 Dual functions via Cauchy identities

Our main results concern the following dual forms of GP_λ and GQ_λ .

Definition 3.1. The *dual K -theoretic Schur P - and Q -functions* gp_λ and gq_λ are the unique elements of $\mathbb{Z} \beta \llbracket x_1, x_2, \dots \rrbracket$ indexed by strict partitions λ satisfying the Cauchy identities

$$\sum_{\lambda} GQ_{\lambda} \mathbf{x} \, gp_{\lambda} \mathbf{y} = \sum_{\lambda} GP_{\lambda} \mathbf{x} \, gq_{\lambda} \mathbf{y} = \prod_{i,j \geq 1} \frac{1 - \bar{x}_i y_j}{1 - x_i y_j} \quad \text{where } \bar{x} : \frac{-x}{1 - \beta x}. \quad (3.1)$$

The power series gp_λ and gq_λ are special cases of Nakagawa and Naruse’s *dual universal factorial Schur P - and Q -functions*, which are defined via a more general version of (3.1) [20, Def. 3.2]. Both $\{gp_\lambda\}$ and $\{gq_\lambda\}$ are families of linearly independent functions that are symmetric in the x_i variables [20, Thm. 3.1] and homogeneous if $\deg \beta = 1$ [2, Prop. 6.1]. These properties let one define the following conjugate symmetric functions, which were first considered in [2].

Definition 3.2. Write ω for the $\mathbb{Z} \beta$ -linear involution of the ring of symmetric functions acting on Schur functions as $\omega s_\mu = s_{\mu^\top}$. The *conjugate dual K -theoretic Schur P - and Q -functions* of a strict partition λ are given by $jp_\lambda : \omega gp_\lambda$ and $jq_\lambda : \omega gq_\lambda$.

4 Shifted plane partition generating functions

Our first main result is a generating function formula for gp_λ and gq_λ that was predicted in [20]. A *shifted plane partition* of strict partition shape λ is a filling of SD_λ by elements of $\{1' < 1 < 2' < 2 < \dots\}$ with weakly increasing rows and columns. Examples include

$$\begin{array}{ccccc} & & & 3 \\ & & 2 & 3 \\ & 1 & 2' & 2 \\ 1' & 1' & 1 & 1 & 5' \end{array} \quad \text{and} \quad \begin{array}{ccccc} & & & 3' \\ & & 2' & 2 \\ & 1 & 2' & 2 \\ 1 & 1 & 1 & 1 & 5 \end{array}$$

which both have shape $5, 3, 2, 1$. Given such a filling T , let c_i be the number of distinct columns of T containing i and let r_i be the number of distinct rows of T containing i' . Then define $|\text{wt}_{\text{pp}} T| = c_1 - r_1 - c_2 + r_2 - \dots$ and $\mathbf{x}^{\text{wt}_{\text{pp}} T} = \prod_{i \geq 1} x_i^{c_i - r_i}$. Both examples above have $|\text{wt}_{\text{pp}} T| = 9$ and $\mathbf{x}^{\text{wt}_{\text{pp}} T} = x_1^4 x_2^3 x_3 x_5$.

Theorem 4.1. Let $\text{ShPP}_Q \lambda$ be the set of all shifted plane partitions of shape λ , and let $\text{ShPP}_P \lambda$ be the subset of such fillings with no **unprimed** diagonal entries. Then

$$gp_\lambda = \sum_{T \in \text{ShPP}_P \lambda} -\beta^{|\lambda| - |\text{wt}_{\text{PP}} T|} \mathbf{x}^{\text{wt}_{\text{PP}} T} \quad \text{and} \quad gq_\lambda = \sum_{T \in \text{ShPP}_Q \lambda} -\beta^{|\lambda| - |\text{wt}_{\text{PP}} T|} \mathbf{x}^{\text{wt}_{\text{PP}} T}.$$

This result was conjectured by Nakagawa and Naruse as [20, Conj. 5.1]. These formulas make it possible to compute gp_λ and gq_λ , which is not straightforward from (3.1).

Example 4.2. If $\lambda = 2, 1$ then $\text{ShPP}_P \lambda$ consists of

$$\begin{array}{|c|c|} \hline & c' \\ \hline a' & b \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline & c' \\ \hline a' & b' \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline & b' \\ \hline a' & a \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline & b' \\ \hline a' & b' \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline & a' \\ \hline a' & a' \\ \hline \end{array}, \quad \text{and} \quad \begin{array}{|c|c|} \hline & b' \\ \hline a' & a' \\ \hline \end{array}$$

for all positive integers $a < b < c$, so Theorem 4.1 asserts that

$$gp_{2,1} = 2 \sum_{a < b < c} x_a x_b x_c = \sum_{a < b} x_a^2 x_b = x_a x_b^2 - \beta \sum_a x_a^2 - \beta \sum_{a < b} x_a x_b = s_{2,1} - \beta s_2.$$

When $\lambda = 2, 1$, adding primes to the diagonal is a weight-preserving 4-to-1 map $\text{ShPP}_Q \lambda \rightarrow \text{ShPP}_P \lambda$ so Theorem 4.1 also tells us that $gq_{2,1} = 4s_{2,1} - 4\beta s_2$.

5 Shifted bar tableaux generating functions

Our second main result is a generating function formula for jp_λ and jq_λ that was predicted in [2]. Continue to let λ be a strict integer partition. Suppose V is a shifted tableau¹ of shape λ with no unprimed entries repeated in any column and no primed entries repeated in any row. Let Π be a partition of the diagram SD_λ into (disjoint, nonempty) subsets of adjacent boxes containing the same entry in V . Each block of Π is a contiguous “bar” of positions in the same row or column, and we refer to the pair $T = (V, \Pi)$ as a *shifted bar tableau* of shape λ .

If V is semistandard in the sense of having weakly increasing rows and columns, then we say that T is also *semistandard*. We draw shifted bar tableaux as pictures like

$$\begin{array}{|c|c|c|} \hline 2 & & 3' \\ \hline 1 & 1 & 3 \\ \hline \end{array} \quad \text{to represent} \quad T = (V, \Pi) = \left(\begin{array}{|c|c|c|c|c|} \hline 2 & 2 & 3' & & \\ \hline 1 & 1 & 1 & 3' & 3 \\ \hline \end{array}, \begin{array}{|c|c|c|c|c|} \hline \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \end{array} \right).$$

These objects are the shifted analogues of what are called *valued-set tableaux* in [9]. The word “valued-set” is just a formal transposition of “set-valued”; we believe that the name “bar tableau” is more intuitive and descriptive.

Given a shifted bar tableau $T = (V, \Pi)$ let $|T| : |\Pi|$ and $\mathbf{x}^T : \prod_{i \geq 1} x_i^{b_i}$ where b_i is the number of blocks in Π containing i or i' . In our example, $|T| = 5$ and $\mathbf{x}^T = x_1^2 x_2 x_3^2$.

¹That is, a shifted set-valued tableau whose entries are all singleton sets.

Theorem 5.1. Let $\text{ShBT}_Q \lambda$ be the set of all semistandard shifted bar tableaux of shape λ . Let $\text{ShBT}_P \lambda$ be the subset of such tableaux with no **primed** diagonal entries. Then

$$jp_\lambda = \sum_{T \in \text{ShBT}_P \lambda} -\beta^{|\lambda| - |T|} \mathbf{x}^T \quad \text{and} \quad jq_\lambda = \sum_{T \in \text{ShBT}_Q \lambda} -\beta^{|\lambda| - |T|} \mathbf{x}^T.$$

This result was conjectured by Chiu and the second author as [2, Conj. 7.2].

Example 5.2. Suppose $\lambda = 2, 1$. Then $\text{ShBT}_P \lambda$ consists of

$$\begin{array}{|c|c|} \hline c \\ \hline a & b \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline c \\ \hline a & b' \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline b \\ \hline a & a \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline b \\ \hline a & b' \\ \hline \end{array}, \quad \text{and} \quad \begin{array}{|c|} \hline b \\ \hline a \\ \hline \end{array}$$

for all positive integers $a < b < c$, so Theorem 5.1 asserts that

$$jp_{21} = 2 \sum_{a < b < c} x_a x_b x_c - \sum_{a < b} x_a^2 x_b - x_a x_b^2 - \beta \sum_{a < b} x_a x_b = s_{21} - \beta s_{11} = \omega gp_{21}.$$

As in Example 4.2, there is a weight-preserving 4-to-1 map $\text{ShBT}_Q \lambda \rightarrow \text{ShBT}_P \lambda$; this is given by either removing all diagonal primes or applying the map

$$\left\{ \begin{array}{|c|c|} \hline b \\ \hline a & b \\ \hline \end{array}, \begin{array}{|c|c|} \hline b' \\ \hline a & b' \\ \hline \end{array}, \begin{array}{|c|c|} \hline b' \\ \hline a & b \\ \hline \end{array}, \begin{array}{|c|c|} \hline b' \\ \hline a' & b' \\ \hline \end{array} \right\} \mapsto \left\{ \begin{array}{|c|} \hline b \\ \hline a \\ \hline \end{array} \right\}.$$

Thus Theorem 5.1 also tells us that $jq_{21} = 4s_{21} - 4\beta s_{11} = \omega gq_{21}$.

6 Application to conjectures of Ikeda and Naruse

Consider the modules consisting of all infinite $\mathbb{Z} \beta$ -linear combinations of the functions $\{GP_\lambda\}$ and $\{GQ_\lambda\}$, with λ ranging over all strict partitions. Ikeda and Naruse proved that these modules are both rings [8, Props. 3.4 and 3.5]. This means concretely that $GP_\lambda GP_\mu$ (respectively, $GQ_\lambda GQ_\mu$) always expands as a (possibly infinite) $\mathbb{Z} \beta$ -linear combination of GP_ν 's (respectively, GQ_ν 's).

Ikeda and Naruse conjectured that these expansions are actually finite [8, Conj. 3.1 and 3.2], meaning that the **finite** linear spans

$$\mathbf{GP} : \mathbb{Z} \beta\text{-span}\{GP_\lambda\} \quad \text{and} \quad \mathbf{GQ} : \mathbb{Z} \beta\text{-span}\{GQ_\lambda\} \quad (6.1)$$

(with λ ranging over all strict partition) are also rings. The fact that $\mathbf{GP}$ is a ring follows from results of Clifford, Thomas, and Yong in [3]; for other proofs see [7, §4], [15, §1.2], and [22, §8]. The problem of showing $\mathbf{GQ}$ is a ring appears to still be open, however.

Building off [2], we are able to resolve this problem. Specifically, [2, Cor. 7.5] is exactly the assertion that the ring property for the GQ -functions follows from Theorem 5.1 (or more precisely from its skew version, to be given as Theorem 8.4). It follows that:

Theorem 6.1. The $\mathbb{Z} \beta$ -module $\mathbf{GQ}$ is a subring of $\mathbf{GP}$. Thus, each product $GQ_\lambda GQ_\mu$ is a **finite** $\mathbb{Z} \beta$ -linear combination of GQ_ν 's.

Our proof of this theorem could be adapted to give another proof that $\mathbf{GP}$ is a ring, but in either case our arguments are nonconstructive. By contrast, [3, Thm. 1.2] gives an explicit Littlewood–Richardson rule for products of GP -functions. It is an open problem to find such a rule for the GQ -functions, as well as for the gp - and gq -functions, which span two other subrings of symmetric functions.

7 Comparison with unshifted versions

Our main results are shifted analogues of “classical” theorems, which we summarize here for comparison.

Throughout this section, let $\lambda = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$ be an arbitrary integer partition, which is not necessarily strict. The (*unshifted*) *diagram* of λ is the set of pairs $D_\lambda : \{i, j \in k \times \mathbb{Z} : 1 \leq j \leq \lambda_i\}$. A *semistandard (unshifted) set-valued tableau* of shape λ is defined in the same way as the analogous shifted object, except that such a tableau is a filling of D_λ by finite nonempty subsets of $\{1 < 2 < 3 < \cdots\}$.

Definition 7.1. Let $\text{SVT } \lambda$ be the set of semistandard set-valued tableaux of shape λ . The *stable Grothendieck polynomial* of λ is the power series $G_\lambda : \sum_{T \in \text{SVT } \lambda} \beta^{|T| - |\lambda|} \mathbf{x}^T$.

Definition 7.2. The *dual stable Grothendieck polynomials* g_λ are the unique formal power series in $\mathbb{Z} \beta \llbracket x_1, x_2, \dots \rrbracket$ indexed by integer partitions λ satisfying the Cauchy identity

$$\sum_{\lambda} G_{\lambda}(\mathbf{x}) g_{\lambda}(\mathbf{y}) = \sum_{\lambda} s_{\lambda}(\mathbf{x}) s_{\lambda}(\mathbf{y}) = \prod_{i,j \geq 1} \frac{1}{1 - x_i y_j}.$$

Definition 7.3. The *conjugate dual stable Grothendieck polynomial* of λ is $j_\lambda : \omega g_{\lambda^\top}$.

All three families $\{G_\lambda\}$, $\{g_\lambda\}$, and $\{j_\lambda\}$ are linearly independent symmetric functions which coincide with the usual Schur functions $\{s_\lambda\}$ when $\beta = 0$ [1, 9]. Our definition of j_λ involves a transposition of indices compared to [9, §9.8]; this ensures that $j_\lambda|_{\beta=0} = s_\lambda$.

We define *plane partitions* and *semistandard bar tableaux* of shape λ in the same way as our shifted versions, except the relevant objects are fillings of D_λ by positive integers. Let $\text{PP } \lambda$ be the set of plane partitions of shape λ . Let $\text{BT } \lambda$ be the set of semistandard bar tableaux of shape λ ; these objects are called *valued-set tableaux* in [9, §9].

Theorem 7.4 (Lam and Pylyavskyy [9, §9]). For all partitions λ it holds that

$$g_\lambda = \sum_{T \in \text{PP } \lambda} -\beta^{|\lambda| - |\text{wt}_{\text{PP } T}|} \mathbf{x}^{\text{wt}_{\text{PP } T}} \quad \text{and} \quad j_\lambda = \sum_{T \in \text{BT } \lambda} -\beta^{|\lambda| - |T|} \mathbf{x}^T.$$

The stable Grothendieck polynomials G_λ were introduced in Fomin and Kirillov's paper [4] as certain limits of Lascoux and Schützenberger's *Grothendieck polynomials* [11], which are K -theory representatives for Schubert varieties. Buch [1, Thm. 3.1] derived the set-valued tableaux generating function for G_λ given in Definition 7.1, and also proved that the stable Grothendieck polynomials are a $\mathbb{Z} \beta$ -basis for a ring [1, Cor. 5.5]. For another proof of this ring property, see [23].

Lam and Pylyavskyy defined g_λ and j_λ by the formulas in Theorem 7.4 with $\beta = -1$. They then proved that $\{g_\lambda\}$ is the basis for the ring of symmetric functions dual to $\{G_\lambda\}$ via the Hall inner product [9, Thm. 9.15], and showed that $j_\lambda = \omega g_{\lambda^\top}$ [9, Prop. 9.25]. For other proofs of the Cauchy identity in Definition 7.2, see [6, 10, 23].

8 Skew versions

As mentioned in Section 6, there are skew generalizations of Theorems 4.1 and 5.1. We write $\mu \subseteq \lambda$ if μ and λ are partitions with $\mu_i \leq \lambda_i$ for all i . If $\mu \subseteq \lambda$ are strict partitions then the *shifted diagram* of λ/μ is the set difference $SD_{\lambda/\mu} : SD_\lambda - SD_\mu$.

Assume $\mu \subseteq \lambda$ are strict partitions. We define shifted set-valued tableaux, plane partitions, and bar tableaux of skew shape λ/μ in exactly the same way as above, only now the relevant objects are fillings of $SD_{\lambda/\mu}$. The definitions of all related weight statistics like the monomials $\mathbf{x}^T$ are also unchanged. Here is some more relevant notation:

- Let $\text{ShPP}_Q \lambda/\mu$ be the set of shifted plane partitions of shape λ/μ , and define $\text{ShPP}_P \lambda/\mu$ to be the subset of such fillings with no unprimed diagonal entries.
- Let $\text{ShBT}_Q \lambda/\mu$ be the set of semistandard shifted bar tableaux of shape λ/μ , and let $\text{ShBT}_P \lambda/\mu$ be the subset of such tableaux with no primed diagonal entries.

We define these sets to be empty if $\mu \not\subseteq \lambda$. These sets will index the terms in generating functions for the skew analogues of gq_λ , gp_λ , jq_λ , and jp_λ , which we now define.

Let $y_1, y_2, \dots$ be another countable set of commuting variables, and for each $f \in \mathbb{Z} \beta \llbracket x_1, x_2, \dots \rrbracket$ define

$$f \mathbf{x} : f x_1, x_2, \dots \quad f, f \mathbf{y} : f y_1, y_2, \dots, \text{ and } f \mathbf{x}, \mathbf{y} : f x_1, y_1, x_2, y_2, \dots$$

If f is symmetric, then we think of $f \mathbf{x}, \mathbf{y}$ as " $f x_1, x_2, \dots, y_1, y_2, \dots$ " although it is not clear how to interpret this notation for an arbitrary power series.

Definition 8.1. For strict partitions λ and μ , let $gp_{\lambda/\mu}$ and $gq_{\lambda/\mu}$ be the elements of $\mathbb{Z} \beta \llbracket x_1, x_2, \dots \rrbracket$ with $gp_\lambda \mathbf{x}, \mathbf{y} = \sum_\mu gp_\mu \mathbf{x} gp_{\lambda/\mu} \mathbf{y}$ and $gq_\lambda \mathbf{x}, \mathbf{y} = \sum_\mu gq_\mu \mathbf{x} gq_{\lambda/\mu} \mathbf{y}$, where the sums are over all strict partitions μ .

Both $gp_{\lambda/\mu}$ and $gq_{\lambda/\mu}$ are symmetric and homogeneous if $\deg \beta = 1$, but the families of such functions are no longer linearly independent. These power series are defined for all strict λ and μ but are nonzero if and only if $\mu \subseteq \lambda$ [2, Props. 6.4 and 6.6].

Definition 8.2. For strict partitions λ and μ , let $jp_{\lambda/\mu}$ and $jq_{\lambda/\mu}$ be the elements of $\mathbb{Z} \beta \llbracket x_1, x_2, \dots \rrbracket$ with $jp_{\lambda} \mathbf{x}, \mathbf{y} = \sum_{\mu} jp_{\mu} \mathbf{x} jp_{\lambda/\mu} \mathbf{y}$ and $jq_{\lambda} \mathbf{x}, \mathbf{y} = \sum_{\mu} jq_{\mu} \mathbf{x} jq_{\lambda/\mu} \mathbf{y}$, where the sums are over all strict partitions μ .

Equivalently, one could define these skew analogues by the following formula:

Proposition 8.3 ([2, Eq. (7.4)]). It holds that $jp_{\lambda/\mu} : \omega gp_{\lambda/\mu}$ and $jq_{\lambda/\mu} : \omega gq_{\lambda/\mu}$.

Our main theorems extend to this skew setting as follows.

Theorem 8.4. For all strict partitions λ and μ , one has

$$\begin{aligned} gp_{\lambda/\mu} &= \sum_{T \in \text{ShPP}_P \lambda/\mu} -\beta^{|\lambda/\mu| - |\text{wt}_{\text{PP}} T|} \mathbf{x}^{\text{wt}_{\text{PP}} T}, & jp_{\lambda/\mu} &= \sum_{T \in \text{ShBT}_P \lambda/\mu} -\beta^{|\lambda/\mu| - |T|} \mathbf{x}^T, \\ gq_{\lambda/\mu} &= \sum_{T \in \text{ShPP}_Q \lambda/\mu} -\beta^{|\lambda/\mu| - |\text{wt}_{\text{PP}} T|} \mathbf{x}^{\text{wt}_{\text{PP}} T}, & jq_{\lambda/\mu} &= \sum_{T \in \text{ShBT}_Q \lambda/\mu} -\beta^{|\lambda/\mu| - |T|} \mathbf{x}^T. \end{aligned}$$

There are also skew versions of GP_{λ} and GQ_{λ} , which are symmetric by results in [12].

9 Proof ideas

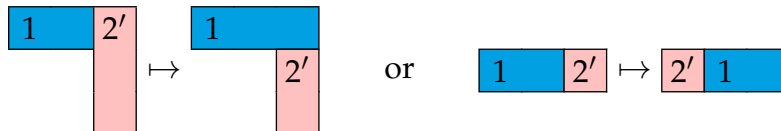
The most difficult part of our proofs of [Theorems 4.1](#), [5.1](#) and [8.4](#) is showing that the desired generating functions are symmetric. We derive this explicitly for jp_{λ} and jq_{λ} by constructing Bender–Knuth involutions for semistandard shifted bar tableaux. We briefly sketch this construction, which may be of independent interest.

The i th Bender–Knuth involution applied to a semistandard bar tableau T produces a new semistandard bar tableau U in which the number of bars filled with i or i' in T is equal to the number of bars filled with $i - 1$ or $i - 1'$ in U , and vice versa. It is enough to consider the case $i = 1$. The involution proceeds in three steps, using an intermediate object called *sorted bar tableaux*.

Essentially, these intermediate objects appear to be semistandard bar tableaux if the order on the entries were $1' < 2' < 1 < 2$ (rather than $1' < 1 < 2' < 2$); however, some special configurations that violate this description may occur near the diagonal. The first step, converting a semistandard shifted bar tableau to a sorted shifted bar tableau, consists of a number of moves of the form

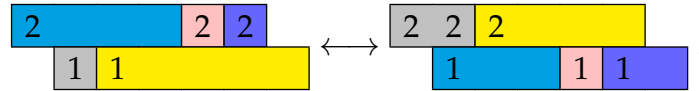


and

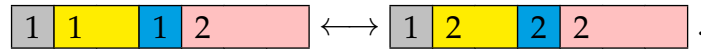


in an appropriate order.

In the second stage, one performs a weight-reversing involution on sorted tableaux. This map is straightforward to describe away from the diagonal. For example, for a collection of bars filled with 1s and 2s forms a configuration like the ones below, we form another two-row-group occupying the same boxes that has a bar division between columns j and $j+1$ in row l (respectively, $l+1$) if and only if a bar division occurs between columns j and $j+1$ in row $l+1$ (respectively, l) in the starting group:



If, instead, a collection of bars filled with 1s and 2s forms a configuration like the ones below, then we form another one-row-group by swapping the number of 1-blocks and 2-blocks (but leaving the bar divisions untouched):



One applies similar moves to the column bars filled with $1'$ and $2'$. These maps obviously reverse the weight, but it is necessary to consider numerous special cases that occur near the diagonal.

Finally, in the third step, one reverses the process from the first step to produce a semistandard shifted bar tableau, whose weight is reversed from the original tableau.

Once symmetry is established, we are able to prove [Theorem 8.4](#) (and so in particular [Theorems 4.1](#) and [5.1](#)) by an inductive algebraic argument. The first step in this argument is to verify [Theorem 8.4](#) when $\lambda = r$ has a single nonzero part. This is carried out in [\[2, §7\]](#) by a direct computation.

For the inductive step, it turns out to be sufficient to show that both sides of the identities in [Theorem 5.1](#) satisfy the same Pieri rule when multiplied by jq_r . The Cauchy identity [\(3.1\)](#) lets us derive Pieri rules expanding $jp_\lambda jq_r$ and $jq_\lambda jq_r$ from the tableau generating functions for GP_λ and GQ_λ . Our proof that the same rules apply to the desired combinatorial formulas for jp_λ and jq_λ relies on the symmetry of those generating functions: we restrict to a finite number of variables, and interpret a single variable x as recording on one hand the smallest number appearing in the tableaux (corresponding to an inner shape and a skewing operation) but also (on the other hand, by symmetry) as recording the largest number appearing in the tableau (corresponding to a border strip on the outer boundary).

10 Hopf algebras

The power series GP_λ , GQ_λ , $gp_{\lambda/\mu}$, $gq_{\lambda/\mu}$, $jp_{\lambda/\mu}$ and $jq_{\lambda/\mu}$ arise when studying certain Hopf algebras of symmetric functions. We discuss these objects here for more context.

Let R be a commutative ring. Write $\otimes = \otimes_R$ for the corresponding tensor product. We have a familiar notion of an (associative, unital) R -algebra: this is an R -module A with an R -linear map $\nabla : A \otimes A \rightarrow A$ (called the *product*) and another R -linear map $\iota : R \rightarrow A$ (called the *unit*) satisfying a few compatibility axioms.

Dually, an R -coalgebra is an R -module with an R -linear map $\Delta : A \rightarrow A \otimes A$, (called the *coproduct*) and a linear map $\varepsilon : A \rightarrow R$ (called the *counit*) subject to a similar set of axioms; see [5, §1] for the complete definitions. An R -algebra that is also a coalgebra is an R -bialgebra if the coproduct and counit maps are algebra homomorphisms.

If A is an R -bialgebra then the set $\text{End } A$ of R -linear maps $f : A \rightarrow A$ is itself an R -algebra for the product $f * g : \nabla \circ f \otimes g \circ \Delta$. The unit element of this *convolution algebra* is the composition $\iota \circ \varepsilon$. The bialgebra A is a *Hopf algebra* if the identity map $A \rightarrow A$ has a two-sided inverse $S : A \rightarrow A$ (called the *antipode*) in $\text{End } A$ relative to $*$.

We specialize to the case when $R = \mathbb{Z}[\beta]$. The objects of interest to us are bialgebras of formal power series in $\mathbb{Z}[\beta][x_1, x_2, \dots]$. In this setting, the counit ε will always be the map setting $x_1 = x_2 = \dots = 0$, and the product and unit will always be the ones inherited from $\mathbb{Z}[\beta][x_1, x_2, \dots]$. The only maps left to specify are Δ and S (if it exists).

For example, let $\text{Sym} = \mathbb{Z}[\beta]\text{-span}\{s_\lambda : \lambda \text{ is any partition}\}$ be the $\mathbb{Z}[\beta]$ -algebra of bounded-degree symmetric functions. Write $c_{\lambda\mu}^\nu \in \mathbb{Z}_{\geq 0}$ for the Littlewood–Richardson coefficients such that $s_\lambda s_\mu = \sum_\nu c_{\lambda\mu}^\nu s_\nu$. It is well-known [5, §2] that Sym is a Hopf algebra for the coproduct Δ and antipode S satisfying

$$\Delta s_\nu = \sum_{\lambda, \mu \subseteq \nu} c_{\lambda\mu}^\nu s_\lambda \otimes s_\mu \quad \text{and} \quad S s_\nu = (-1)^{|\nu|} s_{\nu^\top} \quad s_{\nu^\top} = -x, \quad (10.1)$$

where in the sum λ and μ range over all partitions. It also holds that $s_\nu = x, y \sum_{\lambda, \mu} c_{\lambda\mu}^\nu s_\lambda \otimes s_\mu$, so the map Δ is the composition of $f \mapsto f(x, y)$ with the isomorphism $\text{Sym}(x) \otimes \text{Sym}(y) \xrightarrow{\sim} \text{Sym}$.

Define $\mathbf{gp} = \mathbb{Z}[\beta]\text{-span}\{gp_\lambda\}$ and $\mathbf{gq} = \mathbb{Z}[\beta]\text{-span}\{gq_\lambda\}$ where λ ranges over all strict partitions. These are both submodules of Sym , with $\mathbf{gq} \subseteq \mathbf{gp}$ by results in [2]. The skew and conjugate versions of gp_λ and gq_λ are motivated algebraically by the following:

Theorem 10.1. Both $\mathbf{gp}$ and $\mathbf{gq}$ are Hopf subalgebras of Sym . It holds that

$$\Delta gp_\nu = \sum_{\lambda \subseteq \nu} gp_\lambda \otimes gp_{\nu/\lambda} \quad \text{and} \quad \Delta gq_\nu = \sum_{\lambda \subseteq \nu} gq_\lambda \otimes gq_{\nu/\lambda},$$

where the sums are over strict partitions. Also, $S gp_\nu = jp_\nu = -x$ and $S gq_\nu = jq_\nu = -x$.

Recall the definitions of $\mathbf{GP} \subseteq \mathbf{GQ}$ from (6.1). Although GP_λ and GQ_λ are infinite $\mathbb{Z}[\beta]$ -linear combinations of Schur functions, the formula for Δ in (10.1) extends by “continuous linearity” to a well-defined map $\mathbf{GP} \rightarrow \mathbf{GP} \otimes \mathbf{GP}$, and the following holds:

Theorem 10.2. Both $\mathbf{GP}$ and $\mathbf{GQ}$ are bialgebras, but neither is a Hopf algebra.

Finally, let $\langle \cdot, \cdot \rangle : \text{Sym} \otimes \text{Sym} \rightarrow \mathbb{Z} \beta$ be the $\mathbb{Z} \beta$ -linear form with $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$ for all partitions λ and μ . This extends to a bilinear form on $\text{Sym} \otimes \text{Sym}$ with $\langle f_1 \otimes f_2, g_1 \otimes g_2 \rangle : \langle f_1, g_1 \rangle \langle f_2, g_2 \rangle$. The formulas (10.1) imply that Sym is self-dual with respect to this form, in the sense that $\langle fg, h \rangle = \langle f \otimes g, \Delta h \rangle$ for all $f, g, h \in \text{Sym}$. Analogously:

Theorem 10.3. There is a unique $\mathbb{Z} \beta$ -bilinear form $\langle \cdot, \cdot \rangle : \mathbf{gp} \otimes \mathbf{GP} \rightarrow \mathbb{Q} \beta$ such that

$$gp_\lambda, GQ_\mu = gq_\lambda, GP_\mu = \delta_{\lambda\mu}, \quad \langle fg, H \rangle = \langle f \otimes g, \Delta H \rangle, \quad \text{and} \quad \langle f, HK \rangle = \langle \Delta f, H \otimes K \rangle$$

for all strict partitions λ, μ and all elements $f, g \in \mathbf{gp}$ and $H, K \in \mathbf{GP}$.

Despite this result, $\mathbf{GP}$ and $\mathbf{GQ}$ are not the duals of $\mathbf{gq}$ and $\mathbf{gp}$, because the restricted forms $\langle \cdot, \cdot \rangle : \mathbf{gq} \otimes \mathbf{GP} \rightarrow \mathbb{Z} \beta$ and $\langle \cdot, \cdot \rangle : \mathbf{gp} \otimes \mathbf{GQ} \rightarrow \mathbb{Z} \beta$ are technically degenerate. Instead, the duals of $\mathbf{gq}$ and $\mathbf{gp}$ are the “completions” $\widehat{\mathbf{GP}}$ and $\widehat{\mathbf{GQ}}$, which respectively consist of all infinite $\mathbb{Z} \beta$ -linear combinations of GP - and GQ -functions.

These objects are too large to be Hopf algebras (at least, in the category of $\mathbb{Z} \beta$ -modules), but are examples of what are called *linearly compact Hopf algebras* in [12, 14, 16]. The reason why $\mathbf{GP}$ and $\mathbf{GQ}$ are not Hopf algebras is that the adjoints of the antipodes of $\mathbf{gq}$ and $\mathbf{gp}$ under $\langle \cdot, \cdot \rangle$ only make sense as maps $\widehat{\mathbf{GP}} \rightarrow \widehat{\mathbf{GP}}$ and $\widehat{\mathbf{GQ}} \rightarrow \widehat{\mathbf{GQ}}$.

There are still many open questions related to the symmetric functions in this abstract. For example, Theorems 10.1 and 10.2 imply that products of gp -/ gq -/ GP -/ GQ -functions respectively expand as finite $\mathbb{Z} \beta$ -linear combinations of gp -/ gq -/ GP -/ GQ -functions. It is expected that the coefficients in these expansions are all actually in $\mathbb{Z}_{\geq 0} \beta \subsetneq \mathbb{Z} \beta$. At present, however, this is conjectural outside the case of GP -functions; in the GP case, combinatorial interpretations for the coefficients expanding $GP_\lambda GP_\mu$ in $\bigoplus_v \mathbb{Z}_{\geq 0} \beta GP_v$ appear in [3, 7, 22]. For a list of related open problems, see [16, §4.6].

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Tautological classes of matroids

Andrew Berget¹, Christopher Eur^{*2}, Hunter Spink³, and Dennis Tseng^{†4}

¹Western Washington University, Bellingham, WA, USA

²Harvard University, Cambridge, MA, USA

³Stanford University, Stanford, CA, USA

⁴New York, NY, USA

Abstract. We introduce certain classes on permutohedral varieties which we call "tautological classes of matroids" as a new framework for studying matroids. Using this framework, we unify and extend many recent developments in matroid theory arising from its interaction with algebraic geometry. We achieve this by establishing a cohomological description and a log-concavity property for a 4-variable transformation of the Tutte polynomial, and by establishing an exceptional Hirzebruch-Riemann-Roch formula for permutohedral varieties that bridges K-theory and cohomology theory.

Keywords: Matroids, log-concavity, toric geometry

1 Introduction

Matroid theory has interacted fruitfully with algebraic geometry. See [3] for a survey of recent developments. These developments however had a constraint: They occurred through different geometric models of matroids that were *disjoint*—in the sense that the results and techniques specific to one model could not be easily imported to another.

We introduce “tautological classes of matroids” as a unified framework that overcomes this constraint. The key advantage of this unified approach is that powerful techniques that were previously applicable only within different models can now be used seamlessly in conjunction with each other. This advantage allowed us to recover and extend various recent results in matroid theory, and to resolve open problems. For instance, we establish the following log-concavity for the Tutte polynomial of a matroid.

Theorem 3.3.(iii). Let M be a matroid of rank r on a ground set E , and let $T_M(x, y)$ be its Tutte polynomial. Then, the coefficients of the 4-variable transformation

$$x^{-1} y^{-1} y^{-r} x^{-r} w^{|E|-r} T_M\left(\frac{x}{y}, \frac{y}{z}, \frac{x}{w}, \frac{y}{w}\right)$$

form a “log-concave unbroken array” (see Definition 3.2).

^{*}ceur@math.harvard.edu. C. Eur was partially supported by NSF DMS-2001854.

[†]D. Tseng was partially supported by NSF DMS-2001712.

Our framework also opens new doors: For instance, it has already led to the development of the K -theory of matroids [29], the stellahedral geometry of matroids [22], the Gromov-Witten theory of matroids [35], and the tropical geometry of “type B” generalizations of matroids known as delta-matroids [21].

In this extended abstract of our paper [8], in order to emphasize the combinatorial aspects, we present the framework of “tautological classes of matroids” slightly differently from the presentation in [8]. For example, not emphasized in the presentation here is the method of localization in torus-equivariant geometry, even though it is among crucial tools in our proofs. We assume familiarity with the fundamentals of matroids theory, and point to [34, 37] as standard references. We also assume some familiarity with polyhedral geometry, and point to [38] as a reference.

In Section 2, we review in a purely combinatorial manner some previous studies of matroids via algebraic geometry. In Section 3, we construct the “tautological classes” of a matroid, and explain our main results arising from the framework. In Section 4, we explain the underlying algebraic geometry.

Notations. Let $E = \{1, \dots, n\}$ be a finite set of cardinality n , and let $\mathfrak{S}_E$ the permutation group on E . For a subset $S \subseteq E$, we denote by $\mathbf{e}_S = \sum_{i \in S} \mathbf{e}_i \in \mathbb{R}^E$ the sum of the standard basis vectors indexed by S . Denote by $\langle \cdot, \cdot \rangle$ the standard inner-product on $\mathbb{R}^E$. Let $T = \mathbb{C}^*{}^E = \{x_1, \dots, x_n \in \mathbb{C}^E : x_1 \cdots x_n \neq 0\}$ be the standard torus, and let $\mathbb{P}T = T/\mathbb{C}^*$ be the projectivization of T . All algebraic varieties are over $\mathbb{C}$.

2 Background

Many previous studies of matroids via algebraic geometry fall broadly into one of two camps: the “ K -theoretic” camp and the “cohomological” camp. More precisely, these correspond to two well-studied geometric rings attached to a smooth projective variety X , namely, the Grothendieck K -ring $K(X)$ of vector bundles on X , and the singular cohomology ring $H^\bullet(X)$ of X .

When X is a smooth projective *toric* variety, these two rings admit purely combinatorial descriptions, reviewed in Section 2.1. Specializing to the case of the permutohedral varieties, we highlight in Section 2.2 some previous studies of matroids via these rings.

2.1 A tale of two rings

Let Σ be the normal fan of a simple lattice polytope in $\mathbb{R}^E$ that is unimodular with respect to the lattice $\mathbb{Z}^E$. A *deformation* of Σ is a lattice polytope $P \subset \mathbb{R}^E$ such that the normal fan of P coarsens Σ . We write $P \in \text{Def } \Sigma$ in this case. Let X_Σ be the smooth projective toric variety associated to Σ . In this case, the two rings $K(X_\Sigma)$ and $H^\bullet(X_\Sigma)$ have purely combinatorial descriptions as follows.

For a subset $P \subseteq \mathbb{R}^E$, define the function $\mathbf{1}_P : \mathbb{R}^E \rightarrow \mathbb{Z}$ by $\mathbf{1}_P(x) = 1$ if $x \in P$ and $\mathbf{1}_P(x) = 0$ if otherwise. Let

$$\mathbb{I}(\Sigma) \text{ the subgroup of } \mathbb{Z}^{\mathbb{R}^E} \text{ generated by } \{\mathbf{1}_P : P \in \text{Def } \Sigma\}.$$

Let $\bar{\mathbb{I}}(\Sigma)$ be the quotient of $\mathbb{I}(\Sigma)$ by the subgroup generated by $\{\mathbf{1}_P - \mathbf{1}_{P-v} : P \in \text{Def } \Sigma \text{ and } v \in \mathbb{Z}^E\}$. This group is closely related to McMullen's polytope algebra [31], and has a ring structure with multiplication determined by $\mathbf{1}_P \cdot \mathbf{1}_{P'} = \mathbf{1}_{P+P'}$, where $P+P'$ denotes the Minkowski sum of polytopes. It has the unit $1 = \mathbf{1}_{\text{the origin}}$. See [22, Appendix A] for details.

Theorem 2.1. [22, Theorem A.9] (cf. [33]) The ring $K[X_\Sigma]$ is isomorphic to $\bar{\mathbb{I}}(\Sigma)$.

For $P \in \text{Def } \Sigma$, let $\bar{P}$ denote the class of $\mathbf{1}_P$ in $\bar{\mathbb{I}}(\Sigma) \simeq K[X_\Sigma]$. One can show that $1 - \bar{P}$ is nilpotent in $\bar{\mathbb{I}}(\Sigma)$, and hence $\bar{P}$ admits an inverse. That is, we may consider $\bar{P}^{-1} = \frac{1}{1 - \bar{P}} = 1 + \bar{P} + \bar{P}^2 + \dots$.

On the other hand, let $PP^\bullet(\Sigma)$ be the ring of piecewise polynomial functions in the n -variables $t_1, \dots, t_n$ with integral coefficients. That is, on each cone of Σ we have a polynomial in $\mathbb{Z}[t_1, \dots, t_n]$ such that at an intersection of any two cones, the polynomials agree. It is a graded ring with the i -th graded piece $PP^i(\Sigma)$ consisting of homogeneous degree i piecewise polynomial functions. Let $\overline{PP}^\bullet(\Sigma)$ be the quotient of $PP^\bullet(\Sigma)$ by the ideal generated by the global polynomials (i.e. a piecewise polynomial that is the same polynomial on all cones).

Theorem 2.2. [18, 13] The ring $H^\bullet(X_\Sigma)$ is isomorphic to $\overline{PP}^\bullet(\Sigma)$ as graded rings.¹

For $P \in \text{Def } \Sigma$, let h_P denote the class in $\overline{PP}^1(\Sigma)$ of the piecewise polynomial (in fact, piecewise linear) function $h_P \in PP^1(\Sigma)$ defined by $h_P(x) = \min_{y \in P} \langle x, y \rangle$.

The two rings $K[X_\Sigma]$ and $H^\bullet(X_\Sigma)$ have two natural geometric maps to $\mathbb{Z}$, namely, the sheaf Euler characteristic map $\chi : K[X_\Sigma] \rightarrow \mathbb{Z}$, and the degree map $\deg_\Sigma : H^\bullet(X_\Sigma) \rightarrow \mathbb{Z}$. Under the isomorphisms in Theorems 2.1 and 2.2, these two maps have the following combinatorial characterizations [24]: For any $P \in \text{Def } \Sigma$, the maps χ and $\deg_\Sigma$ satisfy

$$\chi(\bar{P}) = |P \cap \mathbb{Z}^E| \text{ and } \deg_\Sigma(h_P) = \text{volume } P.$$

Volumes are normalized such that a unit n -dimensional simplex in $\mathbb{R}^E$ has volume 1.

¹The isomorphism doubles the grading, in the sense that $H^{2i}(X_\Sigma) \simeq \overline{PP}^i(\Sigma)$ for all i .

2.2 The permutohedral fan and matroids

We now specialize to the *permutohedral fan* Σ_E , the normal fan of the *permutohedron*

$$\Pi_E = \text{convex hull of } \{w \cdot 1, \dots, n : w \in \mathfrak{S}_E\} \subset \mathbb{R}^E.$$

In this case, a lattice polytope $P \subset \mathbb{R}^E$ is a deformation of Σ_E if and only if each edge of P is parallel to $\mathbf{e}_i - \mathbf{e}_j$ for some $i \neq j \in E$. Consequently, we have $-P \in \text{Def } \Sigma$ if and only if $P \in \text{Def } \Sigma$. (Note that $-P \neq -P$). Two notable deformations are

$$\Delta = \text{convex hull of } \{\mathbf{e}_i : i \in E\} \quad \text{and} \quad \nabla = \text{convex hull of } \{-\mathbf{e}_i : i \in E\}.$$

The ring $\overline{PP}^\bullet \Sigma_E$ has the following explicit presentation.

$$\overline{PP}^\bullet \Sigma_E = \frac{\mathbb{Z} x_S : S \text{ a nonempty proper subset of } E}{\langle x_S x_{S'} : S \not\supseteq S' \text{ and } S \not\supseteq S' \rangle \quad \langle \sum_{S \ni i} x_S - \sum_{S' \ni j} x_{S'} : i \neq j \in E \rangle}.$$

We now review some studies of matroids via the rings $\overline{\mathbb{I}} \Sigma$ and $\overline{PP}^\bullet \Sigma$. We highlight only a small portion of such studies, focusing the Tutte polynomial [36, 17], one of the most famous invariant of matroids. A more extensive review can be found in [8, §1.1] or [3]. Throughout, let M be a matroid of rank r on the ground set E .

Model I: Base polytopes

On the “K-theoretic” camp, the classical result of [26] showed that the *base polytope*

$$P_M = \text{convex hull of } \{\mathbf{e}_B : B \subseteq E \text{ a basis of } M\} \subset \mathbb{R}^E$$

is a deformation of Σ_E . Hence, we may consider P_M in the ring $\overline{\mathbb{I}} \Sigma$. We highlight the following “K-theoretic” manifestation of the Tutte polynomial of a matroid. To state it, note that $\chi(P_M - u\Delta - t\nabla)$, the number of lattice points in $P_M - u\Delta - t\nabla$ for u and t nonnegative integers, is a polynomial in u and t by standard Ehrhart theory.

Theorem 2.3. [14, 10] Let $\Psi : \mathbb{Q}[t, u] \rightarrow \mathbb{Q}[x, y]$ be defined as the invertible linear map sending $\binom{t}{i} \binom{u}{j} \mapsto x^i y^j$ for all $i, j \geq 0$. Let $T_M(x, y)$ denote the Tutte polynomial of the matroid M . Then, we have

$$\Psi\left(\chi(P_M - u\Delta - t\nabla)\right) = \sum_{x, y} \binom{-1}{y} \binom{r}{x} x^{1-|E|} T_M\left(\frac{x}{y} \frac{y}{1}, \frac{x}{x} \frac{y}{1}\right).$$

For another “K-theoretic” manifestation of the Tutte polynomial that also involves lattice point counting, see [23]. A powerful technique in the study of matroids in this “K-theoretic” way is that one can exploit the special polytopal properties of P_M , such as the Hopf monoid structure [2] and valuativity [5, 7].

Model II: Bergman classes

On the “cohomological” camp, a matroid M defines a class $\Sigma_M \in \overline{PP}^{n-r} \Sigma_E$, called the *Bergman class* of M , defined as follows. If M has loops, we set $\Sigma_M = 0$. Suppose now M is loopless. One shows via the theory of Minkowski weights that the assignment

$$x_{S_1} \cdots x_{S_{r-1}} \mapsto \begin{cases} 1 & \text{if } S_1 \subsetneq \cdots \subsetneq S_{r-1} \text{ is a chain of nonempty proper flats of } M \\ 0 & \text{otherwise} \end{cases}$$

gives a well-defined element $w_M \in \text{Hom } \overline{PP}^{r-1} \Sigma_E, \mathbb{Z}$ [25, 6]. The dual element w_M then defines the class $\Sigma_M \in \overline{PP}^{n-r} \Sigma_E$, since as a cohomology ring, $\overline{PP}^\bullet \Sigma_E$ is equipped with the Poincaré duality perfect pairing $\overline{PP}^{n-r} \Sigma_E \times \overline{PP}^{r-1} \Sigma_E \rightarrow \mathbb{Z}$ defined by $a, b \mapsto \deg_\Sigma ab$.

We highlight the following “cohomological” manifestation of a specialization of the Tutte polynomial known as the *characteristic polynomial* of a matroid.

Theorem 2.4. [28] Let M be a loopless matroid (of rank r), and $T_M(x, y)$ the Tutte polynomial of M . Then, we have

$$\sum_{i=0}^{r-1} \left(\deg_\Sigma \Sigma_M h_\Delta^{r-i} h_\nabla^i \right) q^{r-1-i} = T_M(q-1, 0) / (q-1).$$

For other “cohomological” manifestations of the characteristic polynomial that also involves the degree map, see [32] and [4]. A powerful technique in the study of matroids in this “cohomological” way is the major breakthrough known as the “Hodge theory of matroids,” paraphrased in the following theorem.

Theorem 2.5. [1, 4] Rings derived from Bergman classes of matroids satisfy the Hodge-Riemann relations. Consequently, the values of $\deg_\Sigma$ involving the Bergman class Σ_M display a log-concavity behavior.

The authors of [1, 4] combined this Hodge theory of matroids with formulas like Theorem 2.4 to resolve several log-concavity conjectures in matroid theory. For a history and a survey of this development see [20].

Question 2.6. Having reviewed these previous geometric studies of matroids, either through base polytopes in K -theory, or through Bergman classes in cohomology theory, one may naturally ask the following questions:

- The formulas in Theorems 2.3 and 2.4 have a remarkable resemblance to one another. How are they actually related? More generally, how do various manifestations of the Tutte polynomial or its specializations in the theorems and in [23, 32, 4] relate to each other?

- Is there a “cohomological” manifestation of the whole Tutte polynomial itself, rather than its specialization? Moreover, is there a log-concavity behavior for the whole Tutte polynomial?

We fully answer both questions by introducing the tautological classes of matroids.

Remark 2.7. To answer the questions, one may attempt to use the classical Hirzebruch-Riemann-Roch theorem, which in our language states the following. There exist a unique ring isomorphism $ch : \bar{H}(\Sigma) \otimes \mathbb{Q} \rightarrow \overline{PP}^\bullet(\Sigma) \otimes \mathbb{Q}$ satisfying $P \mapsto \exp(h_P)$ for all $P \in \text{Def}(\Sigma)$, and a class $Td(\Sigma) \in \overline{PP}^\bullet(\Sigma) \otimes \mathbb{Q}$ called the *Todd class* of Σ , such that

$$\chi(\zeta) = \deg_\Sigma(ch(\zeta) \cdot Td(\Sigma)) \quad \text{for all } \zeta \in \bar{H}(\Sigma) \otimes \mathbb{Q}.$$

However, the Todd class often does not admit sufficiently explicit description for combinatorial computations. In fact, the study of Todd classes of permutohedral fans is a research direction of its own [15]. We will instead introduce an exceptional Hirzebruch-Riemann-Roch formula to bridge the K -theory and cohomology theory ([Theorem 3.3.\(iv\)](#)).

3 Main construction and results

We now introduce the tautological classes of matroids and explain the main results. We will follow the construction in [8, Appendix III], which is equivalent to but looks different from the more geometric description in [8, §3]. Geometric motivation behind the constructions and the main results is explained in [Section 4](#).

Let M be a matroid of rank r on ground set E . Define matroids M_i for $i = 0, \dots, n$ by

the set bases of $M_i = \{B \subseteq E : |B| = i \text{ and } B \text{ contains or is contained in a basis of } M\}$.

The sequence $M_0, \dots, M_n$ is sometimes called the Higgs factorization of M .

Definition 3.1. The *tautological classes* of a matroid M are elements in $\bar{H}(\Sigma_E)$ defined as

$$\mathcal{S}_M = \sum_{i=1}^r -P_{M_{i-1}} - P_{M_i}^{-1} \quad \text{and} \quad \mathcal{Q}_M = \sum_{i=r+1}^n -P_{M_{i-1}} - P_{M_i}^{-1}.$$

The k -th Chern class $c_k(\mathcal{S}_M) \in \overline{PP}^k(\Sigma_E)$ is defined as

$$c_k(\mathcal{S}_M) = k\text{-th elementary symmetric polynomial in } \{h_{-P_{M_{i-1}}} - h_{-P_{M_i}}\}_{i=1, \dots, r}$$

and similarly for $c_k(\mathcal{Q}_M)$.

To state the main results, we will need the following definition. Recall that a nonnegative sequence $a_0, a_1, \dots, a_m$ is *log-concave* if $a_k^2 \geq a_{k-1}a_{k+1}$ for all $1 \leq k \leq m-1$, and has *no internal zeros* if $a_i a_j > 0 \Rightarrow a_k > 0$ for all $0 \leq i \leq k \leq j \leq m$.

Definition 3.2. For a homogeneous polynomial $f \in \mathbb{R}[x_1, \dots, x_N]$ of degree d with nonnegative coefficients, we say that its coefficients form a *log-concave unbroken array* if, for any $1 \leq i < j \leq N$ and a monomial $x^{\mathbf{m}}$ of degree $d' \leq d$, the coefficients of $\{x_i^k x_j^{d-d'-k} x^{\mathbf{m}}\}_{0 \leq k \leq d-d'}$ in f form a log-concave sequence with no internal zeros.

We now state the main results [8, Theorems A, B, C, D].

Theorem 3.3. Let M be a matroid of rank r on ground set E , and T_M its Tutte polynomial.

(i) We have $c_1 \mathcal{Q}_M = h_{-P} M$ and $c_{n-r} \mathcal{Q}_M = \Sigma_M$.

(ii) Define a polynomial $t_M(x, y, z, w)$ by

$$t_M(x, y, z, w) = \sum_{i+j+k+\ell=n} x^i y^j z^k w^\ell T_M\left(\frac{x}{y}, \frac{y}{z}, \frac{x}{w}, \frac{y}{w}\right).$$

Then, we have an equality

$$\sum_{i+j+k+\ell=n} \left(\deg_{\Sigma_E} (h_\Delta^i h_\nabla^j c_k \mathcal{S}_M c_\ell \mathcal{Q}_M) \right) x^i y^j z^k w^\ell = t_M(x, y, z, w).$$

(iii) The coefficients of the polynomial $t_M(x, y, z, w)$ form a log-concave unbroken array. (In fact, it is a denormalized Lorentzian polynomial in the sense of [12]).

(iv) There is a unique ring isomorphism $\zeta : \bar{\mathbb{I}} \Sigma_E \xrightarrow{\sim} \overline{PP}^\bullet \Sigma_E$ satisfying $\zeta = -P M$ $\sum_{k \geq 0} -1^k c_k \mathcal{S}_M$ for all matroids M on E . Moreover, the map ζ satisfies

$$\chi(\zeta) = \deg_{\Sigma_E} \left(\zeta \cdot 1 - h_\Delta - \dots - h_\Delta^{n-1} \right) \quad \text{for all } \zeta \in \bar{\mathbb{I}} \Sigma_E.$$

Some remarks about the main results and their implications follow:

- By appropriately combining parts (i), (ii), and (iv), one easily recovers the various geometric manifestations of the Tutte polynomial such as Theorem 2.3, Theorem 2.4, [23, Theorem 5.2], [32, Theorem 5.4], and [4, Theorem 1.2]. This answers the first question in Question 2.6. The proof of the key formula (ii) does not depend on any of these previous results.
- Part (iii) answers the second question in Question 2.6, and in fact contains as special cases the log-concavity results previously proven in [1] and [4]. See [8, Discussion after Theorem B] for details. The proof of (iii) is an illustration of how our framework can exploit powerful tools in both of the models of matroids reviewed in Section 2.2, since the two key tools in the proof are valuativity (from model I) and the Hodge theory of matroids (from model II).

- Part (iv) differs from the classical Hirzebruch-Riemann-Roch theorem (Remark 2.7) despite the resemblance: The isomorphism in ζ is over integral coefficients and differs from the Chern character map ch , and the class $1 - h_\Delta - \dots - h_\Delta^{n-1}$ differs from the Todd class $Td(\Sigma_E)$.

4 Underlying geometry

We explain the underlying algebraic geometry for those familiar with toric geometry. Let X_E be the projective smooth toric variety associated to the fan Σ_E . It contains the projectivized torus $\mathbb{P}T$ as its open dense torus, and thus the standard torus T acts on X_E . Let the *inverse standard action* of T on $\mathbb{C}^E$ be given by $(t_1, \dots, t_n) \cdot (x_1, \dots, x_n) = (t_1^{-1}x_1, \dots, t_n^{-1}x_n)$.

A *realization* of a matroid M of rank r is an r -dimensional subspace $L \subseteq \mathbb{C}^E$ such that the set of bases of M equals

$$\{B \subseteq E : \text{the composition } L \hookrightarrow \mathbb{C}^E \rightarrow \mathbb{C}^B \text{ is an isomorphism}\}.$$

When M has a realization $L \subseteq \mathbb{C}^E$, the two models of matroids reviewed in Section 2.2 arise geometrically in the following way.

- (I) Let $Gr(r; E)$ be the Grassmannian of r -dimensional subspaces in $\mathbb{C}^E$, with T acting via the inverse standard action. The moment polytope of the torus-orbit-closure $\overline{T \cdot L} \subseteq Gr(r; E)$ is exactly the polytope $-P(M)$. As $-P(M) \in \text{Def}(\Sigma_E)$, we have a canonical map $X_E \rightarrow \overline{T \cdot L}$. Pulling back to X_E the line bundle $\mathcal{O}(1)$ on $Gr(r; E)$ via the composition $X_E \rightarrow \overline{T \cdot L} \hookrightarrow Gr(r; E)$, we obtain the line bundle whose K -class is equal to $-P(M)$ under the isomorphism of Theorem 2.1.
- (II) The *wonderful compactification*, studied in [19], is the closure W_L of $\mathbb{P}L \cap \mathbb{P}T$ inside X_E . As a subvariety of X_E , the variety W_L defines a cohomology class $W_L \in H^\bullet(X_E)$, which equals the Bergman class Σ_M under the isomorphism of Theorem 2.2.

Tautological classes of matroids are modeled after the following vector bundles constructed from realizations of matroids. Let $\underline{\mathbb{C}}^E$ denote the T -equivariant vector bundle $X_E \times \mathbb{C}^E$, where T acts on $\mathbb{C}^E$ via the inverse standard action.

Definition 4.1. Given a realization $L \subseteq \mathbb{C}^E$ of M , define two T -equivariant vector bundles S_L and Q_L on X_E by

- S_L T -equivariant subbundle of $\underline{\mathbb{C}}^E$ whose fiber at the identity of $\mathbb{P}T$ is L , and
- Q_L T -equivariant quotient bundle of $\underline{\mathbb{C}}^E$ whose fiber at the identity of $\mathbb{P}T$ is $\mathbb{C}^E/L$.

Equivalently, consider the map $X_E \rightarrow \overline{T \cdot L} \hookrightarrow Gr(r; E)$ in (I). Pulling back the tautological subbundle $\mathcal{S}$ and the quotient bundle $\mathcal{Q}$ of $Gr(r; E)$ to X_E along this map gives the vector bundles $\mathcal{S}_L$ and $\mathcal{Q}_L$.

The K -classes of these vector bundles only depend on the matroid M that L realizes. More precisely, as elements in $K(X_E)$, we have $\mathcal{S}_L = \mathcal{S}_M$ and $\mathcal{Q}_L = \mathcal{Q}_M$ [8, Proposition 3.7]. The description of the classes $\mathcal{S}_M$ and $\mathcal{Q}_M$ given here arises as follows. Given a realization $L \subseteq \mathbb{C}^E$ of M , we may consider a flag $0 \subsetneq L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_n \subsetneq \mathbb{C}^E$ of subspaces where L_i is a general i -dimensional subspace containing L . This gives rise to filtrations of the vector bundles $\mathcal{S}_L$ and $\mathcal{Q}_L$. The resulting description of their K -classes constitute [Definition 3.1](#).

We now discuss the geometry behind some of the main results listed in [Theorem 3.3](#).

Among the most important invariants of vector bundles are their Chern classes, which loosely speaking measure the dependency loci of global sections. When M has a realization $L \subseteq \mathbb{C}^E$, the statement $c_{n-r}(\mathcal{Q}_M) = \Sigma_M$ in [Theorem 3.3.\(i\)](#) is the reflection of the following geometric statement [8, Theorem 7.10].

Theorem 4.2. Let s_1 be the constant $(1, 1, \dots, 1)$ global section of $\underline{\mathbb{C}}^E$, which defines a section s of $\mathcal{Q}_L$ via $\underline{\mathbb{C}}^E \twoheadrightarrow \mathcal{Q}_L$. Then, the wonderful compactification W_L is exactly the vanishing locus of the section s .

The theorem implies that the normal bundle $\mathcal{N}_{W_L/X_E}$ is isomorphic to the restriction $\mathcal{Q}_L|_{W_L}$. Using this, one can further relate the restriction $\mathcal{S}_L|_{W_L}$ to the log-tangent sheaf $\mathcal{T}_{W_L} = \log \partial W_L$ of the wonderful compactification, and consequently recover the results of [32]. See [8, Section 8] for details.

When a vector bundle is globally generated, like the bundle $\mathcal{Q}_L$, one expects the Chern classes of $\mathcal{Q}_L$ to display “positivity behaviors” in the sense of [30]. The log-concavity statement of [Theorem 3.3.\(iii\)](#), when M has a realization L , is a manifestation of such expectation.

A geometric way to characterize the map ζ in [Theorem 3.3.\(iv\)](#) is that $\zeta : \mathcal{O}_{W_L} \rightarrow W_L$ for any subspace $L \subseteq \mathbb{C}^E$, where $\mathcal{O}_{W_L}$ denotes the structure sheaf of the wonderful compactification W_L [8, Corollary 10.6]. However, phrased in this way, it is not even clear why such a ring map ζ should be well-defined. One geometric explanation for [Theorem 3.3.\(iv\)](#) in this light, as well as a generalization to wonderful compactifications, is given in [29].

Updates and outlook

Let us list some works that the framework of tautological classes of matroids has led to since the appearance of [8]. [Theorem 4.2](#) is a central observation behind the development

of the Gromov-Witten theory of matroids in [35]. A generalization of [Theorem 3.3\(iv\)](#) can be found in [29], where the authors use the generalization to develop the K -theory of matroids. An “augmented” variant of the tautological classes of matroids was introduced and studied in [22], leading to a new log-concavity property of matroids and a better understanding of matroid Schubert varieties studied in [11]. For “type B” generalizations of matroids known as delta-matroids, an “isotropic” variant of tautological classes was introduced and studied in [21], leading to a log-concavity result for delta-matroids.

We speculate that many interesting properties of $\mathcal{S}_M$ and $\mathcal{Q}_M$ remain undiscovered. For instance, one may replace “elementary symmetric polynomials” in [Definition 3.1](#) by Schur polynomials, yielding Schur classes of tautological classes of matroids. Do these have nice numerical or combinatorial properties as did the Chern classes in [Theorem 3.3](#) [8, Question 1.4]? See also [9, Conjecture 9.13, Remark 9.15] for a related conjecture about their positivity, as well as generalizations to double Grothendieck polynomials and equivariant K -theory.

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A Realization of Poset Associahedra

Andrew Sack^{*1}

¹*Department of Mathematics, University of California, Los Angeles*

Abstract. Given any connected poset P we give a simple realization of Galashin's poset associahedron $\mathcal{A}(P)$ as a convex polytope in $\mathbb{R}^P$. The realization is inspired by the description of $\mathcal{A}(P)$ as a compactification of the configuration space of order-preserving maps $P \rightarrow \mathbb{R}$. In addition, we discuss several combinatorially interesting examples of poset associahedra.

Keywords: Poset, associahedron, cyclohedron, realization, configuration space, compactification

1 Introduction

Given a finite connected poset P , the poset associahedron $\mathcal{A}(P)$ is a simple, convex polytope of dimension $|P| - 2$ introduced by Galashin [5]. Poset associahedra arise as a natural generalization of Stasheff's associahedra [6, 12, 17, 18], and were originally discovered by considering compactifications of the configuration space of order-preserving maps $P \rightarrow \mathbb{R}$. These compactifications are generalizations of the Axelrod–Singer compactification of the configuration space of a line [1, 8, 15]. Galashin constructed poset associahedra by performing stellar subdivisions on the polar dual of Stanley's *order polytope* [16], but did not provide an explicit realization.

Poset associahedra bear resemblance to graph associahedra, where the face lattice of each is described by a *tubing criterion*. However, neither class is a subset of the other. When Carr and Devadoss introduced graph associahedra in [3], they distinguish between *bracketings* and *tubings* of a path, where the idea of bracketings does not naturally extend to any simple graph. In the case of poset associahedra, the idea of bracketings *does* extend to every connected poset.

In this paper, we provide a simple realization of $\mathcal{A}(P)$ as an intersection of half spaces, inspired by the compactification description and by a similar realization of graph associahedra due to Devadoss [4]. In independent work [10], Mantovani, Padrol, and Pilaud found a realization of poset associahedra as sections of graph associahedra. The

^{*}andrewsack@math.ucla.edu. This material is based upon work supported by the National Science Foundation Graduate Research Fellowship Program under Grant No. DGE-2034835. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

authors of [10] also generalize from posets to oriented building sets (which combine a building set with an oriented matroid).

Various poset associahedra have already been studied including *permutohedra*, *associahedra*, and *operahedra* [9]. We study two more classes of posets that give rise to previously unstudied polytopes with intriguing combinatorics.

2 Tubes and tubings

2.1 Background

We start by defining the poset associahedron.

Definition 2.1. Let $(P, \preceq)$ be a finite poset. We make the following definitions:

- A subset $\tau \subseteq P$ is *connected* if it is connected as an induced subgraph of the Hasse diagram of P .
- $\tau \subseteq P$ is *convex* if whenever $a, c \in \tau$ and $b \in P$ such that $a \preceq b \preceq c$, then $b \in \tau$.
- A *tube* of P is a connected, convex subset $\tau \subseteq P$ such that $2 \leq |\tau|$.
- A tube τ is *proper* if $|\tau| \leq |P| - 1$.
- Two tubes $\sigma, \tau \subseteq P$ are *nested* if $\sigma \subseteq \tau$ or $\tau \subseteq \sigma$. Tubes σ and τ are *disjoint* if $\tau \cap \sigma = \emptyset$.
- For disjoint tubes σ, τ we say $\tau \prec \sigma$ if there exists $a \in \tau, b \in \sigma$ such that $a \prec b$.
- A *proper tubing* T of P is a set of proper tubes of P such that any pair of tubes is nested or disjoint and the transitive closure of the relation $\prec$ is a partial order on T . That is, whenever $\tau_1, \dots, \tau_k \in T$ with $\tau_1 \prec \dots \prec \tau_k$ then $\tau_k \not\prec \tau_1$. This is referred to as the *acyclic tubing condition*.
- A proper tubing T is *maximal* if adding any tube to T is not a proper tubing.

Figure 1 shows examples and non-examples of proper tubings.

Definition 2.2. For a finite poset P , the *poset associahedron* $\mathcal{A}(P)$ is a simple, convex polytope of dimension $|P| - 2$ whose face lattice is isomorphic to the set of proper tubings ordered by reverse inclusion. That is, if F_T is the face corresponding to T , then $F_S \subset F_T$ if one can make S from T by adding tubes. Vertices of $\mathcal{A}(P)$ correspond to maximal tubings of P .

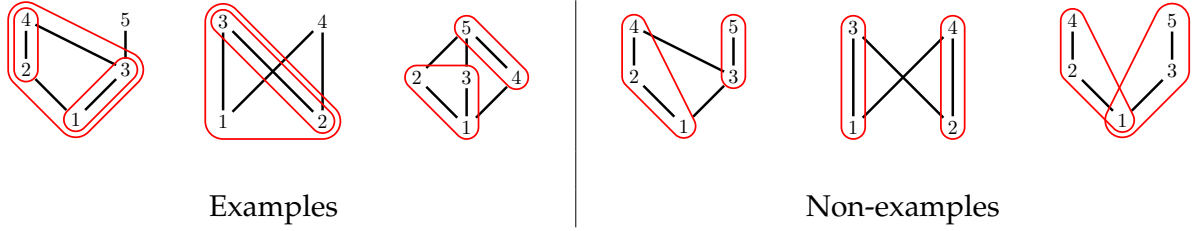


Figure 1: Examples and non-examples of proper tubings.

2.2 Realization

We realize $\mathcal{A}(P)$ as an intersection of half-spaces. We work in the ambient space $\mathbb{R}_{\Sigma=0}^P$, the space of real functions on P that sum to 0. For a subset $\tau \subseteq P$, define a linear function α_τ on $\mathbb{R}_{\Sigma=0}^P$ by

$$\alpha_\tau(p) := \sum_{\substack{i \prec j \\ i, j \in \tau}} p_j - p_i.$$

Here the sum is taken over all covering relations contained in τ . We define the half-space h_τ and the hyperplane H_τ by

$$h_\tau := \left\{ p \in \mathbb{R}_{\Sigma=0}^P \mid \alpha_\tau(p) \geq n^{2|\tau|} \right\} \quad \text{and} \\ H_\tau := \left\{ p \in \mathbb{R}_{\Sigma=0}^P \mid \alpha_\tau(p) = n^{2|\tau|} \right\}.$$

The following is our main result:

Theorem 2.3. *If P is a finite, connected poset, the intersection of H_P with h_τ for all proper tubes τ gives a realization of $\mathcal{A}(P)$.*

2.3 An interpretation of tubings

When P is a chain, $\mathcal{A}(P)$ recovers the classical associahedron. There is a simple interpretation of proper tubings that explains all of the conditions above in terms of *generalized words*.

We can understand the classical associahedron as follows: Let $P = ([n], \leq)$ be a chain. We can think of the chain as a word we want to multiply together with the rule that two elements can be multiplied if they are connected by an edge. A maximal tubing of P is a way of disambiguating the order in which one performs the multiplication. If a pair of adjacent elements x and y have a pair of brackets around them, they contract along the edge connecting them and replace x and y by their product.

Similarly, we can understand the Hasse diagram of an arbitrary poset P as a *generalized word* we would like to multiply together. Again, we are allowed to multiply two elements if they are connected by an edge, but when multiplying elements, we contract

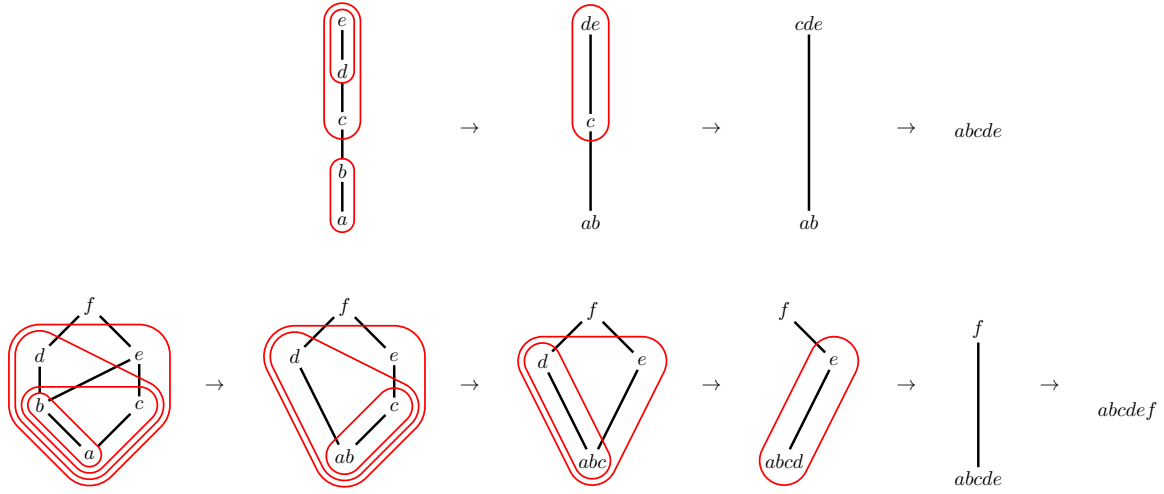


Figure 2: Multiplication of a word and of a generalized word

along the edge connecting them and then take the transitive reduction of the resulting directed graph. That is, we identify the two elements and take the resulting quotient poset. A maximal tubing is again a way of disambiguating the order of the multiplication. See Figure 2 for an illustration of this multiplication order. This perspective is discussed in relation to operahedra in [9, Section 2.1] when the Hasse diagram of P is a rooted tree.

3 Configuration spaces and compactifications

We turn our attention to the relationship between poset associahedra and configuration spaces. For a poset P , the *order cone*

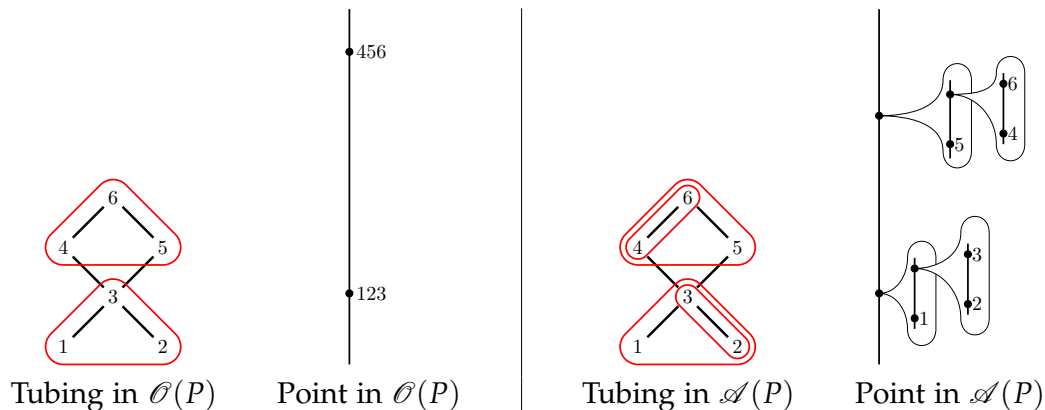
$$\mathcal{L}(P) := \left\{ p \in \mathbb{R}_{\Sigma=0}^P \mid p_i \leq p_j \text{ for all } i \preceq j \right\}$$

is the set of order preserving maps $P \rightarrow \mathbb{R}$ whose values sum to 0.

Fix a constant $c \in \mathbb{R}^+$. The *order polytope*, first defined by Stanley [16] and extended by Galashin [5], is the $(|P| - 2)$ -dimensional polytope

$$\mathcal{O}(P) := \{ p \in \mathcal{L}(P) \mid \alpha_P(p) = c \}.$$

Remark 3.1. When P is *bounded*, that is, has a unique maximum $\hat{1}$ and minimum $\hat{0}$, this construction is projectively equivalent to Stanley's order polytope where we replace the conditions of the coordinates summing to 0 and $\alpha_P(p) = c$ with $p_{\hat{0}} = 0$ and $p_{\hat{1}} = 1$, see [5, Remark 2.5].


 Figure 3: A vertex in $\mathcal{O}(P)$ vs. $\mathcal{A}(P)$

Galashin [5] obtains the poset associahedra by an alternative compactification of $\mathcal{O}^\circ(P)$, the interior of $\mathcal{O}(P)$. We describe this compactification informally, as it serves as motivation for the realization in Theorem 2.3.

A point is on the boundary of $\mathcal{O}(P)$ when any of the inequalities in the order cone achieve equality. The faces of $\mathcal{O}(P)$ are in bijection with proper tubings of P such that all tubes are disjoint. Let T be such a tubing. If p is in the face corresponding to T and $\tau \in T$ then $p_i = p_j$ for $i, j \in \tau$.

We can think of the point p in the face corresponding to T as being “what happens in $\mathcal{O}(P)$ ” when for each $\tau \in T$, the coordinates are infinitesimally close. However, by taking all coordinates in τ to be equal, we lose information about their relative ordering. In $\mathcal{A}(P)$, we still think of the coordinates in τ as being infinitesimally close, but we are still interested in their configuration. Upon zooming in, this is parameterized by the order polytope of the subposet $(\tau, \preceq)$. We iterate this process, allowing points in τ to be infinitesimally closer, and so on. We illustrate this in Figure 3. This idea is a common explanation of the Axelrod–Singer compactification of $\mathcal{O}^\circ(P)$ when P is a chain, see [1, 8, 15].

The idea of the realization in Theorem 2.3 is to replace the notions of *infinitesimally close* and *infinitesimally closer* with being *exponentially close* and *exponentially closer*. For $p \in \mathcal{L}(P)$, α_τ acts a measure of how close the coordinates of $p|_\tau$ are. We can make this precise with the following definition and lemma.

Definition 3.2. For $S \subseteq P$ and $p \in \mathbb{R}^P$, define the *diameter* of p relative to S by

$$\text{diam}_S(p) = \max_{i,j \in S} |p_i - p_j|.$$

That is, $\text{diam}_S(p)$ is the diameter of $\{p_i : i \in S\}$ as a subset of $\mathbb{R}$.

Lemma 3.3. Let $\tau \subseteq P$ be a tube and let $p \in \mathcal{L}(P)$. Then

$$\text{diam}_\tau(p) \leq \alpha_\tau(p) \leq \frac{n^2}{4} \text{diam}_\tau(p).$$

In particular, for $p \in \mathcal{L}(P)$, if $p \in H_\tau$, then $\{p_i \mid i \in \tau\}$ is clustered tightly together compared to any tube containing τ . If $p \in h_\tau$, then $\{p_i \mid i \in \tau\}$ is spread far apart compared to any tube contained in τ .

4 Realizing the poset associahedron

We are now prepared to sketch the proof of [Theorem 2.3](#). Define

$$\mathcal{A}(P) := \bigcap_{\sigma \subset P} h_\sigma \cap H_P$$

where the intersection is over all tubes of P . [Theorem 2.3](#) follows as a result of three lemmas:

Lemma 4.1. *If T is a maximal tubing, then*

$$v^T := \bigcap_{\tau \in T \cup \{P\}} H_\tau$$

is a point.

Lemma 4.2. *If T is a collection of tubes that do not form a proper tubing, then*

$$\bigcap_{\tau \in T} H_\tau \cap \mathcal{A}(P) = \emptyset.$$

Lemma 4.3. *If T is a maximal tubing and $\tau \notin T$ is a proper tube, then $\alpha_\tau(v^T) > n^{2|\tau|}$. That is, v^T lies in the interior of h_τ .*

[Lemma 4.1](#) follows from a standard induction argument.

Proof sketch of [Lemma 4.2](#). If T is not a proper tubing, then there are two cases:

- (1) There is a pair of non-nested and non-disjoint tubes τ_1, τ_2 in T .
- (2) There is a sequence of disjoint tubes $\tau_1, \dots, \tau_k$ such that $\tau_1 \prec \dots \prec \tau_k \prec \tau_1$.

For $S \subseteq P$, define the *convex hull* of S as

$$\text{conv}(\sigma) := \{b \in P \mid \exists a, c \in S : a \leq b \leq c\}.$$

Take $\sigma = \text{conv}(\tau_1 \cup \dots \cup \tau_k)$. One can show that σ is a tube, so [Lemma 3.3](#) tells us that for each τ_i , $\text{diam}_{\tau_i}(p)$ is very small compared to $n^{2|\sigma|}$. As the tubes either intersect or are cyclic, one can show this forces $\text{diam}_\sigma(p)$ to also be small, so $\alpha_\sigma(p) < n^{2|\sigma|}$. $\square$

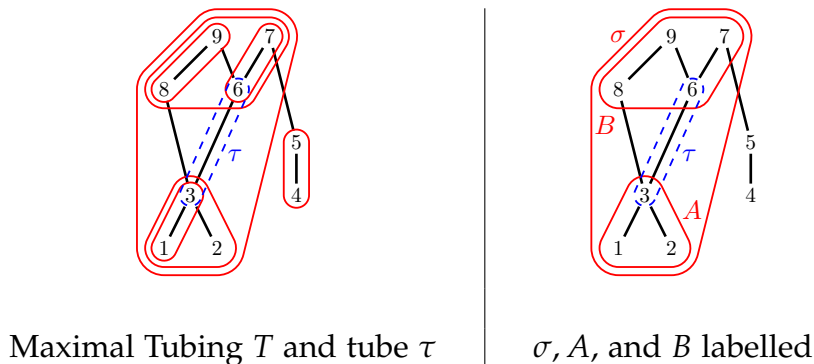


Figure 4: An example illustrating the proof of Lemma 4.3.

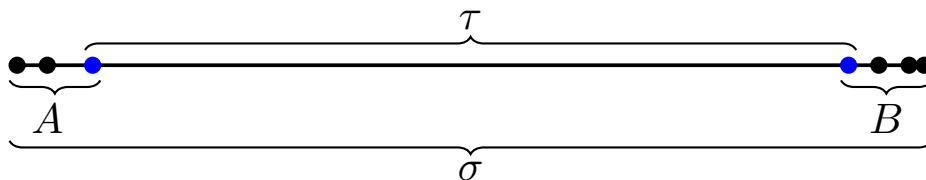


Figure 5: If $\text{diam}_A(p)$ and $\text{diam}_B(p)$ are small and $\text{diam}_\sigma(p)$ is large, then $\text{diam}_\tau(p)$ is large.

Proof sketch of Lemma 4.3. Define the *convex hull* of τ relative to T by

$$\text{conv}_T(\tau) := \min\{\sigma \in T \mid \tau \subset \sigma\}.$$

T partitions σ into a lower set A and an upper set B where A and B are either tubes or singletons. Furthermore, A and B both intersect τ . See Figure 4 for an example illustrating this.

By Lemma 3.3, $\text{diam}_A(v^T)$ and $\text{diam}_B(v^T)$ are both very small compared to $\text{diam}_\sigma(v^T)$. Then for any $a \in A, b \in B$, $|v_a^T - v_b^T|$ must be large. As τ intersects both A and B , $\text{diam}_\tau(v^T)$ must be large and hence $v^T \in h_\tau$. See Figure 5 for an illustration of this. $\square$

Remark 4.4. A similar approach for realizing graph associahedra is taken by Devadoss [4]. For a graph $G = (V, E)$, Devadoss realizes the graph associahedron of G by taking the supporting hyperplane for a graph tube τ to be

$$\left\{ p \in \mathbb{R}^V \mid \sum_{i \in \tau} p_i = 3^{|\tau|} \right\}.$$

One difference is that Devadoss realizes graph associahedra by cutting off slices of a simplex whereas we cut off slices of an order polytope. When the Hasse diagram of P is a tree, the poset associahedron is combinatorially equivalent to the graph associahedron

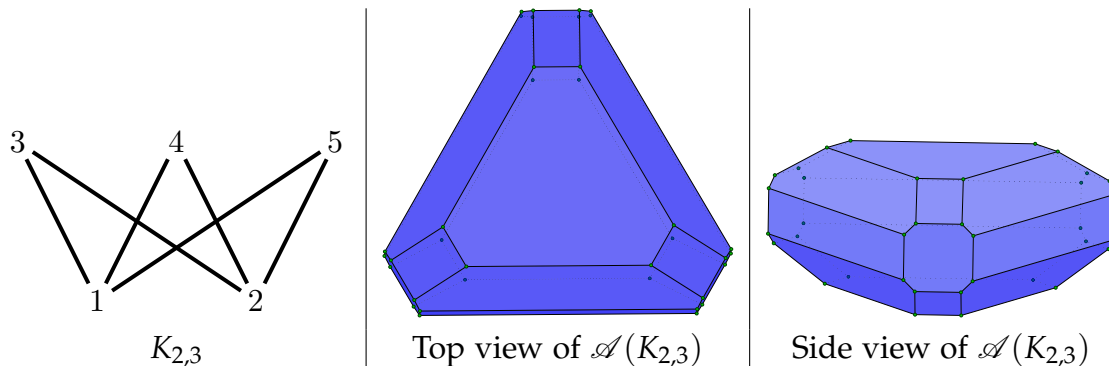


Figure 6: $K_{2,3}$ and $\mathcal{A}(K_{2,3})$

of the line graph of the Hasse diagram. In this case, the two realizations have linearly equivalent normal fans. If the Hasse diagram of P is a path graph, then both realizations have linearly equivalent normal fans to the realization of the associahedron due to Shnider and Sternberg [17].

5 Examples

Several classes of posets produce combinatorially interesting polytopes. Recall that for a simple d -dimensional polytope P , the f -vector, h -vector, and γ -vector of P are $(f_0, \dots, f_d)$, $(h_0, \dots, h_d)$, and $(\gamma_0, \dots, \gamma_{\lfloor d/2 \rfloor})$, where f_i is the number of i -dimensional faces and

$$\sum_{i=0}^d f_i t^i = \sum_{i=0}^d h_i (t+1)^i,$$

$$\sum_{i=0}^d h_i t^i = \sum_{i=0}^{\lfloor d/2 \rfloor} \gamma_i t^i (1+t)^{d-2i}.$$

The h -vector and γ -vector frequently encode interesting combinatorial data such as Eulerian numbers, Narayana numbers, and binomial coefficients. We consider all polytopes in this section only up to combinatorial equivalence. For clarity, we write $h_i(P)$ when P is not clear from context.

5.1 Posets with Hasse diagram $K_{m,n}$

Let $K_{m,n}$ be the poset whose Hasse diagram is the complete bipartite graph $K_{m,n}$ where for each $1 \leq i \leq m < j \leq m+n$ we have $i \prec j$. For example, $\mathcal{A}(K_{1,n})$ is the classical permutohedron. Let $\mathfrak{S}_n$ be the set of permutations of size n . Following the notation of [13], we recall the following definitions. For $w \in \mathfrak{S}_n$, the *descent statistic* is

$$\text{des}(w) := \#\{(i, i+1) \mid w(i) > w(i+1)\}.$$

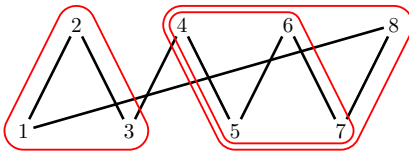


Figure 7: A tubing of a cyclic fence.

A *double descent* is a pair of consecutive descents. We say that w has a *final descent* if $w(n-1) > w(n)$. For $\mathcal{A}(K_{1,n})$, the h -vector is given by the *Eulerian numbers* [12]

$$h_i(\mathcal{A}(K_{1,n})) = \#\{w \in \mathfrak{S}_n \mid \text{des}(w) = i\}.$$

Let $\hat{\mathfrak{S}}_n$ be the set of permutations of size n without any consecutive descents or final descent. Then

$$\gamma_i(\mathcal{A}(K_{1,n})) = \#\{w \in \hat{\mathfrak{S}}_n \mid \text{des}(w) = i\}.$$

These properties generalize to $\mathcal{A}(K_{m,n})$. In particular,

$$\begin{aligned} h_i(\mathcal{A}(K_{m,n})) &= \#\{w \in \mathfrak{S}_{m+n} \mid \text{des}(w) = i, w(1) \leq m, w(m+n) > m\} \quad \text{and} \\ \gamma_i(\mathcal{A}(K_{m,n})) &= \#\{w \in \hat{\mathfrak{S}}_{m+n} \mid \text{des}(w) = i, w(1) \leq m, w(m+n) > m\}. \end{aligned}$$

5.2 Posets whose Hasse diagram is a path graph

When the Hasse diagram is isomorphic to the path graph P_n on n vertices, $\mathcal{A}(P_n)$ recovers the classical associahedron. Let DP_n be the set of Dyck paths with n up-steps. A *peak* is an up-step immediately followed by a down-step. Here, h_i is given by the *Narayana numbers* [12]

$$\begin{aligned} h_i(\mathcal{A}(P_n)) &:= N(n-1, i) \\ &= \frac{1}{n-1} \binom{n-1}{i} \binom{n-1}{i-1} \\ &= \#\{w \in DP_{n-1} \mid \#\text{peaks}(w) = i+1\}. \end{aligned}$$

5.3 Cyclic fences

The *fence*, F_n , is the poset with n elements and alternating covering relations

$$1 \prec 2 \succ 3 \prec 4 \succ \dots$$

As the Hasse diagram is a path, $\mathcal{A}(F_n)$ is the classical associahedron. We define the *cyclic fence* CF_{2n} to be the fence F_{2n} with the additional relation $1 \prec 2n$, see Figure 7.

One may expect that $\mathcal{A}(CF_{2n})$ is equivalent to the *cyclohedron* [2, 11, 14, 17], a cyclic variant of the associahedron. However, this is not the case! The n -dimensional cyclohedron $\mathcal{C}_n$ has $\binom{2n}{n}$ vertices, but $\mathcal{A}(CF_{2(n+1)})$ has $4^n \binom{2n}{n}$ vertices. Despite this, these polytopes are related!

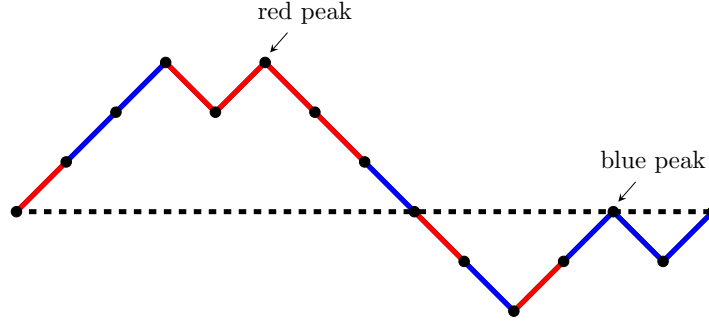


Figure 8: A colored balanced path with 1 red peak and 1 blue peak.

A *balanced path* is a sequence of up-steps and down-steps with an equal number of each. Let BP_n be set of balanced paths with n up-steps. In [14], Simion observes that $|BP_n| = \binom{2n}{n}$ and for the cyclohedron,

$$h_i(\mathcal{C}_n) = \#\{w \in BP_n \mid \# \text{peaks}(w) = i\} = \binom{n}{i}^2.$$

We define a *colored balanced path* to be a balanced path where each step is colored red or blue, and let CBP_n be the set of all colored balanced paths with n up-steps. Define a *red peak* to be a red up-step immediately followed by a red down-step and similarly define a *blue peak*, see Figure 8. Then $|CBP_n| = 4^n \binom{2n}{n}$, and for $\mathcal{A}(CF_{2(n+1)})$,

$$h_i(\mathcal{A}(CF_{2(n+1)})) = \#\{w \in CBP_n \mid \# \text{red peaks}(w) - \# \text{blue peaks}(w) = i - n\}$$

$$\gamma_i(\mathcal{A}(CF_{2(n+1)})) = 4^i \binom{n}{i}^2.$$

6 Open questions

Question 6.1. Define a *colored Dyck path* to be a Dyck path where each step is colored red or blue, and let CDP_n be the set of colored Dyck paths with n up-steps. We can define

$$h_i = \#\{w \in CDP_n \mid \# \text{red peaks}(w) - \# \text{blue peaks}(w) = i - n\}.$$

and calculate $\gamma_i = 4^i N(n-1, i)$. Observe that similarly to the case of colored balanced paths, the γ -vector is 4^i times the h -vector of non-colored paths. Is $(h_0, \dots, h_{2n})$ the h -vector of a polytope related to the associahedron?

Question 6.2. Galashin conjectured that the γ -vector of a poset associahedron is always non-negative, despite poset associahedra not being flag simple in general. We strengthen

this by conjecturing that the polynomial $\sum f_i t^i$ is real-rooted. If poset associahedra are indeed γ -positive, it would be interesting to find a combinatorial interpretation of the γ -vector.

Question 6.3. Postnikov, Reiner, and Williams [13] found a statistic on maximal tubings of graph associahedra of chordal graphs where

$$\sum_T t^{\text{stat}(T)} = \sum h_i t^i.$$

It would be interesting to find a similar statistic on maximal tubings of poset associahedra. For a simple polytope P , one can orient the edges of P according to a generic linear form and take $\text{stat}(v) = \text{outdegree}(v)$ [19, §8.2]. It may be possible to use our realization to find the desired statistic.

Question 6.4. While $h_i(\mathcal{A}(CF_{2(n+1)}))$ is given by a peak statistic, we do not know a bijective proof of this fact. In particular, we would like a bijection that preserves $\text{stat}(T)$ from Question 6.3. There is a known bijection between maximal tubings of $CF_{2(n+1)}$ and CDP_n although it is complicated, see [7].

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Chromatic quasisymmetric functions and noncommutative P -symmetric functions

Byung-Hak Hwang*

Abstract. For a natural unit interval order P , we introduce a combinatorial operation, called a *local flip*, on proper colorings of P . This operation defines an equivalence relation on the proper colorings, and the equivalent relation refines the ascent statistic introduced by Shareshian and Wachs. We also define analogues of noncommutative symmetric functions. They reflect properties of P and local flips, and lead us to positive expansions of the chromatic quasisymmetric functions into several symmetric function bases.

Keywords: Chromatic quasisymmetric function, Noncommutative symmetric function, the Stanley–Stembridge conjecture, the e -positivity conjecture

1 Introduction

1.1 Chromatic quasisymmetric functions

Chromatic quasisymmetric functions are one of the most notable objects in algebraic combinatorics, because of their connections with other fields. In [5, 4], Stanley, and then Shareshian–Wachs, introduced the *chromatic quasisymmetric function* which generalizes the chromatic polynomial. For a graph G on vertex set n , the chromatic quasisymmetric function $X_G(\mathbf{x}, q)$ of G is defined by

$$X_G(\mathbf{x}, q) = \sum_{\kappa} q^{\text{asc } \kappa} x^{\kappa},$$

where $\kappa : n \rightarrow \mathbb{P}$ ranges over all proper colorings of G , $\text{asc } \kappa$ is the number of edges $\{i, j\}$ such that $i < j$ and $\kappa(i) < \kappa(j)$, and $x^{\kappa} = \prod_{v \in V(G)} x_{\kappa(v)}$.

One of the most famous long-standing open problems in algebraic combinatorics is the *e -positivity conjecture*, which is about chromatic quasisymmetric functions (Conjecture 2.1) (the original conjecture was proposed by Stanley–Stembridge, and Shareshian–Wachs gave a refinement of it).

The Shareshian–Wachs quasisymmetric refinement has an advantage for resolving the e -positivity conjecture. The chromatic quasisymmetric function $X_G(\mathbf{x}, q)$ has more

*byunghakhwang@gmail.com.

information about colorings than $X_G(x, 1)$, the chromatic symmetric function, and it is obvious that the conjecture of Shareshian–Wachs implies the original e -positivity conjecture of Stanley–Stembridge. But thanks to the quasisymmetric generalization, we can cluster colorings of G along the ascent statistic, and this clustering makes us focus on certain colorings instead of whole ones. In this sense, the quasisymmetric generalization gives us a hint for the e -positivity conjecture. One can then ask for a natural way which refines the Shareshian–Wachs refinement.

One of the goals of the abstract is to answer this question. We will introduce a combinatorial operation, called a *local flip*. This operation defines an equivalence relation on proper colorings, and this relation refines the refinement of Shareshian and Wachs. Many results for the Shareshian–Wachs refinement still hold for our refinement.

1.2 Positivity of symmetric functions

Whenever a new class of symmetric functions is introduced, a question that naturally arises is positivity of them with respect to various symmetric function bases. To show positivity of a given symmetric function, numerous combinatorial and algebraic tools are developed. One of such well developed tools is the theory of noncommutative symmetric functions. Fomin and Greene [2] introduced noncommutative Schur functions to prove Schur positivity for various symmetric functions. In [1], Blasiak and Fomin gave a more general algebraic framework of this approach.

We devote the second half of this abstract to introducing analogues of noncommutative symmetric functions which reflect properties of a given natural unit interval order P . In the same spirit of [1], these noncommutative symmetric functions provide expansions of the chromatic quasisymmetric function of P in terms of several bases.

The full version of this extended abstract is available at [arXiv:2208.09857](https://arxiv.org/abs/2208.09857).

2 Background

2.1 Natural unit interval orders

Fix a positive integer n , and let $\mathbf{m} = m_1, \dots, m_n$ be a weakly increasing integer sequence satisfying $i \leq m_i \leq n$ for each i . The *natural unit interval order* $P = P(\mathbf{m})$ corresponding to $\mathbf{m}$ is a poset on $n = \{1, 2, \dots, n\}$ whose ordering $<_P$ is given by $i <_P j$ if $m_i < j$. The term “natural unit interval order” arises from the following unit interval model: given a natural unit interval order P , one can assign a unit interval on the real line to each $i \in n$ such that $i <_P j$ if and only if the unit interval assigned to i completely lies on the left of the unit interval assigned to j .

Let P be a natural unit interval order on n . The *incomparability graph* of P is the graph whose vertex set is n , and for $i < j$, i and j are adjacent if and only if $j \leq m_i$. By

abuse of notation, we write the same notation P for the incomparability graph of P .

We also define some notions for words on the alphabet $[n]$ with respect to P . For a word $w = w_1 w_2 \cdots w_d$, we say that w is of type $\mu = \mu_1, \dots, \mu_n$ if the letter i appears μ_i times in w for each i . For $1 \leq i < d$, i is a P -descent of w if $w_i >_P w_{i+1}$, and let

$$\text{Des}_P w = \{i \in [d-1] \mid w_i >_P w_{i+1}\}.$$

For $1 \leq i < j \leq d$, (i, j) is a P -inversion pair of w if w_i and w_j are incomparable in P , and $w_i > w_j$ in the natural order. Denote by $\text{inv}_P w$ the number of P -inversion pairs of w .

2.2 Symmetric and quasisymmetric functions

We assume that the reader is familiar with the basics of the theory of symmetric and quasisymmetric functions. We follow definitions and notations in [6, Chapter 7]. Here, we only discuss some identities which we will use.

Let $\mathbf{x} = x_1, x_2, \dots$ be a sequence of commuting indeterminates, and $\text{Sym} \subset \mathbb{Q}[\mathbf{x}]$ be the $\mathbb{Q}$ -space of symmetric functions. The following identities are well known: for $k \geq 0$ and a partition λ ,

$$h_k(\mathbf{x}) = e_1(\mathbf{x}) h_{k-1}(\mathbf{x}) + \cdots + (-1)^k e_k(\mathbf{x}), \quad \text{and} \quad (2.1)$$

$$s_\lambda(\mathbf{x}) = \det(e_{\lambda'_i - j}(\mathbf{x}))_{i,j=1}^{\lambda_1}, \quad (2.2)$$

where $h_k(\mathbf{x})$, $e_k(\mathbf{x})$ and $s_\lambda(\mathbf{x})$ are complete homogeneous, elementary symmetric function, and Schur functions, respectively, and λ' is the conjugate of λ .

Let $\mathbf{y} = y_1, y_2, \dots$ be another sequence of commuting indeterminates. Define

$$C(\mathbf{x}, \mathbf{y}) = \prod_{j \geq 1} \sum_{\ell \geq 0} x_j^\ell h_\ell(\mathbf{y}) = \sum_{\lambda} m_\lambda(\mathbf{x}) h_\lambda(\mathbf{y}) \in \mathbb{Q}[\mathbf{x}, \mathbf{y}], \quad (2.3)$$

called the *Cauchy product*.

2.3 Chromatic quasisymmetric functions

Let G be a simple graph on the vertex set $[n]$. For a given sequence $\mu = \mu_1, \mu_2, \dots, \mu_n \in \mathbb{N}^n$, a *proper multi-coloring* κ of G of type μ is a function from $[n]$ to the collection of all finite subsets of $\mathbb{P}$ such that for each $i \in [n]$, $|\kappa(i)| = \mu_i$ and $\kappa(j) \cap \kappa(k) = \emptyset$ whenever $\{j, k\} \in E(G)$. The (multi-)chromatic quasisymmetric function $X_G(\mathbf{x}, q; \mu)$ of G is given by

$$X_G(\mathbf{x}, q; \mu) = \sum_{\kappa} q^{\text{asc}_G \kappa} x^\kappa,$$

where the sum is over all proper multi-colorings κ of type μ , $x^\kappa = \prod_{i=1}^n \prod_{k \in \kappa(i)} x_k$ and $\text{asc}_G \kappa = |\{(i, r, j, s) \mid \{i, j\} \in E(G), i < j, r \in \kappa(i), s \in \kappa(j), r < s\}|$. This is a

generalization of chromatic quasisymmetric function $X_G \mathbf{x}, q$ introduced by Shareshian and Wachs [4]. By definition, $X_G \mathbf{x}, q; 1^n = X_G \mathbf{x}, q$.

In general, $X_G \mathbf{x}, q; \mu$ is a quasisymmetric function, but when P is a natural unit interval order, $X_P \mathbf{x}, q; \mu$ is a symmetric function (Theorem 4.2). We close this section with an explicit statement of the (refined) e -positivity conjecture.

Conjecture 2.1 ([7, 5, 4]). *For a natural unit interval order P on n , let*

$$X_P \mathbf{x}, q; 1^n = \sum_{\lambda \vdash n} c_{\lambda} q^{e_{\lambda}} \mathbf{x}^{\lambda}.$$

Then $c_{\lambda} q^{e_{\lambda}}$ is a polynomial with nonnegative integer coefficients.

3 Heaps and local flips

In this section, we review the definition of heaps and define an operation, called a local flip, on heaps. This operation plays a central role in this abstract.

Fix a natural unit interval order P on n , and a nonnegative integer sequence $\mu = \mu_1, \dots, \mu_n \in \mathbb{N}^n$. Let P^{μ} be the graph whose vertex set is $\{v_{a,i} \mid a \in n, 1 \leq i \leq \mu_a\}$, and $v_{a,i}$ and $v_{b,j}$ are adjacent if either $a < b$ or a and b are incomparable in P . A *heap* H of P of type μ is an acyclic orientation of P^{μ} satisfying that for each $a \in n$ and $1 \leq i < j \leq \mu_a$, the direction on the edge between $v_{a,i}$ and $v_{a,j}$ is toward $v_{a,i}$. Clearly a heap of type 1^n is just an acyclic orientation of the incomparability graph of P . We call a vertex of a heap a *piece*, and denote the set of heaps of P of type μ by $\mathcal{H}(P, \mu)$. The terminology “heap” originates from the following diagrammatic realization: take n unit intervals described in Section 2.1. We stack blocks of unit length on top of each interval. For each $i \in n$, the number of blocks stacked on the i -th interval equals μ_i . We will often identify heaps with their diagrammatic realizations.

Each proper multi-coloring κ of P of type μ gives us a heap of P of type μ as follows: for each vertex $a \in n$, list its colors $\kappa(a) = \{c_{a,1} < \dots < c_{a,\mu_a}\}$, and for each edge $\{v_{a,i}, v_{b,j}\}$ assign the direction $v_{a,i} \leftarrow v_{b,j}$ if $c_{a,i} < c_{b,j}$. To see this diagrammatically, consider again n unit intervals on the real line. For each $a \in n$ and $c \in \kappa(a)$, place a block at the position of height c above the a -th interval. After placing all blocks, drop them down as far as gravity takes them, and then we obtain a heap of type μ . For example, let $P = P(2, 3, 3)$ and κ be a proper multi-coloring of P of type $(3, 1, 2)$ given by $\kappa(1) = \{1, 3, 6\}$, $\kappa(2) = \{4\}$, $\kappa(3) = \{2, 6\}$, shown in Figure 1(a), and its corresponding heap is shown in Figure 1(b). For a heap H , let $K_H \mathbf{x} = \sum_{\kappa} \mathbf{x}^{\kappa}$ where the summation ranges over all proper colorings corresponding to H . Since heaps are acyclic orientations, we can also regard them as posets.

Let W_{μ} be the set of all words of type μ . For a heap H of P of type μ , let $f : H \rightarrow d$ be a linear extension of H where $d = \mu_1 + \dots + \mu_n = |H|$. Then define the word

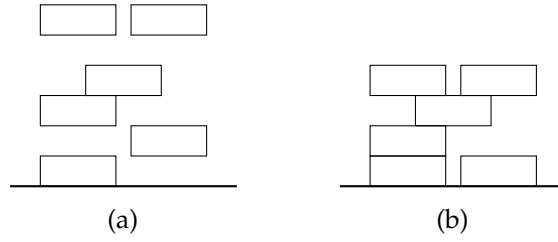


Figure 1

$w_f = w_{f,1} \cdots w_{f,d}$ by $w_{f,k} = a$ if $f^{-1}(k) = v_{a,i} \in H$ for some i . Then w_f is of type μ . Let

$$W(H) = \{w_f \in W(\mu) \mid f \text{ is a linear extension of } H\}.$$

For a heap H , an edge between $v_{a,i}$ and $v_{b,j}$ is *ascent* if the edge is toward $v_{a,i}$ and $a > b$ in the natural order. Let $\text{asc}_P(H)$ denote the number of ascent edges in H .

Theorem 3.1. *We have*

$$\omega X_P(\mathbf{x}, q; \mu) = \sum_{H \in \mathcal{H}(P, \mu)} q^{\text{asc}_P(H)} K_H(\mathbf{x}) = \sum_{w \in W(\mu)} q^{\text{inv}_P(w)} F_{d, \text{Des}_P(w)}(\mathbf{x}),$$

where $F_{d,S}(\mathbf{x})$ is a fundamental quasisymmetric function.

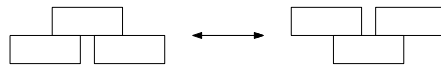
For any distinct pieces p, q and r in H , we call (p, q, r) a *flippable triple* in H if either

- (i) q covers p and r , or
- (ii) q is covered by p and r .

Definition 3.2. Let H be a heap of P and (p, q, r) a flippable triple in H . A *local flip* at (p, q, r) is reversing the directions on the edges $\{p, q\}$ and $\{q, r\}$.

Lemma 3.3. Let H be a heap and (p, q, r) a flippable triple in H . Then the orientation H' obtained from H by local flipping at (p, q, r) is acyclic, so H' is also a heap of the same type.

We can think of a local flip as an operation on diagrammatic realizations of heaps acting by transposing relative positions of blocks as follows:



Using local flips, we can define an equivalence relation on the set of heaps of P of type μ : for two heaps $H, H' \in \mathcal{H}(P, \mu)$, $H \sim H'$ if and only if H' can be obtained from H by applying a finite sequence of local flips. For instance, we illustrate all heaps of $P(2, 3, 4, 5, 5)$ of type 1^5 and their equivalence relations in Figure 2.

The following proposition and theorem tell us why local flips are crucial.

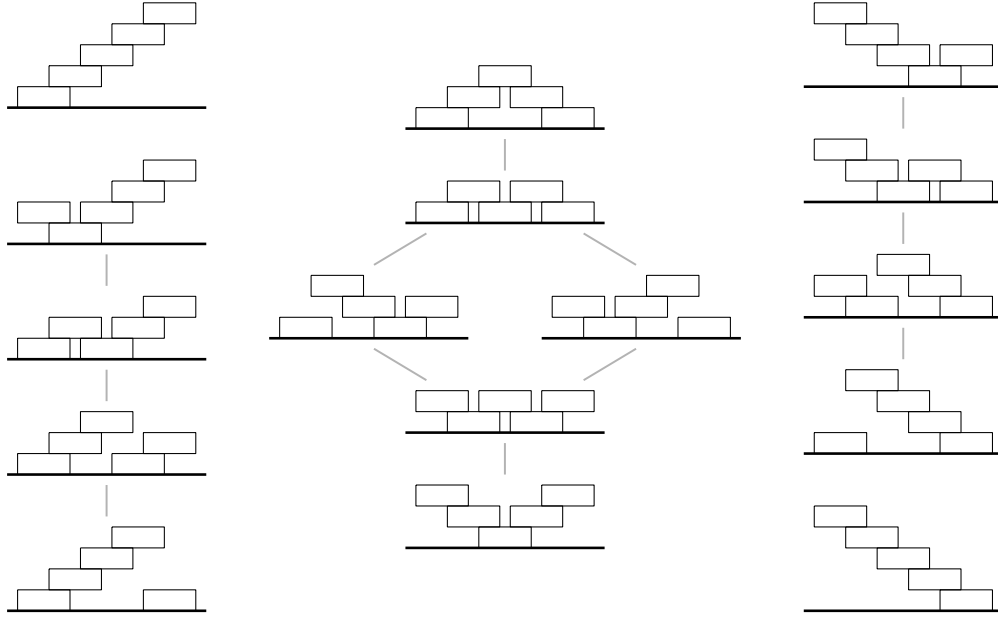


Figure 2: All heaps of $P(2,3,4,5,5)$ of type 1^5 . A gray line between heaps means that they can be transformed to each other via a local flip. Hence each connected component represents an equivalence class.

Proposition 3.4. *Local flips preserve the number of ascents.*

Theorem 3.5. *Let H be an equivalence class in $\mathcal{H}(P, \mu) / \sim$. Then*

$$K_H(\mathbf{x}) : \sum_{H' \in H} K_{H'}(\mathbf{x}) \in \text{Sym}.$$

In particular, $X_G(\mathbf{x}, q; \mu)$ is a symmetric function.

In addition, [Theorem 3.5](#) admits a refinement of the refined e -positivity conjecture.

Conjecture 3.6. *Let P be a natural unit interval order on $[n]$, and $\mu \in \mathbb{N}^n$. For any equivalence class $H \in \mathcal{H}(P, \mu) / \sim$, $K_H(\mathbf{x})$ is h -positive. In particular, $X_P(\mathbf{x}, q; \mu)$ is e -positive.*

We end this section with defining a graph Γ_μ for $\mu \in \mathbb{N}^n$. The vertex set of Γ_μ is the set W_μ of all words of type μ . Two words $w = w_1 \cdots w_d$ and $v = v_1 \cdots v_d$ are adjacent in Γ_μ if there exists an integer i satisfying the one of the following conditions:

- (i) $w_j = v_j$ for $j \notin \{i, i+1\}$, and $\{w_i w_{i+1}, v_i v_{i+1}\} = \{ac, ca\}$ for some $a <_P c$.
- (ii) $w_j = v_j$ for $j \notin \{i-1, i, i+1\}$, and either $\{w_{i-1} w_i w_{i+1}, v_{i-1} v_i v_{i+1}\} = \{bac, acb\}$ or $\{bca, cab\}$ for some $a < b < c$ satisfying $a \not<_P b$, $b \not<_P c$ and $a <_P c$.

The second condition represents how a local flip operates on words.

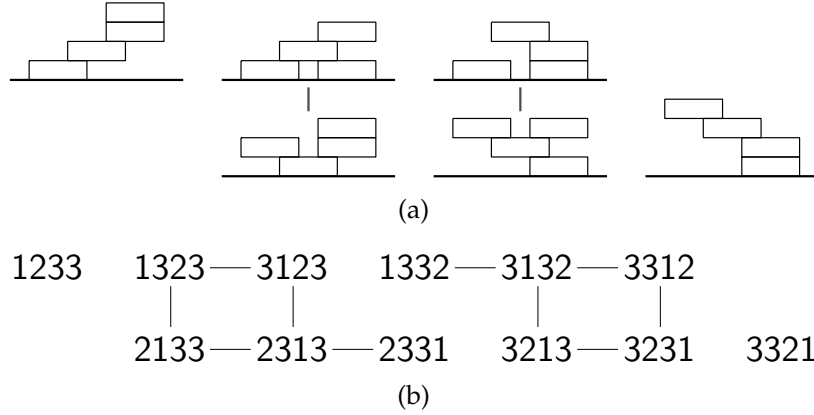


Figure 3: Let $P = [2, 3, 3]$ and $\mu = [1, 1, 2]$. (a) All heaps of P of type μ and their equivalence relations. (b) The graph Γ_μ .

4 Noncommutative P -symmetric functions

In this section, we define noncommutative P -symmetric functions associated with a natural unit interval order P , and present their connection with the chromatic quasisymmetric function of P . Using these, we provide positivity of $X_P(x, q; \mu)$ in several symmetric function bases.

4.1 An analogue of noncommutative symmetric functions

Let P be a natural unit interval graph on $[n]$ and $\mathcal{U}$ the free associative $\mathbb{Z}$ -algebra generated by $\{u_1, \dots, u_n\}$. For simplicity we write $u_w = u_{w_1}u_{w_2}\cdots u_{w_d}$ for a word $w = w_1w_2\cdots w_d$ on the alphabet $[n]$. Let $\mathcal{I}_P$ be the 2-sided ideal of $\mathcal{U}$ generated by the following elements:

$$u_a u_c - u_c u_a \quad (a <_P c), \quad (4.1)$$

$$u_a u_c u_b - u_b u_a u_c \quad (a < b < c, a \not<_P b, b \not<_P c \text{ and } a <_P c). \quad (4.2)$$

The ideal is just an algebraic counterpart of the graph Γ_μ .

For $k \geq 1$, we define the noncommutative P -elementary symmetric function $e_k(\mathbf{u})$ by

$$e_k(\mathbf{u}) = \sum_{i_1 >_P i_2 >_P \cdots >_P i_k} u_{i_1} u_{i_2} \cdots u_{i_k} \in \mathcal{U}. \quad (4.3)$$

By convention, let $e_0(\mathbf{u}) = 1$ and $e_k(\mathbf{u}) = 0$ for any $k < 0$. For a partition $\lambda = \lambda_1, \dots, \lambda_\ell$, define $e_\lambda(\mathbf{u}) = e_{\lambda_1}(\mathbf{u}) \cdots e_{\lambda_\ell}(\mathbf{u})$.

The following property plays a crucial role in the theory of noncommutative symmetric functions.

Theorem 4.1. For any integers $k, \ell \geq 0$, $\mathfrak{e}_k \mathbf{u}$ and $\mathfrak{e}_\ell \mathbf{u}$ commute with each other modulo $\mathcal{I}_P$, that is,

$$\mathfrak{e}_k \mathbf{u} \mathfrak{e}_\ell \mathbf{u} \equiv \mathfrak{e}_\ell \mathbf{u} \mathfrak{e}_k \mathbf{u} \pmod{\mathcal{I}_P}.$$

Similar to the relation (2.1), we define the *noncommutative P -complete homogeneous symmetric functions* $\mathfrak{h}_k \mathbf{u}$ inductively as follows:

$$\mathfrak{h}_k \mathbf{u} = \mathfrak{e}_1 \mathbf{u} \mathfrak{h}_{k-1} \mathbf{u} + \cdots + (-1)^k \mathfrak{e}_k \mathbf{u} \delta_{k,0},$$

with $\mathfrak{h}_0 \mathbf{u} = 1$, and define $\mathfrak{h}_\lambda \mathbf{u} = \mathfrak{h}_{\lambda_1} \mathbf{u} \cdots \mathfrak{h}_{\lambda_\ell} \mathbf{u}$ for a partition $\lambda = \lambda_1, \dots, \lambda_\ell$. Then it is easy to check that

$$\mathfrak{h}_k \mathbf{u} = \sum_{i_1 \not\prec_P i_2 \not\prec_P \cdots \not\prec_P i_k} u_{i_1} u_{i_2} \cdots u_{i_k}. \quad (4.4)$$

By Theorem 4.1, $\mathfrak{h}_k \mathbf{u}$'s also commute with each other in $\mathcal{U}/\mathcal{I}_P$.

Define

$$H x, \mathbf{u} = \sum_{\ell \geq 0} x^\ell \mathfrak{h}_\ell \mathbf{u} \in \mathcal{U}[x] \quad \text{and} \quad \Omega \mathbf{x}, \mathbf{u} = H x_1, \mathbf{u} H x_2, \mathbf{u} \cdots \in \mathcal{U}[\mathbf{x}].$$

Here, x and $\mathbf{x}$ commute with $\mathbf{u}$. We call $\Omega \mathbf{x}, \mathbf{u}$ the *noncommutative P -Cauchy product*. Since $H x_i, \mathbf{u} H x_j, \mathbf{u} \equiv H x_j, \mathbf{u} H x_i, \mathbf{u}$ modulo $\mathcal{I}_P[\mathbf{x}]$, we can write $\Omega \mathbf{x}, \mathbf{u}$ as the usual Cauchy product (2.3):

$$\Omega \mathbf{x}, \mathbf{u} \equiv \sum_{\lambda} m_{\lambda}[\mathbf{x}] \mathfrak{h}_{\lambda} \mathbf{u} \pmod{\mathcal{I}_P[\mathbf{x}]} \quad (4.5)$$

Let $\mathcal{U}^*$ be the free $\mathbb{Z}$ -module generated by words on the alphabet n , and $\langle \cdot, \cdot \rangle$ a canonical pairing between $\mathcal{U}$ and $\mathcal{U}^*$ such that $\langle u_w, v \rangle = \delta_{w,v}$ for words w and v . Let $\mathcal{I}_P^\perp$ be the orthogonal complement of $\mathcal{I}_P$ with respect to the pair, then we have the naturally induced pairing between $\mathcal{U}/\mathcal{I}_P$ and $\mathcal{I}_P^\perp$. Also let $\mathcal{U}_q^* = \mathbb{Z}[q] \otimes_{\mathbb{Z}} \mathcal{U}^*$ and $\mathcal{I}_{P,q}^\perp = \mathbb{Z}[q] \otimes_{\mathbb{Z}} \mathcal{I}_P^\perp$. Then we extend the pairing to $\mathcal{U}_q^*$.

Let

$$\gamma_H = \sum_{w \in W_H} w \in \mathcal{U}^* \quad \text{and} \quad \gamma_H = \sum_{H' \in H} \gamma_{H'} \in \mathcal{U}^*$$

for a heap H . Then one can show $\gamma_H \in \mathcal{I}_P^\perp$. Also let

$$\gamma_{\mu} = \sum_{w \in W_{\mu}} q^{\text{inv}_P w} w = \sum_{H \in \mathcal{H}_{P, \mu} / \sim} q^{\text{asc}_P H} \gamma_H \in \mathcal{I}_{P,q}^\perp \subset \mathcal{U}_q^*.$$

Theorem 4.2. For a nonnegative integer sequence $\mu = \mu_1, \dots, \mu_n$, we have

$$K_H[\mathbf{x}] = \langle \Omega \mathbf{x}, \mathbf{u}, \gamma_H \rangle \quad \text{and} \quad \omega X_P[\mathbf{x}, q; \mu] = \langle \Omega \mathbf{x}, \mathbf{u}, \gamma_{\mu} \rangle.$$

In particular, $K_H[\mathbf{x}]$ is a symmetric function, and so is $X_P[\mathbf{x}, q; \mu]$.

Corollary 4.3. *Suppose that we can write*

$$\Omega \mathbf{x}, \mathbf{u} \equiv \sum_{\lambda} g_{\lambda}(\mathbf{x}) f_{\lambda}(\mathbf{u}) \pmod{\mathcal{I}_P(\mathbf{x})} \quad (4.6)$$

for some symmetric function basis $g_{\lambda}(\mathbf{x})$ and noncommutative P -symmetric functions $f_{\lambda}(\mathbf{u})$. Let $\omega_{X_P(\mathbf{x}, q; \mu)} = \sum_{\lambda} r_{\lambda}(q) g_{\lambda}(\mathbf{x})$. Then for any partition λ , we have

$$r_{\lambda}(q) = \langle f_{\lambda}(\mathbf{u}), \gamma_{\mu} \rangle.$$

Corollary 4.3 offers expansions of $X_P(\mathbf{x}, q; \mu)$ in terms of various bases. As an example, let us consider the noncommutative P -complete homogeneous symmetric functions $h_{\lambda}(\mathbf{u})$, which we already defined. They provide the expansion of $X_P(\mathbf{x}, q; \mu)$ in terms of the forgotten symmetric functions $f_{\lambda}(\mathbf{x})$.

Theorem 4.4. *Let $X_P(\mathbf{x}, q; \mu) = \sum_{\lambda} a_{\lambda}(q) f_{\lambda}(\mathbf{x})$. Then we have*

$$a_{\lambda}(q) = \sum_{\mathbf{w}} q^{\text{inv}_P(\mathbf{w})},$$

where $\mathbf{w}$ ranges over all words of type μ such that when we split $\mathbf{w}$ from left to right into consecutive segments of lengths $\lambda_1, \lambda_2, \dots, \lambda_{\ell}$, each segment has no P -descents.

4.2 Noncommutative P -Schur functions

We define noncommutative P -Schur functions via the dual Jacobi–Trudi identity (2.2).

Definition 4.5. For a partition λ , we define the *noncommutative P -Schur function* $\mathfrak{J}_{\lambda}$ by

$$\mathfrak{J}_{\lambda}(\mathbf{u}) = \sum_{\sigma \in \mathfrak{S}_m} \text{sgn}(\sigma) \epsilon_{\lambda'_1 - \sigma(1) - 1}(\mathbf{u}) \epsilon_{\lambda'_2 - \sigma(2) - 2}(\mathbf{u}) \cdots \epsilon_{\lambda'_m - \sigma(m) - m}(\mathbf{u}), \quad (4.7)$$

where λ' is the conjugation of λ and $m = \lambda_1$.

Proposition 4.6. *We have*

$$\Omega \mathbf{x}, \mathbf{u} \equiv \sum_{\lambda} s_{\lambda}(\mathbf{x}) \mathfrak{J}_{\lambda}(\mathbf{u}) \pmod{\mathcal{I}_P(\mathbf{x})}. \quad (4.8)$$

Similar to ordinary Schur functions, we will provide a combinatorial description of noncommutative P -Schur functions.

Definition 4.7. For a partition λ , a *semistandard P -tableau of shape λ* is a filling of the Young diagram of shape λ with n satisfying that

- (i) each row is non- P -decreasing from left to right, and

(ii) each column is P -increasing from top to bottom.

A semistandard P -tableau T is of type $\mu = \mu_1, \dots, \mu_n$ if each $i \in [n]$ appears μ_i times in T . We denote the set of all semistandard P -tableaux of shape λ by $\mathcal{T}_P \lambda$. The *reading word* $w(T)$ of T is the word obtained by reading T from bottom to top, beginning with the leftmost column of T and working from left to right.

Theorem 4.8. *We have*

$$\mathfrak{J}_\lambda \mathbf{u} \equiv \sum_{T \in \mathcal{T}_P \lambda} u_{w(T)} \pmod{\mathcal{I}_P}. \quad (4.9)$$

Consequently, $\omega X_P \mathbf{x}, q; \mu$ is s -positive and its coefficient of s_λ counts semistandard P -tableaux of shape λ and of type μ . In other words,

$$\omega X_P \mathbf{x}, q; \mu = \sum_T q^{\text{inv}_P T} s_{\text{sh}(T)} \mathbf{x},$$

where T ranges over all semistandard P -tableaux of type μ and $\text{sh}(T)$ denotes the shape of T .

Example 4.9. Let $P = 2, 3, 3$ and $\mu = 1, 1, 2$; see Figure 3. To obtain the coefficient $a_{3,1} q$ of $s_{3,1} \mathbf{x}$ of $\omega X_P \mathbf{x}, q; \mu$, it suffices to find words w such that w is a reading word for some semistandard P -tableaux of shape $(3, 1)$:

$$\begin{array}{ccccccc} 1233 & 1323 & \text{---} \boxed{3123} & 1332 & \text{---} \boxed{3132} & \text{---} 3312 & \\ & | & & | & & | & \\ & 2133 & \text{---} 2313 & \text{---} 2331 & 3213 & \text{---} 3231 & 3321 \end{array}$$

Therefore we have $a_{3,1} q = q^2 + q$.

4.3 Noncommutative P -monomial symmetric functions

Let $N_{\lambda, \mu}$ be the transition matrix between the monomial symmetric functions and the elementary symmetric functions, i.e., $m_\lambda \mathbf{x} = \sum_\mu N_{\lambda, \mu} e_\mu \mathbf{x}$.

Definition 4.10. For a partition λ , the *noncommutative P -monomial symmetric function* $\mathfrak{m}_\lambda \mathbf{u}$ is defined by

$$\mathfrak{m}_\lambda \mathbf{u} = \sum_\mu N_{\lambda, \mu} e_\mu \mathbf{u}. \quad (4.10)$$

Proposition 4.11. *We have*

$$\Omega \mathbf{x}, \mathbf{u} \equiv \sum_\lambda h_\lambda \mathbf{x} \mathfrak{m}_\lambda \mathbf{u} \pmod{\mathcal{I}_P \mathbf{x}}. \quad (4.11)$$

Stanley and Shareshian–Wachs showed that the sum of certain e -coefficients of chromatic quasisymmetric functions are related to acyclic orientations of the graph. For $X_P \mathbf{x}, q; \mu$, we have a similar description of $\sum_{\ell \leq \lambda} c_{\ell, \lambda} q$ in terms of certain heaps.

Theorem 4.12. *Let P be a natural unit interval order. Then for $d, k \geq 1$, we have*

$$\sum_{\substack{\lambda \vdash d \\ \ell(\lambda) = k}} m_\lambda \mathbf{u} \equiv \sum_H u_{w_H} \pmod{\mathcal{I}_P},$$

where H ranges over all heaps of P consisting of d pieces with k sinks. Consequently, let $X_P \mathbf{x}, q; \mu = \sum_\lambda c_\lambda q e_\lambda \mathbf{x}$. Then we have

$$\sum_{\ell(\lambda) = k} c_\lambda q = \sum_H q^{\text{asc}_P H},$$

where H ranges over all heaps of type μ with k sinks.

We now give a positive monomial expression of $m_\lambda \mathbf{u}$ where λ is of 2-column shape. Given a heap H , the *rank* of a piece p is the height of p in the diagrammatic realization of H , denoted by $\text{rank } p$. In particular, $\text{rank } p = 1$ if and only if p is a sink. In addition, we define the *rank* of H by the maximum rank of pieces. We say that a connected heap of rank 2 is of *type W* if the number of piece of rank 1 equals the number of pieces of rank 2 minus 1.

Theorem 4.13. *For $k \geq \ell \geq 0$, we have*

$$m_{2^\ell, 1^{k-\ell}} \mathbf{u} \equiv \sum_H u_{w_H} \pmod{\mathcal{I}_P}, \quad (4.12)$$

where H ranges over all heaps of P such that H consists of k pieces of rank 1 and ℓ pieces of rank 2, and has no connected component of type W. Consequently, let $X_P \mathbf{x}, q; \mu = \sum_\lambda c_\lambda q e_\lambda \mathbf{x}$, then

$$c_{2^\ell, 1^{k-\ell}} q = \sum_H q^{\text{asc}_P H},$$

where H ranges over such heaps of type μ .

Example 4.14. Let $P = P_{2,3,4,5,5}$ and $\mu = 1^5$ (see Figure 2). There are two connected heaps of type μ of rank 2; one is of type W while the other is not. Then by Theorem 4.13, we have $c_{2,2,1} q = q^2$.

Corollary 4.15. *Let P be a natural unit interval order on $[n]$, and $X_P \mathbf{x}, q = \sum_\lambda c_\lambda q e_\lambda \mathbf{x}$. Let n_e and n_o be the numbers of connected components of P consisting of even and odd vertices, respectively. Then for a partition λ of 2-column shape,*

$$c_\lambda q = \begin{cases} q^{n-2n_e-n_o/2-1} q^{n_e} & \text{if } P \text{ is triangle-free and } \lambda' = (n - n_o/2, n - n_o/2), \\ 0 & \text{otherwise.} \end{cases}$$

In particular, the coefficients $c_\lambda q$ where λ are of 2-column shape are unimodal.

We next give a recurrence relation for $m_\lambda \mathbf{u}$, which is equivalent to Harada–Precup’s conjecture [3, Conjecture 8.1]. For a natural unit interval order P , the *height* h of P is the length of longest chains in P . Then, by definition, $e_k \mathbf{u} = 0$ for all $k > h$.

Theorem 4.16 ([3, Conjecture 8.1]). *Let P be a natural unit interval order, and h the height of P . Then, for a partition λ of length $\ell \geq h$, we have*

$$m_\lambda \mathbf{u} \equiv \begin{cases} 0 & \text{if } \ell > h, \\ e_h \mathbf{u} m_{\lambda^-} \mathbf{u} & \text{if } \ell = h, \end{cases}$$

modulo $\mathcal{I}_P$, where $\lambda^- = \lambda_1 - 1, \dots, \lambda_\ell - 1$.

By the same argument as in [3], Theorem 4.16 implies e -positivity of $X_P(\mathbf{x}, q; \mu)$ for natural unit interval orders P of height 2.

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Compact Hyperbolic Coxeter d -Polytopes with $d + 4$ Facets and Related Dimension Bounds

Amanda Burcroff^{*1}

¹*Department of Mathematics, Harvard University, Cambridge, MA, USA*

Abstract. We complete the classification of compact hyperbolic Coxeter d -polytopes with $d + 4$ facets for $d = 4$ and 5 . By previous work of Felikson and Tumarkin, the only remaining dimension where new polytopes may arise is $d = 6$. We derive a new method for generating the combinatorial types of these polytopes via the classification of point set order types. In dimensions 4 and 5 , there are 348 and 51 polytopes, respectively, yielding many new examples for further study.

We furthermore provide new upper bounds on the dimension d of compact hyperbolic Coxeter polytopes with $d + k$ facets for $k \leq 10$. It was shown by Vinberg in 1985 that there are no compact hyperbolic Coxeter polytopes in dimensions higher than 29 , and no better dimension bounds have previously been published for $k \geq 5$. As a consequence of our bounds, we prove that a compact hyperbolic Coxeter 29 -polytope has at least 40 facets.

Keywords: Coxeter polytope, hyperbolic reflection group, affine Gale diagram

1 Introduction

Let $\mathbb{H}^d$ be the d -dimensional real hyperbolic space. A *hyperbolic Coxeter polytope* is a domain in $\mathbb{H}^d$ bounded by a collection of geodesic hyperplanes, such that each intersecting pair of hyperplanes meets at dihedral angle $\frac{\pi}{m}$ for some integer $m \geq 2$. Hyperbolic Coxeter polytopes are precisely the fundamental domains of discrete hyperbolic reflection groups. These polytopes also have relevance to the construction of orbifolds and manifolds, in particular some of minimal volume [17].

We classify the compact hyperbolic Coxeter d -polytopes with $d + 4$ facets for $d = 4$ and 5 , as well as improve some bounds on the dimension of compact Coxeter polytopes with few facets. While Euclidean and spherical Coxeter polytopes were classified by Coxeter in 1934 [5], no complete classification is known in the hyperbolic case. Henceforth, all polytopes are assumed to be hyperbolic unless otherwise specified.

Many of the classification results for compact Coxeter polytopes have been obtained by restricting either the dimension, combinatorial type, or number of facets. A dynamic summary of this progress is maintained on Anna Felikson's webpage [8].

^{*}aburcroff@math.harvard.edu. Amanda Burcroff was supported by the NSF GRFP and the Marshall Commission.

1.1 Classifying compact Coxeter d -polytopes with $d + 4$ facets for $d \neq 6$

We first focus on restricting the number of facets with respect to the dimension. Compact Coxeter simplices, i.e., d -polytopes with $d + 1$ facets, were classified by Lannér in 1950 [18]. These arise only in dimensions 2, 3, and 4. The compact Coxeter d -polytopes with $d + 2$ facets are classified in [16] and [7]; these arise in dimensions 3 through 5 and are simplicial prisms except in dimension 4. Esselmann [6] showed in 1994 that a compact Coxeter d -polytope with $d + 3$ facets must satisfy $d \leq 8$, and that there is a unique polytope of dimension 8 (first constructed by Bugaenko [3]). In 2007, Tumarkin [22] completed the classification of compact Coxeter d -polytopes with $d + 3$ facets, which arise in dimensions 2 through 6 and 8.

The first portion of this work is dedicated to furthering the classification of compact Coxeter d -polytopes with $d + 4$ facets. In 2008, Felikson and Tumarkin [9] showed that such polytopes arise only in dimension at most 7, and furthermore that there is a unique compact Coxeter 7-polytope with 11 facets (originally constructed by Bugaenko [3]). We complete this classification in dimensions 4 and 5, leaving dimension 6 as the only remaining dimension where new polytopes can arise.

Theorem 1. *There are 348 hyperbolic Coxeter 4-polytopes with 8 facets, and 51 hyperbolic Coxeter 5-polytopes with 9 facets. For the full list, see the appendices of [4].*

Remark 2. The classification above includes

- the first known hyperbolic Coxeter polytope in dimension higher than 3 with a dihedral angle of less than $\frac{\pi}{10}$,
- the first and only known hyperbolic Coxeter polytope in dimension higher than 3 with a dihedral angle of $\frac{\pi}{7}$, and
- the first known compact Coxeter polytope in dimension 5 with a dihedral angle of less than $\frac{\pi}{6}$,

along with many new *essential* polytopes.

A polytope is *essential* if it is minimal with respect to the operations of taking the fundamental domain of a finite index reflection subgroup of the corresponding reflection group or gluing two Coxeter polytopes along congruent facets (see [12] for further details). See the Coxeter diagrams in Figure 1 for examples of polytopes having these properties.

The work in the paper on which this extended abstract is based combined with that of Felikson and Tumarkin yields a classification in all dimensions except 6. There is only one known compact Coxeter 6-polytope with 10 facets, which was constructed by Bugaenko [3].

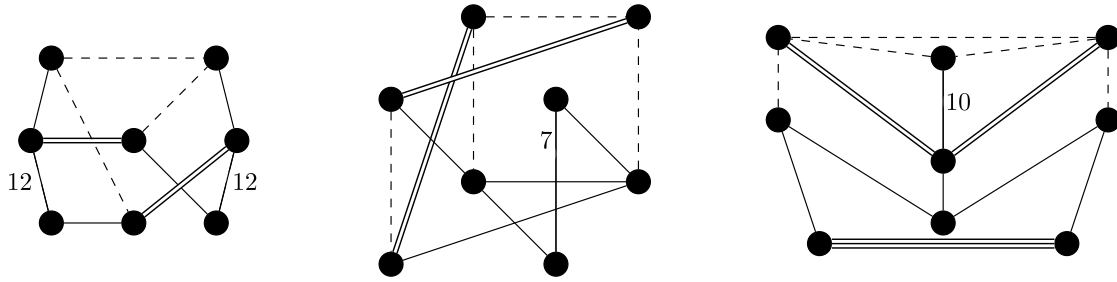


Figure 1: The Coxeter diagrams of three new polytopes from the classification in [Theorem 1](#), all of which are essential. See [subsection 2.1](#) for details on how to interpret Coxeter diagrams.

In order to obtain this classification, we develop a new method for restricting the possible combinatorial types of these polytopes. The Gale diagram of a d -dimensional polytope with n facets is an $n - d - 1$ arrangement of points in Euclidean space that encodes the combinatorial type of the polytope. Hence the combinatorial types of d -polytopes with $d + 4$ facets can be studied in terms of a 3-dimensional point arrangement. Moreover, each Gale diagram can be transformed into an affine Gale diagram, an arrangement of signed points, to further reduce the dimension by 1 [24]. We show that the (affine) Gale diagrams of compact d -dimensional hyperbolic polytopes with $d + 4$ facets can be generated by bipartitioning the points in all simple point set order types with $d + 4$ points (see [Theorem 5](#) for further details). The point set order types of sizes up to 10 have been enumerated and made available through the Point Set Order Type Database [1]. Using this database, we produce a reasonably short list of possible combinatorial types for the polytopes of interest in dimensions 4 and 5. In dimension 6, the same methods can be applied, but the number of point set order types makes the process rather computationally demanding.

Having greatly restricted the possible combinatorial types in dimensions 4 and 5, we then determine whether each combinatorial type can be realised as one or more polytopes. This involves enumerating weighted graphs with restrictions on certain subgraphs and the spectral properties of their adjacency matrices. Though the search space is infinite, combinatorial and linear algebraic techniques (see, e.g., [24]) have previously been successful in reducing this to a computational problem. In particular, Tumarkin handled the analogous task for polytopes with $d + 3$ facets by inspecting local determinants, face structures, and gluings of Lannér diagrams [22]. These methods are only partially effective for the polytopes with $d + 4$ facets, due to the greater complexity of the polytopes and less restrictive combinatorial types. We then utilise Mathematica to check a finite number of cases in order to list all polytopes of a given combinatorial type, a technique which was recently used in classifying the compact Coxeter cubes [15].

Ma and Zheng independently and via different methods classified the compact hy-

perbolic Coxeter 4-polytopes with 8 facets [19] and 5-polytopes with 9 facets [20]. Their work became publicly available within a few months after the release of the author's master's thesis on this topic. The author is very grateful to Ma and Zheng for their communication about this classification, as it helped to correct several minor errors. Due to the sheer volume of data handling required by both our methods and those of Ma and Zheng, there were a few errors in the polytope lists initially announced by both groups, though the methods seem sound. Ma, Zheng, and the current author now agree on the published lists. The existence of two independent methods for obtaining these polytopes may lend some confidence to the accuracy of this rather delicate classification.

1.2 Bounding the dimension of polytopes with few facets

We then shift our focus to bounding the dimension of certain compact Coxeter polytopes. It was shown in 1984 by Vinberg [23] that compact Coxeter polytopes do not arise in dimensions higher than 29. Vinberg proceeded by constructing certain weightings on the edges of the polytopes and utilised a result of Nikulin [21] on the average number of vertices along a 2-dimensional face. A *missing face* of a polytope is a minimal set of facets whose intersection is empty. We show that a slight modification of Vinberg's argument yields a stronger bound for *3-free polytopes*, that is, polytopes having missing faces only of order 2.

Theorem 3. ¹ *Compact hyperbolic Coxeter 3-free polytopes do not arise in dimensions higher than 13.*

Using this property of 3-free polytopes along with the classification results, we improve the bounds on the dimension of compact Coxeter d -polytopes with $d + k$ facets for $5 \leq k \leq 10$. In order to obtain an initial bound, we examine certain faces which must themselves be compact Coxeter polytopes, similar to the methods used by Felikson and Tumarkin [9] to bound the dimension when $k = 4$. We are able to improve these bounds in certain cases, frequently referring to the classification of polytopes with fewer facets. A corollary of our improved bounds is that any compact Coxeter polytopes of dimension 29, i.e., the threshold of Vinberg's bound, must have at least $29 + 11 = 40$ facets.

2 Combinatorial Types of d -Polytopes with $d + 4$ Facets

2.1 Gale and Coxeter diagrams

An important technique in classifying convex polytopes is representing a polytope by a diagram from which one can read off the face structure. We will work with *affine Gale*

¹Since the release of this work, our upper bound of 13 has been improved using similar methods (which, in turn, were inspired by those of Vinberg [23]) to 12 by Alexandrov [2].

diagrams, which are a projection of the more classical *Gale diagrams*, introduced in a 1956 paper by David Gale [13]. Note that while (affine) Gale diagrams are often defined using the vertices of a polytope, we look at the dual construction defined on the facets.

Given a simple d -polytope P with $d + k$ facets, a Gale diagram of P consists of a set of $d + k$ points on the sphere $S^{k-2} \subset \mathbb{R}^{k-1}$ corresponding to the facets of P . In the case $k = 4$, Gale diagrams consist of point configurations on $S^2 \subseteq \mathbb{R}^3$. These seem rather difficult to classify, and thus a crucial step in our analysis is passing to the affine Gale diagrams. These affine variants encode the same information as Gale diagrams but use partitioned sets of points in $\mathbb{R}^{k-2}$.

An *affine Gale diagram* of a d -polytope with $d + k$ facets consists of two (not necessarily disjoint) point sets, called “positive” and “negative”, in $\mathbb{R}^{k-2}$ containing $d + k$ points in total (counted with multiplicity). An affine Gale diagram is obtained from a Gale diagram G by taking a hyperplane H through the origin not containing any points of G , and projecting the points orthogonally onto H , with the projections of points from the open half space H^+ being labelled “positive” and those from H^- labelled “negative”. The face structure of P is determined in the following way: a set of facets $\{f_i : i \in I\}$ of P has a non-trivial intersection if and only if the convex hull of the positive points in $\{f_j : j \notin I\}$ non-trivially intersects the convex hull of the negative points in $\{f_j : j \notin I\}$.

We now define another type of diagram that not only encodes the combinatorial type of a polytope, but also its dihedral angles.

Definition 4. An *abstract Coxeter diagram* is a finite simple graph with weighted ordinary and dashed edges. The weights w_{ij} must be positive and satisfy the following condition: if $w_{ij} < 1$, then $w_{ij} = \cos\left(\frac{\pi}{m_{ij}}\right)$ for some integer $m_{ij} \geq 3$.

The word “abstract” highlights that the diagrams we consider do not always correspond to polytopes. When such a diagram does correspond to a polytope, then the edges of weight $\cos\left(\frac{\pi}{m_{ij}}\right)$ encode that the dihedral angle between facets f_i and f_j is $\frac{\pi}{m_{ij}}$ and non-intersecting facets at distance x are encoded by an edge of weight $\cosh(x)$.

We often denote an abstract Coxeter diagram by the letter Σ . The order $|\Sigma|$ of the diagram Σ is the number of vertices of Σ . A *subdiagram* Σ' of Σ is a vertex-induced subgraph of Σ , where the edge weights are preserved; this relation is written as $\Sigma' \subset \Sigma$. Since we are considering only compact polytopes, there are no edges of weight 1 in our setting. When drawing abstract Coxeter diagrams, we adhere to the following conventions:

- (i) If $w_{ij} < 1$, hence $w_{ij} = \cos\left(\frac{\pi}{m_{ij}}\right)$ for some integer $m_{ij} \geq 2$, then the corresponding edge is drawn as a straight line labelled by m_{ij} . In the special cases where m_{ij} is equal to 3, 4, or 5, we draw the corresponding edge as an unlabelled single, double, or triple line, respectively. If $m_{ij} = 2$, we leave the edge empty.

- (ii) If $w_{ij} > 1$, the corresponding edge is drawn as a dashed line with label w_{ij} . We often omit these weights, since they can be determined from rest via rank conditions.

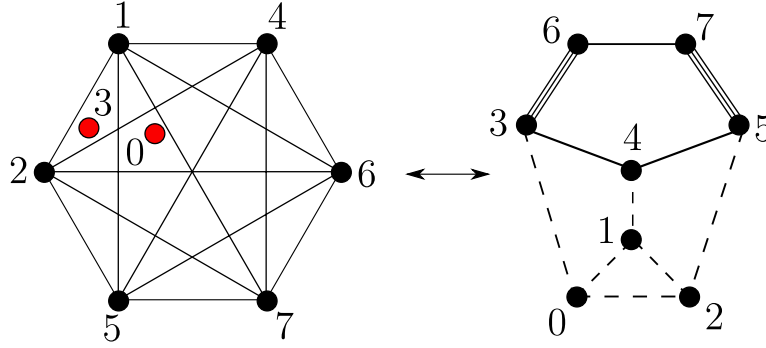


Figure 2: On the left is an affine Gale diagram of a 4-polytope with 8 facets, with the positive points depicted in red and the negative points in black. The line segments between negative points are included to aid in identifying the convex hull of each subset. On the right, we have a Coxeter diagram of polytope realising this combinatorial type.

2.2 Affine Gale diagrams of simple d -polytopes with $d + 4$ facets

In order to determine the compact hyperbolic d -polytopes with $d + 4$ facets in dimension 4 and 5, our methods involve first limiting their possible combinatorial types. This was accomplished for compact hyperbolic d -polytopes with $d + 3$ facets by Tumarkin [22], using *standard Gale diagrams* described by Esselmann [6]. However, Tumarkin's methods do not seem immediately generalisable to Gale diagrams of higher dimension. We develop a new method for generating affine Gale diagrams using a classification of point set order types by Aichholzer, Aurenhammer, and Krasser [1]. These are reduced to a representative list of Gale diagrams, from which one can easily determine the set of missing faces. Later sections are devoted to determining which of these combinatorial types are realisable.

A set of points in $\mathbb{R}^d$ is said to be in *general position* if no m points lie in a subspace of dimension $m - 2$ for $m = 2, \dots, d + 1$. We show that we can slightly perturb the points of the affine Gale diagram so that the points are in general position without changing the combinatorial structure.

Furthermore, it has been shown by Tumarkin and Felikson [10, 11] that for $k \geq 4$, every compact d -polytope with $d + k$ facets has at least two pairs of non-intersecting facets. Given a polytope P with $d + 4$ facets, let f and f' be a non-intersecting pair of facets. In any Gale diagram associated to P , the points corresponding to f and f' can be separated from the remaining points by a hyperplane through the origin. Take the affine Gale diagram obtained by orthogonal projection onto this hyperplane and

choosing the half-space containing f and f' to be positive. Thus, we can obtain an affine Gale diagram associated to P where all points are in general position and with exactly two positive points. In summary, we have the following result.

Theorem 5. *Every simple d -polytope with $d + 4$ facets admits an affine Gale diagram $A = A_+ \cup A_-$, where A_+ and A_- are finite sets of points in $\mathbb{R}^2$ such that*

- (i) *all points of A are in general position,*
- (ii) *$|A_+| = 2$, and*
- (iii) *A_+ is contained in the interior of the convex hull of A_- .*

An example of such an affine Gale diagram and a polytope realising its combinatorial type are depicted in [Figure 2](#). We use these restrictions to show that all affine Gale diagrams of compact d -polytopes with $d + 4$ facets can be obtained by bipartitioning the $d + k$ points of a point set order type.

2.3 Point set order types

In order to enumerate the possible Gale diagrams of compact d -polytopes with $d + 4$ facets, we utilise a classification of the point set order types of size at most 10 [\[1\]](#). The *order type* of a set of points $\{p_1, \dots, p_n\} \subset \mathbb{R}^2$ in general position is a mapping that assigns to each ordered triple (i, j, k) of distinct elements in $\{1, \dots, n\}$ the orientation of the point triple (p_i, p_j, p_k) . Two finite subsets $P, Q \subset \mathbb{R}^2$ are said to be *combinatorially equivalent* if they have the same order type, i.e., if there exists a bijection $f : P \rightarrow Q$ that preserves orientations.

We now obtain our method for generating the desired affine Gale diagrams using the point set order types. Namely, taking a representative of each simple order type and iterating over all choices of two positive points from the interior of the convex hull yields all possible affine Gale diagrams.

Using the order type database for up to 10 points [\[1\]](#), one can classify the combinatorial types of all polytopes with $d + 4$ facets in dimension at most 6. There are 3,315 order types on 8 points, which yield 34 possible combinatorial types of simple 4-polytopes with 8 facets having at least one pair of disjoint facets, listed in [\[4, Appendix A\]](#). The classification in dimension 4 was also completed by Grünbaum and Sreedharan [\[14\]](#) via different methods, though they also considered polytopes with no pairs of disjoint facets. There are 158,817 order types on 9 points, which yield 186 possible combinatorial types of 5-polytopes with 9 facets; the 111 combinatorial types with at least two pairs of disjoint facets are listed in [\[4, Appendix A\]](#). Over 14 million order types on 10 points yield a list of 265 possible combinatorial types of 6-polytopes with 10 facets. Note that not all of these combinatorial types are realized; this is merely a short list of possible types from which we determine those that are realized as compact Coxeter polytopes.

2.4 Computational methods

We now roughly outline the computational steps used to classify compact Coxeter polytopes of a given combinatorial type G . We begin with a complete graph on $d + k$ vertices with unknown edge weights, and describe the process by which we assign edge weights such that every possible Coxeter diagram of a d -polytope with $d + 4$ facets is constructed.

1. Select a set of dihedral angles taking values at most $\frac{\pi}{6}$

For each combinatorial type, we begin by restricting the dihedral angles which can be at most $\frac{\pi}{6}$ using the combinatorial structure and local determinants. In practice, each combinatorial type generated by the methods in the previous section can be limited to having at most five such dihedral angles. We then iterate over all subsets of these angles, and examine the Coxeter diagrams for which this set is precisely the set of dihedral angles of size at most $\frac{\pi}{6}$. Though at this stage we have not fixed the exact size of these angles, knowing that they have size at most $\frac{\pi}{6}$ yields restrictions on the remaining dihedral angles.

2. Assign edge weights within Lannér diagrams of size 4 and 5

The subdiagrams of a Coxeter diagram corresponding to a missing face were classified by Lannér [18] and are known as Lannér diagrams. The maximum size of a Lannér diagram is 5, and there are finitely many Lannér diagrams of size 4 or 5. In particular, if we restrict to only those vertices in the Coxeter diagram contained in Lannér diagrams of size 4 or 5, the resulting subdiagram is obtained by gluing Lannér diagrams from this list. We can thus iterate over the possible Lannér diagrams to assign weightings to edges within such a subdiagram.

3. Assign the dihedral angles of size at least $\frac{\pi}{5}$

There are finitely many assignments of the remaining dihedral angles, i.e., those not contained within a Lannér diagram of size 4 or 5. We iterate over all these possibilities, with the additional restriction that all subdiagrams not containing a missing face must be elliptic. We furthermore require that subdiagram induced by any two Lannér diagrams is connected, lest this subdiagram be superhyperbolic.

4. Solve for the remaining Coxeter diagram edge weights

At this stage, the only quantities which have not been assigned are the dihedral angles of size less than $\frac{\pi}{6}$ and distance between divergent facets. For the former, the weight of the corresponding edge in the Coxeter diagram must be a real number in the range $[\cos(\frac{\pi}{6}), 1) = [\frac{\sqrt{3}}{2}, 1)$. The weight of each edge corresponding to two divergent facets must be a real number in the range $(1, \infty)$. Restricting to these ranges, we find all solutions to an appropriately chosen system of equations, which encodes certain rank conditions. We do so using the computer algebra system *Mathematica*.

5. Check the signs of the eigenvalues of the resulting Gram matrix.

For each of the solutions to the system of equations, we obtain an associated *Gram matrix*. It follows from the theory of hyperbolic polytopes that it is sufficient to check that the resulting matrix has exactly 1 negative eigenvalue and that its rank is $d + 1$. If these conditions are met, we can conclude that this Coxeter diagram corresponds to Coxeter d -polytope with $d + k$ facets of combinatorial type G .

The code implementing the process described above is publicly available at https://github.com/agburcroff/Cox_d-Polytopes_with_dplus4_Facets.git.

3 Coxeter Polytope Dimension Bounds

We first present a bound on compact Coxeter 3-free polytopes, i.e., polytopes where every missing face is of size 2. We utilise this bound along with the classification of compact Coxeter d -polytopes with few facets to place upper bounds on the dimensions of polytopes with $d + k$ facets for $5 \leq k \leq 10$. To our knowledge, the best previous bound on the dimension for each of these classes of polytopes was Vinberg's bound $d \leq 29$ for all compact Coxeter polytopes.

The proof of our dimension bound for 3-free polytopes is refinement of Vinberg's proof that simple Coxeter polytopes have dimension at most 29. This involves studying weightings of the planar angles of such polytopes. A *planar angle* of a polytope is a pair of a vertex and a two-dimensional face containing the vertex. Based on such a weighting, it is possible to derive the following dimension bound.

Proposition 6 ([24, Prop. 6.2]). *Let P be a d -dimensional compact Coxeter polytope and $c > 0$. We assume that the planar angles of P can be endowed with weights such that the sum of the weights of the planar angles at the vertex is at most cd and that the sum of the weights of the planar angles of any 2-dimensional face with k vertices is at least $5 - k$. Then $d < 8c + 6$.*

We construct a weighting on the planar angles of a compact Coxeter 3-free polytope such that the unique nonzero planar angle for each vertex has weight 1 and that each 2-dimensional face with 4 vertices has a nonzero planar angle. Thus we can derive the following dimension bound from [Proposition 6](#) with $c = 1$.

Theorem 7. *There are no compact 3-free Coxeter hyperbolic polytopes of dimension 14 or higher.*

While this result may be of interest in its own right, we presently make use of this result to bound the dimension of polytopes with few facets.

Definition 8. Let $D(k)$ denote the maximum positive integer for which a compact Coxeter $D(k)$ -polytope with $D(k) + k$ facets exists.

By Vinberg's result [23, Theorem 4], we have $D(k) \leq 29$ for all k . For $k \leq 4$, the exact value of $D(k)$ is known.

Theorem 9.² *We have*

- $D(1) = 4$, due to Lannér [18];
- $D(2) = 5$, due to Kaplinskaja [16] and Esselmann [6];
- $D(3) = 8$, due to Esselmann [6];
- $D(4) = 7$, due to Felikson and Tumarkin [9].

Thus, we begin our investigation with $k = 5$, and proceed until the bounds obtained by our methods are no stronger than the general bound of Vinberg. We derive the following improved bounds.

Theorem 10. *We have $D(5) \leq 9$, $D(6) \leq 12$, $D(7) \leq 15$, $D(8) \leq 18$, $D(9) \leq 22$, and $D(10) \leq 26$.*

The argument proceeds by first proving a linear bound on d having slope 4 (with respect to k), then applying the classification of polytopes with fewer facets to slightly improve these bounds in particular cases. The initial linear bound is obtained by first iterating part of the argument used by Felikson and Tumarkin to bound $D(4)$ [9], which yields Lemma 11.

Lemma 11. *If P contains a missing face of size $\ell > 2$, then*

$$D(k) \leq \max_{1 \leq i \leq k-1} D(i) + \ell - 1.$$

Combining these results with our dimension bound of 13 on 3-free polytopes, we obtain the following dimension bound.

Theorem 12. *For $k \geq 5$, we have*

$$D(k) \leq \max \{ \max \{ D(i) + 4 : i < k \}, \min \{ k - 1, 13 \} \}.$$

Proof. Fix a Coxeter d -polytope P with $d + k$ facets. If P is not 3-free, then it contains a missing face of order 3, 4, or 5. The first bound then follows from Lemma 11. $\square$

In order to improve the bounds given by Theorem 12 to those in Theorem 10, we rely heavily on the classification of compact hyperbolic Coxeter d -polytopes with at most $d + 4$ facets. The general form of these fairly technical arguments is as follows:

²Anna Felikson and Pavel Tumarkin have shown in unpublished notes that $D(5) \leq 8$ using a fairly involved argument [Tumarkin, personal communication (2021)].

- Suppose that a compact hyperbolic Coxeter d -polytope P has $d + k$ facets and a missing face of size ℓ . We consider proper subsets Σ_i of the facets corresponding to this missing face.
- Let $P(\Sigma_i)$ be the face of the polytope formed by intersecting the facets in Σ_i . Under certain easily-checked conditions, a face of a Coxeter polytope is itself a Coxeter polytope. We then choose Σ_1 such that $P(\Sigma_1)$ is a Coxeter polytope, and hence it must be a $(d - |\Sigma_1|)$ -polytope with at most $d + k - \ell$ facets.
- Using the classification of polytopes with few facets, we can determine some properties of the polytope $P(\Sigma_i)$. Using a result of Felikson and Tumarkin [9, Corollary 1.1], this yields some restrictions on the dihedral angles between facets not contained in Σ_i . We can then use these restrictions to choose a new subset of facets Σ_{i+1} and again look at the face $P(\Sigma_{i+1})$.
- We repeat the previous step until we have sufficiently many restrictions on the dihedral angles of P to deduce that no such polytope P exists.

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Balanced Shifted Tableaux

Jiyang Gao^{*1}, Shiliang Gao⁺², and Yibo Gao^{‡3}

¹Department of Mathematics, Harvard University, Cambridge, MA 02138

²Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL 61801

³Department of Mathematics, University of Michigan, Ann Arbor, MI 48104

Abstract. We introduce balanced shifted tableaux, as an analogue of balanced tableaux of Edelman and Greene, from the perspective of root systems of type B and C . We show that they are equinumerous to standard Young tableaux of the corresponding shifted shape by presenting an explicit bijection.

Keywords: balanced tableaux, standard Young tableaux, Edelman-Greene insertion

1 Introduction

In their seminal paper [2], Edelman and Greene introduced *balanced tableaux* and showed that they are equinumerous to standard Young tableaux of the same shape. They defined the *Edelman-Greene insertion* which yields a bijective proof of the reduced words of the longest permutation being equinumerous to standard Young tableaux of staircase shape, a result due originally to Stanley [10]. Fomin, Greene, Reiner and Shimozono [3] later generalized this enumeration result to diagrams and related the story to Schubert polynomials.

Shifted tableaux, just as Young tableaux, are also algebraically and combinatorially meaningful (see for example [9, 12]). In this paper, we define *balanced shifted tableaux* (Definition 2.3), as an analogue to balanced tableaux, from the perspective of root systems of type B and C . The following is our main theorem, which says that balanced shifted tableaux are equinumerous to standard shifted tableaux.

Theorem 1.1. *For a shifted shape λ , the number of standard Young tableaux of shape λ equals the number of balanced shifted tableaux of shape λ .*

We prove Theorem 1.1 by presenting an explicit bijection between the two sets of objects, SYT λ and BS λ . Specifically, we have the following chain of bijections:

$$\text{SYT } \lambda \longleftrightarrow \text{SYT } Z^{d,r} \mid_{\lambda} \longleftrightarrow \text{Red } w^{d,r} \mid_{\lambda} \longleftrightarrow \text{BS } Z^{d,r} \mid_{\lambda} \longleftrightarrow \text{BS } \lambda ,$$

^{*}jgao@math.harvard.edu.

⁺sgao23@illinois.edu. Shiliang Gao was partially supported by NSF Graduate Research Fellowship under grant No. DGE-1746047.

[‡]gaoyibo@umich.edu.

where we address each step separately. We defer the definition of $\text{SYT } \lambda$ and $\text{BS } \lambda$ to [Section 2](#) and the definition of $\text{SYT } Z(d, r) \mid_\lambda$, w^λ and $\text{BS } Z(d, r) \mid_\lambda$ to [Section 5](#). Here, $\text{SYT } \lambda \rightarrow \text{SYT } Z(d, r) \mid_\lambda$ and $\text{BS } \lambda \rightarrow \text{BS } Z(d, r) \mid_\lambda$ are the procedures to pad a tableau from shape λ to a large trapezoid $Z(d, r)$, while the middle steps utilize type B Edelman-Greene insertion defined by Kraśkiewicz [6]. Our strategy largely follows the framework of Edelman and Greene [2], with the main difference that double staircases, which are the analogues of staircases in type B , are no longer sufficient for padding purposes.

2 Definitions and Preliminaries

2.1 Strict partitions and shifted tableaux

A *strict partition* λ is a sequence of strictly decreasing positive integers $\lambda_1 > \lambda_2 > \dots > \lambda_d > 0$, where d is the number of (nonzero) parts of λ . We denote $|\lambda| = \sum_{i=1}^d \lambda_i$ as the *size* of λ . For a strict partition λ its corresponding *shifted shape*, consists of λ_i boxes in row i , shifted $d - i - 1$ steps to the left. More specifically, the shifted shape is the diagram

$$D(\lambda) : \{ (i, j - d - i + 1) \mid 1 \leq i \leq d, 1 \leq j \leq \lambda_i \}.$$

For simplicity of notation, we also use λ to denote its shape $D(\lambda)$. Note that for a shifted shape, its columns $-d-1, \dots, 0$ form a staircase shape of length d flipped horizontally. For a shifted shape λ , define a *shifted tableau* T to be a filling of $D(\lambda)$ with non-negative integers. For any shifted tableau T , let $\text{sh } T$ denote its underlying shifted shape.

Throughout the paper, we fix the number d , that is the length of all the shifted shapes we are going to consider. We also write $\bar{i}$ to mean $-i$.

Definition 2.1. A shifted tableau T of shape λ is called a *standard Young tableau* if it is a filling of $1, 2, \dots, |\lambda|$ that is increasing in rows and columns.

The set of standard Young tableaux of shape λ is denote $\text{SYT } \lambda$ and its cardinality is denoted f^λ . The number f^λ can be computed via the hook length formula as we explain here. For a box $(i, j) \in \lambda$ with $j \geq 0$, its *hook* $H(i, j)$ consists of all the boxes in row i to the right of (i, j) , all the boxes in column j below (i, j) and the box (i, j) itself. For a box $(i, \bar{j}) \in \lambda$ with $j > 0$, its *hook* $H(i, \bar{j})$ consists of all the boxes in row i to the right of $(i, \bar{j})$, all the boxes in column $\bar{j}$ below $(i, \bar{j})$, the box $(i, \bar{j})$ itself and all the boxes in row $d - j - 1$. Let $h(i, j) = |H(i, j)|$ be the size of the hook.

Theorem 2.2. [11] For a shifted shape λ , $f^\lambda = |\lambda|! / \prod_{x \in \lambda} h(x)$.

To define an analogous notion of balanced tableaux, as in [2], for shifted shapes, we need some more notions. For a filling B of shape λ , its *extended filling* $\tilde{B}$ is a filling of the extended shape

$$\tilde{\lambda} = \lambda \cup \{ (1, \bar{d}), (2, \overline{d-1}), \dots, (d, \bar{1}) \}$$

which agrees with B on λ and equals $B(i, 0)$ on the newly added box $(i, -d-1-i)$. The *extended hook* is defined as $\tilde{H}(i, j) = H(i, j)$ for $j \geq 0$, and $\tilde{H}(i, \bar{j}) = H(i, j) \cup \{(d-1-j, \bar{j})\}$ for $j > 0$. See [Example 2.5](#) for visualization.

For a box $(i, j) \in \lambda$, we also define its *rank function* $\text{rk}(i, j)$. If $j \geq 0$, let $\text{rk}(i, j)$ be the number of boxes in row i of $H(i, j)$, and let $\text{rk}(i, \bar{j})$ be 2 plus the number of boxes in $H(i, j)$ with positive column index. More formally,

$$\text{rk}(i, j) = \begin{cases} \lambda_i - d - i - j & \text{if } j \geq 0, \\ \lambda_i - d - i - \lambda_{d-1-j} - j - 1 & \text{if } j < 0. \end{cases}$$

We can now introduce our main object of study:

Definition 2.3. A shifted tableau B of shape λ is called a *balanced shifted tableau* if it is a filling of $1, 2, \dots, |\lambda|$ such that $B(i, j)$ is the $\text{rk}(i, j)$ -th largest entry in the extended hook $\tilde{H}(i, j)$ of $\tilde{B}$ for all $(i, j) \in \lambda$. Define $\text{BS}(\lambda)$ to be the set of balanced shifted tableaux of shape λ .

Remark 2.4. We remark that our definition of balanced tableaux is different from the balanced filling in Section 6 of [4] and the standard w -tableau as in [7]. One can derive a result similar to [Theorem 1.1](#) using their definition, but we note that the two results are fundamentally different. The difference can be interpreted loosely as studying the balanced tableaux of dominant permutation v.s. Grassmannian permutation in the framework of [3].

Example 2.5. Let $\lambda = 6, 2, 1$ and consider the balanced shifted tableau in [Figure 1](#). The hook $H(1, -1)$ contains the colored boxes so $h(1, -1) = 7$, while the extended hook $\tilde{H}(1, -1)$ contains one more box at coordinate $(3, -1)$, which is circled and filled with 1. As this hook contains 3 boxes with positive column index, we have $\text{rk}(1, -1) = 5$. The balanced condition is now satisfied at coordinate $(1, -1)$ as 3 is indeed the 5-th largest numbers among the numbers in the extend hook, $9, 5, 2, 4, 3, 7, 1, 1$.

4	6	3	4	2	5	9
	8	7	8			
		①	1			

Figure 1: A balanced shifted tableau of shape $6, 2, 1$

2.2 Root systems and Weyl groups

Readers are referred to [5] for detailed exposition on root systems and Weyl groups. Let $\Phi \subset V \simeq \mathbb{R}^d$ be a finite crystallographic root system of rank d , with a chosen set

of positive roots Φ^+ which corresponds to a set of simple roots $\Delta = \{\alpha_0, \alpha_1, \dots, \alpha_{d-1}\}$. Let s_α be the reflection across the hyperplane normal to α , and write s_i for the simple reflections s_{α_i} . Let $W = \langle \Phi^+ \rangle \subset GL(V)$ be the finite Weyl group, defined to be generated by $s_0, \dots, s_{d-1}$.

For $w \in W$, let $\ell(w)$ denote its Coxeter length, which equals the size of its (left) inversion set $\text{Inv}(w) = \Phi^+ \cap w\Phi^-$. For any sequence $\mathbf{a} = a_1, a_2, \dots, a_{\ell(w)}$, we say $\mathbf{a}$ is a reduced word of w if $w = s_{a_1}s_{a_2}\dots s_{a_{\ell(w)}}$. Let $\text{Red}(w)$ be the set of reduced words of w . For each reduced word $\mathbf{a} \in \text{Red}(w)$, its (total) reflection order is an ordering $\text{ro } \mathbf{a} = \gamma_1, \dots, \gamma_{\ell(w)}$ of $\text{Inv}(w)$ where $\gamma_j = s_{a_1}\dots s_{a_{j-1}}\alpha_j \in \Phi^+$. Let

$$\text{ro } w = \{\text{ro } \mathbf{a} : \mathbf{a} \in \text{Red}(w)\}.$$

The following proposition is classical and very useful, which follows immediately from the biconvexity classification of inversion sets. See for example Proposition 3 of [1].

Proposition 2.6. *Let $\gamma = \gamma_1, \dots, \gamma_{\ell(w)}$ be an ordering of $\text{Inv}(w)$. Then $\gamma \in \text{ro } w$ if and only if for all the triples $\alpha, \beta, \alpha - \beta \in \Phi^+$ such that $\alpha, \alpha - \beta \in \text{Inv}(w)$,*

1. *if $\beta \notin \text{Inv}(w)$, then α appears before $\alpha - \beta$ in this sequence;*
2. *and if $\beta \in \text{Inv}(w)$, then $\alpha - \beta$ appears in the middle of α and β .*

We are primarily concerned with root systems of type B_n , and adopt the following convention, where e_i is the i -th coordinate vector:

- $\Phi_{B_n} = \{\pm e_j \pm e_i \mid 1 \leq i < j \leq n\} \cup \{\pm e_i \mid 1 \leq i \leq n\}$;
- $\Phi^-_{B_n} = \{e_j \pm e_i \mid 1 \leq i < j \leq n\} \cup \{e_i \mid 1 \leq i \leq n\}$;
- $\Delta = \{\alpha_0 = e_1, \alpha_1 = e_2 - e_1, \dots, \alpha_{n-1} = e_n - e_{n-1}\}$;
- $W_{B_n} = \{\text{permutation } w \text{ on } 1, \dots, n, \bar{1}, \dots, \bar{n} \mid w(i) = -w(\bar{i}), \forall i\}$.

The type B_n Weyl group W_{B_n} is called the group of *signed permutations*. For a signed permutation w , its one-line notation is written as $w \ 1 \ w \ 2 \ \dots \ w \ n$. For example, $w = 3\bar{4}2\bar{1} \in W_{B_4}$ means that $w \ 1 = 3$, $w \ 2 = -4$, $w \ 3 = 2$ so that $w \ -3 = -2$ and $w \ 4 = -1$. A reduced word of $w \in W_{B_n}$ can be viewed as going from $\text{id} = 12\dots n$ to w by swapping adjacent entries (and their negatives) one step at a time, while the corresponding reflection order records $e_j - e_i$ if the values j and i are swapped (and records e_i if i and $\bar{i}$ are swapped).

Example 2.7. Consider $w = 1\bar{3}42 \in W_{B_4}$ with a reduced word $\mathbf{a} = 21031 \in \text{Red}(w)$. We compute its reflection order to be $e_3 - e_2, e_3 - e_1, e_3, e_4 - e_2, e_3 - e_1$, which can be seen as follows:

$$1234 \xrightarrow{e_3 - e_2} 1324 \xrightarrow{e_3 - e_1} 3124 \xrightarrow{e_3} \bar{3}124 \xrightarrow{e_4 - e_2} \bar{3}142 \xrightarrow{e_3 - e_1} 1\bar{3}42.$$

3 Bijection between $\text{BS } Z_{d,r}$ and $\text{Red } w^{d,r}$ via reflection order

A crucial shape for our analysis is the *trapezoid*

$$Z_{d,r} : \quad r \quad 2d-1, r \quad 2d-3, \dots, r \quad 3, r \quad 1$$

with height d and base lengths $r \quad 2d-1$ and $r \quad 1$. In particular, $Z_{d,0}$ is the *double staircase* and every shifted shape is contained in some trapezoid of the same height.

Set $e_{-j} = -e_j$ for all $j > 0$ and $e_0 = 0$, and consider the labeling $f : Z_{d,r} \rightarrow \Phi_{B_{d-r}}$ where

$$f(i, j) = \begin{cases} e_{d-1-i} - e_j & \text{if } j \leq 0, \\ e_{d-1-i} - e_{j-d} & \text{if } 0 < j \leq r, \\ e_{d-1-i} - e_{j-r} & \text{if } j > r. \end{cases} \quad (3.1)$$

Define the permutation $w^{d,r} \in W_{B_{d-r}}$ associated to $Z_{d,r}$ by

$$w^{d,r}(i) = \begin{cases} d-i & \text{if } 0 < i \leq r, \\ \overline{i-r} & \text{if } i > r. \end{cases} \quad (3.2)$$

Proposition 3.1. *For all $d > 0$ and $r \geq 0$, $f : Z_{d,r} \rightarrow \text{Inv } w^{d,r}$.*

The labeling f can also be extended to a labeling $\tilde{f} : \tilde{Z}_{d,r} \rightarrow \Phi_{B_{d-r}}$ where

$$\tilde{Z}_{d,r} = Z_{d,r} \cup \{1, \bar{d}, 2, \overline{d-1}, \dots, d, \bar{1}\}$$

is the extended shape of $Z_{d,r}$ with d extra boxes as defined in [Section 2](#). The extended labeling is given by

$$\tilde{f}(i, j) = \begin{cases} 2e_{d-1-i} & \text{if } j = \overline{d-1-i}, \\ f(i, j) & \text{otherwise.} \end{cases}$$

Example 3.2. For $d = 3$ and $r = 2$, we have $Z_{d,r} = 7, 5, 3$ and $w^{3,2} = 45\bar{1}\bar{2}\bar{3} \in W_{B_5}$. See [Figure 2](#) for the extended labeling $\tilde{f}$ in this case.

The filling of a balanced shifted tableau $B \in \text{BS } Z_{d,r}$ can be viewed as a map $B : Z_{d,r} \rightarrow \mathbb{N}$ by sending a box to its entry. Then the composition $Bf^{-1} : \text{Inv } w^{d,r} \rightarrow \mathbb{N}$ encodes an ordering of the roots in $\text{Inv } w^{d,r}$. We will show that this actually gives a reflection order in $\text{ro } w^{d,r}$.

Proposition 3.3. *The map $B \mapsto Bf^{-1}$ is a bijection between $\text{BS } Z_{d,r}$ and $\text{ro } w^{d,r}$, and thus induces a bijection between $\text{BS } Z_{d,r}$ and $\text{Red } w^{d,r}$.*

$2e_3$	$e_3 \ e_2$	$e_3 \ e_1$	e_3	$e_4 \ e_3$	$e_5 \ e_3$	$e_3 - e_1$	$e_3 - e_2$
$2e_2$	$e_2 \ e_1$	e_2	$e_4 \ e_2$	$e_5 \ e_2$	$e_2 - e_1$		
$2e_1$		e_1	$e_4 \ e_1$	$e_5 \ e_1$			

Figure 2: The extended labeling $\tilde{f}$ of $\tilde{Z} \ 3, 2$

Since there is a natural bijection between $\text{ro } w$ and $\text{Red } w$, [Proposition 3.3](#) implies:

Corollary 3.4. *The map $B \mapsto \text{ro}^{-1} B f^{-1}$ is a bijection between $\text{BS } Z \ d, r$ and $\text{Red } w^{d, r}$.*

Example 3.5. Assume we started with the following balanced tableau of shape $Z \ 3, 2$.

B	4	8	7	10	13	5	15
		3	2	6	9	1	
			11	12	14		

The corresponding reflection order $B f^{-1}$ is given as follows:

$$\begin{aligned}
 12345 &\xrightarrow{e_2 - e_1} 21345 \xrightarrow{e_2} \bar{2}1345 \xrightarrow{e_2 \ e_1} 1\bar{2}345 \xrightarrow{e_3 \ e_2} 13\bar{2}45 \xrightarrow{e_3 - e_1} 31\bar{2}45 \\
 &\xrightarrow{e_4 \ e_2} 314\bar{2}5 \xrightarrow{e_3} \bar{3}14\bar{2}5 \xrightarrow{e_3 \ e_1} 1\bar{3}4\bar{2}5 \xrightarrow{e_5 \ e_2} 1\bar{3}45\bar{2} \xrightarrow{e_4 \ e_3} 14\bar{3}5\bar{2} \\
 &\xrightarrow{e_1} \bar{1}4\bar{3}5\bar{2} \xrightarrow{e_4 \ e_1} 4\bar{1}\bar{3}5\bar{2} \xrightarrow{e_5 \ e_3} 4\bar{1}5\bar{3}\bar{2} \xrightarrow{e_5 \ e_1} 45\bar{1}\bar{3}\bar{2} \xrightarrow{e_3 - e_2} 45\bar{1}\bar{2}\bar{3}.
 \end{aligned}$$

Therefore, we can read off a reduced word $\mathbf{a}$ of $w^{3,2} = 45\bar{1}\bar{2}\bar{3}$ as

$$\mathbf{a} = \text{ro}^{-1} B f^{-1} = 101213014201324 \in \text{Red } w^{3,2}.$$

4 Bijection between $\text{SYT } Z \ d, r$ and $\text{Red } w^{d, r}$ via the Kraśkiewicz's insertion

We will follow the notations as recorded in Section 1.3 of [8]. For a shifted tableau T of shape $\lambda = \lambda_1, \dots, \lambda_d$, define $\pi T = T_d T_{d-1}, \dots, T_1$ to be the reading word of T obtained by reading left to right along rows and from bottom to top, where T_i represents the i -th row. For a unimodal sequence of integers

$$\mathbf{R} = r_1 > r_2 > \dots > r_k < r_{k+1} < \dots < r_m,$$

we define the decreasing part of $\mathbf{R}$ to be $\mathbf{R}^\downarrow : r_1 > r_2 > \dots > r_k$, and the increasing part of $\mathbf{R}$ to be $\mathbf{R}^\uparrow : r_{k+1} < r_{k+2} < \dots < r_m$. Note that we include the minimal integer of the sequence in $\mathbf{R}^\downarrow$.

Let $w \in W_{B_n}$ and $\mathbf{a} = a_1 a_2 \dots a_{\ell(w)} \in \text{Red } w$, we define the *Kraśkiewicz's insertion algorithm* recursively. Set $P^0, Q^0 : \emptyset, \emptyset$, for any $i \in \ell(w)$, define the insertion

$$P^{i-1}, Q^{i-1} \leftarrow a_i : P^i, Q^i$$

as follows:

Step 1: Set $\mathbf{R}$ to be the first row of P^{i-1} and $a = a_i$.

Step 2: Insert a into $\mathbf{R}$ as follows:

- Case 1 ($\mathbf{R}a$ is unimodal): Append a to the right of $\mathbf{R}$ to obtain P^i . Then define Q^i from Q^{i-1} by adding i to the unique box in P^i / P^{i-1} . Stop.
- Case 2 ($\mathbf{R}a$ is not unimodal): Let b be the smallest number in $\mathbf{R}^\uparrow$ such that $b \geq a$.
 - Case 2.1 ($a = 0$ and $\mathbf{R}$ contains 101 as a subsequence): We leave $\mathbf{R}$ unchanged and return to start of Step 2 with $a = 0$ and $\mathbf{R}$ equals the next row.
 - Case 2.2 (otherwise): Replace b with a . Set $c = b$ if $a \neq b$ or $c = b - 1$ if $a = b$.

We now insert c into $\mathbf{R}^\downarrow$. Let d be the largest integer such that $d \leq c$. This number always exists since $\mathbf{R}^\downarrow$ contains the smallest number in the row. Replace d with c . Set $a' = d$ if $c \neq d$ or $a' = d - 1$ if $c = d$.

Step 3: Repeat Step 2 with $a = a'$ and $\mathbf{R}$ the next row.

Define $P(\mathbf{a}) = P^{\ell(w)}$ to be the *insertion tableau* and $Q(\mathbf{a}) = Q^{\ell(w)}$ to be the *recording tableau*.

Example 4.1. Let $w = w^{3,2} = 45\bar{1}2\bar{3}$ as in Example 3.2. Consider the reduced word $\mathbf{a} = 010121012342312 \in \text{Red } w$. Following the above insertion algorithm, we obtain

$$\begin{array}{l}
 P^0 = \emptyset \xrightarrow{0} \boxed{0} \xrightarrow{1} \boxed{0 \mid 1} \xrightarrow{0} \boxed{1 \mid 0} \xrightarrow{1} \boxed{1 \mid 0 \mid 1} \xrightarrow{2} \boxed{1 \mid 0 \mid 1 \mid 2} \xrightarrow{1} \boxed{2 \mid 0 \mid 1 \mid 2} \xrightarrow{0} \boxed{2 \mid 1 \mid 0 \mid 2} \xrightarrow{1} \\
 \begin{array}{c} \boxed{2 \mid 1 \mid 0 \mid 1} \\ \boxed{1 \mid 0 \mid 1} \\ \boxed{0} \end{array} \xrightarrow{2} \begin{array}{c} \boxed{2 \mid 1 \mid 0 \mid 1 \mid 2} \\ \boxed{1 \mid 0 \mid 1} \\ \boxed{0} \end{array} \xrightarrow{3} \begin{array}{c} \boxed{2 \mid 1 \mid 0 \mid 1 \mid 2 \mid 3} \\ \boxed{1 \mid 0 \mid 1} \\ \boxed{0} \end{array} \xrightarrow{4} \begin{array}{c} \boxed{2 \mid 1 \mid 0 \mid 1 \mid 2 \mid 3 \mid 4} \\ \boxed{1 \mid 0 \mid 1} \\ \boxed{0} \end{array} \xrightarrow{2} \begin{array}{c} \boxed{3 \mid 1 \mid 0 \mid 1 \mid 2 \mid 3 \mid 4} \\ \boxed{1 \mid 0 \mid 1 \mid 2} \\ \boxed{0} \end{array} \xrightarrow{3} \\
 \begin{array}{c} \boxed{4 \mid 1 \mid 0 \mid 1 \mid 2 \mid 3 \mid 4} \\ \boxed{1 \mid 0 \mid 1 \mid 2 \mid 3} \\ \boxed{0} \end{array} \xrightarrow{1} \begin{array}{c} \boxed{4 \mid 2 \mid 0 \mid 1 \mid 2 \mid 3 \mid 4} \\ \boxed{2 \mid 0 \mid 1 \mid 2 \mid 3} \\ \boxed{0 \mid 1} \end{array} \xrightarrow{2} \begin{array}{c} \boxed{4 \mid 3 \mid 0 \mid 1 \mid 2 \mid 3 \mid 4} \\ \boxed{3 \mid 0 \mid 1 \mid 2 \mid 3} \\ \boxed{0 \mid 1 \mid 2} \end{array}.
 \end{array}$$

We then have

$$P \mathbf{a} \begin{array}{|c|c|c|c|c|c|c|} \hline 4 & 3 & 0 & 1 & 2 & 3 & 4 \\ \hline & 3 & 0 & 1 & 2 & 3 & \\ \hline & & 0 & 1 & 2 & & \\ \hline \end{array}, Q \mathbf{a} \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 2 & 4 & 5 & 9 & 10 & 11 \\ \hline & 3 & 6 & 8 & 12 & 13 & \\ \hline & & 7 & 14 & 15 & & \\ \hline \end{array},$$

and the reading word $\pi P \mathbf{a} = 012301234301234$.

Definition 4.2. For a shifted tableau T with m rows, we say T is a *standard decomposition tableau* of $w \in W_{B_n}$ if

1. $\pi T = T_m T_{m-1} \dots T_1$ is a reduced word of w ,
2. T_i is a unimodal subsequence of maximal length in $T_m T_{m-1} \dots T_i$.

Define the set of all such tableaux to be $\text{SDT } w$.

Theorem 4.3 (Theorem 5.2, [6]). *The Kraśkiewicz's insertion gives a bijection between $\{\mathbf{a} \in \text{Red } w\}$ and the pairs of tableaux $P \mathbf{a}, Q \mathbf{a}$ where $P \mathbf{a} \in \text{SDT } w$ and $Q \mathbf{a}$ is a standard tableau of the same shape.*

Lemma 4.4. $\text{SDT } w^{d,r}$ consists of exactly one shifted tableau. In other words, any $\mathbf{a} \in \text{Red } w^{d,r}$ has the same P tableau.

Corollary 4.5. *By restricting to the recording tableau, Kraśkiewicz's insertion gives a bijection $\mathbf{a} \mapsto Q \mathbf{a}$ between $\text{Red } w^{d,r}$ and $\text{SYT } Z^{d,r}$.*

Combining Corollary 3.4 and Corollary 4.5, we derive Theorem 1.1 for the trapezoid shape $Z^{d,r}$.

5 The general case

In this Section, we prove our main result, Theorem 1.1, in full generality. Fix a strict partition $\lambda \subset Z^{d,r}$ such that $\lambda_d > 0$ and set $N = |\lambda|$. Let $\ell = |Z^{d,r}| - \ell w^{d,r}$ and set $\mu_0 = \sigma_0 = 0$, $\mu_i = Z^{d,r} - \lambda_i$ and $\sigma_i = \sum_{k=1}^i \mu_k$ for all $i \in [d]$. Recall our main framework

$$\text{SYT } \lambda \longleftrightarrow \text{SYT } Z^{d,r} - \lambda \longleftrightarrow \text{Red } w^{d,r} - \lambda \longleftrightarrow \text{BS } Z^{d,r} - \lambda \longleftrightarrow \text{BS } \lambda.$$

The first arrow of bijection is immediate (Definition 5.1). In each of the subsequent subsections, we prove one remaining bijection respectively. A complete example is given at the end of this section in Example 5.11. Readers are encouraged to refer to this example for intuition.

5.1 Bijection between $\text{SYT } Z(d, r) \downarrow_\lambda$ and $\text{Red } w^{d, r} \downarrow_\lambda$

Definition 5.1. For any tableau $T \in \text{SYT } \lambda$, define $T \in \text{SYT } Z(d, r)$ to be the tableau obtained from T by assigning $N - 1, \dots, \ell$ to the cells in $Z(d, r) \setminus \lambda$ from left to right along rows and from top to bottom. Define $\text{SYT } Z(d, r) \downarrow_\lambda$ to be the set of all such T obtained from some $T \in \text{SYT } \lambda$.

Example 5.2. Let $\lambda = 6, 2, 1 \subset Z(3, 2)$ and

$$T = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 5 & 6 & 9 \\ \hline & 4 & 7 & & & \\ \hline & & 8 & & & \\ \hline \end{array}, \text{ then } T = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 5 & 6 & 9 & 10 \\ \hline & 4 & 7 & 11 & 12 & 13 & \\ \hline & & 8 & 14 & 15 & & \\ \hline \end{array}.$$

Definition 5.3. Define the word $\mathbf{a}^\lambda = \mathbf{a}_1^\lambda \cdots \mathbf{a}_d^\lambda$ where

$$\mathbf{a}_i^\lambda = d - r - i - \mu_i - 1, \dots, d - r - i. \quad (5.1)$$

Define $\text{Red } w^{d, r} \downarrow_\lambda$ to be the set of reduced words ending with $\mathbf{a}^\lambda$.

Proposition 5.4. The bijection $\mathbf{a} \mapsto Q(\mathbf{a})$ between $\text{Red } w^{d, r}$ and $\text{SYT } Z(d, r)$ in [Corollary 4.5](#) induces a well-defined bijection between $\text{Red } w^{d, r} \downarrow_\lambda$ and $\text{SYT } Z(d, r) \downarrow_\lambda$.

A key ingredient in the proof of [Proposition 5.4](#) is the existence of the reverse of Kraśkiewicz's insertion.

Lemma 5.5 (Lemma 1.25, [8]). Given $P(\mathbf{a}), Q(\mathbf{a})$ for some $\mathbf{a} = a_1 a_2 \cdots a_\ell w \in \text{Red } w$ and $w \in W_{B_n}$, let Q' be obtained by removing the largest entry in $Q(\mathbf{a})$. Then there is a unique $a \in [0, n-1]$ and a unique $P' \in \text{SDT } ws_a$ such that $P' \leftarrow a \leftarrow P$, and $\text{sh } P' = \text{sh } Q'$. In fact, we have $a = a_\ell w$.

5.2 Bijection between $\text{BS } \lambda$ and $\text{BS } Z(d, r) \downarrow_\lambda$

Recall some notations from the beginning of this section: $\mu_i = Z(d, r)_i - \lambda_i$, $\sigma_i = \sum_{k=1}^i \mu_k$ and $N = |\lambda|$.

Definition 5.6. Define $\text{BS } Z(d, r) \downarrow_\lambda$ to be the set of balanced tableaux T of shape $Z(d, r)$ such that for all $i \in [d]$ and any $k \in [N - \sigma_{i-1} - 1, N - \sigma_i]$, k appears in row i of T .

Lemma 5.7. Let $B \in \text{BS } \lambda$ and fix some $i \in [d]$ such that either $i = 1$ or $\lambda_{i-1} \geq \lambda_i - 3$. Denote $\lambda^\#$ the shifted diagram obtained from λ by adding a box in the i -th row. Let j be the column index of the box $\lambda^\# \setminus \lambda$. Let $B^\#$ be the tableau obtained from B by

1. interchange column j and $j - 1$ of B ,

2. define $B^\#_{i,j} = N - 1$.

Then $B^\#$ is a balanced tableau and the following map is a bijection:

$$\begin{aligned} f_i : \text{BS } \lambda &\longrightarrow \{T \in \text{BS } \lambda^\# : T_{i,j} = N - 1\} \\ B &\longmapsto B^\#. \end{aligned} \quad (5.2)$$

Lemma 5.8. $\text{BS } Z_{d,r} \mid_\lambda$ is the image of $\text{BS } \lambda$ under the composition of maps $F = f_d^{a_d} \circ f_{d-1}^{a_{d-1}} \circ \dots \circ f_1^{a_1}$ with each f_i defined as in (5.2). As a result, F is a bijection between $\text{BS } \lambda$ and $\text{BS } Z_{d,r} \mid_\lambda$.

5.3 Bijection between $\text{BS } Z_{d,r} \mid_\lambda$ and $\text{Red } w^{d,r} \mid_\lambda$

Proposition 5.9. Let $\mathbf{a} \in \text{Red } w^{d,r}$ be a reduced word. Then $\text{ro } \mathbf{a}$ gives a balanced tableau in $\text{BS } Z_{d,r} \mid_\lambda$ if and only if the ending segment of $\mathbf{a}$ is the same as $\mathbf{a}^\lambda$ as in Definition 5.3. Consequently, this induces a bijection between $\text{BS } Z_{d,r} \mid_\lambda$ and $\text{Red } w^{d,r} \mid_\lambda$.

Combining Proposition 5.4, Lemma 5.8 and Proposition 5.9, we get a bijection between $\text{SYT } \lambda$ and $\text{BS } \lambda$.

Proposition 5.10. The bijection $\text{SYT } \lambda \longleftrightarrow \text{BS } \lambda$ in Theorem 1.1 does not depend on the parameter r .

Example 5.11. We now work out an example in the case where $\lambda = 6, 2, 1$, $d = 3$ and $r = 2$. Assume we start with a balanced tableau $B \in \text{BS } \lambda$ shown here:

$$B = \begin{array}{|c|c|c|c|c|c|} \hline 6 & 3 & 4 & 1 & 5 & 9 \\ \hline & 7 & 8 & & & \\ \hline & & 2 & & & \\ \hline \end{array}.$$

We can complete it to $B \in \text{BS } Z_{3,2}$ using the algorithm in Lemma 5.8

$$\begin{array}{l} B = \begin{array}{|c|c|c|c|c|c|} \hline 6 & 3 & 4 & 1 & 5 & 9 \\ \hline & 7 & 8 & & & \\ \hline & & 2 & & & \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|c|c|c|c|} \hline 6 & 3 & 4 & 1 & 5 & 9 & \textcircled{10} \\ \hline & 7 & 8 & & & & \\ \hline & & 2 & & & & \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|c|c|c|c|} \hline 6 & 3 & 4 & 5 & 1 & 9 & \textcircled{10} \\ \hline & 7 & 8 & \textcircled{11} & & & \\ \hline & & 2 & & & & \\ \hline \end{array} \rightarrow \\ \begin{array}{|c|c|c|c|c|c|c|c|} \hline 6 & 3 & 4 & 5 & 9 & 1 & \textcircled{10} \\ \hline & 7 & 8 & \textcircled{11} & \textcircled{12} & & & \\ \hline & & 2 & & & & & \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|c|c|c|c|c|} \hline 6 & 3 & 4 & 5 & 9 & \textcircled{10} & 1 \\ \hline & 7 & 8 & \textcircled{11} & \textcircled{12} & \textcircled{13} & & \\ \hline & & 2 & & & & & \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|c|c|c|c|c|} \hline 6 & 3 & 4 & 9 & 5 & \textcircled{10} & 1 \\ \hline & 7 & 8 & \textcircled{12} & \textcircled{11} & \textcircled{13} & & \\ \hline & & 2 & \textcircled{14} & & & & \\ \hline \end{array} \rightarrow \\ \begin{array}{|c|c|c|c|c|c|c|c|} \hline 6 & 3 & 4 & 9 & \textcircled{10} & 5 & 1 \\ \hline & 7 & 8 & \textcircled{12} & \textcircled{13} & \textcircled{11} & & \\ \hline & & 2 & \textcircled{14} & \textcircled{15} & & & \\ \hline \end{array} = B \end{array}$$

Now B gives a reflection order of $w^{d,r}$ as follows

$$\begin{aligned}
 12345 &\xrightarrow{e_3-e_2} 13245 \xrightarrow{e_1} \bar{1}3245 \xrightarrow{e_3} 3\bar{1}245 \xrightarrow{e_1} \bar{3}\bar{1}245 \xrightarrow{e_3-e_1} \bar{1}\bar{3}245 \\
 &\xrightarrow{e_3} \bar{1}\bar{2}\bar{3}45 \xrightarrow{e_2} 2\bar{1}\bar{3}45 \xrightarrow{e_1} \bar{2}\bar{1}\bar{3}45 \xrightarrow{e_2} 2\bar{1}\bar{3}45 \xrightarrow{e_4} \bar{2}\bar{1}4\bar{3}5 \xrightarrow{e_3} \bar{2}\bar{1}45\bar{3} \\
 &\xrightarrow{e_5} \bar{1}\bar{2}45\bar{3} \xrightarrow{e_4-e_1} \bar{1}\bar{2}45\bar{3} \xrightarrow{e_4} \bar{1}4\bar{2}5\bar{3} \xrightarrow{e_2} \bar{1}4\bar{2}5\bar{3} \xrightarrow{e_5} \bar{1}45\bar{2}\bar{3} \xrightarrow{e_4} 4\bar{1}5\bar{2}\bar{3} \xrightarrow{e_1} 45\bar{1}\bar{2}\bar{3}.
 \end{aligned}$$

We can read off the reduced word $\mathbf{a} = 201012103412312 \in \text{Red } w^{3,2}$. We can confirm that the reduced word ends with $\mathbf{a}^\lambda = 412312$. Now we perform the Kraśkiewicz's insertion on $\mathbf{a}$ described in Section 4 and we get

$$P \mathbf{a} = \begin{array}{|c|c|c|c|c|c|c|} \hline 4 & 3 & 0 & 1 & 2 & 3 & 4 \\ \hline & 3 & 0 & 1 & 2 & 3 & \\ \hline & & 0 & 1 & 2 & & \\ \hline \end{array}, \quad Q \mathbf{a} = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 5 & 6 & 9 & 10 \\ \hline & 4 & 7 & 11 & 12 & 13 & \\ \hline & & 8 & 14 & 15 & & \\ \hline \end{array},$$

Finally, let $T = Q \mathbf{a} \in \text{SYT } w^{3,2}$, and $T \in \text{SYT } \lambda$ is obtained from T by deleting the largest entries until $|\lambda|$ entries are left:

$$T = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 5 & 6 & 9 & 10 \\ \hline & 4 & 7 & 11 & 12 & 13 & \\ \hline & & 8 & 14 & 15 & & \\ \hline \end{array}, \quad T = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 5 & 6 & 9 \\ \hline & 4 & 7 & & & \\ \hline & & 8 & & & \\ \hline \end{array}.$$

Acknowledgements

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The tropical critical points of an affine matroid

Federico Ardila-Mantilla^{*1}, Christopher Eur⁺², and Raul Penaguiao^{‡3}

¹Department of Mathematics, San Francisco State University and Universidad de Los Andes

²Department of Mathematics, Harvard University

³Max Planck Institute for Mathematics in the Sciences, Leipzig

Abstract. We prove that the maximum likelihood degree of a matroid M equals its beta invariant $\beta(M)$. For an element e of M that is neither a loop nor a coloop, this is defined to be the degree of the intersection of the Bergman fan of M, e and the inverted Bergman fan of $N = M/e^\perp$. Equivalently, for a generic vector $w \in \mathbb{R}^{E-e}$, this is the number of ways to find weights $0, x$ on M and y on N with $x \leq y \leq w$ such that on each circuit of M (resp. N), the minimum x -weight (resp. y -weight) occurs at least twice.

Keywords: matroid, Bergman fan, maximum likelihood degree

1 Introduction

During the Workshop on Nonlinear Algebra and Combinatorics from Physics at the Center for the Mathematical Sciences and Applications at Harvard University in April 2022, Sturmfels [15] posed one of those combinatorial problems that is deceptively simple to state, but whose answer requires a deeper understanding of the objects at hand.

Problem 1.1. [15] (Combinatorial version) Let M be a matroid on ground set E . Let e be an element that is neither a loop nor a coloop. Let M/e be the contraction of M by e and let $N = M/e^\perp$ be its dual matroid. Given a vector $w \in \mathbb{R}^{E-e}$, find weight vectors $0, x \in \mathbb{R}^E$ on M (where e has weight 0) and $y \in \mathbb{R}^{E-e}$ on N such that

- on each circuit of M , the minimum x -weight occurs at least twice,
- on each circuit of N , the minimum y -weight occurs at least twice, and
- $w \leq x \leq y$.

Can this always be done? What is the number of solutions for generic w ?

Theorem 1.2. (Geometric Version) *Problem 1.1 can always be solved. If $w \in \mathbb{R}^{E-e}$ is generic, the number of solutions equals the beta invariant $\beta(M)$ of the matroid.*

We now restate Theorem 1.2 in tropical terms; see relevant definitions in Section 2.3.

^{*}federico@sfsu.edu. Partially supported by National Science Foundation Grant DMS-2154279.

[†]ceur@math.harvard.edu. Partially supported by National Science Foundation Grant NSF DMS-2001854.

[‡]raul.penaguiao@mis. Partially supported by Schweizerischer Nationalfonds Grant P2ZHP2 191301.

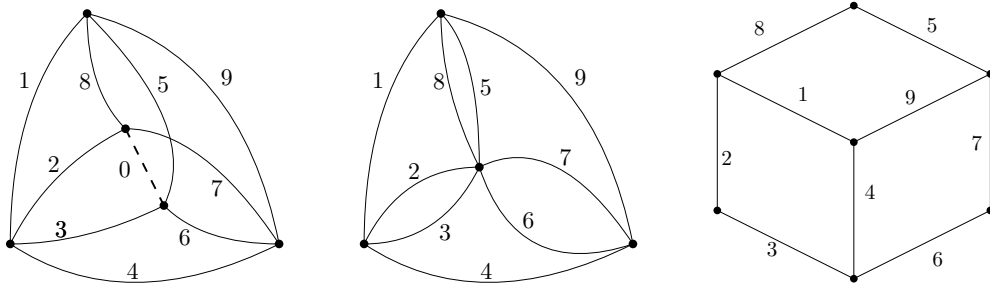


Figure 1: A graph G , its contraction $G/0$, and its dual $H = G/0^\perp$.

Theorem 1.3. *Let M be a matroid on E , and let $e \in E$ be an element that is neither a loop nor a coloop. Let $M_{,e}$ be the affine matroid of M with respect to e , and let $N = M/e^\perp$. Then the degree of the intersection of the Bergman fan $\Sigma_{M_{,e}}$ and the inverted Bergman fan $-\Sigma_N$ is*

$$\deg \Sigma_{M_{,e}} \cdot -\Sigma_N = \beta(M).$$

Agostini, Brysiewicz, Fevola, Kühne, Sturmfels, and Telen [1] first encountered (a special case of) [Problem 1.1](#) in their study of the maximum likelihood estimation for linear discrete models. Using algebro-geometric results of Huh and Sturmfels [10], which built on work of Varchenko [16], they proved [Theorem 1.2](#) and [Theorem 1.3](#) for matroids realizable over the real numbers.

We prove the equivalent [Theorem 1.2](#) and [Theorem 1.3](#) for all matroids. Following the original motivation, we call the answer to [Problem 1.1](#) the maximum likelihood degree of a matroid; our main result is that it equals the beta invariant.

We first prove [Theorem 1.2](#) combinatorially, relying on the tropical geometric fact that all generic w give the same intersection degree. We show that when the entries of w are super-increasing with respect to some order $<$ on E , the solutions to [Problem 1.1](#) are naturally in bijection with the β -nbc bases of the matroid with respect to $<$. We then sketch a proof of [Theorem 1.3](#) that relies on the theory of tautological classes of matroids of Berget, Eur, Spink, and Tseng [6]. This is an extended abstract of our results in [5].

2 Notation and preliminaries

2.1 The lattice of set partitions

A set partition λ of a set E is a collection of subsets, called blocks, of E , say $\lambda = \{\lambda_1, \dots, \lambda_\ell\}$, that cover E and the pairwise intersection is empty. We write $\lambda \mid E$. We let $|\lambda| = \ell$ be the number of blocks of λ . If $e \in E$ and $\lambda \mid E$, we write $\lambda \mid e$ for the block of λ that contains e .

We define the *linear space of a set partition* $\lambda = \{\lambda_1, \dots, \lambda_\ell\} \mid E$ to be

$$\begin{aligned} \mathbb{L} \lambda &:= \text{span}\{e_{\lambda_1}, \dots, e_{\lambda_\ell}\} \subseteq \mathbb{R}^E \\ &\quad \{x \in \mathbb{R}^E \mid x_i = x_j \text{ whenever } i, j \text{ are in the same block of } \lambda\}, \end{aligned}$$

where $\{e_i : i \in E\}$ is the standard basis of $\mathbb{R}^E$, $e_S = \sum_{s \in S} e_s$ for $S \subseteq E$. Notice that $\dim \mathbb{L} \lambda = |\lambda|$. The map $\lambda \mapsto \mathbb{L} \lambda$ is a bijection between the set partitions of E and the flats of the *braid arrangement*, which is the hyperplane arrangement in $\mathbb{R}^E$ given by the hyperplanes $x_i = x_j$ for $i \neq j$ in E .

If $\lambda \mid 0 \sqcup E$ then we write $\mathbb{L} \lambda \mid_{x_0=0} = \{x \in \mathbb{R}^E : 0, x \in \mathbb{L} \lambda \subseteq \mathbb{R}^{0 \sqcup E}\}$.

2.2 The intersection graph of two set partitions

The following construction from [3] will play an important role.

Definition 2.1. Let $\lambda \mid 0 \sqcup E$ and $\mu \mid E$ be set partitions. The intersection graph $\Gamma = \Gamma_{\lambda, \mu}$ is the bipartite graph with vertex set $\lambda \sqcup \mu$ and edge set E , where the edge e connects the parts $\lambda \mid e$ of λ and $\mu \mid e$ of μ containing e . On this graph, the vertex corresponding to $\lambda \mid 0$ is marked with a hollow point.

The intersection graph may have several parallel edges connecting the same pair of vertices. Notice that the label of a vertex in Γ is just the set of labels of the edges incident to it. Therefore we can remove the vertex labels, and simply think of Γ as a bipartite multigraph on edge set E . This is illustrated in Figure 2.

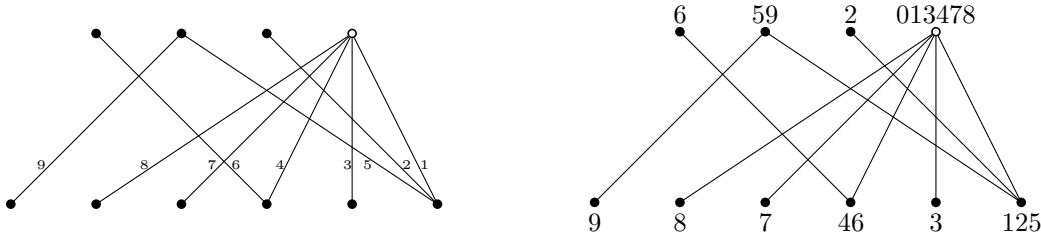


Figure 2: The intersection graph of $\{6, 59, 2, 013478\} \mid 0, 9$ and $\{9, 8, 7, 46, 3, 125\} \mid 9$, omitting brackets for easier legibility. **Left:** The elements of $0, 9$ are labelling the edges. **Right:** the vertices are labelled by parts of the set partitions.

Lemma 2.2. Let $\lambda \mid 0 \sqcup E$ and $\mu \mid E$ be set partitions and $\Gamma_{\lambda, \mu}$ be their intersection graph.

1. If $\Gamma_{\lambda, \mu}$ has a cycle, then $\mathbb{L} \lambda \mid_{x_0=0} \cap w - \mathbb{L} \mu = \emptyset$ for generic¹ $w \in \mathbb{R}^E$.

¹This means that this property holds for all w outside of a set of measure 0.

2. If $\Gamma_{\lambda,\mu}$ is disconnected, then $L \lambda|_{x_0=0} \cap w - L \mu$ is not a point for any $w \in \mathbb{R}^E$.

3. If $\Gamma_{\lambda,\mu}$ is a tree, then $L \lambda|_{x_0=0} \cap w - L \mu$ is a point for any $w \in \mathbb{R}^E$.

Proof. Let $x \in L \lambda$ and $y \in L \mu$ such that $x - y = w$. Write $x_{\lambda_i} : x_i$ and $y_{\mu_j} : y_j$ for simplicity. The subspace $L \lambda|_{x_0=0} \cap w - L \mu$ is cut out by the equalities

$$\begin{aligned} x_{\lambda_i} - y_{\mu_i} &= w_i & \text{for } i \in E, \\ x_{\lambda_0} &= 0. \end{aligned}$$

This system has $|E| - 1$ equations and $|\lambda| - |\mu|$ unknowns. The linear dependences among these equations are controlled by the cycles of the graph $\Gamma_{\lambda,\mu}$. More precisely, the first $|E|$ linear functionals $\{x_{\lambda_i} - y_{\mu_i} : i \in E\}$ gives a realization of the graphical matroid of $\Gamma_{\lambda,\mu}$. The last equation is clearly linearly independent from the others.

If $\Gamma_{\lambda,\mu}$ has a cycle with edges $i_1, i_2, \dots, i_{2k}$ in that order, then the above equalities imply that $w_{i_1} - w_{i_2} + w_{i_3} - \dots - w_{i_{2k}} = 0$. For a generic w , this equation does not hold, so $L \lambda|_{x_0=0} \cap w - L \mu = \emptyset$.

If $\Gamma_{\lambda,\mu}$ is disconnected, let A be the set of edges in a connected component not containing the vertex λ_0 . If $x \in L \lambda$ and $y \in L \mu$ satisfy $x - y = w$ and $x_0 = 0$, then $x - re_A \in L \lambda$ and $y - re_A \in L \mu$ also satisfy those equations for any real number r . Therefore $L \lambda|_{x_0=0} \cap w - L \mu$ is not a point.

Finally, if $\Gamma_{\lambda,\mu}$ is a tree, then its number of vertices is one more than the number of edges, that is, $|E| - 1 = |\lambda| - |\mu|$, so the system of equations has equally many equations and unknowns. Also, these equations are linearly independent since $\Gamma_{\lambda,\mu}$ is a tree. It follows that the system has a unique solution. $\square$

When $\Gamma_{\lambda,\mu}$ is a tree, we call λ and μ an *arboreal pair*.

Lemma 2.3. *Let $\lambda|_{x_0=0} \sqcup E$ and $\mu|_{x_0=0} \sqcup E$ be an arboreal pair of set partitions and let $\Gamma_{\lambda,\mu}$ be their intersection tree. Let $w \in \mathbb{R}^E$. The unique vectors $x \in L \lambda$ and $y \in L \mu$ such that $x - y = w$ and $x_0 = 0$ are given by*

$$\begin{aligned} x_{\lambda_i} &= w_{e_1} - w_{e_2} + \dots \pm w_{e_k} & \text{where } e_1 e_2 \dots e_k \text{ is the unique path from } \lambda_i \text{ to } \lambda_0 \\ y_{\mu_j} &= w_{f_1} - w_{f_2} + \dots \pm w_{f_l} & \text{where } f_1 f_2 \dots f_l \text{ is the unique path from } \mu_j \text{ to } \lambda_0 \end{aligned}$$

for any i and j .

Proof. This follows readily from the fact that, for each $1 \leq i \leq k$, the values of $x_{\lambda_{e_i}}$ and $y_{\mu_{e_i}}$ on the vertices incident to edge i have to add up to w_{e_i} . $\square$

Example 2.4. *The set partitions $\lambda = \{6, 59, 2, 013478\} |_{x_0=0, 9}$, $\mu = \{9, 8, 7, 46, 3, 125\} |_{x_0=0, 9}$ form an arboreal pair, whose intersection tree is shown in Figure 2. We have, for example,*

$y_9 = w_9 - w_5 = w_1$ because the path from $\mu = 9 = \{9\}$ to $\lambda = 0 = \{013478\}$ uses edges $9, 5, 1$ in that order. The remaining values are:

$$\begin{aligned} x_6 &= w_6 - w_4, & x_{59} &= w_5 - w_1, & x_2 &= w_2 - w_1, & x_{13478} &= 0, \\ y_9 &= w_9 - w_5 = w_1, & y_8 &= w_8, & y_7 &= w_7, & y_{46} &= w_4, & y_3 &= w_3, & y_{1235} &= w_1. \end{aligned}$$

Definition 2.5. A vector $w \in \mathbb{R}^{n-1}$ is super-increasing if $w_{i-1} > 3w_i > 0$ for $1 \leq i \leq n$.

Lemma 2.6. Let w be super-increasing. For any $1 \leq a < b \leq n-1$ and any choice of ϵ_i s and δ_j s in $\{-1, 0, 1\}$, we have $\omega_a = \sum_{i=1}^{a-1} \epsilon_i \omega_i < \omega_b = \sum_{j=1}^{b-1} \delta_j \omega_j$.

Proof. After verifying inductively that $\omega_c > 2 \sum_{i=1}^{c-1} \omega_i$ for all c , we see that

$$\omega_a = \sum_{i=1}^{a-1} \epsilon_i \omega_i \leq \omega_a = \sum_{i=1}^{a-1} \omega_i < \frac{3}{2} \omega_a < \frac{1}{2} \omega_b < \omega_b - \sum_{j=1}^{b-1} \omega_j \leq \omega_b - \sum_{j=1}^{b-1} \delta_j \omega_j$$

as desired. $\square$

Definition 2.7. Given a super-increasing vector $w \in \mathbb{R}^{n-1}$ and a real number x , we will say x is near w_i and write $x \approx w_i$ if $w_i - w_1 - \dots - w_{i-1} \leq x \leq w_i - w_1 - \dots - w_i$. Note that if $x \approx w_i$ and $y \approx w_j$ for $i < j$ then $x < y$.

2.3 Matroids, Bergman fans, and tropical geometry

We assume familiarity with basic notions in matroid theory; for definitions and proofs, see [13, 17]. We also state here some facts from tropical geometry that we will need; see [11, 12] for a thorough introduction.

Let M be a matroid on E . The dual matroid $M^\perp$ is the matroid on E whose set of bases is $\{B^\perp \mid B \text{ is a basis of } M\}$, where $B^\perp = E - B$.

The following lemma is useful to how M and $M^\perp$ interact; see [13, Proposition 2.1.11].

Lemma 2.8. [2, Lemma 3.14] Let M be a matroid, and let F be a flat of M and G be a flat of $M^\perp$. Then $|F \cup G| \geq |E| - 1$.

Definition 2.9. Fix a linear order $<$ on M . A broken circuit is a set of the form $C - \min_{<} C$ where C is a circuit of M . An nbc-basis of M is a basis of M that contains no broken circuits. A β nbc-basis of M is an nbc-basis B such that $B^\perp \cup 0 \setminus 1$ is an nbc-basis of $M^\perp$.

Theorem 2.10. [7] The number of β nbc-bases of M is the beta invariant $\beta(M)$ which is given by $\beta(M) = |\chi'_M(1)|$, where χ_M is the characteristic polynomial of M :

$$\chi_M(t) = \sum_{X \subseteq E} (-1)^{|X|} t^{r(M) - r(X)}.$$

Assume that $B = \{b_1 > \dots > b_r\}$ is a basis of the matroid M . We define the flats $F_i := \text{cl}_M\{b_1, \dots, b_i\}$ and the flag $\mathcal{F}_M B := \{F_i\}$. The following characterization of nbc-basis will be useful.

Lemma 2.11. *Let M be a matroid of size $n - 1$ and rank $r - 1$, and B a basis of M . Then B is an nbc-basis of M if and only if $b_i = \min F_i$ for $i = 1, \dots, r - 1$.*

Proof. Omitted. □

An *affine matroid* M, e on E is a matroid M on E with a chosen element $e \in E$.

Definition 2.12. [14] *The Bergman fan of a matroid M on E is*

$$\Sigma_M = \{x \in \mathbb{R}^E \mid \min_{c \in C} x_c \text{ is attained at least twice for any circuit } C \text{ of } M\}.$$

The Bergman fan of an affine matroid M, e on E is

$$\Sigma_{M,e} = \{x \in \mathbb{R}^{E-e} \mid 0, x \in \Sigma_M\}.$$

The motivation for this definition comes from tropical geometry. A subspace $V \subset \mathbb{R}^E$ determines a matroid M_V , and the tropicalization of V is precisely the Bergman fan of M_V . Similarly, an affine subspace $W \subset \mathbb{R}^{E-e}$ determines an affine matroid $M_{W,e}$ consisting of a matroid M_W on E and a special element e , which represents the hyperplane at infinity. The tropicalization of W is the Bergman fan $\Sigma_{M_{W,e}}$.

Theorem 2.13. [4] *The Bergman fan of a matroid M is a tropical fan equal to the union of the cones*

$$\sigma_{\mathcal{F}} = \text{cone } e_{F_1}, \dots, e_{F_{r-1}} \\ \{x \in \mathbb{R}^E \mid x_a > x_b \text{ whenever } a \in F_i \text{ and } b \notin F_i \text{ for some } 1 \leq i \leq r - 1\}$$

for the complete flags $\mathcal{F} = \{\emptyset = F_0 \subsetneq F_1 \subsetneq \dots \subsetneq F_r \subsetneq F_{r-1} = E\}$ of flats of M .

If Σ_1 and Σ_2 are tropical fans of complementary dimensions, then Σ_1 and $\Sigma_2 + v$ intersect transversally at a finite set of points for generic vectors $v \in \mathbb{R}^n$. Each intersection point p is equipped with a weight w_p that depends on the respective intersecting cones. It is a general fact that the quantity

$$\deg \Sigma_1 \cdot \Sigma_2 := \sum_{p \in \Sigma_1 \cap v + \Sigma_2} w_p$$

is constant for generic v ; this is the degree of the intersection [12, Proposition 4.3.3, 4.3.6].

In all the intersections that arise in this paper, one can verify that the weight w_p is equal to 1 for every intersection point p . Therefore the degree of the intersection will be simply the number of intersection points:

$$\deg \Sigma_{M,e} \cdot -\Sigma_N := |\Sigma_{M,e} \cap v - \Sigma_N|$$

for generic $v \in \mathbb{R}^{E-e}$.

3 A combinatorial proof of the main theorem

Let M be a matroid on $[0, n]$ of rank $r - 1$ such that 0 is not a loop nor a coloop. Then $M/0$ has rank r , and $N = M/0^\perp$ has rank $n - r$. For any basis B of M containing 0 , $B^\perp = [0, n] - B$ is a basis of $N = M/0^\perp$. Conversely, every basis of N equals $B^\perp$ for a basis B of M containing 0 .

The following Lemma constructs an intersection point in $\Sigma_M|_{x_0=0} \cap w - \Sigma_N$ for each β -nbc basis B of M .

Lemma 3.1. *Let M be a matroid on $E = [0, n]$ of rank $r - 1$ such that 0 is not a coloop, and let $N = M/0^\perp$. Let $w \in \mathbb{R}^n$ be super-increasing. For any β -nbc basis B of M , there exist unique vectors $[0, x] \in \sigma_{\mathcal{F}_M|B}$ and $y \in \sigma_{\mathcal{F}_N|B^\perp}$ such that $x + y = w$.*

Proof. First we show that the set partitions π of $\mathcal{F}_M|B$ and $\pi^\perp$ of $\mathcal{F}_N|B^\perp$ form an arboreal pair. Since they have sizes $|B| = r - 1$ and $|B^\perp| = n - r$, respectively, their intersection graph has $n - 1$ vertices and n edges. Therefore it is sufficient to prove that the intersection graph $\Gamma_{\pi, \pi^\perp}$ is connected.

Assume contrariwise, and let A be a connected component not containing the edge 1 . Let $a > 1$ be the smallest edge in A . Then a is the smallest element of its part $\pi|_a$ in π , and since B is nbc in M , this implies $a \in B$. Similarly, since $B^\perp$ is nbc in N , this also implies $a \in B^\perp$. This is a contradiction.

It follows from Lemma 2.2 that there exist unique $[0, x] \in \mathcal{L}(\pi)$ and $y \in \mathcal{L}(\pi^\perp)$ such that $x + y = w$. It remains to show that $[0, x] \in \sigma_{\mathcal{F}}$ and $y \in \sigma_{\mathcal{F}^\perp}$.

Lemma 2.3 provides formulas for x and y in terms of the paths from the various vertices of the tree of $\Gamma_{\pi, \pi^\perp}$ to $\pi|_0$. To understand those paths, let us give each edge e an orientation as follows:

$$\begin{aligned} \pi|_e &\longrightarrow \pi^\perp|_e & \text{if } \min \pi|_e > \min \pi^\perp|_e, \\ \pi|_e &\longleftarrow \pi^\perp|_e & \text{if } \min \pi|_e < \min \pi^\perp|_e. \end{aligned}$$

We never have $\min \pi|_e = \min \pi^\perp|_e$, because as above, that would imply $e \in B \cap B^\perp$.

We claim that every vertex other than $\pi|_0$ has an outgoing edge under this orientation. Consider a part $\pi_i \neq \pi|_0$ of π ; let $\min \pi_i = b$. Edge b connects π_i to $\pi|_b \ni b$, and we cannot have $\min \pi^\perp|_b > b = \min \pi|_b$, so we must have $\pi_i \rightarrow \pi|_b$. The same argument works for any part $\pi_j^\perp$ of $\pi^\perp$.

Now, every element $b \in B$ is minimum in $\pi|_b$, so there is a directed path that starts at $\pi|_b$ and can only end at $\pi|_0$ whose first edge is b . Furthermore, by the definition of the orientation, the labels of the edges decrease along this path. Thus in the alternating sum $x_b = w_b \pm \dots$ given by Lemma 2.3, the first term dominates, and $x_b \approx w_b$. Similarly, $y_c \approx w_c$ for all $c \in B^\perp$.

Therefore, if we write $B = \{b_1 > \dots > b_r > b_{r+1} = 0\}$, since w is super-increasing, it follows that $x_{b_1} > x_{b_2} > \dots > x_{b_r} > x_{b_{r+1}} = 0$, so indeed $[0, x] \in \sigma_{\mathcal{F}}$. Similarly, if we

write $B^\perp = E - B = \{c_1 > \dots > c_{n-r} > c_{n-r-1} = 1\}$, then $y_{c_1} > y_{c_2} > \dots > y_{c_{n-r-1}}$, so $y \in \sigma_{\mathcal{F}^\perp}$. The desired result follows. $\square$

Example 3.2. The graphical matroid M of the graph G in Figure 1 has six β -nbc bases: 0256, 0257, 0259, 0368, 0378, and 0379. Let us compute the intersection point in $\Sigma_{M,0} \cap w - \Sigma_N$ associated to 0257 for the super-increasing vector $w = 10^0, 10^1, \dots, 10^8 \in \mathbb{R}^9$.

For $B = 0257$, we have $B^\perp = 134689$. Then

$$\begin{aligned} \mathcal{F}_M|_B &= \{\emptyset \subsetneq 7 \subsetneq 57 \subsetneq 2457 \subsetneq 0123456789\} \\ \mathcal{F}_N|_{B^\perp} &= \{\emptyset \subsetneq 9 \subsetneq 89 \subsetneq 689 \subsetneq 46789 \subsetneq 346789 \subsetneq 123456789\}, \end{aligned}$$

give rise to the corresponding set compositions

$$\pi = 7|5|24|013689, \quad \pi^\perp = 9|8|6|47|3|125.$$

This is indeed an arboreal pair, as evidenced by their intersection graph in Figure 3.

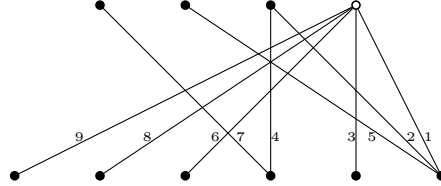


Figure 3: The intersection graph of $\pi = 7|5|24|013689$ and $\pi^\perp = 9|8|6|47|3|125$.

Lemma 3.1 gives us the unique points $0, x \in \mathcal{F}_\pi$ and $y \in \mathcal{F}_\tau$ such that $x + y = w$; they are given by the paths to the special vertex π_0 in the intersection tree $\Gamma_{\pi, \pi^\perp}$. For example $x_7 = 10^6 - 10^3 - 10^1 - 10^0 = 999009$ and $y_4 = 10^3 - 10^1 - 10^0 = 991$ are given by the paths 7421 and 421 from $\pi_7 = \pi_1$ and $\pi^\perp_7 = \pi^\perp_4$ to π_0 , respectively. In this way we obtain:

x	0	9	0	9	9999	0	999009	0	0
y	1	1	100	991	1	100000	991	10000000	100000000
w	1	10	100	1000	10000	100000	1000000	10000000	100000000

and x is in the intersection $\Sigma_{M,0} \cap w - \Sigma_N$.

The following lemma indicates that any intersection point between $\Sigma_{E,e}$ and $v - \Sigma_N$ is an intersection between the cones described above. That is, cones corresponding to a β -nbc basis.

Lemma 3.3. Let M be a matroid on $E = 0, n$ of rank $r - 1$, such that 0 is not a loop nor a coloop, and $N = M/0^\perp$. Let $w \in \mathbb{R}^n$ be generic and super-increasing. Let

$$\begin{aligned} \mathcal{F} &= \{\emptyset \subsetneq F_0 \subsetneq F_1 \subsetneq \dots \subsetneq F_r \subsetneq F_{r-1} \subsetneq E\} \\ \mathcal{G} &= \{\emptyset \subsetneq G_0 \subsetneq G_1 \subsetneq \dots \subsetneq G_{n-r-1} \subsetneq G_{n-r} \subsetneq E - 0\} \end{aligned}$$

be complete flags of the matroids M and N , respectively, such that $\Sigma_{M,0}$ and $w - \Sigma_N$ intersect at $\sigma_{\mathcal{F}}$ and $w - \sigma_{\mathcal{G}}$. Then there exists a β -nbc basis B such that $\mathcal{F} = \mathcal{F}_M \setminus B$ and $\mathcal{G} = \mathcal{F}_N \setminus B^\perp$.

Proof. By Lemma 2.2, the set compositions π and τ of $\mathcal{F}$ and $\mathcal{G}$ form an arboreal pair. In particular, $\pi_a \cap \tau_b = F_a - F_{a-1} \cap G_b - G_{b-1}$ cannot have more than one element for any a and b . We proceed in several steps.

1. Our first step will be to show that in the intersection tree $\Gamma_{\pi,\tau}$, the top right vertex π_{r-1} contains 0 and 1, the bottom right vertex τ_{n-r} contains 1, and thus the edge 1 connects these two rightmost vertices.

Each G_i is a flat of $N = M^\perp - 0$, so $G_i^\bullet : \text{cl}_{M^\perp} G_i \in \{G_i, G_i \cup 0\}$ is a flat of $M^\perp$. Consider the flag of flats of $M^\perp$

$$\mathcal{G}^\bullet : \{\emptyset \subsetneq G_0^\bullet \subsetneq G_1^\bullet \subsetneq \cdots \subsetneq G_{n-r-1}^\bullet \subsetneq G_{n-r}^\bullet \subseteq E\},$$

where $G_{n-r}^\bullet = E$ because 0 is not a coloop of $M^\perp$ and $G_0^\bullet = \emptyset$ because 0 is not a loop of $M^\perp$. Let M be the minimal index such that $0 \in G_M^\bullet$, so

$$\mathcal{G}^\bullet : \{\emptyset \subsetneq G_0 \subsetneq G_1 \subsetneq \cdots \subsetneq G_{M-1} \subsetneq G_M \cup 0 \subsetneq \cdots \subsetneq G_{n-r-1} \cup 0 \subsetneq G_{n-r} \cup 0 \subseteq E\},$$

Consider the unions of the flat F_r with the coflats in $\mathcal{G}^\bullet$; let j be the index such that

$$F_r \cup G_{j-1}^\bullet \subsetneq E, \quad F_r \cup G_j^\bullet = E$$

Since it is the union of a flat and a coflat, the former cannot have size $|E| - 1$, so $F_r \cup G_j^\bullet = F_r \cup G_{j-1}^\bullet \cup E - F_r \cap G_j^\bullet = F_r \cup G_{j-1}^\bullet$ has size at least 2. But $\mathcal{F}$ and $\mathcal{G}$ are arboreal so $\pi_{r-1} \cap \tau_j = E - F_r \cap G_j - G_{j-1}$ has size at most 1. This has two consequences:

- a) $G_j^\bullet = G_j \cup 0$ and $G_{j-1}^\bullet = G_{j-1}$, that is, $j = M$.
- b) $0 \in E - F_r = \pi_{r-1}$.

Similarly, consider the unions of the coflat $G_{n-r-1}^\bullet$ with the flats in $\mathcal{F}$; let i be the index such that

$$F_{i-1} \cup G_{n-r-1}^\bullet \subsetneq E, \quad F_i \cup G_{n-r-1}^\bullet = E.$$

Analogously, we get that $F_i - F_{i-1} \cap E - G_{n-r-1}^\bullet$ has size at least 2, whereas $\pi_i \cap \tau_{n-r} = F_i - F_{i-1} \cap E - 0 - G_{n-r-1}$ has size at most 1. This has three consequences:

- c) $G_{n-r-1}^\bullet = G_{n-r-1}$, that is, $M = n - r$.
- d) $0 \in F_i - F_{i-1}$, which in light of b) implies that $i = r - 1$.

e) $F_i - F_{i-1} \cap E - 0 - G_{n-r-1} = \pi_{r-1} \cap \tau_{n-r} = \{e\}$ for some element $e \in E - 0$. But $e \in \pi_{r-1}$ means that $x_e = 0$ is minimum among all x_i s for any $0, x \in \sigma_{\mathcal{F}}$, and $e \in \tau_{n-r}$ means that y_e is minimum among all y_i s for any $y \in \sigma_{\mathcal{G}}$. Since $w = x + y$ for some such x and y , $w_e = x_e + y_e$ is minimum among all w_i s, and since w is super-increasing, $e = 1$.

It follows that in the intersection tree $\Gamma_{\pi,\tau}$, the top right vertex π_{r-1} contains 0 and 1 by d) and e), the bottom right vertex τ_{n-r} contains 1 by e), and thus 1 connects them.

2. Next we claim that for any path in the tree $\Gamma_{\pi,\tau}$ that ends with the edge 1, the first edge has the largest label.² Assume contrariwise, and consider a containment-minimal path P that does not satisfy this property; its edges must be labelled $e < f > f_2 > \dots > f_k$ sequentially. If edge e goes from π_e to τ_e , Lemma 2.3 gives $x_e = w_e - w_f \pm (\text{terms smaller than } w_f) \approx -w_f < 0 = x_1$, contradicting that $0, x \in \sigma_{\mathcal{F}}$. If e goes from bottom to top, we get $y_e = w_e - w_f \pm (\text{terms smaller than } w_f) \approx -w_f < w_1 = y_1$, contradicting that $y \in \sigma_{\mathcal{G}}$.

3. Now define

$$\begin{aligned} b_i &: \min F_i - F_{i-1} \quad \text{for } i = 1, \dots, r-1, \\ c_j &: \min G_j - G_{j-1} \quad \text{for } j = 1, \dots, n-r. \end{aligned}$$

Then $B : \{b_1, \dots, b_{r-1}\}$ and $C : \{c_1, \dots, c_{n-r}\}$ are bases of M and N , and $\mathcal{F} = \mathcal{F}_M \cup B$ and $\mathcal{G} = \mathcal{F}_N \cup C$. We claim that B is β -nbc and $C = B^\perp$.

To do so, we first notice that the path from vertex $\pi_i = F_i - F_{i-1}$ (resp. $\tau_j = G_j - G_{j-1}$) to edge 1 must start with edge b_i (resp. c_j): if it started with some larger edge $b' \in F_i - F_{i-1}$, then the path from edge b_i to edge 1 would not start with the largest edge. This has two consequences:

f) The sets B and C are disjoint. If we had $b_i = c_j = e$, then edge e , which connects vertices $\pi_i = F_i - F_{i-1}$ and $\tau_j = G_j - G_{j-1}$, would have to be the first edge in the paths from both of these vertices to edge 1; this is impossible in a tree. We conclude that B and C are disjoint. Since $|B| = r-1$ and $|C| = n-r$, we have $C = B^\perp$.

g) For each i we have $x_{b_i} \approx w_{b_i}$, because the path from τ_i to vertex 0 – which is the path from τ_i to edge 1, with edge 1 possibly removed – starts with the largest edge b_i , so Lemma 2.3 gives $x_{b_i} = w_{b_i} \pm (\text{smaller terms}) \approx w_{b_i}$. Similarly $y_{c_i} \approx w_{c_i}$. Now, $0, x \in \sigma_{\mathcal{F}}$ gives $x_{b_1} > \dots > x_{b_{r-1}}$, which implies $w_{b_1} > \dots > w_{b_{r-1}}$, which in turn gives

$$b_1 > \dots > b_r > b_{r-1}; \quad \text{and analogously,} \quad c_1 > \dots > c_{n-r-1} > c_{n-r} = 1.$$

The former implies that B is nbc in M by Lemma 2.11. The latter, combined with c), implies that $c_1 > \dots > c_{n-r-1} > 0$ respectively are the minimum elements of the flats $G_1^\bullet, \dots, G_{n-r-1}^\bullet, G_{n-r}^\bullet = E$ that they sequentially generate, so $C \cup 0 \setminus 1 = B^\perp \cup 0 \setminus 1$ is nbc in $M^\perp$. It follows that B is β -nbc in M .

We conclude that B is β -nbc in M , $\mathcal{F} = \mathcal{F}_M \cup B$, and $\mathcal{G} = \mathcal{F}_N \cup B^\perp$, as desired. $\square$

Combinatorial proof of Theorem 1.2. This follows from the previous two lemmas. $\square$

²It follows that the edge labels decrease along any such path, but we will not use this in the proof.

4 A proof via torus-equivariant geometry

We sketch a proof of [Theorem 1.3](#) using the framework of *tautological classes* of matroids of Berget, Eur, Spink, and Tseng. See [\[5\]](#) for details on what follows.

In this framework, one works with the permutohedral fan Σ_E , which is the Bergman fan of the Boolean matroid on E . Toric geometry equips Σ_E with two “cohomology” rings: The torus-equivariant Chow ring $A_T^\bullet \Sigma_E$, and the non-equivariant Chow ring $A^\bullet \Sigma_E$, which is a quotient of $A_T^\bullet \Sigma_E$ by an explicitly described ideal I .

First, one defines certain elements $\Sigma_{M,e}$ and $-\Sigma_{M/e}^\perp$ of $A^\bullet \Sigma_E$ by using the fact that the fans $\Sigma_{M,e}$ and $-\Sigma_{M/e}^\perp$ are subfans of Σ_E that satisfy a balancing condition in the sense of Minkowski weights [\[8\]](#). These elements have the following property: The ring $A^\bullet \Sigma_E$ is equipped with a degree map $\deg_{\Sigma_E} : A^\bullet \Sigma_E \rightarrow \mathbb{Z}$, which agrees with the map $\deg$ in [Theorem 1.3](#) in the sense that

$$\deg \Sigma_{M,e} \cap -\Sigma_{M/e}^\perp = \deg_{\Sigma_E} \Sigma_{M,e} \cdot -\Sigma_{M/e}^\perp.$$

For a survey of these facts, see [\[9, Section 4\]](#) or [\[6, Section 7.1\]](#).

Second, one notes that [\[6\]](#) provided a distinguished representative in $A_T^\bullet \Sigma_E$ of the class $\Sigma_{M,e} \in A^\bullet \Sigma_E = A_T^\bullet \Sigma_E / I$, and similarly for the class $-\Sigma_{M/e}^\perp$. In more detail, for a matroid M of rank $r - 1$ on a ground set E of size $n - 1$, [\[6, Definition 3.9\]](#) defined *tautological Chern classes* of M as certain elements $\{c_i^T \mathcal{S}_M^\vee\}_{i=0,\dots,r-1}$ and $\{c_j^T \mathcal{Q}_M\}_{j=0,\dots,n-r}$ in the equivariant Chow ring $A_T^\bullet \Sigma_E$, and established the following properties.

Lemma 4.1. [\[6, Theorem 7.6, Propositions 5.11, 5.13\]](#) *Let M be a matroid of rank $r - 1$ on a ground set E of size $n - 1$. Define elements $\Sigma_{M,e}^T$ and $-\Sigma_{M/e}^\perp{}^T$ in $A_T^\bullet \Sigma_E$ by $\Sigma_{M,e}^T = c_{n-r}^T \mathcal{Q}_M$ and $-\Sigma_{M/e}^\perp{}^T = c_r^T \mathcal{S}_{M/e \oplus U_{0,e}}^\vee$. Then, their images in the quotient $A^\bullet \Sigma_E$ are exactly $\Sigma_{M,e}$ and $-\Sigma_{M/e}^\perp$, respectively.*

Finally, to prove [Theorem 1.3](#), one begins with [\[6, Theorem 6.2\]](#) which states that

$$\deg_{\Sigma_E} (\Sigma_{M,e} \cdot c_r \mathcal{S}_M^\vee) = \beta_M.$$

Thus, the desired statement $\deg_{\Sigma_E} \Sigma_{M,e} \cdot -\Sigma_{M/e}^\perp = \beta_M$ will follow once one shows that $\Sigma_{M,e} \cdot (c_r \mathcal{S}_M^\vee - -\Sigma_{M/e}^\perp) = 0$ in $A^\bullet \Sigma_E$. For this end, one considers the distinguished representative $\Sigma_{M,e}^T \cdot (c_r^T \mathcal{S}_M^\vee - -\Sigma_{M/e}^\perp{}^T)$ of this product in $A_T^\bullet \Sigma_E$. An explicit description of this representative straightforwardly displays that it belongs to the ideal I .

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Barrett–Johnson inequalities for totally nonnegative matrices

Mark Skandera^{*1} and Daniel Soskin²

¹Department of Mathematics, Lehigh University

²Department of Mathematics, The University of Notre Dame

Abstract. Given a matrix A , let $A_{I,J}$ denote the submatrix of A determined by rows I and columns J . The Barrett–Johnson Inequalities relate sums of products of principal minors of positive semidefinite (PSD) matrices, when orders of the minors are given by integer partitions $\lambda = \lambda_1, \dots, \lambda_r$, $\mu = \mu_1, \dots, \mu_s$ of n . Specifically, we have

$$\lambda_1! \cdots \lambda_r! \sum_{I_1, \dots, I_r} \det A_{I_1, I_1} \cdots \det A_{I_r, I_r} \geq \mu_1! \cdots \mu_s! \sum_{J_1, \dots, J_s} \det A_{J_1, J_1} \cdots \det A_{J_s, J_s},$$

for all PSD $n \times n$ matrices A , where sums are over ordered set partitions of $\{1, \dots, n\}$ satisfying $|I_k| = \lambda_k$, $|J_k| = \mu_k$, if and only if λ is majorized by μ . We show that these inequalities hold for totally nonnegative matrices as well.

1 Introduction

A matrix $A \in \text{Mat}_{n \times n} \mathbb{C}$ is called *Hermitian* if it satisfies $A^* = A$ where $*$ denotes conjugate transpose. Such a matrix is called *Hermitian positive semi-definite* (HPSD) if we have $x^* A x \geq 0$ for all $x \in \mathbb{C}^n$. For $A \in \text{Mat}_{n \times n} \mathbb{R}$, the Hermitian property reduces to symmetry $A^T = A$, and the positive semidefinite (PSD) property reduces to $x^T A x \geq 0$ for all $x \in \mathbb{R}^n$. A matrix $A \in \text{Mat}_{n \times n} \mathbb{R}$ is called *totally nonnegative* (TNN) if each minor (determinant of a square submatrix) is nonnegative. One can deduce that the entries of all $n \times n$ HPSD and/or TNN matrices satisfy certain polynomial inequalities. In some cases, inequalities for the two classes of matrices are precisely the same. (See [14, §1].)

Given an $n \times n$ matrix $A = (a_{i,j})$ and subsets $I, J \subseteq [n] := \{1, \dots, n\}$, define the submatrix $A_{I,J} = (a_{i,j})_{i \in I, j \in J}$, and define the set $I^c := [n] \setminus I$. Hadamard [7] showed that for A HPSD we have

$$\det A \leq a_{1,1} \cdots a_{n,n}, \quad (1.1)$$

and Koteljanskii [9] showed that this holds for A TNN as well. Fischer [5] strengthened (1.1) by showing that for all $I \subseteq [n]$ we have

$$\det A \leq \det A_{I,I} \det A_{I^c, I^c}, \quad (1.2)$$

^{*}mark.skandera@gmail.com.

and Ky Fan showed that this holds for A TNN as well (unpublished; see [2]). Barrett and Johnson [1] showed that for A PSD, averages of the products of pairs of minors appearing in (1.2) increase as the cardinality difference between I and I^c decreases. Furthermore, they proved a similar result for averages of products of many minors: given integer partitions $\lambda = \lambda_1, \dots, \lambda_r$, $\mu = \mu_1, \dots, \mu_s$ of n , we have

$$\sum_{I_1, \dots, I_r} \frac{\det A_{I_1, I_1} \cdots \det A_{I_r, I_r}}{\binom{n}{\lambda_1, \dots, \lambda_r}} \geq \sum_{J_1, \dots, J_s} \frac{\det A_{J_1, J_1} \cdots \det A_{J_s, J_s}}{\binom{n}{\mu_1, \dots, \mu_s}}, \quad (1.3)$$

if and only if λ is majorized by μ , where the sums are over *ordered set partitions* of n of types λ and μ , i.e., sequences of subsets of n satisfying having cardinalities

$$I_1 \uplus \cdots \uplus I_r = J_1 \uplus \cdots \uplus J_s = n, \quad |I_k| = \lambda_k, \quad |J_k| = \mu_k. \quad (1.4)$$

We will show that these inequalities also hold for TNN matrices.

2 The symmetric group, its traces, and symmetric functions

The *symmetric group algebra* $\mathbb{C} \mathfrak{S}_n$ is generated over $\mathbb{C}$ by $s_1, \dots, s_{n-1}$, subject to relations

$$\begin{aligned} s_i^2 &= e && \text{for } i = 1, \dots, n-1, \\ s_i s_j s_i &= s_j s_i s_j && \text{for } |i-j| = 1, \\ s_i s_j &= s_j s_i && \text{for } |i-j| \geq 2. \end{aligned}$$

We define the *one-line notation* $w_1 \cdots w_n$ of $w \in \mathfrak{S}_n$ by letting any expression for w act on the word $1 \cdots n$, where each generator $s_j = s_{j,j-1}$ acts on an n -letter word by swapping the letters in positions j and $j-1$, i.e., $s_j \circ v_1 \cdots v_n = v_1 \cdots v_{j-1} v_j v_{j-1} v_{j+1} \cdots v_n$. Whenever $s_{i_1} \cdots s_{i_\ell}$ is a reduced (short as possible) expression for $w \in \mathfrak{S}_n$, we call ℓ the *length* of w and write $\ell = \ell(w)$. It is known that $\ell(w)$ is equal to $\text{inv}(w)$, the number of inversions in $w_1 \cdots w_n$. It is also known that conjugacy classes in $\mathfrak{S}_n$ are precisely the set of permutations that have the same cycle type. We name cycle type by integer partitions of n , the weakly decreasing positive integer sequences $\lambda = \lambda_1, \dots, \lambda_\ell$ satisfying $\lambda_1 + \cdots + \lambda_\ell = n$. The $\ell = \ell(\lambda)$ components of λ are called its *parts*, and we let $|\lambda| = n$ and $\lambda \vdash n$ denote that λ is a partition of n . Given $\lambda \vdash n$, we define the *transpose* partition $\lambda^\top = \lambda_1^\top, \dots, \lambda_{\lambda_1}^\top$ by

$$\lambda_i^\top = \#\{j \mid \lambda_j \geq i\}.$$

Sometimes it is convenient to name a partition with exponential notation, omitting parentheses and commas, so that $4^2 1^6 : 4, 4, 1, 1, 1, 1, 1, 1$. Sometimes it is convenient to partially order partitions of n by the *majorization* (or *dominance*) order,

$$\lambda \preceq \mu \quad \text{iff} \quad \lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i \quad (2.1)$$

for all i . It is known that if λ is covered by μ in this partial order then λ, μ have equal parts except in two positions $i < j$ where we have

$$\mu_i = \lambda_i + 1, \quad \mu_j = \lambda_j - 1. \quad (2.2)$$

Let Λ be the ring of symmetric functions in $x_1, x_2, \dots$ having integer coefficients, and let Λ_n be the $\mathbb{Z}$ -submodule of homogeneous functions of degree n . This submodule has rank equal to the number of partitions of n . Five standard bases of Λ_n consist of the monomial $\{m_\lambda \mid \lambda \vdash n\}$, elementary $\{e_\lambda \mid \lambda \vdash n\}$, (complete) homogeneous $\{h_\lambda \mid \lambda \vdash n\}$, power sum $\{p_\lambda \mid \lambda \vdash n\}$, and Schur $\{s_\lambda \mid \lambda \vdash n\}$ symmetric functions. (See, e.g., [15, Ch. 7] for definitions.) The change-of-basis matrix relating $\{e_\lambda \mid \lambda \vdash n\}$ and $\{m_\lambda \mid \lambda \vdash n\}$ is given by the equations

$$e_\lambda = \sum_{\mu \preceq \lambda^\top} M_{\lambda, \mu} m_\mu, \quad (2.3)$$

where $M_{\lambda, \mu}$ equals the number of column-strict Young tableaux of shape $\lambda^\top$ and content μ . That is, $M_{\lambda, \mu}$ is the number of histograms having λ_i boxes in column i for all i , filled with μ_1 ones, μ_2 twos, etc., with column entries strictly increasing from bottom to top.

Let $\mathcal{T}_n$ be the $\mathbb{Z}$ -module of $\mathbb{Z} \wr \mathfrak{S}_n$ -traces (equivalently, $\mathfrak{S}_n$ -class functions), linear functionals $\theta : \mathbb{Z} \wr \mathfrak{S}_n \rightarrow \mathbb{Z}$ satisfying $\theta(gh) = \theta(hg)$ for all $g, h \in \mathbb{Z} \wr \mathfrak{S}_n$. Like the $\mathbb{Z}$ -module Λ_n , the trace space $\mathcal{T}_n$ has dimension equal to the number of integer partitions of n . The Frobenius $\mathbb{Z}$ -module isomorphism (2.4)

$$\begin{aligned} \text{Frob} : \mathcal{T}_n &\rightarrow \Lambda_n \\ \theta &\mapsto \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \theta(w) p_{\text{ctype } w} \end{aligned} \quad (2.4)$$

defines bijections between standard bases of Λ_n and $\mathcal{T}_n$. Schur functions correspond to irreducible characters, while elementary and homogeneous symmetric functions correspond to induced sign and trivial characters,

$$s_\lambda \leftrightarrow \chi^\lambda, \quad e_\lambda \leftrightarrow \epsilon^\lambda = \text{sgn} \uparrow_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n}, \quad h_\lambda \leftrightarrow \eta^\lambda = \text{triv} \uparrow_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n},$$

where $\mathfrak{S}_\lambda$ is the Young subgroup of $\mathfrak{S}_n$ indexed by λ . The power sum and monomial bases of Λ_n correspond to bases of $\mathcal{T}_n$ which are not characters. We call these the *power sum* $\{\psi^\lambda \mid \lambda \vdash n\}$ and *monomial* $\{\phi^\lambda \mid \lambda \vdash n\}$ traces, respectively. These are the bases related to the irreducible character bases by the same matrices of character evaluations and inverse Kostka numbers that relate power sum and monomial symmetric functions to Schur functions,

$$\begin{aligned} p_\lambda &= \sum_{\mu} \chi^\mu \cdot \lambda \cdot s_\mu, & \psi^\lambda &= \sum_{\mu} \chi^\mu \cdot \lambda \cdot \chi^\mu, \\ m_\lambda &= \sum_{\mu} K_{\lambda, \mu}^{-1} s_\mu, & \phi^\lambda &= \sum_{\mu} K_{\lambda, \mu}^{-1} \chi^\mu, \end{aligned} \quad (2.5)$$

where $\chi^\mu \cdot \lambda := \chi^\mu(w)$ for any $w \in \mathfrak{S}_n$ having ctype $w \cdot \lambda$.

3 Immanants and totally nonnegative polynomials

Each of the inequalities stated in Section 1 may be stated in terms of a polynomial in matrix entries. In particular, let $x = (x_{i,j})_{i,j \in n}$ be a matrix of n^2 indeterminates, and for $p(x) \in \mathbb{C}[x] : \mathbb{C}[x_{i,j}]_{i,j \in n}$ and $A = (a_{i,j})$ an $n \times n$ matrix, define $p(A) = p(a_{1,1}, a_{1,2}, \dots, a_{n,n})$. While few of the polynomial inequalities in Section 1 specifically mention the symmetric group, all of them involve polynomials which are linear combinations of monomials of the form $\{x_{1,w_1} \cdots x_{n,w_n} \mid w \in \mathfrak{S}_n\}$. Following [10], [16], we call such polynomials *immanants*. Specifically, given $f : \mathfrak{S}_n \rightarrow \mathbb{C}$ define the f -*immanant* to be the polynomial

$$\text{Imm}_f(x) : \sum_{w \in \mathfrak{S}_n} f(w) x_{1,w_1} \cdots x_{n,w_n} \in \mathbb{C}[x]. \quad (3.1)$$

The sign character ($w \mapsto (-1)^{\ell(w)}$) immanant and trivial character ($w \mapsto 1$) immanant are the determinant and permanent,

$$\det x = \sum_{w \in \mathfrak{S}_n} (-1)^{\ell(w)} x_{1,w_1} \cdots x_{n,w_n}, \quad \text{per } x = \sum_{w \in \mathfrak{S}_n} x_{1,w_1} \cdots x_{n,w_n}.$$

Simple formulas for the induced sign and trivial character immanants are due to Littlewood–Merris–Watkins [10], [12]: for $\lambda = \lambda_1, \dots, \lambda_r \vdash n$, we have

$$\begin{aligned} \text{Imm}_{\epsilon^\lambda}(x) &= \sum_{I_1, \dots, I_r} \det x_{I_1, I_1} \cdots \det x_{I_r, I_r}, \\ \text{Imm}_{\eta^\lambda}(x) &= \sum_{I_1, \dots, I_r} \text{per } x_{I_1, I_1} \cdots \text{per } x_{I_r, I_r}, \end{aligned} \quad (3.2)$$

where the sums are over all ordered set partitions of n of type λ (1.4).

Some current interest in immanants and their connection to TNN matrices was inspired by Lusztig's work with canonical bases of quantum groups. (See, e.g., [11].) In particular, one quantum group has an interesting basis whose elements can be described in terms of immanants which evaluate nonnegatively on TNN matrices. Call a polynomial $p(x)$ *totally nonnegative* (TNN) if $p(A) \geq 0$ whenever A is a totally nonnegative matrix. There is no known procedure to decide if a given polynomial is TNN.

The formula (3.2) makes it obvious that $\text{Imm}_{\epsilon^\lambda}(x)$ is a TNN polynomial for each partition $\lambda \vdash n$. A stronger result [17, Cor. 3.3] asserts that irreducible character immanants $\text{Imm}_{\chi^\lambda}(x)$ are TNN as well. It is clear that the $\mathfrak{S}_n$ -trace immanants

$$\{\text{Imm}_\theta(x) \mid \theta \in \mathcal{T}_n, \text{Imm}_\theta(x) \text{ is TNN}\}$$

form a cone, i.e., are closed under real nonnegative linear combinations. Stembridge has conjectured [18, Conj. 2.1] that the extreme rays of this cone are generated by the monomial trace immanants

$$\{\text{Imm}_{\phi^\lambda}(x) \mid \lambda \vdash n\}, \quad (3.3)$$

and has shown [18, Prop.2.3] that the cone of TNN $\mathfrak{S}_n$ -trace immanants lies inside of the cone generated by (3.3).

Proposition 3.1. *Each immanant of the form $\text{Imm}_\theta x$ with $\theta \in \mathcal{T}_n$ is a totally nonnegative polynomial only if it is equal to a nonnegative linear combination of monomial trace immanants.*

Thus it is conjectured that an $\mathfrak{S}_n$ -trace immanant is TNN if and only if it is equal to a nonnegative linear combination of monomial trace immanants. Indeed it is known that some monomial trace immanants generate extremal rays of the cone of TNN $\mathfrak{S}_n$ -trace immanants [3, Thm.10.3]. (See [Theorem 4.6](#).)

4 The Temperley-Lieb algebra and 2-colorings

Given a complex number ζ , we define the *Temperley-Lieb algebra* $T_n \zeta$ to be the $\mathbb{C}$ -algebra generated by elements $t_1, \dots, t_{n-1}$ subject to the relations

$$\begin{aligned} t_i^2 &= \zeta t_i, & \text{for } i &= 1, \dots, n-1, \\ t_i t_j t_i &= t_i, & \text{if } |i-j| &= 1, \\ t_i t_j &= t_j t_i, & \text{if } |i-j| &\geq 2. \end{aligned}$$

When $\zeta = 2$ we have the isomorphism $T_n 2 \sim \mathbb{C} \mathfrak{S}_n / \langle s_1^2 = s_2^2 = s_1 s_2 = s_2 s_1 = s_1 s_2 s_1 \rangle$. (See e.g. [4], [6, Sec.2.1, Sec.2.11], [19, Sec.7].) Specifically, the isomorphism is given by

$$\begin{aligned} \sigma : \mathbb{C} \mathfrak{S}_n &\rightarrow T_n 2, \\ s_i &\mapsto t_i - 1. \end{aligned} \tag{4.1}$$

Let $\mathcal{B}_n$ be the multiplicative monoid generated by $t_1, \dots, t_{n-1}$ when $\zeta = 1$, also called the standard basis of $T_n \zeta$. It is known that $|\mathcal{B}_n|$ is the n th Catalan number $C_n = \frac{1}{n+1} \binom{2n}{n}$. Diagrams of the basis elements of $T_n \zeta$, made popular by Kauffman [8, Sec.4] are (undirected) graphs with $2n$ vertices and n edges. The identity and generators $1, t_1, \dots, t_{n-1}$ are represented by

$$\begin{array}{cccc} \equiv & \equiv & \equiv & \bowtie \\ \vdots & \vdots & \vdots & \vdots \\ \equiv & \overline{\bowtie} & \bowtie, \dots, & \equiv \end{array}$$

and multiplication of these elements corresponds to concatenation of diagrams, with cycles contributing a factor of ζ . For instance, the fourteen basis elements of $T_4 \zeta$ are

$$\equiv, \bowtie, \overline{\bowtie}, \equiv, \bowtie, \overline{\bowtie}, \bowtie, \overline{\bowtie}, \bowtie, \overline{\bowtie}, \bowtie, \overline{\bowtie}, \bowtie, \overline{\bowtie}$$

and the equality $t_3 t_2 t_3 t_1 = \xi t_1 t_3$ in T_4 is represented by

$$\text{diagram} = \xi \text{diagram} \quad (4.2)$$

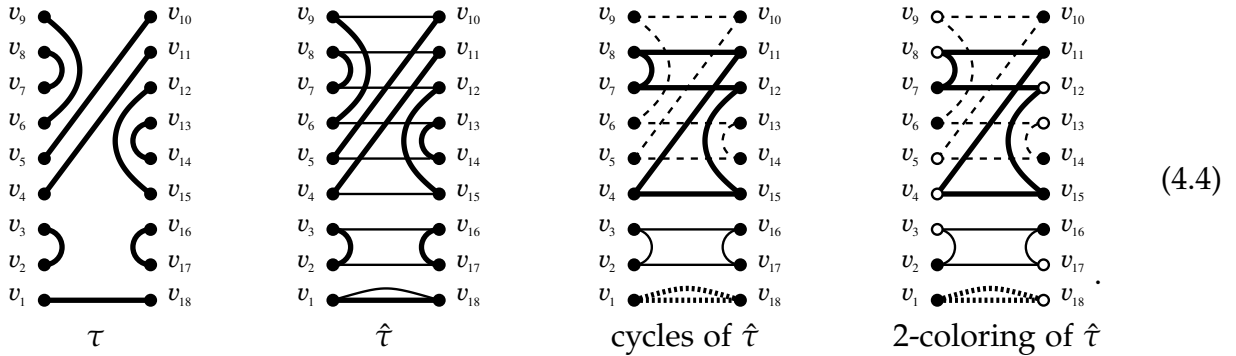
with the "bubble" becoming the scalar multiple ξ . We will identify each element $\tau \in \mathcal{B}_n$ with its Kauffman diagram, and will label vertices $v_1, \dots, v_{2n}$, clockwise from the lower left. We define the height of a vertex by

$$\text{hgt } v_i = \begin{cases} i & \text{if } 1 \leq i \leq n, \\ 2n - 1 - i & \text{if } n + 1 \leq i \leq 2n. \end{cases}$$

For instance, (4.4) shows the element $t_7 t_6 t_8 t_5 t_7 t_4 t_6 t_5 t_2 \in \mathcal{B}_9$ on the left, with vertex labels. Vertices have heights $1, \dots, 9$, from bottom to top. It is easy to see that for all $\tau \in \mathcal{B}_n$, each edge v_i, v_j satisfies

$$\text{hgt } v_i - \text{hgt } v_j \equiv \begin{cases} 1 \pmod{2} & \text{if } i, j \leq n \text{ or } i, j \geq n + 1, \\ 0 \pmod{2} & \text{otherwise.} \end{cases} \quad (4.3)$$

Define $\hat{\tau}$ to be the graph obtained from τ by adding edges v_i, v_{2n+1-i} for all i (even if such an edge already exists). Since each vertex in $\hat{\tau}$ has degree 2, it is clear that this graph is a disjoint union of cycles. For example, corresponding to the element τ on the left of (4.4) we have $\hat{\tau}$ to its right, the decomposition of this graph into four disjoint cycles, and a proper 2-coloring of this graph.

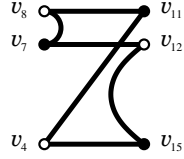


The Temperley-Lieb algebra T_n sometimes arises in the 2-coloring of combinatorial objects. Define a *principal coloring* of $\tau \in \mathcal{B}_n$ to be a map $\kappa : \text{vertices } \tau \rightarrow \{\text{black, white}\}$ which is a *proper coloring* of $\hat{\tau}$, i.e.,

$$\begin{aligned} \text{color } v_i &\neq \text{color } v_{2n+1-i} && \text{for } i = 1, \dots, n, \\ \text{color } v_i &\neq \text{color } v_j && \text{if } v_i \text{ and } v_j \text{ are adjacent in } \tau. \end{aligned}$$

Let τ, κ denote the graph τ with its vertices colored by κ .

Principal colorings of τ are closely related to cycles in $\hat{\tau}$. It is clear that colors must alternate along any one cycle of $\hat{\tau}$. It is also true that vertex colors alternate as one views the vertices in clockwise order, ignoring the edges of that cycle. For example, consider a 2-coloring of the cycle $v_4, v_{11}, v_8, v_7, v_{12}, v_{15}$ of $\hat{\tau}$ in (4.4),



Proposition 4.1. *If $\hat{\tau}$ is a single cycle, then there are two principal colorings of τ . In each, vertices of odd index and of even index have opposite colors.*

Proof. Omitted. □

Clearly if κ is a principal coloring of $\tau \in \mathcal{B}_n$ and if $\hat{\tau}$ is a single cycle, then we have

$$\left| \# \left(\begin{array}{c} \text{white vertices on} \\ \text{left of } \tau, \kappa \end{array} \right) - \# \left(\begin{array}{c} \text{white vertices on} \\ \text{right of } \tau, \kappa \end{array} \right) \right| = \begin{cases} 0 & \text{if } n \text{ even,} \\ 1 & \text{if } n \text{ odd.} \end{cases} \quad (4.5)$$

In this situation we call τ, κ *balanced* if n is even, and *unbalanced* otherwise. More specifically, we call τ, κ *left-unbalanced* (*right-unbalanced*) if it has more white vertices on the left (right). Now consider $\tau \in \mathcal{B}_n$ with $\hat{\tau}$ a disjoint union of cycles $C_1, \dots, C_d$, and κ a principal coloring of τ . Define

$$\begin{aligned} \alpha &= \alpha_{\tau, \kappa} : \# \{i \mid \tau_{C_i}, \kappa \text{ right unbalanced}\}, \\ \beta &= \beta_{\tau, \kappa} : \# \{i \mid \tau_{C_i}, \kappa \text{ left unbalanced}\}. \end{aligned} \quad (4.6)$$

For example, in (4.4), the proper 2-coloring κ of $\hat{\tau}$ corresponds to a principal coloring of τ with $\alpha_{\tau, \kappa} = 1$, $\beta_{\tau, \kappa} = 2$, and $d = 4$. Also note that there is one balanced cycle.

It is easy to characterize the colorings κ of a given Temperley–Lieb basis element τ for which the numbers α, β (4.6) are constant.

Lemma 4.2. *Let $\tau, \kappa, \tau, \kappa'$ be principal colorings with j white vertices on the left, for some j , $0 \leq j \leq \lfloor \frac{n}{2} \rfloor$. Then we have $\alpha_{\tau, \kappa} = \alpha_{\tau, \kappa'}$ and $\beta_{\tau, \kappa} = \beta_{\tau, \kappa'}$.*

Proof. Omitted. □

By this lemma, we may write

$$\alpha_{\tau, j} : \alpha_{\tau, \kappa}, \quad \beta_{\tau, j} : \beta_{\tau, \kappa} \quad (4.7)$$

if there exists a principal coloring of τ in which j vertices on the left are white.

Just as $T_n - 2$ is related to 2-coloring, it is related to total nonnegativity of polynomials in the subspace

$$\text{span}_{\mathbb{R}}\{\det x_{I,I} \det x_{I^c,I^c} \mid I \subseteq [n]\} \quad (4.8)$$

of immanants. We define an immanant $\text{Imm}_{\tau} x$ for each $\tau \in \mathcal{B}_n$ in terms of the function

$$\begin{aligned} f_{\tau} : \mathbb{C}^{\mathfrak{S}_n} &\rightarrow \mathbb{R} \\ w &\mapsto \text{coefficient of } \tau \text{ in } \sigma(w), \end{aligned} \quad (4.9)$$

(extended linearly). To economize notation, we will write Imm_{τ} instead of $\text{Imm}_{f_{\tau}}$,

$$\text{Imm}_{\tau} x = \sum_{w \in \mathfrak{S}_n} f_{\tau}(w) x_{1,w_1} \cdots x_{n,w_n}.$$

For example, consider the case $n = 3$ and $\tau = t_1 \in \mathcal{B}_3$. Extracting the coefficients of t_1 in the expressions

$$\begin{aligned} \sigma = e & \quad 1, & \sigma = s_1 & \quad t_1 - 1, & \sigma = s_2 & \quad t_2 - 1, \\ \sigma = s_1 s_2 & \quad t_1 - 1 \quad t_2 - 1 & \quad t_1 t_2 - t_1 - t_2 & \quad 1, \\ \sigma = s_2 s_1 & \quad t_2 - 1 \quad t_1 - 1 & \quad t_2 t_1 - t_1 - t_2 & \quad 1, \\ \sigma = s_1 s_2 s_1 & \quad t_1 - 1 \quad t_2 - 1 \quad t_1 - 1 & \quad t_1 \quad t_2 - t_1 t_2 - t_2 t_1 - 1, \end{aligned}$$

(where we have used $t_1 t_2 t_1 = t_1$ and $t_1^2 = 2t_1$ to obtain the last expression), we have $f_{t_1}(e) = 0$, $f_{t_1}(s_1) = 1$, $f_{t_1}(s_2) = 0$, $f_{t_1}(s_1 s_2) = -1$, $f_{t_1}(s_2 s_1) = -1$, $f_{t_1}(s_1 s_2 s_1) = 1$, and

$$\text{Imm}_{t_1} x = x_{1,2} x_{2,1} x_{3,3} - x_{1,3} x_{2,1} x_{3,2} - x_{1,2} x_{2,3} x_{3,1} - x_{1,3} x_{2,2} x_{3,1}.$$

Note that in the special case $\tau = 1$, the function f_{τ} maps a permutation w to $-1^{\text{inv } w}$. Thus the determinant is a Temperley-Lieb immanant,

$$\det x = \text{Imm}_1 x.$$

It was shown in [13] that Temperley-Lieb immanants are a basis of the space (4.8), and that they are TNN. Furthermore, they are the extreme rays of the cone of TNN immanants in this space [13, Thm. 10.3].

Proposition 4.3. *Each immanant of the form*

$$\text{Imm}_f x = \sum_{\substack{I, J \subseteq [n] \\ |I| = |J|}} c_{I,J} \det x_{I,I} \det x_{I^c,J^c} \quad (4.10)$$

is a totally nonnegative polynomial if and only if it is equal to a nonnegative linear combination of Temperley-Lieb immanants.

In fact each complementary product of minors is a 0-1 linear combination of Temperley–Lieb immanants [13, Prop. 4.4].

Theorem 4.4. *For $I \subseteq [n]$ we have*

$$\det x_{I,I} \det x_{I^c,I^c} = \sum_{\tau \in \mathcal{B}_n} b_\tau \text{Imm}_\tau x, \quad (4.11)$$

where

$$b_\tau = \begin{cases} 1 & \text{if there is a principal coloring of } \tau \text{ with } \{v_i \mid i \in I\} \text{ white, } \{v_i \mid i \in [n] \setminus I\} \text{ black,} \\ 0 & \text{otherwise.} \end{cases} \quad (4.12)$$

By (3.2) we have that for each two-part partition $\lambda = n - j, j$ of n , the corresponding induced sign character immanant belongs to (4.8). Furthermore, we have the following explicit expansion of these in terms of the Temperley–Lieb immanant basis.

Theorem 4.5. *For $j = 0, \dots, \lfloor \frac{n}{2} \rfloor - 1$, we have*

$$\text{Imm}_{e^{n-j,j}} x = \sum_{\tau \in \mathcal{B}_n} d_{j,\tau} \text{Imm}_\tau x,$$

where $d_{j,\tau}$ is the number of principal colorings of τ having j white vertices on the left. Explicitly, assuming such a coloring exists, this is $2^{d-\alpha-\beta} \alpha^\alpha \beta^\beta$, where d is the number of cycles of $\hat{\tau}$, and $\alpha = \alpha(\tau, j)$, $\beta = \beta(\tau, j)$ are defined as in (4.7).

Proof. Omitted. □

Combining (2.3) and (3.2), we see that monomial immanants $\text{Imm}_{\phi^\mu} x$ indexed by partitions of the form $\mu = 2^c 1^d \vdash n$ belong to the space (4.8) as well. To expand these in the Temperley–Lieb immanant basis, we define for each $\mu = 2^c 1^d \vdash n$ the set P_μ of all $\tau \in \mathcal{B}_n$ such that there exists a principal coloring of τ with $c = d$ white vertices on the left and no principal coloring of τ with $c = d - 1$ white vertices on the left.

Theorem 4.6. *For $\mu = 2^c 1^d \vdash n$, we have that $\text{Imm}_{\phi^\mu} x$ is a totally nonnegative polynomial. In particular we have*

$$\text{Imm}_{\phi^\mu} x = \sum_{\tau \in P_\mu} b_{\mu,\tau} \text{Imm}_\tau x,$$

where $b_{\mu,\tau} = 2^{\#\text{cycles of } \hat{\tau} \text{ of cardinality } 0 \pmod{4}}$.

Proof. Omitted. □

5 Main results

Fischer's inequalities (1.2) naturally lead to the questions of how products

$$\det A_{I,I} \det A_{I^c,I^c} \quad (5.1)$$

of complementary pairs of minors compare to one another, and of whether a greater cardinality difference $|I^c| - |I|$ tends to make the product (5.1) greater or smaller. This second question led Barrett and Johnson [1] to consider the average value of such products when cardinalities are fixed,

$$\frac{1}{n} \sum_{\substack{k \\ I \subseteq n \\ |I|=k}} \det A_{I,I} \det A_{I^c,I^c} . \quad (5.2)$$

They found that for PSD matrices, a smaller cardinality difference makes the average product of minors greater [1, Thm. 1]. We give two proofs that the same is true for TNN matrices.

Theorem 5.1. *For all TNN matrices A and for $k = 0, \dots, \lfloor \frac{n}{2} \rfloor - 1$, we have*

$$\frac{1}{n} \sum_{\substack{k \\ |I|=k}} \det A_{I,I} \det A_{I^c,I^c} \leq \frac{1}{n} \sum_{\substack{k=1 \\ |I|=k-1}} \det A_{I,I} \det A_{I^c,I^c} . \quad (5.3)$$

First proof (idea). By (3.2) it is equivalent to show that the polynomial

$$\frac{\text{Imm}_{\epsilon^{n-k-1,k-1}} x}{\binom{n}{k-1}} - \frac{\text{Imm}_{\epsilon^{n-k,k}} x}{\binom{n}{k}} \quad (5.4)$$

is totally nonnegative. By Theorem 4.5, this difference belongs to the span of the Temperley-Lieb immanants. Multiplying by $n! / (k! (n-k-1)!)$ we obtain

$$(k-1) \text{Imm}_{\epsilon^{n-k-1,k-1}} x - (n-k) \text{Imm}_{\epsilon^{n-k,k}} x = \sum_{\tau \in \mathcal{B}_n} c_\tau \text{Imm}_\tau x , \quad (5.5)$$

where $c_\tau = (k-1) d_{k-1,\tau} - (n-k) d_{k,\tau}$, and $d_{k-1,\tau}, d_{k,\tau}$ are defined in terms of proper colorings of τ as in Theorem 4.5. Straightforward computations show that $c_\tau \geq 0$. $\square$

Second proof (idea). Again we multiply the polynomial (5.4) by $n! / (k! (n-k-1)!)$. Expanding in the monomial immanant basis of the trace immanant space, we have

$$(n-k) \text{Imm}_{\epsilon^{k,n-k}} x - (k-1) \text{Imm}_{\epsilon^{k-1,n-k-1}} x = \sum_{\mu \vdash n} c_\mu \text{Imm}_\mu x \quad (5.6)$$

where the integers $\{c_\mu \mid \mu \vdash n\}$ satisfy $(n-k) e_{k,n-k} - (k-1) e_{k-1,n-k-1} = \sum_{\mu \vdash n} c_\mu m_\mu$. The special case of (2.3) for two-part partitions, implies that each partition μ appearing with nonzero coefficient in (5.6) has the form $\mu = 2^a 1^{n-2a}$. Straightforward computations show that we have $c_\mu \geq 0$, and thus Theorem 4.6 completes the proof. $\square$

Now we may state and prove our main theorem.

Theorem 5.2. Fix $n > 0$ and partitions $\lambda = \lambda_1, \dots, \lambda_r \vdash n$, $\mu = \mu_1, \dots, \mu_s \vdash n$ with $\lambda \preceq \mu$. For all TNN matrices we have

$$\lambda_1! \cdots \lambda_r! \sum_{I_1, \dots, I_r} \det A_{I_1, I_1} \cdots \det A_{I_r, I_r} \geq \mu_1! \cdots \mu_s! \sum_{J_1, \dots, J_s} \det A_{J_1, J_1} \cdots \det A_{J_s, J_s}, \quad (5.7)$$

Proof. By (2.2) it suffices to consider λ, μ having equal parts except $\mu_i = \lambda_i - 1$ and $\mu_j = \lambda_j + 1$ for some $i < j$. (Thus we may assume $s = r$ and will allow $\mu_s = 0$.) Let $\nu = \nu_1, \dots, \nu_{r-2}$ be the partition of $n - \lambda_i - \lambda_j$ consisting of all other parts.

As in the proofs of Theorem 5.1, we observe that each sum of products of minors is an induced sign character immanant (3.2). Thus the inequality (5.7) is equivalent to the total nonnegativity of the polynomial

$$\lambda_1! \cdots \lambda_r! \text{Imm}_{\epsilon^\lambda} x - \mu_1! \cdots \mu_r! \text{Imm}_{\epsilon^\mu} x. \quad (5.8)$$

Dividing this by $\nu_1! \cdots \nu_{r-2}! \lambda_i! \mu_j!$ and collecting terms, we obtain

$$\begin{aligned} & \lambda_j \sum_{\substack{J \subseteq n \\ |J| = |\nu|}} \text{Imm}_{\epsilon^\nu} x_{J,J} \text{Imm}_{\epsilon^{\lambda_i, \lambda_j}} x_{J^c, J^c} - \mu_i \sum_{\substack{J \subseteq n \\ |J| = |\nu|}} \text{Imm}_{\epsilon^\nu} x_{J,J} \text{Imm}_{\epsilon^{\lambda_i - 1, \lambda_j - 1}} x_{J^c, J^c} \\ & \sum_{\substack{J \subseteq n \\ |J| = |\nu|}} \text{Imm}_{\epsilon^\nu} x_{J,J} \left(\lambda_j \text{Imm}_{\epsilon^{\lambda_i, \lambda_j}} x_{J^c, J^c} - \mu_i \text{Imm}_{\epsilon^{\lambda_i - 1, \lambda_j - 1}} x_{J^c, J^c} \right). \end{aligned}$$

By Theorem 5.1, or more precisely (5.5), this polynomial is TNN and so is (5.8). $\square$

It would be interesting to extend the Barrett–Johnson inequalities (1.3) to all HPSP matrices, and to state a permanental analog as originally suggested in [1]. Using (3.2) we may state these problems as follows.

Problem 5.3. Characterize the pairs λ, μ of partitions of n for which we have

1. $\lambda_1! \cdots \lambda_r! \text{Imm}_{\epsilon^\lambda} A \leq \mu_1! \cdots \mu_r! \text{Imm}_{\epsilon^\mu} A$ for all A HPSP,
2. $\lambda_1! \cdots \lambda_r! \text{Imm}_{\eta^\lambda} A \geq \mu_1! \cdots \mu_r! \text{Imm}_{\eta^\mu} A$ for all A HPSP or PSD,
3. $\lambda_1! \cdots \lambda_r! \text{Imm}_{\eta^\lambda} A \geq \mu_1! \cdots \mu_r! \text{Imm}_{\eta^\mu} A$ for all A TNN.

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From Kreweras to Gessel: A walk through patterns in the quarter plane

Andrei Asinowski^{*1}, Cyril Banderier^{†2}, and Sarah J. Selkirk^{‡1}

¹*Institut für Mathematik, Alpen-Adria-Universität Klagenfurt, Klagenfurt, Austria*

²*Laboratoire d'Informatique de Paris Nord, Université Sorbonne Paris Nord, France*

Abstract. We initiate a study of pattern avoidance in quarter-plane lattice walks. First we demonstrate surprising links between Kreweras excursions (avoiding a pattern of length 2) and some famous lattice walk models, such as Gessel, Gouyou-Beauchamps, and Pólya excursions. Next we explore the nature (algebraic, hypergeometric, D-finite) of the corresponding generating functions. In particular, we show that pattern avoidance does not necessarily preserve algebraicity or D-finiteness.

Keywords: lattice walks, pattern avoidance, bijections, generating functions

1 Introduction

In this paper we study pattern avoidance in quarter-plane lattice walks. A *lattice walk* is a word w over a *stepset* — an alphabet $\mathcal{S}$, whose elements (*steps*) are interpreted as vectors in the plane, and then w is visualized as these vectors concatenated to each other to form a polygonal line. A pattern in this context is a fixed word p , and we deal with enumeration of walks that avoid p (that is, p is not a consecutive subsequence of w).

The case of *directed walks* (the models where all the steps have a positive x -coordinate) which avoid a given pattern was studied in [1]. Therein, the vectorial kernel method was developed in order to obtain an explicit expression of the corresponding generating functions, which are systematically algebraic.

In this work, we initiate the study of pattern avoidance for walks in the quarter plane $\mathbb{N}^2$. We use the term *excursion* to indicate a walk in $\mathbb{N}^2$ that starts and ends at $(0,0)$, and the term *meander* to indicate a walk in $\mathbb{N}^2$ that starts at $(0,0)$ and ends anywhere in the quarter plane. As a first step, in this article we focus on some combinatorial surprises which then occur. Firstly, we show that some models of pattern-avoiding *Kreweras excursions* are in bijection with some famous walk models which do not involve any pattern avoidance. Secondly, we comment on the nature of the corresponding generating functions and prove some cases of (non)-algebraicity or D-finiteness.

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2 Historical background: Kreweras, Gessel, Pólya, and Gouyou-Beauchamps models

We briefly survey the history of five noteworthy models which will play a rôle in the next section for our study of patterns in lattice walks. Each model is characterized by its allowed steps; for example, the stepset of Kreweras' model is $\mathcal{S} = \{\downarrow, \leftarrow, \nearrow\}$ (see Figure 1).

				
Kreweras	Gessel	Pólya	Gouyou-Beauchamps	Diagonal

Figure 1: The stepsets of five important models of lattice walks.

2.1 Kreweras walks

In his PhD thesis [27], Kreweras tackled the question of the enumeration of variants of solid partitions (initially studied by MacMahon in [29]), and of other combinatorial structures (like Young tableaux) related to natural posets. He applied this to the *m-candidate ballot problem*: the number of ways that n people can vote for candidates $C_1, \dots, C_m$ such that C_1 remains in the lead (or tied) with respect to the other candidates during the ballot process. For 3 candidates, any such process corresponds to a Kreweras meander. If C_1 remained in the lead but ties with all other candidates at the end, this corresponds to a Kreweras excursion of length $3n$. They are counted (applying [27, Section 3.2]) by

$$e_{3n} = \frac{4^n}{2n-1} \frac{1}{n-1} \binom{3n}{n}. \quad (2.1)$$

This is the sequence [A006335](#) in the On-Line Encyclopedia of Integer Sequences (OEIS). Kreweras proved (2.1) by a guess-and-prove approach, which was then simplified in collaboration with Niederhausen using hypergeometric identities [28, 31]. Gessel later proved (2.1) in [18] with a probabilistic approach, rephrasing the problem in terms of walks in $\mathbb{N}^2$. For other stepsets deeper mathematical tools are required. In fact, these more general walk models correspond to the evolution of two queues in parallel, and probabilists were interested in the stationary distribution of the corresponding infinite Markov chain. To this aim, they solved these models with the machinery of boundary-value problems and Riemann surfaces (see e.g. [14]). For Kreweras walks, Flatto and Hahn [17] proved that the generating function of this stationary distribution is algebraic. This method was revisited combinatorially by Bousquet-Mélou in [8], and later with Mishna in [11], where they performed a classification of almost all models of walks with small steps as algebraic, D-finite, etc. (the classification was completed since).

The algebraicity of Kreweras walks can also be established via a bijection with planar maps (see Bernardi [4]), but many combinatorialists still hope for an even simpler proof, e.g. based on a link with the inherent algebraicity of tree-like structures (see [2, 9, 23]).

2.2 Gessel walks

In 2001, Ira Gessel conjectured that, for the stepset $\{\rightarrow, \leftarrow, \nearrow, \swarrow\}$, the number e_n of excursions of length n is hypergeometric (and given by the OEIS sequence [A135404](#)):

$$e_{2n} = \frac{4^{2n}}{2^n} \frac{1/2}{5/3} \frac{n}{n}, \text{ where } a_m : a_{m-1} \cdots a_1. \quad (2.2)$$

This startling conjecture motivated the name “Gessel walks” since attached to this model.

The hypergeometricity of Gessel excursions was first proven by Kauers, Koutschan, and Zeilberger in [25] using computer algebra techniques. Later, Bostan and Kauers [6] proved that the trivariate generating function (length, final location) is algebraic. These two proofs are nice examples of the “guess and prove” art via computer algebra. Puzzled by a discrepancy between the simplicity of the excursion formula (2.2) and the enormous size of algebraic equations required for its proof, several authors looked for a more “human” approach. Bostan, Kurkova, and Raschel [7] found such a proof using complex analysis, and Bousquet-Mélou [10] found another proof using generating function manipulations. It still remains a challenge to find an elementary combinatorial proof.

2.3 Pólya walks and diagonal walks

Pólya’s drunkard problem asks for the return probability of a random walk on the $\mathbb{Z}^2$ lattice (with equiprobable steps $\downarrow, \uparrow, \rightarrow, \leftarrow$). This probability tends to 1 (as proven by Pólya [32]). A natural question is which conditions on walks in $\mathbb{N}^2$ imply this property. This is answered in [15], where conditions for (null-)recurrence are given.

Pólya walks in $\mathbb{N}^2$ were also studied by Guy, Krattenthaler, and Sagan [21], who gave bijective proofs for the number of walks between any two points. In particular, the number of Pólya excursions in $\mathbb{N}^2$ of length $2n$ is given by the OEIS sequence [A005568](#):

$$e_{2n} = C_n C_{n-1}, \quad (2.3)$$

where C_n is the n -th Catalan number. These walks have connections to planar maps and shuffles of parenthesis systems, as studied by Cori, Dulucq, and Viennot in [12]. The diagonal walks (see [Figure 1](#)) are even simpler to enumerate: $e_{2n} = C_n^2$ (OEIS [A001246](#)).

2.4 Gouyou-Beauchamps walks

In [19] Gouyou-Beauchamps studied Pólya walks in the quarter plane which stay weakly below the line $y = x$, leading to what is now known as Gouyou-Beauchamps walks. Gouyou-Beauchamps excursions in $\mathbb{N}^2$ are counted by the OEIS sequence [A005700](#):

$$e_{2n} = C_n C_{n-2} - C_{n-1}^2. \quad (2.4)$$

Gouyou-Beauchamps walks of length $2n$ ending on the x -axis are in bijection with Pólya excursions of length $2n$, and with Young tableaux of size $2n$ and height ≤ 4 [20].

3 Pattern avoidance in Kreweras excursions: bijections

In this section, we consider avoidance of patterns of length 2 in Kreweras walks. Recall that we deal with consecutive patterns, and note that several occurrences of a pattern can overlap (e.g., the word ABBBABABB has precisely three occurrences of the pattern BB).

Due to the symmetry of the Kreweras stepset with respect to the diagonal $y = x$, there are just five patterns of length 2 that yield non-equivalent pattern-avoiding walk models. We show that the corresponding pattern-avoiding *excursion* models exhibit surprising links to the well-known models mentioned in Section 2: Gessel, Gouyou-Beauchamps, Pólya, and diagonal excursions. These pattern-avoiding Kreweras excursion models are listed in Table 1, along with their respective enumerating sequences, their OEIS entries, and equinumerous quarter-plane excursion models without any forbidden patterns.

	Pattern p	# {Kreweras excursions of length $3n$ avoiding p }	OEIS	In bijection with	Proven in
1		1, 2, 11, 85, 782, 8004, ...	A135404	Gessel excursions	Thm. 4
2		1, 2, 11, 85, 782, 8004, ...	A135404	Gessel excursions	Prop. 5
3		1, 1, 5, 37, 332, 3343, ...	None	Gessel excursions ending with $\downarrow$	Cor. 8
4		1, 2, 10, 70, 588, 5544, ...	A005568 ¹	Pólya excursions	Prop. 9
5		1, 1, 4, 25, 196, 1764, ...	A001246 ¹	Diagonal excursions	Thm. 10

Table 1: Summary of results concerning quarter-plane models in bijection with pattern-avoiding Kreweras excursions. Models 1, 2, and 3 are algebraic, while models 4 and 5 are D-finite, but not algebraic (see Section 4).

Most notably, the first two entries of Table 1 relate pattern-avoiding Kreweras excursions to Gessel excursions. In particular, one deduces that the patterns  and  are equidistributed among Kreweras excursions. In the following theorem, we prove a stronger result about the *joint* statistics of these two patterns.

Theorem 1. *The number of Kreweras excursions of length $3n$ with k occurrences of  and ℓ occurrences of  is equal to the number of Kreweras excursions of length $3n$ with ℓ occurrences of  and k occurrences of .*

Proof. We provide an autobijection on the set of Kreweras excursions of length $3n$ that switches the occurrences of the patterns  and . In this bijection, the patterns can be independently considered despite the possible overlap of a $\leftarrow$ step in both patterns.

¹It is footnoteworthy that [A005568](#) (resp. [A001246](#)) also enumerates Gouyou-Beauchamps walks of even (resp. odd) length ending on the y -axis, and Young tableaux of height ≤ 4 of even (resp. odd) size; see [20].

Let w be a Kreweras excursion with k occurrences of the pattern $\leftarrow\rightarrow$ and ℓ occurrences of the pattern $\nearrow$. We mark each $\leftarrow$ which is followed by a $\leftarrow$, and separately each $\nearrow$ which is preceded by a $\leftarrow$. Note that any $\leftarrow$ step followed by a $\nearrow$ step and preceded by a $\leftarrow$ step (that is, any $\leftarrow$ step being at the overlap of the two patterns) is never marked.

We define the *index* of a step to be its ordinal number, from 1 to n , among the steps of the same kind, in the order as they occur in the excursion.

1. Let $a_1, a_2, \dots, a_k$ be the indices of the *marked* $\leftarrow$ steps, in the order as they occur in w . For each $i = 1, 2, \dots, k$, in this order, we remove the $\leftarrow$ step with the index a_i , and insert it immediately before the $\nearrow$ step with the index $a_i + 1$. This transformation yields a valid excursion, since the section of the walk up until the $a_i + 1$ -th $\nearrow$ step can have up to a_i $\leftarrow$ steps. The newly obtained excursion is denoted by w' .
2. Next, let $b_1, b_2, \dots, b_\ell$ be the indices of the *marked* $\nearrow$ steps, in the order as they occur in w' . (Note that we only consider $\nearrow$ steps marked before Step 1.) For each $j = \ell, \ell - 1, \dots, 1$, in this order, we remove the $\leftarrow$ step that occurs immediately before a *marked* $\nearrow$ step with index b_j , and insert it immediately before the $\leftarrow$ step with index b_j . As above, one can routinely show that the resulting walk is an excursion; we denote it by w'' .

Refer to Figure 2 for an example, where the marked $\leftarrow$ steps are coloured red, and the marked $\nearrow$ steps are coloured blue. It is easily seen directly that $w \mapsto w''$ is an involution. Therefore, it is a bijection.

Finally, we show that w'' has ℓ occurrences of $\leftarrow\rightarrow$ and k occurrences of $\nearrow$. At Step 1, k occurrences of $\leftarrow\rightarrow$ yield k distinct “new” occurrences of $\nearrow$ in w' . The only way in which the patterns can overlap in w' , is a string $\leftarrow\rightarrow$ where the first $\leftarrow$ was adjacent to the (marked) $\nearrow$ in w , and the second $\leftarrow$ is an inserted one. Then, at Step 2, ℓ “old” occurrences of $\nearrow$ yield distinct occurrences $\leftarrow\rightarrow$ in w'' . In particular, in each string $\leftarrow\rightarrow$ the step $\nearrow$ is marked, therefore it yields both $\leftarrow\rightarrow$ and $\nearrow$ in w'' . All in all, it follows that $w \mapsto w''$ swaps the numbers of occurrences of $\leftarrow\rightarrow$ and $\nearrow$. $\square$

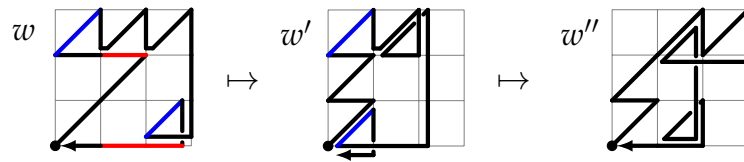


Figure 2: An example of the mapping $w \mapsto w''$ in the proof of Theorem 1.

For $k = 0$ we immediately obtain the following result.

Corollary 2. *There is a bijection between $\leftarrow\rightarrow$ -avoiding Kreweras excursions of length $3n$ and $\nearrow$ -avoiding Kreweras excursions of length $3n$.*

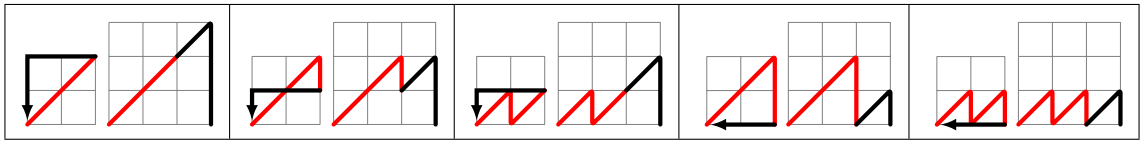
The following table shows the number of Kreweras excursions of length $3n$ with precisely k occurrences of the pattern $\leftarrow\rightarrow$ (or, due to [Theorem 1](#), of the pattern $\nearrow$).

	k	0	1	2	3
n	1	2			
	2	11	5		
	3	85	93	14	
	4	782	1432	560	42

Of course, the row sums in this table are Kreweras numbers. As noted in [Table 1](#) (the first two entries), the first column, corresponding to avoidance, contains Gessel numbers: this will be proven in [Theorem 4](#). On the other hand, the maximum possible number of occurrences of $\leftarrow\rightarrow$ is $n - 1$, and the Kreweras excursions that have that many occurrences of this pattern make a nice cameo of Catalan numbers, as we show now.

Proposition 3. *The number of Kreweras excursions of length $3n$ with $n - 1$ occurrences of the pattern $\leftarrow\rightarrow$ (or, equivalently, $\nearrow$) is the $n - 1$ -th Catalan number.*

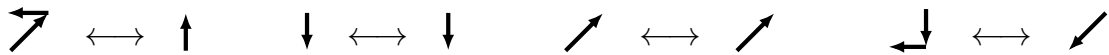
Proof. The only way in which such an excursion can have $n - 1$ occurrences of the pattern $\leftarrow\rightarrow$ is when all $\leftarrow$ steps occur consecutively, and this must happen after all of the $\nearrow$ steps. Therefore the walk ends with a consecutive sequence of n $\leftarrow$ steps followed by a (possibly empty) consecutive sequence of $\downarrow$ steps to return to $(0,0)$. Since it is possible to reconstruct the end of the excursion by knowing where the first $\leftarrow$ step is, we remove this end-section of the excursion and replace it with a $\nearrow$ step followed by $\downarrow$ steps to reach the x -axis. The obtained walk then consists of $n - 1$ steps of type $\nearrow$ and $n - 1$ steps of type $\downarrow$, which is a slanted Dyck path of length $2n - 2$. This correspondence is easily seen to be a bijection, and it is demonstrated in the following figure. □



We now prove the links between constrained Kreweras excursions and Gessel excursions.

Theorem 4 ([Table 1](#), Entry 1). *There is a bijection between $\leftarrow\rightarrow$ -avoiding Kreweras excursions of length $3n$ and Gessel excursions of length $2n$.*

Proof. Consider the following correspondence between steps of $\leftarrow\rightarrow$ -avoiding Kreweras excursions (left) and Gessel excursions (right):



Note that in $\leftarrow\rightarrow$ -avoiding Kreweras excursions, a step immediately before a $\leftarrow$ step can be either $\nearrow$ or $\downarrow$. Therefore, “short-cutting” $\nearrow$ and $\downarrow$ steps followed by $\leftarrow$, and leaving all other steps unchanged, directly yields the desired bijection. □

Proposition 5 (Table 1, Entry 2). *There is a bijection between $\nearrow$ -avoiding Kreweras excursions of length $3n$ and Gessel excursions of length $2n$.*

Proof. This follows directly from Theorems 1 and 4. $\square$

Next we note that in the bijection from Theorem 4, $\downarrow$ steps (not followed by a $\leftarrow$ step) are preserved. Hence we have the following generalization.

Proposition 6. *There is a bijection between Gessel excursions of length $2n$ with at least m final $\downarrow$ steps and $\longleftrightarrow$ -avoiding Kreweras excursions of length $3n$ which end with at least m $\downarrow$ steps.*

The following theorem will be one key ingredient for proving some of the bijective links of Table 1.

Theorem 7. *There is a bijection between $\nearrow$ -avoiding Kreweras excursions of length $3n$ and $\longleftrightarrow$ -avoiding Kreweras excursions of length $3n$ whose last step is $\downarrow$.*

Proof. Let w be a $\nearrow$ -avoiding Kreweras excursion of length $3n$. Consider the n -tuple $\alpha_1, \dots, \alpha_n$, where α_i is the number of $\leftarrow$ steps that immediately follow the i -th $\downarrow$ step. Note that we have $\alpha_1 + \dots + \alpha_n = n$. Transform $\alpha_1, \dots, \alpha_n$ into a $\{0, 1\}$ -sequence of length $2n - 1$, where the gap between entries contributes 0, and the entry $\alpha_i \geq 0$ contributes α_i 1s. (This is a classical bijection, popularized by Feller [16, Chapter II.5], often called the “balls and bars” bijection.) Now erase from w all the $\leftarrow$ steps, thus obtaining a sequence $\tilde{w}$ of n $\nearrow$ steps and n $\downarrow$ steps in the same order as they were in w , and insert $\leftarrow$ steps into $\tilde{w}$ after precisely those positions where we have 1 in the $\{0, 1\}$ -sequence constructed above. Refer to Figure 3 for an illustration, where the $\leftarrow$ steps are coloured blue. The new Kreweras walk w' avoids $\longleftrightarrow$ and has last step $\downarrow$, and it is routine to prove that it is an excursion, and that this mapping is a bijection. $\square$

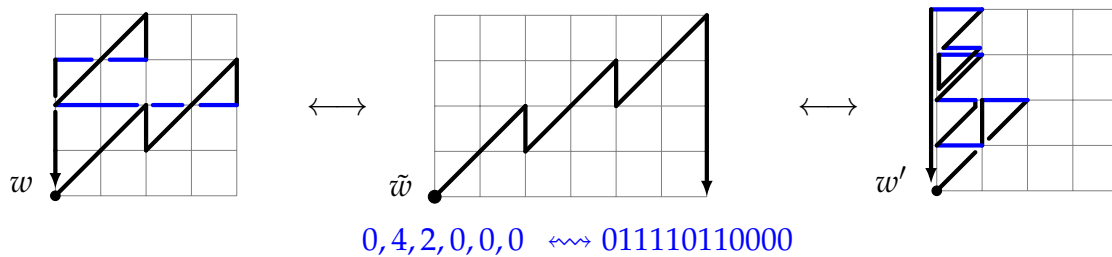


Figure 3: An example of the bijection $w \leftrightarrow w'$ in the proof of Theorem 7.

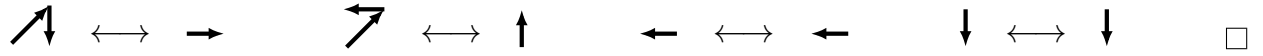
Now, Proposition 6 and Theorem 7 yield directly the following correspondence.

Corollary 8 (Table 1, Entry 3). *There is a bijection between $\nearrow$ -avoiding Kreweras excursions of length $3n$ and Gessel walks which end with at least one $\downarrow$ step.*

We now prove the link with Pólya excursions.

Proposition 9 (Table 1, Entry 4). *There is a bijection between $\nearrow$ -avoiding Kreweras excursions of length $3n$ and Pólya excursions of length $2n$.*

Proof. Similarly to the proof of Theorem 4, the correspondence of steps of $\nearrow$ -avoiding Kreweras (left) and Pólya (right) excursions is given by the following rules.



Finally, we prove the link between $\lrcorner$ -avoiding Kreweras excursions and diagonal excursions.

Theorem 10 (Table 1, Entry 5). *There is a bijection between $\lrcorner$ -avoiding Kreweras excursions of length $3n$ and diagonal excursions of length $2n$.*

Proof. Any Kreweras excursion which avoids the pattern $\lrcorner$ is uniquely determined by the positions of the $\nearrow$ steps in the excursion: All steps in between $\nearrow$ steps are a sequence of $\leftarrow$ steps followed by a sequence of $\downarrow$ steps. Therefore we can uniquely decompose the excursions into pairs of Dyck paths D_1, D_2 , where D_1 consists of all $\nearrow$ and $\downarrow$ steps (in order) in the excursion, and D_2 all $\nwarrow$ and $\leftarrow$ steps in the excursion. The original Kreweras excursion can be obtained from pairs by placing all $\leftarrow$ and $\downarrow$ steps between a given pair of $\nearrow$ steps with $\leftarrow$ steps first, followed by $\downarrow$ steps.

From the pair D_1, D_2 of two Dyck paths of length $2n$, we consider the i -th step in $D_1 = d_{1,1}d_{1,2}\dots d_{1,2n}$ and $D_2 = d_{2,1}d_{2,2}\dots d_{2,2n}$ simultaneously, $1 \leq i \leq 2n$, and form the i -th step g_i of the diagonal excursion as follows:

- If $d_{1,i} = \nearrow$ and $d_{2,i} = \nwarrow$, then $g_i = \nearrow$.
- If $d_{1,i} = \nwarrow$ and $d_{2,i} = \leftarrow$, then $g_i = \nwarrow$.
- If $d_{1,i} = \downarrow$ and $d_{2,i} = \nearrow$, then $g_i = \searrow$.
- If $d_{1,i} = \downarrow$ and $d_{2,i} = \leftarrow$, then $g_i = \swarrow$.

Now, diagonal excursions are decomposed into pairs of Dyck paths by considering the projection of the excursion on the x - and y -axis. These Dyck paths correspond to D_1, D_2 as described above. See Figure 4 for an example. $\square$

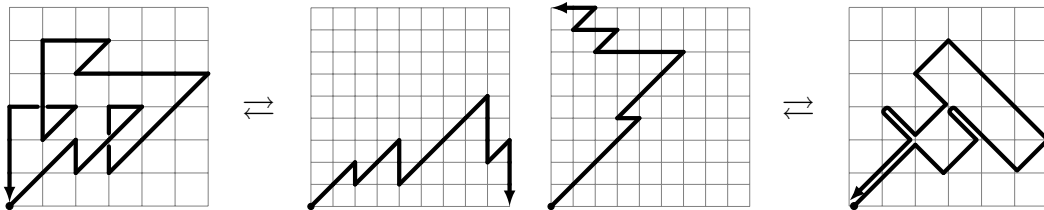


Figure 4: An example for the bijection from Theorem 10.

4 Pattern avoidance and nature of generating functions

In this section, we tackle the question of the (non)-algebraicity of pattern-avoiding walks.

To this aim, it is useful to first recall the notion of directed walks. A walk is called *directed* if all its steps x, y have $x \geq 0$. Since such walks can be encoded by context-free grammars, their generating functions are systematically algebraic; see e.g. [3, 13].

What happens for more general stepsets, when one additionally forbids a pattern, and constrains the domain? This is summarized in the following theorem.

Theorem 11 ((Non)-algebraicity of generating functions). *For any fixed stepset, let $F(t, x, y, v)$ be the generating function of walks constrained in a domain, where t, x, y , and v encode respectively the length of the walk, its final x and y coordinates, and the number of occurrences of a given pattern p . The nature of this generating function satisfies:*

- a) *For walks in $\mathbb{Z}^2$ avoiding a pattern, F is rational.*
- b) *For walks in $\mathbb{N} \times \mathbb{Z}$ avoiding a pattern, F is algebraic.*
- c) *For directed walks in $\mathbb{N}^2$ avoiding a pattern, F is algebraic.*
- d) *For non-directed walks in $\mathbb{N}^2$ avoiding a pattern, F is not necessarily algebraic.*

Proof. a) The walks in $\mathbb{Z}^2$ avoiding a pattern p with stepset $\mathcal{S}$ can be encoded by the complement of a regular expression, namely $\{\mathcal{S}^* p \mathcal{S}^*\}^c$, and thus have a rational generating function.

b) These pattern-avoiding walks in $\mathbb{N} \times \mathbb{Z}$ are encodable by a pushdown automaton with a single stack (encoding the distance to the y -axis), and thus by a context-free grammar [3]; therefore, they have an algebraic generating function.

c) If the walk is directed, then, by the vectorial kernel method (as developed in [1]), $F(t, x, y, v)$ is algebraic. In particular, setting $v = 0$, the generating function of directed walks avoiding a given pattern is algebraic.

d) For non-directed walks, one can have $F(t, x, y, 1)$ algebraic, but $F(t, x, y, 0)$ not algebraic. It is e.g. the case for Entry 4 of Table 1. Indeed, these walks are counted by $C_n C_{n-1} \sim 4 \frac{16^n}{\pi n^3}$. Such asymptotics involving an n^{-3} factor are not compatible with the rather constrained asymptotics of algebraic function coefficients [2, 24]. $\square$

It is natural to ask to what extent forbidding a pattern impacts the nature of the generating function. Figure 5 illustrates the drastic impact it can have.

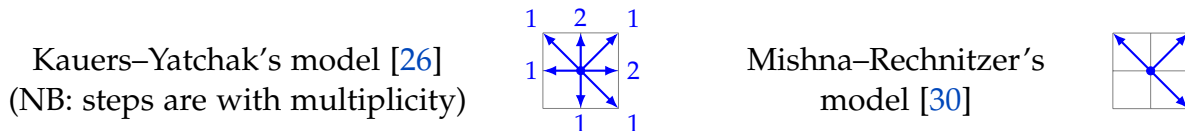


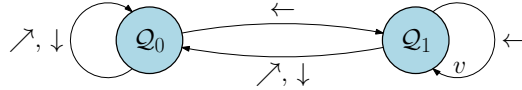
Figure 5: Forbidding some steps in the left (algebraic) model leads to the right (hyper-transcendental) model (as follows by the work of Singer and Hardouin [22]).

The models of Kreweras excursions of Section 3 are thus much more structured than what could be expected, as proven in the following proposition.

Proposition 12. *For each pattern p of length 2, the generating function for p -avoiding Kreweras excursions is D-finite (and not algebraic for the patterns $\nearrow$, $\nwarrow$, and $\downarrow$).*

Proof. Due to the properties of the models discussed in Section 2, the only case which is left open is Entry 3 in Table 1. Yet, its generating function is just the extraction of the coefficient x^0y^1 in a trivariate algebraic generating function (the one of Gessel walks). By classical closure properties [33] this gives a D-finite function, and it can be checked that it satisfies the same differential equation as an algebraic function of degree 4. $\square$

The classification for *meanders* is still open; let us explain what the obstacles are if one tries to extend the kernel method approaches. Kreweras walks with each occurrence of $p = a_1a_2$ is marked by v can be generated by an automaton with two states: The walk is in state Q_1 if its last step is a_1 , and in state Q_0 otherwise. For example, for $p = \leftarrow\rightarrow$:



In this example, the corresponding transition matrix is $A = \begin{pmatrix} xy & y^{-1} & x^{-1} \\ xy & y^{-1} & vx^{-1} \end{pmatrix}$. Now, let $Q_i(t, x, y, v)$ be the generating function of walks starting in state Q_i (for short, we denote it hereafter by $Q_i(x, y)$). One thus has the following matrix equation

$$Q_0(x, y), Q_1(x, y) = \begin{pmatrix} 1, 0 \\ t\{x^{\geq 0}y^{\geq 0}\} \end{pmatrix} Q_0(x, y), Q_1(x, y) A,$$

which, in turn, is equivalent to the following system of two equations:

$$\begin{aligned} (1 - txy - \bar{y})Q_0(x, y) - txy - \bar{y}Q_1(x, y) &= 1 - t\bar{y}(Q_0(x, 0) - Q_1(x, 0)), \\ -t\bar{x}Q_0(x, y) - 1 - tv\bar{x}Q_1(x, y) &= -t\bar{x}(Q_0(0, y) - vQ_1(0, y)). \end{aligned}$$

More generally, a pattern of length m leads similarly to a system of m equations, with $3m$ unknowns and m^2 “kernels” (the coefficients in front of the $Q_i(x, y)$ ’s). In most cases, trying to obtain new equations via variants of the algebraic and vectorial kernel methods (see [1, 8]) does not solve this system, except in a few noteworthy cases which possess more symmetries. This allows us to show, for example, that Pólya walks avoiding the pattern $\leftarrow\rightarrow$ are counted by a pullback of a hypergeometric function (similarly to [5]). We plan to detail these aspects in our forthcoming article.

This concludes our investigation of patterns for lattice walks in $\mathbb{N}^2$: we focused here on patterns of length 2 leading to nice combinatorial features, so many natural extensions are possible. For example, it is possible to tackle longer or more complex patterns (e.g. some regular expressions), to attach weights or multiplicities to the steps, to include border interactions, to add parameters which mark how often a walk visits a given set of sites, to consider other domains than $\mathbb{N}^2$, to follow statistics such as the height or the (signed) area, to establish the corresponding asymptotics and limit laws... The study of lattice path generating functions still has good days ahead!

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Refined canonical stable Grothendieck polynomials and their duals

Byung-Hak Hwang^{*1}, Jihyeug Jang^{†1}, Jang Soo Kim^{‡1},
Minho Song^{§1}, and U-Keun Song^{¶1}

¹*Department of Mathematics, Sungkyunkwan University, Suwon, South Korea*

Abstract. In this extended abstract we introduce refined canonical stable Grothendieck polynomials and their duals with two infinite sequences of parameters. These polynomials unify several generalizations of Grothendieck polynomials including canonical stable Grothendieck polynomials due to Yeliussizov, refined Grothendieck polynomials due to Chan and Pflueger, and refined dual Grothendieck polynomials due to Galashin, Liu, and Grinberg. We give Jacobi–Trudi-like formulas, combinatorial models, Schur expansions, Schur positivity, and dualities of these polynomials. We also consider flagged versions of Grothendieck polynomials and their duals with skew shapes.

Keywords: Grothendieck polynomials, symmetric functions, Jacobi–Trudi formulas, Schur positivity

1 Introduction

1.1 History of Grothendieck polynomials

Grothendieck polynomials were introduced by Lascoux and Schützenberger [8] for studying the Grothendieck ring of vector bundles on a flag variety. When focusing on the Grothendieck ring of vector bundles on Grassmannians rather than that of flag varieties, the stable Grothendieck polynomials are indexed by partitions. These stable Grothendieck polynomials form a basis for the (connective) K -theory ring of Grassmannians, see [12, Section 2.3] and references therein. In this context, Lenart [9] gave the Schur expansion of $G_\lambda x$ and showed that $\{G_\lambda x : \lambda \text{ is a partition}\}$ is a basis for (a completion of) the space of symmetric functions. Buch [1] showed that the stable

^{*}byunghakhwang@gmail.com.

[†]4242ab@gmail.com. Supported by NRF grants #2022R1A2C101100911.

[‡]jangsookim@skku.edu. Supported by NRF grants #2022R1A2C101100911 and #2016R1A5A1008055.

[§]smh3227@skku.edu. Partially supported by NRF-2019R1I1A1A01058099.

[¶]sukeun319@gmail.com. Supported by NRF grants #2022R1A2C101100911 and #2016R1A5A1008055.

Grothendieck polynomial $G_\lambda \mathbf{x}$ is equal to a generating function for semistandard set-valued tableaux of shape λ .

After Buch's work [1], many generalizations of $G_\lambda \mathbf{x}$ have been studied from various viewpoints. In [7], Lam and Pylyavskyy introduced the dual stable Grothendieck polynomials $g_\lambda \mathbf{x}$, and found a combinatorial interpretation for them in terms of reverse plane partitions. Similar to the Grothendieck polynomial case, the dual Grothendieck polynomials form a basis for the space of symmetric functions.

Yeliussizov [15] introduced the canonical stable Grothendieck polynomials $G_\lambda^{\alpha, \beta} \mathbf{x}$ with two parameters α and β , and defined the dual canonical stable Grothendieck polynomials $g_\lambda^{\alpha, \beta} \mathbf{x}$ by the relation $\langle G_\mu^{-\alpha, -\beta} \mathbf{x}, g_\lambda^{\alpha, \beta} \mathbf{x} \rangle = \delta_{\lambda, \mu}$, where $\langle -, - \rangle$ is the Hall inner product. Yeliussizov [15] showed that these polynomials behave as nicely as Schur functions under the involution ω . Moreover, he found combinatorial interpretations for $G_\lambda^{\alpha, \beta} \mathbf{x}$ and $g_\lambda^{\alpha, \beta} \mathbf{x}$.

There are also generalizations of Grothendieck polynomials and their duals with infinite parameters; see [2, 3, 6, 13]. In this extended abstract, motivated by these previous works, we introduce refined canonical stable Grothendieck polynomials $G_\lambda \mathbf{x}_n; \alpha, \beta$ and their duals $g_\lambda \mathbf{x}_n; \alpha, \beta$ with infinite parameters $\alpha = \alpha_1, \alpha_2, \dots$ and $\beta = \beta_1, \beta_2, \dots$, and furthermore their flagged versions with skew shapes. Our generalization unifies all generalizations of Grothendieck polynomials mentioned above.

1.2 Refined canonical Grothendieck polynomials and their duals

Let $\mathbf{x} = x_1, x_2, \dots, x_n$ and $\mathbf{x}_n = x_1, \dots, x_n$ be sequences of variables, and $\alpha = \alpha_1, \alpha_2, \dots$, $\beta = \beta_1, \beta_2, \dots$ sequences of parameters.

We generalize the canonical stable Grothendieck polynomials as follows. See Section 2 for precise definitions of the notations.

Definition 1.1. For a partition λ with at most n parts, the *refined canonical stable Grothendieck polynomial* $G_\lambda \mathbf{x}_n; \alpha, \beta$ is defined by

$$G_\lambda \mathbf{x}_n; \alpha, \beta = \frac{\det \left(x_j^{\lambda_i - n + i} \frac{1 - \beta_1 x_j \cdots 1 - \beta_{i-1} x_j}{1 - \alpha_1 x_j \cdots 1 - \alpha_{\lambda_i} x_j} \right)_{1 \leq i, j \leq n}}{\prod_{1 \leq i < j \leq n} (x_i - x_j)}. \quad (1.1)$$

Using the notation $\ominus$ we can rewrite (1.1) as follows:

$$G_\lambda \mathbf{x}_n; \alpha, \beta = \frac{\det (h_{\lambda_i - n + i} x_j \ominus A_{\lambda_i} - B_{i-1})_{1 \leq i, j \leq n}}{\prod_{1 \leq i < j \leq n} (x_i - x_j)}, \quad (1.2)$$

where $A_k = \alpha_1 \cdots \alpha_k$ and $B_k = \beta_1 \cdots \beta_k$ for $k \geq 1$, and $A_k = B_k = 0$ for $k \leq 0$.

We define our generalization of dual Grothendieck polynomials via the following bialternant formula.

Definition 1.2. For a partition λ with at most n parts, the *refined dual canonical stable Grothendieck polynomial* $g_\lambda \mathbf{x}_n; \alpha, \beta$ is defined by

$$g_\lambda \mathbf{x}_n; \alpha, \beta = \frac{\det h_{\lambda_i - n + i} x_j - A_{\lambda_i - 1} B_{i-1} \quad 1 \leq i, j \leq n}{\prod_{1 \leq i < j \leq n} x_i - x_j}. \quad (1.3)$$

Our generalizations $G_\lambda \mathbf{x}; \alpha, \beta$ and $g_\lambda \mathbf{x}; \alpha, \beta$ generalize several well-studied variations of Grothendieck polynomials as follows. Here $\mathbf{0} = (0, 0, \dots)$, $\mathbf{1} = (1, 1, \dots)$, $\alpha_0 = (\alpha, \alpha, \dots)$ and $\beta_0 = (\beta, \beta, \dots)$.

Variations of G or g	introduced in	how to specialize
$G_{v/\lambda} \mathbf{x}$	Buch [1]	$G_{v/\lambda} \mathbf{x}; \mathbf{0}, \mathbf{1}$
$g_{\lambda/\mu} \mathbf{x}$	Lam–Pylyavskyy [7]	$g_{\lambda/\mu} \mathbf{x}; \mathbf{0}, \mathbf{1}$
$G_\lambda^{\alpha, \beta} \mathbf{x}$	Yeliussizov [15]	$G_\lambda \mathbf{x}; \alpha_0, -\beta_0$
$g_\lambda^{\alpha, \beta} \mathbf{x}$	Yeliussizov [15]	$g_\lambda \mathbf{x}; -\alpha_0, \beta_0$
$RG_\sigma \mathbf{x}; \beta$	Chan–Pflueger [2]	$G_\sigma \mathbf{x}; \mathbf{0}, \beta$
$\tilde{g}_{\lambda/\mu} \mathbf{x}; \beta$	Galashin–Grinberg–Liu [3]	$g_{\lambda/\mu} \mathbf{x}; \mathbf{0}, \beta$
$G_{\lambda/\mu, f/g} \mathbf{x}$	Matsumura [11]	$G_{\lambda/\mu}^{\text{row } g, f} \mathbf{x}; \mathbf{0}, -\beta_0$
$\tilde{g}_{\lambda/\mu}^{\text{row } r, s} \mathbf{x}; \beta$	Grinberg [4], Kim [6]	$\tilde{g}_{\lambda/\mu}^{\text{row } r, s} \mathbf{x}; \mathbf{0}, \beta$

The remainder of the extended abstract is organized as follows. In [Section 2](#), we briefly recall necessary definitions. In [Section 3](#), we give Schur expansions and Jacobi–Trudi-like formulas for $G_\lambda \mathbf{x}_n; \alpha, \beta$ and $g_\lambda \mathbf{x}_n; \alpha, \beta$ and prove their duality. In [Section 4](#), we give tableau models for $G_\lambda \mathbf{x}; \alpha, \beta$ and $g_\lambda \mathbf{x}; \alpha, \beta$ and obtain Schur positivity of $G_\lambda \mathbf{x}; \alpha, -\beta$ and $g_\lambda \mathbf{x}; -\alpha, \beta$. In [Sections 5](#) and [6](#), we extend refined canonical stable Grothendieck polynomials and their duals to skew shapes with certain flag conditions. In [Section 7](#), we give skew Schur expansions for $G_{\lambda/\mu} \mathbf{x}; \alpha, \beta$ and $g_{\lambda/\mu} \mathbf{x}; \alpha, \beta$, and they behave nicely under the involution ω . The full version of this extended abstract is [5], where the missing proofs can be found.

2 Preliminaries

In this section, we set up notations and give the necessary background. Throughout this extended abstract, we denote by $\mathbb{Z}$ (resp. $\mathbb{Z}^+$) the set of integers (resp. positive integers).

2.1 Partitions and tableaux

A *partition* is a weakly decreasing sequence $\lambda = (\lambda_1, \dots, \lambda_\ell)$ of positive integers. Each λ_i is called a *part*, and the *length* $\ell = \ell(\lambda)$ of λ is the number of parts in λ . We denote by Par_n the set of partitions with at most n parts, and by Par the set of all partitions.

We will identify a partition λ with its Young diagram. Each element $i, j \in \lambda$ is called a *cell* of λ . The Young diagram of λ is visualized by placing a square i, j for each cell i, j in λ using the matrix coordinates. The *transpose* λ' of λ is the partition whose Young diagram is obtained from that of λ by reflecting across the diagonal. The *content* $c_{i,j}$ of a cell $i, j \in \lambda$ is $j - i$.

For two partitions $\lambda, \mu \in \text{Par}$ with $\mu \subseteq \lambda$, the *skew shape* λ/μ is defined to be $\lambda - \mu$ $\{ i, j : 1 \leq i \leq \ell(\lambda), \mu_i < j \leq \lambda_i \}$. We denote by $|\lambda/\mu|$ the number of cells in λ/μ .

A *semistandard Young tableau* of shape λ/μ is a filling T of the cells in λ/μ with positive integers such that the integers are weakly increasing in each row and strictly increasing in each column. Let $\text{SSYT } \lambda/\mu$ be the set of semistandard Young tableaux of shape λ/μ . A *reverse plane partition* of shape λ/μ is a filling T of the cells in λ/μ with positive integers such that the integers are weakly increasing in each row and in each column. Let $\text{RPP } \lambda/\mu$ be the set of reverse plane partitions of shape λ/μ .

2.2 Symmetric functions

Let Λ be the $\mathbb{Q}$ -algebra of all symmetric functions in $\mathbb{Q}[x_1, x_2, \dots]$. The *completion* $\hat{\Lambda}$ of Λ is the set of symmetric functions with possibly unbounded degree. For $n \geq 1$, the *complete homogeneous symmetric function* $h_n \mathbf{x}$ and the *elementary symmetric function* $e_n \mathbf{x}$ are defined to be

$$h_n \mathbf{x} = \sum_{i_1 \leq \dots \leq i_n} x_{i_1} \cdots x_{i_n}, \quad \text{and} \quad e_n \mathbf{x} = \sum_{i_1 < \dots < i_n} x_{i_1} \cdots x_{i_n}.$$

Define an endomorphism $\omega : \Lambda \rightarrow \Lambda$ of the $\mathbb{Q}$ -algebra Λ by $\omega(h_n \mathbf{x}) = e_n \mathbf{x}$.

An important basis is the Schur function basis. For a partition $\lambda \in \text{Par}_n$, the *Schur polynomial* $s_\lambda \mathbf{x}_n$ of shape λ is defined by

$$s_\lambda \mathbf{x}_n = \frac{\det(x_j^{\lambda_i - n + i})_{1 \leq i, j \leq n}}{\prod_{1 \leq i < j \leq n} (x_i - x_j)}.$$

The *Schur function* $s_\lambda \mathbf{x}$ is defined to be the coefficient-wise limit of $s_\lambda \mathbf{x}_n$ as $n \rightarrow \infty$. The Schur function $s_{\lambda/\mu} \mathbf{x}$ of shape λ/μ is defined by $s_{\lambda/\mu} \mathbf{x} = \sum_{T \in \text{SSYT } \lambda/\mu} \mathbf{x}^T$, where $\mathbf{x}^T = x_1^{m_1} x_2^{m_2} \cdots$ and m_i is the number of appearances of i in T .

The *Hall inner product* is the inner product on Λ defined by $\langle s_\lambda \mathbf{x}, s_\mu \mathbf{x} \rangle = \delta_{\lambda, \mu}$. This inner product is naturally extended to $\hat{\Lambda} \times \Lambda$ because every element $f \in \hat{\Lambda}$ can be written as $f = \sum_{\lambda \in \text{Par}} c_\lambda s_\lambda \mathbf{x}$.

The Jacobi–Trudi formula and its dual give expansions of Schur functions in terms of $h_n \mathbf{x}$ and $e_n \mathbf{x}$ as determinants: for partitions $\mu \subseteq \lambda$,

$$s_{\lambda/\mu} \mathbf{x} = \det \left(h_{\lambda_i - \mu_j - i + j} \mathbf{x} \right)_{i, j=1}^{\ell(\lambda)} \quad \text{and} \quad s_{\lambda'/\mu'} \mathbf{x} = \det \left(e_{\lambda_i - \mu_j - i + j} \mathbf{x} \right)_{i, j=1}^{\ell(\lambda)}.$$

In this extended abstract, we consider the completion of $\Lambda_{\mathbb{Q}[\alpha, \beta]} = \mathbb{Q}[\alpha, \beta] \otimes \Lambda$ instead of Λ .

2.3 Plethystic substitutions

We define plethystic substitution via the *power sum symmetric functions* $p_n \mathbf{x}$. For a formal power series $Z \in \mathbb{Q}[z_1, z_2, \dots]$, the *plethystic substitution* $p_n Z$ of Z into $p_n \mathbf{x}$ is defined to be the formal power series obtained from Z by replacing each z_i by z_i^n . By definition, $f(z_1, z_2, \dots) = f(p_1 z_1, p_2 z_2, \dots)$. For more properties of plethystic substitution, we refer the reader to [10].

For integers r, s , and n , we define

$$X_n = x_1^n + x_2^n + \dots, \quad X_n = x_1^n + x_2^n + \dots + x_n^n, \quad X_{r,s} = x_r^s + x_{r+1}^s + \dots + x_s^s,$$

$$A_n = \alpha_1^n + \alpha_2^n + \dots + \alpha_n^n, \quad B_n = \beta_1^n + \beta_2^n + \dots + \beta_n^n,$$

where $X_n = A_n = B_n = 0$ if $n \leq 0$ and $X_{r,s} = 0$ if $r > s$.

In this extended abstract, we only consider the plethystic substitution into $h_n \mathbf{x}$ and $e_n \mathbf{x}$. We will use the following properties, which follows from [10, Theorem 6].

Proposition 2.1. For $n \geq 0$ and $Z \in \mathbb{Q}[z_1, z_2, \dots]$, we have

$$h_n(-Z) = (-1)^n e_n Z \quad \text{and} \quad e_n(-Z) = (-1)^n h_n Z.$$

For $Z_1, Z_2 \in \mathbb{Q}[z_1, z_2, \dots]$,

$$h_n(Z_1 + Z_2) = \sum_{a+b=n} h_a Z_1 h_b Z_2 \quad \text{and} \quad e_n(Z_1 + Z_2) = \sum_{a+b=n} e_a Z_1 e_b Z_2.$$

We introduce a novel notation $\ominus$ as follows.

Definition 2.2. For $n \in \mathbb{Z}$ and $Z_1, Z_2 \in \mathbb{Q}[z_1, z_2, \dots]$ with no constant terms,

$$h_n(Z_1 \ominus Z_2) = \sum_{a-b=n} h_a Z_1 h_b Z_2 \quad \text{and} \quad e_n(Z_1 \ominus Z_2) = \sum_{a-b=n} e_a Z_1 e_b Z_2.$$

3 Schur expansions, Jacobi–Trudi-like formulas, and duality

In this section, we give Schur expansions and Jacobi–Trudi-like formulas for $G_\lambda \mathbf{x}; \alpha, \beta$ and $g_\lambda \mathbf{x}; \alpha, \beta$ using the Cauchy–Binet theorem. We also give the duality between $G_\lambda \mathbf{x}; \alpha, \beta$ and $g_\lambda \mathbf{x}; \alpha, \beta$ with respect to the Hall inner product.

For $\lambda, \mu \in \text{Par}_n$, define

$$\begin{aligned} C_{\lambda, \mu}(\alpha, \beta) &= \det \left(h_{\mu_i - \lambda_j - i - j} A_{\lambda_j} - B_{j-1} \right)_{i, j=1}^n, \\ c_{\lambda, \mu}(\alpha, \beta) &= \det \left(h_{\lambda_i - \mu_j - i - j} - A_{\lambda_i-1} \quad B_{i-1} \right)_{i, j=1}^n. \end{aligned}$$

We expand $G_\lambda(x_n; \alpha, \beta)$ and $g_\lambda(x_n; \alpha, \beta)$ into Schur functions as follows.

Theorem 3.1. *Let $\lambda \in \text{Par}_n$. We have*

$$\begin{aligned} G_\lambda(x_n; \alpha, \beta) &= \sum_{\mu \supseteq \lambda} C_{\lambda, \mu}(\alpha, \beta) s_\mu(x_n), \\ g_\lambda(x_n; \alpha, \beta) &= \sum_{\mu \subseteq \lambda} c_{\lambda, \mu}(\alpha, \beta) s_\mu(x_n). \end{aligned}$$

We can apply the Cauchy–Binet theorem again to the Schur expansions in [Theorem 3.1](#) to obtain Jacobi–Trudi-like formulas.

Theorem 3.2. *For $\lambda \in \text{Par}_n$, we have*

$$\begin{aligned} G_\lambda(x_n; \alpha, \beta) &= \det \left(h_{\lambda_i - i - j} X_n \ominus A_{\lambda_i} - B_{i-1} \right)_{i, j=1}^n, \\ g_\lambda(x_n; \alpha, \beta) &= \det \left(h_{\lambda_i - i - j} X_n - A_{\lambda_i-1} \quad B_{i-1} \right)_{i, j=1}^n. \end{aligned}$$

Using the Schur expansions of $G_\lambda(x; \alpha, \beta)$ and $g_\lambda(x; \alpha, \beta)$, we show the following duality with respect to the Hall inner product, which justifies the name “refined dual canonical stable Grothendieck polynomial” for $g_\lambda(x; \alpha, \beta)$.

Theorem 3.3. *For $\lambda, \mu \in \text{Par}$, we have*

$$\langle G_\lambda(x; \alpha, \beta), g_\mu(x; \alpha, \beta) \rangle = \delta_{\lambda, \mu}.$$

4 Combinatorial models and Schur positivity

In this section we give combinatorial models for the Schur coefficients $C_{\lambda, \mu}(\alpha, \beta)$ and $c_{\lambda, \mu}(\alpha, \beta)$ of $G_\lambda(x; \alpha, \beta)$ and $g_\lambda(x; \alpha, \beta)$. As a consequence we show that $G_\lambda(x; \alpha, -\beta)$ and $g_\lambda(x; -\alpha, \beta)$ are Schur-positive. In order to state our results we need the following tableaux, which are modified versions of elegant tableaux [\[7, 9\]](#).

Definition 4.1. A $\mathbb{Z}$ -elegant tableau of shape λ/μ is a filling T of the cells in λ/μ with integers such that the rows are weakly increasing, the columns are strictly increasing, and

$$\min \{i - \mu_i, 1\} \leq T(i, j) < i \quad \text{for all } (i, j) \in \lambda/\mu.$$

Let $\text{ET}_{\mathbb{Z}}(\lambda/\mu)$ denote the set of all $\mathbb{Z}$ -elegant tableaux of shape λ/μ .

Definition 4.2. A $\mathbb{Z}$ -inelegant tableau of shape μ/λ is a filling T of the cells in μ/λ with integers such that the rows are weakly decreasing, the columns are strictly decreasing, and

$$\min \mu_i - i, 0 < T(i, j) \leq \lambda_i \quad \text{for all } (i, j) \in \mu/\lambda.$$

Let $\text{IET}_{\mathbb{Z}} \mu/\lambda$ denote the set of all $\mathbb{Z}$ -inelegant tableaux of shape μ/λ .

We now give combinatorial interpretations for the Schur coefficients $C_{\lambda, \mu}(\alpha, \beta)$ and $c_{\lambda, \mu}(\alpha, \beta)$ of $G_{\lambda}(x; \alpha, \beta)$ and $g_{\lambda}(x; \alpha, \beta)$ in terms of tableaux.

Theorem 4.3. Let $\lambda, \mu \in \text{Par}$ with $\lambda \subseteq \mu$. We have

$$\begin{aligned} C_{\lambda, \mu}(\alpha, \beta) &= \sum_{T \in \text{IET}_{\mathbb{Z}} \mu/\lambda} \prod_{(i, j) \in \mu/\lambda} \alpha_{T(i, j) - \beta_{T(i, j) - c(i, j)}}, \\ c_{\lambda, \mu}(\alpha, \beta) &= \sum_{T \in \text{IET}_{\mathbb{Z}} \lambda/\mu} \prod_{(i, j) \in \lambda/\mu} -\alpha_{T(i, j) - c(i, j)} \beta_{T(i, j)}, \end{aligned}$$

where $\alpha_m = \beta_m = 0$ for $m \leq 0$. In particular, $G_{\lambda}(x; \alpha, -\beta)$ and $g_{\lambda}(x; -\alpha, \beta)$ are Schur-positive, that is, $C_{\lambda, \mu}(\alpha, -\beta)$ and $c_{\lambda, \mu}(-\alpha, \beta)$ are polynomials in α and β with nonnegative integer coefficients.

For the proof of [Theorem 4.3](#), we express $C_{\lambda, \mu}(\alpha, \beta)$ and $c_{\lambda, \mu}(\alpha, \beta)$ in terms of nonintersecting lattice paths using the Lindström–Gessel–Viennot lemma. The nonintersecting lattice paths are then easily transformed into the desired tableaux.

5 Flagged Grothendieck polynomials

In this section we give a combinatorial model for $G_{\lambda}(x; \alpha, \beta)$ using marked multiset-valued tableaux. More generally, we consider two flagged versions of $G_{\lambda}(x; \alpha, \beta)$ and extend the partition λ to a skew shape. To begin with, we introduce the following definition.

Definition 5.1. A *marked multiset-valued tableau* of shape λ/μ is a filling T of λ/μ with multisets such that

- $T(i, j)$ is a nonempty (finite) multiset $\{a_1 \leq \dots \leq a_k\}$ of positive integers, in which each integer a_i may be marked if $i \geq 2$ and $a_{i-1} < a_i$, and
- $\max T(i, j) \leq \min T(i, j+1)$ and $\max T(i, j) < \min T(i+1, j)$ if $(i, j), (i, j+1) \in \lambda/\mu$ and $(i, j), (i+1, j) \in \lambda/\mu$, respectively.

Let $\text{MMSVT} \lambda/\mu$ denote the set of marked multiset-valued tableaux of shape λ/μ .

	1, 2*, 2	2, 2, 4*
1	3, 3	

Figure 1: An example of $T \in \text{MMSVT } 3, 2 / 1$, where the marked integers are indicated with *. The weight of T is given by $\text{wt } T = x_1^2 x_2^4 x_3^2 x_4 \alpha_2^2 \alpha_3 - \beta_1^2$.

For $T \in \text{MMSVT } \lambda/\mu$, let $\mathbf{x}^{T(i,j)} = x_1^{m_1} x_2^{m_2} \cdots$, where m_k is the total number of appearances of k and k^* in $T(i,j)$, and let $\text{unmarked } T(i,j)$ (resp. $\text{marked } T(i,j)$) denote the number of unmarked (resp. marked) integers in $T(i,j)$. We define

$$\text{wt } T = \prod_{(i,j) \in \lambda/\mu} \mathbf{x}^{T(i,j)} \alpha_j^{\text{unmarked } T(i,j) - 1} - \beta_i^{\text{marked } T(i,j)}.$$

See Figure 1.

The main goal of this section is to prove the following combinatorial interpretation for $G_\lambda(\mathbf{x}; \alpha, \beta)$.

Theorem 5.2. *For a partition λ , we have*

$$G_\lambda(\mathbf{x}; \alpha, \beta) = \sum_{T \in \text{MMSVT } \lambda} \text{wt } T.$$

Our strategy for proving Theorem 5.2 is to introduce a flagged version which allows us to use induction. We note that a row-flagged version with partition shape is sufficient to prove this theorem, but for completeness we also consider a column-flagged version and we generalize the partition shape to any skew shape.

Definition 5.3. Let $\mathbf{r} = r_1, r_2, \dots$, $\mathbf{s} = s_1, s_2, \dots$ be in $\mathbb{Z}^n$. Let $\text{MMSVT}^{\text{row } \mathbf{r}, \mathbf{s}} \lambda/\mu$ denote the set of $T \in \text{MMSVT } \lambda/\mu$ such that $r_i \leq \min T(i,j)$ and $\max T(i,j) \leq s_i$ for all $(i,j) \in \lambda/\mu$. Similarly, $\text{MMSVT}^{\text{col } \mathbf{r}, \mathbf{s}} \lambda/\mu$ denotes the set of $T \in \text{MMSVT } \lambda/\mu$ such that $r_j \leq \min T(i,j)$ and $\max T(i,j) \leq s_j$ for all $(i,j) \in \lambda/\mu$. We define the row-flagged and column-flagged refined canonical stable Grothendieck polynomials by

$$G_{\lambda/\mu}^{\text{row } \mathbf{r}, \mathbf{s}}(\mathbf{x}; \alpha, \beta) = \sum_{T \in \text{MMSVT}^{\text{row } \mathbf{r}, \mathbf{s}} \lambda/\mu} \text{wt } T,$$

$$G_{\lambda/\mu}^{\text{col } \mathbf{r}, \mathbf{s}}(\mathbf{x}; \alpha, \beta) = \sum_{T \in \text{MMSVT}^{\text{col } \mathbf{r}, \mathbf{s}} \lambda/\mu} \text{wt } T.$$

Now we give Jacobi–Trudi-like formulas for $G_{\lambda/\mu}^{\text{row } r,s}(\mathbf{x}; \boldsymbol{\alpha}, \boldsymbol{\beta})$ and $G_{\lambda/\mu}^{\text{col } r,s}(\mathbf{x}; \boldsymbol{\alpha}, \boldsymbol{\beta})$.

Theorem 5.4. *Let $\lambda, \mu \in \text{Par}_n, \mathbf{r}, \mathbf{s} \in \mathbb{Z}^n$ with $\mu \subseteq \lambda$. If $r_i \leq r_{i-1}$ and $s_i \leq s_{i-1}$ whenever $\mu_i < \lambda_{i-1}$ for $1 \leq i \leq n-1$, then*

$$G_{\lambda/\mu}^{\text{row } r,s}(\mathbf{x}; \boldsymbol{\alpha}, \boldsymbol{\beta}) = C \det \left(h_{\lambda_i - \mu_j - i - j} X_{r_j, s_i} \ominus A_{\lambda_i} - A_{\mu_j} - B_{i-1} \quad B_j \right)_{i,j=1}^n, \quad (5.1)$$

where $C = \prod_{i=1}^n \prod_{l=1}^{s_i} (1 - \beta_i x_l)$.

We prove Theorem 5.4 by showing that the both sides of (5.1) satisfy the same recurrence relations and initial conditions. The idea is based on a proof of the Jacobi–Trudi-like formula for flagged Schur functions due to Wachs [14]. We prove the following theorem in a similar way.

Theorem 5.5. *Let $\lambda, \mu \in \text{Par}_n, \mathbf{r}, \mathbf{s} \in \mathbb{Z}^n$ with $\mu \subseteq \lambda$. If $r_i - \mu_i \leq r_{i-1} - \mu_{i-1}$ and $s_i - \lambda_i \leq s_{i-1} - \lambda_{i-1} - 1$ whenever $\mu_i < \lambda_{i-1}$ for $1 \leq i \leq n-1$, then*

$$G_{\lambda'/\mu'}^{\text{col } r,s}(\mathbf{x}; \boldsymbol{\alpha}, \boldsymbol{\beta}) = D \det \left(e_{\lambda_i - \mu_j - i - j} X_{r_j, s_i} \ominus A_{i-1} - A_j - B_{\lambda_i} \quad B_{\mu_j} \right)_{i,j=1}^n,$$

where $D = \prod_{i=1}^n \prod_{l=1}^{s_i} (1 - \alpha_i x_l)^{-1}$.

6 Flagged dual Grothendieck polynomials

In this section we give a combinatorial model for the refined dual canonical stable Grothendieck polynomials $g_\lambda(\mathbf{x}; \boldsymbol{\alpha}, \boldsymbol{\beta})$ using marked reverse plane partitions. To this end we introduce a flagged version of $g_\lambda(\mathbf{x}; \boldsymbol{\alpha}, \boldsymbol{\beta})$ using marked reverse plane partitions and prove a Jacobi–Trudi-like formula for this flagged version. More generally, we consider two flagged versions of $g_\lambda(\mathbf{x}; \boldsymbol{\alpha}, \boldsymbol{\beta})$ and extend the partition λ to a skew shape.

Definition 6.1. A *left-marked reverse plane partition* (or simply *marked reverse plane partition*) of shape λ/μ is a reverse plane partition T of shape λ/μ in which every entry $T(i, j)$ with $T(i, j) > T(i, j-1)$ may be marked. We denote by $\text{MRPP } \lambda/\mu$ the set of marked reverse plane partitions of shape λ/μ . For $T \in \text{MRPP } \lambda/\mu$, define

$$\text{wt } T = \prod_{i,j \in \lambda/\mu} \text{wt } T(i, j),$$

where

$$\text{wt } T(i, j) = \begin{cases} -\alpha_j & \text{if } T(i, j) \text{ is marked,} \\ \beta_{i-1} & \text{if } T(i, j) \text{ is not marked and } T(i, j) > T(i-1, j), \\ x_{T(i, j)} & \text{if } T(i, j) \text{ is not marked and } T(i, j) \leq T(i-1, j). \end{cases}$$

Here, the equality $T(i, j) \leq T(i-1, j)$ means that their underlying integers are equal. See Figure 2.

			1	2	4*	4
		1*	1	3	5	
		1	1			
3*	3*	3				

Figure 2: An example of $T \in \text{MRPP } \lambda/\mu$, where $\lambda = 6, 5, 3, 3$ and $\mu = 2, 1, 1$. We have the weight $\text{wt } T = x_1 x_2 x_3^2 x_4 x_5 = -\alpha_1 - \alpha_2^2 - \alpha_5 \beta_1 \beta_2^2$.

The main goal of this section is to prove the following combinatorial interpretation for $g_\lambda(x; \alpha, \beta)$.

Theorem 6.2. *We have*

$$g_\lambda(x; \alpha, \beta) = \sum_{T \in \text{MRPP } \lambda} \text{wt } T. \quad (6.1)$$

Similar to our approach in the previous section, in order to prove [Theorem 6.2](#) we introduce a flagged version. For completeness we again consider row- and column-flagged versions on skew shapes.

Definition 6.3. Let $r = r_1, r_2, \dots \in \mathbb{Z}^n$ and $s = s_1, s_2, \dots \in \mathbb{Z}^n$. We denote by $\text{MRPP}^{\text{row } r, s} \lambda/\mu$ (resp. $\text{MRPP}^{\text{col } r, s} \lambda/\mu$) the set of $T \in \text{MRPP } \lambda/\mu$ such that $r_i \leq T(i, j) \leq s_i$ (resp. $r_j \leq T(i, j) \leq s_j$) for all $i, j \in \lambda/\mu$. We define *row-flagged* and *column-flagged refined dual canonical stable Grothendieck polynomials* by

$$g_{\lambda/\mu}^{\text{row } r, s}(x; \alpha, \beta) = \sum_{T \in \text{MRPP}^{\text{row } r, s} \lambda/\mu} \text{wt } T,$$

$$g_{\lambda/\mu}^{\text{col } r, s}(x; \alpha, \beta) = \sum_{T \in \text{MRPP}^{\text{col } r, s} \lambda/\mu} \text{wt } T.$$

Now we give Jacobi–Trudi-like formulas for the two flagged versions of $g_\lambda(x; \alpha, \beta)$.

Theorem 6.4. *Let $\lambda, \mu \in \text{Par}_n$, $r, s \in \mathbb{Z}^n$ such that $r_i \leq r_{i-1}$ and $s_i \leq s_{i-1}$ whenever $\mu_i < \lambda_{i-1}$. Then we have*

$$g_{\lambda/\mu}^{\text{row } r, s}(x; \alpha, \beta) = \det \left(h_{\lambda_i - \mu_j - i - j} X_{r_j, s_i} - A_{\lambda_i - 1} \quad A_{\mu_j} \quad B_{i-1} - B_{j-1} \right)_{i, j=1}^n,$$

$$g_{\lambda'/\mu'}^{\text{col } r, s}(x; \alpha, \beta) = \det \left(e_{\lambda_i - \mu_j - i - j} X_{r_j, s_i} - A_{i-1} \quad A_{j-1} \quad B_{\lambda_i - 1} - B_{\mu_j} \right)_{i, j=1}^n.$$

The proof is based on the ideas in [6].

7 Schur expansions and the omega involution for skew shapes

When we set $r_i = 1$ and $s_i = \infty$ for each i in [Theorems 5.4, 5.5 and 6.4](#), we have the refined canonical stable Grothendieck polynomial $G_{\lambda/\mu}(x; \alpha, \beta)$ of skew shape λ/μ , and its dual $g_{\lambda/\mu}(x; \alpha, \beta)$.

We define a *generalized partition* of length n to be a sequence $\lambda_1, \dots, \lambda_n \in \mathbb{Z}^n$ satisfying $\lambda_1 \geq \dots \geq \lambda_n$. Denote by GPar_n the set of generalized partitions of length n . We expand $G_{\lambda/\mu}(x; \alpha, \beta)$ and $g_{\lambda/\mu}(x; \alpha, \beta)$ in terms of skew Schur functions (up to some factor) using the Cauchy–Binet Theorem.

Theorem 7.1. For $\lambda, \mu \in \text{Par}_n$ with $\mu \subseteq \lambda$, we have

$$G_{\lambda/\mu}(x; \alpha, \beta) = C \sum_{\substack{\rho, \nu \in \text{GPar}_n \\ \rho \subseteq \mu \subseteq \lambda \subseteq \nu}} C_{\lambda, \nu}(\alpha, \beta) s_{\nu/\rho}(x) C'_{\rho, \mu}(\alpha, \beta), \quad (7.1)$$

where $C = \prod_{i=1}^n \prod_{l=1}^{\infty} (1 - \beta_i x_l)$ and

$$\begin{aligned} C_{\lambda, \nu}(\alpha, \beta) &= \det \left(h_{\nu_i - \lambda_j - i - j} A_{\lambda_j} - B_{j-1} \right)_{i, j=1}^n, \\ C'_{\rho, \mu}(\alpha, \beta) &= \det \left(h_{\mu_i - \rho_j - i - j} - A_{\mu_i} B_j \right)_{i, j=1}^n. \end{aligned}$$

Theorem 7.2. For $\lambda, \mu \in \text{Par}_n$, we have

$$g_{\lambda/\mu}(x; \alpha, \beta) = \sum_{\substack{\rho, \nu \in \text{Par}_n \\ \mu \subseteq \rho \subseteq \nu \subseteq \lambda}} c_{\lambda, \nu}(\alpha, \beta) s_{\nu/\rho}(x) c'_{\rho, \mu}(\alpha, \beta),$$

where

$$\begin{aligned} c_{\lambda, \nu}(\alpha, \beta) &= \det \left(h_{\lambda_i - \nu_j - i - j} - A_{\lambda_i-1} B_{j-1} \right)_{i, j=1}^n, \\ c'_{\rho, \mu}(\alpha, \beta) &= \det \left(h_{\rho_i - \mu_j - i - j} A_{\mu_j} - B_{j-1} \right)_{i, j=1}^n. \end{aligned}$$

Finally, we obtain the effect of the involution ω on $G_{\lambda/\mu}(x; \alpha, \beta)$ and $g_{\lambda/\mu}(x; \alpha, \beta)$.

Theorem 7.3. For $\lambda, \mu \in \text{Par}$ with $\mu \subseteq \lambda$, we have

$$\begin{aligned} \omega G_{\lambda/\mu}(x; \alpha, \beta) &= G_{\lambda'/\mu'}(x; -\beta, -\alpha), \\ \omega g_{\lambda/\mu}(x; \alpha, \beta) &= g_{\lambda'/\mu'}(x; -\beta, -\alpha). \end{aligned}$$

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Quasisymmetric harmonics of the exterior algebra

Nantel Bergeron^{*1}, Kelvin Chan^{†1}, Farhad Soltani^{‡1}, and Mike Zabrocki^{§1}

¹*Department of Mathematics and Statistics, York University, Toronto, Ontario M3J 1P3
Canada*

Abstract. We study *fermionic* quasisymmetric polynomials in the polynomial ring R_n with n anticommuting variables. The main results of this paper are that the quasisymmetric polynomials in R_n form a commutative sub-algebra of R_n , there is a basis of the quotient of R_n by the ideal I_n generated by the quasisymmetric polynomials in R_n that is indexed by ballot sequences, and there is a basis of the ideal generated by quasisymmetric polynomials that is indexed by sequences that break the ballot condition.

Résumé. Nous étudions les polynômes quasisymétriques fermioniques de l’anneau des polynômes R_n à n variables anti-commutatives. Les principaux résultats de cet article sont que les polynômes quasi-symétriques dans R_n forment une sous-algèbre commutative de R_n ; qu’il existe une base du quotient de R_n par l’idéal I_n engendré par les polynômes quasi symétriques dans R_n qui est indexée par des séquences de scrutin; et qu’il existe une base de l’idéal généré par les polynômes quasi symétriques qui est indexée par des séquences qui brisent la condition de scrutin.

Keywords: Quasisymmetric Polynomials, Fermionic Variables, Exterior Algebra, Ballot Sequences, Polynomial Harmonics

1 Introduction

The study of coinvariants of groups dates back to Shephard-Todd and Chevalley [26, 5] and has fruitfully produced many connections between algebra, combinatorics and physics. Motivated by recent developments in coinvariants of symmetric groups and symmetric functions theory incorporating anticommuting (also known as fermionic) variables, we study a coinvariant-like quotient of an exterior algebra obtained by the quotient of the ideal generated by quasisymmetric functions in anticommuting variables. The quotient has a dimension that can be interpreted as the number of ballot sequences (or other interpretations, see for instance the OEIS [25] sequences [A008315](#) and [A001405](#)).

^{*}bergeron@yorku.ca.

[†]ktychan@yorku.ca.

[‡]farhad.soltani91@gmail.com.

[§]zabrocki@yorku.ca.

Many coinvariant quotients feature rich combinatorial connections. One well-known example is the coinvariant ring of the symmetric group whose dimension is $n!$. It is the quotient of the polynomial ring $\mathbb{Q}[x_1, \dots, x_n]$ in commuting variables by the ideal generated by the symmetric polynomials with no constant term. As an $\mathcal{S}_n$ representation, this quotient is naturally graded and is well-known to be isomorphic to the regular representation. Its graded Hilbert series is the q -factorial. Many useful bases of this space have been found by studying combinatorics related to permutations. For more details, see the nice surveys of [3, 13, 23, 24].

This line of inquiry inspired Garsia and Haiman [12, 17] to consider the ring of diagonal harmonics whose dimension is $(n-1)!(n-1)$, the number of parking functions. It is a similar quotient in two sets of commuting variables as an $\mathcal{S}_n$ module. Haiman's work [18] showed that the diagonal harmonics have a deep connection to the theory of Macdonald polynomials. A combinatorial expression for the Frobenius image of the diagonal harmonics known as the Shuffle Conjecture [15] showed that the module structure is closely related to the combinatorics of parking functions and can be described in terms of certain labelled Catalan paths. This connection relating the symmetric functions and the combinatorial expression was proven in [4] and is now known as the Shuffle Theorem.

The connection between the combinatorics and the symmetric function expressions of the Shuffle Theorem have been generalized [16] and proven [8] to an expression known as the Delta Conjecture. The last author with the group at the Fields Institute [29] proposed a deformation of diagonal harmonics to two sets of commuting variables and one set of anticommuting variables. In this case, the connection of representation theoretic interpretation to the symmetric function expression remains open. The symmetric function expressions and representation theoretic interpretation was extended further to include the quotient of two sets of commuting and two sets of anticommuting variables in [7] to what is known as the Theta Conjecture. At present, this also remains an open conjecture, but progress has been made on some special cases [20, 21, 27, 28].

The ring of quasisymmetric polynomials $QSym_n$ contains the ring of symmetric polynomials Sym_n . Many combinatorial structures of $QSym_n$ parallel that of Sym_n . The Temperley-Lieb algebra TL_n can be obtained as a quotient of the symmetric group algebra. Hivert described a TL_n action on $\mathbb{Q}[x_1, \dots, x_n]$ making $QSym_n$ an isotypic trivial representation of TL_n [19]. In 2003, Aval, F. Bergeron, and the first author studied $QSym$ coinvariant spaces obtained by replacing the ideal of non-constant symmetric functions with the ideal of non-constant quasisymmetric functions [1, 2]. Surprisingly they found that dimensions of $QSym$ coinvariants are equal to the Catalan numbers. At the heart of their argument is a recursion built from Catalan paths. Li extended this argument to study some components of coinvariant spaces of diagonally quasisymmetric functions [22].

Desrosiers, Lapointe, and Mathieu [10, 9] introduced symmetric functions with one

set of commuting and one set of anticommuting variables known as symmetric functions in superspace. The aforementioned proposal [29] of a representation theoretic interpretation of the Delta Conjecture can be thought of as a generalization of the classic coinvariant quotients to superspace. Then a natural direction for exploration is to expand the study of “super” symmetric coinvariants to “super” quasisymmetric coinvariants.

Our study of quasisymmetric coinvariant spaces in one set of anticommuting variables is a first step in that study. We denote polynomials in a set of n anticommuting variables by R_n . The main results of this paper show the following interesting facts about symmetric and quasisymmetric functions in anticommuting variables:

1. The quasisymmetric polynomials in R_n form a commutative sub-algebra of R_n (Proposition 2.1).
2. That R_n is free over the ring of symmetric polynomials (Proposition 2.2).
3. There is a basis of the quotient of R_n by the ideal I_n generated by the quasisymmetric polynomials in R_n that is indexed by ballot sequences (Proposition 2.5). The Hilbert series of the quotient is given by

$$\text{Hilb}_{R_n/I_n} q = \sum_{k=0}^{\lfloor n/2 \rfloor} f^{n-k,k} q^k, \quad (1.1)$$

where $f^{n-k,k}$ is the number of standard tableaux of shape $n-k, k$ (Corollary 4.2).

4. There is a basis of the ideal generated by quasisymmetric polynomials that is indexed by sequences that break the ballot condition (Theorem 4.1) and a minimal Gröbner basis that is a subset of this basis (Corollary 4.4).

2 Quasisymmetric invariants on the exterior algebra

Fix n a positive integer and let $R_n = \mathbb{Q}[\theta_1, \theta_2, \dots, \theta_n]$ be the polynomial ring in anticommuting variables. The ring R_n is isomorphic to the exterior algebra of a vector space of dimension n . The variables of this ring satisfy the relations

$$\theta_i \theta_j = -\theta_j \theta_i \text{ if } 1 \leq i < j \leq n \quad \text{and} \quad \theta_i^2 = 0 \text{ for } 1 \leq i \leq n.$$

Since these conditions impose that a monomial in R_n has no repeated variables, the monomials are in bijection with subsets of $\{1, 2, \dots, n\}$ and the dimension of R_n is therefore equal to 2^n .

Denote $[n] = \{1, 2, \dots, n\}$ and let $A = \{a_1 < a_2 < \dots < a_r\} \subseteq [n]$. We define $\theta_A = \theta_{a_1} \theta_{a_2} \dots \theta_{a_r}$, then the set of monomials $\{\theta_A\}_{A \subseteq [n]}$ is a basis of R_n .

We define an action on monomials of R_n and extend this action linearly. For each integer $1 \leq i < n$, let π_i be an operator on R_n that is defined by

$$\pi_i \theta_A = \begin{cases} \theta_A & \text{if } i, i-1 \in A \text{ or } i, i-1 \notin A \\ \theta_{A \cup \{i-1\} \setminus \{i\}} & \text{if } i \in A \text{ and } i-1 \notin A \\ \theta_{A \cup \{i\} \setminus \{i-1\}} & \text{if } i-1 \in A \text{ and } i \notin A \end{cases}. \quad (2.1)$$

These operators instead of exchanging an i for an $i-1$ like the symmetric group action, have the effect of shifting the indices of the variables (if possible). They are therefore known as quasisymmetric operators. They were studied in depth by Hivert [19]. The operators are not multiplicative on R_n in general since, for example,

$$\pi_1 \theta_1 \theta_2 = \theta_1 \theta_2 = -\pi_1 \theta_1 \pi_1 \theta_2.$$

The corresponding operators are also not multiplicative when they act on the polynomial ring in commuting variables.

A polynomial that is invariant under the action of quasisymmetric operators is said to be quasisymmetric invariant (or just ‘quasisymmetric’). The quasisymmetric invariants of R_n are linearly spanned by the elements:

$$F_{1^r} \theta_1, \theta_2, \dots, \theta_n : \sum_{\substack{A \subseteq [n] \\ |A| = r}} \theta_A. \quad (2.2)$$

The symbols F_{1^r} for the elements borrows the notation from the polynomial ring in commuting variable invariants known as the ‘fundamental quasisymmetric polynomials.’ The commuting polynomial quasisymmetric invariants are indexed by compositions. We choose to use F_{1^r} instead of M_{1^r} to maintain consistency with [2, 1] since our methods are similar to theirs.

2.1 Quasisymmetric functions generate a commutative subalgebra

In [11], the authors showed that the quasisymmetric functions in one set of commuting variables and one set of anticommuting variables forms a bi-graded Hopf algebra. This implies that the quasisymmetric functions in one set of anticommuting variables are closed under multiplication and the space is spanned by one element at each non-negative degree. The product is not commutative in general, but it is in the case of this subalgebra even though the variables are anticommuting.

In the notation of [11], $F_{1^r} = M_{\vec{0}^r} = L_{\vec{0}^r}$ where $\vec{0}^r = \vec{0}, \vec{0}, \dots, \vec{0}$ a composition of length r . The degree of F_{1^r} is exactly r .

Proposition 2.1. *The subalgebra generated by quasisymmetric invariants $\{F_{1^r} | r \geq 0\}$ is commutative. Moreover, $F_{1^r} F_{1^s} = a_{r,s} F_{1^{r+s}}$, where for $r + s \leq n$,*

$$a_{r,s} = \begin{cases} \binom{\lfloor \frac{r+s}{2} \rfloor}{\lfloor \frac{r}{2} \rfloor} & \text{if } rs \equiv 0 \pmod{2} \\ 0 & \text{otherwise} \end{cases}.$$

A remark brought to our attention by D. Grinberg [14] shows that $a_{r,s}$ is equal to the q -binomial coefficient $\begin{bmatrix} r+s \\ r \end{bmatrix}_q$ evaluated at $q \rightarrow -1$.

2.2 The ideal generated by symmetric invariants

The symmetric invariants Sym_{R_n} of R_n are very small since a basis consists of only two elements 1 and $F_1 \theta_1, \theta_2, \dots, \theta_n$. Therefore the ideal generated by the invariants of non-zero degree, which we shall denote J_n , is generated by a single element F_1 . We begin by considering the symmetric coinvariants of R_n , the quotient ring R_n/J_n . Because the ideal J_n is principal we can understand this quotient with much more detail. This quotient ring is a special case of the ring recently studied in [20, 21].

Recall that $\dim R_n = 2^n$, and if we consider the quotient R_n/J_n it is isomorphic to R_{n-1} since in this algebra $\theta_n = -\theta_1 - \theta_2 - \dots - \theta_{n-1}$. Let $A \subseteq [n-1]$ and $A' = A \cup \{n\}$, then the map which sends $\theta_{A'}$ to

$$-\theta_A \theta_1 \theta_2 \dots \theta_{n-1} \otimes 1 - \theta_A \otimes F_1 \in R_n/J_n \otimes \text{Sym}_{R_n}$$

and θ_A to

$$\theta_A \otimes 1 \in R_n/J_n \otimes \text{Sym}_{R_n}$$

is an algebra isomorphism. The multiplication in the tensor algebra has a sign twist. For example, $1 \otimes F_1 \theta_2 \otimes 1 = -\theta_2 \otimes F_1$ because the θ_2 has to commute past F_1 . Since this map describes the image for each monomial in R_n , we have the following proposition.

Proposition 2.2. *For each $n \geq 1$,*

$$R_n \sim R_n/J_n \otimes \text{Sym}_{R_n},$$

as an algebra. That is, R_n is free over Sym_{R_n} .

2.3 The ideal generated by the quasisymmetric invariants

Define an ideal of R_n generated by the quasisymmetric invariants as

$$I_n = \langle F_{1^r} \theta_1, \theta_2, \dots, \theta_n : 1 \leq r \leq n \rangle.$$

The *exterior quasisymmetric coinvariants*¹ are defined to be $EQC_n : R_n/I_n$. Similar to its commutative variant, the ideal I_n is not invariant under the action of π . So the quotient EQC_n is not closed under the action of π .

2.4 Differential operators on the exterior algebra

We can define a set of differential operators on R_n which will permit us to define the orthogonal complement to the ideal and a notion of quasisymmetric harmonics.

The operators ∂_{θ_i} act on monomials in R_n by

$$\partial_{\theta_i} \theta_A = \begin{cases} -1^{\#\{j \in A: j < i\}} \theta_{A \setminus \{i\}} & \text{if } i \in A \\ 0 & \text{if } i \notin A \end{cases}.$$

The operators can equally be characterized by the action that $\partial_{\theta_i} 1 = 0$ and the commutation relations

$$\begin{aligned} \partial_{\theta_i} \partial_{\theta_j} &= -\partial_{\theta_j} \partial_{\theta_i} \text{ if } 1 \leq i < j \leq n & \text{and} & \quad \partial_{\theta_i}^2 = 0 \text{ for } 1 \leq i \leq n \\ \partial_{\theta_i} \theta_j &= -\theta_j \partial_{\theta_i} \text{ if } 1 \leq i < j \leq n & \text{and} & \quad \partial_{\theta_i} \theta_i = 1 \text{ for } 1 \leq i \leq n. \end{aligned}$$

For a monomial $\theta_A = \theta_{a_1} \theta_{a_2} \cdots \theta_{a_r}$, let $\overline{\theta_A} = \theta_{a_r} \theta_{a_{r-1}} \cdots \theta_{a_1}$ represent reversing the order of the variables in the monomial. Extend this notation to both differential operators and polynomials (and polynomials of differential operators) by extending the notation linearly. We can define an inner product on R_n by setting for $p, q \in R_n$.

$$\langle p, q \rangle = \overline{p \partial_{\theta_1} \partial_{\theta_2} \cdots \partial_{\theta_n} q} \theta_1 \theta_2 \cdots \theta_n \mid_{\theta_1 = \theta_2 = \cdots = \theta_n = 0}.$$

The monomials of R_n form an orthonormal basis of the space with respect to this inner product.

Define the orthogonal complement to I_n with respect to the inner product as the set

$$EQH_n : \{q \in R_n : \langle p, q \rangle = 0 \text{ for all } p \in I_n\} \quad (2.3)$$

$$\{q \in R_n : p \partial_{\theta_1} \partial_{\theta_2} \cdots \partial_{\theta_n} q = 0 \text{ for all } p \in I_n\}. \quad (2.4)$$

The second equality follows from the fact that I_n is an ideal and shows that EQH_n is also the solution space of a system of differential equations. We refer to EQH_n as the *exterior quasisymmetric harmonics*.² The inner product is positive definite. It follows that, as graded vector spaces, $EQC_n \simeq EQH_n$ for all $n \geq 1$.

¹We borrow the name ‘coinvariant’ space even though the generators, and not the whole ideal, is invariant under the quasisymmetric operators.

²The harmonics and diagonal harmonics borrows the name from the physics literature because the harmonic operator $\partial_1^2 \partial_2^2 \cdots \partial_n^2$ is symmetric in the differential operators. In the case of the exterior algebra, this operator acts as zero and yet we persist by borrowing the name from the analogous spaces of commuting variables.

We will conclude this section by constructing a set of linearly independent elements inside EQH_n , which will give us a lower bound on the dimension of EQC_n . In [Section 4](#) we will see that this is also an upper bound, thus concluding that our set is in fact a basis. To compute EQH_n we need to solve the differential equations in [Equation \(2.4\)](#). Remark first that since I_n is an ideal, we do not need to take all $p \in I_n$, but it is enough to solve for the generators $p = F_{1r}$ for $1 \leq r \leq n$. We can reduce that further using [Proposition 2.1](#) as noted in the following lemma.

Lemma 2.3. *For $n \geq 2$ we have I_n is the ideal generated by F_1 and F_{12} .*

Proof. Clearly we have that the ideal generated by F_1, F_{12} is contained in I_n . For the converse we note that for each $k \geq 1$ there are non-zero coefficients a and a' such that

$$aF_{12k} = F_{12}^k \quad \text{and} \quad a'F_{12k-1} = F_{12}^{k-1}F_1$$

hence all of the generators of I_n are contained in the ideal generated by F_1, F_{12} . $\square$

From this we conclude that

$$EQH_n = \left\{ q \in R_n : \sum_{1 \leq i \leq n} \partial_{\theta_i} q = 0 \quad \text{and} \quad \sum_{1 \leq i < j \leq n} \partial_{\theta_j} \partial_{\theta_i} q = 0 \right\}. \quad (2.5)$$

Given $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$, a non-crossing pairing of length k is a list $C_1, C_2, \dots, C_k$ with

$$C_r = (i_r, j_r) \text{ for } 1 \leq i_r < j_r \leq n \text{ for each } 1 \leq r \leq k \text{ and,} \\ \text{either } i_r < j_r < i_s < j_s \text{ or } i_s < i_r < j_r < j_s \text{ for any } 1 \leq r < s \leq k.$$

Given a non-crossing pairing $C = C_1, C_2, \dots, C_k$, we define

$$\Delta_C = (\theta_{j_1} - \theta_{i_1})(\theta_{j_2} - \theta_{i_2}) \cdots (\theta_{j_k} - \theta_{i_k}). \quad (2.6)$$

Here $\Delta_C = 1$ if $k = 0$. Remark that $j_1 < j_2 < \cdots < j_k$. The following proposition shows that there is a relationship between the non-crossing partition condition and the differential equations from [Equation \(2.5\)](#).

Proposition 2.4. *The set*

$$\mathcal{D}'_n = \left\{ \Delta_C : C = C_1, C_2, \dots, C_k \text{ non-crossing pairing and } 0 \leq k \leq \lfloor \frac{n}{2} \rfloor \right\}$$

is contained in EQH_n .

The set $\mathcal{D}'_n$ is not linearly independent, for example for $n = 3$ and $k = 1$, we have the following three non-crossing pairing: $(1, 2)$, $(1, 3)$ and $(2, 3)$, but

$$\Delta_{(1,2)} - \Delta_{(1,3)} - \Delta_{(2,3)} = 0.$$

We want to select a linearly independent subset of $\mathcal{D}'_n$. We proceed as follows: consider a sequence $\alpha = a_1, a_2, \dots, a_n \in \{0, 1\}^n$ such that $\sum_{i=1}^r a_i \leq r/2$ for all $1 \leq r \leq n$. Such sequences are known as *ballot sequences*. If ever it is the case that $\sum_{i=1}^r a_i > r/2$ then we say that α *breaks the ballot condition* at position r .

Given a ballot sequence α we build a non-crossing pairing $C \alpha$ by first replacing all 0s by open parentheses $0 \mapsto '('$, and all 1s by close parentheses $1 \mapsto ')'$, and then do the natural maximal pairing of parenthesis. The positions of the pairings give us in lexicographic order a non-crossing pairing which we shall denote $C \alpha$. Since α is a ballot sequence, every closed parenthesis is matched and some open parentheses might remain unpaired. The natural pairing of parenthesis guarantees that the result will be non-crossing. For example,

$$\alpha = 0010001101 \quad \mapsto \quad C \alpha = 2, 3, 6, 7, 5, 8, 9, 10.$$

The total number of ballot sequences of size n is equal to the central binomial coefficient $\binom{n}{\lfloor n/2 \rfloor}$ (see [25, A001405]).

Given this construction we have the following Proposition.

Proposition 2.5. *The set*

$$\mathcal{D}_n = \{\Delta_{C \alpha} : \alpha \in \{0, 1\}^n \text{ a ballot sequence}\}$$

is contained in EQH_n and is linearly independent.

Proof. The first statement follows from Proposition 2.4 since $\mathcal{D}_n \subseteq \mathcal{D}'_n \subseteq EQH_n$. To show the linear independence, fix α a ballot sequence and let $C \alpha = i_1, j_1, \dots, i_k, j_k$ be its non-crossing pairing. We remark that the sequence of numbers $j_1 < j_2 < \dots < j_k$ corresponds to the position of the 1s in α . Using the monomial ordering described in Section 3 and by inspection of the product in Equation (2.6), we observe that the term $\theta_{j_1} \theta_{j_2} \dots \theta_{j_k}$ is the smallest lexicographic monomial in $\Delta_{C \alpha}$. For different ballot sequences α we get different positions of the 1s in α and thus different smallest lexicographic monomials, which shows the independence of $\mathcal{D}_n$. $\square$

Ballot sequences with k number of 1's are in bijection with standard tableaux of shape $n - k, k$. It is tempting to try describe $\mathcal{D}_n$ in terms of polynomials directly analogous to Specht polynomials in commuting variables. However, it can be shown that such a direct analogy does not work.

3 A linear basis of the ring

Again let n be a fixed positive integer and $R_n = \mathbb{Q}[\theta_1, \theta_2, \dots, \theta_n]$. We have thus far represented the basis for R_n as the elements θ_A with $A \subseteq [n]$. Define $\alpha_A \in \{0, 1\}^n$ to

be the sequence $a_1 a_2 a_3 \cdots a_n$ with $a_i = 1$ if $i \in A$ and $a_i = 0$ if $i \notin A$ so that

$$\theta_A = \theta_1^{a_1} \theta_2^{a_2} \cdots \theta_n^{a_n} : \theta^{\alpha_A}.$$

For such a sequence $\alpha \in \{0, 1\}^n$, let $m_1(\alpha) = \sum_{i=1}^n a_i$ represent the number of 1s in the string. This will also be the degree of the monomial θ^α .

For sequences $\alpha \in \{0, 1\}^n$, define elements G_α by

$$G_{1^s 0^{n-s}} = F_{1^s} \quad (3.1)$$

and if $\alpha \neq 1^s 0^{n-s}$, then α is of the form $u 0 1^s 0^{n-k-s}$ for some string u of length $k-1$ and we recursively define

$$G_{u 0 1^s 0^{n-k-s}} = G_{u 1^s 0^{n-k-s-1}} - (-1)^{m_1(u)} \theta_k G_{u 1^{s-1} 0^{n-k-s-2}}. \quad (3.2)$$

The challenge in arriving at this definition was finding the correct recursion which yields a set of polynomials $G_\alpha \in I_n$ whose leading term with respect to some monomial order is θ^α .

We will show below that the recurrence for the G_α is defined so that they are S -polynomials [6] for elements of the ideal I_n . In commutative variables, similar polynomials were defined by Aval-Bergeron-Bergeron [1, 2] as a (complete) subset of S -polynomials needed to compute all possible S -polynomials in the Buchberger algorithm for a Gröbner basis. It is not given that one can easily describe such a set of S -polynomials and here we have adapted the definition for working in the exterior algebra.

Example 3.1. For $\alpha = 010110$ and $\beta = 001100$, we compute the elements G_α and G_β using the definition.

$$\begin{aligned} G_{010110} &= G_{011100} - \theta_3 G_{011000} - G_{111000} - \theta_1 G_{110000} - \theta_3 G_{110000} - \theta_1 G_{100000} \\ &\quad - \theta_2 \theta_4 \theta_5 - \theta_2 \theta_4 \theta_6 - \theta_2 \theta_5 \theta_6 - 2\theta_3 \theta_4 \theta_5 - 2\theta_3 \theta_4 \theta_6 - 2\theta_3 \theta_5 \theta_6 - \theta_4 \theta_5 \theta_6 \end{aligned}$$

and we have that

$$\begin{aligned} G_{001100} &= G_{011000} - \theta_2 G_{010000} - G_{110000} - \theta_1 G_{100000} - \theta_2 G_{100000} - \theta_1 G_{000000} \\ &\quad - \theta_3 \theta_4 - \theta_3 \theta_5 - \theta_3 \theta_6 - \theta_4 \theta_5 - \theta_4 \theta_6 - \theta_5 \theta_6. \end{aligned}$$

Proposition 3.2. *The largest lexicographic term in G_α is θ^α .*

The proof of Proposition 3.2 follows by induction on the length of α and from a lemma that is analogous to Lemma 3.3 of [2]. The recursion in this result is really the origin of the definition of G_α because Equation (3.2) was adapted so that this lemma holds. It follows that the set $\{G_\alpha\}_{\alpha \in \{0,1\}^n}$ is a basis for R_n .

4 A basis for the quotient

The elements G_α are defined so that we could use them to identify a nice basis of the ideal I_n . This is our main theorem.

Theorem 4.1. *The set $A_n : \{G_\alpha : \alpha \in \{0,1\}^n \text{ breaks the ballot condition}\}$ is a $\mathbb{Q}$ -linear basis of the ideal I_n .*

The proof of this theorem uses our understanding of the harmonic space $EQH_n \sim EQC_n$. The challenge here was to show that $G_\alpha \in I_n$ if and only if α breaks the ballot condition. In Proposition 2.5 we found that $\dim EQH_n = \dim EQC_n$ and is at least the number of ballot sequences. We first establish a small lemma about a spanning set for the quotient EQC_n showing that the dimension is at most the number of ballot sequences. Therefore we have equality and the set $\mathcal{D}_n$ in Proposition 2.5 is in fact a basis of EQH_n .

There are several straightforward consequence of this theorem which we state here.

Corollary 4.2. *The Hilbert series of EQH_n is given by*

$$\text{Hilb}_{R_n/I_n} q = \sum_{k=0}^{\lfloor n/2 \rfloor} f^{n-k,k} q^k,$$

where $f^{n-k,k}$ counts the number of standard Young tableaux of shape $n-k, k$.

Corollary 4.3. *The set $\mathcal{D}_n$ is a basis of EQH_n and of EQC_n .*

Corollary 4.4. *The set A_n is a (non-reduced, non-minimal) Gröbner basis of I_n . A minimal Gröbner basis for I_n is given by*

$$\{G_\alpha : \alpha \in \{0,1\}^n \text{ breaks the ballot condition only at the rightmost 1 of } \alpha\}.$$

These results give a fairly complete description of the ideal I_n and the quotient R_n/I_n . To demonstrate the results above, we present the following example for a small value of n .

Example 4.5. Let $n = 4$, then $\dim R_4 = 16$. There are 6 ballot sequences, $\{0000, 0001, 0010, 0011, 0100, 0101\}$ and so R_4/I_4 is spanned by $\{1, \theta_2, \theta_3, \theta_4, \theta_3\theta_4, \theta_2\theta_4\}$. The ideal I_4 is spanned by the 10 elements,

$G_{0110} \quad \theta_2\theta_3 \quad \theta_2\theta_4 \quad \theta_3\theta_4$	$G_{0111} \quad \theta_2\theta_3\theta_4$
$G_{1000} \quad \theta_1 \quad \theta_2 \quad \theta_3 \quad \theta_4$	$G_{1001} \quad \theta_1\theta_4 \quad \theta_2\theta_3 \quad 2\theta_2\theta_4 \quad 2\theta_3\theta_4$
$G_{1010} \quad \theta_1\theta_3 \quad \theta_1\theta_4 \quad 2\theta_2\theta_3 \quad 2\theta_2\theta_4 \quad \theta_3\theta_4$	$G_{1011} \quad \theta_1\theta_3\theta_4 \quad \theta_2\theta_3\theta_4$
$G_{1100} \quad \theta_1\theta_2 \quad \theta_1\theta_3 \quad \theta_1\theta_4 \quad \theta_2\theta_3 \quad \theta_3\theta_4$	$G_{1101} \quad \theta_1\theta_2\theta_4 \quad 2\theta_1\theta_3\theta_4 \quad 2\theta_2\theta_3\theta_4$
$G_{1110} \quad \theta_1\theta_2\theta_3 \quad \theta_1\theta_2\theta_4 \quad \theta_1\theta_3\theta_4 \quad \theta_2\theta_3\theta_4$	$G_{1111} \quad \theta_1\theta_2\theta_3\theta_4$

A reduced Gröbner basis for I_4 consists of the elements $\{G_{0110}, G_{1000}\}$.

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Perm_n via QSym_n

Philippe Nadeau^{*1} and Vasu Tewari⁺²

¹Univ Lyon, Université Claude Bernard Lyon 1, CNRS UMR 5208, Institut Camille Jordan,
43 blvd. du 11 novembre 1918, F-69622 Villeurbanne cedex, France

²Department of Mathematics, University of Hawaii at Manoa, Honolulu, HI 96822, USA

Abstract. We give a combinatorial interpretation for the expansion of the cohomology class of the permutahedral variety Perm_n in Schubert classes. This requires understanding Schubert polynomials modulo the ideal QSym_n of positive degree quasisymmetric polynomials. We introduce a new basis for the polynomial ring that we call forest polynomials, together with a bijective correspondence. Both constructions are of independent interest.

Résumé. Nous donnons une interprétation combinatoire au développement de la classe de cohomologie de la variété permutaédrale Perm_n sur les classes de Schubert. Pour cela, il faut comprendre la réduction des polynômes de Schubert modulo l'idéal QSym_n des polynômes quasi-symétriques de degré positif. Nous introduisons une nouvelle base pour l'anneau de polynômes que nous appelons polynômes forestiers, ainsi qu'une correspondance bijective, toutes deux ayant un intérêt pour elles-mêmes.

Keywords: permutahedral variety, quasisymmetric polynomials, binary trees, linear extensions.

1 Introduction

Schubert calculus is the study of certain enumerative problems in algebraic geometry. These can be often phrased as finding intersection numbers of subvarieties subject to certain conditions. Littlewood–Richardson numbers arise for instance as generic triple intersection numbers of Schubert classes in the Grassmannian, and are known to count various families of objects. The same intersection problem in the flag variety has been the study of intensive research for at least the past forty years.

Here we study a particular problem in the flag variety, namely the intersection of the so-called permutahedral (sub)variety Perm_n with a generic Schubert variety X^w . If $w \in \mathcal{S}_n$ has length $n - 1$, the cardinality $a_w = |\text{Perm}_n \cap X^w|$ of intersection points is a well-defined nonnegative integer. Several properties of these numbers were obtained by the authors in [8], as well as their evaluation in special cases, all of which point to rich combinatorics.

^{*}nadeau@math.univ-lyon1.fr. P.N. was supported by the project ANR19-CE48-011-01 (COMBINÉ).

⁺vtewari@math.hawaii.edu. V.T. was supported by Simons Grant #855592.

In this work we complete these results with a combinatorial interpretation for a_w . Recall that $i_1 i_2 \dots i_k$ is a *reduced word* for a permutation w if $w = s_{i_1} s_{i_2} \dots s_{i_k}$ where s_i is the transposition $i \leftrightarrow i+1$, and k is minimal, given by the length $\ell(w)$. We denote the set of reduced words for w by $\text{Red } w$. We also need a particular parking procedure in order to state our main result.

WF-parking procedure: Consider parking spots indexed by $\mathbb{Z}$, initially all empty. Cars $1, 2, \dots$ arrive successively, with car i having preferred spot v_i , and want to park at (empty) spots. Assume inductively that $i-1$ cars have already parked. If spot v_i is empty, then car i parks there. Otherwise, $v_i \in [a, b]$ an interval of occupied spots with spots $a-1$ and $b+1$ being free.¹ Define v_j to be the preferred spot of the car that parked last in $[a, b]$; that is, $j < i$ is maximal such that $v_j \in [a, b]$. Then, car i parks in $b+1$ if $v_i \geq v_j$, while it parks in $a-1$ if $v_i < v_j$.

After k cars have parked they occupy a k -subset $I = \{v_1, \dots, v_k\} \subset \mathbb{Z}$. A preference word $v_1 \dots v_k$ is called a *WF-parking word* if $I = \{1, \dots, k\}$.

Theorem 1. *If $w \in S_n$ has length $n-1$, then a_w is the number of reduced words of w^{-1} that are also WF-parking words.*

Example 2. Consider $w = 21543 \in S_5$ with $\ell(w) = 4$. To compute a_w , we need to compute WF-parking words in $\text{Red } w^{-1} = \text{Red } w$. These are given explicitly in (5.1), and only the first four of them are WF-parking words. It follows $a_{21543} = 4$.

We will explain the main steps of the proof leading to this result, and stress in particular the rôle played by the *forest polynomials* that we introduce in Section 3. Let us first recall the geometrical context and previous results.

We will be brief on the geometry and refer to [8] for more information. The *complete flag variety* Fl_n is the set of complete flags $V_\bullet = V_0 \subseteq V_1 \subseteq V_2 \subseteq \dots \subseteq V_n$ where each V_i is a linear subspace of $\mathbb{C}^n$ of dimension i . It is a smooth projective variety of dimension $\binom{n}{2}$ over $\mathbb{C}$. Inside Fl_n we have first and foremost the *Schubert varieties* X^w indexed by $w \in S_n$, whose cohomology classes $\sigma_w : X^w$ form a linear basis of the cohomology ring $H^*(\text{Fl}_n, \mathbb{Q})$. In this work we consider the *permutahedral variety* Perm_n . It is an ubiquitous toric variety, encoded by the normal fan of the standard permutahedron, and occurs in Fl_n as a generic torus orbit or a regular semisimple Hessenberg variety.

Now by standard cohomology theory, the intersection numbers a_w defined above are known to be given by the coefficients in the expansion of the class $[\text{Perm}_n]$ in terms of Schubert classes. Using a result of Anderson–Tymoczko [1], the authors were led in [8] to an algebraic expression for a_w that we now recall.

Henceforth, set $\mathbb{Q}[x_n] = \mathbb{Q}[x_1, \dots, x_n]$. Let *divided symmetrization* (DS), introduced under that name by Postnikov [10, Section 3], be the linear form that takes a homogeneous

¹Recall that in the classical parking algorithm, car i simply parks in spot $b+1$.

polynomial $f(x_1, \dots, x_n) \in \mathbb{Q}[x_n]$ of degree $n - 1$ as input and outputs $\langle f \rangle_n \in \mathbb{Q}$ as:

$$\langle f \rangle_n : \sum_{w \in \mathbb{S}_n} w \cdot \left(\frac{f(x_1, \dots, x_n)}{\prod_{1 \leq i \leq n-1} (x_i - x_{i+1})} \right). \quad (1.1)$$

For $w \in \mathbb{S}_n$ with length $n - 1$, we have [8, Theorem 3.2.1]

$$a_w = \langle \mathfrak{S}_w \rangle_n. \quad (1.2)$$

Here $\mathfrak{S}_w$ is a Schubert polynomial, see (2.1) for a definition. Before we explain how to go from the expression (1.2) to Theorem 1, note that the authors had previously obtained results on a_w using the algebra of $\mathbb{S}_n$ -invariants of the cohomology ring $H^*(\text{Perm}_n, \mathbb{Q})$. This was based on work of Klyachko [6] in a crucial way. The main result was an explicit formula for a_w as a positive *rational* linear combination of *mixed Eulerian numbers* [8, Theorem 5.1.2]. The formula is reminiscent of Macdonald's famous reduced word identity and this aspect is explored in [7].

A rational expression, even if positive, is unsatisfying since we know $a_w \in \mathbb{N}$. Theorem 1 thus provides a pleasant answer in that respect. Surprisingly the road to the interpretation therein goes through the quotient $\mathbb{Q}[x_n] / \text{QSym}_n$, first studied by Aval–Bergeron–Bergeron [4]. We quickly introduce the relevant notions (see Section 2 for QSym_n). Let $\mathcal{C}_n$ denote the set of weak compositions $c_1, \dots, c_n$ such that for all $1 \leq j \leq n$ we have $c_j + \dots + c_n \leq n - j$. We call its elements *ABB compositions*. We refer to the monomials $\mathbf{x}^c : x_1^{c_1} \cdots x_n^{c_n}$ where $c = c_1, \dots, c_n \in \mathcal{C}_n$ as *ABB monomials*. Aval–Bergeron–Bergeron showed the direct sum decomposition:

$$\mathbb{Q}[x_n] = \mathbb{Q}\{\mathbf{x}^c \mid c \in \mathcal{C}_n\} \oplus \text{QSym}_n, \quad (1.3)$$

from which they inferred that $\dim \mathbb{Q}[x_n] / \text{QSym}_n = \text{Cat}_n : \frac{1}{n+1} \binom{2n}{n}$. The following result connects the ABB decomposition with DS.

Theorem 3 ([9, Theorem 1.3]). *Express a degree $n - 1$ homogeneous polynomial f as $f = g + h$ where $g \in \mathbb{Q}\{\mathbf{x}^c \mid c \in \mathcal{C}_n\}$ and $h \in \text{QSym}_n$. Then $\langle f \rangle_n = g(1, 1, \dots, 1)$.*

Given Formula (1.2), we are led to consider the decomposition of $\mathfrak{S}_w$ as in Theorem 3. The corresponding polynomial g will turn out to be a $\mathbb{N}$ -combination of ABB monomials, thus showing $a_w \in \mathbb{N}$.

We define an $\mathbb{N}$ -vector to be a sequence of nonnegative integers c_i $i \in \mathbb{Z}_{\geq 1}$ where all but finitely many c_i are 0. The *support* of an $\mathbb{N}$ -vector c is the set of indices $i \in \mathbb{Z}_{>0}$ such that $c_i > 0$. We denote the set of $\mathbb{N}$ -vectors by Codes. Occasionally we truncate an $\mathbb{N}$ -vector to a finite sequence $c_1, \dots, c_n$ for some positive integer n . It is then understood that $c_i = 0$ for all $i > n$.

We propose a family of polynomials $\{\mathfrak{p}_c\}_{c \in \text{Codes}}$ which we call *forest polynomials*; the reason will become clear in the sequel. They turn out to enjoy several nice combinatorial

properties, and form a $\mathbb{Z}$ -basis of the space of all polynomials. Our interest in them stems from the following theorem; its first part relies on the *Word-Forest Correspondence* that we introduce in [Section 4](#).

Theorem 4. *The following hold.*

- (i) *Every Schubert polynomial is a nonnegative integral combination of forest polynomials.*
- (ii) *Let us restrict our attention to forest polynomials that only use variables x_1 through x_n . We then have $\mathfrak{p}_c$ modulo $\text{QSym}_n / 0$ if and only if $c \in \mathcal{C}_n$. In the latter case, $\mathfrak{p}_c$ is a nonnegative integral combination of ABB monomials.*

These two results, given in [Sections 3](#) and [4](#) respectively, will help give a manifestly nonnegative integral expression for a_w that can eventually be interpreted as in [Theorem 1](#); see [Section 5](#). We close by mentioning some other applications in a final section.

2 Combinatorial preliminaries

Quasisymmetric polynomials. A strong composition $\alpha : \alpha_1, \dots, \alpha_k$ is a sequence of positive integers. We use $\ell \alpha : k$ to denote the number of *parts* in α . A polynomial $f \in \mathbb{Q} \mathbf{x}_n$ is *quasisymmetric* if for every strong composition α the coefficient of $x_1^{\alpha_1} \cdots x_k^{\alpha_k}$ equals that of $x_{i_1}^{\alpha_1} \cdots x_{i_k}^{\alpha_k}$ for every $1 \leq i_1 < \cdots < i_k \leq n$. Clearly every polynomial symmetric in x_1 through x_n is also quasisymmetric. A linear basis for the ring of quasisymmetric polynomials in x_1 through x_n is given by the *fundamental quasisymmetric polynomials* $L_\alpha x_1, \dots, x_n$ with $\ell \alpha \leq n$. We are interested in the ideal QSym_n generated by positive degree quasisymmetric polynomials in $\mathbb{Q} \mathbf{x}_n$. Equivalently, it is the ideal generated by all $L_\alpha x_1, \dots, x_n$ for $\ell \alpha > 0$.

Schubert polynomials. Given $\mathbf{i} = i_1 \dots i_k \in \mathbb{Z}^*$, let $\text{Comp } \mathbf{i}$ denote the set of sequences $a_1 \geq \cdots \geq a_k$ satisfying $1 \leq a_j \leq i_j$ for all $j = 1, \dots, k$ and $a_j > a_{j-1}$ if $i_j > i_{j-1}$ for $j = 1, \dots, k-1$. Define $\mathfrak{f} \mathbf{i}$ by

$$\mathfrak{f} \mathbf{i} = \sum_{a_1, \dots, a_k \in \text{Comp } \mathbf{i}} x_{a_1} \cdots x_{a_k}.$$

If the sum on the right is empty, then $\mathfrak{f} \mathbf{i} = 0$. Otherwise $\mathfrak{f} \mathbf{i}$ can always be written as a *slide polynomial* $\mathfrak{f}_c$ [\[3\]](#).² As an example, $\mathfrak{f} 423 = x_2^2 x_4 + x_2^2 x_3 + x_1 x_2 x_4 + x_1 x_2 x_3 + x_1^2 x_4 + x_1^2 x_3 + x_1^2 x_2$, which by the convention in [\[3\]](#) is $\mathfrak{f}_{0,2,0,1}$.

Now given $w \in \mathbb{S}_n$, let $\text{Red } w$ denote the set of *reduced words* of w . We then have the following celebrated description of $\mathfrak{S}_w$ due to Billey–Jockusch–Stanley [\[5\]](#):

$$\mathfrak{S}_w = \sum_{\mathbf{i} \in \text{Red } w^{-1}} \mathfrak{f} \mathbf{i}. \quad (2.1)$$

²Assaf–Searles index slides by $\mathbb{N}$ -vectors, and the latter can be read off from the exponent vector of the revlex leading monomial.

For example consider $w = 14253 \in S_5$. Then $\text{Red } w^{-1} = \{243, 423\}$. We just computed $\mathfrak{F} 423$ above, and $\mathfrak{F} 243 = x_1 x_2^2 = \mathfrak{F} 1,2$. Thus $\mathfrak{S}_{14253} \mathfrak{F} 243 = \mathfrak{F} 423$ is given by

$$\mathfrak{S}_{14253} = x_1 x_2^2 + x_1^2 x_2 + x_1^2 x_3 + x_1 x_2 x_3 + x_2^2 x_3 + x_1^2 x_4 + x_1 x_2 x_4 + x_2^2 x_4. \quad (2.2)$$

Indexed forests. An *indexed forest* F is a set of complete binary trees, each with leaves labeled from left to right by an interval in $\mathbb{Z}_{>0}$. These intervals must moreover partition $\mathbb{Z}_{>0}$, and all but a finite number of them are singletons, i.e. the corresponding tree is a leaf. See Figure 1 (ignoring red labels for now), where we identify leaves with $\mathbb{Z}_{>0}$.

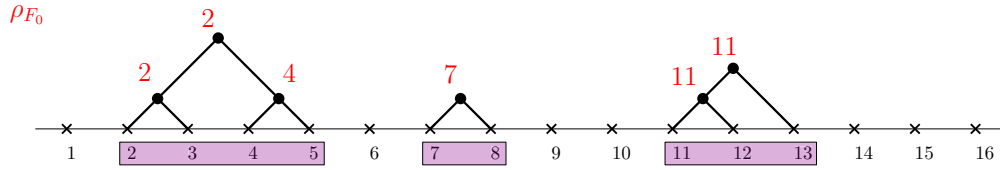


Figure 1: An indexed forest F_0 with size 6

Let $\text{IN } F$ be the set of internal nodes of F , and let $|F|$ be its size. We call a node $v \in \text{IN } F$ *terminal* if both its children are leaves. Let $\text{Term } F$ denote the set of terminal nodes in F . The *support* $\text{Supp } F$ of F is the set of labels of leaves attached to nontrivial trees. The *left support* $\text{LSupp } F$ of F is the set of labels of left leaves attached to nontrivial trees. The indexed forest in Figure 1 has size 6, $\text{Supp } F = \{2, 3, 4, 5, 7, 8, 11, 12, 13\}$, and $\text{LSupp } F = \{2, 4, 7, 11\}$. Furthermore, it has 4 terminal nodes.

Define ρ_F on $\text{IN } F$ by $\rho_F v$: the leaf label reached by following left edges down from v ; see red labels in Figure 1. Define c_i to be the number of nodes labeled i by ρ_F to get a vector $c_F = c_1, c_2, \dots \in \text{Codes}$.

Lemma 5. $F \mapsto c_F$ is a bijection between indexed forests and Codes.

For F in Figure 1 we get $c_F = 0, 2, 0, 1, 0, 0, 1, 0, 0, 0, 2, 0, 0, 0, 0, \dots$.

Words. Given any alphabet $\mathcal{A}$, we denote the set of words in $\mathcal{A}$ by $\mathcal{A}^*$. We denote the *length* of any word $w \in \mathcal{A}^*$ by ℓw . We let ϵ denote the *empty word*. The set of *injective words* in $\mathcal{A}^*$ consists of words comprising solely distinct letters. We denote this set by $\text{Inj } \mathcal{A}$. Our ordered alphabet of interest is obtained by “augmenting” $\mathbb{Z}$: we define the alphabet $\overline{\mathbb{Z}}$ of letters i^j where $i \in \mathbb{Z}$ and $j \in \mathbb{Z}_{>0}$. These letters extend the linear order on $\mathbb{Z}$ by $i < i^1 < i^2 < i^3 < \dots < i - 1$ for all i .

3 Forest polynomials

We are ready to introduce forest polynomials. Given the bijection $F \mapsto c_F$, we can index them by either $\mathbb{N}$ -vectors or by indexed forests. We will do this without prior warning, omitting parentheses, commas, and trailing zeros in writing c .

Definition 6. The forest polynomial $\mathfrak{p}_F \in \mathbb{Z}[x_1, x_2, \dots]$ is defined as

$$\mathfrak{p}_F = \sum_{\kappa} \prod_{v \in \text{IN } F} x_{\kappa(v)}$$

where the sum is over all labelings $\kappa : \text{IN } F \rightarrow \mathbb{Z}_{>0}$ that are bounded above by ρ_F , weakly increasing down left edges and strictly increasing down right edges.

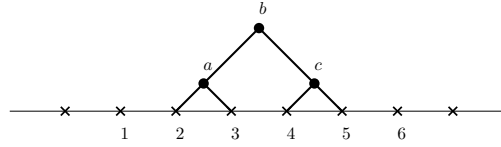


Figure 2: An indexed tree T_0 with $c(T_0) = 0, 2, 0, 1, 0, \dots$

Example 7. Consider the indexed tree T_0 in Figure 2. Then

$$\mathfrak{p}_{T_0} = \sum_{\substack{2 \geq a \geq b \\ 4 \geq c \geq b}} x_a x_b x_c = x_2^2 x_4 + x_1 x_2 x_4 + x_1^2 x_4 + x_2^2 x_3 + x_1 x_2 x_3 + x_1^2 x_3 + x_1^2 x_2 + x_1 x_2^2. \quad (3.1)$$

which coincides with the Schubert polynomial $\mathfrak{S}_{14253}$, cf. (2.2). Note that in general $\mathfrak{p}_F$ is the product of $\mathfrak{p}_T$ over all indexed trees T composing F . Thus for F_0 in Figure 1, we get $\mathfrak{p}_{F_0} = \sum_{i=1}^7 x_i \sum_{1 \leq i \leq j \leq 11} x_i x_j$.

Example 8. (Fundamental quasisymmetric polynomials L_α are forest polynomials) Let F be an indexed forest with $\text{LSupp}(F) = [a, b]$ for some $a \leq b$. A moment's thought should convince the reader that such an F must in fact be a “linear tree”. In fact this condition is equivalent to $\text{Supp}(c(F))$ being an interval in $\mathbb{Z}_{>0}$. Figure 3 shows two instances. The one on the left gives $\mathfrak{p}_F = L_{121}(x_1, \dots, x_4)$ and that on the right $\mathfrak{p}_F = L_2(x_1, x_2)$. In general, let $c(F) = c_1, c_2, \dots$. Then $\mathfrak{p}_F = L_\alpha(x_1, \dots, x_b)$ with $\alpha = c_b, \dots, c_a$.

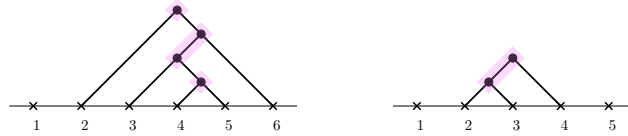


Figure 3: Two indexed forests with left support an interval

The next proposition is a consequence of the observation that the revlex leading term in $\mathfrak{p}_F$ is x^c . It is interesting to compare it with the fact that Schubert polynomials indexed by permutations with last descent in position at most n (respectively, in S_n) form a basis of $\mathbb{Z}[x_n]$ (respectively, of $\mathbb{Q}\{x^c \mid c_i \leq n - i\}$).

Proposition 9. *Given $n \in \mathbb{Z}_{>0}$, the set $\{\mathfrak{p}_F \mid \text{LSupp } F \subseteq n\}$ is a basis of $\mathbb{Z} \mathbf{x}_n$, while the set $\{\mathfrak{p}_F \mid \text{Supp } F \subseteq n\}$ is a basis of $\mathbb{Q}\{\mathbf{x}^c \mid c \in \mathcal{C}_n\}$.*

Definition 6 essentially describes forest polynomials as a generating function for certain *flagged* P, ω, ρ -partitions in the sense of Assaf–Bergeron [2]. Let $\text{Dec } F$ denote the set of *decreasing forests* with shape F , that is bijections $\text{IN } F \rightarrow \{1, \dots, |F|\}$ decreasing down branches. For a fixed F , one can identify $L \in \text{Dec } F$ with a word $\rho_F v_1 \cdots \rho_F v_{|F|}$ where v_i is the node with label $L v_i = i$. The next expansion is a special case of results in *loc. cit.*

$$\mathfrak{p}_F = \sum_{L \in \text{Dec } F} \mathfrak{F}(L). \quad (3.2)$$

Thus any forest polynomial is a sum of slide polynomials. For T_0 in [Example 7](#) and [Figure 2](#), the expansion in (3.2) specializes to $\mathfrak{F}(242) + \mathfrak{F}(422) + \mathfrak{F}(1,2) + \mathfrak{F}(0,2,0,1)$, which as expected coincides with the one defining $\mathfrak{S}_{14253} = \mathfrak{p}_{T_0}$.

Reduction modulo QSym_n. Henceforth, given $f \in \mathbb{Q} \mathbf{x}_n$, we take $f \bmod \text{QSym}_n$ to mean the unique g satisfying $f = g + h$ with $h \in \text{QSym}_n$ and $g \in \mathbb{Q}\{\mathbf{x}^c \mid c \in \mathcal{C}_n\}$. Our aim is to give a straightforward description for $\mathfrak{p}_F \bmod \text{QSym}_n$; see [Theorem 12](#). Despite the simplicity of the statement, the proof is technical (and omitted in this abstract). It relies on the recurrence in [Lemma 10](#).

We discuss two operations on indexed forests that will allow us to describe this recurrence. The first is the *shift* map τ which, given an indexed forest F , simply shifts $\text{Supp } F$ one unit to the right. We abuse notation and define $\tau^i \mathfrak{p}_F = \mathfrak{p}_{\tau^i F}$ for $i \in \mathbb{Z}$, making note that if $\text{Supp } \tau^i F \not\subseteq \mathbb{Z}_{>0}$, then $\mathfrak{p}_{\tau^i F}$ equals 0.

The second operation is *trimming*, which corresponds to removing a terminal node. Explicitly, consider $v \in \text{Term } F$. It lies in one of the indexed trees T composing F , say with support a, b . We obtain $\text{trim}_v F$ from F by replacing T with $T' = T \setminus \{v\}$ with new support $a, b - 1$, and leaving the other trees unaltered. If $a = b - 1$, then T' has no internal nodes (i.e it is trivial), and we omit it.

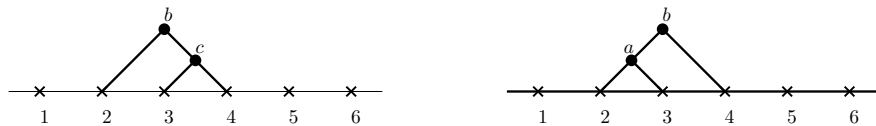


Figure 4: The two indexed forests resulting from trimming T_0 in [Figure 2](#)

Lemma 10. *Given an indexed forest F we have*

$$\mathfrak{p}_F = \mathfrak{p}_{\tau^{-1} F} + \sum_{v \in \text{Term } F} x_{\rho_F v} \mathfrak{p}_{\text{trim}_v F}. \quad (3.3)$$

One can rewrite this recurrence by indexing forest polynomials by $\mathbb{N}$ -vectors. For the tree T_0 in Figure 2, using Lemma 10 we get: $p_{0201} = p_{201} x_2 p_{011} x_4 p_{02}$. Now, $p_{201} = x_1^2 x_2 = x_1^2 x_3$, $p_{011} = e_2 x_1, x_2, x_3$, and $p_{02} = h_2 x_1, x_2$, and substituting these gives the same expansion as in (3.1).

Lemma 10 has several consequences; we record two that concern us most. First, we have a nice interpretation for the sum of coefficients of p_F in a special case.

Proposition 11. *If T is an indexed tree with support $\{1, \dots, |T|\}$, we have*

$$p_T = 1, \dots, 1 \quad |\text{Dec } T|. \quad (3.4)$$

One can extend this to a result for all p_F but we will not need it here. The second consequence of Lemma 10 is Theorem 4(ii) which we reformulate here.

Theorem 12. *Let F be an indexed forest satisfying $\text{LSupp } F \subseteq n$. Then*

$$p_F \bmod \text{QSym}_n = \begin{cases} 0 & \text{if } \text{Supp } F \not\subseteq n \text{ equivalently, } c(F) \notin \mathcal{C}_n \\ p_F & \text{if } \text{Supp } F \subseteq n \text{ equivalently, } c(F) \in \mathcal{C}_n. \end{cases}$$

As an example, note that Theorem 12 says that the forest polynomial p_{0201} in (3.1) is in QSym_4 as $\text{Supp } T_0 \subseteq 4$. We invite the reader to check that

$$p_{0201} = L_{2,1} x_1, \dots, x_4 = L_{1,2} x_1, \dots, x_4 - x_3 x_4 L_{1,1} x_1, \dots, x_4$$

and thus clearly belongs in QSym_4 . On the other hand, $\tau^{-1} T_0$ has support exactly 4 , and $p_{2010} = x_1^2 x_2 = x_1^2 x_3$; each monomial here is an ABB monomial.

4 Word-Forest correspondence

We now proceed to describe the main insertion algorithm in this work, the *Word-Forest correspondence*. We will need labeled versions of indexed forests. A *labeling* of F is a map from $\text{IN } F$ to some alphabet $\mathcal{A}$. Furthermore, we allow our indexed forests to be supported on $\mathbb{Z}$ and not just $\mathbb{Z}_{>0}$. When we consider forest polynomials subsequently, only those that are supported on $\mathbb{Z}_{>0}$ shall matter.

The input is an injective word $W \in \text{Inj } \overline{\mathbb{Z}}$, while the output is a pair $\mathcal{WF} W = P, Q$ of labeled indexed forests where the underlying unlabeled forests are the same. Following conventions, we call P (respectively Q) the *insertion forest* (respectively *recording forest*). We define $\mathcal{WF} W$ inductively via an insertion procedure.

A convenient bookkeeping device while executing our algorithm is the *root list* $\text{rl } W$ for W . It is a totally ordered subset of $\mathbb{Z} \sqcup \overline{\mathbb{Z}}$ that records the list of root labels in P : trivial trees are encoded by their index in $\mathbb{Z}$, other root labels are certain letters in W .

The WF-correspondence. Let $W \in \text{Inj } \overline{\mathbb{Z}}$. If W is empty, P and Q are trivial forests. We thus set $\text{rl } \epsilon = \mathbb{Z}$. Now suppose $W = W'b$ and $\mathcal{WF} W' = P', Q'$ by induction. Then $\mathcal{WF} W$ is determined as follows.

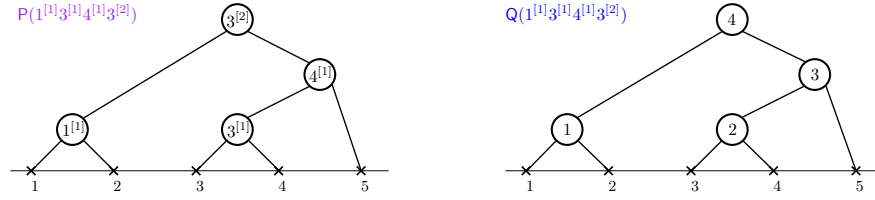


Figure 5: The insertion and recording forest for $W = 1^1 3^1 4^1 3^2$

1. Let a (respectively c) be the greatest (respectively least) elements in $\text{rl } W'$ such that $a < b < c$.³
2. Merge the two trees in P' with roots labeled by a and c into a single tree by adding a new root with label b as their parent. This gives us $P \cdot W$. The new root list is then $\text{rl } W = \text{rl } W' \setminus \{a, c\} \cup \{b\}$.
3. $Q \cdot W$ records this merging procedure by performing a corresponding merge in Q' and having $|W|$ as the new root label.

See Figure 5 for the P and Q that result when one applies the preceding procedure. Note that internal nodes in P are labeled by letters in W whereas those in Q are labeled by $\{1, \dots, |W|\}$. One can say more— P is a *local binary search forest*, i.e. the label of any node is strictly greater (lesser) than the label of its left (respectively right) child, while Q is a *decreasing forest* as already defined.

Fix a finite set $B \subseteq \overline{\mathbb{Z}}$. Denote by S_B the set of all permutations of B (in one-line notation). Clearly $|S_B| = |B|!$ and $S_B \subset \text{Inj } \overline{\mathbb{Z}}$. Denote by $\mathcal{PF}(B)$ the set of all ordered pairs P, Q where P, Q have the same underlying complete indexed forest and additionally

- internal nodes in P are bijectively labeled by letters in B so that the result is a local binary search forest;
- internal nodes in Q are bijectively labeled by letters in $\{1, \dots, |B|\}$ so that the result is a decreasing forest.

We are ready for our main procedure that governs the combinatorics.

Theorem 13 (Word-Forest correspondence). *The map $\mathcal{WF} : S_B \rightarrow \mathcal{PF}(B)$ is a bijection.*

Proof (sketch). The forward direction produces an element in $\mathcal{PF}(B)$ as discussed earlier. The inverse correspondence simply consists in reading the labels of P according to the order determined by Q , and this is injective as well. The claim follows. $\square$

The WF-correspondence specializes to the “Sylvester correspondence” when B has the form $\{1^1, 2^1, \dots, n^1\}$, but does not reduce to it in general. Our main use for this correspondence will be to group together all words that have the same labeled forest P .

The WF-equivalence. We say that words $W_1, W_2 \in \text{Inj } \overline{\mathbb{Z}}$ are *WF-equivalent* if $P \cdot W_1 = P \cdot W_2$. We denote this by $W_1 \stackrel{\text{WF}}{=} W_2$. For instance the reader can check that $3^1 1^1 4^1 3^2$

³Recall that our inputs are injective words

and $1^1 3^1 4^1 3^2$ produce the same P (shown in Figure 5), and thus $3^1 1^1 4^1 3^2 \equiv_{WF} 1^1 3^1 4^1 3^2$. It is helpful to compare the Q symbol for these two words as well. Observe that two *incomparable* nodes swap their labels. We can actually give a ‘local’ description to WF-equivalence which generates $\equiv_{WF}$ as its transitive closure. We say that $W_1 \sim_{WF} W_2$ if we can write $W_1 = UabV$, $W_2 = UbaV$ and there are at least two elements c, d of in the rootlist $rl(U)$ such that

$$\min(a, b) < c, d < \max(a, b). \quad (4.1)$$

Note that this entails that W_1 and W_2 are in the same *commutation class*: if we denote $a = i^{\cdot}, b = j^{\cdot}$, then necessarily $|j - i| \geq 2$.

Lemma 14. $\equiv_{WF}$ is the equivalence relation generated by $\sim_{WF}$.

5 a_w by expanding Schuberts into forests

We now discuss the decomposition of Schubert polynomials into forest polynomials. To this end we will need to apply the WF-correspondence to reduced words. We thus need to transform words in $\mathbb{Z}^*$ as elements in $\text{Inj } \overline{\mathbb{Z}}$.

For $w \in \mathbb{Z}^*$, define the word $\text{stan } w$ in $\overline{\mathbb{Z}}^*$ by labeling the occurrences of the letter i in w from left to right by $i^1, i^2, \dots$, for any $i \in \mathbb{Z}$. For instance $\text{stan } 1221625 = 1^1 2^1 2^2 1^2 6^1 2^3 5^1$. It is clear $w \mapsto \text{stan } w$ embeds $\mathbb{Z}^* \hookrightarrow \text{Inj } \overline{\mathbb{Z}}$. We will implicitly apply constructions of the previous section to words via these standardized versions.

Definition 15. Given a WF-equivalence class C , we define its *shape* $\text{sh } C$ to be the underlying unlabeled indexed forest F of $P(W)$ for any element in $W \in C$. We define $\text{Supp } C$ to be equal to $\text{Supp } F$.

Recall that $\text{Red } w$ is the set of reduced words for a permutation w . For instance, if $w = 21543$ in one-line notation, then

$$\text{Red } w = \{1343, 3143, 3413, 3431, 1434, 4134, 4314, 4341\}. \quad (5.1)$$

Those with the same color form an equivalence class under $\equiv_{WF}$. The corresponding shapes from left to right are shown in Figure 6. In particular we note that $\text{Red } 21543$ is closed under $\equiv_{WF}$. This is in fact true for any set $\text{Red } w$: more generally, commutation classes are always closed under $\equiv_{WF}$ by Lemma 14 and the remark before it.

We are ready to make Theorem 4(i) explicit.

Theorem 16. Fix a permutation w . Let $\text{Red } w^{-1} / \equiv_{WF}$ denote the set of WF-equivalence classes decomposing $\text{Red } w^{-1}$, and let C_1 through C_k be those satisfying $\text{Supp } C_i \subset \mathbb{Z}_{>0}$. Then

$$\mathfrak{S}_w = \sum_{1 \leq i \leq k} p_{\text{sh } C_i}. \quad (5.2)$$

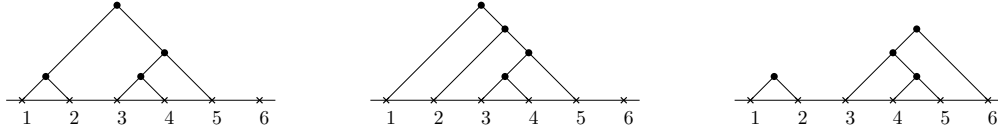


Figure 6: Shapes of equivalence classes for Red 21543

Proof (sketch). By definition of $\mathfrak{S}_w$, it equals the sum over $C \in \text{Red } w^{-1} / \equiv_{\text{WF}}$ of the polynomials $\mathfrak{p}_C : \sum_{W \in C} \mathfrak{F}(W)$. It is easy to show that all terms $\mathfrak{F}(W)$ vanish in $\mathfrak{p}_C$ when $\text{Supp } C \not\subseteq \mathbb{Z}_{>0}$. Then a key lemma is to show that one always has $\mathfrak{p}_C = \mathfrak{p}_{\text{sh } C}$, which we omit in this abstract. $\square$

We now return to our motivating question—describing a_w combinatorially via [Theorem 1](#). We call a WF-equivalence class C a *tree class* if $\text{sh } C$ is an indexed tree T supported on an initial interval n for some n . Going back to [Figure 6](#), we see that $\{1343, 3143, 3413\}$ and $\{3431\}$ are tree classes, whereas $\{1434, 4134, 4314, 4341\}$ is not.

Proof of Theorem 1. [Theorem 16](#) gives us the expansion for $\mathfrak{S}_w$ into forest polynomials. Here w is a permutation in S_n of length $n - 1$, which implies that any forest polynomial $\mathfrak{p}_F$ that appears in (5.2) necessarily satisfies $|F| = n - 1$ and $\text{LSupp } F \subseteq n$. It follows from [Theorem 12](#) that only forest polynomials indexed by trees with support n survive when reducing modulo QSym_n . We thus obtain

$$\mathfrak{S}_w \bmod \text{QSym}_n = \sum_C \mathfrak{p}_{\text{sh } C},$$

where C runs through tree classes in $\text{Red } w^{-1} / \equiv_{\text{WF}}$. Now by [Theorem 3](#) we know that

$$a_w = \sum_C \mathfrak{p}_{\text{sh } C} = 1, \dots, 1.$$

[Proposition 11](#) says that each summand $\mathfrak{p}_{\text{sh } C} = 1, \dots, 1$ equals $|C|$, so we have that a_w is the number of words in $\text{Red } w^{-1}$ belonging to a tree class.

To conclude, one needs to compare the recursive descriptions of the WF-parking procedure and the WF-correspondence. A careful analysis shows that the former is but a “shadow” of the latter, and the set $I = v_1 \cdots v_k$ at the end of the WF-parking procedure is exactly the common support of $P = v_1 \cdots v_k$ and $Q = v_1 \cdots v_k$.

In particular a word in $\mathbb{Z}^*$ belongs to a tree class if and only if it is a WF-parking word, which completes the proof. $\square$

6 Other applications

We have defined several notions in this work in order to solve a particular problem, but these notions are of independent interest. We close with a non-exhaustive list of some

notable properties.

- The strategy developed to determine a_w can be used to compute the DS of other polynomials. It can for instance give automatically a combinatorial interpretation for the remixed Eulerian numbers introduced in [7].
- Forest polynomials enjoy a positive multiplication rule which can be determined combinatorially via the WF-correspondence.
- The WF-correspondence and WF-parking deserve further study. For instance, while the combinatorial interpretation in [Theorem 1](#) implies several of the results from [8], it is not obvious to the authors why $a_w = a_{w^{-1}}$, though this is indeed known to hold [*loc. cit.*, Proposition 5.1.1].

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Jack characters as generating series of bipartite maps and proof of Lassalle's conjecture

Houcine Ben Dali^{*1} and Maciej Dołęga^{†2}

¹Université de Lorraine, CNRS, IECL, F-54000 Nancy; Université de Paris, CNRS, IRIF, F-75006 Paris, France

²Institute of Mathematics, Polish Academy of Sciences, ul. Śniadeckich 8, 00-956 Warszawa, Poland

Abstract. We give an explicit formula for the power-sum expansion of Jack polynomials. We deduce it from a more general formula that we provide here, that interprets Jack characters in terms of bipartite maps. Finally, we prove Lassalle's conjecture from 2008 on integrality and positivity of Jack characters in Stanley's coordinates.

Keywords: Jack polynomials, Jack characters, maps, Lassalle's conjecture, Kerov polynomials

1 Introduction

1.1 Jack polynomials and Jack characters

Jack polynomials J_λ^α are symmetric functions, indexed by an integer partition λ and a deformation parameter α . They interpolate, up to scaling factors, between Schur functions for $\alpha = 1$ and zonal polynomials for $\alpha = 2$. Stanley initiated in his seminal work [18] the combinatorial analysis of these symmetric functions with connection with various objects of algebraic combinatorics, such as partitions, tableaux, paths and maps [14, 7, 9, 3, 15, 2].

Knop and Sahi have given in [9] a combinatorial interpretation for the coefficients of the Jack polynomial J_λ^α in the monomial basis in terms of tableaux of shape λ . Existence of a combinatorial expression for the expansion of the Jack polynomial J_λ^α in the power-sum basis was conjectured by Hanlon [8]. In this paper, we give such a combinatorial interpretation in terms of bipartite maps.

Roughly speaking, a map is a graph drawn on a locally orientable surface (see [Section 2.3](#) for a precise definition). The study of maps is a well developed area with strong connections with analytic combinatorics, mathematical physics and probability [11]. The

^{*}houcine.ben-dali@univ-lorraine.fr.

[†]mdolega@impan.pl. MD is supported by Narodowe Centrum Nauki, grant 2021/42/E/ST1/00162.

relationship between generating series of maps and the theory of symmetric functions was first noticed via a character theoretic approach, and has then been developed to include other techniques such as matrix integrals and differential equations [7, 10, 4, 2].

In fact, the aforementioned combinatorial interpretation follows from a more general result that we prove: a combinatorial interpretation for Jack characters. For a partition μ , the Jack character θ_μ^α is the function on Young diagrams defined by:

$$\theta_\mu^\alpha : \lambda \mapsto \begin{cases} 0 & \text{if } |\lambda| < |\mu|, \\ \frac{|\lambda| - |\mu|}{m_1 \mu} p_{\mu, 1^{n-m}} J_\lambda^\alpha & \text{if } |\lambda| \geq |\mu|. \end{cases}$$

where p_μ is the extraction symbol with respect to the variable $\mathbf{p}$ and $m_1 \mu$ is the number of parts equal to 1 in the partition μ .

Jack characters play an important role in studying asymptotic behaviour of random Young diagrams [3]. When $\alpha = 1$, they correspond to the normalized irreducible characters of the symmetric group, and have a combinatorial interpretation in terms of permutation factorizations which has been conjectured by Stanley and proved by Féray (see [5] for the details). An analogous combinatorial expression has also been established in [6] for Zonal characters, which correspond to θ_μ^α with $\alpha = 2$. In this paper, we give a combinatorial interpretation for θ_μ^α for general α in terms of maps. The expression we obtain interpolates between the expressions given in [5] and [6] for $\alpha = 1$ and $\alpha = 2$.

1.2 Multi-layer maps

Definition 1.1. Fix $k \geq 0$. We say that a bipartite map M is a k -layer map if its black vertices (resp. white vertices) are partitioned into k sets (which may be empty), called the layers of the map; $\mathcal{V}_\circ M = \bigcup_{1 \leq i \leq k} \mathcal{V}_\circ^i M$ (resp. $\mathcal{V}_\bullet M = \bigcup_{1 \leq i \leq k} \mathcal{V}_\bullet^i M$), which satisfy the following condition: if v is a white vertex in a layer i , then all its neighbors are in layers $j \leq i$, and it has at least one neighbor in the layer i .

For $1 \leq i \leq k$, we define the partition $\mathcal{V}_\bullet^i M$ obtained by reordering the degree distribution of the black vertices in the layer i . A k -layer map is labelled if:

- in each layer $1 \leq i \leq k$, the black vertices having the same degree j are numbered by $1, 2, \dots, m_j(\mathcal{V}_\bullet^i M)$, where $m_j(\mathcal{V}_\bullet^i M)$ denotes the number of parts of size j in $\mathcal{V}_\bullet^i M$.
- each black vertex has a distinguished oriented corner.

An example of a 2-layer map is given in the left part of [Figure 1](#).

Note that a k -layer map can be seen as $k - 1$ -layer map with an empty layer $k - 1$. We call then a *multi-layer map* a k -layer map for some $k \geq 1$. We denote by $\mathcal{M}^k$ (resp.

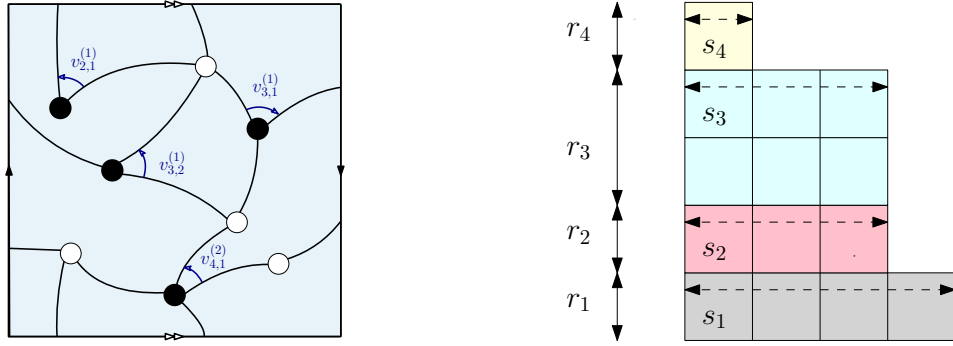


Figure 1: *Left:* A 2-layer map on the Klein bottle, represented here by a square whose left side should be glued to the right one (with a twist) and the top side should be glued to the bottom one (without a twist), as indicated by the arrows. $v_{j,k}^{(i)}$ denotes the black vertex of degree j numbered by k in the layer i . *Right:* The Young diagram of the partition $4, 3, 3, 1$ as the union of 4 rectangles, with $\mathbf{s} = 4, 3, 3, 1$ and $\mathbf{r} = 1, 1, 2, 1$ as multirectangular coordinates.

$\mathcal{M}^\infty$) the set of all labelled k -layer maps (resp. labelled multi-layer maps). Similarly, we denote $\mathcal{M}_\mu^k$ and $\mathcal{M}_\mu^\infty$ those of face-type μ (see Section 2.3 for the definition). The following definition has been introduced by Goulden and Jackson.

Definition 1.2 ([7]). *A statistic of non-orientability on bipartite maps is a statistic ϑ with non-negative integer values, such that $\vartheta(M) = 0$ if and only if M is orientable.*

In practice, a statistic of non-orientability is supposed to "measure" the non-orientability of a map by counting the number of edges which contribute to its non-orientability, following a given algorithm of decomposition of the map. Several examples of such statistics have been introduced in previous works [10, 4, 2].

1.3 First main result

Theorem 1.3. *There exists a statistic of non-orientability ϑ on multi-layer maps such that for any partitions μ and λ , we have*

$$\theta_\mu^{\alpha, \lambda} = (-1)^{|\mu|} \sum_{M \in \mathcal{M}_\mu^\infty} \frac{b^{\vartheta(M)}}{2^{|\mathcal{V}_\bullet(M)| - cc(M)} \alpha^{cc(M)}} \prod_{i \geq 1} \frac{-\alpha \lambda_i^{|\mathcal{V}_\circ^i(M)|}}{z_{\mathbf{v}_\bullet^i(M)}^i}, \quad (1.1)$$

where b is the parameter related to α by $b = \alpha - 1$, and $z_{\mathbf{v}_\bullet^i(M)}^i$ is the classical normalization factor (see Sections 2.1 and 2.3 for the precise definitions and notation).

Note that the product here is finite since each multi-layer map has a finite number of non empty layers. We give in [Section 4](#) the key ingredients of the proof of [Theorem 1.3](#). As a direct consequence of [Theorem 1.3](#), we obtain the following interpretation of Jack polynomials in the basis of power-sum functions.

Theorem 1.4. *Let n be a positive integer and let λ be a partition of n . Then*

$$J_{\lambda}^{\alpha} = -1^n \sum_M p_{v_{\diamond} M} \frac{b^{\vartheta M}}{2^{|\mathcal{V}_{\bullet} M|} (-cc M) \alpha^{cc M}} \prod_{1 \leq i \leq \ell_{\lambda}} \frac{-\alpha \lambda_i^{|\mathcal{V}_{\circ}^i M|}}{z_{v_{\bullet}^i M}^{\alpha}},$$

where the sum is taken over all ℓ_{λ} -layer maps M of size n , and $v_{\diamond} M$ denotes the face-type of M (see [Section 2.3](#)).

1.4 Lassalle's conjecture and second main result.

Stanley has used [\[19\]](#) the following definition of multirectangular coordinates.

Definition 1.5 ([\[19\]](#)). *Let $k \geq 1$ and let $s_1 \geq s_2 \geq \dots \geq s_k \geq 1$ and $r_1, \dots, r_k$ be two sequences of nonnegative integers. We say that $s_1, \dots, s_k$ and $r_1, \dots, r_k$ are multirectangular coordinates for a partition λ and we denote $\lambda = \mathbf{s}^{\mathbf{r}}$, if λ is the union of k rectangles of sizes $s_i \times r_i$, or equivalently $\lambda = s_1^{r_1} \dots s_k^{r_k}$, see the right part of [Figure 1](#) for an example.*

Since we do not require that the sequence $\mathbf{s}$ should be strictly decreasing, the multirectangular coordinates are not unique in general. If λ is a partition of multirectangular coordinates $s_1, \dots, s_k$ and $r_1, \dots, r_k$, we write for any partition μ

$$\tilde{\theta}_{\mu}^{\alpha}(\mathbf{s}, \mathbf{r}) : \theta_{\mu}^{\alpha}(\lambda),$$

where $\mathbf{r} = r_1, \dots, r_k, 0, \dots$ and $\mathbf{s} = s_1, \dots, s_k, 0, \dots$. In the case $b = 0$, which corresponds to the irreducible characters of the symmetric group, Stanley found [\[19\]](#) an explicit formula for $\tilde{\theta}_{\mu}^{\alpha}(\mathbf{s}, \mathbf{r})$ when λ is a rectangle, and he conjectured a formula for general $\mathbf{r}, \mathbf{s}$ that was soon after proved by Féray [\[5\]](#). An analogous formula was found in the case $b = 1$ [\[6\]](#). A conjecture that suggests positivity and integrality of Jack characters in multirectangular coordinates for arbitrary b was stated by Lassalle [\[12\]](#), and it remained unproven for nearly 15 years¹. Our second main result is the proof of this conjecture.

Theorem 1.6. *The normalized Jack characters $-1^{|\mu|} z_{\mu} \tilde{\theta}_{\mu}^{\alpha}$ are polynomials in the variables $b, -s_1, -s_2, \dots, r_1, r_2, \dots$ with non-negative integer coefficients, where $b = \alpha - 1$.*

¹It has been proved in the particular case $\mathbf{s} = s, 0, \dots$ and $\mathbf{r} = r, 0, \dots$ in [\[1\]](#).

Interestingly, the proof of [Theorem 1.6](#) consists of two parts that are proved using very different techniques. In the first part we deduce positivity as a consequence of the combinatorial expression of Jack characters obtained in [Theorem 1.3](#) (see [Section 5](#)). In the second part, we obtain the integrality using integrable system of Nazarov–Sklyanin [\[16\]](#). We relate their theory with Jack characters by proving an explicit combinatorial formula expressing certain basis of shifted symmetric functions in terms of normalized Jack characters. We conclude by showing that the transition matrix between these two bases is invertible over $\mathbb{Z}$. Consequently, we prove that Kerov polynomials for Jack characters have integer coefficients, which was an open problem posed by Lassalle in [\[13\]](#) (see [Section 5.2](#) for details).

2 Notation

For the definitions and notation introduced in [Sections 2.1](#) and [2.2](#) we refer to [\[18, 14\]](#).

2.1 Partitions

A *partition* $\lambda = \lambda_1, \dots, \lambda_\ell$ is a weakly decreasing sequence of positive integers $\lambda_1 \geq \dots \geq \lambda_\ell > 0$. We denote by $\mathbb{Y}$ the set of all integer partitions. The integer ℓ is called the *length* of λ and is denoted $\ell(\lambda)$. The size of λ is the integer $|\lambda| = \lambda_1 + \lambda_2 + \dots + \lambda_\ell$. If n is the size of λ , we say that λ is a partition of n and we write $\lambda \vdash n$. The integers $\lambda_1, \dots, \lambda_\ell$ are called the *parts* of λ . For $i \geq 1$, we denote $m_i(\lambda)$ the number of parts of size i in λ . We set then

$$z_\lambda = \prod_{i \geq 1} m_i(\lambda) i^{m_i(\lambda)}.$$

We denote by $\leq$ the *dominance partial* ordering on partitions, defined by

$$\mu \leq \lambda \iff |\mu| \leq |\lambda| \text{ and } \mu_1 + \dots + \mu_i \leq \lambda_1 + \dots + \lambda_i \text{ for } i \geq 1.$$

We identify a partition λ with its *Young diagram*, defined by

$$\lambda = \{ (i, j), 1 \leq i \leq \ell(\lambda), 1 \leq j \leq \lambda_i \}.$$

2.2 Symmetric functions and Jack polynomials

We fix an alphabet $\mathbf{x} = x_1, x_2, \dots$. We denote by $\mathcal{S}$ the algebra of symmetric functions in $\mathbf{x}$ with coefficients in $\mathbb{Q}$. For every partition λ , we denote m_λ the monomial function and p_λ the power-sum function associated with the partition λ . We consider the associated alphabet of power-sum functions $\mathbf{p} = p_1, p_2, \dots$.

Let $\mathcal{S}_\alpha$ be the algebra of symmetric functions with coefficients in $\mathbb{Q}[\alpha]$. We denote by $\langle \cdot, \cdot \rangle_\alpha$ the α -deformation of the Hall scalar product defined by

$$\langle p_\lambda, p_\mu \rangle_\alpha = z_\lambda \alpha^{\ell(\lambda)} \delta_{\lambda, \mu}, \text{ for any partitions } \lambda, \mu.$$

Macdonald [14, Chapter VI.10] has proved that there exists a unique family of symmetric functions J_λ^α $\lambda \in \mathbb{Y}$ in $\mathcal{S}_\alpha$ indexed by partitions, satisfying the following properties:

$$\begin{cases} \text{Orthogonality:} & \langle J_\lambda, J_\mu \rangle_\alpha = 0, \text{ for } \lambda \neq \mu, \\ \text{Triangularity:} & m_\mu J_\lambda = 0, \text{ unless } \mu \leq \lambda, \\ \text{Normalization:} & m_{1^n} J_\lambda = n!, \text{ for } \lambda \vdash n, \end{cases}$$

where $m_\mu J_\lambda$ denotes the coefficient of m_μ in J_λ , and 1^n is the partition with n parts equal to 1. These functions are known as the *Jack polynomials*.

2.3 Maps

A *connected map* is a connected graph embedded into a surface such that all the connected components of the complement of the graph are simply connected (see [11, Definition 1.3.7]). These connected components are called the *faces* of the map. We consider maps up to homeomorphisms of the surface. A connected map is *orientable* if the underlying surface is orientable. In this paper², a *map* is an unordered collection of connected maps. A map is orientable if each one of its connected components is orientable. Finally, the *size* of a map is its number of edges.

In this paper, all the maps considered are *bipartite*; i.e their vertices are colored in two colors, white and black, such that each edge connects two vertices of different colors. Note that in a bipartite map, all faces have even degree. We define then the *face-type* of a bipartite map M of size n , as the partition of n obtained by reordering the half degrees of the faces and we denote it $\nu_\diamond M$. We also denote its set of white and black vertices by $\mathcal{V}_\circ M$ and $\mathcal{V}_\bullet M$ respectively.

3 The combinatorial model

The purpose of this subsection is to define a family of statistic of non-orientability (see Definition 1.2) on multi-layer maps. We start by some general definitions related to non-orientable maps.

Let M be a bipartite map and let c_1 and c_2 be two corners of M of different colors. Then we have two ways to add an edge to M between these two corners (see Figure 2). We denote by e_1 and e_2 these edges. We say that the pair (e_1, e_2) is a *pair of twisted edges*

²This is not the standard definition of a map; usually a map is connected.

on the map M and we say that e_2 is obtained by twisting e_1 . Note that if M is connected and orientable, then exactly one of the maps $M \cup \{e_1\}$ and $M \cup \{e_2\}$ is orientable. For a given map with a distinguished edge M, e , we denote $\tilde{M}, \tilde{e}$ the map obtained by twisting the edge e .

We recall that b is the parameter related to the Jack parameter α by $b = \alpha - 1$. We now give the definition of a measure of non-orientability due to La Croix.

Definition 3.1. [10, Definition 4.1] We call a measure of non-orientability (MON) a function ρ defined on the set of connected maps M, e with a distinguished edge, with values in $\{1, b\}$, satisfying the following conditions:

- if e connects two corners of the same face of $M \setminus \{e\}$, and the number of the faces increases by 1 by adding the edge e on the map $M \setminus \{e\}$, then $\rho(M, e) = 1$. In this case we say that e is a diagonal.
- if e connects two corners in the same face $M \setminus \{e\}$, and the number of the faces of $M \setminus \{e\}$ is equal to the number of faces of M , then $\rho(M, e) = b$. In this case we say that e is a twist.
- if e connects two corners of two different faces lying in the same connected component of $M \setminus \{e\}$, then ρ satisfies $\rho(M, e) = \rho(\tilde{M}, \tilde{e}) + 1 - b$. In this case we say that e is a handle. Moreover, if M is orientable then $\rho(M, e) = 1$.
- if e connects two faces lying in two different connected components, then $\rho(M, e) = 1$. In this case, we say that e is a bridge.

Let M be a bipartite map and let $e_1, e_2, \dots, e_d$ be d distinct edges of M . For $1 \leq i \leq d$, we denote M_i be the map obtained by deleting the edges $e_1, e_2, \dots, e_i$ from M . We define $\rho(M, e_1, e_2, \dots, e_d)$ as the weight obtained by deleting the edges e_j successively:

$$\rho(M, e_1, e_2, \dots, e_d) : \prod_{1 \leq j \leq d} \rho(M_j, e_j).$$

Given a MON ρ , one can associate a b -weight $\rho(M)$ to a multi-layer map M by choosing an order on the edges of M . We start by defining an order on black vertices.

We fix an integer $k \geq 1$ and a labelled k -layer map M . To each black vertex v of M we associate the triplet of positive integers (i, n, j) , where i is the layer containing the vertex, n is its degree, and j is the number given to v by the labelling of the map. By definition, the triplets associated with two distinct black vertices are different. We define then a linear order $\prec_M$ on the black vertices of M given by the lexicographic order on the triplets (i, n, j) . In other terms, the maximal black vertex with respect to $\prec_M$ is the vertex contained in the layer of smallest index, and having maximal degree and maximal label. Note that when we delete the maximal vertex from a k -layer map and all the edges incident to it, the map obtained is also a k -layer map.

We now introduce a statistic of non-orientability on k -layer labelled maps which will have a key role in this paper.

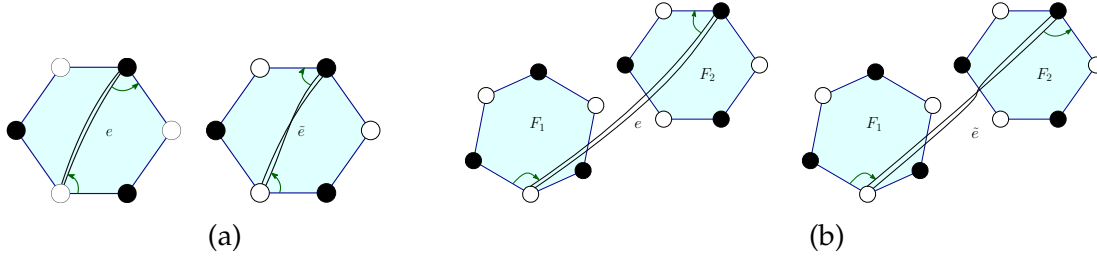


Figure 2: The different ways of adding an edge on a map. The added edge is represented each time by a band, that can be twisted at most once. The arrows indicate how the added edge connects the two respective corners. **Figure 2a:** a pair of twisted edges $e, \tilde{e}$ between two corners of the same face; e is a diagonal while $\tilde{e}$ is a twist. **Figure 2b:** A pair of twisted edges $e, \tilde{e}$ between two corners of different faces F_1 and F_2 . If F_1 and F_2 are on the same connected component then the edges e and $\tilde{e}$ are handles, otherwise they are bridges.

Definition 3.2 (Statistic of non-orientability on k -layer maps). *Let ρ be a MON and let M be a connected labelled k -layer map of size n . We define the b -weight ρM of M , as the weight obtained by decomposing M recursively as following. If M is the empty map then $\rho M = 1$. Otherwise, let v be the maximal black vertex of M with respect to $\prec_M$, and let c be its root and d its degree. We denote by $e_1, \dots, e_d$ the edges incident to v as they appear when we turn around v starting from the root c . Let M' be the map obtained from M by deleting v and all the edges incident to it. Then, by definition $\rho M = \rho M, e_1, \dots, e_d \cdot \rho M'$. We extend this definition to disconnected maps by multiplicativity; if M is a labelled k -layer map with m connected components $M_1, \dots, M_m$, then $\rho M = \prod_{1 \leq i \leq m} \rho M_i$. Finally, we define the statistic ϑ_ρ on labelled multi-layer maps with values in non-negative integers, defined for every M by $\rho M = b^{\vartheta_\rho M}$.*

It follows from the definitions that ϑ_ρ is a statistic of non-orientability over labelled bipartite maps. We prove that [Theorem 1.3](#) holds for every statistic of the form ϑ_ρ .

4 Idea of the proof of [Theorem 1.3](#)

The starting point of the proof of [Theorem 1.3](#) is a characterization of Jack characters due to Féray (see [Section 4.1](#)).

4.1 Féray's characterization of Jack characters

Roughly, an α -shifted symmetric function $f_{s_1, s_2, \dots}$ (a shifted function for short) is a function which is symmetric in the variables $s_i - i/\alpha$. If f is a shifted function and

λ a partition, then we denote $f^\lambda : f^{\lambda_1, \lambda_2, \dots, \lambda_{\ell(\lambda)}, 0, \dots}$. A shifted function is completely determined by its evaluation on Young diagrams $f^\lambda_{\lambda \in \mathbb{Y}}$, which gives an identification between shifted functions and functions on Young diagrams. In particular, Jack characters defined as functions on Young diagrams in [Section 1.1](#) can be seen as shifted functions. The starting point of the proof of the main theorem is the following characterization of the Jack characters θ_μ^α which is due to Valentin Féray (see [\[17, Theorem A.2\]](#)).

Theorem 4.1. *Fix a partition μ . The Jack character θ_μ^α is the unique α -shifted symmetric function of degree $|\mu|$ with top homogeneous part $\alpha^{|\mu|-\ell(\mu)} / z_\mu \cdot p_\mu$, such that $\theta_\mu^\alpha \lambda = 0$ for any partition $|\lambda| < |\mu|$.*

4.2 The generating series of k -layer maps

Fix $k \geq 1$. We define the generating series of k -layer maps:

$$F^k(t, \mathbf{p}, s_1, \dots, s_k) : \sum_{M \in \mathcal{M}^k} \frac{-t^{|M|} p_{v_\diamond M} b^{\theta_\rho M}}{2^{|\mathcal{V}_\bullet M| - cc M} \alpha^{cc M}} \prod_{1 \leq i \leq k} \frac{-\alpha s_i^{|\mathcal{V}_\circ^i M|}}{z_{v_\bullet^i M}^{|\mathcal{V}_\circ^i M|}}. \quad (4.1)$$

We also consider the projective limit $F^\infty : \varprojlim F^k$.

Note that the normalizing factors which appear in this definition are related to the definition of labelling we use here³. This labelling allows to deal with maps with trivial automorphism groups and also plays an important role in the definition of statistic of non-orientability [Section 3](#).

To obtain [Theorem 1.3](#), we prove that $t^{|\mu|} p_\mu F^\infty_{\lambda_1, \dots, \lambda_{\ell(\lambda)}, 0, \dots}$ satisfies the conditions of [Theorem 4.1](#). To this purpose we express the functions F^k using the differential operators introduced in [\[2\]](#).

5 Lassalle's conjecture

5.1 Positivity - ideas

We consider two sequences of variables $\mathbf{s} = s_1, s_2, \dots$ and $\mathbf{r} = r_1, r_2, \dots$. We introduce a bivariate version of the generating series of k -layer maps

$$\tilde{F}^\infty \left(t, \mathbf{p}, \begin{matrix} s_1 & s_2 & \dots \\ r_1 & r_2 & \dots \end{matrix} \right) = \sum_{M \in \mathcal{M}^\infty} \frac{-t^{|M|} p_{v_\diamond M} b^{\theta_\rho M}}{2^{|\mathcal{V}_\bullet M| - cc M} \alpha^{cc M}} \prod_{i \geq 1} \frac{r_i^{|\mathcal{V}_\bullet^i M|} - \alpha s_i^{|\mathcal{V}_\circ^i M|}}{z_{v_\bullet^i M}^{|\mathcal{V}_\circ^i M|}}. \quad (5.1)$$

³the factor $2^{|\mathcal{V}_\bullet M| - cc M}$ can be omitted by using a finer labelling

Hence the functions F^∞ defined in Equation (4.1) are obtained as a specialization of $\tilde{F}^k$:

$$F^\infty(t, \mathbf{p}, s_1, s_2, \dots) = \tilde{F}^\infty\left(t, \mathbf{p}, \begin{smallmatrix} s_1 & s_2 & \dots \\ 1 & 1 & \dots \end{smallmatrix}\right).$$

Proposition 5.1. *Let λ be a partition, and let $\begin{pmatrix} s_1 & \dots & s_k \\ r_1 & \dots & r_k \end{pmatrix}$ be multirectangular coordinates of λ . Then*

$$\tilde{F}^\infty\left(t, \mathbf{p}, \begin{smallmatrix} s_1 & \dots & s_k & 0 & \dots \\ r_1 & \dots & r_k & 0 & \dots \end{smallmatrix}\right) = F^\infty(t, \mathbf{p}, \lambda_1, \dots, \lambda_{\ell(\lambda)}, 0, \dots).$$

As a consequence of Theorem 1.3 and Proposition 5.1, we obtain that for any partition μ

$$\tilde{\theta}_\mu^\alpha(\mathbf{s}, \mathbf{r}) = t^{|\mu|} p_\mu \tilde{F}^\infty\left(t, \mathbf{p}, \begin{smallmatrix} s_1 & \dots & s_k & 0 & \dots \\ r_1 & \dots & r_k & 0 & \dots \end{smallmatrix}\right), \quad (5.2)$$

where $\mathbf{s} = s_1, \dots, s_k, 0, \dots$ and $\mathbf{r} = r_1, \dots, r_k, 0, \dots$. We now prove the positivity in Lassalle's conjecture.

Proof of the positivity in Theorem 1.6. Comparing Equation (5.1) and Equation (5.2) and taking the limit on k we get that for every MON ρ

$$\tilde{\theta}_\mu^\alpha(\mathbf{s}, \mathbf{r}) = -1^{|\mu|} \sum_{M \in \mathcal{M}_\mu^\infty} \frac{b^{\vartheta_\rho(M)}}{2^{|\mathcal{V}_\bullet(M)| - cc(M)} \alpha^{cc(M)}} \prod_{i \geq 1} \frac{r_i^{|\mathcal{V}_\bullet^i(M)|} - \alpha s_i^{|\mathcal{V}_\circ^i(M)|}}{z_{\mathbf{v}_\bullet^i(M)}^{|\mathcal{V}_\bullet^i(M)|}}. \quad (5.3)$$

Since every connected component of a bipartite map component contains at least one white vertex, the α -term which appears in the denominator is compensated. This concludes the proof of the positivity. $\square$

5.2 Integrality - ideas

Let $\text{Ch}_\mu^\alpha : \frac{z_\mu}{\sqrt{\alpha}^{|\mu| - \ell(\mu)}} \tilde{\theta}_\mu^\alpha$. For a positive integer k define free cumulant R_{k-1}^α as

$$R_{k-1}^\alpha : 2k \sum_{\substack{T \in \mathcal{M}_k^\infty \\ |\mathcal{V}_\bullet(T)| = |\mathcal{V}_\circ(T)| = k-1}} \prod_{i \geq 1} \left(\frac{r_i}{2\sqrt{\alpha}} \right)^{|\mathcal{V}_\bullet^i(T)|} \frac{-\sqrt{\alpha} s_i^{|\mathcal{V}_\circ^i(T)|}}{z_{\mathbf{v}_\bullet^i(T)}^{|\mathcal{V}_\bullet^i(T)|}}.$$

The constraints for the summation index says that the sum runs over all bicolored trees with extra labelings and rootings. In particular, the normalization factors $2^{|\mathcal{V}_\bullet^i(T)|} |z_{\mathbf{v}_\bullet^i(T)}^{|\mathcal{V}_\bullet^i(T)|}$ will be compensated by the number of labelings and rootings and in effect it can be

shown that the free cumulants are polynomials in $-\sqrt{\alpha}s_1, -\sqrt{\alpha}s_2, \dots, \frac{r_1}{\sqrt{\alpha}}, \frac{r_2}{\sqrt{\alpha}}, \dots$ with integer coefficients, see [4].

We prove the following theorem that is a weaker version of another conjecture of Lassalle about the structure of Kerov polynomials for Jack characters [13].

Theorem 5.2. *For every partition μ there exists a polynomial $K_\mu \in \mathbb{Z}[x_1, x_2, x_3, x_4, \dots]$ such that $\text{Ch}_\mu^\alpha = K_\mu(\sqrt{\alpha}^{-1} - \sqrt{\alpha}, R_2^\alpha, R_3^\alpha, \dots)$. In particular $z_\mu \tilde{\theta}_\mu^\alpha$ is a polynomial in $b, -s_1, -s_2, \dots, r_1, r_2, \dots$ with integer coefficients.*

The polynomial K_μ is called *Kerov polynomial*, and the last sentence in the above theorem follows from the bound on the degree of K_μ and its parity proved in [3][Proposition 3.10].

We deduce Theorem 5.2 from an explicit combinatorial formula for the moments expressed as a linear combination of Jack characters. A classical relation between moments and free cumulants reads:

$$M_k^\alpha = \sum_{\pi \in \text{NC}(1..k)} \prod_{B \in \pi} R_{|B|}^\alpha, \quad (5.4)$$

where we sum over all non-crossing partitions of the set $1..k$. For any partition μ define $\mu \pm 1 : \mu_1 \pm 1, \dots, \mu_\ell \pm 1$ and $M_\mu^\alpha : \prod_{i=1}^\ell M_{\mu_i}^\alpha$. We prove the following formula:

$$M_{\mu-1}^\alpha = \text{Ch}_\mu^\alpha \sum_{\nu < \mu-1} a_{\nu-1}^\mu \text{Ch}_{\nu-1}^\alpha + \sum_{i=1}^{|\mu|-\ell} \sum_{\nu: |\nu|=|\mu|-1-i} \gamma^i a_\nu^\mu \text{Ch}_\nu^\alpha, \quad (5.5)$$

where a_ν^μ are positive integers with an explicit combinatorial interpretation as numbers of some lattice paths (that we won't describe in details due to lack of space). In particular Equation (5.5) defines a unitriangular system of equations over $\mathbb{Z}$, therefore it can be inverted and then Theorem 5.2 follows applying (5.4). The proof of Equation (5.5) relies on the analysis of the integrable system developed by Nazarov–Sklyanin in [16].

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Limit shapes for skew Howe duality

Dan Betea^{*1}, Anton Nazarov⁺², and Travis Scrimshaw^{‡3}

¹*CNRS, LAREMA, Université d'Angers*

²*Department of High Energy and Elementary Particle Physics, St. Petersburg State University*

³*Department of Mathematics, Hokkaido University*

Abstract. We study large random partitions boxed into a rectangle and coming from skew Howe duality, or alternatively from dual Schur measures. As the sides of the rectangle go to infinity, we obtain: 1) limit shape results for the profiles generalizing the Vershik–Kerov–Logan–Shepp curve; and 2) universal edge asymptotic results for the first parts in the form of the Tracy–Widom distribution, as well as less-universal critical regime results introduced by Gravner, Tracy and Widom. We do this for a large class of Schur parameters going beyond the Plancherel or principal specializations previously studied in the literature, parametrized by two real valued functions f and g . Connections to a Bernoulli model of (last passage) percolation are explored.

Keywords: random partition, limit shape, skew Howe duality, Tracy–Widom distribution, q -Krawtchouk ensemble

1 Introduction

Motivation. The study of random partitions has been, for the past 50 years, a fruitful area of research at the interface of algebraic combinatorics, representation theory, probability, and mathematical physics. Two cornerstone results are the limit shape result of Vershik–Kerov–Logan–Shepp and the Baik–Deift–Johansson Theorem providing random matrix-like asymptotic fluctuations for the longest increasing subsequence of random permutations. See [10, Ch. 1 and 2] and references therein.

In this extended abstract we generalize both of the above by looking at a class of measures called dual Schur measures with arbitrary parameters. Such measures fall into the category of Schur measures first introduced by Okounkov [9], and we use his techniques along the way. The novelty here is twofold. On one hand, we derive our asymptotic results keeping the two parameter sets generic and settle on a class indexed by two real functions. On the other hand, because we are using dual Schur measures, the partitions are forced to be contained within a certain rectangle, and we see this introduces special edge behavior in the so-called critical scaling regime case (discovered

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[†]antonnaz@gmail.com. Partially supported by RSF grant No. 21-11-00141.

[‡]tcscrim@gmail.com. Partially supported by Grant-in-Aid for JSPS Fellows 21F51028.

by Gravner, Tracy, and Widom in [2]), while allowing for universal fluctuations of the Tracy–Widom GUE type away from this regime. As our parameters are reasonably generic, this can be seen as a form of universality.

Combinatorial description. The Cauchy identity is a classical identity in algebraic combinatorics that has a representation-theoretic interpretation as characters computed in two different ways on the $\mathrm{GL}_n \times \mathrm{GL}_k$ representation $\mathrm{Sym} \mathbb{C}^n \boxtimes \mathbb{C}^k$. Howe duality [4] is the statement that this has a multiplicity free decomposition indexed by a single partition (rather than a pair). Using this, we can form a probability measure on partitions, a form of Okounkov’s Schur measure [9], coming from two positive specialization of Schur polynomials $s_\lambda X$ (see [11, Ch. 7] for more on them). The Schur measure and the associated combinatorics have been well-studied from various perspectives and with a broad range of applications (e.g., [1, 9]). In particular, it is a quintessential example of a determinantal point process, where the joint correlation kernel is described by a determinant of pairwise correlations.

There is another $\mathrm{GL}_n \times \mathrm{GL}_k$ representation $\wedge \mathbb{C}^n \boxtimes \mathbb{C}^k$ that has a multiplicity free irreducible decomposition [4]. This is known as *skew Howe duality*. Taking characters yields the classical dual Cauchy identity, which can be described by applying the involution ω , defined by $\omega s_\lambda X = s_{\lambda'} X$, to the Cauchy identity. This leads to the central object of our study, the measure on partitions given by

$$\mu_{n,k}^\lambda |X, Y = \prod_{i,j=1}^{n,k} (1 - x_i y_j)^{-1} s_\lambda X s_{\lambda'} Y, \quad (1.1)$$

where $X = x_1, x_2, \dots, x_n$ and $Y = y_1, y_2, \dots, y_k$. Now the partition λ is constrained to a $n \times k$ rectangle. Various specializations of this measure have appeared previously under different names: the q -Krawtchouk ensembles (see, e.g., [5, 7, 8]) or oriented digital boiling (see, e.g., [3, 2]).

Main results. In this paper we study the asymptotic behavior of $\mu_{n,k}^\lambda$ in the following sense. We first study the limit shape profile of λ as $n, k \rightarrow \infty$ with $\lim k/n \in [0, \infty]$ and then the asymptotic behavior of the largest part λ_1 (or a closely associated quantity if $\lambda_1 \ll k$ by our constraints). Our main results depend on two real-valued functions $f, g: [0, 1] \rightarrow [0, \infty]$ — see (1.2), which we take smooth¹ and satisfying certain integrability conditions, see Section 3 and Theorem 3.1.

Theorem 1.1. *Let $k, n \rightarrow \infty$ with $k/n \rightarrow c \in [0, \infty]$ and consider parameters*

$$x_i = f(i/n), \quad y_j = g(j/k). \quad (1.2)$$

¹This condition can be severely loosened, to the point we can take f, g to be step functions.

Modulo extra technical assumptions detailed in [Section 3](#), the rescaled Russian-notation profile of a random λ sampled from $\mu_{n,k} \cdot |X, Y$ converges, point-wise in probability, to an explicit 1-Lipschitz function $\Omega(t)$ supported on an explicit interval $x_-, x_+ \subseteq [-1, c]$.²

Our form for X and Y roughly means their histograms converge to continuous densities: $n^{-1} \sum_i \delta_{x_i}$ converges to the absolutely continuous (with respect to Lebesgue) measure $f(s) ds$ (and likewise for the case of the y 's and $g(s) ds$) as $k, n \rightarrow \infty$ with $k/n \rightarrow c \in (0, \infty)$, where δ is the Dirac point mass. A picture of a limit shape is depicted in [Figure 1](#).

We now turn to fluctuations around the limit shape. We note that the version $g \equiv 1$ of [Theorem 1.2](#) below appeared in [\[3\]](#) and the case $f \equiv \alpha, g \equiv 1$ of [Theorem 1.3](#) appeared in [\[2\]](#), with the authors calling the latter the *critical regime*.

Theorem 1.2. *With setup as in [Theorem 1.1](#), consider the case when $x_+ < c$ (not full support).*

Let $L = \begin{cases} \lambda_1, & \text{if } \Omega \text{ convex around } x_+ \\ n - |\{i \mid \lambda_i = k\}|, & \text{if } \Omega \text{ concave around } x_+ \end{cases}$. Then, with F_{GUE} the Tracy–Widom GUE distribution [\[12\]](#) and for some explicit constant σ , we have:

$$\lim_{n \rightarrow \infty} \mathbb{P} \left(\frac{L - x_+}{\sigma^{-1} n^{1/3}} \right) = F_{\text{GUE}}(s). \quad (1.3)$$

Theorem 1.3. *With the scaling of [Theorem 1.1](#), in the critical full support case of $x_+ = c$ and for $\Delta \in \mathbb{N}$, we have*

$$\lim_{n \rightarrow \infty} \mathbb{P}(\lambda_1 - nc \leq -\Delta) = \det_{0 \leq i, j \leq \Delta-1} \delta_{i,j} - K_{\text{crit}}(i, j) \quad (1.4)$$

with the matrix $K_{\text{crit}}(i, j) = \sum_{\ell=0}^{\Delta-j-1} \frac{1}{2} \begin{cases} \frac{1}{2\pi \ell!} \sin \frac{\pi(j-i)}{2} \Gamma(\ell - \frac{j-i}{2}), & \text{if } \ell - \frac{j-i}{2} \notin \mathbb{Z}_{\leq 0}, \\ \frac{1}{2} \frac{(-1)^\ell}{\ell! \frac{i-j-\ell}{2}!}, & \text{if } \ell - \frac{j-i}{2} \in \mathbb{Z}_{\leq 0}. \end{cases}$

Connections to last passage percolation. The statistic λ_1 for λ from $\mu_{n,k} \cdot |X, Y$ has the following interpretation. Consider the discrete (i, j) -grid $\{1, \dots, n\} \times \{1, \dots, k\}$. At each point (i, j) place a Bernoulli $0, 1$ random variable ω_{ij} distributed as $\mathbb{P}(\omega_{ij} = 1) = \frac{x_i y_j}{1 - x_i y_j}$. In [Figure 1](#) (bottom left) the non-zero 1's are represented as solid dots. Let G be the length of the maximizing up-right path from $(0, 0)$ to $(n-1, k-1)$ ³ in the following sense: the path is only allowed to go strictly up and weakly to the right, and has to maximize the number of 1's (solid dots in *fig. cit.*) encountered. Such a model is a variant of the Bernoulli last passage percolation in a random environment. Then the Robinson–Schensted–Knuth correspondence and Schensted's theorem [\[11\]](#) yields the following.

Proposition 1.4. $G = \lambda_1$ in distribution, where λ is distributed according to $\mu_{n,k} \cdot |X, Y$.

²It means that $\Omega(t)$ is a straight line with slope ± 1 depending on the parameters for t outside $[x_-, x_+]$.

³The start and end points are dummy locations for convenience.

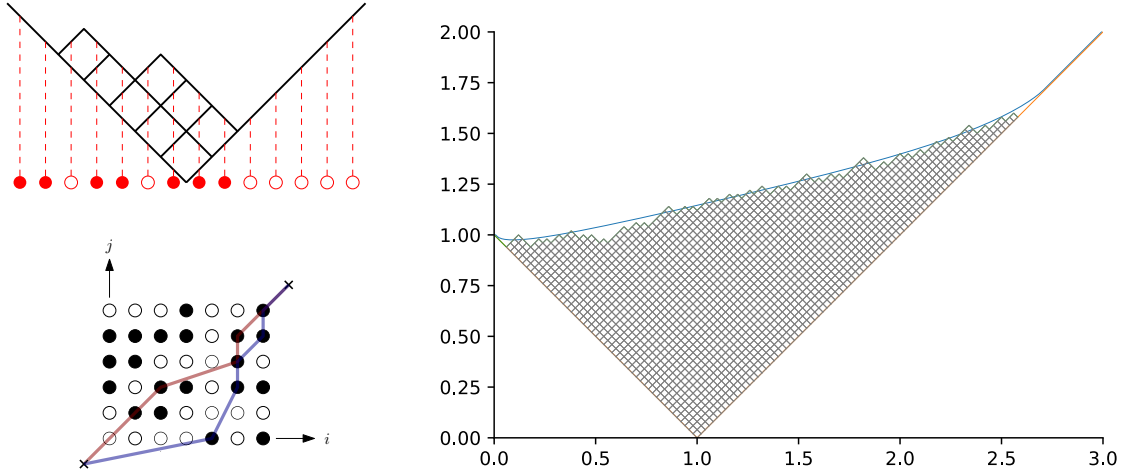


Figure 1: (Top left) Russian and Maya diagram for partition $2^3, 1^2$; (bottom left) two maximizing paths ($G = 5$) in the Bernoulli percolation model described in the introduction (the random variables ω_i are solid dots); (right) a random λ from $\mu_{n,k}(\lambda|X, Y)$, the limit shape (green) superimposed ($n = 50, k = 100, f(x) = 3x, g(x) = x$).

2 Free fermions and finite-size correlations

We briefly recall the free fermion Fock space and half vertex operators as a method to construct the skew Schur functions as matrix elements. Using this, we derive the correlation kernel through an application of Wick's theorem.

A *partition* is an infinite sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ such that $\lambda_i \geq \lambda_{i+1} \geq 0$ for all $i \geq 1$ with only finitely many positive entries. *Fermionic Fock space* (of charge 0) $\mathcal{F}_0$ is a span of certain vectors indexed by partitions $\lambda = (\lambda_1, \lambda_2, \dots)$ defined by $|\lambda\rangle = v_{a_1} \wedge v_{a_2} \wedge v_{a_3} \cdots$, where $a_i = \lambda_i - i - \frac{1}{2}$, $i \geq 1$ is the *Maya sequence* of λ , in a semi-infinite wedge space. These vectors are obtained from the vacuum vector $|0\rangle = -\frac{1}{2} \wedge -\frac{3}{2} \wedge -\frac{5}{2} \wedge \cdots$ by the sequence of actions of the fermionic creation operators ψ_m that adjoin $m \in \mathbb{Z} - \frac{1}{2}$ to the Maya sequence with the proper sign, and fermionic annihilation operators ψ_m^* that remove m . They satisfy the canonical anti-commutation relations $\{\psi_m, \psi_\ell^*\} = \delta_{m,\ell}$.⁴ Let $\langle \mu |$ denote the dual basis element with the natural pairing $\langle \mu | \lambda \rangle = \delta_{\mu,\lambda}$. From these operators we construct their generating series $\psi(z) = \sum_{m \in \mathbb{Z} - 1/2} \psi_m z^m, \psi^*(w) = \sum_{m \in \mathbb{Z} - 1/2} \psi_m^* w^{-m}$, as well as the half-vertex operators $\Gamma_\pm(X) = \exp \sum_{j \geq 1} \frac{p_j X^{\alpha_{\pm j}}}{j}$ with $p_j(X) = \sum_i x_i^j$ and

⁴Here $\{a, b\} = ab - ba$, and all other anti-commutators vanish: $\{\psi_m, \psi_\ell\} = \{\psi_m^*, \psi_\ell^*\} = 0$.

$\alpha_j = \sum_{m \in \mathbb{Z}} \psi_{m-j} \psi_m^*$. The following properties are useful:

$$\begin{aligned} \Gamma_- X, \Gamma_- Y &= \Gamma_- X, \Gamma_- Y = 1, \quad \Gamma_- X, \Gamma_- Y = H X; Y \\ \Gamma_- X |0\rangle &= |0\rangle, \quad \langle 0 | \Gamma_- Y = \langle 0 |, \\ \text{Ad}_{\Gamma_{\pm} X} \psi(z) &= H X; z^{\pm} \psi(z), \quad \text{Ad}_{\Gamma_{\pm} X} \psi^{\dagger}(w) = H X; w^{\pm 1} \psi^{\dagger}(w), \\ \langle 0 | \psi(z) \psi^{\dagger}(w) | 0 \rangle &= \sum_{\ell=0}^{\infty} z^{-1/2-\ell} w^{1/2-\ell} \frac{\sqrt{zw}}{z-w} \quad \text{for } |z| < |w| \end{aligned} \quad (2.1)$$

where $H X; Y = \prod_{i,j} \frac{1}{1-x_i y_j}$, $u, v = uvu^{-1}v^{-1}$, and $\text{Ad}_u x = uxu^{-1}$. We denote $\Gamma'_{\pm} Y = \Gamma_{\pm}^{-1} - Y$ and $E X; Y = \prod_{i,j} \frac{1}{1-x_i y_j} = H X; -Y^{-1}$.

Wick's theorem allows us to write matrix elements $\langle \mu | \Theta | \lambda \rangle$, for some special operators Θ on $\mathcal{F}_0$, as determinants. In particular, Schur polynomials (and functions) are matrix elements of the $\Gamma_{\pm}$ operators: $\langle \lambda | \Gamma_- X | 0 \rangle = \langle 0 | \Gamma_- X | \lambda \rangle = s_{\lambda} X$ and $\langle \lambda | \Gamma'_- X | 0 \rangle = \langle 0 | \Gamma'_- Y | \lambda \rangle = s_{\lambda'} Y$ (the determinants are the (dual) Jacobi–Trudi formulas for the Schur functions). Finally, using (2.1), the dual Cauchy identity becomes

$$\sum_{\lambda} s_{\lambda} X s_{\lambda'} Y = \langle 0 | \Gamma_- X \Gamma'_- Y | 0 \rangle = E X; Y. \quad (2.2)$$

Furthermore, again using Wick's theorem, we obtain the following form of Okounkov's correlations for the dual Schur measure:

Proposition 2.1 (Correlation kernel, [9]). *Let λ be random sampled from $\mu_{n,k} | \lambda | X, Y$. Fix $a_i \in \mathbb{Z} - \frac{1}{2}$, $1 \leq i \leq p$. The associated particles (Maya sequence) form a determinantal ensemble:*

$$\mu_{n,k} | \lambda \text{ has a particle at position } a_i, 1 \leq i \leq p | X, Y = \det[\mathcal{K}(a_i, a_j)]_{i,j=1}^p, \quad (2.3)$$

with the correlation kernel $\mathcal{K}(m, m')$ given by the integral representation

$$\mathcal{K}(m, m') = \oint_{|w| < |z|} \oint \frac{dz}{2\pi i z} \frac{dw}{2\pi i w} \frac{F(z)}{F(w)} \frac{w^{m'}}{z^m} \frac{\sqrt{zw}}{z-w}, \quad F(z) = \prod_{i=1}^n \frac{1}{1-x_i z} \prod_{j=1}^k \frac{1}{1-y_j/z} \quad (2.4)$$

where both contours encircle 0 and the z contour encircles all $-y_j$'s while excluding all $1/x_i$'s.

3 Asymptotics for the limit shape

We discuss how to obtain the limit shape alluded to in Theorem 1.1. We look at the behavior of $\mu_{n,k} | \lambda | X, Y$ as we take the limit $n, k \rightarrow \infty$ such that $\lim_{n,k} \frac{k}{n} = c \in [0, \infty]$. The partition, drawn in Russian notation, becomes a piecewise-linear function with slope -1 (resp. 1) if a particle is in (resp. not in) the interval $[i, i+1]$. Rescaling by n^{-1} , our

partition lies in the interval $[-1, c]$ since $k \sim nc$. The limit density of particles turns out to be well-defined and gives the desired limit curve Ω .⁵ Thus if we show that there exists a limiting density $\rho(t)$, then we can compute the limit shape by

$$\Omega(u) = 1 - \int_{-1}^u dt (1 - 2\rho(t)). \quad (3.1)$$

We demonstrate this by showing that outside of an interval we call the support, the limit density is constant (0 or 1), while inside it converges to a discrete version of the random matrix sine kernel.

We make the following assumptions on the functions f, g : 1) $f, g: [0, 1] \rightarrow [0, \infty]$ are smooth; 2) the Riemann integrals $\int_0^1 \frac{ds}{1 - f(s/z)^2}$, $\int_0^1 \frac{ds}{z - g(s/z)^2}$ converge for $z \in \mathbb{C}$ away from the poles; and 3) the equation

$$\int_0^1 ds \frac{f(s/z)}{1 - f(s/z)^2} - c \int_0^1 ds \frac{g(s/z)}{z - g(s/z)^2} = 0 \quad (3.2)$$

has two roots $z_{\pm} \in \mathbb{C} \cup \{\infty\}$. Furthermore we set

$$x_{\pm} = \int_0^1 ds \frac{f(s/z_{\pm})}{1 - f(s/z_{\pm})^2} - c \int_0^1 ds \frac{g(s/z_{\pm})}{z_{\pm} - g(s/z_{\pm})^2}. \quad (3.3)$$

Theorem 3.1 (Bulk asymptotics). *With parameters $x_i = f(i/n)$, $y_j = g(j/k)$, as $k, n \rightarrow \infty$ with $k/n \rightarrow c \in [0, \infty]$, and assuming the conditions 1)–3) above on f, g . Then for $t \in [x_-, x_+] \subseteq [-1, c]$ and $m, m' \in \mathbb{Z} - \frac{1}{2}$ we have*

$$\lim_{n \rightarrow \infty} \mathcal{K}(nt - m, nt - m') = \begin{cases} \frac{\sin(\varphi \cdot (m - m'))}{\pi (m - m')} & \text{if } m \neq m', \\ \frac{\varphi}{\pi} & \text{if } m = m', \end{cases} \quad (3.4)$$

where $\varphi(t) = \arg z(t)$ is given by the argument of the solution of

$$\int_0^1 ds \frac{f(s/z)}{1 - f(s/z)^2} - c \int_0^1 ds \frac{g(s/z)}{z - g(s/z)^2} - t = 0. \quad (3.5)$$

Outside of $[x_-, x_+]$, $\lim_{n \rightarrow \infty} \mathcal{K}(nt - m, nt - m') \in \{0, 1\}$.

We then define $\rho(t) = \lim_{n \rightarrow \infty} \mathcal{K}(nt, nt)$ for $t \in [x_-, x_+]$; for $t < -1$ (resp. $t > c$), we have $\rho(t) = 1$ (resp. $\rho(t) = 0$); for $t \in [-1, x_-] \cup [x_+, c]$ the situation is more complicated as it depends on the functions and c , but $\rho(t)$ is still identically 0 or 1.

⁵The convergence of the corresponding (rescaled) functions to the limit shape is a more difficult question that we do not consider here; see e.g., [10, Ch. 1] for the case of the Plancherel measure.

We prove [Theorem 3.1](#) by using $\mathcal{K}(nt, nt)$ $\oint_{|w| < |z|} \frac{dz}{2\pi iz} \frac{dw}{2\pi iw} e^{n \int_{|w| < |z|} S(z) - S(w) \frac{\sqrt{zw}}{z-w}}$ for the action $S(z) = \frac{1}{n} \ln F(z) - t \ln z$, which is asymptotically

$$S(z) \approx - \int_0^1 ds \ln(1 - f(s)z) - c \int_0^1 ds \ln(1 - g(s)/z) - x \ln z. \quad (3.6)$$

Only critical points of the action contribute to the integral giving $\mathcal{K}(nt, nt)$. If we have two complex conjugate roots $z, \bar{z}$ of the equation

$$z \partial_z S(z) = \int_0^1 ds \frac{f(s)z}{1 - f(s)z} - c \int_0^1 ds \frac{g(s)}{z - g(s)} - t = 0, \quad (3.7)$$

then $\rho(t) = \frac{1}{\pi} \arg z$. The support of the density x_-, x_+ is determined by finding the solutions $t = x_{\pm}$ when $z = \bar{z}$ is a double critical point of the action. There are two such z 's by our assumption, called $z_{\pm}$, and obtained by solving

$$z \partial_z^2 S(z) = \int_0^1 ds \frac{f(s)z}{(1 - f(s)z)^2} - c \int_0^1 ds \frac{g(s)}{(z - g(s))^2} = 0. \quad (3.8)$$

Plugging into (3.7) and solving for t , we obtain the two ends of the support $t = x_{\pm}$.

Example 3.2 (Equal parameters). As our first example we consider $x_i = \alpha, y_j = 1$ (and hence $f \equiv \alpha, g \equiv 1$) for all i, j with fixed $\alpha \in \mathbb{R}_{>0}$. The measure (1.1) becomes

$$\mu_{n,k}(\lambda) = \alpha^{|\lambda|} (1 - \alpha)^{-nk} \dim V_{\mathrm{GL}_n}(\lambda) \dim V_{\mathrm{GL}_k}(\lambda'), \quad (3.9)$$

and was considered in [2] in relation to a stochastic growth process. We note the ensemble for λ is Johansson's Krawtchouk orthogonal polynomial ensemble [5]. Substituting f, g into (3.7), (3.8) we get $x_{\pm} = \frac{\alpha(c-1) \pm 2\sqrt{\alpha c}}{\alpha-1}$ and roots

$$z, \bar{z} = \frac{\alpha(c-1) - t(1-\alpha) \pm \sqrt{4\alpha t(1-t-c) - (\alpha(c-1) - t(1-\alpha))^2}}{2\alpha t - 1}. \quad (3.10)$$

Therefore the limit density for $t \in x_-, x_+$ is $\rho(t) = \frac{1}{\pi} \arg z = \frac{1}{\pi} \arccos \left(\frac{\alpha(c-1) - t(1-\alpha)}{2\sqrt{\alpha(c-t)(t-1)}} \right)$.

Example 3.3 (q -weights I). Consider now the principal specialization of characters $x_i = q^{i-1}, y_j = q^{j-1}$ related to q -dimensions of the corresponding representations. We have

$$\mu_{n,k}(\lambda) = \frac{q^{||\lambda||} \dim_q V_{\mathrm{GL}_n}(\lambda) \cdot q^{||\bar{\lambda}'||} \dim_q (V_{\mathrm{GL}_k}(\bar{\lambda}'))}{\prod_{i=1}^n \prod_{j=1}^k q^{i-1} q^{j-1}}, \quad (3.11)$$

where $||\lambda|| = \sum_{i=1}^n (i-1) \lambda_i$ and $\bar{\lambda}$ means the complement of λ inside the $n \times k$ rectangle. The q -dimension $\dim_q$ of the irreducible GL_n representation is defined in, e.g., [6, §10.10].

Using explicit formulas for q -dimensions, one can show that the point ensemble for λ is the so-called q -Krawtchouk orthogonal polynomial ensemble; see [7] for details. To compute the limit shape, we fix $\gamma > 0$ and set $q = e^{-\gamma/n} \rightarrow 1$ as $n, k \rightarrow \infty$ (i.e., $f(s) = g(s) = e^{-\gamma s}$). The rescaled profile for a random λ from $\mu_{n,k}|\lambda|X, Y$ converges point-wise in probability to the limit shape given by (3.1) with $\rho(t)$ and $x_{\pm}$ given by

$$\rho(t) = \frac{1}{\pi} \arccos \left(\operatorname{sgn}(-\gamma) \frac{e^{\gamma - \frac{\gamma}{2}t}}{2} \frac{1 - e^{\gamma(c-1)}}{\sqrt{(1 - e^{\gamma(t-1)})(1 - e^{\gamma(c-t)})}} \right), \quad (3.12)$$

$$x_{\pm} = -\frac{\operatorname{sgn}(\gamma)}{\gamma} \ln \frac{3e^{c-1}\gamma - e^{c\gamma} - e^{\gamma}}{1 - e^{c\gamma/2}} \frac{3 \mp 2\sqrt{2} \sqrt{e^{\gamma} - 1} \sqrt{e^{c\gamma} - 1} e^{c-1}\gamma}{1}. \quad (3.13)$$

Example 3.4 (q -weights II). We can also consider $x_i = q^{i-1}, y_j = q^{j-1}$; μ has a similar form as above: $\mu_{n,k}|\lambda|q, q^{-1} = \frac{q^{\|\lambda\|} \dim_q V_{\text{GL}_n}(\lambda) \cdot q^{-\|\lambda'\|} \dim_{1/q} V_{\text{GL}_k}(\bar{\lambda}')}{\prod_{i=1}^n \prod_{j=1}^k q^{i-1} q^{j-1}}$. In the same regime as above as $q \rightarrow 1$ the limit density and support are given by

$$\rho(t) = \frac{1}{\pi} \arccos \left(\operatorname{sgn}(-\gamma) \frac{e^{\frac{\gamma}{2}t - 1 - c}}{2} \frac{1 - e^{\gamma c} - e^{\gamma(c-t-1)}}{\sqrt{(1 - e^{\gamma(t-1)})(1 - e^{\gamma(c-t)})}} \right), \quad (3.14)$$

$$x_{\pm} = -1 - \frac{1}{\gamma} \ln 2 \pm \frac{1}{\gamma} \ln \left(1 - e^{\gamma(c-1)} \pm \sqrt{e^{2\gamma c} - 1} \sqrt{e^{2\gamma} - 1} \right). \quad (3.15)$$

Example 3.5. Take $f(s) = \alpha s, g(s) = s$. We do not have closed-form formulas anymore since the equations (3.7), (3.8) take the form: $-\frac{1}{\alpha z} \ln(1 - \alpha z) - 1 - cz \ln\left(\frac{z}{1-z}\right) - c - t = 0$, $\frac{\alpha z - 1}{\alpha z} \ln \frac{1 - \alpha z}{\alpha z - 1} - c \left(z \ln\left(\frac{z}{z-1}\right) - \frac{z}{z-1} \right) = 0$. Yet by solving these numerically we can still find the support x_-, x_+ and the limit shape as shown for $\alpha = 3$ in Figure 1 (right).

For Examples 3.2, 3.3 and 3.4, limit shapes for various values of γ are presented in Figure 2. For $\gamma = 0$ we recover the constant specialization $x_i = y_j = 1$.

4 Boundary asymptotics

To study boundary asymptotics, we zoom in around x_- and see what happens to the process locally. In the asymptotic analysis described in the previous section, the two roots $z, \bar{z}$ coincide to a double critical point we called z_- .

Sketch of proof of Theorem 1.2. Before we proceed further, we assume for simplicity c is such that $\int_0^1 ds f(s) < c \int_0^1 ds \frac{1}{g(s)}$. This means, *a posteriori*, in the notation of Theorem 1.2, that we are looking at fluctuations for $\lambda_1 \ll k$. In other words, the limit shape

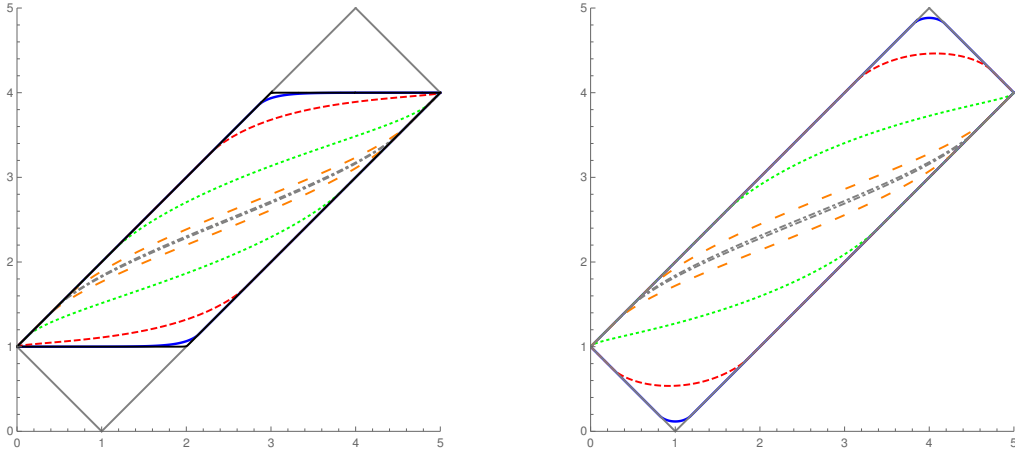


Figure 2: Plots of the limit shapes for Young diagrams corresponding to the densities (3.12) (on the left) and (3.14) (on the right) for $c = 4$ and the values of γ (bottom to top): -10 (solid blue), -2 (dashed red), -0.5 (dotted green), -0.1 (sparsely dashed orange), -0.01 (dot-dashed gray), 0.01 (dot-dashed gray), 0.1 (sparsely dashed orange), 0.5 (dotted green), 2 (dashed red), 10 (solid blue). Solid black lines on the left panel correspond to $\gamma = \pm\infty$ ($q = \text{const}$). We have shifted $t \mapsto t - 1$.

$\Omega(t)$ is convex around x , and in turn $x < c$, so we do not have full support for Ω . In Figure 2, we are dealing with a limit shape *sitting under the diagonal of the rectangle*.

Near the double critical point $z \in \mathbb{R} - \{0\}$ we have $S(z) = S'''(z)z - z^3/6 \mathcal{O}(z - z^4)$. Scaling around $z = z e^{\sigma\zeta n^{-1/3}} \approx z(1 + \sigma\zeta n^{-1/3}) \dots$ as $n \rightarrow \infty$, with σ a yet-to-be-determined constant, we see that we need to scale the matrix entries of $K(m, m')$ as $m, m' \approx x n^{\frac{1}{3}} \zeta, \eta n^{1/3} \sigma^{-1}$. Let $\sigma^{-1} = 2^{-1/3} S'''(z)^{1/3} z > 0$. Then one can show

$$n^{1/3} \mathcal{K}(m, m') \rightarrow \mathcal{K}_{\text{Airy}}(\zeta, \eta) : -\frac{1}{2\pi i} \int_{i\mathbb{R}-\epsilon} d\omega \int_{i\mathbb{R}-\epsilon} d\zeta \frac{\exp(\zeta^3/3 - \zeta\zeta - \omega^3/3 - \eta\omega)}{\zeta - \omega}, \quad (4.1)$$

where $0 < \epsilon \ll 1$ and $\mathcal{K}_{\text{Airy}}(\zeta, \eta)$ is an integral representation of the Airy kernel [12].

For a quick justification notice that

$$n(S(z) - S(w)) \approx \frac{1}{6} \sigma^3 z^3 S'''(z) (\zeta^3 - \omega^3) - \frac{1}{6} \sigma^3 \left(z \partial_z^3 S(z) \Big|_{z=z} \right) (\zeta^3 - \omega^3) \quad (4.2)$$

for $z = z e^{\sigma\zeta n^{-1/3}}$, $w = z e^{\sigma\eta n^{-1/3}}$ in the vicinity of z , and we are therefore interested in

$$z \partial_z^3 S(z) \Big|_{z=z} = \int_0^1 ds \left(\frac{2f^2(s) z^2}{1 - f(s) z^3} - \frac{2cg(s) z^2}{g(s) z^3} \right), \quad (4.3)$$

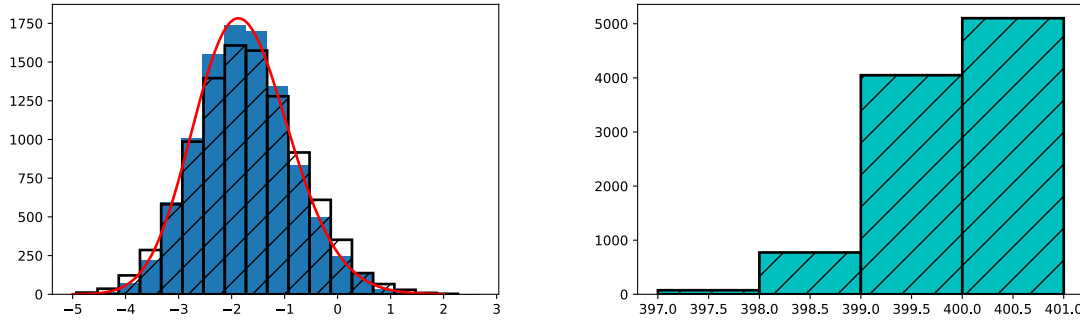


Figure 3: (Left) Illustration of [Theorem 1.2](#): histogram of first row lengths in black ($n = 200, k = 400, f \equiv g \equiv 1$, 10000 samples), Tracy–Widom distribution (red), and Tracy–Widom histogram (blue, 10000 samples). (Right) Histogram of 10000 samples of first row lengths in the critical case of [Theorem 1.3](#) ($n = 200, k = 400, f = \frac{3}{2}x^2, g = x - 2x^{-1}$).

to determine σ . Assuming this integral converges, we can conclude the convergence from (4.1). Further technical estimates along the same lines show that Fredholm determinants converge to Fredholm determinants, and the Tracy–Widom distribution is a Fredholm determinant of the Airy kernel on $\mathcal{L}^2(s, \infty)$. Therefore we conclude that the distribution of $\frac{\lambda_1 - x}{\sigma^{-1}n^{1/3}}$ is asymptotically the Tracy–Widom GUE distribution. See [Figure 3](#) (left) for an illustration.

Remark 4.1. In the case of [Example 3.2](#) ($f \equiv \alpha, g \equiv 1$), we have $x_{\pm} = \frac{\alpha}{\alpha - 1} \frac{c - 1 \pm 2\sqrt{\alpha c}}{1}$ and $z = \frac{\alpha}{\alpha} \frac{c}{\alpha c - 1} \frac{1 - \alpha}{\alpha c - 1} \frac{1}{\sqrt{\alpha c}}$. This yields (recall $c, \alpha > 0$) $\sigma = \frac{\alpha}{\frac{1}{\alpha^6} \sqrt{c} - \sqrt{\alpha}} \frac{1}{\frac{2}{3} - 1} \frac{c^{\frac{1}{6}}}{\sqrt{\alpha c}^{\frac{2}{3}}}$. We see that $\sigma < \infty$ whenever $\alpha \neq c$.

Remark 4.2. Continuing on the previous remark, $\alpha = c$ implies $\sigma = \infty$. Formally this would yield an infinitely thin Tracy–Widom distribution; in practice, we obtain the critical case of [Theorem 1.3](#) (first discussed in [2, Section 3.2], as the models are the same). This happens when λ_1 hits the east-most corner of the rectangle bounding it; from the point of view of asymptotic analysis, the critical point considered is $z = 0$, a singularity of the action.

Sketch of proof of [Theorem 1.3](#). Now we turn to [Theorem 1.3](#). We land in this case when there is an obstruction to our asymptotic analysis of the generic case. This happens when the double critical point z of the action under consideration is $z = 0$. In this situation the constant $\sigma = \infty$, and the integral in (4.3) diverges. Combinatorially, it

happens when $\lambda_1 = nc - o(1)$ (when λ_1 just hits the east-most corner of the rectangle bounding it). In terms of our functions f and g , we get the following condition:

$$\int_0^1 ds f(s) = c \int_0^1 \frac{ds}{g(s)}. \quad (4.4)$$

Notice that $x = c$ is a root of (3.7) for $z = 0$, and then (3.8) gives the above condition.

Now we cannot pass the contours through the double critical point at 0 anymore (being a singularity of the action). This also implies that our limiting distribution will be discrete: we will not scale the m, m' coordinates continuously. Instead we take $m = cn - h, m' = cn - h'$ for some $h, h' \in \mathbb{N} - \frac{1}{2}$ and consider the asymptotics of $K_{m, m'}$ as such. The contours encircle 0 and satisfy $|w| < |z|$. The w contour can be made an arbitrary small circle around 0. Since the contour for z encircles all $-y_j$'s and does not contain all $1/x_i$'s, we can deform it to be from $-i\infty$ to $i\infty$ with a small bump to the right of 0 that contains contour for w . We first integrate over w . Since we have a critical point of the action at 0, $S'(0) = 0$, we can approximate the action as $-nS(w) \approx -nS(0) - \frac{n}{2}S''(0)w^2$; we thus need to compute the integral $\oint_{C_\varepsilon} e^{-\frac{n}{2}S''(0)w^2} w^{-h'} \frac{\sqrt{zw}}{z-w} \frac{dw}{2\pi i w}$. As $|w| < |z|$, we can expand $\frac{1}{z-w}$ and $e^{-\frac{n}{2}S''(0)w^2}$ into power series in w and obtain

$$\oint_{C_\varepsilon} \frac{dw}{2\pi i w} e^{-\frac{n}{2}S''(0)w^2} w^{-h'} \frac{\sqrt{zw}}{z-w} = \oint_{C_\varepsilon} \frac{dw}{2\pi i w} \left(\sum_{\ell=0}^{\infty} \frac{-1}{2^\ell \ell!} \frac{n^\ell S''(0)^\ell}{2^\ell \ell!} w^{2\ell} \right) w^{-h'} \frac{1}{z} \left(\sum_{j=0}^{\infty} \frac{w^j}{z^j} \right).$$

After the residue calculation we get $\sum_{\ell=0}^{\lfloor h'/2-1/4 \rfloor} \frac{-1}{2^\ell \ell!} \frac{n^\ell S''(0)^\ell}{2^\ell \ell!} z^{-h'-2\ell}$. To integrate over z , we note that for purely imaginary z we have $\operatorname{Re} S'(z) < 0$ and therefore as $n \rightarrow \infty$ main contribution comes from the bump around 0. We again can approximate $nS(z) \approx nS(0) - \frac{n}{2}S''(0)z^2$ and we cancel $nS(0)$ from the w -integral, so we need to compute the integral $\int_{-i\infty}^{i\infty} \frac{dz}{2\pi i z} e^{\frac{n}{2}S''(0)z^2} z^h \left(\sum_{\ell=0}^{\lfloor h'/2-1/4 \rfloor} \frac{-1}{2^\ell \ell!} \frac{n^\ell S''(0)^\ell}{2^\ell \ell!} z^{-h'-2\ell} \right)$. We do a substitution $z = \frac{i\sqrt{2t}}{\sqrt{nS''(0)}}$; the new integration contour starts at ∞ , loops around 0 and then returns to ∞ and we have:

$$\mathcal{K}_n(h, h') = \frac{1}{2\pi} \sum_{\ell=0}^{\lfloor h'/2-1/4 \rfloor} \frac{-1}{nS''(0)} \frac{h-h'-1/2}{2^{\ell} \ell!} \int_0^\infty dt t^{h-h'/2-\ell-1} e^{-t}. \quad (4.5)$$

If $h - h' - \ell \geq 0$, last integral is Hankel's integral representation for the Γ -function $\Gamma(z) = -\frac{1}{2i \sin \pi z} \int_0^\infty dt -t^{z-1} e^{-t}$; otherwise we obtain $-\frac{1}{\left(\frac{h-h'}{2}-\ell\right)!} \frac{2\pi i}{\left(\frac{h-h'}{2}-\ell\right)!}$ from the residue calculation. Thus the correlation kernel coincides with the formula [2, (3.19)] up to a normalization. To find the probability $\mathbb{P}(\lambda_1 - nc \leq -\Delta)$, we need to compute the determinant $\det \delta_{i,j} - \mathcal{K}(i-1/2, j-1/2)$ by inclusion-exclusion⁶; the prefactor powers with exponent proportional to $i-j$ cancel and we conclude our proof.

⁶We are looking at a gap probability.

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Hilbert–Poincaré series of matroid Chow rings and intersection cohomology

Luis Ferroni^{*1}, Jacob P. Matherne^{+2,3},
Matthew Stevens^{‡2}, and Lorenzo Vecchi^{§4}

¹ *Department of Mathematics, KTH Royal Institute of Technology, Sweden*

² *Mathematical Institute, University of Bonn, Germany*

³ *Max Planck Institute for Mathematics, Bonn, Germany*

⁴ *Dipartimento di Matematica, Università di Bologna, Italy*

Abstract. We study the Hilbert–Poincaré series of three algebraic objects arising in the Chow-theoretic and Kazhdan–Lusztig framework of matroids. These are, respectively, the Hilbert–Poincaré series of the Chow ring, the augmented Chow ring, and the intersection cohomology module. We develop and highlight an explicit parallelism between the Kazhdan–Lusztig polynomial of a matroid and the Hilbert–Poincaré series of its Chow ring that extends naturally to the Hilbert–Poincaré series of both the intersection cohomology module and the augmented Chow ring.

Keywords: Matroids, Chow rings, Kazhdan–Lusztig polynomials, Hilbert–Poincaré series, Binomial Eulerian polynomials, Real-rootedness, γ -positivity

1 Introduction

Starting from a loopless matroid M , one can construct various algebraic objects having good properties that in turn can be used to answer purely combinatorial questions. Three notable examples of such objects are the **Chow ring** $\underline{\text{CH}} M$, the **augmented Chow ring** $\text{CH} M$, and the **intersection cohomology module** $\text{IH} M$. These three algebraic structures possess a number of remarkable features and play an instrumental role in the proofs of the log-concavity of the Whitney numbers of the first kind [1], and the top-heaviness of the Whitney numbers of the second kind of the lattice of flats $\mathcal{L} M$ [7]. Our main objects of study in this article are the coefficients of the respective Hilbert–Poincaré series of each of these three structures.

^{*}ferroni@kth.se. LF is supported by the Swedish Research Council grant 2018-03968.

⁺jacobm@math.uni-bonn.de. JM is supported by the Deutsche Forschungsgemeinschaft (DFG) under Germany’s Excellence Strategy - GZ 2047/1, Projekt-ID 390685813.

[‡]s6mastev@uni-bonn.de.

[§]lorenzo.vecchi6@unibo.it.

The intersection cohomology module $\mathrm{IH}^*(M)$ is particularly relevant in the Kazhdan–Lusztig theory of matroids. Its Hilbert–Poincaré series is known in the literature as the **Z-polynomial** of the matroid M and is denoted by $Z_M(x)$. The history is as follows. First, in [23], Proudfoot, Xu, and Young introduced the Z-polynomial using a purely combinatorial language (i.e., via the displayed equation in [Theorem 1 \(iii\)](#) below), without making reference to any notion of intersection cohomology for arbitrary matroids (at that point). Later, in [7], Braden, Huh, Matherne, Proudfoot, and Wang introduced the intersection cohomology module $\mathrm{IH}^*(M)$ of a matroid M and showed that the combinatorially-defined $Z_M(x)$ is equal to the Hilbert–Poincaré series of $\mathrm{IH}^*(M)$.

In fact, the Z-polynomial is strongly related to the **Kazhdan–Lusztig polynomial** of M , denoted $P_M(x)$ and first studied by Elias, Proudfoot, and Wakefield in [11]. The following can be used as a simultaneous recursive definition for both the Kazhdan–Lusztig and Z-polynomials of matroids.

Theorem 1 ([23, 8]). *There is a unique way to assign to each loopless matroid M a polynomial $P_M(x) \in \mathbb{Z}[x]$ such that the following properties hold:*

- (i) *If $\mathrm{rk} M = 0$, then $P_M(x) = 1$.*
- (ii) *If $\mathrm{rk} M > 0$, then $\deg P_M(x) < \frac{1}{2} \mathrm{rk} M$.*
- (iii) *For every matroid M , the polynomial*

$$Z_M(x) := \sum_{F \in \mathcal{L}(M)} x^{\mathrm{rk} F} P_{M/F}(x)$$

*is palindromic.*¹

The original definition of the Kazhdan–Lusztig polynomials of Elias, Proudfoot, and Wakefield is given by the following result, which provides a recursion that defines them uniquely in terms of characteristic polynomials of restrictions $M|_F$ and Kazhdan–Lusztig polynomials of contractions M/F for flats $F \in \mathcal{L}(M)$. This does not make any reference to the Z-polynomial.

Theorem 2 ([11]). *There is a unique way to assign to each loopless matroid M a polynomial $P_M(x) \in \mathbb{Z}[x]$ such that the following conditions hold:*

- (i) *If $\mathrm{rk} M = 0$, then $P_M(x) = 1$.*
- (ii) *If $\mathrm{rk} M > 0$, then $\deg P_M(x) < \frac{1}{2} \mathrm{rk} M$.*

¹In this context, we say that $p(x)$ is palindromic if $p(x) = x^d p(x^{-1})$ where $d = \deg p(x)$. For example, the polynomial $q(x) = x^2 + x$ is *not* palindromic, even though it satisfies $x^3 q(x^{-1}) = q(x)$.

(iii) For every matroid M , the following recursion holds:

$$x^{\text{rk } M} P_M x^{-1} = \sum_{F \in \mathcal{L}_M} \chi_{M|_F} x P_{M/F} x.$$

These results provide compact and purely combinatorial definitions of $P_M x$ and $Z_M x$. However, many properties of these families of polynomials are not easy to deduce from such statements. For instance, a non-obvious property that these polynomials possess is the nonnegativity of their coefficients. In fact, another important result of [7] precisely establishes this nonnegativity by proving that the coefficients of these polynomials are given by certain graded dimensions; more precisely, $Z_M x = \text{Hilb } \text{IH } M_{\emptyset}, x$ and $P_M x = \text{Hilb } \text{IH } M_{\emptyset}, x$ (see [7, Theorem 1.9]).

We now turn history on its head. Recall that in the Kazhdan–Lusztig setting, the combinatorially-defined Kazhdan–Lusztig and Z -polynomials came first, and their descriptions as Hilbert–Poincaré series of $\text{IH } M_{\emptyset}$ and $\text{IH } M$ came later. For the Chow-theoretic setting, the Chow ring $\underline{\text{CH}} M$ and augmented Chow ring $\text{CH } M$ were first introduced in [12] and [6], respectively. In what follows, we give an “intrinsic” combinatorial definition of their Hilbert–Poincaré series by mirroring Theorem 1 and Theorem 2. (Theorem 5 below asserts that these polynomials are indeed the correct Hilbert–Poincaré series.) These combinatorial definitions can be seen as the point of departure of our study.

Theorem 3. *There is a unique way to assign to each loopless matroid M a palindromic polynomial $\underline{H}_M x \in \mathbb{Z}[x]$ such that the following properties hold:*

- (i) *If $\text{rk } M = 0$, then $\underline{H}_M x = 1$.*
- (ii) *If $\text{rk } M > 0$, then $\deg \underline{H}_M x = \text{rk } M - 1$.*
- (iii) *For every matroid M , the polynomial*

$$\underline{H}_M x := \sum_{F \in \mathcal{L}_M} x^{\text{rk } F} \underline{H}_{M/F} x$$

is palindromic.

Similar to the case of the Kazhdan–Lusztig polynomials and the Z -polynomials, a non-trivial property that one is able to observe is that the polynomials $\underline{H}_M x$, and therefore also the polynomials $H_M x$, appear to have nonnegative coefficients. On the other hand, given the resemblance between Theorem 3 and Theorem 1, it is reasonable to ask for a counterpart for Theorem 2 in this alternative setting.

Theorem 4. *There is a unique way to assign to each loopless matroid M a polynomial $\underline{H}_M x \in \mathbb{Z}[x]$ such that the following conditions hold:*

- (i) If $\text{rk } M = 0$, then $\underline{H}_M(x) = 1$.
- (ii) For every matroid M , the following recursion holds:

$$\underline{H}_M(x) = \sum_{\substack{F \in \mathcal{L}(M) \\ F \neq \emptyset}} \bar{\chi}_{M|_F}(x) \underline{H}_{M/F}(x).$$

As it is stated, the proof of this fact is an easy induction. What is much less evident is that the polynomials $\underline{H}_M(x)$ arising from this result coincide with those arising from [Theorem 3](#). In particular, it is already a non-trivial and interesting conclusion that the polynomials as defined in [Theorem 4](#) are palindromic, as this is not at all hinted by the recursion. A slight difference with respect to [Theorem 2](#) that deserves a comment is that the polynomial $\underline{H}_M(x)$ is defined recursively via a convolution of itself with the *reduced* characteristic polynomial. Again, the nonnegativity of the coefficients of $\underline{H}_M(x)$ is not easy to deduce from this recursion and, moreover, the fact that the reduced characteristic polynomial has coefficients that alternate in sign introduces an additional issue.

The polynomials $\underline{H}_M(x)$ and $H_M(x)$ are less manageable than $P_M(x)$ and $Z_M(x)$ from a computational point of view. For example neither $\underline{H}_M(x)$ nor $H_M(x)$ behave well under direct sums of matroids; one possible explanation for this is that the reduced characteristic polynomial of a direct sum is not the product of the reduced characteristic polynomials of the summands and, in particular, the argument that Elias, Proudfoot, and Wakefield used in [[11](#), Proposition 2.7] to prove that $P_{M_1 \oplus M_2}(x) = P_{M_1}(x) P_{M_2}(x)$ does not hold.

The key step towards deducing that the coefficients of these polynomials are always nonnegative integers is given by the following connection with the Chow ring and the augmented Chow ring.

Theorem 5. *Let M be a loopless matroid. The polynomial $\underline{H}_M(x)$ is the Hilbert–Poincaré series of the Chow ring $\underline{\text{CH}}(M)$ of M . The polynomial $H_M(x)$ is the Hilbert–Poincaré series of the augmented Chow ring $\text{CH}(M)$. In particular, both of them have nonnegative coefficients.*

The proof relies essentially on a construction of Feichtner and Yuzvinsky [[12](#)] of a certain Gröbner basis for the Chow ring of atomic lattices with respect to arbitrary building sets. The strategy is to start with a raw expression for the Hilbert–Poincaré series of both the Chow ring and the augmented Chow ring and prove that they satisfy the recursions of both [Theorem 3](#) and [Theorem 4](#); this is very much in resemblance to the proof of the positivity of the coefficients of the Kazhdan–Lusztig and the Z -polynomials in [[7](#), Theorem 1.2].

Remark 6. *The reduced characteristic polynomial is a Tutte–Grothendieck invariant of the matroid, i.e., the Tutte polynomial of the matroid determines it. A natural question that one could ask is whether the Hilbert–Poincaré series of the Chow ring has the same property. As can be*

seen from [Theorem 4](#), the map $M \mapsto \underline{H}_M x$ is essentially the inverse of the map $M \mapsto \bar{\chi}_M x$ in the incidence algebra of the lattice of flats. In spite of that, we can find two matroids M_1 and M_2 of rank 4 on 7 elements having the same Tutte polynomial but whose Chow rings have different Hilbert–Poincaré series. Precisely, consider the matroids M_1^* and M_2^* depicted in [Figure 1](#) (the reason for depicting the duals instead of the original matroids is that they have rank 3):

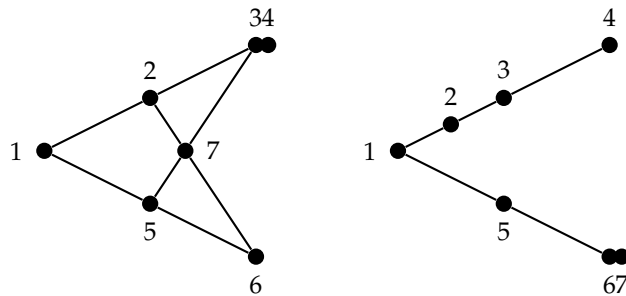


Figure 1: The duals of the matroids M_1 and M_2

The matroids M_1 and M_2 have the same Tutte polynomial:

$$T_{M_1}(x, y) = T_{M_2}(x, y) = x^4 + 3x^3 + 2x^2y + xy^2 + y^3 + 4x^2 + 5xy + 3y^2 + 2x + 2y.$$

However, we have

$$\begin{aligned} \underline{H}_{M_1}(x) &= x^3 + 30x^2 + 30x + 1, \\ \underline{H}_{M_2}(x) &= x^3 + 31x^2 + 31x + 1. \end{aligned}$$

Moreover, the same example shows that the Hilbert–Poincaré series of the augmented Chow ring is not an evaluation of the Tutte polynomial because

$$\begin{aligned} H_{M_1}(x) &= x^4 + 37x^3 + 98x^2 + 37x + 1, \\ H_{M_2}(x) &= x^4 + 38x^3 + 102x^2 + 38x + 1. \end{aligned}$$

We also mention explicitly the fact that neither $\underline{H}_M(x)$ nor $H_M(x)$ determines the other.

2 Real-rootedness and gamma-positivity

As mentioned earlier, one can view [Theorem 3](#) and [Theorem 4](#) as “intrinsic” combinatorial definitions of the Hilbert–Poincaré series of the rings $\underline{\text{CH}} M$ and $\text{CH} M$, avoiding their explicit construction.

Given the similarity that we have with the counterpart definitions in the Kazhdan–Lusztig framework, some statements (many of which are still conjectural) regarding

the Kazhdan–Lusztig polynomial and the Z -polynomial of matroids now admit natural analogues for the Chow ring and the augmented Chow ring.

Two outstanding conjectures posed in [16] and [23] assert the **real-rootedness** of both $P_M x$ and $Z_M x$. A well known fact (see, e.g., [9]) is that the real-rootedness of a polynomial with positive coefficients implies that the coefficients form an **ultra log-concave sequence** and, in the case the polynomial is palindromic, that it is **γ -positive**. Each of these two properties, i.e., ultra log-concavity or γ -positivity, is also known to be stronger than the **unimodality** of the coefficients of the polynomial.

The similarity between the defining recursions of the polynomials arising in [Theorem 1](#) and [Theorem 3](#), and a significant amount of experimentation, make it plausible to postulate that these desirable properties of the coefficients might hold for $\underline{H}_M x$ and $H_M x$ as well. Indeed, these conjectural properties had already been observed in the literature.

Conjecture 7. *For every matroid M , the following polynomials have only real roots.*

- ([14, Conjecture 10.19]) *The polynomial $\underline{H}_M x$, i.e., the Hilbert–Poincaré series of the Chow ring $\underline{CH} M$.*
- ([25, Conjecture 4.3.3]) *The polynomial $H_M x$, i.e., the Hilbert–Poincaré series of the augmented Chow ring $CH M$.*
- ([23, Conjecture 5.1]) *The polynomial $Z_M x$, i.e., the Hilbert–Poincaré series of the intersection cohomology module $IH M$.*
- ([16, Conjecture 3.2]) *The polynomial $P_M x$, i.e., the Hilbert–Poincaré series of the stalk $IH M_\emptyset$ at the empty flat of $IH M$ ².*

The main result of Adiprasito, Huh, and Katz [1, Theorem 1.4] asserts the validity of the **Kähler package** for the Chow rings of matroids. In particular, **Poincaré duality** and the **hard Lefschetz theorem** are valid, and they imply respectively the palindromicity and the unimodality of the coefficients of the Hilbert–Poincaré series of the Chow ring. On the other hand, the Kähler package was proved for the augmented Chow ring by Braden, Huh, Matherne, Proudfoot, and Wang in [6, Theorem 1.6] and for the intersection cohomology module in [7, Theorem 1.6], so one can conclude analogous statements for the Hilbert–Poincaré series of $CH M$ and $IH M$. Less is known regarding the Hilbert–Poincaré series of $IH M_\emptyset$, i.e., the Kazhdan–Lusztig polynomial $P_M x$. Although its coefficients are nonnegative, it is not known whether they are always unimodal.

²We note that $CH M$ and $IH M$ are both modules over the graded Möbius algebra $H M$. We make this point only to be able to define $IH M_\emptyset$, which depends on the $H M$ -module structure. As the purpose of this paper is to study various Hilbert–Poincaré series, we only need to view $\underline{H} M$, $H M$, $\underline{CH} M$, $CH M$, $IH M$, and $IH M_\emptyset$ as graded vector spaces.

Another of our contributions will be to provide a proof of the γ -positivity for three of the families of polynomials above (the palindromic ones, those for which this statement is meaningful).

Theorem 8. *For every matroid M , the polynomials $\underline{H}_M x$, $H_M x$, and $Z_M x$ are γ -positive.*

This result gives further evidence to part of [Conjecture 7](#), and resolves a problem posed in [\[13, Conjecture 5.6\]](#) by Ferroni, Nasr, and Vecchi that was known to hold only in certain cases. The proofs of the γ -positivity of each of these families are surprisingly simple, and depend only on a recursion witnessed by the validity of the semi-small decompositions of [\[6\]](#) for the Chow ring and the augmented Chow ring. For the intersection cohomology, a recursion found by Braden and Vysogorets [\[8\]](#) does the job.

Remark 9. *Two natural questions that are reasonable to formulate are whether the Kazhdan–Lusztig polynomial $P_M x$ is a “non-symmetric γ -positive” polynomial, in the sense of [\[3, Section 5.1\]](#), and whether the equivariant Z -polynomial (see [\[23, Section 6\]](#)) is “equivariant γ -positive”, in the sense of [\[3, Section 5.2\]](#). The answer to both of these questions is negative, and it is possible to provide explicit counterexamples.*

3 Uniform matroids, binomial Eulerian polynomials, and matroid polytopes

The case of uniform matroids is of particular interest both in the Kazhdan–Lusztig and Chow-theoretic frameworks.

For the Z -polynomial and the Kazhdan–Lusztig polynomial of uniform matroids, some explicit formulas are known (see e.g. [\[15\]](#)); we point out that the real-rootedness of $P_M x$ and $Z_M x$ for $M \sim U_{k,n}$ are still open problems.

For the Hilbert–Poincaré series of both the Chow ring and the augmented Chow ring, the situation is also complicated. The polynomials $\underline{H}_M x$ for arbitrary uniform matroids were addressed by Hameister, Rao, and Simpson in [\[18\]](#); they give a nice description of $\underline{H}_{U_{k,n}} x$ in terms of statistics of permutations, but this description turns out to be quite intricate from a computational point of view.

For Boolean matroids, the polynomial $H_M x$ has been studied in detail recently in [\[22, 4, 24, 17, 19, 10\]](#). To be precise, for a Boolean matroid $M \sim U_{n,n}$ one has that $\underline{H}_M x = A_n x$, the n -th Eulerian polynomial (named this way in [\[24\]](#)), whereas $H_M x = \tilde{A}_n x$ is the n -th binomial Eulerian polynomial; notice that the recursion in [Theorem 3 \(iii\)](#) asserts the identity

$$\tilde{A}_n x = \sum_{j=0}^n \binom{n}{j} x^j A_{n-j} x.$$

One can see the Hilbert–Poincaré series of Chow rings and augmented Chow rings as vast generalizations of these polynomials, and within this broader framework one can derive some interesting identities between them.

The polynomials $P_M(x)$, $Z_M(x)$, $\underline{H}_M(x)$, and $H_M(x)$ are defined recursively. In particular, computing them for fairly small matroids is already a challenging computational task. Towards simplifying their computation, a further contribution we make is extending a result of Ardila and Sanchez [2, Theorem 8.9] by proving the following result for $\underline{H}_M(x)$ and $H_M(x)$.

Theorem 10. *The maps $M \mapsto \underline{H}_M(x)$ and $M \mapsto H_M(x)$ are valutive under matroid polytope subdivisions.*

In other words, one can compute the Hilbert–Poincaré series of the augmented Chow ring of a matroid M by subdividing its base polytope $\mathcal{P}(M)$ into smaller matroid polytopes and computing it for those smaller pieces. As a consequence, by using the notion of relaxations of stressed subsets of Ferroni and Schröter [14], one proves that the Hilbert–Poincaré series of the Chow ring and augmented Chow ring behave well under relaxations of stressed subsets (in particular, stressed hyperplanes). This in turn yields fast ways of computing $\underline{H}_M(x)$ and $H_M(x)$ for certain conjecturally “predominant” classes of matroids.³ In particular, for the class of paving matroids, we obtain the following expressions in analogy to [13, Theorem 1.4].

Theorem 11. *Let M be a paving matroid of rank k and cardinality n . Then,*

$$\begin{aligned}\underline{H}_M(x) &= \underline{H}_{U_{k,n}}(x) - \sum_{h \geq k} \lambda_h \left(\underline{H}_{U_{k,h-1}}(x) - \underline{H}_{U_{k-1,h} \oplus U_{1,1}}(x) \right), \\ H_M(x) &= H_{U_{k,n}}(x) - \sum_{h \geq k} \lambda_h \left(H_{U_{k,h-1}}(x) - H_{U_{k-1,h} \oplus U_{1,1}}(x) \right),\end{aligned}$$

where λ_h denotes the number of stressed hyperplanes of size h in M .

In order to make these formulas useful in practice, we prove a conjecture of Hameister, Rao, and Simpson [18, Conjecture 6.2]; this and its analogue in the augmented case yield an explicit formula for $\underline{H}_{U_{k,n}}(x)$ and $H_{U_{k,n}}(x)$ for arbitrary uniform matroids. We state an equivalent reformulation now, which also highlights the connection with (binomial) Eulerian polynomials.

³In the terminology of Mayhew et al. [21], we mean that asymptotically almost every matroid lies in the alleged class.

Theorem 12. *The Hilbert–Poincaré series of the Chow ring and the augmented Chow ring of a uniform matroid of rank k and cardinality n are given by*

$$\begin{aligned} \underline{H}_{U_{k,n}}(x) &= \sum_{j=0}^{k-1} \binom{n}{j} A_j(x) \bar{\chi}_{U_{k-j,n-j}}(x), \\ H_{U_{k,n}}(x) &= \sum_{j=0}^{k-1} (-1)^{k-1-j} \binom{n}{j} \binom{n-1-j}{k-1-j} \tilde{A}_j(x) \frac{1-x^{k-j-1}}{1-x}. \end{aligned}$$

These formulas can be plugged into the equations in [Theorem 11](#) to provide a very fast way of computing the Hilbert–Poincaré series of Chow rings and augmented Chow rings of all paving matroids. In particular, we can test with a computer the validity of [Conjecture 7](#) for sparse paving matroids with up to 40 elements.⁴

Proposition 13. *Let M be a sparse paving matroid on a ground set with at most 40 elements. Then both $\underline{H}_M(x)$ and $H_M(x)$ are real-rooted polynomials.*

4 Monotonicity on coefficients

An unpublished conjecture posed by Gedeon asserts that the maximum of the coefficients of the Kazhdan–Lusztig and Z -polynomial of a matroid of rank k and size n is attained when the matroid is isomorphic to $U_{k,n}$; this conjecture first appeared in print in [20, Conjecture 1.2]. It is known to hold for all paving matroids [13, Theorem 1.5]. Here we prove that its counterpart for the polynomials $\underline{H}_M(x)$ and $H_M(x)$ indeed holds for all matroids.

Theorem 14. *Let M be a loopless matroid of rank k on a ground set of size n . The following coefficient-wise inequalities hold:*

$$\begin{aligned} \underline{H}_M(x) &\preceq \underline{H}_{U_{k,n}}(x), \\ H_M(x) &\preceq H_{U_{k,n}}(x). \end{aligned}$$

In other words, uniform matroids maximize coefficient-wisely the coefficients of the Hilbert–Poincaré series of Chow rings and augmented Chow rings among all matroids with fixed rank and size.

The key gadget to prove this conjecture is given by the construction of an injective map from the barycentric subdivision of the lattice of flats of each matroid of rank k

⁴We observe that, using a rough estimation [5, Equation (10)], there are approximately $2^{2^{40}/60} \approx 10^{5000000000}$ sparse paving matroids on a ground set of cardinality 40.

Table 1: Examples of $\underline{H}_M x$ and $H_M x$ for some uniform matroids M

(a) Examples of $\underline{H}_{U_{k,k-1}} x$								(b) Examples of $H_{U_{k,k-1}} x$							
k	1	2	3	4	5	6	7	k	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
x		1	7	21	51	113	239	x^1	1	4	11	26	57	120	247
x^2			1	21	161	813	3361	x^2		1	11	66	302	1191	4293
x^3				1	51	813	7631	x^3			1	26	302	2416	15619
x^4					1	113	3361	x^4				1	57	1191	15619
x^5						1	239	x^5					1	120	4293
x^6							1	x^6						1	247
								x^7							1

(c) Examples of $\underline{H}_{U_{k,k-2}} x$								(d) Examples of $H_{U_{k,k-2}} x$							
k	1	2	3	4	5	6	7	k	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
x^1		1	11	36	92	211	457	x^1	1	5	16	42	99	219	466
x^2			1	36	337	1877	8269	x^2		1	16	117	610	2641	10204
x^3				1	92	1877	20155	x^3			1	42	610	5637	40444
x^4					1	211	8269	x^4				1	99	2641	40444
x^5						1	457	x^5					1	219	10204
x^6							1	x^6						1	466
								x^7							1

and cardinality n to the barycentric subdivision of the lattice of flats of $U_{k,n}$. This can be combined with some of our formulas for $\underline{H}_M x$ and $H_M x$, which show that these two polynomials can be written as a sum of polynomials with positive coefficients indexed by chains of flats.

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A Murnaghan–Nakayama rule for quantum cohomology of the flag manifold

C. Benedetti^{*1}, N. Bergeron^{†2}, L. Colmenarejo^{‡3},
F. Saliola^{§4} and F. Sottile^{¶5}

¹Universidad de Los Andes, Bogotá, Colombia

²York University, Toronto, Canada

³North Carolina State University, Raleigh, USA

⁴Université du Québec à Montréal, Montréal, Canada

⁵Texas A&M University, College Station, USA

Abstract. We give a rule for the multiplication of a Schubert class by a tautological class in the (small) quantum cohomology ring of the full flag manifold. As an intermediary step, we establish a formula for the multiplication of a Schubert class by a quantum Schur polynomial indexed by a hook partition.

Résumé. On décrit une règle pour la multiplication d’une classe de Schubert par une classe tautologique dans le (petit) anneau de cohomologie quantique de la variété de drapeaux. En cours de route, on obtient une formule pour la multiplication d’une classe de Schubert par un polynôme de Schur quantique correspondant à une équerre.

Keywords: Murnaghan–Nakayama Formula, Schubert Polynomials.

The *Murnaghan–Nakayama rule* is a method for computing values of the irreducible characters of symmetric groups. Under the identification of the character ring with the ring of symmetric functions, it translates into a rule for the multiplication of a Schur symmetric function s_λ by a Newton power sum p_r . As explained in [15, Sect. 2], in the cohomology ring of the full flag manifold, the Newton power sums correspond to tautological classes that are components of the Chern character. Accordingly, a formula describing multiplication by a tautological class is also called a *Murnaghan–Nakayama rule* since it corresponds to multiplication by a Newton power sum.

Here, in [Corollary 2.3](#) we describe a rule for multiplication by a tautological class in the small quantum cohomology ring of the full flag manifold. As an intermediary step,

^{*}c.benedetti@uniandes.edu.co. C. Benedetti was partially supported by the grant FAPA of the Faculty of Science of U. de Los Andes.

[†]Bergeron@yorku.ca. N. Bergeron was partially supported by NSERC and York Research Chair.

[‡]lcolmen@ncsu.edu.

[§]saliola.franco@uqam.ca. F. Saliola was partially supported by an NSERC Discovery Grant.

[¶]sottile@tamu.edu. F. Sottile was partially supported by NSF grant DMS-2201005.

we show in [Theorem 2.2](#) a formula for multiplication by a hook quantum Schur polynomial. This entails a detailed analysis of chains and intervals in the quantum Bruhat order. Making use of results by N. C. Leung and C. Li [\[10\]](#) and A. Postnikov [\[17\]](#), it turns out that quantum products by hook Schur polynomials can be reduced to the classical product from [\[18\]](#).

1 A Murnaghan–Nakayama rule for (classic) cohomology

We begin by reviewing combinatorial models for the cohomology ring of the flag manifold. (We omit the precise definitions of the flag manifold and its cohomology as the combinatorial models will suffice for our purposes; the interested reader can consult [\[7\]](#).) Then we will derive a formula for the product of a Schubert polynomial and a Newton power sum, which will be instructive in deriving our quantum analogue.

Let $\mathbb{F}\ell_n$ denote the manifold of complete flags in $\mathbb{C}^n$. Its cohomology ring $H^*\mathbb{F}\ell_n$ with coefficients in $\mathbb{Z}$ is a $\mathbb{Z}$ -module with a distinguished basis $\{\mathfrak{S}_w\}_{w \in S_n}$ whose elements $\mathfrak{S}_w$ are called *Schubert classes* and are indexed by elements of the symmetric group S_n .

Borel [\[4\]](#) gave a presentation of $H^*\mathbb{F}\ell_n$ as a quotient of a polynomial ring: specifically, $H^*\mathbb{F}\ell_n \simeq \mathbb{Z}[x_1, \dots, x_n] / I_n$, where $I_n = \langle e_l(x_1, \dots, x_n) \mid l \in [n] \rangle$ where e_l is the l th elementary symmetric polynomial and $[n] = \{1, \dots, n\}$.

Lascoux and Schützenberger [\[9\]](#) defined a set of polynomials in $\mathbb{Z}[x]$ called *Schubert polynomials* that correspond to the Schubert classes $\mathfrak{S}_w$ under Borel’s isomorphism. We write $\mathfrak{S}_w(x)$ for the Schubert polynomial associated with $\mathfrak{S}_w$ on the set of variables $x = \{x_1, \dots, x_n\}$. Every Schur polynomial $s_\lambda(x_1, \dots, x_k)$ is a Schubert polynomial, as follows. Let $w \in S_n$ be a permutation with a unique descent at k . Then the sequence $w(k-1), \dots, w(2), w(1)$ is a partition contained in the rectangular partition $R_{k, n-k}$ of $k + n - k$ into k parts with each part of size $n - k$. This is a bijection and for $\lambda \subseteq R_{k, n-k}$ we write $v(\lambda, k) \in S_n$ for the corresponding permutation. Then $\mathfrak{S}_{v(\lambda, k)}(x) = s_\lambda(x_1, \dots, x_k)$.

Open Problem 1.1. Find a combinatorial rule expressing $\mathfrak{S}_u \cdot \mathfrak{S}_v$ in terms of the basis $\{\mathfrak{S}_w\}_{w \in S_n}$ of $H^*\mathbb{F}\ell_n$. That is, find a combinatorial rule for the coefficients $c_{u,v}^w \in \mathbb{Z}$ in the product

$$\mathfrak{S}_u(x) \cdot \mathfrak{S}_v(x) = \sum_w c_{u,v}^w \mathfrak{S}_w(x). \quad (1.1)$$

A remarkable property of the Schubert polynomials is that the computation in [Open Problem 1.1](#) is performed in the polynomial ring, not in the quotient. The resulting coefficients $c_{u,v}^w$ in the quotient will be the same.

Although [Problem 1.1](#) remains open in general, several special cases are known. [Monk’s Formula](#) [\[13\]](#) treats the case where v is the simple transposition $(k, k+1)$: for any $u \in S_n$, we have $\mathfrak{S}_u \cdot \mathfrak{S}_{(k, k+1)} = \sum \mathfrak{S}_{u \cdot i, j}$ with the sum ranging over all transpositions (i, j) satisfying $i \leq k < j \leq n$ and $\ell(u \cdot i, j) = \ell(u) + 1$, where $\ell(u) = \#\{i < j \mid u(i) > u(j)\}$.

The terms appearing in Monk's Formula define a partial order on S_n graded by ℓu . The *k -Bruhat order* $\leq_k$ is defined by the covering relation $u \leq_k u_{i,j}$ if $i \leq k < j \leq n$ and $\ell u = 1 + \ell u_{i,j}$; that is, $u \leq_k u_{i,j}$ if $\mathfrak{S}_{u_{i,j}}$ appears in the product $\mathfrak{S}_u x \cdot \mathfrak{S}_{k,k-1}$.

Write u, w_k for the interval between u and w in the k -Bruhat order. Label each cover relation $u \leq_k u_{i,j}$ by $u \xrightarrow{i}$ and write $u \xrightarrow{u^i} u_{i,j}$. The maximal element of a (saturated) chain $\gamma: u \xrightarrow{a_1} u_1 \xrightarrow{a_2} \cdots \xrightarrow{a_r} u_r$ is denoted $\text{end } \gamma = u_r$ and this chain has *length* r . If $a_1 > \cdots > a_l < a_{l+1} < \cdots < a_r$, then γ is a *peakless chain of height l and length r* .

The following case of [Open Problem 1.1](#) was proved in [18].

Proposition 1.2. *Let $u \in S_n$, $l \leq k$, and $m \leq n-k$. Then*

$$\mathfrak{S}_u x \cdot s_{m,1^{l-1}} x_1, \dots, x_k = \sum_{\gamma} \mathfrak{S}_{\text{end } \gamma} x \quad (1.2)$$

summing over all peakless chains of height l and length $m + l - 1$ in the labeled k -Bruhat order.

To describe the multiplicities in (1.2), we define the *Grassmannian–Bruhat order* on S_n by $\eta \preceq \zeta$ if there are a permutation $u \in S_n$ and an integer k with $u \leq_k \eta u \leq_k \zeta u$. This is a graded partial order on S_n with minimal element the identity permutation e . Its rank function is $\mathcal{L} \zeta = \ell \zeta u - \ell u$ for any $u \in S_n$ and $k \in \mathbb{N}$ with $u \leq_k \zeta u$. The cover relations also inherit the labelling from the k -Bruhat orders. As shown in [1], neither $\preceq$, $\mathcal{L}$ nor the labelling depend on the choice of k or u .

We say that a permutation ζ is *minimal* if $\mathcal{L} \zeta = \#\text{supp } \zeta - s_{\zeta}$, where $\text{supp } \zeta = \{i \in [n] \mid \zeta(i) \neq i\}$ and s_{ζ} is the number of nontrivial cycles in the factorization of ζ into disjoint cycles. The *height* of a minimal permutation ζ is $\text{ht } \zeta = \#\{i \in [n] \mid i < \zeta(i)\}$. Using results from [3], the following characterizes minimal permutations in terms of the Grassmannian–Bruhat order.

Lemma 1.3. *A permutation ζ is minimal if and only if the interval $[e, \zeta]_{\preceq}$ has a peakless chain. The number of peakless chains of height l in $[e, \zeta]_{\preceq}$ is the binomial coefficient $\binom{s_{\zeta}-1}{\text{ht } \zeta - l}$.*

[Proposition 1.2](#) now reads as follows.

Proposition 1.4. *Let $u \in S_n$, $l \leq k$ and $m \leq n-k$. Then*

$$\mathfrak{S}_u x \cdot s_{m,1^{l-1}} x_1, \dots, x_k = \sum \binom{s_{\zeta}-1}{\text{ht } \zeta - l} \mathfrak{S}_{\zeta u} x, \quad (1.3)$$

summing over all minimal permutations $\zeta \in S_n$ such that $u \leq_k \zeta u$ and $\mathcal{L} \zeta = m + l - 1$.

We use this to describe the product of a Schubert polynomial and a power sum symmetric polynomial. Let $p_r x_1, \dots, x_k : x_1^r \cdots x_k^r$ denote the r th power sum polynomial. Expanded in terms of Schur polynomials [11], we have

$$p_r x_1, \dots, x_k = s_{r-1,1} x_1, \dots, x_k + \cdots + (-1)^{r-1} s_{1^r} x_1, \dots, x_k. \quad (1.4)$$

Corollary 1.5. *Let $u \in S_n$. Then*

$$\mathfrak{S}_u x \cdot p_r x_1, \dots, x_k = \sum -1^{\text{ht } \zeta - 1} \mathfrak{S}_{\zeta u} x, \quad$$

summing over all minimal cycles $\zeta \in S_n$ such that $u \leq_k \zeta u$ and $\mathcal{L}(\zeta) = r$.

Proof. Expanding the product $\mathfrak{S}_u x \cdot p_r x_1, \dots, x_k$ using Equations (1.4) and (1.3) shows that the coefficient of $\mathfrak{S}_{\zeta u}$ for $u \leq_k \zeta u$ is zero unless $\mathcal{L}(\zeta) = r$ and ζ is minimal. When ζ is the product of s ζ -minimal cycles whose supports are disjoint, this coefficient is

$$-1^{\text{ht } \zeta - 1} \sum_{l=1}^r -1^{l - \text{ht } \zeta} \binom{s \zeta - 1}{\text{ht } \zeta - l} = \begin{cases} 0, & \text{if } s \zeta \neq 1, \\ -1^{\text{ht } \zeta - 1}, & \text{if } s \zeta = 1. \end{cases} \quad \square$$

This proof differs from that given in [14, 15].

2 A Murnaghan–Nakayama rule for quantum cohomology

The (small) quantum cohomology ring $qH^*\mathbb{F}\ell_n$ of the flag manifold is a deformation of the cohomology ring $H^*\mathbb{F}\ell_n$ whose product encodes the three-point Gromov–Witten invariants of $\mathbb{F}\ell_n$. To simplify notation, write $\mathbb{Z} q$ for $\mathbb{Z} q_1, \dots, q_{n-1}$, where $q_1, \dots, q_{n-1}$ are indeterminates. As a $\mathbb{Z} q$ -module, we have $qH^*\mathbb{F}\ell_n = \mathbb{Z} q \otimes_{\mathbb{Z}} H^*\mathbb{F}\ell_n$. It follows that the Schubert classes $\{\mathfrak{S}_w\}_{w \in S_n}$ in $H^*\mathbb{F}\ell_n$ form a $\mathbb{Z} q$ -basis of $qH^*\mathbb{F}\ell_n$. We write $*$ to distinguish this “*quantum product*” from the “*classical product*” in $H^*\mathbb{F}\ell_n$.

Open Problem 2.1. *Find a combinatorial rule for the coefficients $C_{u,v}^w \in \mathbb{Z} q$ in the product*

$$\mathfrak{S}_u * \mathfrak{S}_v = \sum_{w \in S_n} C_{u,v}^w \mathfrak{S}_w. \quad (2.1)$$

Givental and Kim [8] proved an analogue of Borel’s presentation, expressing $qH^*\mathbb{F}\ell_n$ as a quotient of $\mathbb{Z} q[x]$. We write $\psi : \mathbb{Z} q[x] \rightarrow qH^*\mathbb{F}\ell_n$ for the corresponding surjection. Fomin, Gelfand, and Postnikov [5] defined *quantum Schubert polynomials* $\mathfrak{S}_w^q x \in \mathbb{Z} q[x]$ so that the quantum product of Schubert classes satisfies $\mathfrak{S}_u * \mathfrak{S}_v = \psi(\mathfrak{S}_u^q x \cdot \mathfrak{S}_v^q x)$. To simplify notation, for $f(x) \in \mathbb{Z} q[x]$, we will write $\mathfrak{S}_u * f(x)$ for $\psi(\mathfrak{S}_u^q x \cdot f(x))$.

As we already mentioned, if $\lambda \subseteq R_{k,n-k}$, then $s_\lambda x_1, \dots, x_k = \mathfrak{S}_{v_{\lambda,k}} x$. Analogously, for $\lambda \subseteq R_{k,n-k}$, define the *quantum Schur polynomials* as

$$s_\lambda^q x_1, \dots, x_k : \mathfrak{S}_{v_{\lambda,k}}^q x,$$

and following Equation (1.4), define the *quantum power sum* as:

$$p_r^q x_1, \dots, x_k : s_r^q x_1, \dots, x_k = s_{r-1,1}^q x_1, \dots, x_k + \dots + (-1)^{r-1} s_{1^r}^q x_1, \dots, x_k. \quad (2.2)$$

We derive a formula for $\mathfrak{S}_u * p_r^q x$ via intermediary formulas for $\mathfrak{S}_u * s_{b,1^{a-1}}^q x$, as in the proof of [Corollary 1.5](#).

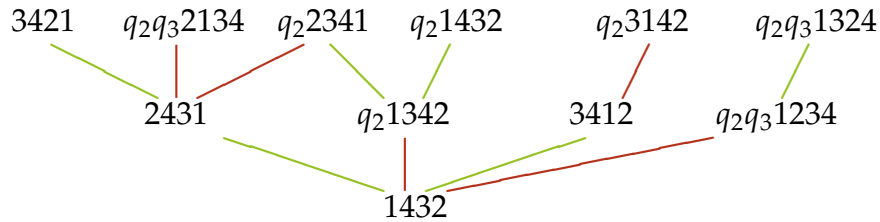
We begin with the [quantum Monk Formula](#) of Fomin, Gelfand, and Postnikov [5]: for $u \in S_n$ and $1 \leq k < n$, $\mathfrak{S}_u * \mathfrak{S}_{k,k-1} = \sum \mathfrak{S}_{u_{i,j}} + \sum q_{i,j} \mathfrak{S}_{u_{i,j}}$, where the first sum is given by the classical Monk Formula and the second sum is over transpositions i, j with $i \leq k < j$ and $\ell(u_{i,j}) = \ell(u) - 2$, and $q_{i,j} = q_i q_{i-1} \cdots q_{j-1}$. For example,

$$\mathfrak{S}_{1432} * \mathfrak{S}_{2,3} = \mathfrak{S}_{2431} + \mathfrak{S}_{3412} + q_2 \mathfrak{S}_{1342} + q_{2,4} \mathfrak{S}_{1234}.$$

The terms appearing in the quantum Monk Formula define a ranked partial order on $S_n(q)$: $\{q^\alpha u : u \in S_n \text{ and } q^\alpha \text{ is a monomial in } q_1, \dots, q_{n-1}\}$ as follows. The [quantum \$k\$ -Bruhat order](#) $\leq_k^q$ on $S_n(q)$ is defined by the following cover relations:

1. $u \leq_k^q u_{i,j}$ if $i \leq k < j$ and $\ell(u_{i,j}) = \ell(u) - 1$, so that $u = u_{i,j} s_{i,j}$;
2. $u \leq_k^q q_{i,j} u_{i,j}$ if $i \leq k < j$ and $\ell(u_{i,j}) = \ell(u) - 2$;
3. extend q -multiplicatively: $u \leq_k^q w$ if and only if $q^\alpha u \leq_k^q q^\alpha w$ for $u, w \in S_n(q)$.

We display two levels of the quantum 2-Bruhat order on $S_4(q)$ above 1432.



The rank function ℓ on the quantum k -Bruhat order is given by $\ell(q^\alpha u) = 2 \deg q^\alpha + \ell(u)$, where $\deg q^\alpha$ is the usual degree of the monomial q^α in $\mathbb{Z}[q]$. As far as we know, there is no analogue of the Grassmannian–Bruhat order related to the quantum k -Bruhat order. Consequently, rather than speak of minimal permutations (as in the classical case), we will define minimal intervals in the quantum k -Bruhat order. Suppose that $u, w \in S_n$ are permutations and $u \leq_k^q q^\alpha w$. We say that the interval $[u, q^\alpha w]_k^q$ is [minimal](#) if its rank $\ell(q^\alpha w) - \ell(u)$ equals $\#\text{supp } wu^{-1} - s(wu^{-1})$. When $q^\alpha = 1$, the interval is minimal if and only if wu^{-1} is a minimal permutation.

We state our two main results.

Theorem 2.2. *Let $u \in S_n$, $l \leq k$ and $m \leq n - k$. Then*

$$\mathfrak{S}_u * s_{m,1^{l-1}}^q(x_1, \dots, x_k) = \sum \binom{s(wu^{-1}) - 1}{\text{ht } wu^{-1} - l} q^\alpha \mathfrak{S}_w,$$

summing over all minimal intervals $[u, q^\alpha w]_k^q$ such that $\ell(q^\alpha w) - \ell(u) = m - l + 1$.

The following quantum Murnaghan–Nakayama formula was conjectured in [14]. It follows from [Theorem 2.2](#) in the same way that [Corollary 1.5](#) follows from [Proposition 1.4](#).

Corollary 2.3. *Let $u \in S_n$. Then*

$$\mathfrak{S}_u * p_r^q(x_1, \dots, x_k) = \sum -1^{\text{ht } w^{-1}u - 1} q^\alpha \mathfrak{S}_w,$$

summing over all minimal intervals $[u, q^\alpha w_k^q]$ of rank r such that $w^{-1}u$ is a single cycle.

For example, in $qH^*\mathbb{F}\ell_8$, if $u = 38254671$, then the product $\mathfrak{S}_u * p_4^q(x_1, \dots, x_4)$ is

$$\begin{aligned} \mathfrak{S}_{2,5,6,3,4} u &= \mathfrak{S}_{2,3,5,6,4} u - \mathfrak{S}_{2,5,7,6,4} u - \mathfrak{S}_{3,5,7,6,4} u - q_4 \mathfrak{S}_{2,4,7,6,5} u - q_4 \mathfrak{S}_{3,4,7,6,5} u \\ &\quad - q_4 \mathfrak{S}_{2,3,4,6,5} u - q_{2,5} \mathfrak{S}_{2,4,6,5,8} u - q_{2,5} \mathfrak{S}_{2,5,6,4,8} u - q_{2,5} \mathfrak{S}_{2,3,4,5,8} u \\ &\quad - q_{4,8} \mathfrak{S}_{1,7,6,5,4} u - q_{2,8} \mathfrak{S}_{1,7,6,5,8} u - q_{2,8} \mathfrak{S}_{1,8,2,3,4} u - q_4 q_{2,8} \mathfrak{S}_{1,8,2,4,5} u. \end{aligned}$$

3 Sketch of Proof and Remarks

We prove [Theorem 2.2](#) in two parts. In the first part we show that if $[u, q^\alpha w_k^q]$ is a minimal interval, then there is a hook partition λ with $|\lambda| = \ell(q^\alpha w) - \ell(u)$ such that $C_{v, \lambda, k, u}^{q^\alpha w}$ is nonzero. This uses a study of chains in the interval $[u, q^\alpha w_k^q]$ in terms of certain left operators modeled on those for the ordinary k -Bruhat order as developed in [\[2\]](#). This is sketched in [Section 3.1](#).

Our second part of the proof uses a Corollary to the proof of [Theorem 1.2](#) in [\[10\]](#), the result of Leung and Li that “quantum equals classical”. We discuss this in [Section 3.2](#). A consequence is that if $C_{v, \lambda, k, u}^{q^\alpha w} \neq 0$ for some partition λ , then there are $y, z \in S_n$ with $y \leq_k z$, $\ell(z) - \ell(y) = \ell(q^\alpha w) - \ell(u) = |\lambda|$, and $zy^{-1} = wu^{-1}$ such that for all partitions μ with $|\mu| = |\lambda|$, $C_{v, \mu, k, u}^{q^\alpha w} = q^\alpha c_{v, \mu, k, y}^z$. If λ is a hook partition, then [Proposition 1.4](#) implies that the interval $[u, q^\alpha w_k^q]$ is minimal. Then [Theorem 2.2](#) follows from these results and [Proposition 1.4](#).

We point out [Theorem 15](#) of [\[12\]](#) also claims that quantum equals classical for hook partitions, and from which one could in principle deduce our [Theorem 2.2](#). Unfortunately, its proof invokes a lemma ([Lemma 13](#) in [\[12\]](#)) whose hypotheses do not apply to the case (hook partitions) that is invoked. We sketch this in [Section 3.3](#). This necessitates the alternative proof we provide.

3.1 Left Operators

We now sketch a proof of the first part in the following result.

Theorem 3.1. *Let $u, w \in S_n$ and $q^\alpha w \in S_n$. Then $[u, q^\alpha w_k^q]$ is a minimal quantum interval if and only if there is a hook partition λ with $|\lambda| = \ell(q^\alpha w) - \ell(u)$ such that $C_{v, \lambda, k, u}^{q^\alpha w} \neq 0$.*

A cover $y \xrightarrow{q} q^\alpha z$ in the quantum k -Bruhat order gives a transposition $a, b \rightarrow zy^{-1}$. Therefore, a chain in $u, q^\alpha w \xrightarrow{q}$ provides a factorization of $\zeta : wu^{-1}$ into transpositions

$$\zeta = a_r, b_r \cdots a_2, b_2 a_1, b_1, \quad (3.1)$$

which in turn defines a (multi) graph with vertex set n whose edges are a_i, b_i for each transposition above.

Lemma 3.2. Suppose that $u, q^\alpha w \xrightarrow{q}$ is minimal and (3.1) is a factorization of $\zeta = wu^{-1}$ corresponding to a chain in $u, q^\alpha w \xrightarrow{q}$. Let F be the corresponding multigraph. Then

1. F is a forest: it is acyclic without multiple edges and its connected components are trees.
2. The vertices in connected components of F form a noncrossing partition of n .
3. ζ is a product of disjoint cycles, one for each tree in F .

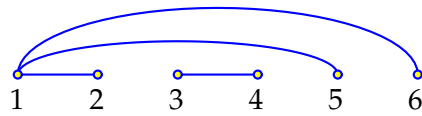
Example 3.3. We have the following chain in the quantum 3-Bruhat order:

$$\begin{aligned} 635214 & \xrightarrow{q} u \xrightarrow{q} q_{1,5} 16 u \xrightarrow{q} q_{1,5} 12 16 u \xrightarrow{q} q_{1,5} 235164 \\ & \xrightarrow{q} q_{1,5} 34 12 16 u \xrightarrow{q} q_{1,5} q_3 15 34 12 16 u \xrightarrow{q} q_1 q_2 q_3^2 q_4 241563. \end{aligned}$$

The interval $635214, q_1 q_2 q_3^2 q_4 241563 \xrightarrow{q}$ has rank four, as

$$\ell(q_1 q_2 q_3^2 q_4 241563) = 10, \quad \ell(635214) = 11.$$

Set $\zeta = 1, 5 \rightarrow 3, 4 \rightarrow 1, 2 \rightarrow 1, 6 \rightarrow 3, 4 \rightarrow 1, 6, 2, 5$. Its support has six elements and ζ consists of two cycles, and thus this interval is minimal. The supports of the two cycles in ζ are non-crossing. We draw its corresponding forest.



Postnikov [17] observed that the quantum Bruhat order (union over k of all quantum k -Bruhat orders) has a cyclic symmetry. We describe this for the quantum k -Bruhat order. Write $\bullet$ for the n -cycle $1, 2, \dots, n$. If $i > j$, define $q_{i,j} : q_{j,i}^{-1}$, a Laurent monomial. For $u, w \in S_n$, define $q(u, w) : q_{w^{-1}n, u^{-1}n}$.

Lemma 3.4. Let $u, w \in S_n$. We have $u <_k^q q^\alpha w$ if and only if $\bullet u <_k^q q(u, w) q^\alpha \bullet w$.

For $a < b$ in n , define an operator $\mathbf{v}_{a,b}$ on $S_n \setminus \{\bullet\}$ by $\mathbf{v}_{a,b} \bullet = \bullet$, and for $u \in S_n$,

$$\mathbf{v}_{a,b} \bullet u : \begin{cases} a, b \bullet u & \text{if } u \xrightarrow{q} a, b \bullet u, \\ q_{i,j} b, a \bullet u & \text{if } u \xrightarrow{q} q_{i,j} b, a \bullet u \text{ with } u^{-1} b = i \leq k < u^{-1} a = j, \\ \bullet & \text{otherwise,} \end{cases}$$

and extend this action multiplicatively. We will say that $\mathbf{v}_{a,b}$ is a *classical operator* if the first case holds, and that it is a *quantum operator* when the second case holds.

A composition of operators $\mathbf{v} = \mathbf{v}_{a_r, b_r} \cdots \mathbf{v}_{a_1, b_1}$ is zero if for all $u \in S_n$, $\mathbf{v} \bullet u = \mathbf{0}$ and otherwise it is nonzero. If $\mathbf{v}' = \mathbf{v}_{c_r, d_r} \cdots \mathbf{v}_{c_1, d_1}$ is another composition of operators, then it is *equivalent to $\mathbf{v}$* if for all $u \in S_n$, $\mathbf{v} \bullet u = \mathbf{v}' \bullet u$. When $\mathbf{v}$ is nonzero, there are $u, w \in S_n$ and $q^\alpha w \in S_n$ such that $\mathbf{v} \bullet u = q^\alpha w$. In this case, $u \leq_k^q q^\alpha w$, $\mathbf{v}$ corresponds to a chain in $u, q^\alpha w$, and $\mathbf{v}$ corresponds to a factorization (3.1) of $\zeta = wu^{-1}$. Then the chains in $u, q^\alpha w$ correspond to compositions $\mathbf{v}'$ of operators equivalent to $\mathbf{v}$.

By Lemma 3.4, *cyclic shift* (multiplication by $\circ$) is an isomorphism of intervals $u, q^\alpha w$, and it acts (via conjugation) on chains in intervals and on the operators $\mathbf{v}_{a,b}$, preserving relations. If $b \neq n$, then $\circ \mathbf{v}_{a,b} \circ^{-1} = \mathbf{v}_{a-1, b-1}$ and this preserves classical/quantum operators. However, if $b = n$, then $\circ \mathbf{v}_{a,n} \circ^{-1} = \mathbf{v}_{1,a-1}$, and it interchanges classical operators with quantum operators. Key to our proof of the forward implication of Theorem 3.1 is showing that a chain in a minimal interval may be cyclically shifted (conjugated by some power of $\circ$) to become a chain of classical operators.

The relations among classical operators were determined in [2], and they are all of degree two or three. In particular, given two classical operators $\mathbf{v}_{a,b}$ and $\mathbf{v}_{c,d}$:

- they commute if a, b, c, d is non-crossing, in the sense of Lemma 3.2(2);
- their product is nonzero and they do not commute if $b = c$ or $a = d$; and
- their product is zero if the partition $\{a, b\} \sqcup \{c, d\}$ is crossing or if $a = c$ or $b = d$.

There is an interesting relation of degree three that we do not describe. If we also consider quantum operators, we do not have a complete set of relations, but do know those of degrees two and three, for they are induced from the classical relations by the cyclic symmetry of Lemma 3.4.

A composition $\mathbf{v} = \mathbf{v}_{a_r, b_r} \cdots \mathbf{v}_{a_1, b_1}$ is a *connected (classical) row* if $b_1 = a_2, b_2 = a_3, \dots, b_{r-1} = a_r$. It is a *connected (classical) column* if $a_1 = b_2, \dots, a_{r-1} = b_r$. The graph with vertices $\{a_1, b_1, \dots, a_r, b_r\}$ and edges $a_i b_i$ is a path in either case. A *classical row* is a composition of connected rows in which every pair has noncrossing support (a noncrossing composition), and the same for a *classical column*. A composition is a *row (column)* if it may be cyclically shifted to a classical row (column).

Given these definitions, we may express Postnikov's quantum Pieri formula [16] as

$$\mathfrak{S}_u * s_m^q(x_1, \dots, x_k) = \sum q^\alpha \mathfrak{S}_v,$$

summing over all $q^\alpha v = \mathbf{R} \bullet u$, where $\mathbf{R}$ is a row of length m . For quantum multiplication by $s_m^q(x_1, \dots, x_k)$, replace row by column. A consequence of the Pieri formula and Lemma 3.4 is that if $u \leq_k^q q^\alpha w$ and λ is a partition with $|\lambda| = \ell$, $q^\alpha w = \ell u$, then

$$q(u, w) C_{v, \lambda, k, u}^{q^\alpha w} = C_{v, \lambda, k, ou}^{q(u, w) q^\alpha ow}. \quad (3.2)$$

A composition $\mathbf{T} = \mathbf{v}_{a_t, b_t} \cdots \mathbf{v}_{a_1, b_1}$ is a *tree* if the graph with vertices $\{a_t, b_t, \dots, a_1, b_1\}$ given by the corresponding factorization is a tree. By Lemma 3.2, if $u, q^\alpha w$ is a minimal

quantum interval, then any chain has corresponding composition of operators that is a *noncrossing forest*: a composition of trees in which every pair has noncrossing support. We state our main technical result.

Lemma 3.5. *Suppose that $\mathbf{v}$ is a nonzero noncrossing forest. Then $\mathbf{v}$ is equivalent to the product of a row $\mathbf{R}$ and a column $\mathbf{C}$, and the product $\mathbf{R} \cdot \mathbf{C}$ may be cyclically shifted to the product of a classical row and a classical column.*

Proof of the forward implication of Theorem 3.1. Suppose that $u, q^\alpha w_k^q$ is a minimal interval. By Lemma 3.2, there is a noncrossing forest $\mathbf{F}$ of left operators of length $\ell(q^\alpha w) - \ell(u)$ with $q^\alpha w = \mathbf{F} \bullet u$. By Lemma 3.5, there are a row $\mathbf{R}$ and a column $\mathbf{C}$ of lengths m and l , respectively, with $m + l = \ell(q^\alpha w) - \ell(u)$ and $q^\alpha w = \mathbf{R} \cdot \mathbf{C} \bullet u$. By Lemma 3.5 again, $\mathbf{R} \cdot \mathbf{C}$ may be cyclically shifted to the product of a classical row and a classical column. If r is that shift, then $\mathfrak{S}_{\sigma^r w}$ appears in

$$s_{m+1} x_1, \dots, x_k \cdot s_{1^l} x_1, \dots, x_k \cdot \mathfrak{S}_{\sigma^r u} \\ s_{m, 1^l} x_1, \dots, x_k \quad s_{m+1, 1^{l-1}} x_1, \dots, x_k \cdot \mathfrak{S}_{\sigma^r u},$$

which is a product in $H^* \mathbb{F} \ell_n$. By (3.2), at least one of $C_{v, m, 1^l, k, u}^{q^\alpha w}$ or $C_{v, m+1, 1^{l-1}, k, u}^{q^\alpha w}$ is nonzero. $\square$

3.2 Quantum equals Classical

Leung and Li [10] gave a different identity among the quantum coefficients than (3.2). In it, the exponent of q^α is reduced. For $i \in [n-1]$ let e_i be the exponent of q_i , so that $q^{e_i} = q_i$. We state the results of Leung and Li, in the form that we use.

Proposition 3.6. *Suppose that $u, w \in S_n$, $q^\alpha w \in S_n q$, and that λ is a partition such that $C_{v, \lambda, k, u}^{q^\alpha w} \neq 0$. Then there is an $i \in [n-1]$ such that one of the following two conditions hold:*

1. *We have $i \neq k$ and $1 - 2\alpha_i - \alpha_{i-1} - \alpha_{i+1}, \ell(w, i, i-1) = \ell(w) - 1, \ell(u, i, i-1) = \ell(u) - 1$, and $q^{\alpha - d_i w} i, i-1 \geq_q^u u, i, i-1$.*
2. *We have $i = k$ and $2 - 2\alpha_k - \alpha_{k-1} - \alpha_{k+1}, \ell(w, k, k-1) = \ell(w) - 1, \ell(u, k, k-1) = \ell(u) - 1$, and $q^{\alpha - e_k w} k, k-1 \geq_q^u u, k, k-1$.*

Proposition 3.7. *Suppose that $u, w \in S_n$, $q^\alpha w \in S_n q$, and $i \in [n-1]$, and either condition of Proposition 3.6 holds. Then for any partition λ , $C_{v, \lambda, k, u}^{q^\alpha w} = q_i C_{v, \lambda, k, u}^{q^{\alpha - e_i w} i, i-1}$.*

In [10] these are formulated in a weaker form; the stronger result of Proposition 3.7 is not stated. The journal version of this abstract will contain a complete explanation.

Observe that in the statement of Proposition 3.7, $wu^{-1} = w, i, i-1 = u, i, i-1^{-1}$. An induction on $\deg(q^\alpha)$ using Propositions 3.6 and 3.7 shows that if $C_{v, \lambda, k, u}^{q^\alpha w} \neq 0$, then

there exist $y, z \in S_n$ with $y \leq_k z$, $|\lambda| = \ell z - \ell y = \ell q^\alpha w - \ell u$, $zy^{-1} = wu^{-1}$, and $q^\alpha c_{v, \lambda, k, y}^z = C_{v, \lambda, k, u}^{q^\alpha w} / 0$. If λ is a hook partition, then this implies that zy^{-1} is a minimal permutation and thus $u, q^\alpha w \stackrel{q}{\sim}_k$ is minimal, which is the reverse implication in [Theorem 3.1](#).

Proof of Theorem 2.2. By [Theorem 3.1](#), $u, q^\alpha w \stackrel{q}{\sim}_k$ is a minimal interval if and only if there is a hook partition $\lambda \vdash m, 1^{l-1}$ with $C_{v, \lambda, u}^{q^\alpha w} \neq 0$.

It remains to compute the coefficient $C_{v, \lambda, k, u}^{q^\alpha w}$. By the induction sketched above, and using the same notation, this coefficient equals $q^\alpha c_{v, \lambda, k, y}^z$. Now by [Proposition 1.4](#), this equals $q^\alpha \begin{pmatrix} s_{wu^{-1}} - 1 \\ \text{ht } wu^{-1} - l \end{pmatrix}$, as $wu^{-1} = zy^{-1}$. $\square$

3.3 Remarks

[Proposition 1.4](#) may be deduced from a stronger result of Mészáros, Panova, and Postnikov [[12](#), Thm. 8]. There, they establish a formula for a ‘lift’ of the Schur polynomial $s_{m, 1^{l-1}}(x_1, \dots, x_k)$ to the Fomin–Kirillov [[6](#)] algebra $\mathcal{E}_n$, which is a subalgebra (generated by certain Dunkl elements θ_i) of the quotient of a free associative algebra with generators $x_{i,j}$ for $1 \leq i < j \leq n$ by certain relations, which include $x_{i,j}^2 = 0$. Their paper contains another result [[12](#), Thm. 15] which is a quantum analog of [[12](#), Thm. 8], giving a formula for a ‘quantum lift’ $s_{m, 1^{l-1}}^q(x_1, \dots, x_k)$ in the quantum Fomin–Kirillov algebra $\mathcal{E}_n^q$ [[16](#)]. This formula would imply [Theorem 2.2](#). Unfortunately, as we now sketch, there is a gap in the proof of [[12](#), Thm. 15] (we have communicated this to the authors).

In [[12](#)], the result [[12](#), Thm. 15] is deduced from the proof of [[12](#), Thm. 8] using a technical result [[12](#), Lem. 13] which roughly asserts the following: If an identity involving positive terms holds in $\mathcal{E}_n$, and if no relations $x_{i,j}^2 = 0$ were used to deduce the identity, then the same identity holds in $\mathcal{E}_n^q$. We are convinced this lemma is correct, but it unfortunately does not apply in the case invoked to prove [[12](#), Thm. 15].

In addition to using relations (but not $x_{i,j}^2 = 0$), the proof of [[12](#), Thm. 8] uses in a fundamental way [[12](#), Cor. 10], that $e_a \theta_1, \dots, \theta_a h_b \theta_1, \dots, \theta_a = 0$ in $\mathcal{E}_{a+b}$ (in their notation). However, this identity can only be proven directly in $\mathcal{E}_{a+b}$ using the relations $x_{i,j}^2 = 0$. For example, when $a = b = 1$, as $\theta_1 = x_{1,2}$, it becomes $0 = e_1 x_{1,2} h_1 x_{1,2} = x_{1,2}^2$.

We can prove a stronger version of [Theorem 3.1](#) which is nearly a quantum analog of [Proposition 1.2](#). Its arguments involve a significantly more detailed study of the left operators and their relations than our proof of [Theorem 3.1](#).

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On the A_2 Andrews–Schilling–Warnaar identities

Shashank Kanade^{*1}

¹*Department of Mathematics, University of Denver, USA*

Abstract. In a groundbreaking work, Andrews–Schilling–Warnaar invented an A_2 generalization of the A_1 Bailey machinery and discovered many identities related to the principal characters of standard modules for $\widehat{\mathfrak{sl}}_3$, or equivalently, for the vertex operator algebra $\mathcal{W}_{3/2, p'}$. Jointly with Russell, we have given conjectures for completing this set of identities and proved these conjectures for small values of p' . In another direction, character of $\mathcal{W}_{r, p, p'}$ has been related to an appropriate limit of certain $\mathfrak{sl}_r$ coloured Jones polynomials of torus knots $T(p, p')$ under some restrictions on r, p, p' . This note summarizes these developments.

Keywords: Rogers–Ramanujan-type identities, Affine Lie algebras, Vertex operator algebras, Jones polynomials

1 Introduction

1.1 Rogers–Ramanujan identities and their origins

Let us begin by recalling the celebrated pair of Rogers–Ramanujan identities:

$$\sum_{n \geq 0} \frac{q^{n^2}}{q^n} = \frac{1}{\theta(q; q^5)}, \quad \sum_{n \geq 0} \frac{q^{n^2 + n}}{q^n} = \frac{1}{\theta(q^2; q^5)}. \quad (1.1)$$

where we use the standard notations for the Pochhammer symbols q_n (and more generally $a; q_n$) for $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ as found in [2] and we use the notation:

$$\theta(a_1, a_2, \dots, a_k; q) = \prod_{1 \leq i \leq k} a_i; q_\infty / a_i; q_\infty. \quad (1.2)$$

Rogers–Ramanujan identities, their various generalizations, and analogues appear in many branches of mathematics. Most importantly for us, they appear in the study of affine Lie algebras and vertex operator algebras (henceforth, VOAs). Secondly, they also appear in relation to the coloured Jones polynomials of knots and links.

^{*}shashank.kanade@du.edu. S. Kanade is supported by Collaboration Grant for Mathematicians #636937 awarded by the Simons Foundation.

The identities (1.1) were first proved completely representation-theoretically by Lepowsky and Wilson [18, 19, 20, 21] by considering standard (i.e., integrable, highest-weight) level 3 modules for the affine Lie algebra $\widehat{\mathfrak{sl}}_2$. Identities (1.1) also arise by considering the representation theory of Virasoro VOA $\text{Vir } 2,5$. In general, representation theory of VOAs is an enormously rich source of Rogers–Ramanujan-type identities having the “sum=product” shape of (1.1).

On the knot-theoretical side, the second Rogers–Ramanujan identity in (1.1) seems to have been proved first by Armond and Dasbach [6] by considering tails of Jones polynomials for torus knots $T_{2,5}$ coloured with finite dimensional irreducible $\mathfrak{sl}_2$ modules. Here, the product side is a result of using the Rosso–Jones formula [24] as explained by Morton, [23]. The sum side is calculated using a combinatorial walk model [7]. More generally, Armond and Dasbach proved one Andrews–Gordon identity at each modulus $2k+1$ by considering the torus knot $T_{2,2k+1}$.

1.2 Bailey machinery

In the world of q -series, Bailey machinery provides a highly productive mechanism to prove Rogers–Ramanujan-type identities. By Bailey machinery, we mean the framework that takes as input a Bailey pair (as defined in [1], for instance) and modifies this pair iteratively (using the Bailey lemma, the Bailey lattice, or any other suitable transformations). The modified pairs then lead to requisite identities by taking appropriate limits.

From the point of view of representation theory and knot theory, this mechanism is especially well suited to handle cases that are closely related to the Lie algebra $\mathfrak{sl}_2$. In particular, it produces identities related to principal characters (see (2.3) below) for standard modules for affine Lie algebras $\widehat{\mathfrak{sl}}_2 = A_1^1$ and A_2^2 , rational Virasoro algebras $\text{Vir } p, q$, etc. This machinery has also been successfully used (see [15], for instance) in understanding Jones polynomials of links coloured with irreducible, finite-dimensional $\mathfrak{sl}_2$ modules.

For these and some more reasons explained in [3], we refer to the classical Bailey machinery by the designation $\mathfrak{sl}_2$ or A_1 .

1.3 Andrews–Schilling–Warnaar’s A_2 Bailey machinery

In [3], Andrews–Schilling–Warnaar invented an A_2 generalization of the A_1 Bailey machinery. Specifically, they defined a notion of the A_2 Bailey pair, found the unit A_2 Bailey pair, and gave a transformation law for A_2 Bailey pairs. Note that this transformation law is not yet as general as the one for A_1 Bailey pairs.

In [3], the authors then used this mechanism to discover “sum=product” identities related to principal characters of certain standard modules at every level $\ell \in \mathbb{Z}_{>0}$ for the

affine Lie algebra $\widehat{\mathfrak{sl}}_3$. At levels $\ell \in \mathbb{Z}_{>0}$ coprime to 3, these principal characters also arise by considering characters of modules for the rational VOA $\mathcal{W}_3(3, \ell - 3)$ (see [3, 12]).

The actual number of possible identities at level $\ell \in \mathbb{Z}_{>0}$ grows as a quadratic in ℓ , but the number of identities found in [3] grows linearly in ℓ . Consequently, a major subset of identities were yet to be found.

Our recent work [17] gives precise conjectures for all these missing identities. We are further able to prove our conjectures at levels 3 and 7. A further family of results valid for all levels divisible by 3 is detailed in Section 3 below.

1.4 Certain $\mathfrak{sl}_r$ coloured Jones polynomials of torus knots

Morton’s work [23] effectively relates characters of $\text{Vir}(p, q)$ VOA to $\mathfrak{sl}_2$ coloured Jones polynomials of torus knots $T(p, q)$ where $2 \leq p < q$ are coprime. Now, $\text{Vir}(p, q)$ VOA is the $r = 2$ member in the family of principal $\mathcal{W}_r$ algebras of type A. It is thus natural to investigate whether characters of $\mathcal{W}_r(p, q)$ algebras are analogously related to torus knots. In [16] we show that limits of certain $\mathfrak{sl}_r$ coloured invariants of torus knots $T(p, q)$ similarly give rise to characters of $\mathcal{W}_r(p, q)$ VOAs. For precise conditions and statements, see in Section 4. Importantly, our hope is that generalizing the work of Armond and Dasbach [6] to $\mathfrak{sl}_3$ coloured invariants of torus knots $T(3, p - 3)$ will put the Andrews–Schilling–Warnaar identities in a very different light.

1.5 Recent and related developments

Recently, Andrews–Schilling–Warnaar’s A_2 identities have gained much attention. Especially, the interested reader may look at [10], [29], [26], [27] and [30].

2 Foundations

In this section, we begin by collecting the basic setup required for this note.

2.1 Notation regarding the Lie algebras $\mathfrak{sl}_r$

As usual, we let the simple roots of $\mathfrak{sl}_r$ be $\alpha_1, \dots, \alpha_{r-1}$. The set of positive roots will be denoted by Φ_r and the set of all roots will be Φ_r . The fundamental weights will be denoted by $\Lambda_1, \dots, \Lambda_{r-1}$. Q_r and P_r will denote the root and the weight lattices, respectively. The Weyl group will be denoted by $\mathfrak{S}_r$. The lattices $Q_r \subsetneq P_r$ come equipped with a symmetric, positive definite and $\mathfrak{S}_r$ -invariant bilinear form $(\cdot, \cdot)$ such that $(\alpha_i, \alpha_i) = 2$ for all $0 \leq i \leq r - 1$. We will let $\|\cdot\|$ be the norm corresponding to $(\cdot, \cdot)$. The Weyl vector will be denoted by δ . Recall that $w \mapsto (-1)^{\ell(w)}$ is the sign representation of $\mathfrak{S}_r$ where $\ell(\cdot)$ is the standard length function on $\mathfrak{S}_r$.

2.2 Affine Lie algebras $\widehat{\mathfrak{sl}}_r$

For $\widehat{\mathfrak{sl}}_r$, we will reuse some notation from $\mathfrak{sl}_r$ case. We shall provide adequate clarification in the cases of potential confusion.

For $0 \leq i \leq r-1$, let α_i be the simple roots and let Λ_i be the fundamental weights of the affine Lie algebra $\widehat{\mathfrak{sl}}_r$. Fix $\ell \in \mathbb{Z}_{\geq 0}$ (called the level) and let $\lambda = c_0\Lambda_0 + \cdots + c_{r-1}\Lambda_{r-1}$, where $c_i \in \mathbb{Z}_{\geq 0}$ and $c_0 + \cdots + c_{r-1} = \ell$. By $L(\lambda)$ we denote the irreducible highest-weight $\widehat{\mathfrak{sl}}_r$ module with highest-weight λ . The character of this module is denoted by $\text{ch } L(\lambda)$ and it belongs to:

$$\text{ch } L(\lambda) \in e^\lambda \mathbb{Z}[e^{-\alpha_0}, \dots, e^{-\alpha_{r-1}}]. \quad (2.1)$$

Here, e^λ and e^{α_i} are formal symbols. We define the *principally specialized character* and the *principal character*¹ of $L(\lambda)$ to be respectively –

$$\chi_{L(\lambda)} = \left(e^{-\lambda} \text{ch } L(\lambda) \right) \Big|_{e^{-\alpha_0} \mapsto q, \dots, e^{-\alpha_{r-1}} \mapsto q'} \quad (2.2)$$

$$\chi_{\Omega(\lambda)} = \frac{\chi_{L(\lambda)}}{\chi_{L(\Lambda_0)}} = \frac{q^\infty}{q^r; q^r}_\infty \chi_{L(\lambda)} \quad (2.3)$$

Due to the Dynkin diagram symmetries in the $\widehat{\mathfrak{sl}}_r$ case, we also have that:

$$\chi_{\Omega(\lambda)} = \chi_{\Omega(\sigma\lambda)}, \quad (2.4)$$

where σ is any dihedral permutation of $c_0, c_1, \dots, c_{r-1}$. In the case of $\widehat{\mathfrak{sl}}_3$, we have the following for $c_0, c_1, c_2 \in \mathbb{Z}_{\geq 0}$, $c_0 + c_1 + c_2 = \ell$, $m = \ell - 3$ (see [12, 28], etc.):

$$\chi_{\Omega(c_0\Lambda_0 + c_1\Lambda_1 + c_2\Lambda_2)} = \frac{q^m; q^m}_q \frac{q^{\frac{2}{\infty}}}{q^{\frac{2}{\infty}}} \theta(q^{c_0-1}; q^m) \theta(q^{c_1-1}; q^m) \theta(q^{c_2-1}; q^m). \quad (2.5)$$

2.3 Principal $\mathcal{W}_r$ algebras

Let $r \leq p, p'$ be a pair of coprime integers. Now, principal $\mathcal{W}_r(p, p')$ algebras are certain simple vertex operator algebras (VOAs) that can be obtained as the quantum Hamiltonian reductions of affine VOAs based on $\widehat{\mathfrak{sl}}_r$ at levels $\frac{p}{p'} - r$. For deep and foundational properties of these VOAs, see [4, 5].

From [22] and [12], we recall characters of $\mathcal{W}_r(p, p')$ VOAs and their modules.

1. In the notation of [12], the character of the VOA $\mathcal{W}_r(p, p')$ is $\chi_{0,0}^{r,p,p'}$ and up to a pure power of q it equals the following normalized character:

$$\overline{\chi}_{0,0}^{r,p,p'} = q^{\frac{1-r}{\infty}} \sum_{\alpha \in Q_r} \sum_{\sigma \in \mathfrak{S}_r} (-1)^{\ell(\sigma)} q^{\frac{1}{2} p p' \|\alpha\|^2 - p' \langle \alpha, \delta \rangle - p \langle \alpha, \sigma \delta \rangle - \delta, \sigma \delta - \delta}. \quad (2.6)$$

¹Note that the first equality in (2.3) as a definition of principal characters is only valid for the affine Lie algebras of type X_r^t with $X = A, D, E$ and $t \in \{1, 2, 3\}$.

2. When $p \geq r$, inequivalent irreducible modules of $\mathcal{W}_r(r, p')$ are enumerated by $\mathfrak{sl}_r$ weights $\zeta = c_1\Lambda_1 + \cdots + c_{r-1}\Lambda_{r-1} \in P_r$ such that $c_1 + \cdots + c_{r-1} \leq p' - r$. The character of the corresponding irreducible $\mathcal{W}_r(r, p')$ module is denoted by $\chi_{0,\zeta}^{r,r,p'}$ [12] and up to a pure power of q it equals the normalized character:

$$\bar{\chi}_{0,\zeta}^{r,r,p'} = q^{\frac{1-r}{2}} \sum_{\alpha \in Q_r} \sum_{\sigma \in \mathfrak{S}_r} -1^{\ell(\sigma)} q^{\frac{1}{2}rp' \|\alpha\|^2 - p' \alpha, \delta - r \alpha, \sigma \zeta - \delta - \delta, \sigma \zeta - \delta - \zeta - \delta}. \quad (2.7)$$

2.4 Cylindric partitions

We let $r \geq 1$ and $\ell \geq 1$.

A *composition* of ℓ of length r is an (ordered) sequence of non-negative integers $c_0, c_1, \dots, c_{r-1}$ that adds up to ℓ . Note crucially that ℓ stands for the level and r for rank as in Section 2.2.

A *cylindric partition* of profile c is defined to be an (ordered) sequence of partitions $\Lambda = \lambda^{(0)}, \dots, \lambda^{(r-1)}$ such that:

1. Each $\lambda^{(j)} = \lambda_1^{(j)} \lambda_2^{(j)} \cdots$ is written in a non-increasing order and is assumed to continue indefinitely by appending an infinite string of zeroes.
2. For all $0 \leq i \leq r-2$ and all j , $\lambda_j^{(i)} \geq \lambda_{j+c_{i-1}}^{(i-1)}$ and $\lambda_j^{(r-1)} \geq \lambda_{j+c_0}^{(0)}$.

Given a composition $c = c_0, \dots, c_{r-1}$ of ℓ , define its set of non-negative indices by

$$I_c = \{0 \leq i \leq r-1 \mid c_i > 0\}.$$

For $\emptyset \subsetneq J \subseteq I_c$, define $c \setminus J = c_0 \setminus J, \dots, c_{r-1} \setminus J$ as

$$c_i \setminus J = \begin{cases} c_i - 1 & i \in J, i-1 \notin J \\ c_i + 1 & i \notin J, i-1 \in J \\ c_i & \text{otherwise,} \end{cases} \quad (2.8)$$

where we think of the indices $0, \dots, r-1$ (and consequently elements of the sets I_c and J) as elements of the cyclic group $\mathbb{Z}_r$. Note that if c is a composition of ℓ , then so is $c \setminus J$.

Let $\mathcal{C}_c$ be the collection of all cylindric partitions of profile c and define:

$$H_c(z, q) = \frac{zq^\infty}{q^{-\infty}} \sum_{\Lambda \in \mathcal{C}_c} z^{\max \Lambda} q^{\text{wt } \Lambda}. \quad (2.9)$$

Now, the following properties of cylindric partitions and the functions H_c are crucial.

1. The functions H_c are governed by the Corteel–Welsh recurrence [11]. For a fixed r and ℓ , the functions H_c as c varies over length r compositions of ℓ are the unique solutions in $\mathbb{Z}[z, q]$ to the following finite system of recurrences and initial conditions²:

$$H_c(z, q) = \sum_{\emptyset \subsetneq J \subseteq I_c} (-1)^{|J|-1} zq^{-|J|} H_{c|_J}(zq^{|J|}, q), \quad (2.10)$$

$$H_c(0, q) = q^{-1}, \quad H_c(z, 0) = 1. \quad (2.11)$$

2. H_c appear as characters as follows ([8, 25, 14, 12]):

$$q^{-\infty} H_{c_0, \dots, c_{r-1}}(1, q) = \chi(\Omega_{c_0 \Lambda_0} \cdots c_{r-1} \Lambda_{r-1}) \bar{\chi}_{0, c_1 \Lambda_1}^{r, r, \ell} \cdots c_{r-1} \Lambda_{r-1}. \quad (2.12)$$

3. H_c functions have certain symmetries. The first follows from the definition of cylindric partitions. The second follows from the previous relation to principal characters, but can also be deduced from the properties of cylindric partitions.

$$H_{c_0, \dots, c_{r-1}}(z, q) = H_{c_1, c_2, \dots, c_{r-1}, c_0}(z, q), \quad (2.13)$$

$$H_{c_0, \dots, c_{r-1}}(1, q) = H_{c_{r-1}, c_{r-2}, \dots, c_1, c_0}(1, q). \quad (2.14)$$

Due to (2.13), we cyclically permute c and assume that c_0 is the largest part.

3 Completing the A_2 identities from [3]

In this section, we will now solely focus on the $r = 3$ case. That is, we will only consider principal characters of standard $\widehat{\mathfrak{sl}}_3$ modules of level $\ell \in \mathbb{Z}_{>0}$. Equivalently, these are the characters $\bar{\chi}_{0, \zeta}^{3, 3, \ell}$ for irreducible modules for the VOA $\mathcal{W}_3^{3, 3, \ell}$ whenever ℓ is coprime to 3, however, we will stick with the former description.

In [3], using an appropriate generalization of the usual Bailey machinery to the A_2 root system, “sum=product” identities involving certain principal characters were found. For identities found in [3], the triple (c_0, c_1, c_2) appearing in (2.5) satisfies:

$$\begin{aligned} & c_{\sigma(0)} = 1, c_{\sigma(1)} = 1, c_{\sigma(2)} = 1 \\ & \in \{ (i, i, 3k - 2i - s), (1, k - t, 2k - t - 1 \pm 1) \mid k \geq 2; 1 \leq i \leq k; t = 0, 1; s = -1, 0, 1 \}. \end{aligned} \quad (3.1)$$

where σ is some permutation of the indices $0, 1, 2$ (recall (2.4)). This implies that a majority of identities at every level were yet to be found. In [17], jointly with Russell, we were able to conjecture all of the missing identities and prove our conjectures at levels 3 and 7. In fact, our conjectures pertain to the two-variable generating functions $H_c(z, q)$. Upon setting $z = 1$, one gets conjectures for the principal characters of standard modules (see (2.12)).

²The referee has kindly pointed out that $H_c(z, 0) = 1$ from (2.11) is redundant.

3.1 The arrangement of modules

Fix $\ell > 0$ and let $m = \ell - 3$ and let k be such that $m - \ell - 3 \in 3\mathbb{Z} \setminus \{-1, 0, 1\}$.

We now arrange length 3 compositions of ℓ in a triangular array. Suppose that $c = c_0, c_1, c_2$ is a composition of ℓ . Due to the cyclic symmetry (2.13), we assume that c_0 is the largest part of c . Let $\lambda_0, \lambda_1, \lambda_2$ be the sequence obtained by permuting c_0, c_1, c_2 in a weakly decreasing order. Now place the composition c in row λ_1 and column λ_2 . This gives rise to a triangular arrangement of compositions. Lastly, we draw a horizontal line after row $k - 1$. For example, here is the arrangement for $\ell = 11, m = 14, k = 5$:

$$\begin{array}{rcll}
 \text{row 0 :} & 11, 0, 0 & & \\
 \text{row 1 :} & 10, 1, 0 / 10, 0, 1 & 9, 1, 1 & \\
 \text{row 2 :} & 9, 2, 0 / 9, 0, 2 & 8, 2, 1 / 8, 1, 2 & 7, 2, 2 \\
 \text{row 3 :} & 8, 3, 0 / 8, 0, 3 & 7, 3, 1 / 7, 1, 3 & 6, 3, 2 / 6, 2, 3 & 5, 3, 3 \\
 \text{row 4 :} & 7, 4, 0 / 7, 0, 4 & 6, 4, 1 / 6, 1, 4 & 5, 4, 2 / 5, 2, 4 & 4, 4, 3 \\
 \hline
 \text{row 5 :} & 6, 5, 0 / 6, 0, 5 & 5, 5, 1 & &
 \end{array} \tag{3.2}$$

Theorem 1. [17, Thm 5.1] Fix $\ell > 0$. Suppose that expressions for all $H_c(z, q)$ are known where c runs over length 3 compositions of ℓ that lie above the line, i.e., $\lambda_1 \leq k - 1$. Then, the expressions for $H_c(z, q)$ where c is a length 3 composition of ℓ that lies below the line, i.e., $\lambda_1 \geq k$ are determined by the Corteel–Welsh recursions (2.10).

This theorem implies that it is enough to produce conjectures for the compositions that lie above the line.

3.2 Conjectures for compositions above the line

Let us continue to let ℓ, m, k as in the previous subsection. For $0 \leq j \leq k - 1$, define:

$$e_j = \overbrace{0, \dots, 0}^j, 1, \dots, 1 \in \mathbb{Z}^{k-1}, \quad e_{-1} = 2, 1, \dots, 1 \in \mathbb{Z}^{k-1}. \tag{3.3}$$

For $\rho, \sigma \in \mathbb{Z}^{k-1}$, define (with the convention that $q^{-1} = 0$ if $n < 0$):

$$\begin{aligned}
 S_{3k-1}(\rho | \sigma) &= \sum_{r, s \in \mathbb{Z}_{\geq 0}^{k-1}} \frac{z^{r_1} q^{\sum_{1 \leq i \leq k-1} r_i^2 - r_i s_i} s_i^2}{\prod_{1 \leq i \leq k-2} q^{r_i - r_{i-1}} q^{s_i - s_{i-1}}} \cdot \frac{q^{2r_{k-1} s_{k-1}}}{q^{r_{k-1}} q^{s_{k-1}} q^{r_{k-1} s_{k-1} - 1}} \\
 S_{3k-1}(\rho | \sigma) &= \sum_{r, s \in \mathbb{Z}_{\geq 0}^{k-1}} \frac{z^{r_1} q^{\sum_{1 \leq i \leq k-1} r_i^2 - r_i s_i} s_i^2}{\prod_{1 \leq i \leq k-2} q^{r_i - r_{i-1}} q^{s_i - s_{i-1}}} \cdot \frac{1}{q^{r_{k-1}} q^{s_{k-1}} q^{r_{k-1} s_{k-1} - 1}} \\
 S_{3k}(\rho | \sigma) &= \sum_{r, s \in \mathbb{Z}_{\geq 0}^{k-1}} \frac{z^{r_1} q^{\sum_{1 \leq i \leq k-1} r_i^2 - r_i s_i} s_i^2}{\prod_{1 \leq i \leq k-2} q^{r_i - r_{i-1}} q^{s_i - s_{i-1}}} \cdot \frac{1}{q^{r_{k-1} s_{k-1}} q^{r_{k-1} s_{k-1} - 1}} \begin{bmatrix} r_{k-1} & s_{k-1} \\ r_{k-1} & r_{k-1} \end{bmatrix}_{q^3}
 \end{aligned} \tag{3.4}$$

Conjecture 1. [17, Conj. 5.1] Fix ℓ and let $c = c_0, c_1, c_2$ be a composition above the line (i.e., $\lambda_1 \leq k - 1$). We then have:

$$H_{c_0, c_1, c_2}(z, q) = \begin{cases} S_{\ell-3} e_{c_1} | e_{c_2} - q S_{\ell-3} e_{c_1-1} | e_{c_2-1} & c_1, c_2 > 0 \\ S_{\ell-3} e_{c_1} | e_0 & c_2 = 0 \\ S_{\ell-3} e_0 | e_{c_2} - q(1-z) S_{\ell-3} e_{-1} | e_{c_2-1} & c_1 = 0, c_2 \neq 0 \end{cases} \quad (3.5)$$

Theorem 2. [17, Thm 5.3] The conjectures above satisfy the initial conditions (2.11), i.e., $H_c(0, q) = q^{-1}$ and $H_c(z, 0) = 1$.

3.3 Proving the conjectures at low levels and other results

We were able to prove our conjectures in some specific (small) levels:

Theorem 3. [17] *Conjecture 1 is true for $\ell \in \{2, 3, 4, 5, 7\}$ (the cases $\ell \in \{3, 7\}$ are new).*

Idea of proof. Broadly, the proof has two steps. First we find certain fundamental (recurrence) relations satisfied by the S functions defined in (3.4). This step is straight-forward (see also [9]). The next step is to show that the Corteel–Welsh recursions (2.10) are a consequence of these fundamental relations. This step requires a non-trivial amount of computer assistance, especially in levels 5 and 7. $\square$

Recently, Uncu has extended the theorem above to include two further cases.

Theorem 4. [27] *Conjecture 1 is true for $\ell \in \{8, 10\}$.*

In an important development, S. O. Warnaar has given a far reaching enhancement of the A_2 Bailey machinery of [3] and proved the following.

Theorem 5. [30] *Conjecture 1 is true with $z = 1$.*

In the case of $3 \mid \ell$ and with $z = 1$, we are in fact able to improve Theorem 1. In this case, we prove concrete formulas for $H_c(1, q)$ when c is below the line. This process crucially uses an identity of Weierstraß (see [17, Lem 6.1]). Note that the original formulation in [17] of the theorem below was conditional upon the validity of Conjecture 1 with $z = 1$, which in turn has been proved in [30].

Theorem 6. [17, Thm. 6.4], [30]³ Let $k \geq 3$. Let $c = c_0, c_1, c_2$ with $c_0 \geq c_1 \geq c_2 \geq 0$, $c_0 - c_1 - c_2 = \ell - 3k - 3$, and $c_1 \geq k$. Then:

$$H_c(1, q) = X(1, q) - q^{c_2-1} Y(1, q), \quad (3.6)$$

³Here we correct two minor errors in [17]: In equations (6.10) and (6.11) of [17], S_{3m} are supposed to be S_{3k} . Additionally, in equations (6.7) and (6.10) of [17], the second case never occurs.

where

$$X(z, q) = S_{3k}(\mathbf{e}_{2k-c_1-2} \mid \mathbf{e}_{2k-c_0-2}) - q S_{3k}(\mathbf{e}_{2k-c_1-3} \mid \mathbf{e}_{2k-c_0-3}), \quad (3.7)$$

$$Y(z, q) = \begin{cases} S_{3k}(\mathbf{e}_{c_0-k} \mid \mathbf{e}_{c_1-k}) - q S_{3k}(\mathbf{e}_{c_0-k-1} \mid \mathbf{e}_{c_1-k-1}), & c_1 > k \\ S_{3k}(\mathbf{e}_{c_0-k} \mid \mathbf{e}_0) & c_1 = k. \end{cases} \quad (3.8)$$

3.4 Examples

As examples, we single out two new identities that emerge from the theorems above. In both, $\sum$ stands for a sum over $r_1, r_2, s_1, s_2 \geq 0$.

$$\sum \frac{q^{r_1^2-r_1s_1} s_1^2 r_2^2-r_2s_2 s_2^2 r_2 s_2 1 - 2q^{r_1 s_1} \begin{bmatrix} r_2 & s_2 \\ r_2 & \end{bmatrix}_{q^3}}{q^{r_1-r_2} q^{s_1-s_2} q^{r_2 s_2} q^{r_2 s_2} 1} \frac{1}{q^\infty \theta} \frac{1}{q, q^2, q^2, q^3, q^3; q^9} \frac{\chi \Omega 3\Lambda_0 3\Lambda_1}{q^\infty}, \quad (3.9)$$

$$\sum \frac{q^{r_1^2-r_1s_1} s_1^2 r_2^2-r_2s_2 s_2^2 q^{-r_1 s_1 s_2} - q^{r_2 s_2} q^{r_1 r_2 s_1 s_2}}{q^{r_1-r_2} q^{s_1-s_2} q^{r_2 s_2} q^{r_2 s_2} 1} \frac{1}{q^\infty \theta} \frac{1}{q, q^2, q^2, q^3, q^3, q^4; q^{10}} \frac{\chi \Omega 4\Lambda_0 3\Lambda_1}{q^\infty}. \quad (3.10)$$

Identity (3.9) arises from Theorem 6 with $c_0, c_1, c_2 = 3, 3, 0$. (3.10) is one of the identities in Theorem 3 with $\ell = 7$.

4 Coloured $\mathfrak{sl}_r$ invariants of torus knots

Let us begin by collecting a few further facts about the representation theory of $\mathfrak{sl}_r$.

Recall that the Weyl denominator associated with $\mathfrak{sl}_r$ is:

$$\Delta_r = \sum_{w \in \mathfrak{S}_r} (-1)^{\ell(w)} e^{w \cdot \delta},$$

where $e^{w \cdot \delta}$ is a formal exponential, to be thought of as an element of the group algebra of the weight lattice P_r . We denote:

$$\text{qdim } \Delta_r = \sum_{w \in \mathfrak{S}_r} (-1)^{\ell(w)} q^{w \cdot \delta}.$$

Weyl denominator formula implies that:

$$\text{qdim } \Delta_r = \prod_{\alpha \in \Phi_r} (q^{\frac{1}{2} \delta, \alpha} - q^{-\frac{1}{2} \delta, \alpha}).$$

The function qdim can be defined more generally by $\text{qdim } e^\mu = q^{\mu, \delta}$ for $\mu \in P_r$. In particular, for a finite-dimensional irreducible module L , we may define $\text{qdim } L$ as simply $\text{qdim } \text{ch } L$ where the character $\text{ch } L$ belongs to the group algebra of P_r .

Let $L_r \ n\Lambda_1$ denote the (finite-dimensional) irreducible $\mathfrak{sl}_r$ module of highest weight $n\Lambda_1$ and let $\Pi_{r,n}$ be the set of weights of $L_r \ n\Lambda_1$.

Given a knot K with a fixed framing f , a finite-dimensional simple Lie algebra $\mathfrak{g}$ along with a finite-dimensional irreducible module L , let $J_K^f L$ denote the framing-dependent, un-normalized, L coloured Jones invariant of K . Here, un-normalized means that for the unknot with zero framing this invariant equals $\text{qdim } L$.

Let p, p' be a pair of coprime positive integers, and let $T_{p,p'}$ be the torus knot obtained by the (p, p') cabling of the unknot of zero framing, see [23] for details. In this case, $T_{p,p'}$ has writhe pp' . We will fix (and then forget) this framing of $T_{p,p'}$. Now, the Rosso–Jones formula [24] (as explained by [23]) in conjunction with some elementary properties of $\Pi_{r,n}$ leads to the following theorem.

Theorem 7. *Let p, p' be a pair of positive coprime integers.*

1. [16, Thm. 5.1] *The framing-dependent, un-normalized invariants of torus knots satisfy:*

$$J_{T_{p,p'}} L_r \ n\Lambda_1 = \frac{q^{-\frac{p'}{2p}\|\delta\|^2}}{\text{qdim } \Delta_r} \sum_{\lambda \in \Pi_{r,n}, w \in \mathfrak{S}_r} (-1)^{\ell(w)} q^{p\lambda - w\delta, \delta - \frac{p'}{2p}\|p\lambda - w\delta\|^2}.$$

2. [16, Thm. 6.1] *We have:*

$$\lim_{n \rightarrow \infty} J_{T_{p,p'}} L_r \ nr\Lambda_1 = \frac{q_{\infty}^{r-1}}{\prod_{\alpha \in \Phi_r} (1 - q_{\infty}^{\alpha, \delta})} \bar{\chi}_{0,0}^{r,p,p'}. \quad (4.1)$$

3. [16, Prop. 3.4] *If $p < p'$ and $1 \leq p \leq r-1$, then, $\bar{\chi}_{0,0}^{r,p,p'} = 0$ and the limit in (4.1) is 0.*

In part 3 above, the reason the limit equals 0 is that the lowest degree of the corresponding Jones invariant approaches ∞ as $n \rightarrow \infty$. Thus, it is interesting to understand limits of the shifted invariants $\hat{J}_K^f L$ which equal $J_K^f L$ divided by its trailing monomial. We have the following conjecture:

Conjecture 2. [16, Conj. 6.6] *Let $2 \leq p < p'$ be a pair of coprime positive integers such that $p < r$. Letting $\Phi_1 = \emptyset$ and denoting the Weyl vector of $\mathfrak{sl}_r$ by δ_r , we have:*

$$\lim_{n \rightarrow \infty} \hat{J}_{T_{p,p'}} L_r \ n\Lambda_1 = \frac{\prod_{\alpha \in \Phi_{r-p}} (1 - q^{\alpha, \delta_{r-p}})}{\prod_{\alpha \in \Phi_r} (1 - q^{\alpha, \delta_r})} q_{\infty}^{p-1} \bar{\chi}_{0,0}^{p,p,p'}.$$

Note that unlike (4.1) which involves $L_r \ nr\Lambda_1$, this conjecture involves $L_r \ n\Lambda_1$.

The $r = 3$ case of this conjecture holds due to the work [13] (see also [16]).

Our hope is that this connection of $\mathcal{W}_3$ algebra characters to knot theory will provide useful insights on the corresponding Andrews–Schilling–Warnaar identities.

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Hook length bias in odd versus distinct partitions

Cristina Ballantine^{*1}, Hannah Burson^{†2}, William Craig^{‡3},
Amanda Folsom^{§4}, and Boya Wen^{¶5}.

¹*Department of Mathematics and Computer Science, College of the Holy Cross, Worcester, MA 01610, USA*

²*School of Mathematics, University of Minnesota, Twin Cities, Minneapolis, MN 55455, USA*

³*Department of Mathematics, University of Virginia, Charlottesville, VA 22904, USA*

⁴*Department of Mathematics and Statistics, Amherst College, Amherst, MA 01002, USA*

⁵*Department of Mathematics, University of Wisconsin-Madison, Madison, WI 53706, USA*

Abstract. Motivated in part by hook-content formulas for certain restricted partitions in representation theory, we consider the total number of hooks of fixed length in odd versus distinct partitions. We show that there are more hooks of length 2, respectively 3, in all odd partitions of n than in all distinct partitions of n , and make the analogous conjecture for arbitrary hook length $t \geq 2$. To this end, we establish very general linear inequalities for the number of distinct partitions, which is also of independent interest. We also establish additional related partition bias results.

Keywords: hook length, distinct partitions, odd partitions

1 Introduction

Background. Connections between representation theory and the theory of integer partitions are well-known. For example, the irreducible polynomial representations of $\mathrm{GL}_n \mathbb{C}$ may be indexed by partitions of length at most n ; moreover, the conjugacy classes of the symmetric group S_n , and therefore the number of non-equivalent irreducible complex representations, may be indexed by the partitions of n . *Hook lengths* of partitions play particularly important roles in establishing these connections. Namely, such irreducible representations can be analyzed via partition *Young tableaux*. The dimension of a representation of S_n (respectively $\mathrm{GL}_n \mathbb{C}$) corresponding to a particular

^{*}cballant@holycross.edu.

[†]hburson@umn.edu.

[‡]wlc3vf@virginia.edu. Craig was partially supported by his advisor's grants, namely the Thomas Jefferson Fund and the NSF (DMS-1601306 and DMS-2055118) and received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No. 101001179).

[§]afolsom@amherst.edu. Folsom was partially supported by NSF Grant DMS-2200728.

[¶]bwlen25@wisc.edu.

partition is given by a *hook length formula* (respectively a *hook-content formula*). (For more on these topics, see e.g. [17].)

One of many celebrated results in this subject is the Nekrasov-Okounkov formula for arbitrary powers of Euler's infinite product in terms of hook numbers, as well as Han's extension [9], which unifies the Macdonald identities in representation theory and *t-core partition* generating functions. *Restricted* partitions also play important roles in this context. For example, Han and Xiong have established hook-content formulas for distinct partitions in [10, 11]. See also [12, 16] for more in this direction. Here, we further study hook lengths associated to distinct partitions versus odd partitions. Before stating our results, we briefly define some terminology (and refer the reader to [1] for more background on partitions).

Terminology. Recall that a *partition* $\lambda = \lambda_1, \lambda_2, \dots, \lambda_j$ of size $n \in \mathbb{N}_0$ is a non-increasing sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_j$ called *parts* that add up to n . By convention, the empty partition is the only partition of size 0. We write $|\lambda|$ for the size of λ . We denote by $p(n)$ the number of partitions of n . We define the multiplicity $m_\lambda(i)$ of a part i of a partition λ to be the number of times i appears as a part of λ . We define the length $\ell(\lambda)$ of a partition λ to be the number of parts of λ , and define $\ell_1(\lambda)$ (resp. $\ell_2(\lambda)$) to be the number of parts λ_i of λ with $\lambda_i - \lambda_{i+1} = 1$ (resp. $\lambda_i - \lambda_{i+1} = 2$). We assume $\lambda_k = 0$ if $k > \ell(\lambda)$.

Each partition is naturally equipped with a *Young diagram*, a left-justified vertical array of boxes with rows corresponding to parts (see Figure 1). We abuse notation, and refer to partitions as their Young diagrams, with their parts referred to as rows. The *conjugate* of a partition λ is the partition λ' whose Young diagram has the columns of λ as rows. Each box in a Young diagram of λ may be labeled with a *hook number*, also called *hook length*, which, informally, is the number of boxes in the upside-down-L-shaped portion of the diagram with the box appearing as its corner. More precisely, for a box in the i -th row and j -th column of the Young diagram of a partition λ , its hook length is defined as $h(i, j) = \lambda_i + \lambda'_j - i - j + 1$ (see Figure 1).

8	7	4	3	1
6	5	2	1	
3	2			
2	1			

Figure 1: Young diagram with hook lengths for the partition $\lambda = 5, 4, 2, 2$.

In what follows, we refer to a partition into odd parts as an *odd partition* and to a partition into distinct parts as a *distinct partition*. We denote by $\mathcal{O}(n)$, respectively $\mathcal{D}(n)$, the set of odd, respectively distinct, partitions of n . Euler's identity [1, Corollary 1.2] states that $|\mathcal{O}(n)| = |\mathcal{D}(n)|$ for all $n \geq 0$.

Results. Let $a_t n$ (respectively $b_t n$) be the total number of hooks of length t in all odd (respectively distinct) partitions of n .

For a partition λ , a box in its Young diagram has hook length 1 if and only if it is at the end of a row and there is no box directly below it. Thus, in a partition λ , the number of hooks of length 1 equals the number of different part sizes in λ .

The next result was conjectured by Beck [13] and proved analytically by Andrews [2].

Theorem 1 (Theorem 2, [2]). *The difference between the total number of parts in all distinct partitions of n and the total number of different part sizes in all odd partitions of n equals $c n$, the number of partitions of n with exactly one part occurring three times while all other parts occur only once.*

Corollary 2. *For $n \geq 0$, $b_1 n - a_1 n = c n$.*

Thus, there are at least as many hooks of length 1 in all distinct partitions of n as there are in all odd partitions of n . On the other hand, Euler's identity yields

$$\sum_{t \geq 1} a_t n = \sum_{t \geq 1} b_t n.$$

It is natural to study the relationship between $a_t n$ and $b_t n$ for any fixed $t \geq 1$. For $t = 1$ the above shows that $b_1 n \geq a_1 n$. We conjecture that this bias reverses for $t \geq 2$: eventually (for large enough n), we conjecture that there are at least as many hooks of length t in all odd partitions of n as there are in all distinct partitions of n . More specifically, we make the following conjecture.

Conjecture 3. *For every integer $t \geq 2$, there exists an integer N_t such that for all $n > N_t$, we have $a_t n \geq b_t n$, and $a_t n - b_t n \rightarrow \infty$ as $n \rightarrow \infty$. Moreover, for $4 \leq t \leq 10$, we conjecture the following values of N_t :*

t	2	3	4	5	6	7	8	9	10
N_t	0	7	8	18	16	34	34	56	59

Figure 2: Conjectural values for N_t .

Data supporting this conjecture was obtained by enumerating partitions and not from generating functions; this is because the generating functions for $a_t n$ and $b_t n$ are difficult to derive explicitly. We are, however, able to write down generating functions for $t = 2$ and $t = 3$, which we use to prove [Conjecture 3](#) for $t = 2$ and $t = 3$.

Theorem 4. *[Conjecture 3](#) is true for $t = 2$ and $t = 3$ with $N_2 = 0$ and $N_3 = 7$, respectively.*

Analogous to [Corollary 2](#), for $t = 2$, we prove that $a_2 n - b_2 n = c_2 n$, where $c_2 n$ is the number of overpartitions λ of n into odd parts with $m_\lambda 1 \equiv 0, 3 \pmod{4}$, and

exactly one part greater than 1 overlined. An overpartition of n is a partition of n in which the first occurrence of a part may be overlined [5].

We give complete details for the case $t = 2$ of [Theorem 4](#). The case $t = 3$ is more complicated, and requires tools from combinatorics as well as analytic number theory. For space considerations, we broadly sketch the main idea used in the proof for $t = 3$, and will publish the complete proof in a forthcoming work [3].

2 Hooks of length 2

Let $a_2(m, n)$ (respectively $b_2(m, n)$) be the number of odd (respectively distinct) partitions of n with m hooks of length 2. To prove [Theorem 4](#) for $t = 2$, we first establish the generating functions for the sequences $a_2(n)$ and $b_2(n)$ by finding the bivariate generating functions $\sum_{m,n \geq 0} a_2(m, n) z^m q^n$ and $\sum_{m,n \geq 0} b_2(m, n) z^m q^n$, differentiating with respect to z , and evaluating at $z = 1$. We follow with the analysis of $a_2(n) - b_2(n)$.

Odd partitions. Let λ be an odd partition. The number of hooks of length 2 in λ is equal to the number of different part sizes of λ that are greater than 1 plus the number of different parts of λ that occur at least twice. Thus,

$$F_2(z; q) := \sum_{m,n \geq 0} a_2(m, n) z^m q^n = \left(1 - q - \frac{zq^2}{1-q}\right) \prod_{n=2}^{\infty} \left(1 - zq^{2n-1} - \frac{z^2 q^{2 \cdot 2n-1}}{1 - q^{2n-1}}\right),$$

and hence

$$\sum_{n \geq 0} a_2(n) q^n = \left. \frac{\partial}{\partial z} \right|_{z=1} F_2(z; q) = \frac{1}{q; q^2_{\infty}} \left(q^2 - \sum_{n \geq 2} q^{2n-1} - q^{2 \cdot 2n-1} \right).$$

Here and throughout, the q -Pochhammer symbol is defined for $n \in \mathbb{N}_0 \cup \{\infty\}$ by

$$a; q_n := \prod_{j=0}^{n-1} (1 - aq^j).$$

Distinct partitions. If λ is a distinct partition, the number of hooks of length 2 in λ equals the number of parts λ_i such that $\lambda_i - \lambda_{i-1} \geq 2$. Thus, the number of hooks of length 2 in λ can be calculated as follows: starting with the Young diagram of λ , remove a staircase of length $\ell(\lambda)$ (i.e. subtract 1 from the smallest part, 2 from the second to last part, etc.) to obtain a partition μ . Then, we count the number of different part sizes in μ . We note that the number of different part sizes in μ is equal to the number of different part sizes in its conjugate μ' . Let $u_n(t, m)$ be the number of partitions of m with t different part sizes and all parts at most n . Then

$$\sum_{m,t \geq 0} u_n(t, m) z^t q^m = \prod_{j=1}^n \left(1 - \frac{zq^j}{1-q^j}\right),$$

and

$$G_2(z; q) := \sum_{n, m \geq 0} b_{2, n} z^m q^n = \sum_{n \geq 1} q^{n-1/2} \prod_{j=1}^n \left(1 - \frac{zq^j}{1-q^j} \right).$$

Then, differentiating with respect to z and evaluating at $z = 1$, we obtain

$$\sum_{n \geq 0} b_{2, n} q^n \left. \frac{\partial}{\partial z} \right|_{z=1} G_2(z; q) = \sum_{n \geq 1} q^{n-1/2} \frac{1}{q; q}_n \frac{q - q^{n-1}}{1 - q}.$$

Using the well-known limiting case of the q -binomial theorem (see e.g. [1, (2.2.6)])

$\sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} z^n / (q; q)_n = -zq; q_{\infty}$ with $z = q$, this can be re-written as:

$$\sum_{n \geq 0} b_{2, n} q^n = \frac{q}{1-q} \sum_{n \geq 1} \frac{q^{\frac{n^2-n}{2}}}{q; q}_{n-1} = \frac{q^2}{1-q} \sum_{n \geq 0} \frac{q^{\frac{n^2-3n}{2}}}{q; q}_n = \frac{q^2}{1-q} - q^2; q_{\infty}.$$

Hook bias for $t = 2$. We consider the generating function for $a_{2, n} - b_{2, n}$. We have

$$\begin{aligned} \sum_{n \geq 0} a_{2, n} - b_{2, n} q^n &= \frac{1}{q; q^2_{\infty}} \left(q^2 \sum_{n \geq 2} q^{2n-1} - q^{2 \cdot 2n-1} \right) - \frac{q^2}{1-q} - q^2; q_{\infty} \\ &= -q; q_{\infty} \frac{q^2}{1-q^2} \left(\frac{1}{1} \frac{q}{q^2} - 1 \right) = -q; q_{\infty} \frac{q^2}{1-q^2} \left(\frac{q - q^2}{1} \frac{q^3}{q^2} \right) \\ &= q^3 \frac{1}{1-q^2} - q^3; q_{\infty} = \frac{q^3}{1-q^2} - q^3; q_{\infty} = q^{1-2} - q^3; q_{\infty}. \end{aligned} \quad (2.1)$$

In the second line, we used Euler's identity [1, (1.2.5)] $1 / (q; q^2_{\infty}) = -q; q_{\infty}$. Clearly, the expression in (2.1), expanded as a q -series, has non-negative coefficients which can be interpreted combinatorially as follows. The expression $-q^3; q_{\infty}$ is the generating function for the number of distinct partitions of n that do not have 1 and 2 as parts. Glaisher's bijection [7], which splits each even part in a distinct partition into equal odd parts, maps these partitions bijectively to odd partitions λ of n with $m_{\lambda}(1) \equiv 0 \pmod{4}$. Similarly, $q^{1-2} - q^3; q_{\infty}$ is the generating function for the number of distinct partitions of n that have both 1 and 2 as parts. Glaisher's bijection maps these partitions bijectively to odd partitions λ of n with $m_{\lambda}(1) \equiv 3 \pmod{4}$. Moreover, $q^3 / (1 - q^2) = q^3 + q^5 + \dots$ is the generating function of the number of partitions of n consisting of a single odd part greater than 1, which we overline. Thus, the right hand side of (2.1) is the generating function for the number of overpartitions of n into odd parts with $m_{\lambda}(1) \equiv 0, 3 \pmod{4}$ and exactly one part overlined, which is greater than 1.

3 Hooks of length 3

In this section, we briefly sketch the main ideas of the proof of Theorem 4 for $t = 3$. The complete proof is very involved and not suited for an extended abstract. It will be

published in our forthcoming paper [3].

As in Section 2, we define $a_3 m, n$ (respectively $b_3 m, n$) to be the number of odd (respectively distinct) partitions of n with m hooks of length 3, and use the same strategy to find the generating functions for the sequences $a_3 n$ and $b_3 n$.

Odd partitions. Let λ be an odd partition. Among parts equal to 1, there is a hook of length 3 only in the third to last part equal to 1 (thus, only if the multiplicity of 1 is at least three). For all other parts, there is a hook of length 3 in the second to last and third to last occurrence of the part size (if the multiplicity of the part permits). There is also a hook of length 3 in the last occurrence of the part size if that last occurrence is in row i with $\lambda_i - \lambda_{i-1} \geq 2$. Therefore $a_3 m, n$ is the number of odd partitions λ of n such that

$$\delta_{m_\lambda \geq 3} \left(\sum_{u \geq 2} \delta_{m_\lambda u \geq 1} \delta_{m_\lambda u \geq 2} \delta_{m_\lambda u \geq 3} \right) - \ell_2 \lambda = m.$$

Here, δ_ρ denotes the Kronecker delta symbol, which evaluates to 1 if property ρ is true, and 0 if not. If $x k, n$ is the number of odd partitions λ of n such that

$$\delta_{m_\lambda \geq 3} \sum_{u \geq 2} \delta_{m_\lambda u \geq 1} \delta_{m_\lambda u \geq 2} \delta_{m_\lambda u \geq 3} = k,$$

and $y k, n$ is the number of odd partitions λ of n with $\ell_2 \lambda = k$, then

$$a_3 n = \sum_{m \geq 0} m a_3 m, n = \sum_{k \geq 0} k x k, n = \sum_{k \geq 0} k y k, n.$$

Let $\mathcal{F}_1 z; q := \sum_{n, k \geq 0} x k, n z^k q^n$ and $\mathcal{F}_2 z; q := \sum_{n, k \geq 0} y k, n z^k q^n$. Then,

$$\sum_{n \geq 0} a_3 n q^n = \left. \frac{\partial}{\partial z} \right|_{z=1} \mathcal{F}_1 z; q = \left. \frac{\partial}{\partial z} \right|_{z=1} \mathcal{F}_2 z; q.$$

We have

$$\mathcal{F}_1 z; q = \left(1 - q - q^2 - \frac{zq^3}{1-q} \right) \prod_{n \geq 1} \left(1 - zq^{2n-1} - z^2 q^{2 \cdot 2n-1} - \frac{z^3 q^{3 \cdot 2n-1}}{1 - q^{2n-1}} \right).$$

To find $\mathcal{F}_2 z; q$, we consider the conjugate of the 2-modular diagram of a partition. In the conjugate diagram, the boxes in the top row are filled with 1 and the boxes in all other rows are filled with 2. We need to count rows with multiplicity 1 among the rows filled with 2 that are less than the first row, which we record in the exponent of z . Thus,

$$\mathcal{F}_2 z; q = \sum_{n \geq 1} \frac{q^n}{1 - q^{2n}} \prod_{j=1}^{n-1} \left(1 - zq^{2j} - \frac{q^{2 \cdot 2j}}{1 - q^{2j}} \right).$$

Furthermore,

$$\begin{aligned} \frac{\partial}{\partial z} \Big|_{z=1} \mathcal{F}_1(z; q) &= \frac{1}{q; q^2_\infty} \left(q^3 \sum_{n \geq 1} q^{2n-1} - q^{2 \cdot 2n-1} - q^{3 \cdot 2n-1} \right) \\ &= -q; q_\infty \left(q^3 - \frac{q^3}{1-q^2} - \frac{q^6}{1-q^4} - \frac{q^9}{1-q^6} \right) \end{aligned}$$

and

$$\frac{\partial}{\partial z} \Big|_{z=1} \mathcal{F}_2(z; q) = \sum_{n \geq 1} q^n \frac{1}{q^2; q^2_n} \sum_{j=1}^{n-1} q^{2j} (1 - q^{2j}).$$

Using the well-known identity $\sum_{n \geq 0} z^n / (q; q)_n = 1 / (z; q)_\infty$ (see e.g. [1, (2.2.5)]) and further simplifications similar to those in the calculation of $\frac{\partial}{\partial z} \Big|_{z=1} G_2(z; q)$, we obtain

$$\frac{\partial}{\partial z} \Big|_{z=1} \mathcal{F}_2(z; q) = \frac{1}{q; q^2_\infty} \frac{q^4}{1-q^4}.$$

Finally, we have

$$\begin{aligned} \sum_{n \geq 0} a_3(n) q^n &= \frac{\partial}{\partial z} \Big|_{z=1} \mathcal{F}_1(z; q) - \frac{\partial}{\partial z} \Big|_{z=1} \mathcal{F}_2(z; q) \\ &= -q; q_\infty \left(q^3 - \frac{q^3}{1-q^2} - \frac{q^6}{1-q^4} - \frac{q^9}{1-q^6} - \frac{q^4}{1-q^4} \right) \\ &= -q^3; q_\infty \frac{q^3(1-q^3)}{1-q^2} - q; q_\infty \left(\frac{q^6}{1-q^4} - \frac{q^3}{1-q^6} \right). \end{aligned} \quad (3.1)$$

Distinct partitions. Let λ be a distinct partition. There is a hook of length 3 in every row $\lambda_i > 1$ except when $\lambda_i - \lambda_{i-1} = 2$. Therefore $b_3(m, n)$ is the number of distinct partitions of n such that $\ell(\lambda) - m_\lambda = 1 - \ell_2(\lambda) = m$. If $u(k, n)$ is the number of distinct partitions of n with exactly k parts greater than 1, and $v(k, n)$ is the number of distinct partitions λ of n with $\ell_2(\lambda) = k$, then

$$b_3(n) = \sum_{m \geq 0} m b_3(m, n) = \sum_{k \geq 0} k u(k, n) - \sum_{k \geq 0} k v(k, n).$$

Let $\mathcal{G}_1(z; q) := \sum_{n, k \geq 0} u(k, n) z^k q^n$ and $\mathcal{G}_2(z; q) := \sum_{n, k \geq 0} v(k, n) z^k q^n$. Then,

$$\sum_{n \geq 0} b_3(n) q^n = \frac{\partial}{\partial z} \Big|_{z=1} \mathcal{G}_1(z; q) - \frac{\partial}{\partial z} \Big|_{z=1} \mathcal{G}_2(z; q).$$

We have

$$\mathcal{G}_1(z; q) = 1 - q - zq^2; q_\infty.$$

To find $\mathcal{G}_2(z; q)$, let λ be a partition with distinct parts and, as in the case $t = 2$, remove a staircase of length $\ell(\lambda)$ from λ to obtain a partition μ with at most $\ell(\lambda)$ parts. Recall that $\ell_1(\mu)$ is the number of parts μ_i of μ such that $\mu_i - \mu_{i-1} = 1$. Note that parts μ_i in μ such that $\mu_i - \mu_{i-1} = 1$ correspond to parts λ_i in λ such that $\lambda_i - \lambda_{i-1} = 2$. Thus, $\ell_1(\mu) = \ell_2(\lambda)$. For a partition ν , denote by $\tilde{\ell}(\nu)$ the number of parts of ν with multiplicity one. Since $\ell_1(\mu) = \tilde{\ell}(\mu')$, we have

$$\mathcal{G}_2(z; q) = \sum_{m \geq 1} q^{\frac{m(m-1)}{2}} \prod_{j=1}^m \left(1 - zq^j - \frac{q^{2j}}{1 - q^j} \right).$$

We compute

$$\frac{\partial}{\partial z} \Big|_{z=1} \mathcal{G}_1(z; q) = -q; q_\infty = \sum_{m \geq 2} \frac{q^m}{1 - q^m}$$

and

$$\begin{aligned} \frac{\partial}{\partial z} \Big|_{z=1} \mathcal{G}_2(z; q) &= \sum_{n \geq 1} \frac{q^{\frac{n(n-1)}{2}}}{q; q_n} \sum_{j=1}^n q^j (1 - q^j) = \sum_{n \geq 1} \frac{q^{\frac{n(n-1)}{2}}}{q; q_n} \left(\frac{1 - q^{n-1}}{1 - q} - \frac{1 - q^{2n-1}}{1 - q^2} \right) \\ &= \frac{q}{1 - q^2} \sum_{n \geq 1} \frac{q^{\frac{n(n-1)}{2}}}{q; q_{n-1}} (1 - q^{n-1}) = \frac{q}{1 - q^2} \left(\sum_{n \geq 0} \frac{q^{\frac{n(n-1)}{2}}}{q; q_n} - \sum_{n \geq 0} \frac{q^{\frac{n(n-1)}{2} - 2n - 3}}{q; q_n} \right) \\ &= \frac{q}{1 - q^2} (q - q^2; q_\infty - q^3 - q^3; q_\infty) = \frac{q^2}{1 - q^2} - q^3; q_\infty. \end{aligned}$$

For the second to last equality we have again used the q -binomial theorem.

Finally, we have

$$\begin{aligned} \sum_{n \geq 0} b_3(n) q^n &= \frac{\partial}{\partial z} \Big|_{z=1} \mathcal{G}_1(z; q) - \frac{\partial}{\partial z} \Big|_{z=1} \mathcal{G}_2(z; q) \\ &= -q; q_\infty = \sum_{m \geq 2} \frac{q^m}{1 - q^m} - \frac{q^2}{1 - q^2} - q^3; q_\infty. \end{aligned} \tag{3.2}$$

Hook bias for $t = 3$. Next we consider the difference $a_3(n) - b_3(n)$. Establishing non-negativity for $n > 7$ is more complex than for $a_2(n) - b_2(n)$ in [Section 2](#), so we provide a sketch of the proof, leaving the full details to our forthcoming paper [\[3\]](#).

Sketch of proof of [Theorem 4](#) for $t = 3$. We begin by subtracting the generating functions established in (3.1) and (3.2). After a lengthy series of manipulations, calculations, and simplifications, we prove that, to establish the desired non-negativity result, it is sufficient to establish certain linear inequalities for the (restricted) partition numbers

$$q(n) : p(n) \mid \text{distinct parts} \quad \text{and} \quad \rho(n, m) : p(n) \mid \text{distinct parts at least } m.$$

An intermediate ingredient we use is the following result of Erdős-Nicolas-Szalay relating $q(n)$ and $\rho(n, m)$.

Theorem 5 (Theorem 1, [6]). *For all n and m satisfying $1 \leq m \leq n$, we have that*

$$\frac{q(n)}{2^{m-1}} \leq \rho(n, m) \leq \frac{q(n) - m(m-1)/2}{2^{m-1}}.$$

To complete our proof of Theorem 4 for $t = 3$, we prove the following general theorem on linear inequalities for $q(n)$, which is also of independent interest.

Theorem 6 (Theorem 1.11, [3]). *Suppose $\sum_{k=1}^r \alpha_k < \sum_{\ell=1}^s \beta_\ell$, where $\{\alpha_k\}_{k=1}^r, \{\beta_\ell\}_{\ell=1}^s \subset \mathbb{N}$, and $r, s \in \mathbb{N}$. Moreover, let $\{\mu_k\}_{k=1}^r, \{\nu_\ell\}_{\ell=1}^s \subset \mathbb{N}$, where $\mu_k \nmid \mu_j$ and $\nu_j \nmid \nu_k$ for $j \neq k$. Then for sufficiently large n , we have that*

$$\sum_{k=1}^r \alpha_k q(n - \mu_k) \leq \sum_{\ell=1}^s \beta_\ell q(n - \nu_\ell). \quad (3.3)$$

Remark 7. In [3], we make Theorem 6 effective, and establish a lower bound on n (depending on the sequences given in the hypothesis of the theorem) sufficient to guarantee (3.3).

Sketch of Proof of Theorem 6. We first show that it suffices to establish

Proposition 8 (Proposition 2.1, [3]). *For $L \in \mathbb{N}$, and $\varepsilon > 0$, there exists an $N_{\varepsilon, L}$ such that for $n > N_{\varepsilon, L}$*

$$0 < \frac{q(n - L)}{q(n)} - 1 < \varepsilon.$$

Continuing our sketch of proof of Theorem 6, we recall the following beautiful exact formula for $q(n)$ as an infinite sum of Kloosterman sums multiplied by Bessel functions due to Hagis [8], reminiscent of the celebrated exact formula for $p(n)$ by Hardy-Ramanujan-Rademacher [15] obtained using the Circle Method.

Theorem 9 (Theorem 4, [8]). *For $n \in \mathbb{N}$, we have that*

$$q(n) = \frac{\pi}{24n - 1/2} \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} k^{-1} \left(\sum'_{h \bmod k} \chi(h, k) \exp(-2\pi i n h / k) \right) I_1 \left(\frac{\pi}{12k} (48n - 1/2)^{1/2} \right).$$

Above, I_1 is the Bessel function of the first order, χ is an explicit exponential function, and the sum on h is taken over $h \bmod k$ relatively prime to k . Using Theorem 9, Beckwith-Bessenrodt establish the following asymptotic estimate on $q(n)$, an important ingredient in our proof of Theorem 6.

Theorem 10 (Theorem 2.3, [4]). Let $\mu : \frac{\pi}{6\sqrt{2}}\sqrt{24n-1}$. Then for $n \in \mathbb{N}$, we have that

$$q^n = \frac{\pi}{\sqrt{24n-1}} I_1(\mu) + E(n),$$

where

$$|E(n)| \leq \frac{0.9\pi}{\sqrt{24n-1}} \cdot \frac{e^\mu}{\mu} \left(1 - 5\mu^2 e^{-\mu}\right).$$

Given the above, a last key ingredient in our proof of [Theorem 6](#) is certain effective estimates on the I_1 -Bessel function.

Proposition 11 (Exercise 7.13.2, [14]). Let z be a complex number with $|\arg z| \leq \frac{\pi}{2}$ and let $s \in \mathbb{C}$. Then we have

$$I_1(z) = \frac{e^z}{\sqrt{2\pi z}} (1 + \delta(z)) + i \frac{e^{-z}}{\sqrt{2\pi z}} (1 + \gamma(z)),$$

where we have the bounds

$$|\gamma(z)| \leq \frac{3}{4|z|} \exp\left(\frac{3}{4|z|}\right), \quad |\delta(z)| \leq \frac{3\pi}{8|z|} \exp\left(\frac{3\pi}{8|z|}\right).$$

Then, a lengthy calculation using [Theorem 10](#) and [Proposition 11](#) proves [Proposition 8](#). We refer the interested reader to [3] for full details of the proof of [Theorem 4](#) for $t = 3$ and [Theorem 6](#) using the ingredients given here.

4 Further bias results

Recall that $\ell_2(\lambda)$ is the number of parts λ_i of λ with $\lambda_i - \lambda_{i-1} = 2$. Note that this partition statistic appeared in our calculations of both $a_3(n)$ and $b_3(n)$. Thus, it is natural to investigate a possible bias in the total number of gaps of size exactly 2 in odd versus distinct partitions. We prove that such a bias exists.

Theorem 12. For $n \in \mathbb{N}_0$, we have that

$$\sum_{\lambda \in \mathcal{O}(n)} \ell_2(\lambda) - \sum_{\lambda \in \mathcal{D}(n)} \ell_2(\lambda)$$

is non-negative except for $n = 2$ and $n = 6$ in which cases it equals -1 .

Proof. From [Section 3](#),

$$\sum_{\lambda \in \mathcal{D}(n)} \ell_2(\lambda) = -q^3; q \infty \frac{q^2}{1-q^2} \quad \text{and} \quad \sum_{\lambda \in \mathcal{O}(n)} \ell_2(\lambda) = -q; q \infty \frac{q^4}{1-q^4}.$$

Thus, we need to show that, the coefficients of q^n , $n \neq 2, 6$, in

$$H(q) : -q; q \infty \frac{q^4}{1-q^4} - -q^3; q \infty \frac{q^2}{1-q^2}$$

are non-negative. A direct calculation shows that the coefficients of q^2 and q^6 in $H(q)$ are both equal to -1 .

After some q -series manipulations, we have that

$$\begin{aligned} H(q) - q^2 - q^6 &= -q^3; q \infty \frac{q^5}{1-q^2} - q^5 - q^4; q \infty - q^2 - 1 - q^4 \sum_{m \geq 5} q^m - q^{m-1}; q \infty \\ &= -q^4; q \infty \frac{q^7}{1-q} - q^2 - 1 - q^4 \sum_{m \geq 5} q^m - q^{m-1}; q \infty \\ &= -q^4; q \infty \sum_{k \geq 7} q^k - 1 - q^4 \sum_{k \geq 7} q^k - q^{k-1}; q \infty \\ &= 1 - q^4 \sum_{k \geq 7} q^k \left(-q^5; q \infty - -q^{k-1}; q \infty \right). \end{aligned}$$

Moreover, for $k \geq 7$, we have

$$-q^5; q \infty - -q^{k-1}; q \infty = \sum_{j \geq 5} q^j - q^{j-1}; q \infty - \sum_{j \geq k-1} q^j - q^{j-1}; q \infty = \sum_{j=5}^{k-2} q^j - q^{j-1}; q \infty.$$

Thus,

$$H(q) - q^2 - q^6 = 1 - q^4 \sum_{k \geq 7} q^k \sum_{j=5}^{k-2} q^j - q^{j-1}; q \infty, \quad (4.1)$$

which clearly has non-negative coefficients and completes the proof. $\square$

We interpret (4.1) as the generating function for $|\mathcal{H}(n)|$, where $\mathcal{H}(n)$ is the set of partitions λ of n satisfying all of the following conditions:

- (i) $m_{\lambda}(3) \geq 2$, and if $m_{\lambda}(3) = 2$, then $\lambda_{\ell(\lambda)} < 3$,
- (ii) 3 is the only repeated part, and 1 and 2 cannot both occur as parts in λ ,
- (iii) λ has part greater than 4 and the smallest part s greater than 4 satisfies $5 \leq s \leq (\sum_{\lambda_i \leq 3} \lambda_i) - 2$.

To see this, write $k \geq 7$ as $k = 3d + r$, $0 \leq r \leq 2$ and interpret q^k as generating d parts equal to 3 and one part equal to r . Moreover, $\sum_{j=5}^{k-2} q^j - q^{j-1}; q \infty$ generates non-empty distinct partitions with smallest part between 5 and $k-2$ (inclusive). Then, in analogy to Theorem 1, for $n \neq 2, 6$,

$$\sum_{\lambda \in \mathcal{O}(n)} \ell_2(\lambda) = \sum_{\lambda \in \mathcal{D}(n)} \ell_2(\lambda) = |\mathcal{H}(n)|.$$

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Horizontal-strip LLT polynomials

Foster Tom^{*1}

¹*Massachusetts Institute of Technology*

Abstract. Lascoux, Leclerc, and Thibon defined a remarkable family of symmetric functions that are q -deformations of products of skew Schur functions. These LLT polynomials $G_\lambda(x; q)$ can be indexed by a tuple λ of skew diagrams. When each skew diagram of λ is a row, we define a weighted graph $\Pi(\lambda)$ associated to λ . We show that a horizontal-strip LLT polynomial is determined by this weighted graph. When $\Pi(\lambda)$ has no triangles, we establish a combinatorial Schur expansion of $G_\lambda(x; q)$. We also explore a connection to extended chromatic symmetric functions.

Keywords: chromatic symmetric function, cocharge, Hall–Littlewood polynomial, LLT polynomial, Schur-positive, symmetric function

1 Introduction

LLT polynomials are symmetric functions $G_\lambda(x; q)$ indexed by a sequence λ of skew diagrams, and are actively studied in algebraic combinatorics and representation theory. Horizontal-strip LLT polynomials generalize Hall–Littlewood polynomials, which are the Frobenius series of cohomology rings of certain subsets of the flag variety [7]. The Shuffle Theorem [3] of Carlsson and Mellit gives a positive combinatorial formula for ∇e_n , the Frobenius series of the space of diagonal harmonics, in terms of LLT polynomials. Haglund, Haiman, and Loehr [6] proved that Macdonald polynomials also expand positively into LLT polynomials, which implies that they are Schur-positive. LLT polynomials are also closely related to chromatic quasisymmetric functions, defined by Shareshian and Wachs [12].

Although LLT polynomials are known to be Schur-positive through Kazhdan–Lusztig theory [5, 11], it remains a predominant open problem to give a combinatorial proof.

Problem 1.1. Find a combinatorial Schur expansion of LLT polynomials of the form

$$G_\lambda(x; q) = \sum_{T \in \mathcal{S}} q^{\text{stat } T} s_{\text{partition } T}. \quad (1.1)$$

Because LLT polynomials generalize products of skew Schur functions, for which characterizing equalities is an area of active research, it is also interesting and challenging to determine when two LLT polynomials are equal.

^{*}ftom@berkeley.edu. The author was supported in part by the Natural Sciences and Engineering Research Council of Canada

Problem 1.2. Characterize those sequences of skew diagrams λ and μ for which we have

$$G_\lambda(x; q) = G_\mu(x; q). \quad (1.2)$$

In the unicellular case, meaning that every skew diagram of λ consists of a single cell, we can associate a unit interval graph $\Gamma \lambda$ to λ . Huh, Nam, and Yoo [8] proved a combinatorial Schur expansion of the LLT polynomial $G_\lambda(x; q)$ whenever $\Gamma \lambda$ is a “melting lollipop”, namely

$$G_\lambda(x; q) = \sum_{T \in \text{SYT}_n} q^{\text{wt}_a T} s_{\text{shape } T}. \quad (1.3)$$

Moreover, the unicellular LLT polynomial $G_\lambda(x; q)$ is related by a change of variables to the chromatic quasisymmetric function of $\Gamma \lambda$, specifically by the plethysm [3]

$$q - 1^{-n} G_\lambda(x; q - 1; q) = X_{\Gamma \lambda}(x; q), \quad (1.4)$$

where n is the number of cells. In particular, this implies that for λ and μ sequences of single cells, we have

$$G_\lambda(x; q) = G_\mu(x; q) \text{ if and only if } X_{\Gamma \lambda}(x; q) = X_{\Gamma \mu}(x; q). \quad (1.5)$$

More generally, we consider the horizontal-strip case, meaning that every skew diagram of λ is a row. We generalize the construction $\Gamma \lambda$ by defining a weighted interval graph $\Pi \lambda$ associated to λ . We prove that a horizontal-strip LLT polynomial is determined by this weighted graph, which provides a powerful sufficient condition for equality of LLT polynomials.

Theorem 1.3. [14, Theorem 2.7] Let λ and μ be sequences of rows.

$$\text{If } \Pi \lambda \sim \Pi \mu, \text{ then } G_\lambda(x; q) = G_\mu(x; q). \quad (1.6)$$

We define a statistic on tableaux, denoted $\text{cocharge}_{\Pi \lambda}$, and we prove the following combinatorial Schur expansion of $G_\lambda(x; q)$ whenever the weighted graph $\Pi \lambda$ has no triangles.

Theorem 1.4. [13, Theorem 4.6] Let λ be a sequence of rows such that $\Pi \lambda$ is triangle-free. Then

$$G_\lambda(x; q) = \sum_{T \in \text{SSYT } \lambda} q^{\text{cocharge}_{\Pi \lambda} T} s_{\text{shape } T}. \quad (1.7)$$

We also show that at $q = 1$, the horizontal-strip LLT polynomial $G_\lambda(x; q)$ is related by a change of variables to the extended chromatic symmetric function $X_{\Pi \lambda}(x)$ defined by Crew and Spirkl [4] for a weighted graph.

Theorem 1.5. Let λ be a sequence of n rows. The plethystically modified LLT polynomial $G_\lambda(x; q-1; q)$ is divisible by $(q-1)^n$, and at $q=1$ we have

$$(q-1)^{-n} G_\lambda(x; q-1; q) \big|_{q=1} = X_{\Pi \lambda}(x). \quad (1.8)$$

In particular, this implies that for λ and μ sequences of rows, we have that

$$\text{if } G_\lambda(x; q) = G_\mu(x; q), \text{ then } X_{\Pi \lambda}(x) = X_{\Pi \mu}(x). \quad (1.9)$$

2 Background

A *partition* σ is a finite nonincreasing sequence of positive integers $\sigma = \sigma_1 \cdots \sigma_\ell$. By convention, we set $\sigma_i = 0$ if $i > \ell$. A *skew diagram* λ is a subset of $\mathbb{Z} \times \mathbb{Z}$ of the form

$$\lambda = \sigma/\tau = \{i, j : i \geq 1, \tau_i - 1 \leq j \leq \sigma_i\} \quad (2.1)$$

for some partitions σ and τ with $\sigma_i \geq \tau_i$ for every i . When τ is empty, we write σ instead of $\sigma/\emptyset$. The elements of λ are called *cells* and the *content* of a cell $u = i, j \in \lambda$ is the integer $c(u) = j - i$. We will focus heavily on *rows*, which are skew diagrams of the form

$$R = a/b = \{1, j : b - 1 \leq j \leq a\} \quad (2.2)$$

for some $a \geq b \geq 0$. We denote by $\ell(R) = b$ and $r(R) = a - 1$ the smallest and largest contents of cells in R respectively. Note that $\ell(R)$ is the content of the leftmost cell in R , not the length of R , which is $|R| = r(R) - \ell(R) + 1$. We also denote by $R^+ = a + 1 / b + 1$ and $R^- = a - 1 / b - 1$ the rows obtained by shifting R right or left respectively by one cell. A *semistandard Young tableau* (SSYT) of shape λ is a function $T : \lambda \rightarrow \{1, 2, 3, \dots\}$ that satisfies

$$T_{i,j} \leq T_{i,j-1} \text{ and } T_{i,j} < T_{i-1,j}, \quad (2.3)$$

where we write $T_{i,j}$ to mean $T(i, j)$. The *weight* of T is the sequence $w(T) = w_1, \dots$, where $w_i = |T^{-1}(i)|$ is the number of times the integer i appears. We denote by SSYT_λ the set of SSYT of shape λ and by SSYT_α the set of SSYT of weight α . We define the *skew Schur function* of shape $\lambda = \sigma/\tau$ to be

$$s_\lambda = \sum_{T \in \text{SSYT}_\lambda} \mathbf{x}^T, \quad (2.4)$$

where $\mathbf{x}^T$ is the monomial $x_1^{w_1} x_2^{w_2} \cdots$. When τ is empty, we call s_λ a *Schur function*.

A *multiskew partition* is a finite sequence of skew diagrams $\lambda = \lambda^{(1)}, \dots, \lambda^{(n)}$. We say that λ is *unicellular* if each $\lambda^{(i)}$ is a single cell and in keeping with the terminology of

Alexandersson and Sulzgruber [1], we say that λ is a *horizontal-strip* if each λ^i is a row. We denote by

$$\text{SSYT}_\lambda = \{T = T^1, \dots, T^n : T^i \in \text{SSYT}_{\lambda^i}\} \quad (2.5)$$

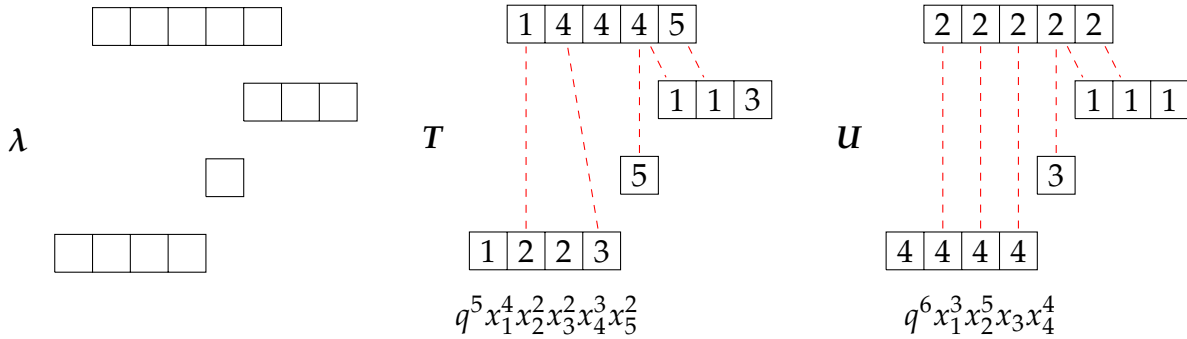
the set of *semistandard multiskew tableaux* of shape λ . Two entries $T^i u$ and $T^j v$ with $i < j$ form an *inversion* if either

- $c u = c v$ and $T^i u > T^j v$, or
- $c u = c v - 1$ and $T^j v > T^i u$.

We denote by $\text{inv } T$ the number of inversions of T . Now we define the *LLT polynomial* [6, 10]

$$G_\lambda(x; q) = \sum_{T \in \text{SSYT}_\lambda} q^{\text{inv } T} x^T. \quad (2.6)$$

Example 2.1. The horizontal-strip $\lambda = 4/0, 5/4, 8/5, 6/1$, drawn so that cells of the same content are aligned vertically, two SSYTs S and T of shape λ with their inversions marked by dashed lines, and the corresponding monomials of the LLT polynomial $G_\lambda(x; q)$ are given below.



Some terms of the Schur function expansion of the LLT polynomial $G_\lambda(x; q)$ are

$$G_\lambda(x; q) = q^6 s_{5431}(x) + q^6 s_{544}(x) + \dots + q^6 (2q^5 s_{733}(x) + \dots + 3q s_{121}(x) + s_{13}(x)). \quad (2.7)$$

Note that $G_\lambda(x; q)$ is *Schur-positive*, meaning that it is an $\mathbb{N}[q]$ -linear combination of Schur functions. In fact, this property holds in general [5, Corollary 6.9].

3 A weighted graph description

We now define our weighted graph $\Pi(\lambda)$.

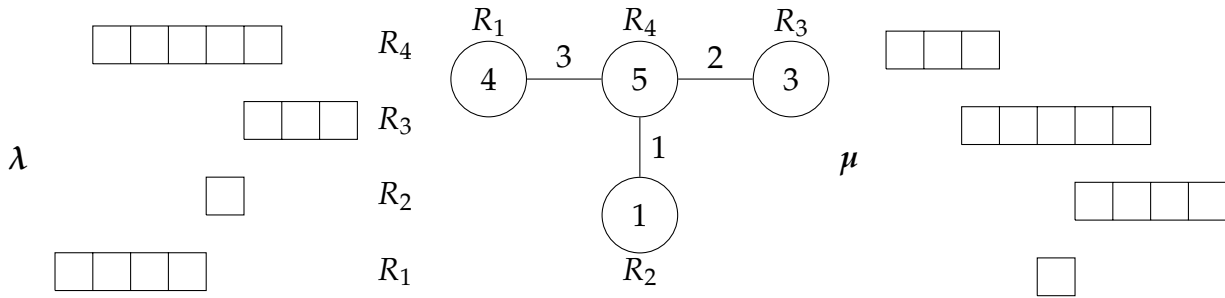
Definition 3.1. [13, Definition 3.1] Let R and R' be rows. We define the integer

$$M(R, R') = \begin{cases} |R \cap R'| & \text{if } \ell(R) \leq \ell(R'), \\ |R \cap R'| - 1 & \text{if } \ell(R) > \ell(R'). \end{cases} \quad (3.1)$$

Note that $0 \leq M(R, R') \leq \min\{|R|, |R'|\}$. We can think of $M(R, R')$ as measuring the extent to which the rows R and R' interact. More precisely, $M(R, R')$ is the maximum number of inversions that a tableau $T \in \text{SSYT}_\lambda$ can have between cells in R and R' [13, Theorem 3.5].

Definition 3.2. [13, Definition 3.2] Let $\lambda = R_1, \dots, R_n$ be a horizontal-strip. We define a *weighted graph* $\Pi \lambda$ whose vertices are the rows of λ . The weight of a row R_i is the number of cells $|R_i|$ and rows R_i and R_j with $i < j$ are joined by an edge of weight $M(R_i, R_j)$. By convention, we omit edges of weight zero.

Example 3.3. The horizontal-strip $\lambda = 4/0, 5/4, 8/5, 6/1$ and the weighted graph $\Pi \lambda$ are given below. We have $M(R_1, R_4) = 3$, $M(R_2, R_4) = 1$, and $M(R_3, R_4) = 2$. We have also drawn the horizontal-strip $\mu = 5/4, 9/5, 7/2, 3/0$, whose weighted graph $\Pi \mu$ is isomorphic to $\Pi \lambda$.



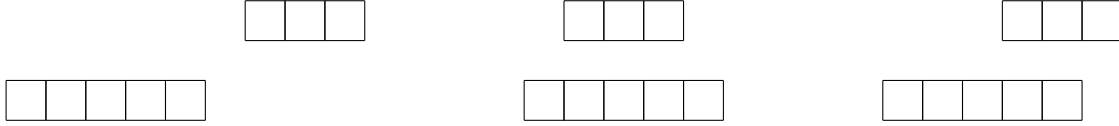
Because $\Pi \lambda \sim \Pi \mu$, it follows from Theorem 1.3 that $G_\lambda(x; q) = G_\mu(x; q)$. Moreover, because the weighted graph $\Pi \lambda$ is triangle-free, Theorem 1.4 gives a combinatorial Schur expansion of $G_\lambda(x; q)$. We will give the details in Example 3.18.

We now describe some of the tools that were used to prove Theorem 1.3 and Theorem 1.4. The following two operations, *cycling* and *commuting*, allow us to move the rows of λ while preserving both the weighted graph $\Pi \lambda$ and the LLT polynomial $G_\lambda(x; q)$.

Proposition 3.4. (Cycling) Let $\lambda = R_1, R_2, \dots, R_n$ be a horizontal-strip and let $\kappa = R_2, \dots, R_n, R_1^-$. Then $G_\kappa(x; q) = G_\lambda(x; q)$.

Definition 3.5. We say that rows R and R' *commute*, denoted $R \leftrightarrow R'$, if we have $M(R, R') = M(R', R)$, and otherwise we write $R \nleftrightarrow R'$.

Example 3.6. The pairs of rows on the left and the middle commute and the pair of rows on the right does not. As a visual description, we have that two rows commute if and only if they are either disjoint and separated by at least one cell, or if one is contained in the other.



Lemma 3.7. [13, Lemma 3.15] (Commuting) Let $\lambda = R_1, \dots, R_n$ be a horizontal-strip and let $\mu = R_1, \dots, R_{i-1}, R_i, \dots, R_n$. If $R_i \leftrightarrow R_{i-1}$, then $G_\lambda(x; q) = G_\mu(x; q)$.

Remark 3.8. We also prove the converse that if $G_\lambda(x; q) = G_\mu(x; q)$, then $R_i \leftrightarrow R_{i-1}$ and therefore $\Pi \lambda \sim \Pi \mu$. In other words, equalities of LLT polynomials in this case are precisely characterized by the associated weighted graphs.

We now show how we can use cycling and commuting to prove [Theorem 1.3](#) in the following very special case, in which $G_\lambda(x; q)$ is the modified Hall–Littlewood polynomial $\tilde{H}_\lambda(x; q)$. This argument captures the spirit of the general proof.

Lemma 3.9. [14, Lemma 3.15] Let $\lambda = R_1, \dots, R_n$ and μ be horizontal-strips with

$$M(R_i, R_j) = \min\{|R_i|, |R_j|\} \quad (3.2)$$

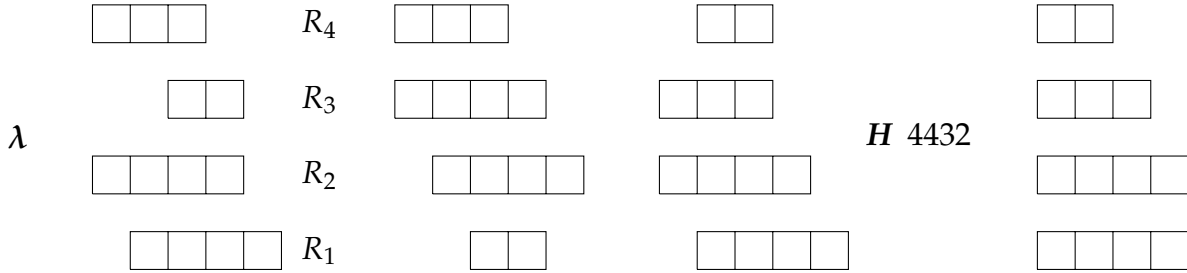
for every $1 \leq i < j \leq n$. If $\Pi \lambda \sim \Pi \mu$, then $G_\lambda(x; q) = G_\mu(x; q)$.

Proof sketch. Let λ denote the partition determined by the row lengths of λ and denote by H_λ the horizontal-strip $\lambda_1/0, \dots, \lambda_n/0$. We show that $G_\lambda(x; q) = G_{H_\lambda}(x; q)$, meaning that the LLT polynomial $G_\lambda(x; q)$ only depends on λ and therefore only on the weighted graph $\Pi \lambda$. By translating all cells by a fixed amount, we may assume without loss of generality that $\min\{\ell(R_i) : 1 \leq i \leq n\} = 0$. Let us also assume that λ has $\sum_{i=1}^n \ell(R_i)$ minimal among such horizontal-strips with the same LLT polynomial.

We now claim that $\ell(R_i) = 0$ for every $1 \leq i \leq n$. If not, let j be such that $\ell(R_j) \geq 1$ is maximal. For every $1 \leq i < j$, the conditions $\ell(R_i) \leq \ell(R_j)$ and $M(R_i, R_j) = \min\{|R_i|, |R_j|\}$ will imply that $R_i \leftrightarrow R_j$, and therefore by commuting and cycling we have that $G_{R_1, \dots, R_n, R_j^-}(x; q) = G_\lambda(x; q)$, contradicting minimality of $\sum_{i=1}^n \ell(R_i)$. Therefore, we have $\ell(R_i) = 0$ for every $1 \leq i \leq n$, and by commuting again we see that $G_\lambda(x; q) = G_{H_\lambda}(x; q)$. This completes the proof. $\square$

Example 3.10. The idea of the proof of [Lemma 3.9](#) is illustrated below. The row R_3 , which has $\ell(R_3)$ maximal, commutes with all rows below, so by commuting and cycling,

we can move it down and then to the left. Continuing in this way, the horizontal-strip λ is shown to be similar to H 4432 on the right.



The general proof of [Theorem 1.3](#) employs this basic technique, along with a comprehensive analysis of several cases, to arrange the rows of λ and μ in order to apply the following recurrence relation of LLT polynomials. We can view it as a deletion-contraction relation of the corresponding weighted graphs.

Lemma 3.11. [[13](#), Lemma 3.17] (Deletion-contraction) Let $\lambda = R_1, \dots, R_n$ be a horizontal-strip with $\ell R_{i-1} < \ell R_i$ and $R_i \leftrightarrow R_{i-1}$. Define the horizontal-strips

$$\lambda' = R_1, \dots, R_{i-1}, R_i, \dots, R_n \text{ and} \quad (3.3)$$

$$\lambda'' = R_1, \dots, R_i \cup R_{i-1}, R_i \cap R_{i-1}, \dots, R_n. \quad (3.4)$$

Then we have

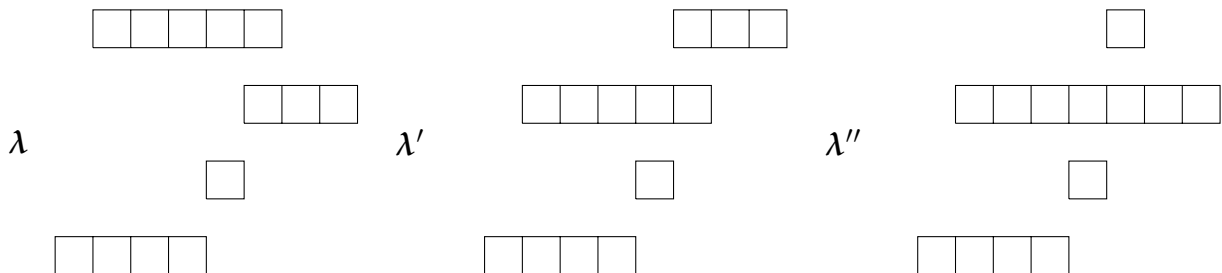
$$G_\lambda(x; q) = qG_{\lambda'}(x; q) - (q-1)G_{\lambda''}(x; q). \quad (3.5)$$

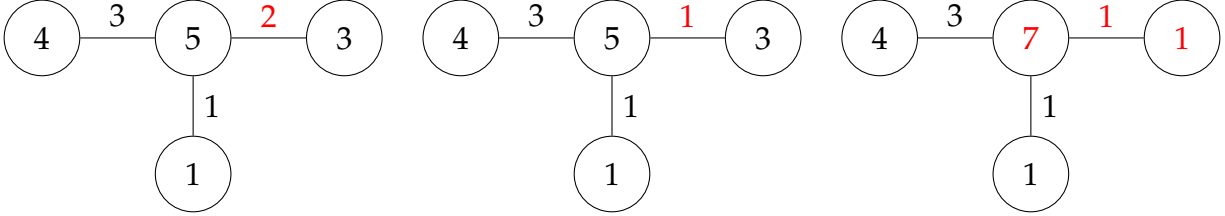
Note that the condition $R_i \leftrightarrow R_{i-1}$ will mean that $R_i \cup R_{i-1}$ is indeed a row.

Example 3.12. Let $\lambda = 4/0, 5/4, 8/5, 6/1$ and note that $\ell R_4 < \ell R_3$ and $R_3 \leftrightarrow R_4$. Therefore, letting $\lambda' = 4/0, 5/4, 6/1, 8/5$ and $\lambda'' = 4/0, 5/4, 8/1, 6/5$, we have that

$$G_\lambda(x; q) = qG_{\lambda'}(x; q) - (q-1)G_{\lambda''}(x; q). \quad (3.6)$$

The horizontal-strips λ , λ' , and λ'' , and their weighted graphs $\Pi \lambda$, $\Pi \lambda'$, and $\Pi \lambda''$ are given below. We can think of $\Pi \lambda'$ and $\Pi \lambda''$ as a deletion and contraction of $\Pi \lambda$.





In order to prove [Theorem 1.4](#), we found a combinatorial formula that also satisfies the deletion-contraction relation [\(3.5\)](#). We now describe the statistic cocharge_{Π} in our formula.

Definition 3.13. Let $T \in \text{SSYT}_{\mu} \alpha$ be a tableau of shape μ , weight α , and with smallest entry i . We define the integer

$$f(T) = \max\{t : 0 \leq t \leq \mu_1 - \mu_2, t \leq \alpha_i, T_{2,j'} > T_{1,j'} - t \text{ for all } 1 \leq j' \leq \mu_2\}. \quad (3.7)$$

Informally, $f(T)$ is the maximum number of i 's that we can remove from T so that no entry moves down when we rectify the resulting skew tableau.

Definition 3.14. Let $T \in \text{SSYT}_{\mu} \alpha$ and let $i < j$. We denote by $T|_{i,j}$ the rectification of the skew tableau obtained by restricting T to the entries x with $i \leq x \leq j$ and we define the integer

$$\text{cocharge}_{i,j}(T) = \alpha_i - f(T|_{i,j}). \quad (3.8)$$

Example 3.15. Two tableaux S and T and their restrictions $S|_{2,4}$ and $T|_{2,4}$ are given below. We have $f(S|_{2,4}) = 3$ and $\text{cocharge}_{2,4}(S) = 5 - 3 = 2$, and we have $f(T|_{2,4}) = 3$ because we must have $t \leq 3$, so $\text{cocharge}_{2,4}(T) = 3 - 3 = 0$.

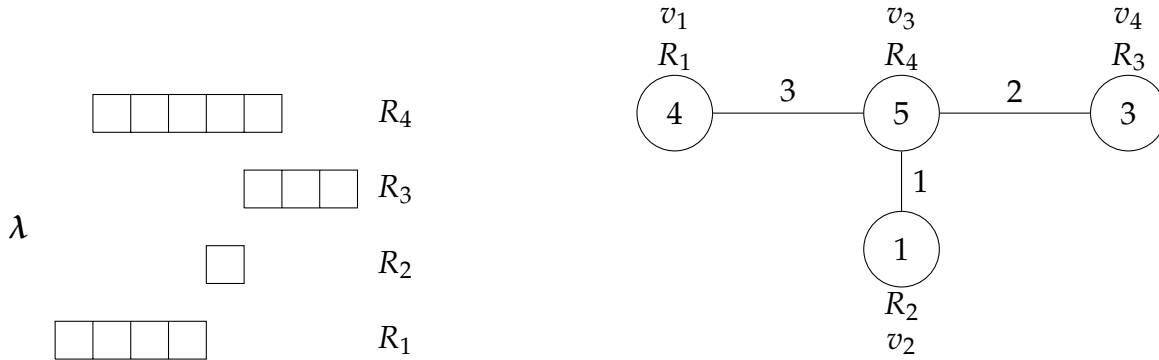
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Example 3.16. In the case where $j = i + 1$, the tableau $T|_{i,j}$ has at most two rows and $\text{cocharge}_{i,j}(T)$ is the number of entries on the second row of $T|_{i,j}$.

Definition 3.17. Let λ be a horizontal-strip such that the weighted graph $\Pi \lambda$ is triangle-free. We can label the vertices of $\Pi \lambda$ as $v_1, \dots, v_n$ so that if $i < j < k$ and v_i is adjacent to v_k , then the weight $M_{j,k}$ of the edge joining v_j and v_k is the weight of the vertex v_j . Let α_i be the weight of the vertex v_i and let $T \in \text{SSYT } \alpha$. We define

$$\text{cocharge}_{\Pi \lambda} T = \sum_{i < j} \min\{M_{i,j}, \text{cocharge}_{i,j} T\}. \quad (3.9)$$

Example 3.18. The horizontal-strip $\lambda = 4/0, 5/4, 8/5, 6/1$ and the weighted graph $\Pi \lambda$ with an appropriate labelling are given below. We have $M_{1,3} = 3$, $M_{2,3} = 1$, and $M_{3,4} = 2$. Note that because v_1 and v_3 are adjacent, $M_{2,3}$ is the weight of v_2 .



To calculate the coefficient of s_{733} , we consider the three tableaux of weight $\alpha = 4153$ and shape 733 below.

T_1	<table><tr><td>4</td><td>4</td><td>4</td></tr><tr><td>3</td><td>3</td><td>3</td></tr><tr><td>1</td><td>1</td><td>1</td></tr></table>	4	4	4	3	3	3	1	1	1	T_2	<table><tr><td>4</td><td>4</td><td>4</td></tr><tr><td>2</td><td>3</td><td>3</td></tr><tr><td>1</td><td>1</td><td>1</td></tr></table>	4	4	4	2	3	3	1	1	1	T_3	<table><tr><td>3</td><td>4</td><td>4</td></tr><tr><td>2</td><td>3</td><td>3</td></tr><tr><td>1</td><td>1</td><td>1</td></tr></table>	3	4	4	2	3	3	1	1	1
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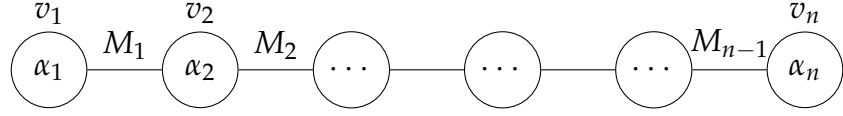
We have

$$\begin{aligned} \text{cocharge}_{\Pi \lambda} T_1 &= \min\{3, \text{cocharge}_{1,3} T_1\} + \min\{1, \text{cocharge}_{2,3} T_1\} + \min\{2, \text{cocharge}_{3,4} T_1\} \\ &= \min\{3, 4-2\} + \min\{1, 1\} + \min\{2, 3\} = 2 + 1 + 2 = 5. \end{aligned} \quad (3.10)$$

Similarly, $\text{cocharge}_{\Pi \lambda} T_2 = 3 + 0 + 2 = 5$ and $\text{cocharge}_{\Pi \lambda} T_3 = 3 + 1 + 2 = 6$. Therefore, the coefficient of s_{733} is $q^6 - 2q^5$.

In the case where $\Pi \lambda$ is a path, our formula (1.7) takes on a more convenient form.

Corollary 3.19. [13, Corollary 4.8] Let λ be a horizontal-strip whose weighted graph $\Pi \lambda$ is the path below.



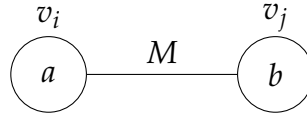
Then the LLT polynomial of λ is

$$G_{\lambda}(x; q) = \sum_{T \in \text{SSYT}_{\alpha}} q^{\text{cocharge}_{\Pi \lambda}(T)} s_{\text{shape}(T)}, \quad (3.11)$$

where

$$\text{cocharge}_{\Pi \lambda}(T) = \sum_{i=1}^{n-1} \min\{M_i, \text{number of entries in the second row of } T|_{i, i-1}\}.$$

Example 3.20. Let λ be a horizontal-strip with exactly two rows, so that $\Pi \lambda$ is



for some $a \geq b \geq M$, where (i, j) is either $(1, 2)$ or $(2, 1)$, so $\alpha = (a, b)$ or $\alpha = (b, a)$ respectively. In either case, for each $0 \leq k \leq b$, there is a unique tableau T_k with content α and shape $(a - b - k, k)$. Therefore, by Corollary 3.19, the LLT polynomial is

$$G_{\lambda}(x; q) = \sum_{k=0}^b q^{\min\{M, k\}} s_{(a-b-k, k)} = s_{(a-b, 0)} \cdots q^M s_{(a-b-M, M)} \cdots q^M s_{ab}. \quad (3.12)$$

4 Further directions

We would like to extend our results toward Problem 1.1 and Problem 1.2 to more general weighted graphs $\Pi \lambda$ or more general multiskew partitions λ . Another direction is to consider expansions into k -Schur functions s_{λ}^k , which are Schur-positive functions introduced by Lapointe, Lascoux, and Morse [9]. They presented several conjecturally equivalent definitions based on different desired properties, and Blasiak, Morse, Pun, and Summers [2] proved that these definitions are equivalent by showing that a class of Catalan symmetric functions satisfies all of these properties.

Conjecture 4.1. Let λ be a multiskew partition whose cells have contents in $\{1, \dots, k\}$ for some k . Then there are polynomials $c_{\mu}(q) \in \mathbb{N}[q]$ such that

$$\omega G_{\lambda}(x; q) = \sum_{\mu} c_{\mu}(q) s_{\mu}^k. \quad (4.1)$$

Alternatively, we could consider Macdonald polynomials, which expand positively into LLT polynomials and therefore into Schur functions, but for which a combinatorial formula is open.

Problem 4.2. Find a combinatorial Schur expansion of Macdonald polynomials of the form

$$\tilde{H}_\lambda(x; q, t) = \sum_{T \in S} q^{\text{stat}_1(T)} t^{\text{stat}_2(T)} s_{\text{partition}(T)}. \quad (4.2)$$

It would also be interesting to explore the connection to chromatic symmetric and quasisymmetric functions. The results (1.4) and Theorem 1.5 suggest that an extended chromatic quasisymmetric function, defined for a weighted graph and incorporating a parameter q , could generalize both results and unify all of these ideas. We pose the following problem.

Problem 4.3. Let Π be a vertex-weighted and edge-weighted graph. Define an extended chromatic quasisymmetric function $X_\Pi(x; q)$ such that

- $X_\Pi(x; q)$ is manifestly quasisymmetric,
- if Π has all vertex weights 1 and all edge weights 0 or 1, we recover the chromatic quasisymmetric function,
- if $q = 1$, we recover the extended chromatic symmetric function,
- $X_\Pi(x; q)$ satisfies a deletion-contraction relation, and
- if $\Pi = \Pi \sqcup \lambda$ for a horizontal-strip $\lambda = R_1, \dots, R_n$, then we have the plethystic relationship

$$X_\Pi(x; q) = (q-1)^{-n} G_\lambda(x; q-1; q), \quad (4.3)$$

which in particular implies that for horizontal-strips λ and μ , we have that

$$G_\lambda(x; q) = G_\mu(x; q) \text{ if and only if } X_{\Pi \sqcup \lambda}(x; q) = X_{\Pi \sqcup \mu}(x; q). \quad (4.4)$$

Note that if $\Pi = \Pi \sqcup \lambda$ for a horizontal-strip λ , then (4.3) can be taken as the definition of $X_\Pi(x; q)$, but it would be interesting to define $X_\Pi(x; q)$ in terms of colourings for general weighted graphs. The flexibility of a weighted graph allows us to consider a deletion-contraction relation, which exists for chromatic polynomials but not for the chromatic symmetric functions of unweighted graphs. This may be a key innovation to the main open problems of chromatic symmetric functions, namely the Stanley–Stembridge conjecture or its refinement, the Shareshian–Wachs conjecture, which we can equivalently state as follows.

Problem 4.4. Let λ be a unicellular multiskew partition. Note that in this case, the chromatic quasisymmetric function $X_{\Pi \lambda}(x; q)$ is in fact symmetric. Find a combinatorial elementary symmetric function expansion of $X_{\Pi \lambda}(x; q)$ of the form

$$X_{\Pi \lambda}(x; q) = \sum_{\theta \in \text{AO}(\Pi \lambda)} q^{\text{asc}(\theta)} e_{\text{partition}(\theta)}(x) \quad (4.5)$$

for some statistic $\text{partition}(\theta)$ on acyclic orientations θ of the graph $\Pi \lambda$.

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Cyclic shuffle-compatibility and cyclic quasisymmetric functions

Jinting Liang^{*1}, Bruce E. Sagan⁺¹, and Yan Zhuang^{‡2}

¹Department of Mathematics, Michigan State University

²Department of Mathematics and Computer Science, Davidson College

Abstract. A permutation statistic st is said to be shuffle-compatible if the distribution of st over the set of shuffles of two disjoint permutations π and σ depends only on $st\pi$, $st\sigma$, and the lengths of π and σ . This notion is implicit in Stanley's work on P -partitions, and was first explicitly studied by Gessel and Zhuang, who developed an algebraic framework for shuffle-compatibility in which quasisymmetric functions play an important role. Later, Domagalski et al. defined a version of shuffle-compatibility for statistics on cyclic permutations. We develop an algebraic framework for cyclic shuffle-compatibility in which the role of quasisymmetric functions is replaced by the cyclic quasisymmetric functions recently introduced by Adin et al.

Keywords: permutation statistics, shuffle-compatibility, cyclic shuffles, quasisymmetric functions, cyclic descents, cyclic peaks

1 Introduction

We say that $\pi = \pi_1\pi_2 \cdots \pi_n$ is a (linear) permutation of length n if it is a sequence of n distinct positive integers, not necessarily from 1 to n . (We refer to these as linear permutations to distinguish them from cyclic permutations, but we will often drop the descriptor “linear” if it is clear from context that we are referring to linear permutations.) For example, 826491 is a permutation of length 6. Let $|\pi|$ denote the length of a permutation π , and $\mathfrak{P}_n$ the set of permutations of length n .

Let $\pi \in \mathfrak{P}_m$ and $\sigma \in \mathfrak{P}_n$ be disjoint permutations, that is, permutations with no letters in common. We say that $\tau \in \mathfrak{P}_{m+n}$ is a shuffle of π and σ if both π and σ are subsequences of τ . The set of shuffles of π and σ is denoted $\pi \sqcup \sigma$. For example, we have $71 \sqcup 25 = \{7125, 7215, 7251, 2715, 2751, 2571\}$.

Following [4], a (linear) permutation statistic is a function st on permutations such that $st\pi = st\sigma$ whenever π and σ have the same relative order. Four well-studied permutation statistics are the descent set Des , descent number des , peak set Pk , and

^{*}liangj26@msu.edu.

⁺sagan@math.msu.edu.

[‡]yazhuang@davidson.edu. YZ was partially supported by an AMS-Simons travel grant.

peak number pk . We say that $i \in [n-1]$ is a *descent* of $\pi \in \mathfrak{P}_n$ if $\pi_i > \pi_{i+1}$. Then the *descent set* and *descent number* of π are defined by

$$\text{Des } \pi : \{i \in [n-1] : \pi_i > \pi_{i+1}\} \quad \text{and} \quad \text{des } \pi : |\text{Des } \pi|,$$

respectively. We say that $i \in \{2, 3, \dots, n-1\}$ is a *peak* of $\pi \in \mathfrak{P}_n$ if $\pi_{i-1} < \pi_i > \pi_{i+1}$; similarly, the *peak set* and *peak number* of π are defined by

$$\text{Pk } \pi : \{i \in \{2, 3, \dots, n-1\} : \pi_{i-1} < \pi_i > \pi_{i+1}\} \quad \text{and} \quad \text{pk } \pi : |\text{Pk } \pi|,$$

respectively.

All four statistics defined above have a remarkable property related to shuffles, called “shuffle-compatibility”. Given a set S of permutations and a permutation statistic st , the *distribution* of st over S is the multiset

$$\text{st } S : \{\{\text{st } \pi : \pi \in S\}\}$$

of all values of st among permutations in S , including multiplicity. Then st is called *shuffle-compatible* if the distribution of st over the shuffles of any two disjoint permutations π and σ depends only on $\text{st } \pi$, $\text{st } \sigma$, and the lengths of π and σ . In other words, st is shuffle-compatible if $\text{st } \pi \sqcup \sigma = \text{st } \pi' \sqcup \sigma'$ whenever $\text{st } \pi = \text{st } \pi'$, $\text{st } \sigma = \text{st } \sigma'$, $|\pi| = |\pi'|$, and $|\sigma| = |\sigma'|$.

The shuffle-compatibility of the descent set and descent number are implicit consequences of Stanley’s theory of P -partitions [7]. Likewise, Stembridge’s [9] work on enriched P -partitions implies that the peak set and peak number are shuffle-compatible. Gessel and Zhuang coined the term “shuffle-compatibility” and initiated the study of shuffle-compatibility per se; in [4], they developed an algebraic framework for shuffle-compatibility based on their notion of the shuffle algebra of a shuffle-compatible statistic, whose multiplication encodes the distribution of the statistic over sets of shuffles.

Gessel’s [3] quasisymmetric functions serve as natural generating functions for P -partitions, and for a family of statistics called “descent statistics”, one can use quasisymmetric functions to characterize shuffle algebras and prove shuffle-compatibility results. Notably, the multiplication rule for fundamental quasisymmetric functions shows that the descent set is shuffle-compatible and that its shuffle algebra is isomorphic to the algebra QSym of quasisymmetric functions. One of Gessel and Zhuang’s main results is a necessary and sufficient condition for shuffle-compatibility of descent statistics which implies that the shuffle algebra of any shuffle-compatible descent statistic is isomorphic to a quotient algebra of QSym .

In the past few years, shuffle-compatibility has become an active topic of research. Most relevant here are the recent papers of Adin–Gessel–Reiner–Roichman [1] and Liang [5] on cyclic quasisymmetric functions and toric $\vec{D}$ -partitions, and of Domagalski–Liang–Minnich–Sagan–Schmidt–Sietsma [2] which defined and investigated a notion of shuffle-compatibility for cyclic permutations.

1.1 Cyclic permutations, statistics, and shuffles

Given a linear permutation $\pi = \pi_1\pi_2 \cdots \pi_n$, let $[\pi]$ be the equivalence class of π under cyclic rotation, that is,

$$[\pi] : \{ \pi_1\pi_2 \cdots \pi_n, \pi_n\pi_1 \cdots \pi_{n-1}, \dots, \pi_2 \cdots \pi_n\pi_1 \}.$$

The sets $[\pi]$ are called *cyclic permutations*. Let the *length* of a cyclic permutation $[\pi]$ be the length of π , which makes sense as all linear permutations in $[\pi]$ have the same length.

Let us define a *cyclic permutation statistic* to be a function cst on cyclic permutations such that $\text{cst } [\pi] = \text{cst } [\sigma]$ whenever π and σ have the same relative order. Two examples of cyclic permutation statistics are the cyclic descent set cDes and the cyclic descent number cdes . First, define the *cyclic descent set* of a linear permutation $\pi \in \mathfrak{P}_n$ by

$$\text{cDes } \pi : \{ i \in [n] : \pi_i > \pi_{i-1} \text{ where } i \text{ is considered modulo } n \};$$

the elements of $\text{cDes } \pi$ are called *cyclic descents* of π . Then the *cyclic descent set* of a cyclic permutation $[\pi]$ is the multiset

$$\text{cDes } [\pi] : \{ \{ \text{cDes } \bar{\pi} : \bar{\pi} \in [\pi] \} \},$$

i.e., the distribution of the linear statistic cDes over all linear permutation representatives of $[\pi]$. For example, we have $\text{cDes } 279358 = \{ \{ \{3,6\}^2, \{1,4\}^2, \{2,5\}^2 \} \}$. Note that $\text{cDes } [\pi]$ can also be characterized as the multiset of cyclic shifts of $\text{cDes } \pi$. More precisely, given $S \subseteq [n]$ and an integer i , define the *cyclic shift* $S \rightarrow i$ by

$$S \rightarrow i : \{ s \rightarrow i : s \in S \}$$

where the values are considered modulo n ; then $\text{cDes } [\pi] = \{ \{ \text{cDes } \pi \rightarrow i : i \in [n] \} \}$.

The *cyclic descent number* on linear and cyclic permutations are defined by

$$\text{cdes } \pi : |\text{cDes } \pi| \quad \text{and} \quad \text{cdes } [\pi] : \text{cdes } \pi,$$

respectively; note that the latter is well-defined because all linear permutations in $[\pi]$ have the same number of cyclic descents. The cyclic peak set cPk and cyclic peak number cpk can be defined in an analogous way, and we will state their definitions in [Section 4](#).

Given disjoint $\pi \in \mathfrak{P}_m$ and $\sigma \in \mathfrak{P}_n$, we say that τ is a *cyclic shuffle* of π and σ if $\tau \in \mathfrak{P}_{m+n}$ and there exist $\bar{\pi} \in [\pi]$ and $\bar{\sigma} \in [\sigma]$ such that $\tau \in \bar{\pi} \sqcup \bar{\sigma}$. Let $[\pi] \sqcup [\sigma]$ denote the set of cyclic shuffles of $[\pi]$ and $[\sigma]$. For instance, we have $63 \sqcup 24 = \{ 6324, 6234, 6243, 6342, 6432, 6423 \}$.

A cyclic permutation statistic cst is called *cyclic shuffle-compatible* if the distribution of cst over all cyclic shuffles of $[\pi]$ and $[\sigma]$ depends only on $\text{cst } [\pi]$, $\text{cst } [\sigma]$, and the lengths of $[\pi]$ and $[\sigma]$. That is, cst is cyclic shuffle-compatible if we have $\text{cst } [\pi] \sqcup [\sigma] = \text{cst } [\pi'] \sqcup [\sigma']$ whenever $\text{cst } [\pi] = \text{cst } [\pi']$, $\text{cst } [\sigma] = \text{cst } [\sigma']$, $|\pi| = |\pi'|$, and $|\sigma| = |\sigma'|$.

The first results in cyclic shuffle-compatibility were implicit in the work of Adin et al. [1], which introduced toric $\vec{D}$ -partitions (a toric poset analogue of P -partitions) and cyclic quasisymmetric functions (which are natural generating functions for toric $\vec{D}$ -partitions). In particular, Adin et al. established a multiplication formula for fundamental cyclic quasisymmetric functions which implies that the cyclic descent set cDes is cyclic shuffle-compatible, as well as another formula which implies that the cyclic descent number cdes is cyclic shuffle-compatible.

In [2], Domagalski et al. formally defined cyclic shuffle-compatibility and proved a result called the “lifting lemma”, which allows one (under certain conditions) to prove that a cyclic statistic is cyclic shuffle-compatible from the shuffle-compatibility of a related linear statistic. They then used the lifting lemma to prove the cyclic shuffle-compatibility of all four statistics cDes , cdes , cPk , and cpk . Most recently, Liang [5] defined and studied enriched toric $\vec{D}$ -partitions, an analogue of enriched P -partitions for toric posets, whose generating functions are “cyclic peak quasisymmetric functions”. She derived a multiplication formula for these cyclic peak quasisymmetric functions which gives a different proof for the cyclic shuffle-compatibility of the cyclic peak set cPk .

The lifting lemma of Domagalski et al. is purely combinatorial, but the work of Adin et al. and Liang suggest that there is an algebraic framework for cyclic shuffle-compatibility à la Gessel and Zhuang, in which the role of quasisymmetric functions is replaced by cyclic quasisymmetric functions. This extended abstract is based on a forthcoming paper by the present authors in which we develop this algebraic framework.

1.2 Outline

We organize this extended abstract as follows. In [Section 2](#), we review the definition of the shuffle algebra of a shuffle-compatible permutation statistic, define the cyclic shuffle algebra of a cyclic shuffle-compatible statistic, and state one of our main results ([Theorem 2.1](#)), which allows one to construct cyclic shuffle algebras from linear shuffle algebras.

In [Section 3](#), we review the role of quasisymmetric functions in the theory of shuffle-compatibility and then summarize our analogous theory in the cyclic realm. [Theorem 2.1](#) is used to construct the non-Escher subalgebra cQSym^- of cyclic quasisymmetric functions from the algebra QSym of quasisymmetric functions, which gives another proof that cDes is cyclic shuffle-compatible with cyclic shuffle algebra isomorphic to cQSym^- . We then give a necessary and sufficient condition for cyclic shuffle-compatibility of cyclic descent statistics which implies that the cyclic shuffle algebra of any cyclic shuffle-compatible cyclic descent statistic is isomorphic to a quotient algebra of cQSym^- .

The theory from [Section 3](#) is used to give explicit descriptions of the cyclic shuffle algebras of cPk , cpk , cdes , cpk , and cdes , which yields algebraic proofs for their cyclic shuffle-compatibility. We end in [Section 4](#) by describing these cyclic shuffle algebras.

Before continuing, we note that not all of our results are included in this extended abstract due to the page limit. For example, we have also shown that if two cyclic permutation statistics are related by a symmetry (such as complementation) and one is cyclic shuffle-compatible, then the other is cyclic shuffle-compatible and they have isomorphic cyclic shuffle algebras. In addition, any linear permutation statistic st induces a multiset-valued cyclic permutation statistic (which we also denote st by a slight abuse of notation) if we let $st \pi : \{\{st \tilde{\pi} : \tilde{\pi} \in \pi\}\}$, the distribution of the linear statistic st over all linear permutations in π . We have investigated the cyclic shuffle-compatibility of these cyclic statistics induced by various linear statistics. Full details are available in [6].

2 Cyclic shuffle algebras

Let st be a permutation statistic. We say that π and σ are *st-equivalent* if $st \pi = st \sigma$ and $|\pi| = |\sigma|$, and we write the st -equivalence class of π as π_{st} . Let $\mathcal{A}_{st}$ denote the $\mathbb{Q}$ -vector space consisting of formal linear combinations of st -equivalence classes of permutations. If st is shuffle-compatible, then we can define a multiplication on $\mathcal{A}_{st}$ by

$$\pi_{st} \sigma_{st} = \sum_{\tau \in \pi \sqcup \sigma} \tau_{st}$$

for any disjoint representatives $\pi \in \pi_{st}$ and $\sigma \in \sigma_{st}$; in fact, this multiplication is well-defined precisely when st is shuffle-compatible. The resulting $\mathbb{Q}$ -algebra $\mathcal{A}_{st}$ is called the (linear) *shuffle algebra* of st . Observe that $\mathcal{A}_{st}$ is graded by length, that is, π_{st} belongs to the n th homogeneous component of $\mathcal{A}_{st}$ if π has length n .

Our definition of cyclic shuffle algebras will be analogous to that of linear ones. Let cst be a cyclic permutation statistic. Then the cyclic permutations π and σ are called *cst-equivalent* if $cst \pi = cst \sigma$ and $|\pi| = |\sigma|$, and we use the notation π_{cst} to denote the cst -equivalence class of the cyclic permutation π . We associate to cst a $\mathbb{Q}$ -vector space $\mathcal{A}_{cst}^{cyc}$ by taking as a basis the set of all cst -equivalence classes of permutations, and then we give this vector space a multiplication by defining

$$\pi_{cst} \sigma_{cst} = \sum_{\tau \in \pi \sqcup \sigma} \tau_{cst}$$

for any disjoint π and σ with $\pi \in \pi_{cst}$ and $\sigma \in \sigma_{cst}$; this multiplication is well-defined if and only if cst is cyclic shuffle-compatible. The resulting $\mathbb{Q}$ -algebra $\mathcal{A}_{cst}^{cyc}$ is called the *cyclic shuffle algebra* of cst , and is also graded by length.

The next theorem—one of our main results—allows us to construct a cyclic shuffle algebra from an associated linear shuffle algebra.

Theorem 2.1. *Let cst be a cyclic permutation statistic and let st be a shuffle-compatible (linear) permutation statistic. Given a cyclic permutation π , let*

$$v_\pi = \sum_{\tilde{\pi} \in \pi} \tilde{\pi}_{\text{st}} \in \mathcal{A}_{\text{st}}.$$

Suppose that $v_\pi = v_\sigma$ whenever π and σ are cst -equivalent, and that $\{v_\pi\}$ (ranging over all cst -equivalence classes) is linearly independent. Then cst is cyclic shuffle-compatible and the map $\psi_{\text{cst}}: \mathcal{A}_{\text{cst}}^{\text{cyc}} \rightarrow \mathcal{A}_{\text{st}}$ given by $\psi_{\text{cst}}(\pi_{\text{cst}}) = v_\pi$ extends linearly to a $\mathbb{Q}$ -algebra isomorphism from $\mathcal{A}_{\text{cst}}^{\text{cyc}}$ to the span of $\{v_\pi\}$, a subalgebra of $\mathcal{A}_{\text{st}}$.

3 Cyclic shuffle-compatibility of cyclic descent statistics

This section is concerned with cyclic shuffle-compatibility of cyclic descent statistics, in which cyclic quasisymmetric functions play a central role. We begin by providing the necessary background on descent compositions and cyclic descent compositions.

3.1 Descent compositions and cyclic descent compositions

Given a subset $S \subseteq [n-1]$ with elements $s_1 < s_2 < \cdots < s_j$, let $\text{Comp } S$ be the composition $(s_1, s_2 - s_1, \dots, s_j - s_{j-1}, n - s_j)$ of n . Also, given a composition $L = (L_1, L_2, \dots, L_k)$, let $\text{Des } L = \{L_1, L_1 + L_2, \dots, L_1 + \cdots + L_{k-1}\}$ be the corresponding subset of $[n-1]$. It is straightforward to verify that Comp and Des are inverse bijections. If $\pi \in \mathfrak{P}_n$ has descent set $S \subseteq [n-1]$, then we say that $\text{Comp } S$ is the *descent composition* of π , which we also denote by $\text{Comp } \pi$. By convention, the empty permutation has descent composition $\emptyset$. Conversely, a permutation with descent composition L has descent set $\text{Des } L$.

Let us now extend the notion of descent compositions to cyclic permutations, which shall require a few more definitions. A *cyclic shift* of a composition $L = (L_1, L_2, \dots, L_k)$ is a composition of the form $(L_j, L_{j+1}, \dots, L_k, L_1, \dots, L_{j-1})$. A *cyclic composition* of n is then the equivalence class of a composition of n under cyclic shift. For example, both $(2, 1, 3)$ and $(1, 3, 2)$ are cyclic compositions. By convention, we'll also allow the empty set $\emptyset$ to be a cyclic composition.

Let us call S a *non-Escher* subset of $[n]$ if S is the cyclic descent set of some linear permutation of length n . When $n = 0$ or $n = 1$, only the empty set is non-Escher, and when $n \geq 2$, all subsets of $[n]$ are non-Escher except for the empty set and $[n]$ itself. For a non-Escher subset $S = \{s_1 < s_2 < \cdots < s_j\} \subseteq [n]$, define the composition $\text{cComp } S$ by

$$\text{cComp } S : \begin{cases} (s_2 - s_1, \dots, s_j - s_{j-1}, n - s_j) & \text{if } n \geq 2, \\ 1 & \text{if } n = 1, \\ \emptyset & \text{if } n = 0. \end{cases}$$

It is easy to see that if S' is a cyclic shift of S , then $\text{cComp } S'$ is a cyclic shift of $\text{cComp } S$. So, if S is the equivalence class of S under cyclic shift, then we can let $\text{cComp } S$ be the cyclic composition given by $\text{cComp } S : \text{cComp } S$.

We say that a cyclic composition is *non-Escher* if it is an image of this induced map cComp ; then cComp is a bijection from equivalence classes of non-Escher subsets of n under cyclic shift to non-Escher cyclic compositions of n . If S is the cyclic descent set of a linear permutation π , then we call $\text{cComp } S$ the *cyclic descent composition* of the cyclic permutation π . We denote the cyclic descent composition of π simply as $\text{cComp } \pi$. For example, $\pi = 179624$ has cyclic descent set $S = \{3, 4, 6\}$, so the cyclic descent composition of π is $\text{cComp } \pi = \text{cComp } S = 1, 2, 3$.

3.2 Cyclic quasisymmetric functions

Given a composition L of n , recall that the *fundamental quasisymmetric function* F_L is defined by

$$F_L : \sum_{\substack{i_1 \leq i_2 \leq \dots \leq i_n \\ i_j < i_{j+1} \text{ if } j \in \text{Des } L}} x_{i_1} x_{i_2} \cdots x_{i_n},$$

and that the set $\{F_L\}$ over all compositions is a basis of QSym . It is sometimes more convenient to index fundamental quasisymmetric functions by sets rather than compositions, in which case we'll use the notation $F_{n,S} : F_{\text{Comp } S}$.

The multiplication rule for the fundamental basis is given by the following theorem, which can be proved using P -partitions; see [8, Exercise 7.93].

Theorem 3.1. *Let $A \subseteq m - 1$ and $B \subseteq n - 1$. Then*

$$F_{m,A} F_{n,B} = \sum_{\tau \in \pi \sqcup \sigma} F_{m+n, \text{Des } \tau}$$

where π is any permutation of length m with descent set A and σ is any permutation (disjoint from π) of length n with descent set B .

From Theorem 3.1, we see that the descent set shuffle algebra $\mathcal{A}_{\text{Des}}$ is isomorphic to QSym ; this is Corollary 4.2 of [4].

We are now ready to discuss cyclic quasisymmetric functions and their role in cyclic shuffle-compatibility. Given a subset S of n where $n \geq 1$, let

$$F_{n,S}^{\text{cyc}} : \sum_{i \in n} F_{n, S \cap i \cap n-1},$$

and let $F_{0,\emptyset}^{\text{cyc}} : 1$; these are the *fundamental cyclic quasisymmetric functions* introduced by Adin, Gessel, Reiner, and Roichman [1]. It is clear that the $F_{n,S}^{\text{cyc}}$ are invariant under cyclic shift; in other words, if $S' = S \cap i$ for some integer i , then $F_{n,S}^{\text{cyc}} = F_{n,S'}^{\text{cyc}}$. As such, if

S is the equivalence class of the set S under cyclic shift, then it makes sense to define $F_{n,S}^{\text{cyc}} : F_{n,S}^{\text{cyc}}$. We can also index fundamental cyclic quasisymmetric functions using compositions; for a composition L of n , let $F_L^{\text{cyc}} : F_{n,\text{cDes } L}^{\text{cyc}}$ and $F_L^{\text{cyc}} : F_L^{\text{cyc}}$.

Let cQSym^- denote the span of $\{F_{n,S}^{\text{cyc}}\}$ over all $n \geq 0$ and all equivalence classes of non-Escher subsets $S \subseteq [n]$. The following theorem by Adin et al. [1, Theorem 3.22] gives a multiplication rule for the fundamental cyclic quasisymmetric functions in cQSym^- .

Theorem 3.2. *Let $A \subseteq [m]$ and $B \subseteq [n]$ be non-Escher subsets. Then*

$$F_{m,A}^{\text{cyc}} F_{n,B}^{\text{cyc}} = \sum_{\tau \in \pi \sqcup \sigma} F_{m+n,\text{cDes } \tau}^{\text{cyc}}$$

where π is any cyclic permutation of length m with cyclic descent set A and σ is any cyclic permutation (with σ disjoint from π) of length n with cyclic descent set B .

Corollary 3.3. *The cyclic descent set cDes is cyclic shuffle-compatible, and the linear map on $\mathcal{A}_{\text{cDes}}^{\text{cyc}}$ defined by $\pi \mapsto F_{\text{cComp } \pi}^{\text{cyc}}$ is a $\mathbb{Q}$ -algebra isomorphism from $\mathcal{A}_{\text{cDes}}^{\text{cyc}}$ to cQSym^- .*

Adin et al. proved Theorem 3.2 using toric $\vec{D}$ -partitions, and this theorem implies that cQSym^- is a graded $\mathbb{Q}$ -subalgebra of QSym . In [6], we give an alternative proof of Theorem 3.2 by constructing cQSym^- from QSym using Theorem 2.1.

Adin et al. also show that the span of $\{F_{0,\emptyset}^{\text{cyc}}, F_{1,\emptyset}^{\text{cyc}}, F_{1,\{1\}}^{\text{cyc}}\} \cup \{F_{n,S}^{\text{cyc}}\}_{n \geq 2, \emptyset \neq S \subseteq [n]}$, denoted cQSym , is a graded $\mathbb{Q}$ -subalgebra of QSym , although this result is less relevant to cyclic shuffle-compatibility. They call cQSym^- the *non-Escher subalgebra* of cQSym .

3.3 A cyclic shuffle-compatibility criterion for cyclic descent statistics

A permutation statistic st is called a *descent statistic* if it depends only on the descent composition, that is, if $\text{Comp } \pi = \text{Comp } \sigma$ implies $\text{st } \pi = \text{st } \sigma$. If st is a descent statistic, then two permutations with the same descent composition are necessarily st -equivalent. Thus, the notion of st -equivalence coarsens to an equivalence relation on compositions.

The theorem below, due to Gessel and Zhuang [4, Theorem 4.3], provides a necessary and sufficient condition for shuffle-compatibility of descent statistics in terms of quasisymmetric functions, and implies that the shuffle algebra of any shuffle-compatible descent statistic is a quotient algebra of QSym .

Theorem 3.4. *A descent statistic st is shuffle-compatible if and only if there exists a $\mathbb{Q}$ -algebra homomorphism $\phi_{\text{st}} : \text{QSym} \rightarrow A$, where A is a $\mathbb{Q}$ -algebra with basis $\{u_\alpha\}$ indexed by st -equivalence classes α of compositions, such that $\phi_{\text{st}} F_L = u_\alpha$ whenever $L \in \alpha$. In this case, the linear map on $\mathcal{A}_{\text{st}}$ defined by $\pi \mapsto u_\alpha$, where $\text{Comp } \pi \in \alpha$, is a $\mathbb{Q}$ -algebra isomorphism from $\mathcal{A}_{\text{st}}$ to A .*

A cyclic permutation statistic cst is called a *cyclic descent statistic* if it depends only on the cyclic descent composition, i.e., if $\text{cComp } \pi = \text{cComp } \sigma$ implies $\text{cst } \pi = \text{cst } \sigma$. As in the linear setting, the notion of cst -equivalence coarsens to an equivalence relation on non-Escher cyclic compositions when cst is a cyclic descent statistic.

The following is another one of our main results: a cyclic analogue of [Theorem 3.4](#).

Theorem 3.5. *A cyclic descent statistic cst is cyclic shuffle-compatible if and only if there exists a $\mathbb{Q}$ -algebra homomorphism $\phi_{\text{cst}}: \text{cQSym}^- \rightarrow A$, where A is a $\mathbb{Q}$ -algebra with basis $\{v_\alpha\}$ indexed by cst -equivalence classes α of non-Escher cyclic compositions, such that $\phi_{\text{cst}} F_L^{\text{cyc}} = v_\alpha$ whenever $L \in \alpha$. In this case, the linear map on $\mathcal{A}_{\text{cst}}^{\text{cyc}}$ defined by $\pi_{\text{cst}} \mapsto v_\alpha$, where $\text{cComp } \pi \in \alpha$, is a $\mathbb{Q}$ -algebra isomorphism from $\mathcal{A}_{\text{cst}}^{\text{cyc}}$ to A .*

Corollary 3.6. *If cst is a cyclic shuffle-compatible descent statistic, then $\mathcal{A}_{\text{cst}}^{\text{cyc}}$ is isomorphic to a quotient algebra of cQSym^- .*

To conclude this section, we state a useful special case of [Theorem 3.5](#) in which the homomorphism ϕ_{cst} is given in terms of the homomorphism ϕ_{st} of a related (linear) descent statistic; c.f. [Theorem 2.1](#).

Theorem 3.7. *Let cst be a cyclic descent statistic and let st be a shuffle-compatible (linear) descent statistic, so there exists a $\mathbb{Q}$ -algebra homomorphism $\phi_{\text{st}}: \text{QSym} \rightarrow A$ satisfying the conditions in [Theorem 3.4](#). Define the $\mathbb{Q}$ -algebra homomorphism $\phi_{\text{cst}}: \text{cQSym}^- \rightarrow A$ by*

$$\phi_{\text{cst}} F_{n,S}^{\text{cyc}} = \sum_{i \in n} \phi_{\text{st}} F_{n, S \setminus i \cap n-1}.$$

Suppose that $\phi_{\text{cst}} F_{n,S}^{\text{cyc}} = \phi_{\text{cst}} F_{n,T}^{\text{cyc}}$ whenever $\text{cComp } S$ and $\text{cComp } T$ are cst -equivalent, so that we can write $\phi_{\text{cst}} F_{n,S}^{\text{cyc}} = v_\alpha$ whenever $\text{cComp } S \in \alpha$, and suppose that $\{v_\alpha\}$ is linearly independent. Then cst is cyclic shuffle-compatible and the linear map on $\mathcal{A}_{\text{cst}}^{\text{cyc}}$ defined by $\pi_{\text{cst}} \mapsto v_\alpha$, where $\text{cComp } \pi \in \alpha$, is a $\mathbb{Q}$ -algebra isomorphism from $\mathcal{A}_{\text{cst}}^{\text{cyc}}$ to the span of $\{v_\alpha\}$, a subalgebra of A .

4 Characterizations of cyclic shuffle algebras

In [6], we use the theory summarized in the previous section to give explicit descriptions of some other cyclic shuffle algebras. After defining the cyclic peak set cPk and the cyclic peak number cpk , we will present our characterizations of the cyclic shuffle algebras of cPk , cpk , cdes , cpk , and cdes . This yields new proofs for the cyclic shuffle-compatibility of the statistics cPk , cpk , and cdes , as well as the first proof for cpk , cdes .

The *cyclic peak set* of a linear permutation $\pi \in \mathfrak{P}_n$ is defined by

$$\text{cPk } \pi : \{i \in n : \pi_{i-1} < \pi_i > \pi_{i+1} \text{ where } i \text{ is considered modulo } n\}.$$

and the elements of $\text{cPk } \pi$ are called *cyclic peaks* of π . Then the *cyclic peak set* of a cyclic permutation π is defined to be the multiset

$$\text{cPk } \pi : \{ \{ \text{cPk } \bar{\pi} : \bar{\pi} \in \pi \} \}.$$

For example, we have $\text{cPk } 184756 = \{ \{ \{2, 4, 6\}^3, \{1, 3, 5\}^3 \} \}$. It is clear from the definitions that, in general, $\text{cPk } \pi$ is the multiset consisting of all cyclic shifts of $\text{cPk } \pi$.

Define the *cyclic peak number* on linear and cyclic permutations by

$$\text{cpk } \pi : |\text{cPk } \pi| \quad \text{and} \quad \text{cpk } \pi : \text{cpk } \pi,$$

respectively. It is easy to see that cPk and cpk are both cyclic descent statistics.

4.1 The cyclic shuffle algebra of cPk

In [6], we construct the cyclic shuffle algebra $\mathcal{A}_{\text{cPk}}^{\text{cyc}}$ from the linear shuffle algebra $\mathcal{A}_{\text{Pk}}$. The latter is known to be isomorphic to Stembridge's [9] *algebra of peaks* Π , a subalgebra of QSym spanned by the *peak quasisymmetric functions* $K_{n,S}$ where n ranges over all non-negative integers and S over all possible peak sets of permutations in $\mathfrak{P}_n$.

The analogue of Stembridge's quasisymmetric peak functions in the cyclic setting are Liang's [5] *cyclic peak quasisymmetric functions* $K_{n,S}^{\text{cyc}}$, which can be defined in terms of the $K_{n,S}$. For brevity, let us say that S is a *cyclic peak set* of n if S is the cyclic peak set of some permutation of length n . Then, if S is a cyclic peak set of n , let

$$K_{n,S}^{\text{cyc}} : \sum_{i \in n} K_{n, S \cup i \setminus \{1,n\}} = \sum_{\bar{\pi} \in \pi} K_{n, \text{Pk } \bar{\pi}}$$

where π is any permutation in $\mathfrak{P}_n$ with cyclic peak set S . We can also write $K_{n,S}^{\text{cyc}} : K_{n,S}^{\text{cyc}}$ since the $K_{n,S}^{\text{cyc}}$ are invariant under cyclic shift.

Liang showed that the $K_{n,S}^{\text{cyc}}$ span a subalgebra Λ of cQSym called the *algebra of cyclic peaks*. The following theorem—which is equivalent to Equation (5.10) of [5]—gives a multiplication rule for the $K_{n,S}^{\text{cyc}}$.

Theorem 4.1. *Let A be a cyclic peak set of m and B a cyclic peak set of n . Then*

$$K_{m,A}^{\text{cyc}} K_{n,B}^{\text{cyc}} = \sum_{\tau \in \pi \sqcup \sigma} K_{m+n, \text{cPk } \tau}^{\text{cyc}}$$

where π is any cyclic permutation of length m with cyclic peak set A and σ is any cyclic permutation (disjoint from π) of length n with cyclic peak set B .

Corollary 4.2. *The cyclic peak set cPk is cyclic shuffle-compatible, and the linear map on $\mathcal{A}_{\text{cPk}}^{\text{cyc}}$ defined by $\pi_{\text{cPk}} \mapsto K_{|\pi|, \text{cPk } \pi}^{\text{cyc}}$ is a $\mathbb{Q}$ -algebra isomorphism from $\mathcal{A}_{\text{cPk}}^{\text{cyc}}$ to Λ .*

While Liang's proof of [Theorem 4.1](#) uses enriched toric $\vec{D}$ -partitions, in [6] we give an alternative proof by constructing Λ from Stembridge's algebra Π using [Theorem 3.7](#).

4.2 The cyclic shuffle algebras of cpk, cdes , cpk , and cdes

In [6], we also use [Theorem 3.7](#) to construct the cyclic shuffle algebra $\mathcal{A}_{\text{cpk,cdes}}^{\text{cyc}}$ from the linear shuffle algebra $\mathcal{A}_{\text{pk,des}}$, for which Gessel and Zhuang [4, Theorem 5.9] gave an explicit description. Below, we write $\mathbb{Q} \langle t \rangle$ for the $\mathbb{Q}$ -algebra of formal power series in t where the multiplication is given by the *Hadamard product* $*$ defined by

$$\left(\sum_{n=0}^{\infty} a_n t^n \right) * \left(\sum_{n=0}^{\infty} b_n t^n \right) : \sum_{n=0}^{\infty} a_n b_n t^n.$$

Theorem 4.3. *The pair cpk, cdes is cyclic shuffle-compatible. Furthermore, let*

$$\begin{aligned} v_{n,j,k}^{\text{cpk,cdes}} &= \frac{j! y^{t-1} (yt)^{1-y} (y-t)^{yt}}{k-j-1! (yt)^{n-j-k-1} (1-y)^{2j-1}} \\ &\times \frac{t^j y^{t^{k-j-1}-1} (yt)^{n-j-k-1} (1-y)^{2j-1}}{1-t^{n-1}} x^n. \end{aligned}$$

Then the linear map on $\mathcal{A}_{\text{cpk,cdes}}^{\text{cyc}}$ defined by

$$\pi_{\text{cpk,cdes}} \mapsto \begin{cases} v_{|\pi|, \text{cpk}, \pi, \text{cdes}}^{\text{cpk,cdes}}, & \text{if } |\pi| \geq 1, \\ 1/(1-t), & \text{if } |\pi| = 0, \end{cases}$$

is a $\mathbb{Q}$ -algebra homomorphism from $\mathcal{A}_{\text{cpk,cdes}}^{\text{cyc}}$ to the span of

$$\left\{ \frac{1}{1-t}, \frac{t-1}{1-t} \frac{y}{2} x \right\} \cup \{v_{n,j,k}^{\text{cpk,cdes}}\}_{n \geq 2, 1 \leq j \leq \lfloor n/2 \rfloor, j \leq k \leq n-j},$$

a subalgebra of $\mathbb{Q} \langle t \rangle \langle x, y \rangle$.

Finally, we present the cyclic shuffle algebras $\mathcal{A}_{\text{cpk}}^{\text{cyc}}$ and $\mathcal{A}_{\text{cdes}}^{\text{cyc}}$, which are homomorphic images of $\mathcal{A}_{\text{cpk,cdes}}^{\text{cyc}}$ obtained by setting $y = 1$ and $y = 0$, respectively. Alternative characterizations of both cyclic shuffle algebras are given in [6].

Theorem 4.4. *The cyclic peak number cpk is cyclic shuffle-compatible. Furthermore, let*

$$v_{n,j}^{\text{cpk}} = \frac{j! (1-t)^{2-n-2j} (4t)^{j-1} (1-t)^{n-2j-1}}{1-t^{n-1}} x^n.$$

Then the linear map on $\mathcal{A}_{\text{cpk}}^{\text{cyc}}$ defined by

$$\pi_{\text{cpk}} \mapsto \begin{cases} v_{|\pi|, \text{cpk}, \pi}^{\text{cpk}}, & \text{if } |\pi| \geq 1, \\ 1/(1-t), & \text{if } |\pi| = 0, \end{cases}$$

is a $\mathbb{Q}$ -algebra isomorphism from $\mathcal{A}_{\text{cpk}}^{\text{cyc}}$ to the span of

$$\left\{ \frac{1}{1-t}, \frac{tx}{1-t^2} \right\} \cup \{v_{n,j}^{\text{cpk}}\}_{n \geq 2, 1 \leq j \leq \lfloor n/2 \rfloor},$$

a subalgebra of $\mathbb{Q}[t^*x]$.

Theorem 4.5. *The cyclic descent number cdes is cyclic shuffle-compatible. Furthermore, the linear map on $\mathcal{A}_{\text{cdes}}^{\text{cyc}}$ defined by*

$$\pi_{\text{cdes}} \mapsto \begin{cases} \frac{\text{cdes } \pi \cdot t^{\text{cdes } \pi} \frac{|\pi| - \text{cdes } \pi}{1-t^{|\pi|-1}} x^{|\pi|}}{1-t^{|\pi|-1}}, & \text{if } |\pi| \geq 1, \\ 1/(1-t), & \text{if } |\pi| = 0, \end{cases}$$

is a $\mathbb{Q}$ -algebra isomorphism from $\mathcal{A}_{\text{cdes}}^{\text{cyc}}$ to the span of

$$\left\{ \frac{1}{1-t}, \frac{tx}{1-t^2} \right\} \cup \left\{ \frac{kt^k}{1-t^{n-1}} x^n \right\}_{n \geq 2, 1 \leq k \leq n-1},$$

a subalgebra of $\mathbb{Q}[t^*x]$.

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Set-valued tableaux for Macdonald polynomials

Zajj Daugherty^{*1} and Arun Ram^{†2}

¹Department of Mathematics and Statistics, Reed College, Portland, OR 97202

²Department of Mathematics and Statistics, University of Melbourne, Parkville VIC 3010 AU

Abstract. Set-valued tableaux formulas play an important role in Schubert calculus. Using the box greedy reduced word for the construction of the Macdonald polynomials, we convert the alcove walk formula to a set-valued tableaux formula for Macdonald polynomials. Our hope is that providing such formulas will help to strengthen the analogies and possible connections between the calculus of Macdonald polynomials and Schubert calculus.

Keywords: Macdonald polynomials, tableaux formulas, Schubert calculus

1 Introduction

The goal of this paper is to give set-valued tableaux formulas for Macdonald polynomials. As evidenced by the work of [2], [9], [11], [14], [17], [18] and others, set-valued tableaux formulas play an important role in Schubert calculus. Although there are many hints from the study of affine Springer fibers [13] and seminfinite flag varieties [8] the concrete connection between Schubert calculus and Macdonald polynomials for general type and general q and t is still elusive. There are many flavors of set-valued tableaux in the Schubert calculus literature and the relationships between them, including the relationship to the set-valued tableaux that we use, remain under-explored. Our hope is to help strengthen the analogies and possible connections between the calculus of Macdonald polynomials and Schubert calculus. See [3] for a version of this paper with additional details and examples.

Our formulas for Macdonald polynomials are derived by making a bijection between set-valued tableaux and alcove walks and converting the alcove walks formula from [16, Theorem 2.2] to a set-valued tableaux formula. We follow the framework of [5, §1 and §2], which gives an exposition of the alcove walks formula in the GL_n context and analyzes the favourite reduced word for the n -periodic permutation u_μ that is used to construct the relative Macdonald polynomial E_μ^z . Then a set-valued tableau formula for the symmetric Macdonald polynomials P_λ , with λ a partition, follows from

$$P_\lambda = (\text{const}) \sum_{z \in S_n \lambda} E_\lambda^z ;$$

^{*}zdaugherty@reed.edu

[†]aram@unimelb.edu.au

see [6, (1.4), (1.5) and (1.10)].

The set-valued tableaux formulas given in [Theorem 2.1](#) (respectively, [Theorem 2.2](#)) are specializable at $q = 0$ and $t = 0$ (respectively, $q = \infty$ and $t = \infty$). The specializations of E_μ^z at $t = 0$ and $t = \infty$ have interpretations in terms of crystals for level 1 and level 0 affine Demazure characters (see, for example, [7, Theorem C and Corollary D] or [12, (2.12), Theorem 1.1 and §2.4]). It would be interesting to write the root operators for these crystals explicitly on the set-valued tableaux (it is likely that the crystals of [18, §4] cover some cases of these crystals).

2 Set-valued tableaux for Macdonald polynomials

We begin with combinatorial definitions necessary for stating the set-valued tableaux formula for Macdonald polynomials. To aid in processing these definitions, we encourage the reader to follow [Example 2.1](#) in tandem.

Fix $n \in \mathbb{Z}_{>0}$ and let S_n denote the symmetric group on n letters. Identify $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}_{\geq 0}^n$ with the set of boxes in μ ,

$$\mu = \{(r, c) \mid r \in \{1, \dots, n\} \text{ and } c \in \{1, \dots, \mu_r\}\}.$$

2.1 Definition of set-valued tableaux

For $(r, c) \in \mu$ define

$$u_\mu(r, c) = \#\{r' \in \{1, \dots, r-1\} \mid \mu_{r'} < c \leq \mu_r\} + \#\{r' \in \{r+1, \dots, n\} \mid \mu_{r'} < c-1 < \mu_i\}.$$

The values $u_\mu(r, c)$ play a similar role to the flagging in the use of set-valued tableaux to give formulas for Grothendieck polynomials (see [17, §2.6 and Theorem 7.1]). A *set-valued tableau* T of shape μ is a choice of subset $T(r, c) \subseteq \{1, \dots, u_\mu(r, c)\}$ for each box $(r, c) \in \mu$. More formally, a set-valued tableau T of shape μ is a function

$$T: \mu \rightarrow \{\text{subsets of } \{1, \dots, n\}\} \quad \text{such that} \quad T(r, c) \subseteq \{1, \dots, u_\mu(r, c)\}.$$

Let $|T|$ denote the total number of entries in T .

2.2 Definition of the shift and height statistics

The minimal length permutation $v_\mu \in S_n$ such that $v_\mu \mu$ is weakly increasing is given by

$$v_\mu(r) = 1 + \#\{r' \in \{1, \dots, r-1\} \mid \mu_{r'} \leq \mu_r\} + \#\{r' \in \{r+1, \dots, n\} \mid \mu_{r'} < \mu_r\},$$

for $r \in \{1, \dots, n\}$. For a box $(r, c) \in \mu$ and $i \in \{1, \dots, u_\mu(r, c)\}$ define

$$\text{sh}(r, c) = \mu_r - c + 1 \quad \text{and} \quad \text{ht}(i, r, c) = v_\mu(r) - i.$$

2.3 Definition of $z_{(r,c)}^T$ and x^T

Using one-line notation, let γ_n be the n -cycle in the symmetric group S_n given by

$$\gamma_n = (n \ 1 \ 2 \ \cdots \ n-1).$$

For positive integers $k_1, \dots, k_\ell$ such that $k_1 + \dots + k_\ell = n$, let

$$\gamma_{k_1} \times \cdots \times \gamma_{k_\ell} \quad \text{be the disjoint product of cycles in} \quad S_{k_1} \times \cdots \times S_{k_\ell} \subseteq S_n.$$

Fix a permutation $z \in S_n$. Order the boxes of μ down columns and then left to right (i.e., in increasing order of the values $r + nc$). Starting from the permutation z , associate a permutation $z_{(r,c)}^T$ to each box (r, c) of μ as follows:

If $(r, c) \in \mu$ and $T(r, c) = \{m_1, \dots, m_p\}$ with $1 \leq m_1 < \dots < m_p \leq u_\mu(r, c)$, define

$$z_{(r,c)}^T = z_{(r',c')}^T (\gamma_{m_1-0} \times \gamma_{m_2-m_1} \cdots \times \gamma_{m_p-m_{p-1}} \times \gamma_{u_\mu(r,c)+1-m_p} \times (\gamma_1)^{\times(n-u_\mu(r,c)-1)}) \gamma_n^{-1},$$

where $(r', c') \in \mu$ denotes the box before (r, c) in μ . In the case that (r, c) is the first box in μ we let $z_{(r',c')}^T = z$. Then define

$$x^T = \prod_{(r,c) \in \mu} x_{z_{(r,c)}^T(n)},$$

where $z_{(r,c)}^T(n)$ is the value of the permutation $z_{(r,c)}^T$ at n .

The permutation $z_{\text{init}}^T = z$ is the *left key* (or *initial direction*) of T and the permutation z_{fin}^T corresponding to the last box in μ is the *right key* (or *final direction*) of T . These left and right keys play important roles in Schubert calculus: for example in [18, §3] and in the statement of Pieri-Chevalley formulas (compare [4, Theorem 1] and [15, Corollary]).

2.4 Definition of the cov and maj statistics.

Keep the notation $T(r, c) = \{m_1, \dots, m_p\}$ with $m_1 < \dots < m_p$. Define

$$\begin{aligned} T_{<}^z(r, c) &= \{m_k \in T(r, c) \mid z_{(r,c)}^T(m_{k-1}) < z_{(r,c)}^T(m_k)\} \quad \text{and} \\ T_{>}^z(r, c) &= \{m_k \in T(r, c) \mid z_{(r,c)}^T(m_{k-1}) > z_{(r,c)}^T(m_k)\}, \end{aligned}$$

where we make the convention that $m_0 = n$. Then define

$$\begin{aligned} \text{maj}_{>}^z(T) &= \sum_{(r,c) \in \mu} \text{sh}(r, c) \cdot |T_{>}^z(r, c)|, \\ \text{cov}_{>}^z(T) &= \left(\sum_{(r,c) \in \mu} \sum_{m \in T_{>}^z(r, c)} \text{ht}(m, r, c) \right) + \frac{1}{2} (\ell(z_{\text{fin}}^T) - \ell(z_{\text{init}}^T v_\mu^{-1}) - |T|), \end{aligned}$$

where $\ell(w)$ denotes the length of the permutation w in S_n .

2.5 Set-valued tableaux formulas for E_μ^z .

The relative Macdonald polynomial $E_\mu^z = E_\mu^z(x_1, \dots, x_n; q, t)$ is what is termed the “permutated basement nonsymmetric Macdonald polynomial” in [1] (see [5, (3.7)] for further references and a definition in terms of Cherednik-Dunkl operators and the double affine Hecke algebra action on polynomials).

Theorem 2.1. *Let $z \in S_n$ and $\mu \in \mathbb{Z}_{\geq 0}^n$. Then the relative Macdonald polynomial E_μ^z is*

$$E_\mu^z = \sum_T q^{\text{maj}_>^z(T)} t^{\text{cov}_>^z(T)} \left(\prod_{(r,c) \in \mu} \prod_{m \in T(r,c)} \frac{1-t}{1-q^{\text{sh}(r,c)} t^{\text{ht}(m,r,c)}} \right) x^T,$$

where the sum is over set-valued tableaux T of shape μ .

Because the powers of q and t in formula in Theorem 2.1 are nonnegative integers, this formula is well-suited to the specializations $q = 0$ and/or $t = 0$. The following theorem provides an alternate formula which is better for identifying the specializations of E_μ^z at $q = \infty$ and/or $t = \infty$ (i.e., specializations at $q^{-1} = 0$ and $t^{-1} = 0$).

$$\begin{aligned} \text{maj}_<^z(T) &= \sum_{(r,c) \in \mu} \text{sh}(r,c) \cdot |T_{<}^z(r,c)|, \\ \text{cov}_<^z(T) &= \left(\sum_{(r,c) \in \mu} \sum_{m \in T_{<}^z(r,c)} \text{ht}(m,r,c) \right) - \frac{1}{2} \left(\ell(z_{\text{fin}}^T) - \ell(z_{\text{init}}^T v_\mu^{-1}) - |T| \right). \end{aligned}$$

Theorem 2.2. *Let $z \in S_n$ and $\mu \in \mathbb{Z}_{\geq 0}^n$. Then the relative Macdonald polynomial E_μ^z is*

$$E_\mu^z = \sum_T q^{-\text{maj}_<^z(T)} t^{-\text{cov}_<^z(T)} \left(\prod_{(r,c) \in \mu} \prod_{m \in T(r,c)} \frac{1-t^{-1}}{1-q^{-\text{sh}(r,c)} t^{-\text{ht}(i,r,c)}} \right) x^T,$$

where the sum is over set-valued tableaux T of shape μ .

The proof of Theorem 2.2 is obtained from Theorem 2.1 by multiplying numerator and denominator of each coefficient by

$$\prod_{(r,c) \in \mu} \prod_{i=1}^{u_\mu(r,c)} q^{-\text{sh}(r,c)} t^{-\text{ht}(i,r,c)}.$$

The proof of Theorem 2.1 is obtained by considering the reduced word for the n -periodic permutation

$$u_\mu = \prod_{(r,c) \in \mu} s_{u_\mu(r,c)-1} \cdots s_2 s_1 \pi \quad (\text{product taken in increasing order of boxes}),$$

studied in [5, Propositions 2.1 and 2.2] (see also [10, (2.4.3)]) and using a straightforward bijection between alcove walks of type u_μ and set-valued tableaux of shape μ (see Section 3.3). This bijection gives a conversion between the alcove walk formula for in E_μ^z in [5, Theorem 1.1(a)] (see also [16, Theorem 2.2] for this alcove walk formula in a root system language) and the set-valued tableaux formula in Theorem 2.1.

Example 2.1. Let $n = 5$ and $\mu = (0, 4, 5, 1, 4)$. Then the box arrangement associated to μ is as follows, with boxes placed on a labeled grid, filled with values $u_\mu(r, c)$, and marked in the lower corner by their cylindrical numbers $r + nc$.

$$\begin{array}{c}
 \text{column } c \\
 \begin{array}{ccccc}
 & 1 & 2 & 3 & 4 & 5 \\
 \hline
 1 & & & & & \\
 2 & 1 & 1 & 2 & 2 & \\
 3 & 1 & 1 & 2 & 2 & 3 \\
 4 & 1 & & & & \\
 5 & 1 & 2 & 2 & 2 &
 \end{array} \\
 \text{row } r
 \end{array}
 \quad (2.1)$$

Box by box, the corresponding factors $\frac{(1-t)}{1-q^{\text{sh}(r,c)}t^{\text{ht}(i,r,c)}}$ are

$$\frac{(1-t)}{1-q^{\text{sh}}t^{\text{ht}}} = \begin{array}{|c|c|c|c|c|}
 \hline
 \frac{(1-t)}{(1-q^4t^2)} & \frac{(1-t)}{(1-q^3t^2)} & \frac{(1-t)}{(1-q^2t^2)} & \frac{(1-t)}{(1-qt^2)} & \\
 \hline
 \frac{(1-t)}{(1-q^5t^4)} & \frac{(1-t)}{(1-q^4t^4)} & \frac{(1-t)}{(1-q^3t^4)} & \frac{(1-t)}{(1-q^2t^4)} & \frac{(1-t)}{(1-qt^4)} \\
 \hline
 \frac{(1-t)}{(1-qt)} & & \frac{(1-t)}{(1-q^3t^3)} & \frac{(1-t)}{(1-q^2t^3)} & \frac{(1-t)}{(1-qt^3)} \\
 \hline
 \frac{(1-t)}{(1-q^4t^3)} & \frac{(1-t)}{(1-q^3t^3)} & \frac{(1-t)}{(1-q^2t^3)} & \frac{(1-t)}{(1-qt^3)} & \\
 \hline
 \frac{(1-t)}{(1-q^3t^2)} & \frac{(1-t)}{(1-q^2t^2)} & \frac{(1-t)}{(1-qt^2)} & &
 \end{array} \cdot$$

One set-valued tableau of shape λ is

$$T = \begin{array}{|c|c|c|c|}
 \hline
 \{1\} & \emptyset & \{1\} & \{2\} \\
 \hline
 \{1\} & \emptyset & \{1,2\} & \emptyset \\
 \hline
 \emptyset & & & \\
 \hline
 \emptyset & \emptyset & \{1,2\} & \emptyset \\
 \hline
 \end{array} = \begin{array}{|c|c|c|c|}
 \hline
 1 & & 1 & 2 \\
 \hline
 1 & & 12 & 13 \\
 \hline
 & & & \\
 \hline
 & & 12 & \\
 \hline
 \end{array} \quad (2.2)$$

which has size $|T| = 1 + 1 + 1 + 2 + 2 + 1 + 2 = 10$ (we shall often omit the set notation and commas as in the right-most picture above). The product in [Theorem 2.1](#) corresponding to this set-valued tableau T is

$$\prod_{(r,c) \in \mu} \prod_{m \in T^z(r,c)} \frac{(1-t)}{1 - q^{\text{sh}(r,c)} t^{\text{ht}(i,r,c)}} =$$

$\frac{(1-t)}{(1-q^4 t^2)}$	1	$\frac{(1-t)}{(1-q^2 t^2)}$	$\frac{(1-t)}{(1-qt)}$	
$\frac{(1-t)}{(1-q^5 t^4)}$	1	$\frac{(1-t)}{(1-q^3 t^4)}$	$\frac{(1-t)}{(1-q^3 t^3)}$	$\frac{(1-t)}{(1-qt^4)}$ $\frac{(1-t)}{(1-qt^2)}$
1				
1	1	$\frac{(1-t)}{(1-q^2 t^3)}$	$\frac{(1-t)}{(1-q^2 t^2)}$	1

Let $z = \text{id}$ be the identity. The *box-by-box permutation sequence* of T with initial direction z (written in 1-line notation $w = (w(1) \cdots w(n))$) is

$$z^T =$$

(<u>2</u> 345 <u>1</u>)	(3245 <u>1</u>)	(<u>1</u> 425 <u>3</u>)	(2314 <u>5</u>)	
(3451 <u>2</u>)	(3451 <u>2</u>)	(<u>4</u> 253 <u>1</u>)	(2435 <u>1</u>)	(<u>5</u> 314 <u>2</u>)
(3512 <u>4</u>)				
(3124 <u>5</u>)	(3412 <u>5</u>)	(<u>2</u> 531 <u>4</u>)	(2351 <u>4</u>)	

(2.3)

In each box we have underlined the positions specified by the entries of T ; these positions are the $m_1, \dots, m_p$ in [Section 2.4](#). The red highlighted entry indicates $z_{(r,c)}^T(n)$, which is used in the formula for x^T in [Section 2.3](#), and also as $z_{(r,c)}^T(m_0)$ in the definition of $T_{<}^z(r, c)$ and $T_{>}^z(r, c)$ in [Section 2.4](#). So we have

$$x^T =$$

<u>x_1</u>	x_1	x_3	<u>x_5</u>
<u>x_2</u>	x_2	x_1	x_1
<u>x_4</u>			
<u>x_5</u>	x_5	x_4	<u>x_4</u>

 $= x_1^4 x_2^3 x_3 x_4^3 x_5^3;$

and, following the definition in [Section 2.4](#) and splitting T into *ascents* and *descents* in the sequence of underlined numbers (preceded by the red number $z_{(r,c)}^T(n)$):

$$T_{<}^z =$$

1			
1			1
		2	

and
 $T_{>}^z =$

		1	2
		2	3
		1	

The initial and final directions of T are

$$z_{\text{init}}^T = \text{id} = (12345) \quad \text{and} \quad z_{\text{fin}}^T = z_{(3,5)}^T = (53142),$$

where $z_{(3,5)}$ indicates the permutation in box $(3,5)$ of z^T . Since $v_\mu = (13524)$ then

$$\ell(z_{\text{fin}}^T) = 4 + 2 + 0 + 1 = 7 \quad \text{and} \quad \ell(z_{\text{init}}^T v_\mu^{-1}) = 3,$$

so that

$$\frac{1}{2}(\ell(z_{\text{fin}}^T) - \ell(z_{\text{init}}^T v_\mu^{-1}) - |T|) = \frac{1}{2}(7 - 3 - 10) = -3.$$

Then

$$q^{\text{maj}_>^z(T)} t^{\text{cov}_>^z(T)} = t^{\frac{1}{2}(7-3-10)} \cdot \begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline 1 & 1 & q^2 t^2 & q t & \\ \hline 1 & 1 & q^3 t^3 & 1 & q t^2 \\ \hline 1 & & & & \\ \hline 1 & 1 & q^2 t^3 & 1 & \\ \hline \end{array} = t^{-3} q^9 t^{11} = q^9 t^8.$$

3 Proof of Theorem 2.1

In this section we describe the conversion from alcove walks to set-valued tableaux. We follow the framework of [5, §1], which gives an exposition of the alcove walks formula in the GL_n context and analyzes the favourite reduced for the n -periodic permutation u_μ that is used to construct the relative Macdonald polynomial E_μ^z .

3.1 Inversions and the box-greedy reduced word

An n -periodic permutation is a bijection $w: \mathbb{Z} \rightarrow \mathbb{Z}$ such that $w(i+n) = w(i) + n$. Since an n -periodic permutation w is determined by the values $w(1), \dots, w(n)$, a permutation $w \in S_n$ extends uniquely to an n -periodic permutation. For $i \in \{1, \dots, n-1\}$ let s_i be the transposition in S_n switching i and $i+1$. Define $\pi: \mathbb{Z} \rightarrow \mathbb{Z}$ by $\pi(i) = i+1$.

An *inversion* of an n -periodic permutation w is an element of the set

$$\text{Inv}(w) = \{(i, k) \mid i \in \{1, \dots, n\}, k \in \mathbb{Z}, i < k \text{ and } w(i) > w(k)\}.$$

If $i, j \in \{1, \dots, n\}$ with $i < j$ and $\ell \in \mathbb{Z}$, then the *shift* and *height* of $(i, j + \ell n)$ are defined to be

$$\text{sh}(i, j + \ell n) = \ell \quad \text{and} \quad \text{ht}(i, j + \ell n) = j - i.$$

Let $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}_{\geq 0}^n$ and let u_μ be the n -periodic permutation defined by

$$u_\mu(i) = v_\mu^{-1}(i) + n\mu_i, \quad \text{for } i \in \{1, \dots, n\}. \quad (3.1)$$

Let $u_\mu(r, c)$ be as defined in Section 2.1. Following [5, Prop. 2.2(a)], the *box-greedy reduced word* for u_μ (an abstract word in symbols $s_1, \dots, s_{n-1}$ and π) is

$$u_\mu^\square = \prod_{\text{boxes } (r, c) \text{ in } \mu} (s_{u_\mu(r, c)} \cdots s_2 s_1 \pi), \quad (3.2)$$

where the product is taken in increasing order of the values $r + nc$. As in [5, (2.12)], this reduced word induces a partition of the elements of $\text{Inv}(u_\mu)$ according to the boxes (r, c) of μ , and an ordering of the inversions in each box. For $(r, c) \in \mu$ and $i \in \{1, \dots, u_\mu(r, c)\}$,

$$\text{the } i\text{th inversion in the box } (r, c) \text{ is } \beta(i, r, c) = (v_\mu(r), i + n \cdot (\mu_r - c + 1)).$$

The inversion $\beta(i, r, c)$ has shift and height

$$\text{sh}(r, c) = \text{sh}(\beta(i, r, c)) = \mu_r - c + 1 \quad \text{and} \quad \text{ht}(i, r, c) = \text{ht}(\beta(i, r, c)) = v_\mu(r) - i. \quad (3.3)$$

Example 3.1. If $\mu = (0, 4, 5, 1, 4)$ then $n = 5$, the box-greedy reduced word for u_μ is

$$u_\mu^\square = (s_1 \pi)^6 (s_2 s_1 \pi)^7 (s_3 s_2 s_1 \pi) = \begin{array}{|c|c|c|c|c|} \hline & s_1 \pi & s_1 \pi & s_2 s_1 \pi & s_2 s_1 \pi \\ \hline s_1 \pi & s_1 \pi & s_2 s_1 \pi & s_2 s_1 \pi & s_3 s_2 s_1 \pi \\ \hline s_1 \pi & & & & \\ \hline s_1 \pi & s_2 s_1 \pi & s_2 s_1 \pi & s_2 s_1 \pi & \\ \hline \end{array} \quad (3.4)$$

(again multiplying boxes in order by cylindrical numbers) and the inversion set of u_μ is

$$\text{Inv}(u_\mu) = \begin{array}{|c|c|c|c|c|} \hline & (3, 1 + 4 \cdot 5) & (3, 1 + 3 \cdot 5) & (3, 1 + 2 \cdot 5) & (3, 1 + 1 \cdot 5) \\ \hline & & & (3, 2 + 2 \cdot 5) & (3, 2 + 1 \cdot 5) \\ \hline (5, 1 + 5 \cdot 5) & (5, 1 + 4 \cdot 5) & (5, 1 + 3 \cdot 5) & (5, 1 + 2 \cdot 5) & (5, 1 + 1 \cdot 5) \\ \hline & & (5, 2 + 3 \cdot 5) & (5, 2 + 2 \cdot 5) & (5, 2 + 1 \cdot 5) \\ \hline & & & & (5, 3 + 1 \cdot 5) \\ \hline (2, 1 + 1 \cdot 5) & & & & \\ \hline (4, 1 + 4 \cdot 5) & (4, 1 + 3 \cdot 5) & (4, 1 + 2 \cdot 5) & (4, 1 + 1 \cdot 5) & \\ \hline & (4, 2 + 3 \cdot 5) & (4, 2 + 2 \cdot 5) & (4, 2 + 1 \cdot 5) & \\ \hline \end{array}$$

3.2 Alcove walks and permutation sequences

Let $z \in S_n$ and choose a subset F of the s_i factors in $u_\mu^\square = w_1 w_2 \cdots w_r$ to cross out. The corresponding *alcove walk of type* $(z, u_\mu^\square)$ is the sequence $p(F) = (p_0, p_1, \dots, p_r)$ of

elements of W given by

$$p_0 = z \quad \text{and} \quad p_k = \begin{cases} p_{k-1}\pi, & \text{if } w_k = \pi, \\ p_{k-1}, & \text{if } w_k \in F \text{ (so that the factor } w_k \text{ is crossed out),} \\ p_{k-1}w_k, & \text{if } w_k \notin F \text{ (so that the factor } w_k \text{ is not crossed out).} \end{cases}$$

Following [5, (1.14)], the *permutation sequence* of $p(F)$ is the sequence of permutations in S_n given by

$$\bar{p}(F) = (\bar{p}_0, \bar{p}_1, \dots, \bar{p}_r), \quad \text{where } \bar{w}(i) = w(i) \bmod n.$$

Note that $\overline{w_1 w_2} = \bar{w}_1 \bar{w}_2$ and $\bar{\pi} = c_n = s_1 \cdots s_{n-1}$ and $\bar{s}_j = s_j$ for $j \in \{1, \dots, n-1\}$.

3.3 Converting to set-valued tableaux

The bijection between alcove walks and set-valued tableaux is

$$\begin{array}{ccc} \{\text{alcove walks of type } u_\mu^\square\} & \longleftrightarrow & \{\text{set-valued tableaux of shape } \mu\} \\ p(F) & \longmapsto & T \end{array}$$

where, in the set-valued tableau T corresponding to the alcove walk $p(F)$, the set $T(r, c)$ in box (r, c) specifies which factors are crossed out in that box: if a box in T contains i , then delete s_i in the corresponding word.

Example 3.2. In the case $\mu = (0, 4, 5, 1, 4)$, where the box-greedy reduced word for u_μ is as given in (3.4), there are $6 + 2 \cdot 7 + 3 = 23$ factors of the form s_i in $u_\mu^\square$ and so there are a total of 2^{23} alcove walks of type $(z, u_\mu^\square)$ (for any fixed permutation $z \in S_n$). Each alcove walk corresponds to a choice of the s_i factors in $u_\mu^\square$ to cross out.

The tableau T in Example 2.1 corresponds to the subset

$$F = \begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline \cancel{s_1}\pi & s_1\pi & s_2\cancel{s_1}\pi & \cancel{s_2}s_1\pi & \\ \hline \cancel{s_1}\pi & s_1\pi & \cancel{s_2}\cancel{s_1}\pi & s_2s_1\pi & \cancel{s_3}s_2\cancel{s_1}\pi \\ \hline s_1\pi & & & & \\ \hline s_1\pi & s_2s_1\pi & \cancel{s_2}\cancel{s_1}\pi & s_2s_1\pi & \\ \hline \end{array}, \quad (3.5)$$

which has alcove walk

$$p(F) = (p_0, p_1, \dots, p_{37}) = (z, z, z\pi, z\pi, z\pi^2, z\pi^2s_1, z\pi^2s_1\pi, z\pi^2s_1\pi s_1, z\pi^2s_1\pi s_1\pi, \dots)$$

(there is a repeat entry in $p(F)$ each time there is an s_i crossed out in F). Using one line notation $w = (w(1) \ w(2) \ \cdots \ w(n))$ for n -periodic permutations (underlining 2-digit terms for emphasis), a box-by-box formulation in the case that $z = \text{id} = (12345)$ is

$$p(F) = \begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline \begin{array}{c} (1\ 2\ 3\ 4\ 5) \\ (2\ 3\ 4\ 5\ 6) \end{array} & \begin{array}{c} (6\ 3\ 7\ 9\ \underline{10}) \\ (3\ 7\ 9\ \underline{10}\ \underline{11}) \end{array} & \begin{array}{c} (3\ \underline{11}\ 9\ \underline{12}\ \underline{15}) \\ (3\ \underline{11}\ 9\ \underline{12}\ \underline{15}) \\ (\underline{11}\ 9\ \underline{12}\ \underline{15}\ 8) \end{array} & \begin{array}{c} (\underline{12}\ \underline{15}\ 8\ \underline{16}\ \underline{14}) \\ (\underline{15}\ \underline{12}\ 8\ \underline{16}\ \underline{14}) \\ (\underline{12}\ 8\ \underline{16}\ \underline{14}\ \underline{20}) \end{array} & \\ \hline \begin{array}{c} (2\ 3\ 4\ 5\ 6) \\ (3\ 4\ 5\ 6\ 7) \end{array} & \begin{array}{c} (7\ 3\ 9\ \underline{10}\ \underline{11}) \\ (3\ 9\ \underline{10}\ \underline{11}\ \underline{12}) \end{array} & \begin{array}{c} (\underline{11}\ 9\ \underline{12}\ \underline{15}\ 8) \\ (\underline{11}\ 9\ \underline{12}\ \underline{15}\ 8) \\ (9\ \underline{12}\ \underline{15}\ 8\ \underline{16}) \end{array} & \begin{array}{c} (\underline{12}\ \underline{16}\ 8\ \underline{14}\ \underline{20}) \\ (\underline{16}\ \underline{12}\ 8\ \underline{14}\ \underline{20}) \\ (\underline{12814}\ \underline{20}\ \underline{21}) \end{array} & \begin{array}{c} (\underline{12}\ 8\ \underline{20}\ \underline{21}\ \underline{19}) \\ (\underline{12}\ \underline{20}\ 8\ \underline{21}\ \underline{19}) \\ (\underline{12}\ \underline{20}\ 8\ \underline{21}\ \underline{19}) \\ (\underline{20}\ 8\ \underline{21}\ \underline{19}\ \underline{17}) \end{array} \\ \hline \begin{array}{c} (4\ 3\ 5\ 6\ 7) \\ (3\ 5\ 6\ 7\ 9) \end{array} & & & & \\ \hline \begin{array}{c} (5\ 3\ 6\ 7\ 9) \\ (3\ 6\ 7\ 9\ \underline{10}) \end{array} & \begin{array}{c} (3\ \underline{10}\ 9\ \underline{11}\ \underline{12}) \\ (\underline{10}\ 3\ 9\ \underline{11}\ \underline{12}) \\ (3\ 9\ \underline{11}\ \underline{12}\ \underline{15}) \end{array} & \begin{array}{c} (9\ \underline{12}\ \underline{15}\ 8\ \underline{16}) \\ (9\ \underline{12}\ \underline{15}\ 8\ \underline{16}) \\ (\underline{12}\ \underline{15}\ 8\ \underline{16}\ \underline{14}) \end{array} & \begin{array}{c} (\underline{12}\ \underline{14}\ 8\ \underline{20}\ \underline{21}) \\ (\underline{14}\ \underline{12}\ 8\ \underline{20}\ \underline{21}) \\ (\underline{12}\ 8\ \underline{20}\ \underline{21}\ \underline{19}) \end{array} & \\ \hline \end{array}$$

The permutation sequence is

$$\bar{p}(F) = (\bar{p}_0, \bar{p}_1, \dots, \bar{p}_{37}) = (z, z, zc_n, zc_n, zc_n^2, zc_n^2s_1, zc_n^2s_1c_n, zc_n^2s_1c_ns_1, zc_n^2s_1c_ns_1c_n, \dots)$$

(obtained by replacing each π by the n -cycle $c_n = s_1 \cdots s_{n-1} = \gamma_n^{-1}$). Namely, $\bar{p}(F)$ is obtained by reducing all the values in $p(F)$ mod n (and $n = 5$ for $\mu = (0, 4, 1, 5, 4)$).

$$\bar{p}(F) = \begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline \begin{array}{c} (\underline{1}2345) \\ (23451) \end{array} & \begin{array}{c} (13245) \\ (32451) \end{array} & \begin{array}{c} (31425) \\ (\underline{3}1425) \\ (14253) \end{array} & \begin{array}{c} (\underline{2}5314) \\ (52314) \\ (23145) \end{array} & \\ \hline \begin{array}{c} (\underline{2}3451) \\ (34512) \end{array} & \begin{array}{c} (23451) \\ (34512) \end{array} & \begin{array}{c} (1\underline{4}253) \\ (\underline{14}253) \\ (42531) \end{array} & \begin{array}{c} (21345) \\ (12345) \\ (24351) \end{array} & \begin{array}{c} (23\underline{5}14) \\ (25314) \\ (\underline{25}314) \\ (53142) \end{array} \\ \hline \begin{array}{c} (43512) \\ (35124) \end{array} & & & & \\ \hline \begin{array}{c} (53124) \\ (31245) \end{array} & \begin{array}{c} (35412) \\ (53412) \\ (34125) \end{array} & \begin{array}{c} (\underline{4}2\underline{5}31) \\ (\underline{42}531) \\ (25314) \end{array} & \begin{array}{c} (24351) \\ (42351) \\ (23514) \end{array} & \\ \hline \end{array} \tag{3.6}$$

The *box-by-box permutation sequence* of F is obtained by recording the last permutation in each box, which is z^T in (2.3). The elements $z_{(r,c)}^T$ in each box of z^T are also obtained from $z = z_{\text{init}}^T$ and T by the formula in Section 2.3.

We have used the blue highlighted numbers in (3.6) to record the positions $i, i + 1$ when there is a repeat entry in the sequence $\bar{p}(F)$ coming from a crossed out s_i . These record the information of F . This same information is translated into the underlines in z^T in (2.3), and the underlines exactly specify the entries of T . The set-valued tableau indicating the positions of the underlined numbers in each box is the tableau in (2.2).

3.4 Conversion of the statistics

The proof of Theorem 2.1 is completed by matching up the statistics in Theorem 2.1 with the statistics that appear in [16, Theorem 3.1]. In Example 3.2, we have used the blue highlighted numbers in (3.6) and the underlines in (2.3) to illustrate how the information of the ‘folds’ in [16, Theorem 3.1] is equivalent to the information of the set-valued tableau T . In the context of [16, (2.36)], the set $T(r, c)$ exactly records the folds coming from the box (r, c) . By (3.3), the factor $\prod_{(r,c) \in \mu} \prod_{m \in T(r,c)} \frac{(1-t)}{1 - q^{\text{sh}(r,c)} t^{\text{ph}(i,r,c)}}$ which appears in the product in Theorem 2.1 corresponds to

$$\text{the factors} \quad \left(\prod_{k \in f^+(p)} \frac{(1 - t_{\beta_k^\vee})}{1 - q^{\langle -\beta_k^\vee, \rho_c \rangle}} \right) \left(\prod_{k \in f^-(p)} \frac{(1 - t_{\beta_k^\vee})}{1 - q^{\langle -\beta_k^\vee, \rho_c \rangle}} \right) \quad \text{in [16, Theorem 3.1].}$$

The sets $T_{<}^z$ and $T_{>}^z$ correspond to the sets $f^+(p)$ and $f^-(p)$ of positive and negative folds in [16, Theorem 3.1]. The permutation z_{fin}^T is the permutation denoted $\varphi(p)$ in [16, Theorem 3.1] (denoted z_r in [5, (1.14)]), and the permutation $z_{\text{init}}^T v_\mu^{-1}$ is the permutation m of [16, Theorem 3.1]. With these conversions, the last term $\frac{1}{2}(\ell(z_{\text{fin}}^T) - \ell(z_{\text{init}}^T v_\mu^{-1}) - |T|)$ in the definition of $\text{cov}_{>}^z(T)$ in Section 2.4 corresponds to the combination of the factors $t_{\varphi(p)}^{\frac{1}{2}}$ and $t_{\beta_k^\vee}^{-\frac{1}{2}}$ in [16, Theorem 3.1] and the factor $t_m^{\frac{1}{2}}$ in [16, Remark 3.2].

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Inequalities for f^* -vectors of Lattice Polytopes

Matthias Beck^{*1}, Danaï Deligeorgaki^{†2},
Max Hlavacek^{‡3}, and Jerónimo Valencia-Porras^{§4}

¹Department of Mathematics, San Francisco State University, San Francisco, CA 94132, USA

²Institutionen för Matematik, KTH, SE10044, Stockholm, Sweden

³Department of Mathematics, UC Berkeley, Berkeley, CA 94720, USA

⁴Department of Combinatorics and Optimization, University of Waterloo, Waterloo, ON, Canada

Abstract. The Ehrhart polynomial $\text{ehr}_P(n)$ of a lattice polytope P counts the number of integer points in the n -th dilate of P . The f^* -vector of P , introduced by Felix Breuer in 2012, is the vector of coefficients of $\text{ehr}_P(n)$ with respect to the binomial coefficient basis $\left\{ \binom{n-1}{0}, \binom{n-1}{1}, \dots, \binom{n-1}{d} \right\}$, where $d = \dim P$. Similarly to h/h^* -vectors, the f^* -vector of P coincides with the f -vector of its unimodular triangulations (if they exist). We present several inequalities that hold among the coefficients of f^* -vectors of polytopes. These inequalities resemble striking similarities with existing inequalities for the coefficients of f -vectors of simplicial polytopes; e.g., the first half of the f^* -coefficients increases and the last quarter decreases. Even though f^* -vectors of polytopes are not always unimodal, there are several families of polytopes that carry the unimodality property. We also show that for any polytope with a given Ehrhart h^* -vector, there is a polytope with the same h^* -vector whose f^* -vector is unimodal.

Keywords: Lattice polytope, Ehrhart polynomial, Gorenstein polytope, f^* -vector, h^* -vector, unimodality.

1 Introduction

For a d -dimensional lattice polytope $P \subset \mathbb{R}^d$ (i.e., the convex hull of finitely many points in $\mathbb{Z}^d$) and a positive integer n , let $\text{ehr}_P(n)$ denote the number of integer lattice points in nP . Ehrhart's famous theorem [12] says that $\text{ehr}_P(n)$ evaluates to a polynomial in n . Similar to the situations with other combinatorial polynomials, it is useful to express $\text{ehr}_P(n)$ in different bases; here we consider two such bases consisting of binomial coefficients:

$$\text{ehr}_P(n) = \sum_{k=0}^d h_k^* \binom{n}{d-k} = \sum_{k=0}^d f_k^* \binom{n-1}{k}. \quad (1.1)$$

^{*}becksfu@gmail.com.

[†]danaide@kth.se. Danaï Deligeorgaki was supported by the Knut and Alice Wallenberg Foundation.

[‡]mhlava@math.berkeley.edu

[§]j2valenc@uwaterloo.ca

We call $f_0^*, f_1^*, \dots, f_d^*$ the f^* -vector and $h_0^*, h_1^*, \dots, h_d^*$ the h^* -vector of P . Stanley [16] proved that the h^* -vector of any lattice polytope is nonnegative (whereas the coefficients of $\text{ehr}_P n$ written in the standard monomial basis can be negative). Breuer [10] proved that the f^* -vector of any lattice polytopal complex is nonnegative (whereas the h^* -vector of a complex can have negative coefficients); his motivation was that various combinatorially-defined polynomials can be realized as Ehrhart polynomials of complexes and so the nonnegativity of the f^* -vector yields a strong constraint for these polynomials. The f^* - and h^* -vector can also be defined through the *Ehrhart series* of P :

$$\text{Ehr}_P z : 1 + \sum_{n \geq 1} \text{ehr}_P n z^n = \frac{\sum_{k=0}^d h_k^* z^k}{1 - z^{d-1}} = 1 + \sum_{k=0}^d f_k^* \left(\frac{z}{1-z} \right)^{k-1}.$$

It is thus sometimes useful to add the definition $f_{-1}^* := 1$. The polynomial $\sum_{k=0}^d h_k^* z^k$ is the h^* -polynomial of P , and its degree is the *degree* of P . For general background on Ehrhart theory, see, e.g., [4]. The f^* - and h^* -vectors share the same relation as f - and h -vectors of polytopes/polyhedral complexes, namely

$$\sum_{k=0}^d h_k^* z^k = \sum_{k=0}^{d-1} f_{k-1}^* z^k (1 - z^{d-k-1}) \quad (1.2)$$

$$h_k^* = \sum_{j=-1}^{k-1} (-1)^{k-j-1} \binom{d-j}{k-j-1} f_j^* \quad (1.3)$$

$$f_k^* = \sum_{j=0}^k \binom{d-j}{k-j} h_j^*. \quad (1.4)$$

The (very special) case that P admits a unimodular triangulation yields the strongest connection between f^*/h^* -vectors and f/h -vectors: in this case the f^*/h^* -vector of P equals the f/h -vector of the triangulation, respectively.

Example 1. Let P be the 2-dimensional cube $[-1, 1]^2$. The unimodular triangulation of P shown in Figure 1, has f -vector $f_0, f_1, f_2 = 9, 16, 8$, as f_i counts its i -dimensional faces. Equivalently,

$$f^* P = 9, 16, 8,$$

and one easily checks that (1.1) yields the familiar Ehrhart polynomial $\text{ehr}_P n = 2n + 1$.

Example 2. The f^* -vector of a d -dimensional unimodular simplex Δ equals

$$\left[\binom{d}{1}, \binom{d}{2}, \dots, \binom{d}{d} \right],$$

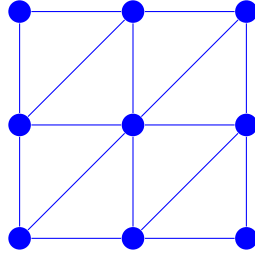


Figure 1: A (regular) unimodular triangulation of the cube $[-1, 1]^2$.

coinciding with the f -vector of Δ considered as a simplicial complex. If we append this vector by $f_{-1}^* = 1$, it gives the only instance of a *symmetric* f^* -vector of a lattice polytope P , since the equality $f_{-1}^* = f_d^*$ implies that $h_i^* = 0$ for all $1 \leq i \leq d$.

There has been much research on (typically linear) constraints for the h^* -vector of a given lattice polytope (see, e.g., [17, 18]). On the other hand, f^* -vectors seem to be much less studied, and our goal is to rectify that situation. Our motivating question is how close the f^* -vector of a given lattice polytope is to being *unimodal*, i.e., the f^* -coefficients increase up to some point and then decrease. Our main results are as follows.

Theorem 3. *Let $d \geq 2$ and let P be a d -dimensional lattice polytope. Then*

- (a) $f_0^* < f_1^* < \cdots < f_{\lfloor \frac{d}{2} \rfloor - 1}^* \leq f_{\lfloor \frac{d}{2} \rfloor}^*$;
- (b) $f_{\lfloor \frac{3d}{4} \rfloor}^* > f_{\lfloor \frac{3d}{4} \rfloor - 1}^* > \cdots > f_d^*$;
- (c) $f_k^* \leq f_{d-1-k}^*$ for $0 \leq k \leq \frac{d-3}{2}$.

Examples 1 and 2 yield cases of polytopes for which the inequalities $f_{\lfloor \frac{3d}{4} \rfloor - 1}^* < f_{\lfloor \frac{3d}{4} \rfloor}^*$ and $f_{\lfloor \frac{d}{2} \rfloor}^* > f_{\lfloor \frac{d}{2} \rfloor - 1}^*$ hold, respectively, and thus the strings of inequalities in ((a)) and ((b)) can, in general, not be extended further. We record the following immediate consequence of Theorem 3.

Corollary 4. *Let P be a d -dimensional lattice polytope. Then for $0 \leq k \leq d$,*

$$f_k^* \geq \min\{f_0^*, f_d^*\}.$$

We remark that one can prove that if P is of degree $d \geq 2$, equal to its dimension, then $f_0^* \leq f_d^*$, except when the h^* -vector of P has entries $h_2^* = \cdots = h_d^* = 1$.

Theorem 5. *The f^* -vector of a d -dimensional lattice polytope, where $1 \leq d \leq 13$, is unimodal. On the other hand, there exists a 15-dimensional lattice simplex with nonunimodal f^* -vector.*

Even though f^* -vectors are quite different from f -vectors of polytopes, the above results resemble striking similarities with existing theorems on f -vectors. Namely, Björner [6, 7, 8] proved that the f -vector of a simplicial d -polytope satisfies all inequalities in [Theorem 3](#) (with the $*$ s removed, and the last coordinate dropped). In fact, Björner also showed that in the f -analogue of [Theorem 3\(\(b\)\)](#) the decrease starts from $\lfloor \frac{3d-1}{4} \rfloor - 1$ instead of $\lfloor \frac{3d}{4} \rfloor$, and that the inequalities in [Theorem 3\(\(a\)\)](#) and [\(\(b\)\)](#) cannot be further extended, by constructing a simplicial polytope with f -vector that peaks at f_j , for any $\lfloor \frac{d}{2} \rfloor \leq j \leq \lfloor \frac{3d-1}{4} - 1 \rfloor$. [Corollary 4](#) compares the entries of the f^* -vector with the minimum between the first and the last entry. Note that a similar relation for f -vectors of polytopes was recently proven by Hinman [15], answering a question of Bárány from the 1990s. (Hinman also proved a stronger result, namely certain lower bounds for the ratios $\frac{f_k}{f_0}$ and $\frac{f_k}{f_{d-1}}$.) The f -analogue of [Theorem 5](#) is again older: Björner [6] showed that the f -vector of any simplicial d -polytope is unimodal for $d \leq 15$ (later improved to $d \leq 19$ by Eckhoff [11]), and he and Lee [5] produced examples of 20-dimensional simplicial polytopes with nonunimodal f -vectors. For a special class of polytopes we can increase the range in [Theorem 3\(\(b\)\)](#). A lattice polytope P is *Gorenstein of index g* if

- nP contains no interior lattice points for $1 \leq n < g$,
- gP contains a unique interior lattice point, and
- $\text{ehr}_P n - g$ equals the number of interior lattice points in nP , for $n > g$.

This is equivalent to P having degree $d - 1 - g$ and a symmetric h^* -vector (with respect to its degree).

Theorem 6. *Let P be a d -dimensional Gorenstein polytope of index g . Then*

$$f_{k-1}^* > f_k^* \quad \text{for } \frac{1}{2} \left(d - 1 - \left\lfloor \frac{d-1-g}{2} \right\rfloor \right) \leq k \leq d.$$

Going even further, for a certain class of polytopes we can prove unimodality of the f^* -vector, a consequence of the following refinement of [Theorem 3\(\(b\)\)](#) for polytopes with degree $< \frac{d}{2}$.

Theorem 7. *Let P be a d -dimensional lattice polytope with positive degree $\leq s$. Then*

$$f_{k-1}^* > f_k^* \quad \text{for } \lceil \frac{d-s}{2} \rceil \leq k \leq d,$$

unless the degree of P is 0, i.e., P is a unimodular simplex with f^* -vector as in [Example 2](#).

This theorem implies that lattice d -polytopes of degree s satisfying $s^2 - s - 1 \leq \frac{d}{2}$ have a unimodal f^* -vector (see [Proposition 9](#) below for details). One family with asymptotically small degree, compared to the dimension, is given by taking iterated pyramids. Given a polytope $P \subset \mathbb{R}^d$, we denote by $\text{Pyr } P \subset \mathbb{R}^{d+1}$ the convex hull of P and the $d+1$ st unit vector. It is well known that P and $\text{Pyr } P$ have the same h^* -vector (ignoring an extra 0), and so we conclude:

Corollary 8. *If P is any lattice polytope then $\text{Pyr}^n P$ has unimodal f^* -vector for sufficiently large n .*

2 Selected proofs

The aim of this section is to offer some insight into the proofs of the results stated in Section 1. See [3] for full details of the proofs.

The main ingredient in the proofs of Theorem 3((a)) and ((c)) and Theorem 7 is the nonnegativity of h^* -vectors; we omit the proof details. Corollary 8 is a consequence of Theorem 7 together with the following proposition, whose proof again relies on the nonnegativity of h^* -vectors.

Proposition 9. *Let P be a d -dimensional lattice polytope that has degree at most s for some $s \geq 1$. If $d \geq 2s^2 - 2s - 2$ then the f^* -vector is unimodal with a (not necessarily "sharp") peak at f_p^* , where $\lfloor \frac{d}{2} \rfloor \leq p \leq \lceil \frac{d+s}{2} \rceil - 1$.*

The next proofs use more than just the nonnegativity of h^* -vectors. The first result needs the following elementary lemma on binomial coefficients.

Lemma 10. *Let j, k, n be positive integers such that $k \leq n - 1 - j$. Then, for $n \geq 2k - 1$,*

$$\left| \binom{n}{k} - \binom{n}{k-1} \right| \geq \left| \binom{n-j}{k} - \binom{n-j}{k-1} \right|.$$

We are now prepared to prove Theorem 3((b)).

Proof of Theorem 3((b)). The inequality $f_{d-1}^* > f_d^*$ holds by Theorem 7. Now, let $\lfloor \frac{3d}{4} \rfloor \leq k < d$. By (1.4),

$$f_{k-1}^* - f_k^* = \sum_{j=0}^{k-1} \left(\binom{d-1-j}{k-j} - \binom{d-1-j}{k-1-j} \right) h_j^*. \quad (2.1)$$

The difference $\binom{d-1-j}{k-j} - \binom{d-1-j}{k-1-j}$ is nonnegative whenever $k-j \geq \lfloor \frac{d-1-j}{2} \rfloor$ and negative otherwise. Hence, the difference is nonnegative whenever $j \leq 2k-d$ and negative whenever $j > 2k-d$. Since $2d-2k < 2k-1-d$ for $\lfloor \frac{3d}{4} \rfloor \leq k$, from (2.1) we obtain

$$f_{k-1}^* - f_k^* \geq \sum_{j=0}^{2d-2k} \left(\binom{d-1-j}{k-j} - \binom{d-1-j}{k-1-j} \right) h_j^* \quad (2.2)$$

$$- \sum_{j=2k-1-d}^{k-1} \left(\binom{d-1-j}{k-j} - \binom{d-1-j}{k-1-j} \right) h_j^* \quad (2.3)$$

where the differences appearing in (2.2) are nonnegative and the ones in (2.3) are negative. Our aim is to compare the sums in (2.2) and (2.3) to conclude that $f_{k-1}^* - f_k^*$ is positive. Using standard identities for binomial coefficients, the right hand-side of (2.2) equals

$$\begin{aligned} & \sum_{j=0}^{2d-2k} \left(\sum_{l=j}^{2d-2k-1} \left(\binom{d-l}{k-l} - \binom{d-l}{k-1-l} \right) \left(\binom{2k-d-1}{3k-2d} - \binom{2k-d-1}{3k-2d-1} \right) \right) h_j^* \\ & \quad \sum_{l=0}^{2d-2k-1} \left(\left(\binom{d-l}{k-l} - \binom{d-l}{k-1-l} \right) \sum_{j=0}^{2d-2k-1-l} h_j^* \right) \\ & \quad \left(\binom{2k-d-1}{3k-2d} - \binom{2k-d-1}{3k-2d-1} \right) \sum_{j=0}^{2d-2k} h_j^*, \end{aligned}$$

whence we conclude that right hand-side of (2.2) is bounded below by

$$\begin{aligned} & \left(\binom{d}{k} - \binom{d}{k-1} \right) h_0^* - \left(\binom{2k-d-1}{3k-2d} - \binom{2k-d-1}{3k-2d-1} \right) \sum_{j=0}^{2d-2k} h_j^* \\ & > \left(\binom{2k-d-1}{3k-2d} - \binom{2k-d-1}{3k-2d-1} \right) \sum_{j=0}^{2d-2k} h_j^* \end{aligned} \quad (2.4)$$

since $\binom{d}{k} - \binom{d}{k-1} > 0$ for $\lfloor \frac{3d}{4} \rfloor - 1 \leq k < d$, and $h_0^* = 1, h_j^* \geq 0$ for $j = 1, \dots, 2d-2k-1$. On the other hand, for the differences appearing in (2.3), using that $2d-2k < j$ and $j \leq k-1$, it follows by Lemma 10 that

$$\left| \binom{d-1-2d-2k}{d-1-k} - \binom{d-1-2d-2k}{d-k} \right| \geq \left| \binom{d-1-j}{d-1-k} - \binom{d-1-j}{d-k} \right|,$$

i.e.,

$$\left| \binom{2k-d-1}{3k-2d} - \binom{2k-d-1}{3k-2d-1} \right| \geq \left| \binom{d-1-j}{k-j} - \binom{d-1-j}{k-1-j} \right|.$$

Hence for $j \geq 2k-1-d$,

$$- \left(\binom{2k-d-1}{3k-2d} - \binom{2k-d-1}{3k-2d-1} \right) \leq \binom{d-1-j}{k-j} - \binom{d-1-j}{k-1-j}.$$

Since both $-\binom{d-1-j}{k-j} + \binom{d-1-j}{k-1-j}$ and h_j^* are nonnegative for $j \geq 2k-1-d$, the sum in (2.3) is bounded below by

$$- \left(\binom{2k-d-1}{3k-2d} - \binom{2k-d-1}{3k-2d-1} \right) \sum_{j=2k-1-d}^d h_j^*. \quad (2.5)$$

Now (2.4) and (2.5) yield

$$f_{k-1}^* - f_k^* > \left(\binom{2k-d-1}{3k-2d} - \binom{2k-d-1}{3k-2d-1} \right) \left(\sum_{j=0}^{2d-2k} h_j^* - \sum_{j=2k-1-d}^d h_j^* \right).$$

Hibi [13] showed that the inequality

$$\sum_{j=0}^{m-1} h_j^* \geq \sum_{j=d-m}^d h_j^* \quad (2.6)$$

holds for $m = 0, \dots, \lfloor \frac{d}{2} \rfloor - 1$. Since $2d - 2k - 1 \leq \lfloor \frac{d}{2} \rfloor - 1$ for $\lfloor \frac{3d}{4} \rfloor - 1 \leq k$, we can use (2.6) to finally obtain

$$f_{k-1}^* - f_k^* > 0. \quad \square$$

Proof of Theorem 5. If $d = 1$ or 2 , there is nothing to prove. If $3 \leq d \leq 6$, then by Theorem 3, either

$$f_0^* \leq \dots \leq f_{\lfloor \frac{d}{2} \rfloor}^* \geq f_{\lfloor \frac{3d}{4} \rfloor}^* \geq \dots \geq f_d^*$$

or

$$f_0^* \leq \dots \leq f_{\lfloor \frac{d}{2} \rfloor}^* \leq f_{\lfloor \frac{3d}{4} \rfloor}^* \geq \dots \geq f_d^*.$$

For $7 \leq d \leq 13$, it suffices to check that if $f_i^* \geq f_{i-1}^*$, then $f_{i-1}^* \geq f_{i-2}^*$, for all $\lfloor \frac{d}{2} \rfloor \leq i \leq \lfloor \frac{3d}{4} \rfloor - 2$. By Theorem 3, this will imply the unimodality of $f_0^*, f_1^*, \dots, f_d^*$. Using the inequality

$$\sum_{j=1}^{m-1} h_j^* - h_{d-1-j}^* > 0, \quad (2.7)$$

which holds for $m = 0, \dots, \lfloor \frac{d}{2} \rfloor - 1$ [17, Remark 1.2], as well as the nonnegativity of h^* -vectors, for each value of d , we were able to verify that $f_{i-1}^* - f_{i-2}^* \geq c(f_i^* - f_{i-1}^*)$ for some nonnegative real c in each case.

To construct a polytope with nonunimodal f^* -vector, we employ a family of simplices introduced by Higashitani [14]. Concretely, denote the j th unit vector by e_j and let

$$\Delta_w : \text{conv}\{0, e_1, e_2, \dots, e_{14}, w\}$$

where

$$w : \underbrace{1, 1, \dots, 1}_7, \underbrace{131, 131, \dots, 131}_7, 132.$$

It has h^* -vector

$$1, \underbrace{0, 0, \dots, 0}_7, 131, \underbrace{0, 0, \dots, 0}_7$$

and, via (1.4), f^* -vector

$$16, 120, 560, 1820, 4368, 8008, 11440, 13001,$$

$$12488, 11676, 11704, 10990, 7896, 3788, 1064, 132 \quad .$$

□

We record the following consequence of [Theorem 5](#), which follows by the nonnegativity of h^* -vectors.

Corollary 11. *Every lattice polytope of degree at most 5 has unimodal f^* -vector.*

Proof of Theorem 6. Let $s = d - 1 - g$. We first consider the case that s is odd; the case s even will be similar. Since $h_j^* = 0$ for $j > s$ and $h_j^* = h_{s-j}^*$,

$$\begin{aligned} f_{k-1}^* - f_k^* &= \sum_{j=0}^s \left(\binom{d-j}{k-j} - \binom{d-j}{k-j-1} \right) h_j^* \\ &= \sum_{j=0}^{\lfloor \frac{s}{2} \rfloor} \left(\binom{d-j}{k-j} - \binom{d-j}{k-j-1} \right) h_j^* - \sum_{j=\lfloor \frac{s}{2} \rfloor+1}^s \left(\binom{d-j}{k-j} - \binom{d-j}{k-j-1} \right) h_j^* \\ &= \sum_{j=0}^{\lfloor \frac{s}{2} \rfloor} \left(\binom{d-j}{k-j} - \binom{d-j}{k-j-1} - \binom{d-s-j}{k-s-j} + \binom{d-s-j}{k-s-j-1} \right) h_j^*. \end{aligned}$$

Because we assume $k \geq \frac{1}{2}d - 1 - \lfloor \frac{s}{2} \rfloor$,

$$\binom{d-j}{k-j} - \binom{d-j}{k-j-1} > 0$$

for $0 \leq j \leq \lfloor \frac{s}{2} \rfloor$. The inequality

$$\binom{d-j}{k-j} - \binom{d-j}{k-j-1} - \binom{d-s-j}{k-s-j} + \binom{d-s-j}{k-s-j-1} > 0$$

follows directly if $\binom{d-s-j}{k-s-j} - \binom{d-s-j}{k-s-j-1} \geq 0$ or $k-s-j-1 < 0$. Otherwise, [Lemma 10](#) implies that, for the same range of j ,

$$\binom{d-j}{k-j} - \binom{d-j}{k-j-1} - \binom{d-s-j}{k-s-j} + \binom{d-s-j}{k-s-j-1} \geq 0.$$

In fact, the last inequality is strict for $k \geq \frac{1}{2}d - 1 - \lfloor \frac{s}{2} \rfloor$ (the proof is analogous to that of [Lemma 10](#)). Finally we use that $h_j^* \geq 0$ and $h_0^* = 1$ to deduce that $f_{k-1}^* - f_k^* > 0$. The

computations in the case s even is very similar. Now we write

$$\begin{aligned}
 f_{k-1}^* - f_k^* &= \sum_{j=0}^s \left(\binom{d-j}{k-j} - \binom{d-j}{k-j-1} \right) h_j^* \\
 &\quad \sum_{j=0}^{\frac{s}{2}-1} \left(\binom{d-j}{k-j} - \binom{d-j}{k-j-1} - \binom{d-s-j}{k-s-j} + \binom{d-s-j}{k-s-j-1} \right) h_j^* \\
 &\quad \left(\binom{d-\frac{s}{2}}{k-\frac{s}{2}} - \binom{d-\frac{s}{2}}{k-\frac{s}{2}-1} \right) h_{\frac{s}{2}}^*
 \end{aligned}$$

and use the same argumentation as in the case s odd. □

3 Concluding Remarks

There are many avenues to explore f^* -vectors, e.g., along analogous studies of h^* -vectors, and we hope the above results form an enticing starting point. The techniques in our proof of [Theorem 5](#) do not offer much insight in the case of 14-dimensional lattice polytopes as there are candidates f^* -vectors with corresponding h^* -vectors that satisfy all inequalities discussed in [\[17\]](#). It is unknown, though, if such polytopes exist.

Higashitani [\[14, Theorem 1.1\]](#) provided examples of d -dimensional polytopes with nonunimodal h^* -vector for all $d \geq 3$. Therefore, by [Theorem 5](#) we have examples of polytopes that have such an h^* -vector but their f^* -vector is unimodal. It would be interesting to know if the opposite can be true, that is, if there exist polytopes with unimodal h^* -vector and nonunimodal f^* -vector. By [Corollary 11](#), such polytopes would need to have degree at least 6.

Whenever one detects that a given given is unimodal, it is natural to ask about the stronger property that the polynomial is log concave or, even stronger, real rooted. Our methods do not yield these properties but it would be interesting if one could extend, e.g., [Corollary 8](#) or [Proposition 9](#) along these lines.

Finally, starting with Stapledon's work [\[17\]](#), there has been much recent attention to symmetric decompositions of h - and h^* -polynomials; see, e.g., [\[1, 2\]](#) and, in particular, [\[9\]](#) where analogous decompositions for f -vectors are discussed. We believe this line of research is worthy of attention with regards to understanding f^* -vectors and the inequalities that hold among their coefficients.

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Degenerations of the Grassmannian via matroidal subdivisions of the hypersimplex

Daniel Corey^{*1} and Dante Luber^{†1}

¹*Technische Universität Berlin*

Abstract. We study initial degenerations of $\text{Gr}_0\ 3,8$ —the open part of the Grassmannian $\text{Gr}\ 3,8$ defined by the nonvanishing of all Plücker coordinates—using matroid subdivisions of the $3,8$ –hypersimplex. We show that all initial degenerations are smooth and, outside of a single symmetry class, irreducible. As an application, we show that the Chow quotient of $\text{Gr}\ 3,8$ by the diagonal torus of $\text{PGL}\ 8$ is the log canonical compactification of the moduli space of 8 lines in the projective plane in linear general position. This fully resolves a conjecture of Hacking, Keel, and Tevelev.

Keywords: matroids, Grassmannians, tropical geometry

1 Introduction

Regular subdivisions of lattice polytopes, as developed by Gelfand, Kapranov, and Zelevinsky [7] with roots in the 1908 work of Voronoi [24], are combinatorial constructions that have deep connections to algebra and geometry. Of particular relevance is the relationship between regular matroidal subdivisions of the hypersimplex and initial degenerations of the Grassmannian. Denote by $\text{Gr}\ d,n$ the Grassmannian of d -planes in $\mathbb{C}^n$ and $\text{Gr}^\circ\ d,n$ the open locus of $\text{Gr}\ d,n$ defined by the nonvanishing of all Plücker coordinates. Given a vector w in the tropicalization $\text{TGr}^\circ\ d,n$ of $\text{Gr}^\circ\ d,n$, we may form its w -initial degeneration $\text{in}_w\ \text{Gr}^\circ\ d,n$. On the combinatorial side, the vector w induces a regular subdivision of the hypersimplex $\Delta\ d,n$ into matroid polytopes, a result which is proved independently by Lafforgue [18] and Speyer [22], see also [11]. From such a subdivision, one may form a finite inverse limit of thin Schubert cells (also known as matroid strata), which we denote by $\text{Gr}\ w$. The first author in [3] shows that there is a closed immersion $\text{in}_w\ \text{Gr}^\circ\ d,n \hookrightarrow \text{Gr}\ w$, which is an isomorphism for $2,n$, $3,6$ and $3,7$.

In this extended abstract we discuss the main results and techniques of *The Grassmannian of 3-planes in $\mathbb{C}^8$ is schön* [4]. We study the initial degenerations of $\text{Gr}^\circ\ 3,8$ via finite inverse limits of thin Schubert cells as discussed in the previous paragraph.

^{*}corey@math.tu-berlin.de. Daniel Corey is supported by the SFB-TRR project “Symbolic Tools in Mathematics and their Application” (project-ID 286237555).

[†]d.luber@tu-berlin.de. Dante Luber was supported by “Facets of Complexity” (GRK 2434, project-ID 385256563)

Theorem 1.1. *The initial degenerations of $\text{Gr}^\circ 3,8$ are smooth.*

In particular, $\text{Gr}^\circ 3,8$ is *schön* in the sense of Tevelev [23] (see the characterization in [9]). We also consider connectedness of the initial degenerations of $\text{Gr}^\circ 3,8$. Surprisingly, we find that they are not all connected.

Theorem 1.2. *There exists an initial degeneration of $\text{Gr}^\circ 3,8$ with two connected components. Up to $\mathfrak{S}_8$ -symmetry, the remaining initial degenerations are connected.*

Given the size of the Plücker ideal, it is challenging to directly prove that the initial degenerations of $\text{Gr}^\circ 3,8$ are smooth, even with computer assistance. By contrast, once the regular subdivision is computed, one may determine “by hand” if the corresponding inverse limit is smooth, irreducible, and compute its dimension. We illustrate this in [Example 5.4](#). Of course, this is only useful if the closed immersions $\text{in}_w \text{Gr}^\circ 3,8 \hookrightarrow \text{Gr } w$ are isomorphisms. It turns out that they are. For example, if $\text{Gr } w$ is smooth, irreducible and has dimension 15, then $\text{in}_w \text{Gr}^\circ 3,8 \hookrightarrow \text{Gr } w$ is an isomorphism. Thus, while these results are geometric, the proofs of [Theorems 1.1](#) and [1.2](#) use techniques from polyhedral geometry, matroid theory, and commutative algebra.

Denote by $X_{d,n}$ the moduli space of n hyperplanes in $\mathbb{P}^{d-1}$ in linear general position, up to projective transformations. Our main application of the previous theorems is to study compactifications of $X_{3,8}$. The diagonal torus $H \subset \text{PGL } n$ acts freely on $\text{Gr}^\circ d,n$, and the quotient $\text{Gr}^\circ d,n / H$ is isomorphic to $X_{3,8}$; this is the famous Gelfand-MacPherson correspondence. The normalization $\overline{X}_{d,n}$ of the Chow quotient $\text{Gr } d,n // H$ compactifies $X_{d,n}$, but it is in general not the log canonical compactification, failing already for $3,9$ and $4,8$ [16]. Hacking, Keel, and Tevelev conjecture in [*loc. cit.*, Conjecture 1.6] that $\overline{X}_{d,n}$ is the log canonical compactification in the cases $2,n$, $3,6$, $3,7$, and $3,8$. This is motivated by the relationship between these spaces and moduli spaces of del Pezzo surfaces as in [8]. The first 3 cases were handled previously by [15], [20], and [3], respectively.

Theorem 1.3. *The normalization $\overline{X}_{d,n}$ of the Chow quotient $\text{Gr } 3,8 // H$ is the log canonical compactification of $X_{3,8}$.*

Code

Our results rely on extensive computations using `polymake.jl` [6, 12] and `OSCAR` [5, 21], both of which run using `julia` [2]. The code can be found at the following link:

<https://github.com/dcorey2814/Gr38Schoen>

Conventions and notation

Given a positive integers $d \leq n$, we write $[n] = \{1, \dots, n\}$ and $\binom{[n]}{d}$ for the collection of d -element subsets of $[n]$. For subsets with a small number of elements, we use juxtaposition to denote the subset they form, e.g., if $a, b \in \lambda$, then we write ab for $\{a, b\}$. Given an abelian group A and a field F , we denote by $A_F = A \otimes_{\mathbb{Z}} F$. We write $\mathbf{1} = 1, 1, \dots, 1$ viewed either in $\mathbb{Z}^n$ or $\mathbb{R}^n$. The symmetric group on $[n]$ is denoted by $\mathfrak{S}_n$.

2 Background

2.1 Regular matroid subdivisions

Given positive integers $d \leq n$, a d, n -matroid is a nonempty subset $Q \subset \binom{[n]}{d}$ satisfying the *basis-exchange axiom*: given distinct $\lambda_1, \lambda_2 \in Q$ and $a \in \lambda_1 \setminus \lambda_2$, there is $b \in \lambda_2$ such that $\lambda_1 \setminus a \cup b \in Q$. Denote by $\epsilon_1, \dots, \epsilon_n$ the standard basis of $\mathbb{R}^n$, and ϵ_λ the indicator vector of $\lambda \subset [n]$. The *polytope* of Q , written ΔQ , is the convex hull of the vectors $\epsilon_\lambda \in \mathbb{R}^n$ for $\lambda \in Q$; in fact, these are the vertices of ΔQ . The *hypersimplex* $\Delta d, n$ is the polytope of the uniform matroid $\binom{[n]}{d}$, and its vertices are naturally labeled by $\binom{[n]}{d}$.

Given $w \in \mathbb{R}^n / \langle \mathbf{1} \rangle$, denote by $\mathcal{Q} w$ the regular subdivision of $\Delta d, n$ induced by w . The subdivision $\mathcal{Q} w$ is *matroidal* if each polytope in $\mathcal{Q} w$ is the polytope of a matroid. The set of $w \in \mathbb{R}^n / \langle \mathbf{1} \rangle$ such that $\mathcal{Q} w$ is matroidal is called the *d, n - Dressian*, denoted $\text{Dr } d, n$. There is a fan structure on $\text{Dr } d, n$, where two vectors w and w' lie in the relative interior of the same cone if and only if $\mathcal{Q} w = \mathcal{Q} w'$; this is called the *secondary fan structure* of $\text{Dr } d, n$, which we denote by $S_{\text{mat}} d, n$.

The *tight span* of $\mathcal{Q} w$, denoted by $\text{TS } w$, is a polyhedral complex dual to $\mathcal{Q} w$ that has a k -dimensional (bounded) cell for each $n - k - 1$ -dimensional cell of $\mathcal{Q} w$ meeting the relative interior of $\Delta d, n$ [10]. The *dual graph* of $\mathcal{Q} w$, denoted by Γw , is the 1-skeleton of $\text{TS } w$, i.e., it has a vertex for each maximal cell of $\mathcal{Q} w$, and two vertices share an edge if and only if their corresponding polytopes share a facet.

2.2 Thin Schubert cells of the Grassmannian

As a set, the Grassmannian $\text{Gr } d, n$ consists of the d -dimensional linear subspaces of $\mathbb{C}^n$. It is given the structure of a $d(n - d)$ -dimensional projective variety via the Plücker embedding $\iota : \text{Gr } d, n \hookrightarrow \mathbb{P}^{\binom{n}{d}-1}$: concretely, if V is the row-span of a full-rank $d \times n$ matrix A , then ιV is the homogeneous vector of maximal minors of A . The components of ιV are called the *Plücker coordinates* of V ; these are only well-defined up to a simultaneous scaling by a nonzero complex number.

Given $V \in \text{Gr } d, n$, its matroid is the set of $\lambda \in \binom{[n]}{d}$ such that the λ -th Plücker coordinate of V is nonzero. Given a $\mathbb{C}$ -realizable d, n -matroid Q , its *thin Schubert cell* is the locally-closed subscheme $\text{Gr } Q$ of $\text{Gr } d, n$ consisting of those V whose matroid is Q . Alternatively, it is the scheme-theoretic intersection $\text{Gr } Q = \text{Gr } d, n \cap \mathbb{C}^*{}^Q$, thus we frequently view $\text{Gr } Q$ as a closed subscheme of $\mathbb{C}^*{}^Q$. The thin Schubert cell of the uniform matroid $\binom{[n]}{d}$ is denoted by $\text{Gr}^\circ d, n$, i.e., it is the open subvariety defined by the nonvanishing of all Plücker coordinates. The decomposition of the Grassmannian into thin Schubert cells refines the decomposition by Schubert cells, see [17, §1.1] for a nice discussion on this.

2.3 Initial Degenerations

Let M, N be a pair of rank m dual lattices, and let $\langle u, v \rangle$ be the natural pairing for $u \in M$ and $v \in N$. Write $x^u \in \mathbb{C}^M$ to denote the monomial corresponding to $u \in M$. Given $f = \sum c_u x^u \in \mathbb{C}^M$ and $w \in N_{\mathbb{R}}$, the w -initial form is

$$\text{in}_w f = \sum_{\substack{\langle u, w \rangle \\ \text{is minimal}}} c_u x^u.$$

Denote by $T = N \otimes \mathbb{C}^* \sim \mathbb{C}^*{}^m$, let $I \subset \mathbb{C}^M$ be an ideal, and $X = V I \subset T$ its vanishing locus. The w -initial ideal of I is $\text{in}_w I = \langle \text{in}_w f : f \in I \rangle$. The *tropicalization* of X is $\text{Trop } X = \{w \in N_{\mathbb{R}} : \text{in}_w I \neq \langle 1 \rangle\}$; it is the underlying set of a N -rational polyhedral fan. Given $w \in \text{Trop } X$, the w -initial degeneration of X is $\text{in}_w X = V \text{in}_w I$.

In this work we study initial degenerations of $\text{Gr}^\circ d, n$. We abbreviate $\text{Trop } \text{Gr}^\circ d, n$ by $\text{TGr}^\circ d, n$. By [11, 22], if $w \in \text{TGr}^\circ d, n$, then $\mathcal{Q}_w$ is matroidal, and by [1], there is a subfan $\mathcal{S}_{\text{trop}} 3, 8$ of $\mathcal{S}_{\text{mat}} 3, 8$ whose support is $\text{TGr}^\circ 3, 8$. The fan $\mathcal{S}_{\text{trop}} 3, 8$ coarsens the Gröbner fan structure on $\text{TGr}^\circ 3, 8$, see [4, §2.3].

2.4 Inverse Limits

The *face order* on the set of d, n -matroids is the relation $Q_1 \preceq Q_2$ whenever ΔQ_1 is a face of ΔQ_2 . When $Q_1 \preceq Q_2$, the coordinate projection $\mathbb{C}^*{}^{Q_2} \rightarrow \mathbb{C}^*{}^{Q_1}$ induces a morphism $\text{Gr } Q_2 \rightarrow \text{Gr } Q_1$ by [18, Proposition I.6]. Given $w \in \text{TGr}^\circ d, n$, the cells of $\mathcal{Q}_w$ and the morphisms induced by the face relations form a finite inverse system, and we denote the corresponding inverse limit by $\text{Gr } w$. In fact, the inverse limit of the subdiagram induced by the tight span is isomorphic to $\text{Gr } w$.

Theorem 2.1. [3] *Given $w \in \text{TGr}^\circ d, n$, there exists a closed immersion $\text{in}_w \text{Gr}^\circ d, n \hookrightarrow \text{Gr } w$. If $\text{Gr } w$ is smooth, irreducible, and of dimension $d(n-d)$, then this closed immersion is an isomorphism.*

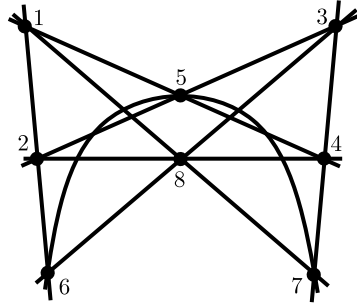


Figure 2.1: The Möbius-Kantor arrangement Q_{mk}

3 A reducible initial degeneration

By Mnëv's universality theorem (see [18, 19]), every singularity type appears in a thin Schubert cell of a rank-3 matroid. However, it is known that the thin Schubert cells of $2, n$, $3, 6$, and $3, 7$ -matroids are smooth and irreducible. We show in [4] that the thin Schubert cells of $3, 8$ -matroids are smooth, and exhibit a matroid whose thin Schubert cell has 2 connected components. We use this matroid to find an initial degeneration of $\text{Gr}^\circ 3, 8$ that is reducible. We sketch these constructions here.

Let Q_{mk} be the $3, 8$ -matroid whose nonbases are given by the colinearities in Figure 2.1. This appears throughout the literature as the *Möbius-Kantor* or *MacLane* matroid. Since Q_{mk} is connected, we have that $\text{Gr } Q_{mk} \sim X_{Q_{mk}} \times H$ as a consequence of the Gelfand-MacPherson correspondence. Up to a projective transformation, the normal vectors to a hyperplane arrangement realizing Q_{mk} can be brought to the columns of the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & a & 1 & a \\ 0 & 0 & 1 & 1 & 1 & 0 & 1-a & 1 \end{bmatrix} \quad (3.1)$$

where $a \in \mathbb{C}$ is one of the two solutions to $x^2 - x - 1 = 0$. In fact,

$$X_{Q_{mk}} \sim \text{Spec} \left(\mathbb{C}[a] / \langle a^2 - a - 1 \rangle \right),$$

and so $\text{Gr } Q_{mk}$ is smooth has 2 connected components.

Proposition 3.1. *If Q is a $\mathbb{C}$ -realizable $3, 8$ -matroid, then $\text{Gr } Q$ is smooth. Additionally, if Q is not isomorphic to Q_{mk} , then $\text{Gr } Q$ is irreducible.*

Given distinct $i, j, k \in [8]$, define matroids

$$Q_{ijk} = \{ \lambda \in \binom{[8]}{3} : |ijk \cap \lambda| \geq 2 \} \quad Q'_{ijk} = \{ \lambda \in \binom{[8]}{3} : |ijk \cap \lambda| = 2 \}.$$

Denote by $w_{mk} \in \mathbb{R}^{\binom{8}{3}}$ the vector $w_{mk}(\lambda) = 3 - \text{rank } \lambda$ for $\lambda \in \binom{8}{3}$. The tight span $\text{TS } w_{mk}$ is a star-graph whose central node corresponds to Q_{mk} and the 8 leaves correspond to Q_{ijk} for $ijk \in \binom{n}{3} \setminus Q_{mk}$. From this, one may show that

$$\text{Gr } w_{mk} \sim \text{Gr } Q_{mk} \times \prod_{ijk} \text{Gr } Q'_{ijk} \prod_{ijk} \text{Gr } Q_{ijk} \sim X(Q_{mk}) \times \mathbb{C}^*{}^{15}.$$

and so $\text{Gr } w_{mk}$ is smooth, 15 dimensional, and has 2 connected components. Denote by $\mathcal{C}_{mk}$ the closed cone of $\text{Trop Gr}^\circ(3, 8)$ containing w_{mk} in its relative interior. This discussion yields the following theorem.

Theorem 3.2. *For any cone $\mathcal{C}$ of $\mathcal{S}_{\text{trop}}(3, 8)$ in the $\mathfrak{S}_8$ -orbit of $\mathcal{C}_{mk}$ and w in the relative interior of $\mathcal{C}$, the initial degeneration $\text{in}_w \text{Gr}^\circ(3, 8)$ is smooth and has 2 connected components.*

Remark 3.3. The subdivision $\mathcal{Q}(w_{mk})$ is the only matroidal subdivision of $\Delta(3, 8)$ that contains $\Delta(Q_{mk})$ as a cell.

4 Decomposing tight spans: leaves, branches, and fins

In light of [Theorem 2.1](#), we focus on inverse limits to deduce information about initial degenerations. In most cases, it is convenient to decompose an inverse system into subsystems, and compute the limit in pieces. As we are taking these limits over diagrams coming from tight spans, this amounts to considering subcomplexes of the polytopal complex $\text{TS } w$. We describe these pieces for a general polytopal complex Σ .

A *leaf* of Σ is a leaf-vertex of the 1-skeleton of Σ . A *leaf-pair* (v, e) is a leaf v , along with its adjacent edge e . Let Σ_L be the subcomplex of the tight span obtained by removing all leaf-pairs from $\text{TS } w$. Observe that $\Sigma_L \subset \Sigma$ may still have leaves. The process of iteratively removing leaf-pairs must terminate, and the result is the subcomplex $\Sigma_{Br} \subset \Sigma$, which is nonempty provided Σ is not a tree. A maximal connected subgraph of $\Sigma \setminus \Sigma_{Br}$ is called a *branch* of Σ . Any vertex in a branch is called a *branch vertex* and any edge contained in a branch, or connecting a branch to the rest of Σ , is called a *branch edge*.

Let F be a closed 2-dimensional cell of Σ with $k \geq 3$ vertices. We say that F is a *fin* if its intersection with $\overline{\Sigma \setminus F}$ is a path of edge-length ℓ with $1 \leq \ell \leq k - 2$ (see [Figure 5.1](#)). We call this the *connecting path* of F . Denote by $V(F)$ and $E(F)$ the vertices and edges of F . We say $v \in V(F)$, respectively $e \in E(F)$, is *exposed* if $v \notin \overline{\Sigma \setminus F}$, respectively $e \notin \overline{\Sigma \setminus F}$. Denote by

$$\mathcal{E}_{\text{vert}}(F) = \{v \in V(F) : v \text{ is exposed}\} \quad \mathcal{E}_{\text{edge}} = \{e \in E(F) : e \text{ is exposed}\}.$$

Given a collection of fins $\mathfrak{F}$ denote by $\Sigma(\mathfrak{F}) \subset \Sigma$ the subcomplex obtained by removing, for each $F \in \mathfrak{F}$, the relatively open cell F° and the exposed vertices and edges of F .

5 Evaluating inverse limits

5.1 Matrix coordinates for thin Schubert cells

In this section, we give a presentation of the coordinate ring of a thin Schubert cell using *matrix coordinates*. Let Q be a d, n -matroid; without loss of generality, suppose $d \in Q$ (see [3, 4] for the more general treatment). Define $\mathbb{C} x_{ij} : \mathbb{C} x_{ij} : i \in d, j \in n-d$ and consider the $\mathbb{C} x_{ij}$ -valued matrix

$$A = \begin{bmatrix} 1 & 0 & \dots & 0 & x_{11} & \dots & x_{1,n-d} \\ 0 & 1 & \dots & 0 & x_{21} & \dots & x_{2,n-d} \\ \vdots & \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 1 & x_{d1} & \dots & x_{d,n-d} \end{bmatrix} \quad (5.1)$$

Define

$$B_Q = \mathbb{C} x_{ij} : d \triangle \{i, d-j\} \in Q$$

where $d \triangle \{i, d-j\}$ denotes the symmetric difference, and define the projection ring homomorphism

$$\pi_Q : \mathbb{C} x_{ij} \rightarrow \mathbb{C} x_{ij} / \langle x_{ij} : d \triangle \{i, d-j\} \in \binom{n}{d} \setminus Q \rangle \sim B_Q.$$

Denote by $\text{col}_i A$ the i -th column of A , and given $\lambda = \{\lambda_1 < \dots < \lambda_d\} \subset \binom{n}{d}$ define the $d \times d$ submatrix $A_\lambda = \text{col}_{\lambda_1} A \cdots \text{col}_{\lambda_d} A$. Define the ideal I_Q and multiplicative semigroup S_Q by

$$I_Q = \langle \pi_Q \det A_\lambda : \lambda \in \binom{n}{d} \setminus Q \rangle, \quad S_Q = \langle \pi_Q \det A_\lambda : \lambda \in Q \rangle_{\text{semigrp}}.$$

The coordinate ring of $\text{Gr } Q$ is isomorphic to $S_Q^{-1} B_Q / I_Q$.

5.2 Coordinate rings and subcomplexes

Let $w \in \text{TGr}^\circ d, n$. For a face C of $\text{TS } w$, denote by Δ_C the cell of $\mathcal{Q } w$ corresponding to C . In particular, if v is a vertex, then Δ_v is a maximal cell of $\text{TS } w$. A connected subcomplex $\Sigma \subset \text{TS } w$ is *vertex-intersecting* if

$$\bigcap_{v \in V \Sigma} \Delta_v \neq \emptyset.$$

where $V \Sigma$ denotes the vertex set of Σ . In other words, the matroids of the maximal cells in $\mathcal{Q } w$ corresponding to $V \Sigma$ share a common basis. The subcomplex Σ is *vertex-connecting* if, for each vertex x of $\Delta d, n$, the subcomplex of Σ consisting of those cells C such that $x \in \Delta_C$ is empty or connected.

Let $\Sigma \subset \text{TS } w$ be a vertex-intersecting connected subcomplex. Given a vertex $v \in \text{TS } w$, denote by Q_v the matroid of the polytope Δ_v . Then we have a basis common to all matroids Q_v with $v \in V \Sigma$. Without loss of generality, suppose that this basis is d . Define a polynomial subring B_Σ of $\mathbb{C}[x_{ij}]$ by

$$B_\Sigma = \mathbb{C}[x_{ij} : d \triangle \{i, d - j\} \in Q_v \text{ for some } v \in V \Sigma]$$

Next, define the ideal and multiplicative semigroup

$$I_\Sigma = \sum_{v \in V \Sigma} I_{Q_v} \cdot B_\Sigma \quad \text{and} \quad S_\Sigma = \langle S_{Q_v} : v \in V \Sigma \rangle_{\text{semigrp}};$$

note that we may view each S_{Q_v} as a subset of B_Σ under the inclusion $B_{Q_v} \subset B_\Sigma$. Finally, set

$$R_\Sigma = S_\Sigma^{-1} B_\Sigma / I_\Sigma \tag{5.2}$$

Proposition 5.1. *If Σ is vertex-intersecting and vertex-connecting, then the coordinate ring of the inverse limit $\varprojlim_\Sigma \text{Gr}$ is isomorphic to R_Σ .*

5.3 A combinatorial sieve of diagrams

In this section we discuss the approach used to prove [Theorems 1.1](#) and [1.2](#) from the introduction.

Theorem 5.2. *Let $w \in \text{TGr}^\circ(d, n)$.*

1. *The inverse limit $\text{Gr } w$ is smooth of dimension 15.*
2. *If w is not in the interior of a cone in the $\mathfrak{S}_8$ orbit of C_{mk} , then $\text{Gr } w$ is irreducible.*
3. *The closed immersion $\text{in}_w \text{Gr}^\circ(3, 8) \hookrightarrow \text{Gr } w$ is an isomorphism.*

The secondary fan structure of the tropical Grassmannian $\text{TGr}^\circ(3, 8)$ is computed in [\[1\]](#). A *combinatorial type* is a $\mathfrak{S}_8$ -orbit cone in $\mathcal{S}_{\text{trop}}(3, 8)$, and we label a combinatorial type by a vector lying in the relative interior of a cone representative. There are 57344 combinatorial types of cones in $\mathcal{S}_{\text{trop}}(3, 8)$. Fix a combinatorial type w and set $\Sigma = \text{TS } w$. Consider the following conditions, ordered by increasing complexity:

1. The tight span $\text{TS } w$ is vertex-intersecting.
2. The dual graph Γw is a tree.
3. The subcomplex $\Sigma_L \subset \text{TS } w$ is vertex-intersecting.
4. The subcomplex $\Sigma_{Br} \subset \text{TS } w$ is vertex-intersecting.
5. The subcomplex of Σ_L obtained by removing all fins of Σ_L whose connecting path had length 1 is vertex-intersecting.
6. The subcomplex of Σ_L obtained by removing all fins of Σ_L is a tree.

While these conditions are not mutually exclusive, we can still use them to separate the combinatorial types into 6 sets: set G_i consists of those combinatorial types satisfying condition i but not j for $j < i$. This sorting is done using OSCAR, and we see that

$$|G_1| = 13641, |G_2| = 215, |G_3| = 28227, |G_4| = 483, |G_5| = 14389, |G_6| = 389.$$

In [4, §6], we prove [Theorem 5.2](#) by employing a different strategy for each set G_i . Sets G_2, G_3, G_4, G_5 all follow a similar strategy, with G_5 requiring the most general technique. For this set, we verify [Theorem 5.2](#) using the following proposition.

Proposition 5.3. *Let $\mathfrak{F}$ be a collection of fins of Σ_L . If each fin $F \in \mathfrak{F}$ is B-maximal and $\varprojlim_{\Sigma_L} \mathfrak{F}$ Gr is smooth and irreducible then $\text{Gr } w$ is smooth and irreducible. Furthermore, the dimension of $\text{Gr } w$ can be computed using Proposition 6.15 of [4].*

The definition of B-maximality is technical, so we refer the reader to the full paper [4, §4.4]. In the following example we demonstrate how one can compute the inverse limit obtained from a matroid subdivision using [Proposition 5.3](#).

Example 5.4. Let e_{ijk} for $\{i < j < k\} \in \binom{[8]}{3}$ denote the standard basis of $\mathbb{R}^{\binom{8}{3}}$ and let

$$w = \begin{matrix} e_{124} & e_{125} & 2e_{126} & 2e_{134} & 3e_{137} & e_{145} & e_{146} & 2e_{147} & e_{148} & e_{156} & 2e_{235} & 2e_{237} \\ e_{245} & e_{246} & e_{256} & 2e_{257} & e_{258} & 3e_{347} & 2e_{357} & 2e_{367} & 2e_{368} & 2e_{378} & 2e_{456} & 3e_{678}. \end{matrix}$$

The tight span $\Sigma = \text{TS } w$ of $\mathcal{Q} = w$ is given in [Figure 5.1](#). The edges given by v_2, v_3 , v_3, v_{14} and v_{12}, v_{13} are leaves. Furthermore the face $F = \langle v_8, v_{11}, v_{12} \rangle$ of Σ is a fin (see [Section 4](#)). Note that F is the unique fin of Σ with connecting path of edge-length 1; let $\mathfrak{F} = \{F\}$. By considering the matroids of the vertices in $\Sigma_L \mathfrak{F}$ one readily verifies that this complex is vertex-intersecting, so $w \in G_5$.

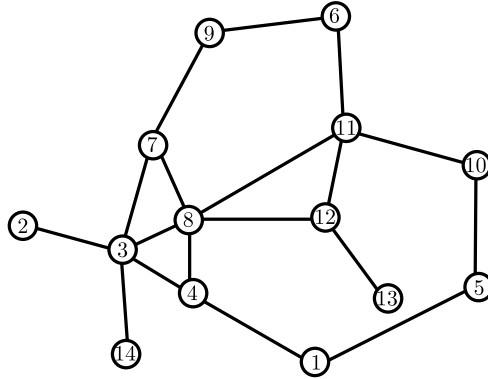
Observe that the pentagonal and hexagonal cells of Σ_L are indeed fins by the general definition in [Section 4](#). However, we have that F is B-maximal (a fact we defer to the full paper). Hence, it suffices to show that $\varprojlim_{\Sigma_L} \mathfrak{F}$ Gr is smooth and irreducible to apply [Proposition 5.3](#).

Denote by Q_i the matroid corresponding to the vertex v_i of Γw . When v_i and v_j share an edge, the matroid of the edge is denoted by $Q_{i,j}$. Direct computation tells us the dimensions of the thin Schubert cells corresponding to the vertices and edges of $\text{TS } w$ are given by

$$\dim \text{Gr } Q_i = 7 \text{ for } 1 \leq i \leq 14 \quad \dim \text{Gr } Q_{i,j} = 6 \text{ for } v_i, v_j \in E \Gamma w$$

Moreover, we have $\dim \text{Gr } Q_F = 5$ for $F = \langle v_8, v_{11}, v_{12} \rangle$ of Σ_L . Finally, we employ the technique described in [Section 5.2](#) to determine $R_{\Sigma_L \mathfrak{F}}$. We compute

$$B_{\Sigma_L \mathfrak{F}} = \mathbb{C} [x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{21}, x_{22}, x_{23}, x_{24}, x_{25}, x_{31}, x_{32}, x_{34}, x_{35}].$$

Figure 5.1: Tight span of $\mathcal{Q} w$

The ideal is

$$I_{\Sigma_L \mathfrak{F}} = \langle x_{13}x_{25} - x_{23}x_{15}, x_{11}x_{32}x_{25} - x_{31}x_{22}x_{15}, x_{11}x_{32}x_{23} - x_{31}x_{22}x_{13} \rangle,$$

and the semigroup $S_{\Sigma_L \mathfrak{F}}$ contains all monomials of $B_{\Sigma_L \mathfrak{F}}$. Let f_1, f_2, f_3 denote the generators of $I_{\Sigma_L \mathfrak{F}}$. The relations $f_i \equiv 0 \pmod{I_{\Sigma_L \mathfrak{F}}}$ (for $i = 1, 2$) yield

$$x_{13} \equiv \frac{x_{23}x_{15}}{x_{25}} \quad \text{and} \quad x_{32} \equiv -\frac{x_{31}x_{22}x_{15}}{x_{25}x_{11}} \pmod{I_{\Sigma_L \mathfrak{F}}}.$$

After these substitutions, the last relation disappears, and so the coordinate ring of $\varprojlim_{\Sigma \mathfrak{F}} \text{Gr}$ is isomorphic to a Laurent polynomial ring in 12 variables (after possibly inverting a finite number of elements). So we have that $\varprojlim_{\Sigma_L \mathfrak{F}} \text{Gr}$ is smooth, irreducible and of dimension 12. By B -maximality of F , we have that $\text{Gr} w$ is smooth and irreducible by Proposition 5.3. One may use Equation 6.5 in [4] to verify that $\dim \text{Gr} w = 15$. Hence, by Theorem 2.1, $\text{in}_w \text{Gr}^\circ_{3,8}$ is smooth and irreducible.

6 The Chow quotient of the Grassmannian

Recall that $X_{d,n}$ is the moduli space n hyperplanes in $\mathbb{P}^{d-1}$ in linear general position up to projective transformation. The normalization of the Chow quotient of $\text{Gr}_{d,n}$ by the diagonal torus $H \subset \text{PGL}(n)$, which we denote by $\overline{X}_{d,n}$, is a compactification of $X_{d,n}$. In this section, we sketch the proof of Theorem 1.3, that $\overline{X}_{3,8}$ is the log canonical compactification of $X_{3,8}$, which fully resolves [16, Conjecture 1.6]. For details, see [4, §7].

We follow an approach outlined by Hacking, Keel, and Tevelev in their construction of the log canonical compactification of moduli spaces of marked del Pezzo surfaces [8]. The first key fact we need is that $\overline{X}_{d,n}$ is the closure of $X_{d,n}$ in the toric variety

of the fan $\mathcal{S}_{\text{trop}}(3,8)/L$, where L is the lineality space of $\mathcal{S}_{\text{trop}}(3,8)$, see [14, 13]. By [8, Theorems 3.1, 9.1], to show that the log-canonical divisor of $X(3,8) \subset \overline{X}(3,8)$ is ample, it suffices to prove the following.

- The moduli space $X(3,8)$ is schön.
- The fan $\mathcal{S}_{\text{trop}}(3,8)/L$ is convexly disjoint in the sense of [8, Definition 1.15].

Schönness of $X(3,8)$ follows from Theorem 1.1 and the fact that $\text{in}_w \text{Gr}^\circ(3,8) \sim \text{in}_{\overline{w}} X(3,8) \times H$ (where $\overline{w}$ is the image of w under the projection $\mathbb{R}^8/\langle \mathbf{1} \rangle \rightarrow \mathbb{R}^8/L$). For the second point, we show that the secondary fan structure on the Dressian (see [11]) $\text{Dr}(3,8)/L$ is convexly disjoint (we do this for a general (d,n)), which shows that the subfan $\mathcal{S}_{\text{trop}}(3,8)/L$ is convexly disjoint.

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We thank Benjamin Schröter for sharing the data of the tropicalization of $\text{Gr}^\circ(3,8)$ with us, and for providing the key insight to the proof that the secondary fan of the Dressian is convexly disjoint. We also thank Michael Joswig, Lars Kastner, Benjamin Lorenz, Sam Payne, and Antony Della Vecchia for helpful conversations.

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Castelnuovo–Mumford regularity for 321-avoiding Kazhdan–Lusztig varieties

Colleen Robichaux^{*1}

¹ *Dept. of Mathematics, University of California, Los Angeles, Los Angeles, CA 90095, USA*

Abstract. We introduce an algorithm to compute the degrees of 321-avoiding unspecialized Grothendieck polynomials. Our result provides an algorithm to compute the Castelnuovo–Mumford regularity of 321-avoiding Kazhdan–Lusztig ideals. This extends the work of an earlier paper of Rajchgot, the author, and Weigandt (2022) which gives a formula in the case of Grassmannian Kazhdan–Lusztig ideals.

Keywords: Kazhdan–Lusztig varieties, Grothendieck polynomials, Castelnuovo–Mumford regularity, excited Young diagrams

1 Introduction

A. Woo and A. Yong [10] introduced *Kazhdan–Lusztig varieties* to study singularities of Schubert varieties. Kazhdan–Lusztig varieties are generalized determinantal varieties which include *Matrix Schubert varieties* [4] as special cases. Another well-studied class of these Kazhdan–Lusztig varieties is the *ladder determinantal varieties*, introduced by S. S. Abhyankar [1].

The *Castelnuovo–Mumford regularity* of a graded module is an invariant used to measure its complexity. In general, this regularity may be computed using the minimal free resolution of the module. Using the fact that Kazhdan–Lusztig varieties are Cohen–Macaulay, one may instead compute their regularities combinatorially in terms of degrees of *unspecialized Grothendieck polynomials*.

Our main results [Theorems 2.8](#) and [3.3](#) extend the work of [8] to provide an algorithm which computes the Castelnuovo–Mumford regularity for Kazhdan–Lusztig varieties indexed by a pair of 321-avoiding permutations. These results continue the work of J. Rajchgot, the author, and A. Weigandt [8] which provides a combinatorial formula to compute the regularity for Kazhdan–Lusztig varieties indexed by a pair of grassmannian permutations. This is an extended abstract of [9].

Due to a correspondence with matrix Schubert varieties in this case, this result in [8] may be recovered using the results of O. Pechenik–D. Speyer–A. Weigandt [7]. The work

^{*}robichaux@ucla.edu. Colleen Robichaux was supported by the NSF GRFP under Grant No. DGE 1746047 and NSF RTG Grant No. DMS 1937241.

in [7] uses different techniques to compute the regularity of arbitrary matrix Schubert varieties. Our paper extends the techniques used in [8] to compute the regularities of certain Kazhdan–Lusztig varieties which are *not* isomorphic to matrix Schubert varieties. That is, the results of this abstract cannot, in general, be recovered using [7].

2 Combinatorial Background

In this section we define the underlying combinatorial objects used for our algorithm.

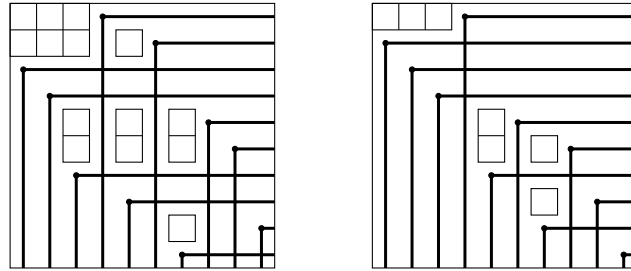
2.1 Pipe complexes

Let S_n denote the symmetric group on n letters. The **Rothe diagram** of $u \in S_n$ is the subset

$$D(u) = \{ (i, j) \in [n] \times [n] \mid u_i > j \text{ and } u_j^{-1} > i \}.$$

We illustrate $D(u)$ as cells remaining in the $n \times n$ grid after placing points in cells (i, u_i) for each $i \in [n]$ and drawing a line through cells which appear weakly south or weakly east of each (i, u_i) . Let $\ell(u) = \#D(u)$ denote the **Coxeter length** of u .

Example 2.1. Below are $D(v)$ and $D(w)$ for $v = 46128935107$ and $w = 41236859710$.



Here $\ell(w) = \#D(w) = 7$.

Define an algebra over $\mathbb{Z}$ generated by $\{e_u \mid u \in S_n\}$ with multiplication such that

$$e_u e_{s_i} = \begin{cases} e_{us_i} & \text{if } \ell(us_i) > \ell(u) \\ e_u & \text{otherwise.} \end{cases}$$

Here s_i is the simple transposition permuting elements i and $i + 1$.

Label the boxes of $D(u)$ along rows so that k th westmost box in row i is assigned the label $i - k + 1$. Given $P \subseteq D(u)$ let word P in $D(u)$ be the sequence formed by reading the labels of P in $D(u)$, moving east to west across rows, starting with the northmost

row and progressing south. Define $I u$: word $D u$ in $D u$. The **Demazure product** of word P , denoted δP , is the permutation determined by

$$e_{s_{i_1}} \cdots e_{s_{i_k}} e_{\delta P},$$

where word $P = i_1, i_2, \dots, i_k$.

Take $v \geq w \in S_n$, where $\geq$ denotes Bruhat order on S_n . Define

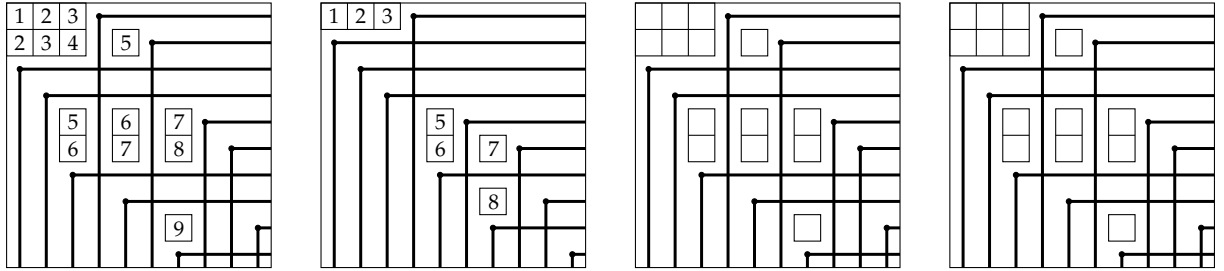
$$\text{Pipes } v, w = \{P \subseteq D v \mid \text{word } P = i_1, i_2, \dots, i_{\ell(w)} \text{ in } D v \text{ and } \delta P = w\}.$$

Similarly, let

$$\overline{\text{Pipes}} v, w = \{P \subseteq D v \mid \delta P = w\}.$$

We illustrate $P \subseteq D v$ by marking $i, j \in D v$ with a $\square$ whenever $i, j \in P$. Lastly, let $D^{NE} v, w \subseteq D v$ be the boxes corresponding to the earliest subsequence of word $D v$ that forms $I w$. Since $v \geq w$, $D^{NE} v, w$ exists.

Example 2.2. The left two diagrams are $D v$ and $D w$ for w, v as in [Example 2.1](#) with $I v$ and $I w$ labeled. This gives $I v = 3, 2, 1, 5, 7, 6, 8$ in $D v$. The third diagram is $D^{NE} v, w \in \text{Pipes } v, w \subseteq \overline{\text{Pipes}} v, w$ and the fourth is another $P \in \overline{\text{Pipes}} v, w$.



As defined by Woo–Yong [11], the **unspecialized Grothendieck polynomial** is

$$\mathfrak{G}_{v,w} \mathbf{t} = \sum_{P \in \overline{\text{Pipes}} v, w} (-1)^{\#P - \ell(w)} \prod_{i,j \in P} t_{ij}. \quad (2.1)$$

By setting $v = w_0 \in S_n$ and specializing variables t_{ij} , these unspecialized Grothendieck polynomials recover the *double Grothendieck polynomials* of [6]. Note that we follow the conventions of [8] for $\mathfrak{G}_{v,w} \mathbf{t}$, which differ from those in [11].

2.2 Skew Excited Young Diagrams

A permutation $u \in S_n$ is **321-avoiding** if there does not exist a 321 *pattern* in u , i.e., indices $i < j < k$ such that $u_k < u_j < u_i$. For example, $u = \underline{1}725\underline{8}34\underline{6}$ is not 321-avoiding; we underlined the positions of a 321 pattern. Let $S_n^{321\text{-av}} = \{u \in S_n \mid u \text{ is 321-avoiding}\}$.

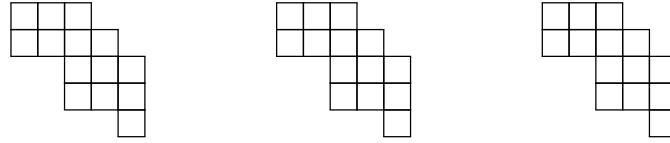
For this subsection assume $v \geq w$ where $v, w \in S_n^{321-\text{av}}$. Let

$$\phi_v : \{P \subseteq D \mid v\} \rightarrow \{S \subset n \times n\}$$

be the map which deletes empty rows and columns of $D \mid v$ from $P \subseteq D \mid v$. The shape $\mathcal{R}_v : \phi_v D \mid v$ is a skew Young diagram, i.e., λ/μ for some partitions $\mu \subseteq \lambda$. Our conventions for drawing Young diagrams reflect the diagrams in English notation across the y -axis. Define $D_{\text{top}} v, w : \phi_v D^{NE} v, w$.

We visualize $D \subseteq \mathcal{R}_v$ by marking $i, j \in n \times n$ with a $\cdot$ when $i, j \in D$. In general, we call a collection of $\cdot$'s inside $\mathcal{R}_v$ a **diagram** in $\mathcal{R}_v$.

Example 2.3. For v, w as in [Example 2.2](#), the left picture is $\mathcal{R}_v$, the middle is $D_{\text{top}} v, w$, and the rightmost diagram is $\phi_v P$ for the rightmost P in [Example 2.2](#).



An **excited move** of a diagram D in $\mathcal{R}_v$ is the operation on a 2×2 subsquare of D such that

$$\begin{array}{|c|c|} \hline \cdot & \cdot \\ \hline \cdot & \cdot \\ \hline \end{array} \mapsto \begin{array}{|c|c|} \hline \cdot & \cdot \\ \hline \cdot & \cdot \\ \hline \end{array}. \quad (2.2)$$

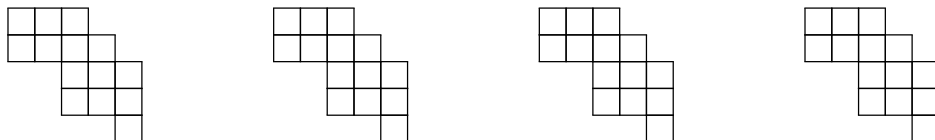
For this move to occur, the subsquare must be contained in $\mathcal{R}_v$. We let $\text{SEYD } v, w$ denote the set of $D \subseteq \mathcal{R}_v$ which can be computed through sequential applications of excited moves to $D_{\text{top}} v, w$. We call $D \in \text{SEYD } v, w$ a **skew excited Young diagram**.

We also may apply **K-theoretic excited moves** to diagrams in $\mathcal{R}_v$

$$\begin{array}{|c|c|} \hline \cdot & \cdot \\ \hline \cdot & \cdot \\ \hline \end{array} \mapsto \begin{array}{|c|c|} \hline \cdot & \cdot \\ \hline \cdot & \cdot \\ \hline \end{array}, \quad (2.3)$$

again, where all cells pictured are contained in $\mathcal{R}_v$. Write $\overline{\text{SEYD}} v, w$ for the set of diagrams which can be obtained by sequential applications of excited and K-theoretic excited moves on $D_{\text{top}} v, w$ in $\mathcal{R}_v$. We say $D \in \overline{\text{SEYD}} v, w$ is a **K-theoretic skew excited Young diagram**. Let $\#D$ denote the number of pluses in D . We say $D \in \overline{\text{SEYD}} v, w$ define to be **maximal** if $D' \in \overline{\text{SEYD}} v, w$ implies $\#D' \leq \#D$.

Example 2.4. Continuing [Example 2.3](#), the left two diagrams are in $\text{SEYD } v, w$. The right two diagrams are both maximal diagrams in $\overline{\text{SEYD}} v, w$.



Using [3] and the fact that $v, w \in S_n^{321-\text{av}}$ we obtain the following:

Proposition 2.5. *For $v \geq w$ where $v, w \in S_n^{321-\text{av}}$, the map ϕ_v restricted to $\overline{\text{Pipes}}_{v,w}$ gives a bijection*

$$\tilde{\phi}_v : \overline{\text{Pipes}}_{v,w} \rightarrow \overline{\text{SEYD}}_{v,w} \quad (2.4)$$

such that for $P \in \overline{\text{Pipes}}_{v,w}$, $\#P = \#\tilde{\phi}_v(P)$.

Combining Proposition 2.5 with Equation (2.4) produces the following result.

Corollary 2.6. *Suppose $v, w \in S_n$. Then*

$$\deg \mathfrak{G}_{v,w}(\mathbf{t}) = \max\{\#D \mid D \in \overline{\text{SEYD}}_{v,w}\}.$$

Example 2.7. Since the rightmost diagram in Example 2.4 is maximal, $\deg \mathfrak{G}_{v,w}(\mathbf{t}) = 8$ by Corollary 2.6.

In Section 4.1 we give an algorithm to compute statistics $\Delta_{v,w}(q)$ from $D_{\text{top}}(v, w)$ for certain $q \in \mathbb{Z}_{>0}$. Using Corollary 2.6, we prove the following.

Theorem 2.8. *Suppose $v \geq w$, where $v, w \in S_n^{321-\text{av}}$. Then if $D_{\text{top}}(v, w) = \bigcup_{q \in s} C_q$ where C_q are the connected components of $D_{\text{top}}(v, w)$,*

$$\deg \mathfrak{G}_{v,w}(\mathbf{t}) = \sum_{q \in s} \Delta_{v,w}(q).$$

A proof sketch for Theorem 2.8 appears in Section 4.2.

3 Castelnuovo–Mumford Regularity of Kazhdan–Lusztig varieties

In this section, we define Castelnuovo–Mumford regularity and Kazhdan–Lusztig varieties. We then recall results of [8] which provide combinatorial interpretations of the Castelnuovo–Mumford regularity of Kazhdan–Lusztig varieties.

3.1 Castelnuovo–Mumford Regularity

Let $S = \mathbb{C}[x_1, \dots, x_n]$ be a polynomial ring with the standard grading and let $I \subseteq S$ be a homogeneous ideal. The **Hilbert series** of S/I is a formal power series

$$H(S/I; t) = \sum_{k \in \mathbb{Z}} \dim_{\mathbb{C}}(S/I)_k t^k = \frac{K(S/I; t)}{1 - t^n}.$$

The numerator of the Hilbert series $K(S/I; t) \in \mathbb{C}[t^{\pm 1}]$ is the **K-polynomial** of S/I . A minimal free resolution of S/I is the complex

$$0 \rightarrow \bigoplus_j S(-j)^{\beta_{l,j}(S/I)} \rightarrow \bigoplus_j S(-j)^{\beta_{l-1,j}(S/I)} \rightarrow \dots \rightarrow \bigoplus_j S(-j)^{\beta_{0,j}(S/I)} \rightarrow S/I \rightarrow 0,$$

where $l \leq n$ and $S(-j)$ is the free S -module shifted by j in degree. The **Castelnuovo-Mumford regularity** of S/I , written $\text{reg } S/I$, is the statistic

$$\text{reg } S/I : \max\{j - i \mid \beta_{i,j}(S/I) > 0\}.$$

In cases where S/I is Cohen-Macaulay,

$$\text{reg } S/I = \deg K(S/I; t) - \text{ht}_S I, \quad (3.1)$$

where $\text{ht}_S I$ denotes the height of the ideal I . For more context, consult [2, Lemma 2.5].

3.2 Kazhdan–Lusztig varieties

For $v \in S_n$, define $M^v = (m_{ij})$ to be the matrix such that for $i, j \in [n]$,

$$m_{ij} = \begin{cases} 1 & \text{if } j = v_i, \\ z_{ij} & \text{if } i, j \in D(v), \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathbb{C}[\mathbf{z}^v] = \mathbb{C}[z_{ij} \mid i, j \in D(v)]$. For $v \geq w \in S_n$ the **Kazhdan–Lusztig ideal** $J_{v,w} \subseteq \mathbb{C}[\mathbf{z}^v]$ is defined by

$$J_{v,w} = \langle r_w(i, j) = 1 - \text{minors in } M_{i,j}^v \mid i, j \in D(w) \rangle,$$

where $M_{I,J}$ denote the submatrix of M with row indices in I and column indices in J for $I, J \subseteq [n]$. As noted in [8] when $v \in S_n^{321-\text{av}}$, $J_{v,w}$ is homogeneous with respect to the standard grading.

Let $B_+, B_- \subset \text{GL}_n(\mathbb{C})$ denote the Borel and opposite Borel subgroups, respectively. As defined in [10], the **Kazhdan–Lusztig variety** is the intersection of the Schubert variety $B_- \backslash \overline{B_- w B_-} \subseteq B_- \backslash \text{GL}_n(\mathbb{C})$ with the opposite Schubert cell $B_- \backslash B_- v B_-$. The coordinate ring of this Kazhdan–Lusztig variety is precisely $\mathbb{C}[\mathbf{z}^v] / J_{v,w}$. Through this fact, $\mathbb{C}[\mathbf{z}^v] / J_{v,w}$ is Cohen-Macaulay. Again we follow the conventions used in [8] rather than those in [10]. For additional context concerning Kazhdan–Lusztig varieties, see the survey [12].

Reformulating [11, Theorem 4.5] for the case $v, w \in S_n^{321-\text{av}}$,

Lemma 3.1. [8, Lemma 6.3] Let $v, w \in S_n^{321-av}$ where $w \leq v$. Then

$$K \mathbb{C} \mathbf{z}^v / J_{v,w}; t = \sum_{P \in \overline{\text{Pipes}}_{v,w}} -1^{\#P-\ell(w)} (1-t)^{\#P}.$$

We apply this Lemma along with Equation (3.1) for the following proposition.

Proposition 3.2. [8, Proposition 6.4] Let $v, w \in S_v^{321-av}$ where $w \leq v$. Then,

$$\deg K \mathbb{C} \mathbf{z}^v / J_{v,w}; t = \deg \mathfrak{G}_{v,w} \mathbf{t}.$$

Furthermore, the Castelnuovo–Mumford regularity of $\mathbb{C} \mathbf{z}^v / J_{v,w}$ is given by

$$\text{reg } \mathbb{C} \mathbf{z}^v / J_{v,w} = \deg \mathfrak{G}_{v,w} \mathbf{t} - \#D(w).$$

By combining Proposition 3.2 and Theorem 2.8, we obtain the following theorem.

Theorem 3.3. Suppose $v \geq w$, where $v, w \in S_n^{321-av}$. Then if $D_{\text{top}}(v, w) = \bigcup_{q \in s} C_q$ where C_q are the connected components of $D_{\text{top}}(v, w)$,

$$\text{reg } \mathbb{C} \mathbf{z}^v / J_{v,w} = \sum_{q \in s} \Delta_{v,w}(q).$$

In [5] S. R. Ghorpade–C. Krattenthaler give an algorithm to compute a related invariant called the a -invariant of certain two-sided ladder determinantal varieties. Two-sided ladder determinantal varieties are Kazhdan–Lusztig varieties indexed by particular $v, w \in S_n^{321-av}$. In this setting, the a -invariant is easily computed from the Castelnuovo–Mumford regularity. As we show in the full version of this abstract, Theorem 3.3 may be applied to generalize [5, Lemma 14].

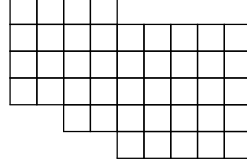
4 Construction and Recurrence for Theorem 2.8

Assume $v \geq w \in S_n^{321-av}$. In Section 4.1 we describe how to compute the statistics $\Delta_{v,w}(q)$ used in Theorem 2.8. In Section 4.2 we sketch the proof of Theorem 2.8.

4.1 Construction for Theorem 2.8

We index $\mathcal{R}_v$ using matrix indexing, identifying the northwest most box in $\mathcal{R}_v$ with $(1, 1)$. Suppose $D_{\text{top}}(v, w) = \bigcup_{q \in s} C_q$ where C_q are the connected components of $D_{\text{top}}(v, w)$. Order C_q such that the indices increase when viewing components from northwest to southeast.

Example 4.1. Consider $D = D_{\text{top}}(v, w)$ below for some $v, w \in S_{15}^{321-\text{av}}$.



Then D has connected components C_1 and C_2 , where $C_1 = \{1, 3, 1, 4, 2, 3, 2, 4, 3, 3\}$ and $C_2 = \{2, 6, 2, 7, 2, 8, 2, 9, 3, 6, 3, 7, 3, 8, 3, 9, 4, 8, 4, 9, 5, 8, 5, 9\}$.

For $q \in s$ in decreasing order, compute $\text{md } C_q = \{d_k^q\}_{k \in \ell_q} \subseteq C_q$, such that $\text{md } C_q$ is the westmost then southmost diagonal of length ℓ_q that minimizes

$$\#(\|\psi_E d_{\ell_q}^q\| = 1 \cap \{\|d_{k'}^{q'}\|\}_{q' > q, k' \in \ell_{q'}}).$$

Here $\|b\| : b_1 \leq b_2$ for $b = b_1, b_2 \in D_{\text{top}}(v, w)$. Boxes in $\text{md } C_q$ are ordered increasingly northwest to southeast.

Set $D_{\text{zip}}^0(v, w) : D_{\text{top}}(v, w)$. For $q \in s$, we define $D_{\text{zip}}^q(v, w)$ iteratively by applying excited moves to certain pluses in $C_q \subseteq D_{\text{zip}}^{q-1}(v, w)$. In $D_{\text{zip}}^{q-1}(v, w)$, set

$$S = \{b \in C_q - \text{md } C_q \text{ weakly southwest of } \text{md } C_q\}.$$

To each in $b \in S$, working west to east and south to north, let b' be the new position of b after applying as many excited moves as possible to b . Let

$$D_{\text{zip}}^q(v, w) : D_{\text{zip}}^{q-1}(v, w) - S \cup \{b' \mid b \in S\}.$$

Define $D_{\text{zip}}(v, w) : D_{\text{zip}}^s(v, w)$. For $b \in \text{md } C_q$, define $\text{trail}_{v, w} b$ such that

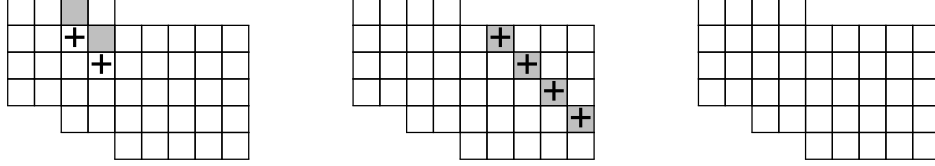
$$\begin{aligned} \text{trail}_{v, w} b : \max\{k \in \{0, 1, \dots, n\} \mid b = k', -k', b = k', 1 - k', \\ b = k' - 1, -k' \in \mathcal{R}_v - D_{\text{zip}}(v, w) \text{ for each } k' \in k\}. \end{aligned}$$

We define the statistic

$$\Delta_{v, w}^q : \sum_{k \in \ell_q} \text{trail}_{v, w} d_k^q.$$

Example 4.2. We continue with v, w as in [Example 4.1](#). Left to right, the diagrams below are $D_{\text{zip}}^0(v, w)$, $D_{\text{zip}}^1(v, w)$, and $D_{\text{zip}}^2(v, w)$, respectively. In $D_{\text{zip}}^0(v, w)$, $\text{Diag}_{v, w} C_1$

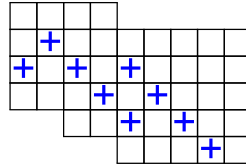
is bolded, and $\text{md } C_1$ is shaded. In $D_{\text{zip}}^1 v, w$, $\text{Diag}_{v,w} C_2$ is bolded, and $\text{md } C_2$ is shaded.



Using $D_{\text{zip}}^0 v, w$ and $\text{Diag}_{v,w} C_1$, we find $\Delta_{v,w} 1 \quad 2 \quad 1$. Similarly using $D_{\text{zip}}^1 v, w$ and $\text{Diag}_{v,w} C_2$, we compute $\Delta_{v,w} 2 \quad 2 \quad 2 \quad 1 \quad 1$. Therefore, [Theorem 2.8](#) determines

$$\deg \mathfrak{G}_{v,w} \mathbf{t} \quad \#D \quad w \quad \Delta_{v,w} 1 \quad \Delta_{v,w} 2 \quad 17 \quad 3 \quad 6 \quad 26.$$

By [Corollary 2.6](#), there exists $D \in \overline{\text{SEYD}} v, w$ where $\#D = 26$. We have drawn such a diagram below. This is computed by applying $\text{trail}_{v,w} d_k^q$ -many K-theoretic excited moves along the antidiagonals of $d_k^q \in \text{md } C_q$ for each $k \in \ell_q, q \in s$. The pluses that result from these K-theoretic excited moves are drawn in blue.



[Theorem 3.3](#) gives $\text{reg } \mathbb{C} \mathbf{z}^v / J_{v,w} = \Delta_{v,w} 1 \quad \Delta_{v,w} 2 \quad 9$. This corresponds precisely with the number of blue pluses in the diagram above.

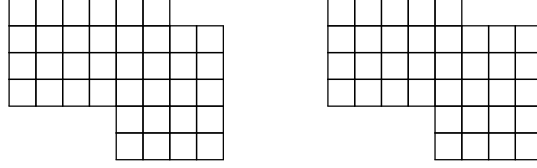
4.2 Proof Sketch for [Theorem 2.8](#)

In proving [Theorem 2.8](#), we utilize a particular recurrence on $\mathfrak{G}_{v,w} \mathbf{t}$. Let a, b be the northmost then eastmost plus in $D_{\text{top}} v, w$. Take a', b' to be the northmost then eastmost box in $\mathcal{R}_v$. Set $i = \text{word } \phi_v^{-1} \{a, b\}$ and $i' = \text{word } \phi_v^{-1} \{a', b'\}$ in $D v$. Define $v_P = s_{i'} v$, which gives $\mathcal{R}_{v_P} = \mathcal{R}_v - \{a', b'\}$. Define $w_P = s_i w$, $w_C = w$, and $v_C = v_P$.

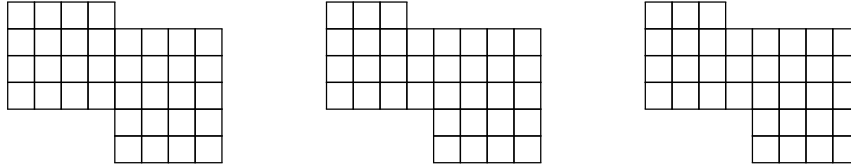
To proceed, we first establish $v_P, w_P \in S_n^{321-\text{av}}$. This follows from the definition of a, b and a', b' as northeast most choices along with the graphical definition of 321-avoiding. That is, $u \in S_n^{321-\text{av}}$ if and only if $\mathcal{R}_u$ is a skew Young diagram.

Example 4.3. Below we have $D_{\text{top}} v, w$ on the left and $D_{\text{top}} v_C, w_C$ on the right for

particular $v, w \in S_{15}^{321-av}$. In this case, $a, b \not\prec a', b'$.



Below are $D_{\text{top}} v', w'$, $D_{\text{top}} v'_C, w'_C$, and $D_{\text{top}} v'_P, w'_P$, listed from left to right, for particular $v', w' \in S_{15}^{321-av}$. In this case, $a, b \prec a', b'$.



We give a general correspondence of these K-theoretic skew excited Young diagrams.

Lemma 4.4. For $v \geq w$ and $v, w \in S_n^{321-av}$, the following hold:

1. If $a, b \not\prec a', b'$,

$$\overline{\text{SEYD}} v, w \not\subseteq \overline{\text{SEYD}} v_C, w_C.$$

2. If $a, b \prec a', b'$,

$$\overline{\text{SEYD}} v, w \subseteq \overline{\text{SEYD}} v_C, w_C \sqcup \left\{ D \cup a, b \mid D \in \overline{\text{SEYD}} v_C, w_C \cup \overline{\text{SEYD}} v_P, w_P \right\}.$$

Combining [Corollary 2.6](#) with [Lemma 4.4](#), we obtain the following corollary.

Corollary 4.5. For $v \geq w$ and $v, w \in S_n^{321-av}$, the following hold:

1. if $a, b \not\prec a', b'$, $\deg \mathfrak{G}_{v, w} \mathbf{t} \not\subseteq \deg \mathfrak{G}_{v_C, w_C} \mathbf{t}$.
2. If $a, b \prec a', b'$, $\deg \mathfrak{G}_{v, w} \mathbf{t} \subseteq \max \deg \mathfrak{G}_{v_P, w_P} \mathbf{t}, \deg \mathfrak{G}_{v_C, w_C} \mathbf{t} + 1$.

With this recurrence established, we now sketch the proof.

Proof sketch of [Theorem 2.8](#):

We proceed by induction on ℓv . For $\ell v = 0$, the statement is trivial since in this case $\text{SEYD } v, w = \emptyset$. Suppose the statement holds for v such that $\ell v = k - 1$ for $k \geq 1$. Consider v such that $\ell v = k$. For brevity let $d_{u_1, u_2} = \sum_{q \in s} \Delta_{u_1, u_2} q$ where $D_{\text{top}} u_1, u_2$ has s components.

If $a, b \not\prec a', b'$, by [Corollary 4.5](#), $\deg \mathfrak{G}_{v, w} \mathbf{t} \not\subseteq \deg \mathfrak{G}_{v_C, w_C} \mathbf{t}$. Using [Lemma 4.4](#), we determine $D_{\text{zip}} v, w = D_{\text{zip}} v_C, w_C$, so $d_{v, w} = d_{v_C, w_C}$. Thus the result follows by the inductive assumption.

Now assume $a, b \neq a', b'$. Then by [Corollary 4.5](#),

$$\deg \mathfrak{S}_{v,w} \leq \max \{ \deg \mathfrak{S}_{v_P, w_P}, \deg \mathfrak{S}_{v_C, w_C} \} + 1.$$

Since $\#D(w) = \#D(w_C)$ and $\#D(w) = \#D(w_P) + 1$, from the inductive assumption,

$$\deg \mathfrak{S}_{v,w} \leq \#D(w) = \max \{ \deg \mathfrak{S}_{v_P, w_P}, \deg \mathfrak{S}_{v_C, w_C} \} + 1.$$

Thus it suffices to prove

$$\deg \mathfrak{S}_{v,w} = \max \{ \deg \mathfrak{S}_{v_P, w_P}, \deg \mathfrak{S}_{v_C, w_C} \} + 1. \quad (4.1)$$

To establish Equation (4.1), we perform a careful case analysis on the position of $a, b \in C_q$ in relation to $\text{md } C_q$. The following claim is useful in this examination.

Claim 4.6. *Suppose $a, b \neq a', b' \in C_q$ where C_q is a connected component in $D_{\text{top}}(v, w)$. Let $R_q = \{d \in C_q \mid d \text{ lies weakly southwest of } a, b\}$. Then*

$$1. D_{\text{top}}(v_P, w_P) = D_{\text{top}}(v, w) - a, b, \text{ and}$$

$$2. D_{\text{top}}(v_C, w_C) = D_{\text{top}}(v, w) - R_q \cup R'_q,$$

where $R'_q = \{d \in C_q \mid d \text{ lies weakly northeast of } a, b\}$.

With Equation (4.1) proven, [Theorem 2.8](#) follows. $\square$

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The algebra of extended peaks

Darij Grinberg^{*1} and Ekaterina A. Vassilieva⁺²

Abstract. Building up on our previous works regarding q -deformed P -partitions, we introduce a new family of subalgebras for the ring of quasisymmetric functions. Each of these subalgebras admits as a basis a q -analogue to Gessel's fundamental quasisymmetric functions where q is equal to a complex root of unity. Interestingly, the basis elements are indexed by sets corresponding to an intermediary statistic between peak and descent sets of permutations that we call extended peak.

Résumé. En nous appuyant sur nos travaux précédents concernant les P -partitions q -déformées, nous introduisons une nouvelle famille de sous-algèbres pour l'anneau des fonctions quasi-symétriques. Chacune de ces sous-algèbres admet comme base un q -analogue aux fonctions quasisymétriques fondamentales de Gessel où q est égal à une racine complexe de l'unité. Il est notable que les éléments de base sont indexés par des ensembles correspondant à une statistique intermédiaire entre les ensembles de pics et de descentes des permutations que nous appelons pic étendu.

Keywords: Quasisymmetric functions, descent set, peak set.

1 Introduction

In [4], we show that a q -deformation of the generating functions for P -partitions leads to a unified framework between classical and enriched P -partitions. In particular, we introduce our q -fundamental quasisymmetric functions that interpolate between I. Gessel's fundamental ([1], $q = 0$) and J. Stembridge's peak ([8], $q = 1$) quasisymmetric functions. When q is not a root of unity, q -fundamentals are a basis of QSym , the ring of quasisymmetric functions, and are indexed by descent sets of permutations. If $q = 1$, they span the subalgebra of QSym named the algebra of peaks. The relevant basis elements are those indexed by peak sets. As it turns out, using other complex roots of unity for q , we are able to build new intermediate subalgebras between the algebra of peaks and QSym , the basis of which are q -fundamentals indexed by a new permutation statistic that lies between peak and descent sets. We call this statistic the extended peak set and the corresponding subalgebras of quasisymmetric functions the algebra of extended peaks. We begin with the required definitions and results from [4]. Then we introduce and prove the new results regarding the algebra of extended peaks.

^{*}darijgrinberg@gmail.com.

[†]katya@lix.polytechnique.fr.

1.1 Permutation statistics

Let $\mathbb{P}$ be the set of positive integers. For $m, n \in \mathbb{P}$, write $m, n = \{m, m+1, \dots, n\}$ and simply $n = \{1, 2, \dots, n\}$. We denote S_n the symmetric group on n . Given $\pi \in S_n$, define its *descent set*

$$\text{Des } \pi = \{1 \leq i \leq n-1 \mid \pi(i) > \pi(i+1)\} \subset [n-1]$$

and its *peak set*

$$\text{Peak } \pi = \{2 \leq i \leq n-1 \mid \pi(i-1) < \pi(i) > \pi(i+1)\}.$$

The peak set of a permutation is said to be *peak-lacunar*, i.e. it neither contains 1 nor contains two consecutive integers.

1.2 Enriched P -partitions and q -deformed generating functions

We recall the main definitions regarding weighted posets, enriched P -partitions and their q -deformed generating functions. The reader is referred to [1, 3, 4, 7, 8] for more details.

Definition 1 (Labelled weighted poset, [3]). A labelled weighted poset is a triple $(n, <_P, \epsilon)$ where $(n, <_P)$ is a labelled poset, i.e., an arbitrary partial order $<_P$ on the set n and $\epsilon : n \rightarrow \mathbb{P}$ is a map (called the weight function).

Each node of a labelled weighted poset is marked with its label and weight (Figure 1).

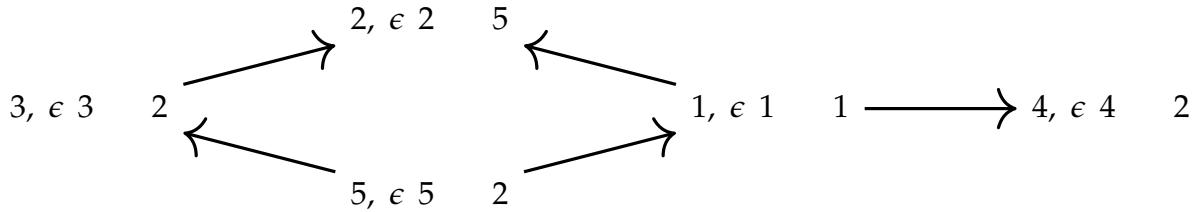


Figure 1: A 5-vertex labelled weighted poset. Arrows show the covering relations.

Definition 2 (Enriched P -partition, [8]). Let $\mathbb{P}^\pm$ be the set of positive and negative integers totally ordered by $-1 < 1 < -2 < 2 < -3 < 3 < \dots$. We embed $\mathbb{P}$ into $\mathbb{P}^\pm$ and let $-\mathbb{P} \subseteq \mathbb{P}^\pm$ be the set of all $-n$ for $n \in \mathbb{P}$. Given a labelled weighted poset $(n, <_P, \epsilon)$, an enriched P -partition is a map $f : n \rightarrow \mathbb{P}^\pm$ that satisfies the two following conditions:

- (i) If $i <_P j$ and $i < j$, then $f(i) < f(j)$ or $f(i) = f(j) \in \mathbb{P}$.
- (ii) If $i <_P j$ and $i > j$, then $f(i) < f(j)$ or $f(i) = f(j) \in -\mathbb{P}$.

We let $\mathcal{L}_{\mathbb{P}^\pm} P$ denote the set of enriched P -partitions.

Definition 3 (q -Deformed generating function, [4]). Consider the set of indeterminates $X = \{x_1, x_2, x_3, \dots\}$, the ring $\mathbb{C}[X]$ of formal power series on X where $\mathbb{C}$ is the set of complex numbers, and let $q \in \mathbb{C}$ be an additional parameter. Given a labelled weighted poset $(n, <_P, \epsilon)$, define its generating function $\Gamma^q(n, <_P, \epsilon) \in \mathbb{C}[X]$ as

$$\Gamma^q(n, <_P, \epsilon) = \sum_{f \in \mathcal{L}_{\mathbb{P}^\pm}(n, <_P, \epsilon)} \prod_{1 \leq i \leq n} q^{f(i) < 0} x_{|f(i)|}^{\epsilon(i)},$$

where $f(i) < 0 = 1$ if $f(i) < 0$ and 0 otherwise.

1.3 Enriched q -monomial and q -fundamental quasisymmetric functions

We state without proofs the required definitions and propositions from [4]. The main building block of this previous work is the q -deformed generating function for enriched P -partitions on labelled weighted chains that we call *universal quasisymmetric functions*.

Definition 4 (Universal quasisymmetric functions). Given a composition, i.e. a sequence of positive integers $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ with n entries, and a permutation $\pi = \pi_1 \dots \pi_n$ of S_n , we let $P_{\pi, \alpha} = (n, <_\pi, \alpha)$ be the labelled weighted poset on the set $[n]$, where the order relation $<_\pi$ is such that $\pi_i <_\pi \pi_j$ if and only if $i < j$ and where α is the weight function sending the vertex labelled π_i to α_i (see Figure 2). Define the q -universal quasisymmetric function

$$U_{\pi, \alpha}^q = \Gamma^q(n, <_\pi, \alpha).$$

$$\pi_1, \alpha_1 \longrightarrow \pi_2, \alpha_2 \longrightarrow \dots \longrightarrow \pi_n, \alpha_n$$

Figure 2: The labelled weighted poset $P_{\pi, \alpha}$.

Universal quasisymmetric functions belong to the subalgebra of $\mathbb{C}[X]$ called the ring of *quasisymmetric functions* (QSym), i.e. for any strictly increasing sequence of indices $i_1 < i_2 < \dots < i_p$ the coefficient of $x_1^{k_1} x_2^{k_2} \dots x_p^{k_p}$ is equal to the coefficient of $x_{i_1}^{k_1} x_{i_2}^{k_2} \dots x_{i_p}^{k_p}$.

They are directly connected to classical bases of QSym, as $L_\pi = U_{\pi, 1^n}^0$ (resp. $K_\pi = U_{\pi, 1^n}^1$) is the Gessel's *fundamental* ([1]) (resp. Stembridge's *peak*, [8]) quasisymmetric function indexed by π . (As usual, 1^n denotes n consecutive 1's: $1, 1, \dots, 1$.) Moreover, if we let $id_n = 1 2 \dots n$ and $\overline{id}_n = n n-1 \dots 1$, then $U_{id_n, \alpha}^0 = M_\alpha$ is the *monomial* ([1]), $U_{id_n, \alpha}^0 = E_\alpha$ the *essential* ([5]) and $U_{id_n, \alpha}^1 = \eta_\alpha$ the *enriched monomial* ([6, 3]) quasisymmetric

functions indexed by α . Moreover universal quasisymmetric functions satisfy the explicit expression

$$U_{\pi, \alpha}^q = \sum_{\substack{i_1 \leq i_2 \leq \dots \leq i_n \\ j \in \text{Peak } \pi \Rightarrow i_{j-1} < i_j}} q^{|\{j \in \text{Des } \pi \mid i_j > i_{j-1}\}|} q^{-1 - |\{i_1, i_2, \dots, i_n\}|} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \dots x_{i_n}^{\alpha_n}. \quad (1.1)$$

As universal quasisymmetric functions are the generating functions of some P -partitions on labelled chains, they admit the closed form product rule of [Proposition 1](#).

Proposition 1 (Product rule). *Let $q \in \mathbb{C}$, let π and σ be two permutations in S_n and S_m , and let $\alpha = \alpha_1, \dots, \alpha_n$ and $\beta = \beta_1, \dots, \beta_m$ be two compositions with n and m entries. The product of two q -universal quasisymmetric functions is given by*

$$U_{\pi, \alpha}^q U_{\sigma, \beta}^q = \sum_{\tau, \gamma \in \pi, \alpha \sqcup \sigma, \beta} U_{\tau, \gamma}^q. \quad (1.2)$$

Here $\pi, \alpha \sqcup \sigma, \beta$ denote the coshuffle of π, α and σ, β , i.e. the set of pairs τ, γ where $\tau \in S_{n+m}$ is a shuffle of π and σ and $n - \sigma = n - \sigma_1, \dots, n - \sigma_m$, and γ is a composition with $n+m$ entries, obtained by shuffling the entries of α and β using the same shuffle used to build τ .

Remark 1 (Coproduct). *Let $n \geq 0$ be an integer, $\pi \in S_n$ and α be a composition with n entries. The coproduct $\Delta : \text{QSym} \rightarrow \text{QSym} \otimes \text{QSym}$ of the Hopf algebra QSym (see [\[2, §5.1\]](#)) acts on the universal quasisymmetric functions as follows:*

$$\Delta U_{\pi, \alpha}^q = \sum_{i=0}^n U_{\text{std } \pi_1 \pi_2 \dots \pi_i, \alpha_1, \alpha_2, \dots, \alpha_i}^q \otimes U_{\text{std } \pi_{i+1} \pi_{i+2} \dots \pi_n, \alpha_{i+1}, \alpha_{i+2}, \dots, \alpha_n}^q.$$

Here, if γ is a sequence of non-repeating integers, $\text{std } \gamma$ is the permutation whose values are in the same relative order as the entries of γ .

Our work relies on two significant specialisations of universal quasisymmetric functions called enriched q -monomial and q -fundamental quasisymmetric functions.

Definition 5 (Enriched q -monomial quasisymmetric functions). *Let $q \in \mathbb{C}$ and α be a composition with n entries. The enriched q -monomial indexed by α is defined as*

$$\eta_\alpha^q = U_{\text{id}_n, \alpha}^q = \sum_{i_1 \leq i_2 \leq \dots \leq i_n} q^{-1 - |\{i_1, i_2, \dots, i_n\}|} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \dots x_{i_n}^{\alpha_n}.$$

As compositions $\alpha = \alpha_1, \dots, \alpha_n$ such that $\alpha_1 + \dots + \alpha_n = s$ are in bijection with subsets of $s-1$, we also use the equivalent expression for $I \subseteq s-1$.

$$\eta_{s, I}^q = \sum_{\substack{i_1 \leq \dots \leq i_s \\ j \in I \Rightarrow i_j > i_{j-1}}} q^{-1 - |\{i_1, \dots, i_s\}|} x_{i_1} \dots x_{i_s}.$$

As an immediate consequence of [Definition 5](#), η_α^0 (resp. η_α^1) is Hoffman's essential ([5]) (resp. enriched monomial, [3]) quasisymmetric function indexed by the composition α . Except the degenerate case $q = -1$, q -enriched monomials are a basis of QSym.

Proposition 2. *Let $q \in \mathbb{C} \setminus \{-1\}$. The family $(\eta_{s,I}^q)_{s \geq 0, I \subseteq [s-1]}$ is a basis of QSym.*

Definition 6 (q -Fundamental quasisymmetric functions). *Let π be a permutation in S_n and $q \in \mathbb{C}$. Define the q -fundamental quasisymmetric function indexed by π as*

$$L_\pi^q = U_{\pi, 1^n}^q.$$

According to Equation (1.1), L_π^q depends only on the descent set of π . As a result, q -fundamentals are naturally indexed by sets and we may denote them $L_{n,I}^q$ $I \subseteq [n-1]$.

The specialisations of L_π^q to $q = 0$ and $q = 1$ are respectively the Gessel's fundamental [1] and Stembridge's peak [8] quasisymmetric functions indexed by permutation π . The expression of q -fundamentals in the basis of enriched q -monomials is of particular importance.

Proposition 3. *Let $I \subseteq [n-1]$ and $q \in \mathbb{C}$. Let also $\text{Peak } I = I \setminus [I-1] \setminus \{1\}$. The q -fundamental quasisymmetric functions may be expressed in the enriched q -monomial basis as*

$$L_{n,I}^q = \sum_{\substack{J \subseteq I \\ K \subseteq \text{Peak } I \\ J \cap K = \emptyset}} (-q)^{|K|} (q-1)^{|J|} \eta_{n, J \cup [K-1] \cup K}^q. \quad (1.3)$$

Unlike q -monomials, q -fundamentals are not always a basis of QSym.

Proposition 4. $(L_{n,I}^q)_{n \geq 0, I \subseteq [n-1]}$ is a basis of QSym if and only if $q \in \mathbb{C}$ is not a root of unity.

2 Extended peaks

According to [Proposition 4](#), q -fundamental quasisymmetric functions are very similar to classical ones when q is not a root of unity and are naturally indexed by descent sets. When $q = 1$, they reduce to peak quasisymmetric functions, are indexed by peak sets and span a very significant subalgebra of QSym. Understanding what subalgebra they span and how to index them when q is another root of unity appears as a natural question. We begin with the introduction of the appropriate sets.

2.1 Extended peak sets

We use the following subsets with constraints on consecutive elements.

Definition 7 (Extended peak set). *Let n and p be two positive integers. We say that $I \subseteq [n-1]$ is a p -extended peak set if $I \cup \{0\}$ doesn't contain more than p consecutive elements (as a result, $1, p \notin I$). We write $I \subseteq_p [n-1]$ for this statement.*

Example 1. Set $n = 9$. One has $\{4, 8\} \subseteq_1 [8]$, $\{1, 4, 5, 8\} \subseteq_2 [8]$, $\{1, 2, 4, 5, 6, 8\} \subseteq_3 [8]$. However $\{1, 2, 4, 5, 8\} \not\subseteq_2 [8]$ as the subsequence $1, 2$ containing 1 has size $2 > p - 1 = 1$.

Remark 2 (Permutation statistics). *Extended peak sets look like an intermediary statistic between peak and descent sets. Any peak set is a 1-extended peak set, and any descent set on permutations of n elements is a p -extended peak set for $p \geq n$. Moreover, given a permutation π in S_n and an integer $p \geq 1$, we can define $\text{Peak}_p \pi$ as*

$$\text{Peak}_p \pi = \{i \in \text{Des } \pi \mid i \leq p-1 \text{ or } \exists 1 \leq j \leq p \text{ such that } i-j \notin \text{Des } \pi\}.$$

On the one hand, $\text{Peak } \pi = \text{Peak}_1 \pi \subseteq \text{Peak}_p \pi \subseteq \text{Des } \pi$. On the other hand, $\text{Peak}_p \pi \subseteq_p [n-1]$. For instance let $\pi = 54163287$. We have $\text{Peak}_1 \pi = \text{Peak } \pi = \{4, 7\} \subseteq \text{Peak}_2 \pi = \{1, 4, 5, 7\} \subseteq \text{Peak}_3 \pi = \{1, 2, 4, 5, 7\} \subseteq \text{Des } \pi$.

We count the number of extended peak sets.

Proposition 5. *Let $n, p \in \mathbb{P}$. Define s_n^p to be the number of p -extended peak sets on n elements. Extend the definition with $s_0^p = 0$ for all positive p . One has*

$$s_n^p = \begin{cases} 2^{n-1} & \text{if } n \leq p, \\ \sum_{k=0}^p s_{n-k-1}^p & \text{if } n > p. \end{cases} \quad (2.1)$$

Proof. The result is immediate for $n \leq p$. For $n > p$, there is a bijection between p -extended peak sets on n elements and the union of p -extended peak sets on $n-1-k$ elements for $k \in [0, p]$. Indeed, given a set $I \subseteq_p [n-1]$, define $k \leq p$ as the integer such that $[n-k, n-1]$ is the maximum sequence of consecutive integers in I containing $n-1$. If $n-1 \notin I$ we define $k = 0$ and assume $[n, n-1] = \emptyset$. We map I to the unique element $J \subseteq_p [n-k-2]$ such that $I = J \cup [n-k, n-1]$. This mapping is clearly one-to-one and the result follows. $\square$

2.2 Extended peak quasisymmetric functions

We proceed with the definition of the relevant subfamilies of q -fundamentals.

Definition 8 (Extended-peak quasisymmetric functions). Let $n, p \in \mathbb{P}$, and let ρ_p denote the root of unity $\rho_p = e^{-i\pi(p-1)/p}$. Thus, $\rho_1 = 1, \rho_2 = e^{-i\pi/3}, \rho_3 = e^{-i\pi/2}, \dots$. Note that $-\rho_p$ is a primitive $p-1$ -th root of unity, i.e., $-\rho_p^{p-1} = 1$ but $-\rho_p^j \neq 1$ for $1 \leq j < p-1$. Given a subset $I \subseteq [n-1]$, we define the p -extended peak quasisymmetric function indexed by I

$$L_{n,I}^p = L_{n,I}^{\rho_p}. \quad (2.2)$$

Define $\mathcal{P}^p \subseteq \text{QSym}$ to be the subalgebra of QSym spanned by $L_{n,I}^p$ $n \geq 0, I \subseteq [n-1]$, and let $\mathcal{P}_n^p \subseteq \text{QSym}_n$ be its subspace composed of quasisymmetric functions of degree n (that is, the vector space spanned by $L_{n,I}^p$ $I \subseteq [n-1]$). We call $\mathcal{P}^p$ the algebra of p -extended peaks.

We remark that $\mathcal{P}^p$ is actually a Hopf subalgebra of QSym (by Remark 1).

Definition 8 gives extended peak functions over all subsets. However, we know from Proposition 4 that they do not span QSym . As a result, for all $p \in \mathbb{P}$, the family $L_{n,I}^p$ $n \geq 0, I \subseteq [n-1]$ is not linearly independent and some indices are redundant. We characterise these set indices. First, for $n, p \in \mathbb{P}$, if a set I is not a p -extended peak set, then $L_{n,I}^p$ may be expressed in terms of other p -extended peak quasisymmetric functions.

Theorem 1 (Extended peak functions over sets that are not p -extended peaks). Let $n, p \in \mathbb{P}$ with $n \geq p-1$. Let i be an integer such that $0 \leq i \leq n-1-p$, and $J \subseteq [n-1]$ be a subset that satisfies $i-1, i-p-1 \in J$ and $i \in J \cup \{0\}$. Then, the set $i-1, i-p \cup J \not\subseteq [n-1]$ as it contains either a sequence of $p-1$ consecutive elements or the sequence $1, p$. Notice further that any set that is not a p -extended peak set may be written as such. We have the equality

$$\sum_{I \subseteq [i-1, i-p]} (-1)^{|I|} L_{n, I \cup J}^p = 0.$$

Secondly, we can compute explicitly the dimension of $\mathcal{P}_n^p$ for $n, p \in \mathbb{P}$.

Theorem 2 (Subspaces dimension). Let $n, p \in \mathbb{P}$ be two positive integers. The dimension of $\mathcal{P}_n^p$ is equal to s_n^p , the number of p -extended peak sets on n elements:

$$\dim \mathcal{P}_n^p = s_n^p.$$

We postpone the proofs of Theorems 1 and 2 respectively to Sections 3.1 and 3.2. Combining them we characterise the subalgebra $\mathcal{P}^p$.

Theorem 3 (Basis for algebra of extended peaks). Let $p \in \mathbb{P}$. The family $L_{n,I}^p$ $n \geq 0, I \subseteq [n-1]$ is a basis of the subalgebra $\mathcal{P}^p$ of QSym .

Proof. Fix $p \in \mathbb{P}$. As p -extended peak quasisymmetric functions are special cases of q -fundamentals, the stability by multiplication is actually a direct consequence of Equation (1.2). Then Theorem 1 shows that only p -extended peak quasisymmetric functions indexed by p -extended peak sets may be linearly independent. Finally, Theorem 2 shows that for all $n \in \mathbb{P}$ the dimension of the finite vector space containing homogeneous quasisymmetric functions of degree n is exactly the number of p -extended peak sets on n elements. $\square$

3 Proofs of Theorems 1 and 2

3.1 Extended peak functions indexed by generic sets

In order to show [Theorem 1](#), compute

$$\sum_{I \subseteq [i-1, i-p]} -1^{|I|} L_{I \cup J}^q$$

using Equation (1.3) for integers i, n, p and set J satisfying the conditions of the theorem. Note that for $I \subseteq [i-1, i-p]$, $i-1 \notin \text{Peak } I \cup J$ (either $i-1 = 1$ or $i \in J$) and that $\text{Peak } I \cup J \cap J = \text{Peak } J$ irrelevant of the choice of I (as $i-p-1 \notin J$). As a result, we can decompose in Equation (1.3) any subset or peak lacunar subset (i.e. 1-extended peak subset) of $I \cup J$ as a (peak lacunar) subset of I and a (peak lacunar) subset of J . Namely,

$$\begin{aligned} & \sum_{I \subseteq [i-1, i-p]} -1^{|I|} L_{I \cup J}^q \\ &= \sum_{\substack{U' \subseteq J \\ V' \subseteq \text{Peak } J \\ U' \cap V' = \emptyset}} -q^{|V'|} (q-1)^{|U'|} \sum_{I \subseteq [i-1, i-p]} -1^{|I|} \sum_{\substack{U \subseteq I \\ V \subseteq \text{Peak } I \setminus \{i-1\} \\ U \cap V = \emptyset}} -q^{|V|} (q-1)^{|U|} \eta_{U' \cup V' - 1 \cup V' \cup U \cup V - 1 \cup V}. \end{aligned}$$

Next, invert summation indices in the last sums.

$$\begin{aligned} & \sum_{I \subseteq [i-1, i-p]} -1^{|I|} L_{I \cup J}^q \\ &= \sum_{\substack{U' \subseteq J \\ V' \subseteq \text{Peak } J \\ U' \cap V' = \emptyset}} -q^{|V'|} (q-1)^{|U'|} \sum_{\substack{U \subseteq [i, i-p] \\ V \subseteq [i-1, i-p] \\ U \cap V = \emptyset \\ U \cap V - 1 = \emptyset}} -q^{|V|} (q-1)^{|U|} \eta_{U' \cup V' - 1 \cup V' \cup U \cup V - 1 \cup V} \sum_{U \cup V \subseteq [i-1, i-p] \setminus V-1} -1^{|I|}. \end{aligned}$$

The last sum is obviously 0 except when $U \cup V = [i-1, i-p] \setminus V-1$. As a result,

$$\begin{aligned} & \sum_{I \subseteq [i-1, i-p]} -1^{|I|} L_{I \cup J}^q \\ &= \sum_{\substack{U' \subseteq J \\ V' \subseteq \text{Peak } J \\ U' \cap V' = \emptyset}} -q^{|V'|} (q-1)^{|U'|} \eta_{U' \cup V' - 1 \cup V' \cup [i, i-p]} \sum_{V \subseteq [i-1, i-p]} -q^{|V|} (q-1)^{p-2|V|} -1^{p-|V|}. \end{aligned}$$

The summands in the sum over all 1-extended peak sets $V \subseteq [i-1, i-p]$ depend only on the cardinality of V . It easy to show (left to the reader) that

$$|\{V \subseteq [i-1, i-p], |V| = v\}| = \binom{p-v}{v}.$$

Subsequently,

$$\sum_{I \subseteq [i-1, i-p]} -1^{|I|} L_{I \cup J}^q - 1^p \sum_{\substack{U' \subseteq J \\ V' \subseteq \text{Peak } J \\ U' \cap V' = \emptyset}} -q^{|V'|} (q-1)^{|U'|} \eta_{U' \cup V' - 1 \cup V' \cup i-1, i-p}^q \sum_{v=0}^p -1^v \binom{p-v}{v} -q^v (q-1)^{p-2v}.$$

We use the following lemma.

Lemma 1 ([9]). *Let $n \in \mathbb{P}$ and $x, y \in \mathbb{C}$, one has*

$$\sum_{k=0}^n -1^k \binom{n-k}{k} x y^k x^{n-2k} y^{n-2k} = \sum_{j=0}^n x^{n-j} y^j.$$

Denote for $n \in \mathbb{P}, c \in \mathbb{C}$, $n_c = 1 - c^n / (1 - c)$. As a direct consequence of [Lemma 1](#),

$$\begin{aligned} \sum_{I \subseteq [i-1, i-p]} -1^{|I|} L_{I \cup J}^q &= \sum_{\substack{U' \subseteq J \\ V' \subseteq \text{Peak } J \\ U' \cap V' = \emptyset}} -q^{|V'|} (q-1)^{|U'|} \eta_{U' \cup V' - 1 \cup V' \cup i-1, i-p}^q \sum_{t=0}^p -q^{p-t}, \\ &= (p-1-q) \sum_{\substack{U' \subseteq J \\ V' \subseteq \text{Peak } J \\ U' \cap V' = \emptyset}} -q^{|V'|} (q-1)^{|U'|} \eta_{U' \cup V' - 1 \cup V' \cup i-1, i-p}^q. \end{aligned}$$

End the proof with

$$(p-1-\rho_p) \frac{1 - (-\rho_p)^{p-1}}{1 - \rho_p} = 0.$$

3.2 Finite subspaces dimension

For $n \in \mathbb{P}$ and $q \in \mathbb{C}$, denote B_n^q the transition matrix between $L_{n,I}^q$ $I \subseteq [n-1]$ and $\eta_{n,J}^q$ $J \subseteq [n-1]$ with coefficients given by Equation (1.3). Columns and rows are indexed by subsets I of $[n-1]$ sorted in reverse lexicographic order. A subset I is before subset J iff the word obtained by writing the elements of I in decreasing order is before the word obtained from J for the lexicographic order. The column indexed by the subset I corresponds to $L_{n,I}^q$ and the row indexed by J to $\eta_{n,J}^q$ (as a direct consequence B_n^q is the transpose of the similar matrix defined in [4]). For $n = 0$, assume B_0^q to be the empty matrix.

Example 2. For $n = 4$, the transition matrix B_4^q between L_I^q $I \subseteq 3$ and η_J^q $J \subseteq 3$ is given by

	$\emptyset$	$\{1\}$	$\{2\}$	$\{2,1\}$	$\{3\}$	$\{3,1\}$	$\{3,2\}$	$\{3,2,1\}$
B_4^q	$\emptyset$	1	1	1	1	1	1	1
	$\{1\}$	0	$q-1$	0	$q-1$	0	$q-1$	$q-1$
	$\{2\}$	0	0	$q-1$	$q-1$	0	$q-1$	$q-1$
	$\{2,1\}$	0	0	$-q$	$q-1$	0	$-q$	$q-1$
	$\{3\}$	0	0	0	0	$q-1$	$q-1$	$q-1$
	$\{3,1\}$	0	0	0	0	$q-1$	0	$q-1$
	$\{3,2\}$	0	0	0	$-q$	$-q$	$q-1$	$q-1$
	$\{3,2,1\}$	0	0	0	0	$-q$	$q-1$	$q-1$

Our goal is to compute the dimension of the kernel of B_n^q to get the dimension of the vector subspace $\mathcal{P}_n^p$ as

$$\dim \mathcal{P}_n^p = \text{rank } B_n^q = 2^{n-1} - \dim \ker B_n^q.$$

We show the following proposition.

Proposition 6. Let $n, p \in \mathbb{P}$ be two positive integers. We have

$$\dim \ker B_n^p = \begin{cases} \sum_{k=1}^p \dim \ker B_{n-k}^p & n > p \\ 0 & \text{for } n \leq p. \end{cases}$$

Proof. The second case is a direct consequence of the fact that the matrix B_n^p is invertible for $n \leq p$ (see [4]). To show the general recurrence, assume that $n > p$. As in [4], notice that the matrix B_n^q is block upper triangular. For each $k \in [n]$, let A_k^q denote the transition matrix from $L_{n,I}^q$ $I \subseteq [n-1], \max I = k-1$ to $\eta_{n,J}^q$ $J \subseteq [n-1], \max J = k-1$ (where $\max \emptyset = 0$); this actually does not depend on n . Note that A_k^q is a $2^{k-2} \times 2^{k-2}$ -matrix if $k \geq 2$, whereas A_1^q is a 1×1 -matrix. We have

$$B_n^q = \begin{pmatrix} A_1^q & * & * & \dots & * \\ 0 & A_2^q & * & \dots & * \\ 0 & 0 & A_3^q & \dots & * \\ 0 & 0 & 0 & \ddots & * \\ 0 & 0 & 0 & 0 & A_n^q \end{pmatrix}.$$

We have the following lemma:

Lemma 2. The matrices $(B_n^q)_n$ and $(A_n^q)_n$ satisfy the following recurrence relations (for $n \geq 1$ and $n \geq 2$, respectively):

$$B_n^q = \begin{pmatrix} B_{n-1}^q & B_{n-1}^q \\ 0 & A_n^q \end{pmatrix}, \quad A_n^q = \begin{pmatrix} q-1 & B_{n-2}^q & q-1 & B_{n-2}^q \\ -qB_{n-2}^q & q-1 & A_{n-1}^q \end{pmatrix}.$$

Consider for $n \geq 2$ and coefficients $\alpha, \beta \in \mathbb{C}$ the kernel of the matrix $\alpha A_{n-1}^q \quad \beta B_n^q$. Let X be a 2^{n-1} vector and denote X^1 and X^2 the two vectors of size 2^{n-2} such that $X = \begin{pmatrix} X^1 \\ X^2 \end{pmatrix}$. We compute

$$\begin{pmatrix} \alpha A_{n-1}^q & \beta B_n^q \end{pmatrix} X = 0 \Leftrightarrow \begin{cases} q-1 \alpha \quad \beta B_{n-1}^q X^1 & X^2 & 0 \\ -q\alpha B_{n-1}^q X^1 & q-1 \alpha \quad \beta A_n^q X^2 & 0 \end{cases} \quad (3.1)$$

Two cases arise from the previous equation. Either $q-1 \alpha \quad \beta \neq 0$ and

$$\begin{pmatrix} \alpha A_{n-1}^q & \beta B_n^q \end{pmatrix} X = 0 \Leftrightarrow \begin{cases} B_{n-1}^q X^1 & X^2 & 0 \\ \begin{pmatrix} q-1 \alpha \quad \beta A_n^q & q\alpha B_{n-1}^q \end{pmatrix} X^2 & 0 \end{cases} \quad (3.2)$$

or $q-1 \alpha \quad \beta = 0$ and

$$\begin{pmatrix} \alpha A_{n-1}^q & \beta B_n^q \end{pmatrix} X = 0 \Leftrightarrow q\alpha B_{n-1}^q X^1 = 0 \quad (3.3)$$

Consider the sequence of coefficients

$$\begin{aligned} & \begin{pmatrix} \alpha_0^q & \beta_0^q \end{pmatrix} = 0 \quad 1 \\ & \begin{pmatrix} \alpha_n^q & \beta_n^q \end{pmatrix} = \begin{pmatrix} \alpha_{n-1}^q & \beta_{n-1}^q \end{pmatrix} \begin{pmatrix} q-1 & q \\ 1 & 0 \end{pmatrix}, \text{ for } n \geq 0. \end{aligned}$$

Solving the recurrence we have for integer $n \geq 1$

$$\begin{aligned} \begin{pmatrix} \alpha_n^q & \beta_n^q \end{pmatrix} &= 0 \quad 1 \begin{pmatrix} q-1 & q \\ 1 & 0 \end{pmatrix}^n \\ &= (-1)^n \begin{pmatrix} 0 & 1 \\ -1 & q \end{pmatrix} \begin{pmatrix} n-1-q & -q \\ -n-q & q \end{pmatrix} \\ &= (-1)^n \begin{pmatrix} n-1-q & -q \\ -n-q & q \end{pmatrix}, \end{aligned}$$

where for integer i and complex number c recall that $i_c = 1 - c^i / 1 - c$. Finally notice that for integer $n \geq 1$

$$\begin{aligned} \alpha_n^q &= q-1 \quad \beta_n^q = (-1)^n \begin{pmatrix} n-1-q & -q \\ -n-q & q \end{pmatrix} \\ &= (-1)^n \begin{pmatrix} n-1-q & -q \\ -n-q & q \end{pmatrix}. \end{aligned}$$

Get back to the case $q = \rho_p$ for some positive integer $p \in \mathbb{P}$. As a direct consequence of the definition of ρ_p in [Definition 8](#), we have

$$\begin{aligned} p - 1 - \rho_p &= 0; \\ n - \rho_p &\neq 0, \quad 1 \leq n \leq p. \end{aligned}$$

As a result, if $q = \rho_p$, one may iterate the recurrence in Equation (3.1) p times with the case of Equation (3.2) and one more time to go to the case of Equation (3.3). Recall that $\beta_p^{\rho_p} = \rho_p - p - \rho_p \neq 0$ to conclude the proof. $\square$

Noticing that for $n > p - 1$, $2^{n-1} = 2^{n-2} + 2^{n-3} + \dots + 2^{n-p-1} + 2 \cdot 2^{n-p-2}$ we can deduce the rank of $B_n^{\rho_p}$ using [Proposition 6](#). We get

$$\text{rank} \left(B_n^{\rho_p} \right) = 2^{n-1} - \dim \ker B_n^{\rho_p} = \begin{cases} \sum_{k=1}^p \text{rank} \left(B_{n-k}^{\rho_p} \right) & \text{for } n > p, \\ 2^{n-1} & \text{for } 1 \leq n \leq p. \end{cases} \quad (3.4)$$

We conclude that the sequence of subspace dimensions $\mathcal{P}_n^p$ follows the same recurrence with the same initial conditions as the sequence of the numbers of p -extended peak sets s_n^p . [Theorem 2](#) follows.

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The BCFW Tiling of the Amplituhedron

Chaim Even-Zohar^{*1}, Tsviqa Lakrec⁺², Ran J. Tessler^{‡3}

¹*Faculty of Mathematics, Technion*

²*Institute of Mathematics, University of Zürich*

³*Department of Mathematics, Weizmann Institute of Science*

Abstract. The amplituhedron $\mathcal{A}_{n,k,A}$ is a geometric object, discovered by Arkani-Hamed and Trnka (2013) in the study of scattering amplitudes in quantum field theories. They conjecture that $\mathcal{A}_{n,k,A}$ admits a decomposition into tiles, which is based on a certain combinatorial structure. The tiles are images of BCFW positroid cells in the nonnegative Grassmannian $\text{Gr}_{k,n}^{\geq}$ that arise from the Britto–Cachazo–Feng–Witten recurrence (2005).

In a recent preprint we prove this conjecture. The full paper is available online at <https://arxiv.org/pdf/2112.02703>. In this extended abstract we review the amplituhedron and its BCFW tiling, and outline the main ideas from the proof.

1 The Amplituhedron

The amplituhedron $\mathcal{A}_{n,k,m}$ is the image of a positive map $\text{Gr}_{k,n}^{\geq} \xrightarrow{\tilde{Z}} \text{Gr}_{k,k-m}$ which “projects” the nonnegative real Grassmannian to a smaller Grassmannian. Here we target the fundamental problem of triangulating the amplituhedron into images of positroid cells of $\text{Gr}_{k,n}^{\geq}$. We first briefly review the definitions of all these objects.

The real Grassmannian $\text{Gr}_{k,n}$ is the variety of k -dimensional linear subspaces in $\mathbb{R}^n$. It is concretely represented as the quotient space $\text{Gr}_{k,n} = \text{GL}_k(\mathbb{R}) \backslash \text{Mat}_{k \times n}^*(\mathbb{R})$. This means full-rank $k \times n$ real matrices modulo row operations performed by invertible $k \times k$ matrices. For a representative $k \times n$ matrix C , every set $I \subseteq \{1, \dots, n\}$ of k columns defines a Plücker coordinate $P_I(C)$ as the determinant of the $k \times k$ minor corresponding to I . Using all Plücker coordinates, the Grassmannian embeds in the $\binom{n}{k} - 1$ -dimensional real projective space. The positive Grassmannian $\text{Gr}_{k,n}^>$ is the subset of points in the Grassmannian with all $P_I > 0$. Its closure is the nonnegative Grassmannian $\text{Gr}_{k,n}^{\geq}$, with all $P_I \geq 0$.

^{*}chaime@technion.ac.il. C.E. thanks the Lloyds Register Foundation’s Alan Turing Institute programme on Data-Centric Engineering for their support during part of the research.

[†]tsviqa@gmail.com. T.L. was supported by ERC 2020 grant HomDyn, grant no. 833423 and by SNSF grant Dynamical Systems, grant no. 188535.

[‡]ran.tessler@weizmann.ac.il. R.T., incumbent of the Lillian and George Lytle Career Development Chair, was supported by the ISF grant no. 335/19 and by a research grant from the Center for New Scientists of Weizmann Institute.

The theory of positivity for algebraic groups and partial flag varieties, and in particular the positive Grassmannian, was developed by Lusztig [34]. It was further studied by Rietsch [42, 41, 40], Marsh and Rietsch [35], Fomin and Zelevinsky [18, 19, 20], and Postnikov [38, 39]. Postnikov developed a rich combinatorial picture of the nonnegative Grassmannian and its cell decomposition into positroid cells. An open *positroid cell* $S \subseteq \text{Gr}_{k,n}^{\geq}$ comprises the points where a certain subset of the Plücker coordinates are positive, i.e., all C such that $P_I C > 0$ for $I \in M$ and $P_I C = 0$ for $I \notin M$, for some set M . These positroid cells admit explicit parametrizations by $[0, \infty)^d$ for some $d = \dim S$. They correspond to various combinatorial objects such as graphs, tableaux, and permutations, which play a role in this work. The positive Grassmannian bears relations and applications to diverse areas, including cluster algebras, tropical geometry, and integrable systems [43, 27, 28, 33, 44]. A recent application to scattering amplitudes in theoretical physics by Arkani-Hamed et. al. [5] motivates the definition of the amplituhedron by Arkani-Hamed and Trnka [11], which is the focus of this work.

The amplituhedron depends on an additional parameter $m \leq n - k$ and a real matrix $Z \in \text{Mat}_{n \times k+m}^>$. This notation means that Z is assumed to be a *positive* matrix, in the sense that each one of its $\binom{n}{k+m}$ maximal minors has a positive determinant. The right multiplication by Z induces a well-defined map $\tilde{Z} : \text{Gr}_{k,n}^{\geq} \rightarrow \text{Gr}_{k,k+m}$. In terms of representative matrices, the image of a point $C \in \text{Gr}_{k,n}^{\geq}$ is $CZ \in \text{Gr}_{k,k+m}$, and the image of a cell $S \subseteq \text{Gr}_{k,n}^{\geq}$ is $\tilde{Z} S = \{CZ : C \in S\}$. The tree *amplituhedron* is defined as the image of the entire nonnegative Grassmannian:

$$\mathcal{A}_{n,k,m} Z = \left\{ CZ : C \in \text{Gr}_{k,n}^{\geq} \right\} \subset \text{Gr}_{k,k+m}.$$

The amplituhedron is known to be a proper subspace of full dimension km . Since many of its structural properties do not seem to depend on the choice of Z , we often denote the amplituhedron by $\mathcal{A}_{n,k,m}$ and argue for any fixed Z . See [11, 5, 24, 13] for an exposition of the topic and physical background. The amplituhedron has been much studied in recent years [12, 25, 32, 23, 26, 31, 36, 37]. Generalizations to non-tree level or other physical theories have been considered [6, 3, 9, 8, 46, 7]. Its mathematical foundations are being studied under the name *positive geometries* [4, 30].

2 Tiling

The structure of the amplituhedron is less understood than that of the nonnegative Grassmannian. It is desirable to establish ways to subdivide $\mathcal{A}_{n,k,m}$ into simple pieces, in analogy to triangulations of a polytope for example. The following definition suggests that this can be achieved using homeomorphic images under Z of positroid cells of the appropriate dimension.

Definition 1 (Bao and He [12]). A *triangulation* of the amplituhedron $\mathcal{A}_{n,k,m}$ is a collection $\mathcal{T}$, of km -dimensional open positroid cells of the nonnegative Grassmannian $\text{Gr}_{k,n}^{\geq}$, that satisfies the following properties for every $Z \in \text{Mat}_{n \times k}^{> m}$:

- *Injectivity*: $S \rightarrow \tilde{Z} S$ is an injective map for every cell $S \in \mathcal{T}$.
- *Separation*: $\tilde{Z} S$ and $\tilde{Z} S'$ are disjoint for every two cells $S \neq S'$ in $\mathcal{T}$.
- *Surjectivity*: $\bigcup_{S \in \mathcal{T}} \tilde{Z} S$ is an open dense subset of $\mathcal{A}_{n,k,m} Z$.

The case $m = 4$ is the most relevant to physics, being applicable to scattering amplitudes in planar $\mathcal{N} = 4$ supersymmetric Yang–Mills theory (SYM). Recurrence relations for computing scattering amplitudes arise from the work of Britto, Cachazo, Feng, and Witten [14, 15]. The BCFW recurrence translates into a recursive definition of a collection of $4k$ -dimensional positroid cells in the nonnegative Grassmannian $\text{Gr}_{k,n}^{\geq}$. These cells correspond to different terms that add up to the scattering amplitude, see [11, 5]. A direct definition, based on a canonical way of applying BCFW, was given by Karp, Williams, and Zhang [26]. This collection is named the *BCFW cells* and denoted here by $\mathcal{BCFW}_{n,k}$ – see definitions in Section 3 below. It contains $\frac{1}{k-1} \binom{n-3}{k} \binom{n-4}{k}$ cells. The relation between scattering amplitudes and the amplituhedron is based on the following conjecture.

Conjecture 2 (Arkani-Hamed and Trnka [11]). *For every $k \geq 1$ and $n \geq k + 4$, the cells $\mathcal{BCFW}_{n,k}$ form a triangulation of the amplituhedron $\mathcal{A}_{n,k,4}$.*

In our full preprint [16] we prove this conjecture. We develop new machinery for analyzing how positroid cells map into the amplituhedron. We devise a scheme of procedures for recursively constructing subsets of the positive Grassmannian, including all BCFW cells. We derive distinctive characteristics of their images, such as functions having constant sign on a given cell, and keep track of their evolution as the construction proceeds. These techniques let us locate preimages in order to show injectivity, compare two cells to tell their images apart, and analyze boundaries between cells. We demonstrate our approach in the case of Conjecture 2, showing that the BCFW cells triangulate the $m = 4$ amplituhedron in the sense of Definition 1. Namely, we prove the following three properties for every $k \geq 1$, $n \geq k + 4$ and positive $Z \in \text{Mat}_{n \times k}^{> 4}$.

Theorem 3. *The map $S \rightarrow \tilde{Z} S$ is injective for every cell $S \in \mathcal{BCFW}_{n,k}$.*

Theorem 4. *The images $\tilde{Z} S$, $\tilde{Z} S'$ are disjoint for each two different cells $S, S' \in \mathcal{BCFW}_{n,k}$.*

Theorem 5. *The union of $\tilde{Z} S$ over $S \in \mathcal{BCFW}_{n,k}$ is an open dense subset of $\mathcal{A}_{n,k,4} Z$.*

Another outcome of our analysis is a characterization of the boundary of the $m = 4$ amplituhedron. We also show that this is a *good* triangulation, in the sense that every internal wall is an image of a boundary stratum between two BCFW cells. For every positive Z , we deduce that the interior of the amplituhedron $\mathcal{A}_{n,k,4} Z$ is homeomorphic to an open ball. We note that Galashin, Karp and Lam [22] proved that $\mathcal{A}_{n,k,m} Z$

is homeomorphic to a closed ball for some specific choice of a positive matrix Z , for every even m . See the full preprint [16, Corollaries 8.7-8.8 and Theorem 9.2] for the precise formulation of our results, which may be steps towards establishing a regular CW decomposition [21] for $\mathcal{A}_{n,k,4} Z$.

Many of the techniques we develop generalize to other values of m and other triangulations and families of positroid cells. These tools are useful for manipulating functions of the amplituhedron's coordinates, showing injectivity of the amplituhedron map, separating between cells, and boundary cancellations. We establish a high-level decomposition of the amplituhedron [16, Theorem 9.6], and in a future paper we obtain as a corollary more general triangulations from the different ways to apply the BCFW recursion.

Related Work: Since [Conjecture 2](#) was posed, triangulations of $\mathcal{A}_{n,k,m}$ have been studied for various values of m and k . Karp and Williams give a triangulation of $\mathcal{A}_{n,k,1}$ [25]. The case of $\mathcal{A}_{n,1,m}$ is a cyclic polytope in projective space [45]. Galashin and Lam introduce the parity duality, which relates triangulations of $\mathcal{A}_{n,k,m}$ with triangulations of $\mathcal{A}_{n,n-m-k,m}$ [23]. Karp, Williams and Zhang prove injectivity and separation in the special case $\mathcal{A}_{n,2,4}$ using domino forms and an exhaustive case analysis [26]. Bao and He prove a triangulation of $\mathcal{A}_{n,k,2}$ based on BCFW-like cells [12]. Parisi, Sherman-Bennett and Williams use twistor coordinates to establish many triangulations of $\mathcal{A}_{n,k,2}$, and relate them to triangulations of the hypersimplex [37]. For other related results, see recent surveys by Williams [48] and Lam [30].

Our methods are inspired by several ideas from previous works on [Conjecture 2](#). We use matrix operations similar to Lam [29] and Bao and He [12] in order to manipulate cells. Our basic ingredient in separating cells is the twistor coordinates, similarly to Parisi, Sherman-Bennett and Williams [37]. Our analysis of boundaries is reminiscent with that of Agarwala and Marcott [1]. We use the domino form by Karp, Williams, Zhang, and Thomas [26] to represent BCFW cells. In particular, we settle their [Conjecture A.7](#) – see [Theorem 8](#) in the next section.

3 BCFW Tiles

In the seminal work of Postnikov [38], several families of combinatorial structures are employed to define and study the cells of the nonnegative Grassmannian. A few objects that are identified with positroid cells are: certain bicolored planar networks, known as *plabic graphs*; certain 0/1-filled Young tableaux, known as $\oplus$ -*diagrams* [47]; and permutations with two kinds of fixed points, known as *decorated permutations*.

More combinatorial representations are available for the special class of BCFW cells. These are given by a particular family of plabic graphs in [5, with a certain rotation by two], and by pairs of *noncrossing lattice walks* in [26], describing additional equivalent

viewpoints using *binary trees*, *Dyck paths* and *domino matrices*. For our purposes in this extended abstract, it is convenient to define the BCFW cells by chord diagrams – yet another combinatorial representation, which is a variant of the *Wilson loop diagrams* used by Agarwala, Marin-Amat and Marcott [2, 1].

Definition 6. A *chord diagram* is a straight horizontal line in the plane, which contains n markers labeled $\{1, 2, \dots, n\}$ left-to-right, and k chords $c_1, c_2, \dots, c_k$ above that line, whose *tail* and *head* lie left-to-right on the line, such that:

- (a) No chord starts or ends on a marker.
- (b) No two chords intersect.
- (c) No chord starts before the marker 1 or ends after $n - 1$.
- (d) No two chords *start* on the same segment between two markers.
- (e) No chord starts and ends on the same segment, nor on adjacent segments.

See Figure 1 for an example of a chord diagram. The number of chord diagrams with n markers and k chords is the Narayana number $\frac{1}{k+1} \binom{n-3}{k} \binom{n-4}{k}$.

For every chord c_l we denote by $i_l, i_l - 1$ the segment that contains its tail, and by $j_l, j_l - 1$ the one that contains its head. We call c_l a *top* chord if it is not nested in another chord. Otherwise c_l is a child of the chord c_m which is immediately above it.

Karp, Williams, Zhang and Thomas [26, Appendix A] suggest representing the points in the BCFW cells by special matrices, whose rows are called *domino bases*. Here we redefine these matrices via their one-to-one correspondence to chord diagrams.

Definition 7. The *domino matrix* that corresponds to a given chord diagram with n markers and k chords is a $k \times n$ real matrix depending on $5k$ real variables. Order the chords left-to-right by their tails, and fill row l using c_l as follows.

- (a) Write the variables $\alpha_l, \beta_l, \gamma_l, \delta_l$ at the tail and head columns $i_l, i_l - 1, j_l, j_l - 1$.
- (b) If c_l is a top chord, write another variable ε_l at the last column n .
- (c) If c_l is a child of c_m , add $\varepsilon_l \alpha_m, \varepsilon_l \beta_m$ at the columns $i_m, i_m - 1$ of the parent's tail.
- (d) In the remaining columns, write zeros.

See Figure 1 for an example of a domino matrix. Note that every row is supported on five or six entries. This form is useful for analyzing the amplituhedron map.

In order to make sure that these domino matrices are nonnegative, we define the following set of *sign rules*. The tail variables α_l and β_l are always positive. The sign of the head variables γ_l and δ_l is -1 below c_l , depending on the number of chords below c_l . The sign of ε_l is -1^{k-l} if c_l is a top chord, and -1^{l-m-1} if it is a child of another chord c_m . Moreover, if a chord c_m and its child c_l end in the same segment between markers then $\delta_l / \gamma_l < \delta_m / \gamma_m$, and if a chord c_l starts in the segment where c_m ends then $\delta_m / \gamma_m < \beta_l / \alpha_l$. Unless stated otherwise, when considering a domino matrix we assume that it satisfies these rules.

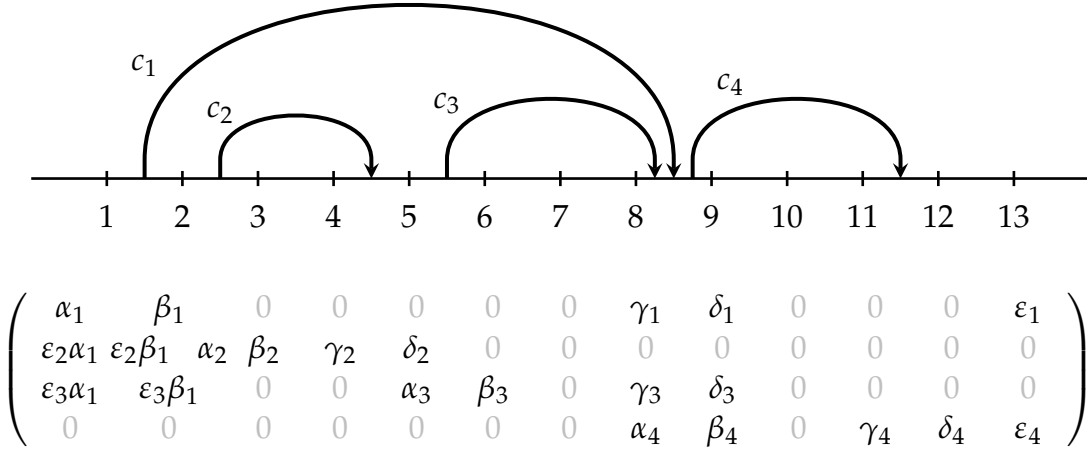


Figure 1: A chord diagram with $n = 13$ markers and $k = 4$ chords, and below it the corresponding domino matrix. The sign rules in this case give: $\alpha_i, \beta_i, \gamma_i, \delta_i > 0$ for all i ; $\varepsilon_2, \varepsilon_4 > 0$ while $\varepsilon_1, \varepsilon_3 < 0$; and $\delta_3/\gamma_3 < \delta_1/\gamma_1 < \beta_4/\gamma_4$.

We prove in [16] the following theorem, asserting that under the sign rules a domino matrix provides a one-to-one parametrization of a $4k$ -dimensional positroid cell in $\text{Gr}_{k,n}^{\geq}$. Moreover, we show that the positroid cells that correspond to all chord diagrams via these matrices are exactly the BCFW cells.

Theorem 8. *Every point in a BCFW cell has a representative domino matrix, which corresponds to the chord diagram associated with that cell. This representation is unique up to rescaling each row by a positive number. Conversely, every domino matrix that satisfies the sign rules represents a point in the corresponding BCFW cell.*

See [16, Theorem 3.37, Proposition 7.3] for a more detailed formulation of this result. For the purposes of this overview, assume Theorem 8 to define the BCFW positroid cells, as those arising from all n -marker k -chord diagrams via Definitions 6-7. For an example, see $\text{BCFW}_{7,2}$ in Figure 2. In [16, Sections 2-3] we show that this definition is equivalent to that of Karp, Williams, and Zhang [26, Section 5] who used plabic graphs based on Arkani-Hamed et al. [5, Section 16].

A key ingredient in the proof of Theorem 8 is a recursive algorithm that gradually constructs a domino matrix by a sequence of simple matrix operations that preserve positivity [16, Sections 3.1-3.2]. In short, these operations are: insertion of a zero column, insertion of a new row and a unit column, and adding a multiple of one column to an adjacent one, with a new positive variable as the coefficient. This explicit iterative construction allows us to apply induction on the structure of the chord diagram also in the subsequent proofs of Theorems 3, 4, and 5.

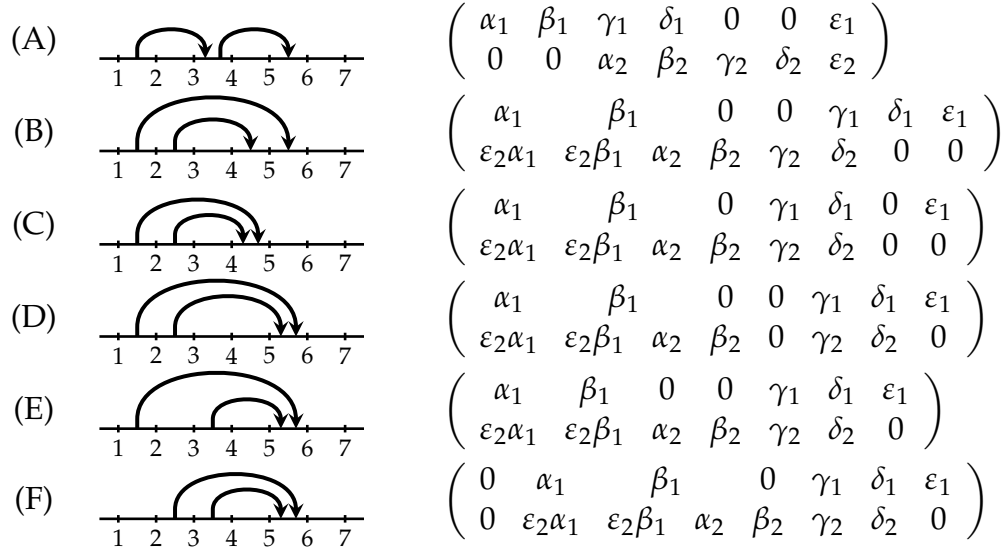


Figure 2: The full triangulation of the amplituhedron $\mathcal{A}_{7,2,4}$ with 6 tiles given as chord diagrams and domino matrices. By the sign rules, all variables are positive, except for ε_1 in tiles (A)-(F) and γ_1, δ_1 in tiles (B)-(F) which are negative. Also, $\delta_1/\gamma_1 < \beta_2/\alpha_2$ in tile (A), and $\delta_2/\gamma_2 < \delta_1/\gamma_1$ in tiles (C)-(F), where two chords occur in one segment.

4 Proof Overview

Theorems 3, 4, and 5 assert that the amplituhedron $\mathcal{A}_{n,k,4}$ is triangulated by the images of the BCFW cells, as characterized above using chord diagrams, domino matrices and Theorem 8. We now sketch the main tools and ideas that appear in the proofs.

We work with the *twistor coordinates* for the amplituhedron, which are a set of real-valued functions that take into account the positive matrix Z . They were introduced by Arkani-Hamed and Trnka [11], and used by Arkani-Hamed, Thomas and Trnka [10] to develop a combinatorial and topological picture of the amplituhedron. Parisi, Sherman-Bennett and Williams [37] used them to characterize the Z -images of a large family of positroid cells giving rise to triangulations of $\mathcal{A}_{n,k,2}$.

Definition 9. Given two matrices $Y \in \text{Mat}_{k \times k+4}$ and $Z \in \text{Mat}_{n \times k+4}$, the twistor of $a, b, c, d \in \{1, \dots, n\}$ is defined as the following $(k+4) \times (k+4)$ determinant:

$$\langle a b c d \rangle = \det \begin{pmatrix} Y_1 & Y_2 & \cdots & Y_k & Z_a & Z_b & Z_c & Z_d \end{pmatrix}$$

Here Y_i and Z_i denote rows in the matrices Y and Z . As usual, Z is the positive matrix that defines the amplituhedron map. The matrix Y represents a point in its image, which lies in $\text{Gr}_{k,k+4}$. Note that the twistors are defined on the amplituhedron up to global rescaling. Since Z and $Y = CZ$ are usually fixed and understood from the context, they are omitted from the twistor notation.

In section 4 of [16] we devise general tools for analyzing the *promotion* of twistors, which is the way they transform when the above-mentioned matrix operations are applied to C , or when a chord is added to the diagram. This machinery is instrumental in the proofs of Theorems 3-5, which rely on the recursive construction of domino matrices using a sequence of matrix operations and their effects on twistors.

Injectivity

Theorem 3 asserts that the amplituhedron map $\tilde{Z} : S \rightarrow \mathcal{A}_{n,k,4}$ is injective for each BCFW cell S . Our proof in [16, Section 5] goes by solving the inverse problem. Given a point in the image $\tilde{Z} S$ we find its unique preimage in S . By Theorem 8 above, it is sufficient to construct a domino matrix C , given the image CZ . This is done based on the hierarchical structure of the corresponding chord diagram, where the nonzero entries of C are recovered in parent-to-child order.

Every top chord c_l corresponds to a domino matrix row C_l supported on five entries $\alpha_l, \beta_l, \gamma_l, \delta_l, \varepsilon_l$, say in the respective columns $a, b, c, d, e \in \{1, \dots, n\}$. By linear algebra [16, Lemma 4.34], it follows that the two following vectors must be proportional:

$$\alpha_l, \beta_l, \gamma_l, \delta_l, \varepsilon_l \propto \langle b c d e \rangle, -\langle a c d e \rangle, \langle a b d e \rangle, -\langle a b c e \rangle, \langle a b c d \rangle$$

Thus we can recover the matrix row C_l unless all five twistors vanish at some point. We rule out the latter possibility based on the domino representation as well.

Now suppose that a chord c_l is a child of another chord c_m whose entries have already been recovered. Then, the row C_l might have six nonzero entries: $\varepsilon_l \alpha_m, \varepsilon_l \beta_m, \alpha_l, \beta_l, \gamma_l, \delta_l$ in columns $a, b, c, d, e, f \in \{1, \dots, n\}$. However, we already know the proportion between α_m and β_m from the parent's row. The contribution of the rows Z_a and Z_b is proportional to the combination $\alpha_m Z_a - \beta_m Z_b$. In the same way as before, we recover an “effective” preimage row of five entries, again verifying that they never vanish. This leads to expressions by twistors for all six entries, as demonstrated in the following example.

Example 10. Consider the BCFW cell (E) from Figure 2. A preimage of this domino form can be expressed using twistors as follows.

$$\begin{pmatrix} -\langle 2567 \rangle & \langle 1567 \rangle & 0 & 0 & -\langle 1267 \rangle & \langle 1257 \rangle & -\langle 1256 \rangle \\ \alpha_1 \langle 3456 \rangle & \beta_1 \langle 3456 \rangle & -\langle X456 \rangle & \langle X356 \rangle & -\langle X346 \rangle & \langle X345 \rangle & 0 \end{pmatrix}$$

where $\alpha_1 = -\langle 2567 \rangle$, $\beta_1 = \langle 1567 \rangle$ and we write formally $X = \alpha_1 1 - \beta_1 2$ and expand linearly. For example, the entry $-\langle X346 \rangle$ equals $\langle 2567 \rangle \langle 1346 \rangle - \langle 1567 \rangle \langle 2346 \rangle$.

As shown in the above example, our solution for the inverse problem expresses the preimage as a domino matrix, where each entry is a homogeneous polynomial in the twistor coordinates. We call such a polynomial in twistors a *functionary*.

Separation

Theorem 4 asserts that every two BCFW cells $S, S' \in \mathcal{BCFW}_{n,k}$ have disjoint images in the amplituhedron $\mathcal{A}_{n,k,4}$. We prove this by finding a *separating functionary*. This means a polynomial in the twistors that attains fixed and opposite signs on $\tilde{Z}^S$ and $\tilde{Z}^{S'}$, for every positive matrix Z . The functionary is combinatorially constructed by induction on the chord diagrams of S and S' . The proof that it separates relies on our study of twistor promotions under matrix operations. To illustrate these promotions, they look similar to **Example 10** where a twistor “transforms” into a quadratic functionary.

To give a taste of how the separation works, we sketch here the inductive definition of the separating functionary, and refer to the full preprint [16, Section 6] for more details and proofs, and to [17] for a Sage implementation. Compare the rightmost top chords of S and S' . Let $I = a, b, c, d, n$ and $I' = a', b', c', d', n$ be the respective supports of their domino rows. Then,

- If $d' < d$, take the following twistor as a separating functionary: $\langle a b c n \rangle$
- If $c, d = c', d'$ but $b \neq b'$, take the quadratic: $\langle a b c n \rangle \langle a' b' d n \rangle - \langle a b d n \rangle \langle a' b' c n \rangle$
- If $I = I'$ and the diagrams differ in the subdiagrams below this common chord, promote the separating functionary of the subdiagrams by plugging simultaneously:

$$d \mapsto \langle a b c n \rangle d - \langle a b d n \rangle c, \quad n \mapsto \langle a b c d \rangle n - \langle a b c n \rangle d - \langle a b d n \rangle c$$
- Otherwise, they differ in some chords to the left of the rightmost top chord, and then promote the separating functionary of these subdiagrams by substituting:

$$b \mapsto \langle a c d n \rangle b - \langle b c d n \rangle a$$

Surjectivity

Theorem 5 asserts that the closure of all BCFW images covers the amplituhedron. Our proof relies on a detailed analysis of the codimension-one adjacencies between cells.

First, we characterize each BCFW cell by inequalities on domino matrix terms and on 2-by-2 minors. We also show that every codimension-one boundary of a BCFW positroid cell either maps to a codimension-one stratum in the interior of $\mathcal{A}_{n,k,4}$ or to the topological boundary $\partial \mathcal{A}_{n,k,4}$. We then verify that every such interior boundary is shared by exactly two BCFW cells. These adjacencies can be characterized in combinatorial terms, such as shifting by one a head or a tail of one chord. For a simple example in **Figure 2**, one can see that $\delta_1 \rightarrow 0$ in (B) matches $\gamma_1 \rightarrow 0$ in (C). Other adjacency rules are a bit more complicated and carefully described in [16, Section 7]. In $\mathcal{A}_{7,2,4}$ they yield boundaries between: (D)-(E), (E)-(F), (B)-(C), (B)-(D), (B)-(E), (A)-(C), (A)-(E), (A)-(F).

The idea of the surjectivity proof in [16, Section 8] is topological – you cannot escape the image of the BCFW cells. Every path between two points in the interior can be perturbed to a transversal path that avoids codimension-two strata of BCFW images. Then,

as you travel along, whenever you exit one BCFW image, there is always another BCFW image from the other side, because every internal codimension-one stratum is shared by two disjoint cells. If there were an interior point outside the closures of all BCFWs, then we could have reached it by path-connectivity and obtain a contradiction.

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Pizza and 2-structures

Richard Ehrenborg^{*1}, Sophie Morel^{†2}, and Margaret Readdy^{‡1}

¹ *Department of Mathematics, University of Kentucky, Lexington, KY 40506, USA*

² *ENS de Lyon, Unité De Mathématiques Pures Et Appliquées, 69342 Lyon Cedex 07, France*

Abstract. Let $\mathcal{H}$ be a Coxeter hyperplane arrangement in n -dimensional Euclidean space. Assume that the negative of the identity map belongs to the associated Coxeter group W . Furthermore assume that the arrangement is not of type A_1^n . Let K be a measurable subset of the Euclidean space with finite volume which is stable by the Coxeter group W and let a be a point such that K contains the convex hull of the orbit of the point a under the group W . In a previous article the authors proved the generalized pizza theorem: that the alternating sum over the chambers T of $\mathcal{H}$ of the volumes of the intersections $T \cap K - a$ is zero. In this paper we give a dissection proof of this result. In fact, we lift the identity to an abstract dissection group to obtain a similar identity that replaces the volume by any valuation that is invariant under affine isometries. This includes the cases of all intrinsic volumes. Apart from basic geometry, the main ingredient is a previous theorem of the authors where we relate the alternating sum of the values of certain valuations over the chambers of a Coxeter arrangement to similar alternating sums for simpler subarrangements called 2-structures, introduced by Herb to study discrete series characters of real reduced groups.

Résumé. Soit $\mathcal{H}$ un arrangement d'hyperplans de Coxeter dans un espace euclidien V de dimension n . On suppose que $\mathcal{H}$ n'est pas de type A_1^n et que l'isométrie $x \mapsto -x$ sur V appartient à son groupe de Coxeter W . Soit K un sous-ensemble mesurable de mesure finie de V qui est stable par W , et soit a un point de V tel que K contienne l'enveloppe convexe de l'orbite de a sous W . Dans un article précédent, nous avons prouvé le "théorème de la pizza généralisé", qui dit que la somme alternée sur les chambres T de $\mathcal{H}$ du volume de $T \cap K - a$ est nulle. Dans cet article, nous donnons une preuve géométrique par dissection de ce résultat. Plus précisément, nous relevons l'identité ci-dessus dans un groupe de dissection abstrait, ce qui implique des identités similaires lorsque le volume est remplacé par n'importe quelle valuation (mesure

^{*}richard.ehrenborg@uky.edu. Partially supported by the Simons Foundation (#429370 to Richard Ehrenborg).

[†]sophie.morel@ens-lyon.fr. Partially supported by the LABEX MILYON (ANR-10-LABX-0070) of Université de Lyon, within the program "Investissements d'Avenir" (ANR-11-IDEX-0007) operated by the French National Research Agency (ANR), and by Princeton University.

[‡]margaret.readdy@uky.edu. Partially supported by the Simons Foundation (#422467 to Margaret Readdy).

finiment additive) invariante par isométries affines. Ceci inclut en particulier le cas des volumes intrinsèques sur les ensembles convexes compacts. Les ingrédients principaux de la preuve sont de la géométrie élémentaire et un théorème précédent des auteurs qui calcule la somme alternée des valeurs d’une valuation sur les chambres d’un arrangement de Coxeter en fonction de sommes alternées similaires associées à des sous-arrangements plus simples appelés 2-structures, introduits à l’origine par Herb pour étudier les caractères des séries discrètes des groupes réductifs réels.

Keywords: Coxeter arrangements, 2-structures, dissections, Pizza Theorem, reflection groups, intrinsic volumes, pseudo-root systems, Bolyai-Gerwien theorem

1 Introduction

The 2-dimensional pizza theorem is the following result: Given a disc in the plane, choose a point on this disc and cut the disc by $2k$ equally spaced lines passing through the point, where $k \geq 2$. The alternating sum of the areas of the resulting slices is then equal to zero. This was first proved by Goldberg [8]. Frederickson [7] gave a dissection proof based on dissection proofs of Carter-Wagon in the case $k = 2$ (see [4]) and of Allen Schwenk (unpublished) in the cases $k = 3, 4$. Frederickson deduced dissection proofs of a similar sharing result for the pizza crust and of the so-called calzone theorem, which is the analogue of the pizza theorem for a ball in $\mathbb{R}^3$ that is cut by one horizontal plane and by $2k$ equally-spaced vertical planes all meeting at one point in the ball.

To generalize the pizza problem, consider a finite central hyperplane arrangement $\mathcal{H}$ in $\mathbb{R}^n$ and fix a base chamber of this arrangement. Each chamber T has a sign -1^T determined by the parity of the number of hyperplanes separating it from the base chamber. If K is a measurable subset of $\mathbb{R}^n$ of finite volume, what can we say about the pizza quantity $\sum_T -1^T \text{Vol } T \cap K$, where the sum runs over all the chambers T of $\mathcal{H}$? The original pizza theorem is the case where $n = 2$, $\mathcal{H}$ has the type of the dihedral arrangement $I_2(2k)$ and K is a disc containing the origin. The calzone theorem is the case where $n = 3$, $\mathcal{H}$ has the type of the product arrangement $I_2(2k) \times A_1$ and K is a ball containing the origin.

The following generalization of the pizza and the calzone was proved in [6, Corollary 6.9] by analytic means. We also recently learned that Brailov had proved independently the same theorem for a ball and an arrangement of type B_n in [3], by somewhat similar methods.

Theorem 1.1 (Ehrenborg, Morel and Readdy). *Let $\mathcal{H}$ be a Coxeter arrangement with Coxeter group W that contains the negative of the identity map, denoted by $-\text{id}$. Assume that $\mathcal{H}$ is not of type A_1^n . Let K be a set of finite measure that is stable by the group W . Then for every point $a \in \mathbb{R}^n$ such that K contains the convex hull of $\{w \cdot a : w \in W\}$, we have*

$$\sum_T -1^T \text{Vol } T \cap K \cdot a = 0.$$

The proof of this result uses an expression for $\sum_T -1^T \text{Vol } T \cap K$ as an alternating sum of pizza quantities over subarrangements of $\mathcal{H}$ of the form $\{e_1^\perp, \dots, e_n^\perp\}$ with $e_1, \dots, e_n$ an orthonormal basis of $\mathbb{R}^n$, in other words, subarrangements that have type A_1^n .

In the paper [5], we study a different sum $\sum_T -1^T \nu(\bar{T})$, where ν is a valuation defined on closed convex polyhedral cones of $\mathbb{R}^n$ that takes integer values. Under the same condition that $\mathcal{H}$ is a Coxeter arrangement, we rewrite this quantity as an alternating sum of similar quantities for certain subarrangements of $\mathcal{H}$ that are products of rank 1 and rank 2 arrangements [5, Theorem 3.2.1], and then deduce an expression for it. These subarrangements, called *2-structures*, were introduced by Herb [11] to study characters of discrete series of real reductive groups. In fact, the identity of [5, Theorem 3.2.1] is valid for any valuation and its proof uses only basic properties of Coxeter systems and closed convex polyhedral cones.

In this paper we use the setting of 2-structures and [5, Corollary 3.2.4] (recalled in Theorem 2.5) to obtain a dissection proof of the higher-dimensional pizza theorem of [6, Corollary 6.9] that is independent of the results and methods of [6] (and of [3]):

Theorem 1.2 (Abstract pizza theorem; see Theorem 3.4.). *With the notation and hypotheses of Theorem 1.1, we have*

$$\sum_T -1^T [\bar{T} \cap K]_a = \sum_T -1^T [T \cap K]_a = 0,$$

where the brackets denote classes in the abstract dissection group of Definition 3.1.

As we take into account lower-dimensional sets when defining our abstract dissection group, this result implies generalizations of the higher-dimensional pizza theorem to all the intrinsic volumes (if K is convex). The idea of the proof of Theorem 1.2 is the following: by expanding the expression using 2-structures, we can reduce to a sum where each term is a similar expression for an arrangement that is a product of arrangements of types A_1 and $I_2^{2^k}$. We then adapt the dissection proof of Frederickson to an arrangement of type $I_2^{2m} \times \mathcal{H}'$. We also explain how to keep track of lower-dimensional regions of the dissection. If our product arrangement contains at least one dihedral factor, then its contribution is zero, and we immediately get a dissection proof of the result. However, if all the product arrangements that appear are of type A_1^n , then their individual contributions are not zero. We need one extra step in the proof to show that the contributions cancel. This step uses a slight refinement of the Bolyai-Gerwien theorem presented in Section 4. An interesting point to note is that the shape of the pizza plays absolutely no role in this proof, as long as it has the same symmetries as the arrangement and contains the convex hull of $\{w \cdot a : w \in W\}$. In particular, we no longer need to assume that it is measurable and of finite volume.

Let us mention some interesting questions that remain open:

- (1) The paper [6] proves the pizza theorem for more general arrangements (the condition is that the arrangement $\mathcal{H}$ is a Coxeter arrangement and that its number of hyperplanes is greater than the dimension n and has the same parity as that dimension), but only in the case of the ball; see [6, Corollary 7.6]. Is it possible to give a dissection proof of this result?
- (2) Mabry and Deiermann [16] show that the two-dimensional pizza theorem does not hold for a dihedral arrangement having an odd number of lines. More precisely, they determine the sign of the quantity $\sum_T -1^T \text{Vol } T \cap K$, where K is a disc containing the origin, and show that it vanishes if and only if the center of K lies on one of the lines. Their methods are analytic. As far as we know, there exists no dissection proof of this result either. The higher-dimensional case where $\mathcal{H}$ is a Coxeter arrangement and the number of its hyperplanes does not have the same parity as n also remains wide open.

2 Review of 2-structures and of the basic identity

Let V be a finite-dimensional real vector space with an inner product $\langle \cdot, \cdot \rangle$. For every $\alpha \in V$, we denote by H_α the hyperplane $\alpha^\perp$ and by s_α the orthogonal reflection in the hyperplane H_α .

We say that a subset Φ of V is a *normalized pseudo-root system* if:

- (a) Φ is a finite set of unit vectors;
- (b) for all $\alpha, \beta \in \Phi$, we have $s_\beta \alpha \in \Phi$ (in particular, taking $\alpha = \beta$, we get that $-\alpha \in \Phi$).

Elements of Φ are called *pseudo-roots*. The *rank* of Φ is the dimension of its span.

We call such objects *pseudo-root systems* to distinguish them from the root systems that appear in representation theory. If Φ' is a root system then $\Phi = \{\alpha/\|\alpha\| : \alpha \in \Phi'\}$ is a normalized pseudo-root system. Not every normalized pseudo-root system arises in this manner; see for instance the pseudo-root systems of type H_3 and H_4 .

We say that a normalized pseudo-root system Φ is *irreducible* if, whenever $\Phi = \Phi_1 \sqcup \Phi_2$ with Φ_1 and Φ_2 orthogonal, we have either $\Phi_1 = \emptyset$ or $\Phi_2 = \emptyset$. Every normalized pseudo-root system can be written uniquely as a disjoint union of irreducible normalized pseudo-root systems. Irreducible normalized pseudo-root systems are classified: they are in one of the infinite families A_n ($n \geq 1$), B_n/C_n ($n \geq 2$),¹ D_n ($n \geq 4$), I_2 ($m \geq 3$) or one of the exceptional types E_6 , E_7 , E_8 , F_4 , H_3 or H_4 , with types I_2 3 and A_2 isomorphic, as well as types I_2 4 and B_2 . (See [10, Chapter 5] or Table 1 in [1, Appendix A].)

¹The pseudo-root systems of types B_n and C_n are identical after normalizing the lengths of the roots.

We say that a subset $\Phi \subset \Phi$ is a *positive system* if there exists a total ordering $<$ on the $\mathbb{R}$ -vector space V such that $\Phi = \{\alpha \in \Phi : \alpha > 0\}$ (see [13, Section 1.3]). The *Coxeter group* of Φ is the group of isometries W of V generated by the reflections s_α for $\alpha \in \Phi$. This group preserves Φ by definition of a normalized pseudo-root system, and it acts simply transitively on the set of positive systems by [13, Section 1.4]. In particular, the Coxeter group W is finite.

Let E be a finite set of unit vectors of V such that $E \cap -E = \emptyset$. The corresponding *hyperplane arrangement* is the set of hyperplanes $\mathcal{H} = \{H_e : e \in E\}$. A *chamber* of $\mathcal{H}$ is a connected component of $V - \bigcup_{e \in E} H_e$; we denote by $\mathcal{T}(\mathcal{H})$ the set of chambers of $\mathcal{H}$. Fix a chamber T_0 to be the *base chamber*. For a chamber $T \in \mathcal{T}(\mathcal{H})$ we denote by $S(T, T_0)$ the set of $e \in E$ such that the two chambers T and T_0 are on different sides of the hyperplane H_e , and define the *sign* of T to be $-1^{|S(T, T_0)|}$.

We say that $\mathcal{H}$ is a *Coxeter arrangement* if it is stable by the orthogonal reflections in each of its hyperplanes. In that case, the set $\Phi = E \cup -E$ is a normalized pseudo-root system. We call its Coxeter group the Coxeter group of the arrangement. The map sending a positive system $\Phi \subset \Phi$ to the set $\{v \in V : \forall \alpha \in \Phi \quad v, \alpha > 0\}$ is a bijection from the positive systems in Φ to the chambers of $\mathcal{H}$. See for example [2, Chapitre V § 4 No. 8 Proposition 9 p. 99] and the discussion following it. Conversely, if $\Phi \subset V$ is a normalized pseudo-root system with Coxeter group W and $\Phi \subset \Phi$ is a positive system, then $\mathcal{H} = \{H_\alpha : \alpha \in \Phi\}$ is a Coxeter hyperplane arrangement, and in that case we always take the base chamber T_0 to be the chamber corresponding to Φ .

Let us define product arrangements. Let V_1 and V_2 be two finite-dimensional real vector spaces equipped with inner products, and suppose that we are given hyperplane arrangements $\mathcal{H}_1$ and $\mathcal{H}_2$ on V_1 and V_2 respectively. We consider the product space $V_1 \times V_2$, where the factors are orthogonal. The *product arrangement* $\mathcal{H}_1 \times \mathcal{H}_2$ is then the arrangement on $V_1 \times V_2$ with hyperplanes $H \times V_2$ for $H \in \mathcal{H}_1$ and $V_1 \times H'$ for $H' \in \mathcal{H}_2$. If $\mathcal{H}_1$ is the empty arrangement, then we write $V_1 \times \mathcal{H}_2$ instead of the confusing $\emptyset \times \mathcal{H}_2$. Similarly, if $\mathcal{H}_2$ is the empty arrangement, we write $\mathcal{H}_1 \times V_2$. If the arrangements $\mathcal{H}_1$ and $\mathcal{H}_2$ arise from normalized pseudo-root systems $\Phi_1 \subset V_1$ and $\Phi_2 \subset V_2$, then their product $\mathcal{H}_1 \times \mathcal{H}_2$ arises from the normalized pseudo-root system $\Phi_1 \times \{0\} \cup \{0\} \times \Phi_2 \subset V_1 \times V_2$. We also denote this pseudo-root system by $\Phi_1 \times \Phi_2$.

The notion of 2-structures was introduced by Herb for root systems to study the characters of discrete series representations; see for example the review article [11]. The definition we give here is Definition B.2.1 of [5]. It has been slightly adapted to work for pseudo-root systems.

Definition 2.1. Let Φ be a normalized pseudo-root system with Coxeter group W . A *2-structure* for Φ is a subset φ of Φ satisfying the following properties:

- (a) The subset φ is a disjoint union $\varphi = \varphi_1 \sqcup \varphi_2 \sqcup \cdots \sqcup \varphi_r$, where the φ_i are pairwise orthogonal subsets of φ and each of them is an irreducible pseudo-root system of

type A_1 , B_2 or I_2 2^k for $k \geq 3$.

(b) Let $\varphi = \varphi \cap \Phi$. If $w \in W$ is such that $w \cdot \varphi = \varphi$ then $\det w = 1$.

We denote by $\mathcal{T}(\Phi)$ the set of 2-structures for Φ .

Proposition 2.2. *Let Φ be a normalized pseudo-root system with Coxeter group W .*

(i) *The group W acts transitively on the set of 2-structures $\mathcal{T}(\Phi)$.*

(ii) *The pseudo-root system Φ and its 2-structures have the same rank if and only if there exists $w \in W$ whose restriction to $\text{Span}(\Phi)$ is equal to $-\text{id}_{\text{Span}(\Phi)}$.*

To each 2-structure $\varphi \subset \Phi$, we can associate a sign $\epsilon_\varphi \in \varphi, \Phi$ (see the start of Section 5 and Lemma 5.1 of [12] and Definition B.2.8 of [5]).

We next introduce the abstract pizza quantity. Let $\mathcal{H}$ be a central hyperplane arrangement on V . Let $\mathcal{C}_{\mathcal{H}}(V)$ be the set of closed convex polyhedral cones in V that are intersections of closed half-spaces bounded by hyperplanes H where $H \in \mathcal{H}$, and let $K_{\mathcal{H}}(V)$ be the quotient of the free abelian group $\bigoplus_{K \in \mathcal{C}_{\mathcal{H}}(V)} \mathbb{Z}K$ on $\mathcal{C}_{\mathcal{H}}(V)$ by the relation $K = K' + K \cup K' - K \cap K'$ for all $K, K' \in \mathcal{C}_{\mathcal{H}}(V)$ such that $K \cup K' \in \mathcal{C}_{\mathcal{H}}(V)$. For $K \in \mathcal{C}_{\mathcal{H}}(V)$, we still denote the image of K in $K_{\mathcal{H}}(V)$ by K .

Definition 2.3. Suppose that we have fixed a base chamber of $\mathcal{H}$. The *abstract pizza quantity* of $\mathcal{H}$ is

$$P(\mathcal{H}) = \sum_{T \in \mathcal{T}(\mathcal{H})} -1^T \bar{T} \in K_{\mathcal{H}}(V).$$

Remark 2.4. By Lemma 3.2.3 of [5], we have

$$P(\mathcal{H}) = \sum_{T \in \mathcal{T}(\mathcal{H})} -1^T T.$$

We use this alternative definition of $P(\mathcal{H})$ in our proofs.

The following result is Corollary 3.2.4 of [5]. It shows how to evaluate the pizza quantity for a Coxeter arrangement in terms of the associated 2-structures.

Theorem 2.5. *Let $\Phi \subset V$ be a normalized pseudo-root system. Choose a positive system $\Phi^+ \subset \Phi$ and let $\mathcal{H}$ be the hyperplane arrangement H_α $\alpha \in \Phi^+$ on V with base chamber corresponding to the positive system Φ^+ . For every 2-structure $\varphi \in \mathcal{T}(\Phi)$, we write $\varphi = \varphi \cap \Phi$ and we denote by $\mathcal{H}_\varphi$ the hyperplane arrangement H_α $\alpha \in \varphi$ with base chamber corresponding to φ . Then we have*

$$P(\mathcal{H}) = \sum_{\varphi \in \mathcal{T}(\Phi)} \epsilon_\varphi P(\mathcal{H}_\varphi).$$

If $\varphi \in \mathcal{T}(\Phi)$ then the closures of the chambers of $\mathcal{H}_\varphi$ are elements of $\mathcal{C}_{\mathcal{H}}(V)$, so $P(\mathcal{H}_\varphi)$ makes sense as an element of $K_{\mathcal{H}}(V)$.

3 A dissection proof of the higher-dimensional pizza theorem

Definition 3.1. Let $\mathcal{C} \subseteq V$ be a non-empty family of subsets of V that is stable by finite intersections and affine isometries and such that, if $C \in \mathcal{C} \subseteq V$ and D is a closed affine half-space of V , then $C \cap D \in \mathcal{C} \subseteq V$. Furthermore, we assume that $\mathcal{C} \subseteq V$ is closed with respect to Cartesian products, that is, if $C_i \in \mathcal{C} \subseteq V_i$ for $i = 0, 1$ then $C_0 \times C_1 \in \mathcal{C} \subseteq V_0 \times V_1$. For example, we could take $\mathcal{C} \subseteq V$ to be the family of all convex subsets of V , or of all closed (or compact) convex subsets, or of all convex polyhedra.

We denote by $K \subseteq V$ the quotient of the free abelian group $\bigoplus_{C \in \mathcal{C} \subseteq V} \mathbb{Z} C$ on $\mathcal{C} \subseteq V$ by the three relations: (i) $\emptyset = 0$; (ii) $C \cup C' = C \cap C' + C - C'$ for all $C, C' \in \mathcal{C} \subseteq V$ such that $C \cup C' \in \mathcal{C} \subseteq V$; and (iii) $g C = C$, for every $C \in \mathcal{C} \subseteq V$ and every affine isometry g of V . For $C \in \mathcal{C} \subseteq V$, we still denote the image of C in $K \subseteq V$ by C .

A *valuation* on $\mathcal{C} \subseteq V$ with values in an abelian group A is a function $\mathcal{C} \subseteq V \rightarrow A$ that can be extended to a morphism of groups $K \subseteq V \rightarrow A$.

Remark 3.2. Define $\mathcal{B} \subseteq V$ to be the relative Boolean algebra generated by $\mathcal{C} \subseteq V$, that is, the smallest collection of subsets of V that contains $\mathcal{C} \subseteq V$ and is closed under finite unions, finite intersections and set differences. Groemer's Integral Theorem states that a valuation on $\mathcal{C} \subseteq V$ can be extended to a valuation on the Boolean algebra $\mathcal{B} \subseteq V$; see [9] and also [15, Chapter 2]. Applying this to the valuation $C \mapsto C$ with values in $K \subseteq V$, we see that we can make sense of C for any $C \in \mathcal{B} \subseteq V$. For instance, we have $C_1 \cup C_2 = C_1 + C_2 - C_1 \cap C_2$ and $C_1 - C_2 = C_1 - C_1 \cap C_2$. Moreover if $\mathcal{C} \subseteq V$ is the set of all convex polyhedra in V , then $\mathcal{B} \subseteq V$ contains all polyhedra (convex or not), and also half-open polyhedra.

Next we have the following straightforward lemma, whose proof we omit, which states that the class symbol is well-behaved with respect to Cartesian products.

Lemma 3.3. *The two class identities $C_0 = D_0$ and $C_1 = D_1$ in $K \subseteq V_0$ and $K \subseteq V_1$ respectively imply that $C_0 \times C_1 = D_0 \times D_1$ in $K \subseteq V_0 \times K \subseteq V_1$.*

Let $\mathcal{H}$ be a central hyperplane arrangement on V with fixed base chamber. If $K \in \mathcal{C} \subseteq V$, we have a morphism of groups $e_K : K_{\mathcal{H}} \subseteq V \rightarrow K \subseteq V$ induced by the map $\mathcal{C}_{\mathcal{H}} \subseteq V \rightarrow \mathcal{C} \subseteq V, C \mapsto C \cap K$.

We denote by $P \subseteq \mathcal{H}, K$ the image of $P \subseteq \mathcal{H}$ by this morphism e_K ; in other words, we have

$$P \subseteq \mathcal{H}, K = \sum_{T \in \mathcal{T} \subseteq \mathcal{H}} -1^T \overline{T} \cap K = \sum_{T \in \mathcal{T} \subseteq \mathcal{H}} -1^T T \cap K,$$

where Remark 2.4 implies the second equality.

We state the main theorem of this paper. First for $u, v \in V$ define the half-open line segment $[u, v)$ by $\{ (1 - \lambda)u + \lambda v : 0 < \lambda \leq 1 \}$. Our main result is the following:

Theorem 3.4 (The Abstract Pizza Theorem). *Let $\mathcal{H}$ be a Coxeter hyperplane arrangement with Coxeter group W in an n -dimensional space V such that $-\text{id}_V \in W$. Let $K \in \mathcal{C}(V)$ and $a \in V$. Suppose that K is stable by the group W and contains the convex hull of the set $\{w \cdot a : w \in W\}$.*

1. *If $\mathcal{H}$ is not of type A_1^n , we have $P(\mathcal{H}, K, a) = 0$ in $K(V)$.*
2. *If $\mathcal{H}$ has type A_1^n , Φ is the normalized pseudo-root system corresponding to $\mathcal{H}$ and $\Phi = \{e_1, \dots, e_n\}$ where $\Phi^+ \subset \Phi$ is the positive system corresponding to the base chamber of $\mathcal{H}$, then the following identity holds:*

$$P(\mathcal{H}, K, a) = \left[\prod_{i=1}^n \langle 0, 2a, e_i \rangle e_i \right]. \quad (3.1)$$

Here we are using Remark 3.2 to make sense of the right-hand side of equation (3.1).

The conditions on K are satisfied if for example K is convex, contained in $\mathcal{C}(V)$, stable by W and $0 \in K - a$. Indeed, the last condition implies that $-a \in K$; as $-\text{id}_V \in W$ by assumption, this in turn implies that $a \in K$, hence that K contains the convex hull of the set $\{w \cdot a : w \in W\}$.

We will give the proof of Theorem 3.4 at the end of the section. This proof does not use Corollary 6.9 of [6], so we obtain a new proof of that result.

Let $V_0, \dots, V_n$ denote the intrinsic volumes on V (see [18, Section 4.2]).

Lemma 3.5. *Let $v_1, \dots, v_k$ be an orthogonal family of vectors in V . Then $V_i = 0$, $v_1 \times \dots \times v_k = 0$ for $0 \leq i \leq k - 1$.*

Corollary 3.6. *We keep the notation and hypotheses of Theorem 3.4. If $\mathcal{H}$ is not of type A_1^n , we have*

$$\sum_{T \in \mathcal{T}(\mathcal{H})} (-1)^{|T|} V_i(T \cap K - a) = 0 \quad (3.2)$$

for every $0 \leq i \leq n$, where K is assumed to be convex if $i \neq n$. If $\mathcal{H}$ has type A_1^n and K is convex then equation (3.2) holds for $0 \leq i \leq n - 1$.

Remark 3.7. Theorem 3.4 immediately implies generalizations to our higher-dimensional case of the “thin crust” and “thick crust” results of Confection 3 and Leftovers 1 of [16] for an even number of cuts.

We obtain the “thin crust” result by evaluating the $(n - 1)$ st intrinsic volume on $P(\mathcal{H}, K - a)$. Note that this result holds for a pizza of any (convex) shape and even in the case where we only make n cuts, where n is the dimension.

To generalize the “thick crust” result, consider two sets $K \subset L$ stable by W and in $\mathcal{C} \setminus V$. If $a \in V$ is such that K contains the convex hull of the set $\{w \cdot a : w \in W\}$, then

$$P(\mathcal{H}, L - K) \cdot a = P(\mathcal{H}, L) \cdot a - P(\mathcal{H}, K) \cdot a = 0,$$

so in particular

$$\sum_{T \in \mathcal{T}(\mathcal{H})} -1^T \text{Vol}(T \cap (L - K)) \cdot a = 0.$$

The case where K and L are balls with the same center is the “thick crust” result.

We now state some lemmas that will be used in the proof of [Theorem 3.4](#).

Lemma 3.8. *Let $\mathcal{H}_i$ be a hyperplane arrangement on V_i for $i = 0, 1$. Assume furthermore that $\mathcal{H}_1 = \{H_e\}_{e \in E_1}$ has type A_1^r and $\dim V_1 = r$. Let $E_1 = \{e_1, \dots, e_r\}$ be the index set of $\mathcal{H}_1$. Let $\mathcal{H}$ and V be the Cartesian products $\mathcal{H}_0 \times \mathcal{H}_1$ and $V_0 \times V_1$ respectively. Then for every $K \in \mathcal{C} \setminus V$ that is stable under the orthogonal reflections in the hyperplanes $V_0 \times H_{e_1}, \dots, V_0 \times H_{e_r}$ and for every $a \in V_1$, if $L = K \cdot a$, we have the identity*

$$P(\mathcal{H}, L) = P(\mathcal{H}_0 \times V_1, L \cap V_0 \times [0, 2] \cdot a, e_1 \cdot e_1 \times \dots \times [0, 2] \cdot a, e_r \cdot e_r),$$

where $\mathcal{H}_0 \times V_1$ is the product of $\mathcal{H}_0$ and the empty hyperplane arrangement on V_1 .

Lemma 3.9. *Suppose that we have $V = V_1^1 \times \dots \times V_1^r \times V_2^1 \times \dots \times V_2^s$, where the factors of the product are pairwise orthogonal, and that $\mathcal{H}$ is a product $\mathcal{H}_1^1 \times \dots \times \mathcal{H}_1^r \times \mathcal{H}_2^1 \times \dots \times \mathcal{H}_2^s$, where each $\mathcal{H}_i^j$ is a hyperplane arrangement on V_i^j . Suppose further that:*

- (a) *If $1 \leq j \leq r$ then V_1^j is 1-dimensional, and we have a unit vector e^j in V_1^j yielding the hyperplane arrangement $\mathcal{H}_1^j = \{0\}$.*
- (b) *If $1 \leq j \leq s$ then V_2^j is 2-dimensional, and the arrangement $\mathcal{H}_2^j$ is of type $I_2^{2m^j}$ for some $m^j \geq 2$.*

Let $a \in V$ and $K \in \mathcal{C} \setminus V$. Suppose that K is stable by the Coxeter group W and contains the convex hull of the set $\{w \cdot a : w \in W\}$. Then the following two statements hold:

- (i) *If $s \geq 1$ we have $P(\mathcal{H}, K) \cdot a = 0$ in $K \setminus V$.*
- (ii) *If $s = 0$ we have in $K \setminus V$ the identity*

$$P(\mathcal{H}, K) \cdot a = \left[[0, 2] \cdot a, e^1 \cdot e^1 \times \dots \times [0, 2] \cdot a, e^r \cdot e^r \right].$$

Proof of Theorem 3.4. Statement (ii) is exactly Lemma 3.9(ii). We now prove statement (i), so we assume that $\mathcal{H}$ is not of type A_1^n . By Theorem 2.5, we have

$$P(\mathcal{H}, K, a) = \sum_{\varphi \in \mathcal{T}(\Phi)} \epsilon(\varphi) P(\mathcal{H}_\varphi, K, a).$$

By definition, any 2-structure for Φ is of type $A_1^r \times \prod_{k \geq 2} I_2^{2^k s_k}$ with $\sum_{k \geq 2} s_k$ finite and, as W acts transitively on the set of 2-structures (Proposition 2.2(i)), the integers r and s_k , for $k \geq 2$, do not depend on the 2-structure but only on Φ . Also, by Proposition 2.2(ii), we have $\dim V = r + \sum_{k \geq 2} 2s_k$, so we are in the situation of Lemma 3.9. Suppose that $\sum_{k \geq 2} s_k \geq 1$. Then by Lemma 3.9(i) we have $P(\mathcal{H}_\varphi, K, a) = 0$ for every $\varphi \in \mathcal{T}(\Phi)$ and hence $P(\mathcal{H}, K, a) = 0$. Assume now that $\sum_{k \geq 2} s_k = 0$, that is, $s_k = 0$ for every k . Statement (ii) of the same lemma implies that

$$P(\mathcal{H}, K, a) = \sum_{\varphi \in \mathcal{T}(\Phi)} \epsilon(\varphi) \left[\prod_{e \in \varphi \cap \Phi} 0, 2^{-a, e} \right]. \quad (3.3)$$

This is an alternating sum of classes of half-open rectangular parallelotopes in V . So we can apply Theorem 4.1 to prove that $P(\mathcal{H}, K, a) = 0$ in $K[V]$. We know that $V_i P(\mathcal{H}, K, a) = 0$ if $0 \leq i \leq n-1$ by Lemma 3.5, so it remains to prove that $V_n P(\mathcal{H}, K, a) = 0$, that is, that the alternating sum of the volumes of the parallelotopes $\prod_{e \in \varphi \cap \Phi} 0, 2^{-a, e}$ is equal to zero. This follows from Corollary 6.9 of the paper [6]. However we can now give a direct proof (that does not use analysis) using the method of that corollary. Let $f : V \rightarrow \mathbb{R}$ be the function defined by

$$f(a) = \sum_{\varphi \in \mathcal{T}(\Phi)} \epsilon(\varphi) \prod_{e \in \varphi \cap \Phi} 2^{-a, e}.$$

Note that f is a polynomial homogeneous of degree n on V . Furthermore equation (3.3) implies that $\text{Vol}(P(\mathcal{H}, K, a)) = f(a)$ for every convex subset K of V of finite volume that is stable by W and every $a \in V$ such that $0 \in K \cap a$. The polynomial f satisfies $f(wa) = \det w f(a)$ for every $w \in W$ and every $a \in V$ (this is easy to see; see for example Corollary 2.3 of [6]), so it vanishes on every hyperplane of $\mathcal{H}$. But if $f \neq 0$, then the vanishing set of f must be of degree at most n , which contradicts the fact that, as $\mathcal{H}$ is not of type A_1^n , we have $|\mathcal{H}| > n$. Hence we must have $f = 0$, and this gives the desired result. $\square$

4 The Bolyai-Gerwien theorem for parallelotopes

The classical Bolyai-Gerwien theorem states that two polygons are scissors congruent if and only if they have the same area. There is also a well-known generalization in higher

dimensions that applies to parallelotopes; it follows from the characterization of translational scissors congruences in arbitrary dimensions, and was proved independently by Jessen-Sah and Sah, see the beginning of Section 7 of [14] or Theorem 1.1 in Chapter 4 of [17]. In this section, we state a slight refinement of this generalization, [Theorem 4.1](#), that keeps track of lower-dimensional faces; in other words, we do not want ignore the boundaries.

As in the previous sections, let V be an n -dimensional real vector space with an inner product $\langle \cdot, \cdot \rangle$. If $v_1, \dots, v_r$ is a linearly independent list of elements of V , we define the parallelotope $P(v_1, \dots, v_r) = \{\sum_{i=1}^r a_i v_i : 0 \leq a_i \leq 1\}$. We denote by $\mathcal{P}(V)$ the set of all convex polytopes in V (including lower-dimensional ones) and by $\mathcal{Z}(V)$ the subfamily of polytopes that are translates of parallelotopes of the form $P(v_1, \dots, v_r)$. The set $\mathcal{P}(V)$ satisfies the conditions of [Definition 3.1](#), so we can define an abelian group $K_{\mathcal{P}}(V)$ as in that definition. We denote by $K_{\mathcal{Z}}(V)$ the subgroup of $K_{\mathcal{P}}(V)$ generated by the classes $[P]$ for $P \in \mathcal{Z}(V)$. [Remark 3.2](#) implies that, if $\mathcal{P}_{\text{ext}}(V)$ is the relative Boolean algebra generated by $\mathcal{P}(V)$, then we can define the class $[P]$ in $K_{\mathcal{P}}(V)$ of any element P in $\mathcal{P}_{\text{ext}}(V)$. We denote by $\mathcal{Z}_{\text{ext}}(V)$ the set of elements P of $\mathcal{P}_{\text{ext}}(V)$ such that $[P] \in K_{\mathcal{P}}(V)$ is in the subgroup $K_{\mathcal{Z}}(V)$. For example, the set $\mathcal{Z}_{\text{ext}}(V)$ contains $\mathcal{Z}(V)$, and it also contains all half-open parallelotopes.

Let $V_0, \dots, V_n$ be the intrinsic volumes on V ; see [18, Section 4.2]. These are valuations on the set of all compact convex subsets of V , and in particular on $\mathcal{P}(V)$, so they induce morphisms of groups from $K_{\mathcal{P}}(V)$ to $\mathbb{R}$, that we still denote by $V_0, \dots, V_n$. Note that V_0 is the Euler-Poincaré characteristic with compact support, so the image of $K_{\mathcal{P}}(V)$ is $\mathbb{Z}$.

The main result of this section is the following isomorphism.

Theorem 4.1. *The morphism $(V_0, V_1, \dots, V_n) : K_{\mathcal{Z}}(V) \longrightarrow \mathbb{Z} \times \mathbb{R}^n$ is an isomorphism. In particular, if $P, P' \in \mathcal{Z}_{\text{ext}}(V)$ are such that $V_i(P) = V_i(P')$ for every $0 \leq i \leq n$, then $[P] = [P']$ in $K_{\mathcal{Z}}(V)$.*

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Robertson's Conjecture in Algebraic Topology

Ben Knudsen^{*1} and Eric Ramos^{†2}

¹*Department of Mathematics, Northeastern University, Boston, MA 02115*

²*Department of Mathematical Sciences, Stevens Institute of Technology, Hoboken, NJ 07030*

Abstract. One of the most famous results in graph theory is that of Kuratowski's theorem, which states that a graph G is non-planar if and only if it contains one of $K_{3,3}$ or K_5 as a topological minor. That is, if some subdivision of either $K_{3,3}$ or K_5 appears as a subgraph of G . In this case we say that the question of planarity is determined by a finite set of forbidden (topological) minors. A conjecture of Robertson, whose proof was recently announced by Liu and Thomas, characterizes the kinds of graph theoretic properties that can be determined by finitely many forbidden minors. In this extended abstract we will present a categorical version of Robertson's conjecture, which we have proven in certain cases. We will then illustrate how this categorification, if proven in all cases, would imply many non-trivial statements in the topology of graph configuration spaces.

Keywords: Configuration Spaces, Representations of Categories, Graph Well-Quasi-Orders, Representation Stability

1 Introduction

Let G denote a finite graph. If we view G as a simplicial complex, then one may define its **configuration space on n points** as the topological space

$$\mathcal{F}_n(G) = \{(x_1, \dots, x_n) \in G^n \mid x_i \neq x_j\}.$$

One also considers the quotient of this space by the coordinate-permuting action of the symmetric group $\mathfrak{S}_n$

$$\mathcal{UF}_n(G) = \mathcal{F}_n(G) / \mathfrak{S}_n.$$

This latter space is called the **unlabeled configuration space on n points**. The study of these spaces has seen a tremendous increase in activity due not only to their connections to physics and robotics [1, 16], but also for their surprising theoretical properties. For the purposes of this work, we take the time to point out the following collection of results from the literature of graph configuration spaces, spanning almost 20 years:

^{*}b.knudsen@northeastern.edu. Ben Knudsen was partially supported by NSF Grant DMS 1906174.

[†]eramos3@stevens.edu. Eric Ramos was partially supported by NSF Grant DMS 2137628.

- (Torsion Boundedness) For any graph G and any $n \geq 1$, the group $H_1(\mathcal{UF}_n(G))$ is either torsion free, or all torsion is 2-torsion. Moreover, it is torsion free if and only if G is planar [9].
- (Universality of generators for trees) If G is a tree, and $i, n \geq 1$, then $H^i(\mathcal{UF}_n(G))$ is generated by products of H^1 classes pushed forward along embeddings of star trees into G [6]. A star tree is a tree with only one vertex of degree above one.
- (Universality of planar generators for H_2) There exists a finite set of graphs, depending only on n , such that for any planar graph G , $H_2(\mathcal{UF}_n(G))$ is generated by push forwards of classes along embeddings of the members of this finite list into G [3].

Each of these results is remarkable in its uniformity. Namely, that one can conclude extremely powerful results about the general “shapes” of these homology and cohomology groups across infinite families of graphs by only understanding much simpler cases.

The purpose of the present abstract is to present a possible framework that unifies and expands upon all of these universality phenomena. This framework accomplishes this by proposing a kind of categorification (see [Conjecture 6](#)) of an extremely powerful and delicate theorem arising from structural graph theory - Robertson’s Conjecture (see [Theorem 2](#)).

The structure of this work proceeds as follows. To start, we outline the necessary graph theory background to describe Robertson’s conjecture. Following this, we turn to graph categories, and explain what is meant by a representation of a category as well as what a categorification of Robertson’s conjecture would necessarily look like. This second section ends by stating the technical heart of our aforementioned framework, [Conjecture 6](#). Finally, we conclude by showing how this technical theorem can be applied to graph configuration spaces, and why each of the above can be thought as arising from this application.

2 Robertson’s conjecture on topological minors

In this section, we outline the necessary background related with the underlying classical graph theory that motivates our study.

Definition 1. In this work, the terminology of **graph** will always refer to a finite *simple* graph (i.e. not permitting loops or multi-edges). A **path** in G is a sequence of vertices of G , $v_1, v_2, \dots, v_r$ such that no vertex is repeated, and each pair of subsequent vertices is connected by an edge, modulo the involution which reverses the beginning and ending point of the sequence.. If G is a graph we will write V_G, E_G , and P_G for the sets of

vertices, edges, and paths of G , respectively. In cases where the graph G is understood from context, we will often drop the subscript in these notation.

If G, G' are two graphs, then a **homeomorphic embedding** from G to G' is a pair $\rho = (\rho_V, \rho_E)$ of maps of sets

$$\rho_V : V_G \rightarrow V_{G'}, \quad \rho_E : E_G \rightarrow P_{G'}$$

satisfying the following:

1. ρ_V is an injection;
2. if $e \in E_G$ has endpoints $\{a, b\}$, then $\rho(e)$ is a path connecting a and b ;
3. if $e \in E_G$ and $v \in V_G$, $\rho_V(v)$ is a member of $\rho_E(e)$ if and only if v is an end point of e .
4. if $e_1, e_2 \in E_G$ are distinct, then $\rho(e_1)$ and $\rho(e_2)$ are disjoint if e_1 and e_2 do not share an endpoint, and otherwise only intersect at the image of this common endpoint under ρ_V ;

Note that a homeomorphic embedding of graphs induces a continuous simplicial embedding between the realizations of the relevant graphs as simplicial complexes. Finally, we write $G \leq_{top} G'$ to indicate that there exists a homeomorphic embedding of G into G' . We refer to the quasi-order $\leq_{top}$ as the **topological minor** relation.

Somewhat notoriously, the topological graph minor relation does permit infinite anti-chains. That is to say, it is not a **well-quasi-order**. This puts it in stark contrast to other orderings on graphs, such as the more commonly used **graph minor** relation, which was very famously proven to not have infinite anti-chains by Robertson and Seymour in [14]. One particular flavor of example of infinite anti-chain will be important enough for us to take some time and describe it in detail.

For $k \geq 1$, let R_k denote the graph which is obtained from a path with $k + 1$ vertices by doubling each edge, and then subdividing to do away with each multi-edge. For instance, the graph R_1 is a triangle, whereas R_2 is two triangles joined at a common vertex. Generally, we visualize R_k as a chain with precisely k links. One can easily build infinite anti-chains on the family $\{R_k\}$ by attaching certain graphs to both ends of the chain. For the simplest example, one can attach a trio of leaves to each sides of the chain. The following conjecture of Robertson essentially claims that the anti-chains arising in this way are the basis for all others. A stronger version of this conjecture that allows for vertex labels has recently had a proof announced by Liu and Thomas [10]

Theorem 2 (Robertson's Conjecture, [10]). *Fix $k \geq 1$, and let $\{G_i\}_{i \geq 1}$ be any infinite collection of graphs with the property that $R_k \not\leq_{top} G_i$ for any i . Then there exists $j < l$ such that $G_j \leq_{top} G_l$. In particular, for any fixed $k \geq 1$, the restriction of $\leq_{top}$ to the class of graphs not containing R_k as a topological minor is a well-quasi-order.*

Observe that, the condition of not containing R_1 as a topological minor is equivalent to being a tree. Robertson's conjecture therefore specializes to **Kruskal's Tree Theorem** in that case. Moreover, the class of graphs not containing R_2 as a topological minor includes all graphs with maximum vertex degree 3. Robertson's conjecture for $k = 2$ therefore implies a conjecture of Vázsonyi that these subcubic graphs do not admit infinite anti-chains in the topological graph minor relation.

3 Categories of graphs

The purpose of this extended abstract is to lay out a potential categorification of Robertson's conjecture, and illustrate how this categorification can be used to prove statements in the topology of graph configuration spaces. While the most general form of this categorification will remain conjectural, we will spend some time outlining what a proof of this conjecture might look like in principal, and also the cases which have already been proven. This work is very much inspired by similar work on categorifying the graph minor theorem and Kruskal's Tree Theorem [11, 12, 13], as well as other categorification efforts such as the theory of Gröbner categories due to Sam and Snowden [15].

Definition 3. Let $k \geq 1$ be a fixed integer. We write $\mathcal{TG}_k$ to denote the category whose objects are graphs not containing R_k as a topological minor, and whose morphisms are homeomorphic embeddings.

Based on everything we saw in the previous section, it should be clear that the categories $\mathcal{TG}_k$ are precisely those that one would want to study in any attempted categorification of Robertson's Conjecture. It therefore only remains to determine what such a categorification would "look like." If $(Q, \leq)$ is any quasi-order, then the condition of being a well-quasi-order is equivalent to saying that the complement of every $\leq$ -closed subset of Q has finitely many $\leq$ -minimal elements. Put another way, it says that containment in these sets is determined by the appearance, or lack-there-of, of some member of a finite list of elements. When thinking about such a characterization in situations arising in more algebraic contexts, one might draw parallels to some kind of finite generation. The condition of being a well-quasi-order then becomes akin to a kind of Noetherian Property. That is to say, that every submodule of a free module is itself finitely generated.

To make all of this precise, we therefore need to start by understanding what one even means by a module in this context.

Definition 4. For any commutative ring A , a $\mathcal{TG}_k$ -**module over A** , or just a $\mathcal{TG}_k$ -**module**, is a covariant functor $M : \mathcal{TG}_k \rightarrow A\text{-mod}$ from $\mathcal{TG}_k$ to the category of *finitely generated* A -modules.

We say that a $\mathcal{TG}_k$ -module M is **finitely generated** if there is a finite list of objects in $\mathcal{TG}_k$, $\{G_i\}$, such that for any object G of $\mathcal{TG}_k$ the A -module $M(G)$ is generated by the images of the maps $M(G_i) \rightarrow M(G)$ induced by maps $G_i \rightarrow G$ of $\mathcal{TG}_k$. In this case we call the objects G_i **generators** of M .

More concretely, a $\mathcal{TG}_k$ -module is a collection of A -modules, $\{M(G)\}_G$, one for every graph G not containing R_k as a topological minor, such that for any homeomorphic embedding $G \rightarrow G'$, one obtains a homomorphism $M(G) \rightarrow M(G')$, that respects composition. In slightly different language, a $\mathcal{TG}_k$ -module is a representation of the quiver with relations underlying the category $\mathcal{TG}_k$.

The notion of finite generation outlined above can also be viewed more concretely in a similar fashion. In the language of quiver representations it becomes the usual notion of finite generation. Heuristically speaking, saying that a $\mathcal{TG}_k$ -module is finitely generated tells one that all of the algebraic content of all of the modules $M(G)$ is determined by only a finite amount of data coming from the modules $M(G_i)$. We will use this heuristic reasoning in the following example.

Example 5. For each object G of $\mathcal{TG}_k$, set $M(G) = \mathbb{Z}$. If $G \rightarrow G'$ is a homeomorphic embedding, then we set the map $M(G) \rightarrow M(G')$ to be the identity map. Then M is a $\mathcal{TG}_k$ -module over $\mathbb{Z}$, and is also seen to be generated by the graph which is just a single vertex.

Now let S be a collection of objects in $\mathcal{TG}_k$ that is closed under the topological minor relation. We may also define a module M_S by the assignments,

$$M_S(G) = \begin{cases} \mathbb{Z} & \text{if } G \notin S \\ 0 & \text{otherwise.} \end{cases}$$

By consequence of the fact that S is minor closed, it follows that M_S is a submodule of M . One may think of M_S as being the submodule of M "generated by" the graphs in the complement of S . The question now becomes whether M_S is also finitely generated.

For any G , the algebraic information encapsulated by $M_S(G)$ precisely encodes whether or not G is a member of S . Therefore, to say that M_S is finitely generated is to say that there is a finite list of graphs that determines containment in S . This is precisely the statement of Robertson's [Theorem 2](#).

We are now ready to state the main technical conjecture of this work.

Conjecture 6 (The Categorical Robertson's Conjecture). *Fix $k \geq 1$. If A is a Noetherian commutative ring, and M is a finitely generated $\mathcal{TG}_k$ -module, then any submodule of M is also finitely generated.*

This conjecture represents the next step in a research program that originated in Sam and Snowden's seminal work [15]. That paper outlines a method by which one can

convert purely combinatorial theorems about well-quasi-orders into algebraic statements about Noetherian properties arising in the representation theory of certain associated categories. For instance, Sam and Snowden categorified Higman’s lemma - that sequences are well-quasi-ordered by the subword relation - in the original work [15], whereas later work of Barter [4] as well as Proudfoot and the second author [12, 13], do something similar for Kruskal’s tree theorem. In [11], Miyata and the second author provide a categorical version of a weakened graph minor theorem as well. From this pre-existing literature we can therefore already say the following in relation to our conjecture.

Theorem 7 (Barter [4]; Proudfoot & Ramos [12, 13]). *If A is a Noetherian commutative ring, and M is a finitely generated $\mathcal{TG}_1$ -module, then any submodule of M is also finitely generated.*

In the originating work [8], we provide an outline of Sam and Snowden’s machinery, and indicate how one would use it to possibly prove our main conjecture. Importantly, this outline argues what one would need to understand about the proof of Theorem 2 to ultimately prove Conjecture 6. To avoid bogging us down with these technicalities in this extended abstract, however, we instead opt to move on to display how the Categorical Robertson’s Conjecture has exciting applications to topology.

Finally, before finishing this section, we note that just as Robertson’s Theorem 2 has a stronger labeled version, so does the categorical Robertson’s Conjecture. It is also the case that this labeled version of the conjecture has already been proven for $k = 1$. Because our primary applications to configuration space do not use the extra data of vertex labels, we have not presented this version here, though it should be noted that [8] use the labeled version of the conjecture to prove a variety of facts about topologies associated to **cographs**. See [8] for more details on this.

4 Graph configuration spaces

In this section we detail some applications of Conjecture 6 to graph configuration spaces, as defined at the beginning of this abstract. The reader should also keep in mind throughout that Conjecture 6 is actually a theorem in the case where $k = 1$, and so everything that follows can be seen as unconditional in that case.

The applications to graph configuration spaces in the present work are fundamentally related with the observation that if $\rho : G \rightarrow G'$ is a homeomorphic embedding, then one has a continuous map from the (labeled or unlabeled) configuration space of G to that of G' . Our goal will be to show that these maps are essentially the mechanisms underlying stability phenomena in the homology groups of these spaces.

To start, we recall the following important construction of Abrams.

Definition 8. Let G be a graph. The **discretized configuration space of G on n points** $D_n(G)$ is the sub-complex of G^n of all cells $\sigma_1 \times \sigma_2 \times \dots \times \sigma_n$, such that for any $i \neq j$,

the endpoints of σ_i are distinct from those of σ_j . As before, $\mathfrak{S}_n$ acts on $D_n(G)$, and we define the **unlabeled discretized configuration space of G on n points** to be the quotient $UD_n(G) = D_n(G)/\mathfrak{S}_n$.

This particular discretization of configuration space is not the only one that has appear in the literature (see, for instance [17]). Generally, the biggest issue with D_n is that it has far more cells than is practical to work with computationally. Because of this, other authors have applied techniques from discrete Morse theory [7] to try to trim it down to something that is more amenable to computer computation. Because our interests are ultimately more on the theoretical side, however, this overabundance of cells will not be an issue for us. Moreover, we note that $D_n(G)$ behaves relatively nicely when one subdivides the edges of G . This will be critical for us. For now we state the following theorem that reinforces our interest in these discretized spaces.

Theorem 9 (Abrams [2]). *Fix $n \geq 2$ and let G be a graph satisfying the **path condition**, that for any two vertices of G , x, y , with degree not equal to 2, every path from x to y has length at least $n + 1$. Then there exists an $\mathfrak{S}_n$ -equivariant homotopy equivalence*

$$D_n(G) \sim \mathcal{F}_n(G).$$

Notice that while the number of cells of $D_n(G)$ is sensitive to subdivision, the configuration space $\mathcal{F}_n(G)$ is not. For this reason the path condition is not going to hinder us too much.

For the remainder of this section we will present our results entirely in terms of $D_n(G)$, despite the fact that analogous statements will hold for the unordered spaces as well. We do this just for expositional clarity. In all cases, the proofs for the unlabeled case are identical.

Observe that any homeomorphic embedding $\rho : G \rightarrow G'$ can be realized as a composition of a map that purely subdivides G , followed by a map that embeds this subdivision into G' . We have already noted that the Abrams model respects subdivision, and it clearly does so for graph embeddings. It follows that if $\rho : G \rightarrow G'$ is a homeomorphic embedding, and $n \geq 2$ is fixed, then one has a cellular map between cubical complexes $D_n(G) \rightarrow D_n(G')$. In particular, one has induced maps on the level of cubical chains. Writing $\mathcal{D}_{n,G,\bullet}$ for the chain complex whose homology computes the homologies of $D_n(G)$, we have just argued for the following lemma.

Lemma 10. *The assignment*

$$G \mapsto \mathcal{D}_{n,G,i}$$

can be extended to a $\mathcal{TG}_k$ – module over $\mathbb{Z}$. Moreover, varying both i and G , $\mathcal{D}_{n,\star,\bullet}$ defines a complex of $\mathcal{TG}_k$ -modules.

Our way forward now becomes clear. All we really need to do is argue that the $\mathcal{TG}_k$ -modules $G \mapsto \mathcal{D}_{n,G,i}$ are finitely generated. The conclusion of [Conjecture 6](#) would then imply the same about the homology groups of the discretized configuration spaces essentially for free! Luckily, finite generation is actually not hard to prove in this particular instance.

If we fix $k, n \geq 1$, and $i \geq 0$, and let $G_{i,n-i}$ be the (disconnected) graph which is a disjoint union of i line segments and $n - i$ isolated vertices, then $\mathcal{D}_{n,\star,i}$ is generated by $\mathcal{D}_{n,G_{i,n-i},i}$. Indeed, for any graph G , any i -cell of $D_n(G)$ can be written as $\sigma_1 \times \dots \times \sigma_n$, where the σ_j are either edges or vertices of G that do not intersect each other at their endpoints. The data of which edges and vertices are present in this cell induces an embedding from $G_{i,n-i}$ to G .

Moving from the homologies of the discretized configuration spaces back to the homologies of the usual configuration spaces is now just a matter of being careful about the path condition.

Theorem 11. *Let $i \geq 0$ and $k, n \geq 1$ be fixed. Assuming [Conjecture 6](#), the $\mathcal{TG}_k$ -module $H_i(\mathcal{F}_n(\bullet))$ is finitely generated.*

Proof. Let $\mathcal{H}$ denote the $\mathcal{TG}_k$ -submodule of $H_i(D_n(\bullet))$ generated by all $H_i(D_n(G))$, where G is a graph satisfying the path condition. By [Conjecture 6](#), we know that $\mathcal{H}$ must be generated by some finite list of graphs $\{G_j\}$, which we may assume all satisfy the path condition. Then $H_i(\mathcal{F}_n(\bullet))$ is generated by this same finite list, where one smooths away any vertices of degree 2. $\square$

Example 12. As mentioned at the beginning of this section, the above theorem is unconditionally true in the case $k = 1$. The category $\mathcal{TG}_1$ is equivalent to the category of trees with homeomorphic embeddings. In this case, work of Chettih and Lütgehetmann [\[5\]](#) implies that $H_1(\mathcal{F}_n(\bullet))$ is generated by the trees that look like the letters Y and H. They also prove similar statements for the higher homologies as well. Other than this theorem, however, very little is known beyond some partial results. This is especially the case for labeled configuration spaces.

Looking at the unlabeled case, a recent theorem of An and the first author [\[3\]](#) says that if you look at the full subcategory of $\mathcal{TG}_k$ of *planar* graphs, then the module

$$H_2(U\mathcal{F}_3(\bullet))$$

is generated by the following list of graphs:

- two disjoint triangles;
- the disjoint union of a triangle and a graph that looks like the letter Y;
- the disjoint union of two graphs that look like the letter Y;

- The theta graph of two vertices connected by four (subdivided) edges.

Importantly, one should observe that all of these generators can already be found in $\mathcal{TG}_2$. In other words, one does not obtain new exotic generators as k increases beyond this point. This observation motivates the next direction of our study: To what extent does the presence of the Robertson chains impact the generating set of the homology of graph configuration space?

To make sense of this question, we must first view our categories $\mathcal{TG}_k$ not as separate entities, but rather as members in a coherent family.

Definition 13. Let $\mathcal{TG}$ denote the category of all graphs with homeomorphic embeddings, and for each $k \geq 1$ let ι_k denote the embedding $\iota_k : \mathcal{TG}_k \rightarrow \mathcal{TG}$. The functor ι_k induces a pair of adjoint functors

$$\begin{aligned} (\iota_k)_! : \mathcal{TG}_k\text{-mod} &\rightarrow \mathcal{TG}\text{-mod}, \\ (\iota_k)^* : \mathcal{TG}\text{-mod} &\rightarrow \mathcal{TG}_k\text{-mod}. \end{aligned}$$

To be more specific, $(\iota_k)^*$ is the **restriction** functor, while $(\iota_k)_!$ is **extension**.

The k th Robertson submodule of $H_i(\mathcal{F}_n(\bullet))$, denoted $R_k H_i(\mathcal{F}_n(\bullet))$, is defined by

$$(\iota_k)_! \iota_k^* H_i(\mathcal{F}_n(\bullet)) \rightarrow H_i(\mathcal{F}_n(\bullet)).$$

In other words, $R_k H_i(\mathcal{F}_n(G))$ is the subgroup of $H_i(\mathcal{F}_n(G))$ obtained by pushing forward classes from the topological minors of G containing no embedded copy of R_k .

For instance, in the simplest case, the group $R_1 H_i(\mathcal{F}_n(G))$ is that which is generated by pushforwards of homology classes coming from the sub-trees of G .

We refer to the resulting filtration of $H_i(\mathcal{F}_n(\bullet))$ as the *Robertson filtration*. For any fixed graph G , the associated filtration of $H_i(\mathcal{F}_n(G))$ is exhaustive as it will stabilize no later than when k matches the largest Robertson chain present in G . What is more interesting is that, for each fixed homological index i , all evidence suggests that the entire Robertson filtration stabilizes.

Conjecture 14. For every $i \geq 0$, there exists an integer $g_i \geq 0$ such that for any $n \geq 1$ we have $R_j H_i(\mathcal{F}_n(\bullet)) = R_{j+1} H_i(\mathcal{F}_n(\bullet))$ whenever $j > g_i$.

In the originating work [8], the authors provide a proof of this fact conditional on a categorical version of the graph minor theorem. There is also some hope of an unconditional argument that instead uses well known spectral sequences relating configuration spaces of a graph to that of the graphs obtained through vertex explosion. In either case, an unconditional proof of the above conjecture would be a critical step in understanding the structure of the homology groups of graph configuration spaces.

As one nice consequence of the boundedness of the Robertson filtration, we can see where the bounded torsion theorem from the introduction is coming from.

Theorem 15. Fix $i, n \geq 1$. Assuming [Conjectures 6 and 14](#), there exists an integer $d_{i,n}$, such that for any graph G , any torsion appearing in $H_i(\mathcal{F}_n(G))$ has order dividing $d_{i,n}$.

Proof. To start, consider the $\mathcal{TG}_k$ -module $H_i(\mathcal{F}_n(\bullet))$. One may take the submodule generated by all torsion classes appearing among all of the homology groups. This forms a $\mathcal{TG}_k$ -submodule, as the torsion classes are sent to torsion classes by the induced maps. By the Categorical Robertson’s Conjecture, this submodule must be finitely generated. By taking the least common multiple of the exponents of the generating homologies, we obtain an integer $d_{i,n,k}$ that all torsion appearing in the $\mathcal{TG}_k$ -module must have order dividing. To remove the dependence on k , one can apply [Conjecture 14](#). $\square$

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Birational rowmotion over noncommutative rings

Darij Grinberg^{*1} and Tom Roby⁺²

¹*Department of Mathematics, Drexel University, Philadelphia, USA*

²*Department of Mathematics, University of Connecticut, USA*

Abstract. We study the dynamics of birational rowmotion over an arbitrary noncommutative ring $\mathbb{K}$. This generalizes the birational rowmotion map in the commutative setting, which itself lifts the well-studied combinatorial rowmotion map on a finite poset. When the underlying poset P is a rectangle (i.e., a product of two chains), this operation has “twisted periodicity” and “reciprocity” properties, surprisingly similar to the commutative setting. We briefly outline proofs of these results (details are on the arXiv) and discuss extensions and variants. In particular, we conjecture similar results for the case when P is a Δ - or ∇ -shaped triangle or a trapezoid. We also conjecture that the results remain valid when $\mathbb{K}$ is a semiring. We further prove some elementary properties of birational rowmotion for general P , and (for the sake of exposition) discuss connections to the octahedron recurrence and Zamolodchikov periodicity (which are not new, but deserve better circulation).

Keywords: birational rowmotion, birational combinatorics, toggles, dynamical algebraic combinatorics, rowmotion, Zamolodchikov periodicity, posets

1 Introduction

Let P be a finite poset, and $\mathcal{J} P$ be the set of order ideals of P . *Combinatorial rowmotion* is an invertible map $\rho : \mathcal{J} P \rightarrow \mathcal{J} P$ which takes each ideal $I \in \mathcal{J} P$ to the order ideal generated by the minimal elements of the complement of I in P . It can also be viewed as a map on the antichains $\mathcal{A} P$ of a poset, via the well-known bijection $\mathcal{J} P \rightarrow \mathcal{A} P$ that sends an order ideal to its set of maximal elements. Rowmotion in this sense was studied in a sporadic sequence of papers by Brouwer and Schrijver [2], and Cameron and Fon-der-Flaass [3], who realized it as a product of “toggling” involutions, one for each element of the poset. Among the core results of these works were formulas for the *order* of ρ for special types of posets, such as products of chains.

In 2012, Striker and Williams [18] defined a related notion of *order-ideal promotion* π , which generalizes some cases of *Schützenberger promotion* on semistandard tableaux. Both ρ and π live in the *toggle group* of P (the group of permutations of $\mathcal{J} P$ generated by the “toggling” involutions), and are conjugate in it, which allows properties of π to

^{*}darijgrinberg@gmail.com.

[†]tom.robby@uconn.edu.

be derived from properties of ρ and vice versa. Armstrong, Stump, and Thomas [1] identified rowmotion for *root posets* with the *Kreweras complementation map* studied by Lie theorists. This rekindled interest in ρ ; a flurry of activity followed, opening up what is now known as *dynamical algebraic combinatorics*. We refer to [18, 17, 10] for some snapshots of progress in this field.

A major avenue of study have been “liftings” (i.e., generalizations) of ρ and π . The map ρ has been lifted three times successively:

- first, to a *piecewise-linear rowmotion* ρ_P on the *order polytope* $\mathcal{O}(P) = \{f : P \rightarrow [0, 1] : f \text{ is order-preserving}\}$ (see [4]),
- then by “detropicalization” to a *birational rowmotion* map R (also known as ρ_B) defined on vertex-labelings of P by rational functions (in commuting variables), as detailed by Einstein and Propp [4],
- and finally to *noncommutative birational rowmotion*, where the rational functions are replaced by elements of an arbitrary ring.

A priori, there was no reason to expect periodicity at all for these lifted maps; indeed, already ρ_P can have infinite order even if the poset P is rather “nice”. However, in [7, 6, 16], it was shown that for a few classes of posets (e.g., root posets and minuscule posets associated with finite-dimensional Lie algebras and *skeletal posets*, built up inductively as a special kind of series-parallel poset), the order of birational rowmotion R is finite and surprisingly small — generally the **same** as for combinatorial rowmotion ρ . For the case where $P = p \times q$ is a product of two chains, proofs of periodicity (i.e., finite order) of R took significant effort [6, 15]. The result was extended to all minuscule posets by Okada [16] in a type-by-type way (with some computer algebra), and the conceptual reasons for the periodicity remain a mystery, although Garver, Patrias and Thomas [5] have found an explanation for the periodicity of the (less general) ρ_P using quiver representations. Periodicity of R for all root posets of coincidental types is still an open question [9, Conj. 4.40], as a proof for “trapezoid-shaped” posets [6, Conj. 75] has not been found.¹

The noncommutative realm remained unexplored until recently. First results were obtained by Joseph and Roby in [13, 14], but (as with any of the three levels of generality) some of the good behavior of R is lost when allowing noncommutativity. Our initial conjecture from 2014 was that the periodicity property of R on $p \times q$ remains true, up to a (predictable and simple) “twist”. This resisted a number of different proof attempts until we finally resolved it [8].

¹To keep the story simple, we are limiting our discussion to ρ (as opposed to π) and to periodicity (as opposed to other properties, such as *homomesies*). The promotion map π has also been lifted (along with the entire toggle group) to the birational and even to the noncommutative realm [12, 13, 14]. Much of the work has been focusing on the antichain variant of rowmotion, which however is equivalent to the order-ideal variant ρ via the *transfer map*.

By well-understood reductions, our proof perforce establishes periodicity of ρ_P and ρ as well (for $P = p \times q$). We feel that this proof is more elementary and simpler than any that we have seen up until now at any level above the combinatorial. This abstract summarizes the results of [8] and briefly outlines the main steps of their proofs, sets them in context and adds some newer observations.

2 Definitions and examples

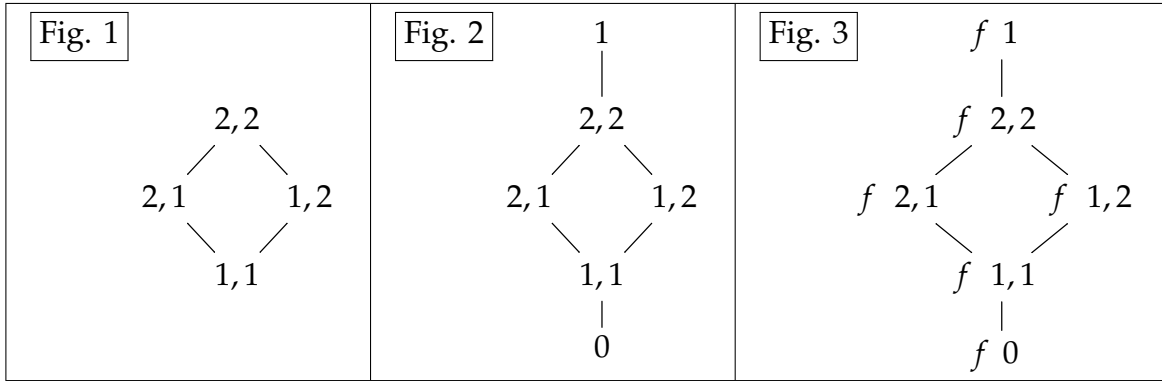
Throughout this work, P denotes a finite poset, and $\mathbb{K}$ denotes a ring (associative and with 1, but not necessarily commutative). We use the notation $\bar{x}$ for the (multiplicative) inverse x^{-1} of an element $x \in \mathbb{K}$.

The simplest type of poset is a k -chain (for $k \in \mathbb{N}$) – i.e., the set $\{1, 2, \dots, k\}$ equipped with its usual total order. We denote it by k .

Our favorite poset will be the $p \times q$ -rectangle (for $p, q \in \mathbb{N}$) – i.e., the Cartesian product $p \times q$ with entrywise partial order. Its Hasse diagram looks like a rectangular grid.

We let $\hat{P}$ denote the poset P with two new elements 0 and 1 adjoined to it, arranged so that every $p \in \hat{P}$ satisfies $0 \leq p \leq 1$ in $\hat{P}$.

A $\mathbb{K}$ -labelling (or, short, *labelling*) of P means a map $f : \hat{P} \rightarrow \mathbb{K}$. Its values $f(p)$ are called its *labels* at the “points” $p \in \hat{P}$, and we draw it by overlaying these labels at the respective points on the Hasse diagram of $\hat{P}$. For example, if P is the 2×2 -rectangle, then the following pictures show P itself (Fig. 1), the extended poset $\hat{P}$ (Fig. 2) and a labelling of P (Fig. 3):



When u and v are two elements of $\hat{P}$, we shall use the notation “ $u \triangleleft v$ ” (or, equivalently, “ $v \triangleright u$ ”) for the statement “ $u < v$, and there is no $w \in \hat{P}$ such that $u < w < v$ ”.

We are now ready to define the main actor of our play: the *birational rowmotion* operation. While it is traditionally defined as a composition of *toggles* (see, e.g., [8, Definition 3.16]), we instead here use a recursive equation (which is equivalent):

Definition 1. *Birational rowmotion* is the partial map $R : \mathbb{K}^{\hat{P}} \dashrightarrow \mathbb{K}^{\hat{P}}$ that transforms each labelling f of P into a new labelling Rf whose values are

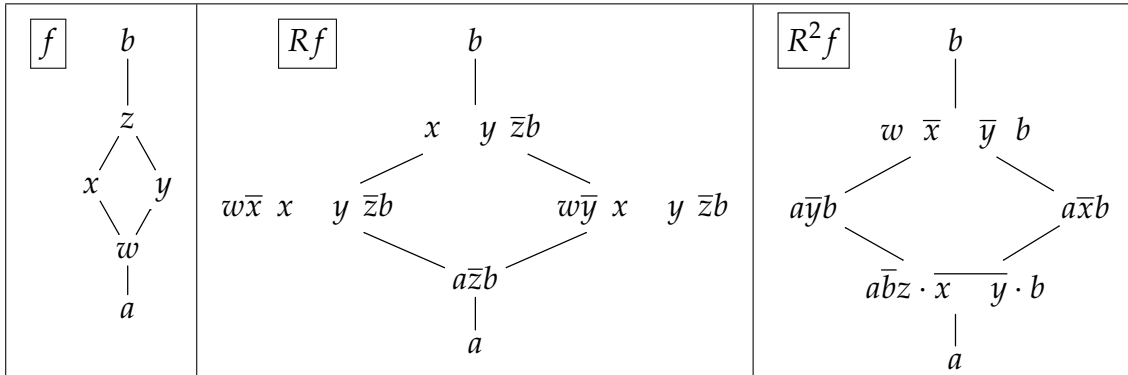
$$Rf|_0 = f|_0, \quad Rf|_1 = f|_1, \quad \text{and} \\ Rf|_v = \left(\sum_{\substack{u \in \hat{P}; \\ u < v}} f|_u \right) \cdot \overline{f|_v} \cdot \overline{\sum_{\substack{u \in \hat{P}; \\ u > v}} Rf|_u} \quad \text{for all } v \in P.$$

Note that the formula for $Rf|_v$ involves several inverses, which may or may not exist. If they don't, Rf is undefined, which is why R is only a **partial** map. Also, this formula is **recursive**: The value it assigns to $Rf|_v$ depends on the values $Rf|_u$ for elements $u \in \hat{P}$ satisfying $u > v$. Since the poset $\hat{P}$ is finite, we can solve this recursion “starting from the top of P ” (using $Rf|_1 = f|_1$ to bootstrap the process). Making this procedure explicit gives precisely the description of R as a composition of toggles that is used as a definition of R in [8]. (See [8, Proposition 3.18] for details.)

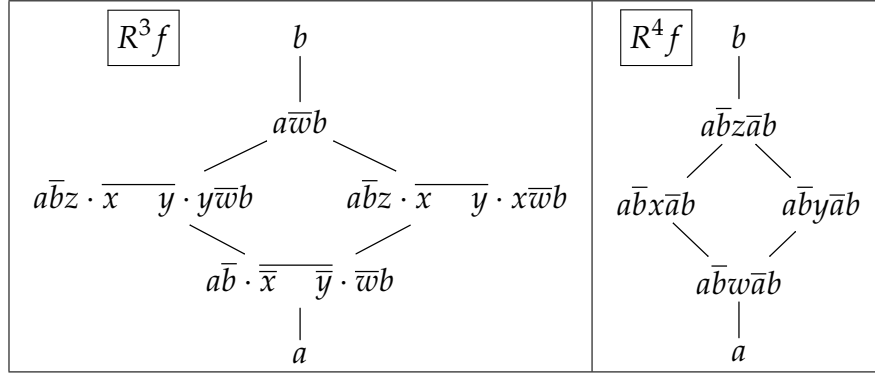
The map R is what we called “noncommutative birational rowmotion” in Section 1 – that is, the third (and ultimate) generalization of combinatorial rowmotion ρ . The second generalization (commutative birational rowmotion) is obtained when $\mathbb{K}$ is a field. The first generalization (piecewise-linear rowmotion ρ_P) is recovered from the second by *tropicalization* (i.e., by setting $\mathbb{K}$ to be a tropical semifield²). The original rowmotion ρ on $\mathcal{J} P$ is the restriction of ρ_P to the vertices of the order polytope $\mathcal{O} P$ (which are the indicator functions of order ideals of P). Therefore, any (positive) result about noncommutative birational rowmotion automatically yields results about all lower levels (commutative, piecewise-linear and combinatorial).

Partial maps can be composed; in particular, R can be composed with itself. As usual, R^k shall mean the composition $R \circ R \circ \cdots \circ R$ with k many R 's.

Example 1. Here are the first four iterations of R for $P = 2 \times 2$, acting on an arbitrary labelling f :



²Of course, a tropical semifield is not a field. However, properties of R that hold for all fields will automatically hold for all semifields (since they reduce to subtraction-free polynomial identities), and thus we can allow $\mathbb{K}$ to be a semifield.



Here, some nontrivial algebra has been used to simplify some of the labels; e.g., the $a\bar{b}y\bar{a}b$ in $R^4 f$ was originally an $a\bar{b} \cdot \bar{x} \cdot \bar{y} \cdot \bar{x} \cdot \bar{y} \cdot y \cdot \bar{x} \cdot \bar{y} \cdot x \cdot y \cdot \bar{a}b$.

3 The main theorems for rectangles

Example 1 illustrates two surprising phenomena that apply to all rectangular posets. One is a “twisted periodicity”, saying that the initial labels of f reappear in $R^4 f$ framed by $a\bar{b}$ on the left and $\bar{a}b$ on the right. (Note that this is not a conjugation, since $a\bar{b}^{-1}b\bar{a} \neq \bar{a}b$ unless a and b commute.) For a generic poset P , nothing like this holds (instead, the labels of $R^k f$ get increasingly complicated as k grows, even if $\mathbb{K}$ is commutative), but it turns out to be true whenever P is a rectangle:

Theorem 1 (Periodicity for the $p \times q$ -rectangle). *Let p and q be two positive integers. Let $P = p \times q$, and let $f \in \mathbb{K}^{\hat{P}}$ be a $\mathbb{K}$ -labelling such that $R^p f$ is well-defined. Set $a = f(0)$ and $b = f(1)$. Then, a and b are invertible, and for any $x \in \hat{P}$ we have*

$$(R^p f)(x) = a\bar{b} \cdot f(x) \cdot \bar{a}b. \quad (3.1)$$

This periodicity actually follows from the following “antipodal reciprocity” property, which is the second phenomenon seen in **Example 1**. This property generalizes the $a\bar{z}b$ label in Rf , the $a\bar{y}b$ and $a\bar{x}b$ labels in $R^2 f$, and the $a\bar{w}b$ label in $R^3 f$. In its general form, it says that the inverse of any label in $R^k f$ appears as a label in $R^\ell f$ (for some $\ell \geq k$), framed by a on the left and by b on the right. The ℓ depends on the position of the label; the precise statement is as follows:

Theorem 2 (Reciprocity for the $p \times q$ -rectangle). *Let p and q be two positive integers. Let $P = p \times q$. Fix $\ell \in \mathbb{N}$, and let $f \in \mathbb{K}^{\hat{P}}$ be a $\mathbb{K}$ -labelling such that $R^\ell f$ is well-defined. Set $a = f(0)$ and $b = f(1)$. Let $i, j \in P$ satisfy $\ell - i - j - 1 \geq 0$. Then,*

$$(R^\ell f)(i, j) = a \cdot (R^{\ell-i-j-1} f)(p-1-i, q-1-j) \cdot b. \quad (3.2)$$

We conjectured these two theorems in 2014, after proving them in the particular case when $\mathbb{K}$ is commutative ([6, §4–§8]).

4 Connections which do not lift

Strikingly, this noncommutative generalization has resisted all approaches that have previously succeeded in the commutative case. The determinantal computations involved in the proof in [6] can be extended to the noncommutative setting using the *quasideterminants* of Gelfand and Retakh, but it seems impossible to make a rigorous proof out of them (lacking, e.g., any useful notation of Zariski topology in this setting, it is not clear what it means for a $\mathbb{K}$ -labelling to be “generic”). The Musiker–Roby proof of commutative periodicity [15] (via a lattice-path formula for $(R^i f)_v$) could not be generalized either. Thus the noncommutative case remained an open problem.

At some point, Glick and Grinberg noticed that the Y -variables in the type- AA Zamolodchikov periodicity theorem of Volkov [19] could be written as ratios of labels under iterated birational rowmotion [17, § 4.4]; this allows the periodicity in one setting to be derived from that in the other (with some work). However, for noncommutative $\mathbb{K}$, Zamolodchikov periodicity fails even in small examples such as $r = r' = 2$ (no matter in which of the 120 possible orders we multiply the factors), while noncommutative birational rowmotion continues to exhibit periodicity. This approach is therefore unavailable in the noncommutative case as well.

Another recent proof of periodicity and reciprocity for commutative $\mathbb{K}$ (Johnson and Liu, [11]) uses a connection to the octahedron recurrence; this connection, too, does not seem to generalize readily to the noncommutative case (although a proper analysis of all options has not been made yet), and anyway the proof in [11] uses determinants, which are not available for general $\mathbb{K}$.

5 Proof outline

We shall now outline the main steps of our proofs of [Theorems 1](#) and [2](#). Full details are available in [8], including justifications of all the inverses being taken (in the following, we just assume they all exist) and proofs of all lemmas.

Step 1: *Observe that [Theorem 1](#) follows from [Theorem 2](#).* Indeed, assume that [Theorem 2](#) is proved. Now, to prove [Theorem 1](#), assume WLOG that $x \in P$ (as opposed to $x = 0$ or $x = 1$, which cases are easy), and write $x = u, v$. Now, apply [Theorem 2](#) first to $\ell, i, j = p, q, u, v$. Then, apply [Theorem 2](#) again to $\ell, i, j = p, q - u - v - 1, p - 1 - u, q - 1 - v$ in order to rewrite the right hand side.

It thus remains to prove [Theorem 2](#).

Step 2: *Introduce notations:* Fix a $\mathbb{K}$ -labelling f of $P = p \times q$. For any element $x = i, j \in P$, we define its *antipode* $x^\sim \in P$ by $x^\sim : p - 1 - i, q - 1 - j$. For instance,

$1, 1 \sim p, q$. For any $x \in \widehat{P}$ and $\ell \in \mathbb{N}$, we write

$$x_\ell : (R^\ell f) x. \quad (5.1)$$

Informally, we think of our subscripts as indicating the “time” of the label (i.e., number of times R was applied to f). For instance, in [Example 1](#), we have $1, 2 \xrightarrow{2} (R^2 f) 1, 2 \xrightarrow{1} a\bar{x}b$.

In this notation, [Theorem 2](#) claims that

$$x_\ell = a \cdot \overline{x_{\ell-i-j-1}^\sim} \cdot b \quad (5.2)$$

for all $x = i, j \in P$ and all $\ell \geq i + j - 1$. Clearly, it suffices to prove this for $\ell = i + j - 1$.

For each $v \in P$ and $\ell \in \mathbb{N}$, we define the two *slack values*³

$$A_\ell^v : v_\ell \cdot \overline{\sum_{u < v} u_\ell} \quad \text{and} \quad V_\ell^v : \overline{\sum_{u > v} u_\ell} \cdot \overline{v_\ell}.$$

Furthermore, when $v \in \{0, 1\}$, we set $A_\ell^v : 1$ and $V_\ell^v : 1$ for all $\ell \in \mathbb{N}$. For instance, in [Example 1](#), we have $A_0^{2,2} = z \cdot \overline{y \cdot x}$ and $V_0^{1,1} = \overline{y \cdot x} \cdot \overline{w}$ and $V_1^{1,1} = \overline{w \cdot y \cdot x} \cdot \overline{z \cdot b} \cdot \overline{w \cdot x} \cdot \overline{y \cdot z \cdot b} \cdot \overline{a \cdot z \cdot b} = \overline{w \cdot a}$ (after simplifications).

A *path* will mean a sequence $v_0, v_1, \dots, v_k$ of elements of $\widehat{P}$ satisfying $v_0 > v_1 > \dots > v_k$. We will call it a *path from v_0 to v_k* . For instance, there are two paths from $2, 2$ to $1, 1$, namely $2, 2 \rightarrow 1, 2 \rightarrow 1, 1$ and $2, 2 \rightarrow 2, 1 \rightarrow 1, 1$.

For any path $\mathbf{p} = v_0, v_1, \dots, v_k$ and any $\ell \in \mathbb{N}$, we define the *slack products*

$$A_\ell^\mathbf{p} : A_\ell^{v_0} A_\ell^{v_1} \dots A_\ell^{v_k} \quad \text{and} \quad V_\ell^\mathbf{p} : V_\ell^{v_0} V_\ell^{v_1} \dots V_\ell^{v_k}$$

If u and v are elements of $\widehat{P}$, and if $\ell \in \mathbb{N}$, then we define the *slack sums*

$$A_\ell^{u \rightarrow v} : \sum_{\mathbf{p} \text{ is a path from } u \text{ to } v} A_\ell^\mathbf{p} \quad \text{and} \quad V_\ell^{u \rightarrow v} : \sum_{\mathbf{p} \text{ is a path from } u \text{ to } v} V_\ell^\mathbf{p}.$$

For instance,

$$A_\ell^{2,2 \rightarrow 1,1} = A_\ell^{2,2 \rightarrow 1,2 \rightarrow 1,1} + A_\ell^{2,2 \rightarrow 2,1 \rightarrow 1,1} = A_\ell^{2,2} A_\ell^{1,2} A_\ell^{1,1} + A_\ell^{2,2} A_\ell^{2,1} A_\ell^{1,1}.$$

The following formula connects the slack values at “time ℓ ” with those at “time $\ell - 1$ ”:

Proposition 1 (Transition equation in A - V -form). *Let $v \in \widehat{P}$ and $\ell \geq 1$. Then, $V_\ell^v = A_{\ell-1}^v$.*

Proof. This is just the formula for $Rf = v$ in [Definition 1](#), rewritten. $\square$

³In both sums, u ranges over elements of $\widehat{P}$.

Step 3: The path formulas. The following formulas allow us to recover the labels u_ℓ from the slack values A_ℓ^v or V_ℓ^v :

Theorem 3 (path formulas for rectangle). *Let $\ell \in \mathbb{N}$. Then, each $u \in P$ satisfies*

$$u_\ell = \overline{V_\ell^{1 \rightarrow u}} \cdot b = \overline{V_\ell^{p,q \rightarrow u}} \cdot b \quad (5.3)$$

and

$$u_\ell = A_\ell^{u \rightarrow 0} \cdot a = A_\ell^{u \rightarrow 1,1} \cdot a. \quad (5.4)$$

Proof. Induction on u (downwards for (5.3), upwards for (5.4)). $\square$

Using (5.4) and (5.3) and [Proposition 1](#), we can already prove that $1, 1 \vdash a \cdot p, q \vdash_0 b$, which confirms (5.2) for $x = 1, 1$. The general case, however, requires more work.

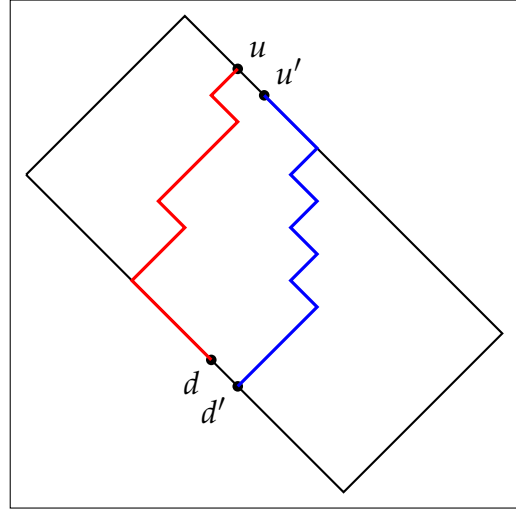
Step 4: The conversion lemma. The following lemma is at the core of our proof:⁴

Lemma 1 (Conversion lemma). *Let u and u' be two elements of the northeastern edge of P satisfying $u \succ u'$ (that is, $u = k, q$ and $u' = k-1, q$ for some $k \in \{2, 3, \dots, p\}$). Let d and d' be two elements of the southwestern edge of P satisfying $d \succ d'$ (that is, $d = i, 1$ and $d' = i-1, 1$ for some $i \in \{2, 3, \dots, p\}$). Then, for each $\ell \in \mathbb{N}$, we have*

$$A_\ell^{u \rightarrow d} = V_\ell^{u' \rightarrow d'}. \quad (5.5)$$

Note that both sides of this equality are slack sums at “time ℓ ” (i.e., defined purely in terms of the labeling $R^\ell f$). Thus, this lemma does not involve birational rowmotion. Nevertheless, its proof is far from trivial. It involves “interpolating” between $A_\ell^{u \rightarrow d}$ and $V_\ell^{u' \rightarrow d'}$ through a “mixed slack sum”. The latter is a sum over “path-jump-paths”, which are like paths but allow a “jump” in the middle. We refer to [8, §9] for the details.

Step 5: The southwestern edge. If $\ell \geq 1$, then (5.5) can be rewritten as $A_\ell^{u \rightarrow d} = A_{\ell-1}^{u' \rightarrow d'}$ (by [Proposition 1](#)). Applying this equality many times (“gliding down” the rectangle), we see that each $i \in [p]$ satisfies $A_{i-1}^{p, q \rightarrow i, 1} = A_0^{p-i-1, q \rightarrow 1, 1}$. However, [Proposition 1](#) yields $V_i^{p, q \rightarrow i, 1} = A_{i-1}^{p, q \rightarrow i, 1} = A_0^{p-i-1, q \rightarrow 1, 1}$. In view of (5.4) and (5.3), we can readily use this to show that $i, 1 \vdash a \cdot p-1-i, q \vdash_0 b$. This proves (5.2) for $x = i, 1$.



⁴The figure on the right illustrates this lemma (where the red path is one of the paths contributing to the sum $A_\ell^{u \rightarrow d}$, while the blue path contributes to $V_\ell^{u' \rightarrow d'}$).

In other words, (5.2) is proved for all x lying on the southwestern edge of P (speaking in terms of the Hasse diagram). A symmetric argument applies when x lies on the southeastern edge.

Step 6: Inducting up the poset. It remains to prove (5.2) for all $x = i, j$ with $i, j \geq 2$. This is done by strong induction on $i + j$, which allows us to apply the induction hypothesis to the four pairs

$$m : i, j-1, \quad u : i-1, j-1, \quad s : i, j-2, \quad t : i-1, j-1$$

in lieu of $x = i, j$ (to the extent that these pairs do belong to P). As a result, we obtain

$$m_\ell = a \cdot \overline{m_{\ell-k}} \cdot b, \quad s_{\ell-1} = a \cdot \overline{s_{\ell-k}} \cdot b, \quad (5.6)$$

$$t_{\ell-1} = a \cdot \overline{t_{\ell-k}} \cdot b, \quad m_{\ell-1} = a \cdot \overline{m_{\ell-k-1}} \cdot b, \quad (5.7)$$

$$u_\ell = a \cdot \overline{u_{\ell-k-1}} \cdot b \quad \text{where } k = i + j - 2. \quad (5.8)$$

On the other hand, the formula for $Rf = v$ in Definition 1 (applied once to $v = m$ and once to $v = m^\sim$) yields

$$m_\ell = s_{\ell-1} \cdot t_{\ell-1} \cdot \overline{m_{\ell-1}} \cdot \overline{u_\ell} \cdot \overline{x_\ell} \quad \text{and} \quad (5.9)$$

$$m_{\ell-k}^\sim = (u_{\ell-k-1}^\sim \cdot x_{\ell-k-1}^\sim) \cdot \overline{m_{\ell-k-1}} \cdot \overline{s_{\ell-k}} \cdot \overline{t_{\ell-k}}^\sim. \quad (5.10)$$

Taking reciprocals on both sides of (5.10), we obtain

$$\overline{m_{\ell-k}} = \left(\overline{s_{\ell-k}} \cdot \overline{t_{\ell-k}}^\sim \right) \cdot m_{\ell-k-1}^\sim \cdot \overline{u_{\ell-k-1}} \cdot \overline{x_{\ell-k-1}}^\sim. \quad (5.11)$$

In view of the five equations (5.6)–(5.8), we see that (up to the a and b factors, which cancel out) each label appearing in (5.9) is the inverse of the corresponding label in (5.11), except for x_ℓ and $x_{\ell-k-1}^\sim$. Since the equations (5.9) and (5.11) can be uniquely solved for x_ℓ and $x_{\ell-k-1}^\sim$, we conclude that x_ℓ must be $a \cdot \overline{x_{\ell-k-1}^\sim} \cdot b$. This completes the induction step, proving (5.2) and with it Theorem 2. Recalling Step 1, we have thus proved Theorem 1 too.

6 Other posets

The rectangles $P = p \times q$ may be the simplest family of posets for which noncommutative birational rowmotion exhibits a (twisted) periodicity.⁵ In the commutative case, the “skeletal posets” of [7] are even simpler, but their periodicity does not extend to the noncommutative case. For example, if P is the 4-element “claw” poset $\mathbb{V}$, then $R^6 = \text{id}$

⁵Apart from the chains m , which are just $m \times 1$ -rectangles, and from the antichains, which are trivial to analyze.

for commutative $\mathbb{K}$, but no periodicity holds for noncommutative $\mathbb{K}$ (not even “twisted”; see [8, §13.4] for details). Experimentally, things look better for “Lie-theoretical” families of posets, although nothing has been proven so far. The family of *type-A positive root posets*, also known as *triangles* Δ_p , may be within reach.

We can easily define these triangles as pieces of a $p \times p$ -rectangle. Indeed, let p be a positive integer. Define two subsets Δ_p and ∇_p of the $p \times p$ -rectangle $p \times p$ by

$$\begin{aligned}\Delta_p &= \{i, j \in p \times p \mid i - j > p - 1\}; \\ \nabla_p &= \{i, j \in p \times p \mid i - j < p - 1\}.\end{aligned}$$

These two subsets Δ_p and ∇_p inherit partial orders from $p \times p$. Now, we claim:

Conjecture 1 (Periodicity for Δ_p and ∇_p). *Let $p \geq 2$ be an integer. Assume that P is the poset Δ_p or ∇_p . Let $f \in \mathbb{K}^{\hat{P}}$ be a $\mathbb{K}$ -labelling such that $R^p f$ is well-defined. Let $a = f(0)$ and $b = f(1)$. Let $x \in \hat{P}$. Let $x' \in \hat{P}$ be defined as follows: If $x = i, j$, then we set $x' = j, i$; otherwise (i.e., if $x = 0$ or $x = 1$), we set $x' = x$. Then,*

$$R^p f(x) = \bar{a} \bar{b} \cdot f(x') \cdot \bar{a} \bar{b}.$$

Further posets P for which R is conjecturally periodic (with a twist) are “right-half triangles” [8, §13.2] and trapezoids [8, §13.3].

7 General posets

As already mentioned, periodicity and reciprocity are features of certain classes of posets, even though the complete identification of these classes is still unsolved even at the piecewise-linear level⁶. Nevertheless, the $i, j = 1, 1$ case of [Theorem 2](#) can be generalized to arbitrary posets in the following form ([8, §14]):

Proposition 2. *Let P be any finite poset. Let $f \in \mathbb{K}^{\hat{P}}$ be a labeling of P such that Rf is well-defined and all labels $Rf(u)$ are invertible. Let $a = f(0)$ and $b = f(1)$. Then,*

$$b \cdot \sum_{\substack{u \in \hat{P}; \\ u > 0}} \overline{Rf(u)} \cdot a = \sum_{\substack{u \in \hat{P}; \\ u < 1}} f(u).$$

This adds to the (so far rather meager) list of properties of R that have been observed for general posets P . (The rest of the list are results by Joseph and Roby [13, §5] that connect R with *birational antichain rowmotion* and the *birational toggle group*.)

Note that [Proposition 2](#) can be used to generalize the w -tuple periodicity [7, Proposition 37] to the case of noncommutative $\mathbb{K}$ (again with a twist).

⁶See [10, Conjecture 5.7] for a sequence of posets V_n for which ρ_P is conjectured to have finite order, but R seems to have infinite order even when $\mathbb{K}$ is commutative.

8 The semiring case

The definition of R is *subtraction-free*: It involves only addition, multiplication and division in $\mathbb{K}$. Thus, we can replace $\mathbb{K}$ by an arbitrary *semiring* (i.e., a “ring without additive inverses”). It is thus natural to wonder:

Question 1. Do [Theorems 1 and 2](#) still hold if $\mathbb{K}$ is only a semiring?

This question is subtler than it may seem. When $\mathbb{K}$ is commutative, the answer is positive for general reasons: A polynomial identity true over all commutative rings is automatically true over all commutative semirings. However, such automatic transfer principles do not exist in the noncommutative realm. Case in point: If $\mathbb{K}$ is a ring, then the identity $a \cdot \overline{a \cdot b} = \overline{b \cdot a} \cdot b = \overline{b \cdot a} \cdot b$ holds for all $a, b \in \mathbb{K}$ for which $a \cdot b$ is invertible; but this is no longer true if $\mathbb{K}$ is merely a semiring (David Speyer, MathOverflow #401273). A negative answer to [Question 1](#) would thus not be entirely unexpected.

The proofs of [Theorems 1 and 2](#) sketched above rely on subtraction in Step 6, so they cannot be applied to semirings unless $\min\{p, q\} \leq 2$ (in which case there is nothing to subtract). We suspect that the smallest “interesting” case is 3×3 for [Theorem 2](#) and 3×4 for [Theorem 1](#). Noncommutative semirings that cannot be embedded in rings yet have many multiplicative inverses do exist⁷, but automated computation inside them is not straightforward, which complicates experimentation. We nevertheless believe that progress can be made in this direction.

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⁷One such semiring consists of all weakly increasing functions from $\mathbb{R}$ to $\mathbb{R}$. Its addition is pointwise maximum, and its multiplication is composition.

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Wreath Macdonald operators

Daniel Orr^{*1,2}, Mark Shimozono^{†2}, and Joshua Jeishing Wen^{‡3}

¹Max Planck Institut für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany

²Department of Mathematics, Virginia Tech, 460 McBryde Hall, Virginia Tech, 225 Stanger St., Blacksburg, VA 24061 USA

³Department of Mathematics, Northeastern University, 463 Lake Hall, 43 Leon St, Boston, MA 02115 USA

Abstract. We construct a novel family of difference-permutation operators and prove that they are diagonalized by the wreath Macdonald P -polynomials. Our operators arise from the action of the horizontal Heisenberg subalgebra in the vertex representation of the quantum toroidal algebra.

Keywords: Macdonald polynomials, quantum toroidal algebras, shuffle algebras

1 Introduction

Let $X_N = \{x_1, \dots, x_N\}$ be a set of variables and let $\mathbb{Y}_N$ be the set of partitions with at most N parts. The Macdonald polynomials $\{P_\lambda(X_N; q, t) \mid \lambda \in \mathbb{Y}_N\}$ are a basis of the ring of symmetric polynomials $\mathbb{Q}[q, t][X_N]^{\mathfrak{S}_N}$ that can be characterized as eigenfunctions of a commuting family of difference operators, the Macdonald operators: for $1 \leq n \leq N$,

$$D_n(X_N; q, t) : t^{\frac{n(n-1)}{2}} \sum_{\substack{J \subset \{1, \dots, N\} \\ |J| = n}} \left(\prod_{\substack{i \in J \\ j \notin J}} \frac{tx_i - x_j}{x_i - x_j} \right) \prod_{i \in J} T_{q, x_i} \quad (1.1)$$

$$D_n(X_N; q, t) P_\lambda(X_N; q, t) = e_n(q^{\lambda_1} t^{N-1}, q^{\lambda_2} t^{N-2}, \dots, q^{\lambda_N}) P_\lambda(X_N; q, t) \quad (1.2)$$

Here T_{q, x_i} is the q -shift operator $T_{q, x_i} x_j = q^{\delta_{i,j}} x_j$ and e_n is the n th elementary symmetric polynomial.

This paper is concerned with wreath Macdonald polynomials, a generalization of the Macdonald polynomials defined by Haiman [1]. Let r be a positive integer and let $I = \mathbb{Z}/r\mathbb{Z}$. Fix an r -core partition γ and a vector $N = (N_0, N_1, \dots, N_{r-1}) \in \mathbb{Z}_{\geq 0}^I$ that

^{*}dorr@math.vt.edu. D.O. was supported by the Simons Foundation (Collaboration Grant for Mathematicians, 638577) and the Max Planck Institute for Mathematics (MPIM Bonn).

[†]mshimo@math.vt.edu.

[‡]j.wen@northeastern.edu. J.J.W. was supported by NSF-RTG grant “Algebraic Geometry and Representation Theory at Northeastern University” (DMS-1645877).

is compatible with γ in the sense given by (3.5) below. Letting $N = |N_\bullet| : \sum_{i \in I} N_i$, we partition the variables $x_1, \dots, x_N$ into r subsets:

$$X_{N_\bullet} : \bigsqcup_{i=0}^{r-1} \{x_l^i \mid 1 \leq l \leq N_i\} = \{x_1, \dots, x_N\}. \quad (1.3)$$

We call the index i the *color* of x_l^i . The product of symmetric groups $\mathfrak{S}_{N_\bullet} : \prod_{i \in I} \mathfrak{S}_{N_i}$ acts on the polynomial ring $\mathbb{Q}[q, t] X_{N_\bullet}$ whereby $\mathfrak{S}_{N_i}$ only permutes the variables of color i . Let $\mathbb{Y}_{N, \gamma}$ be the set of partitions with r -core γ and at most N parts. The wreath Macdonald polynomials $\{P_\lambda X_{N_\bullet}; q, t \mid \lambda \in \mathbb{Y}_{N, \gamma}\}$ are a basis of the space $\mathbb{Q}[q, t] X_{N_\bullet}^{\mathfrak{S}_{N_\bullet}}$ of color-symmetric polynomials. The original Macdonald polynomials $P_\lambda X_N; q, t$ [2, p. VI.4] are the case $r = 1$.

Haiman's definition characterizes $P_\lambda X_{N_\bullet}; q, t$ using a pair of triangularity conditions. In contrast with the usual Macdonald theory, we *a priori* do not have an analogous characterization as the joint eigenfunction of an explicit family of difference operators. The present work remedies this situation. For each $p \in I$ and positive integer n , define the operators $D_{p,n}$ on $\mathbb{Q}[q, t] X_{N_\bullet}$ by

$$\begin{aligned} D_{p,n} X_{N_\bullet}; q, t : & \frac{-1^{\frac{n(n-1)}{2}}}{\prod_{k=1}^n (1 - q^k t^{-k})} \\ & \times \sum_{\underline{J} \in Sh_p^n X_{N_\bullet}} \prod_{a=1}^n \left\{ 1 - q t^{-1} \frac{|J_a|}{|J|} \left(\frac{x_{J_a}^{r-1}}{x_{J_a}^p} \right) \frac{\prod_{l=1}^{N_p} \left(t x_{J_a}^{p-1} - x_l^p \right)}{\prod_{l=1}^{N_p} \left(x_{J_a}^p - x_l^p \right)} \right. \\ & \times \left. \left(\prod_{i \in I \setminus \{p\}} \prod_{\substack{l=1 \\ x_l^i / x_{J_a}^i}}^{N_i} \frac{\left(t x_{J_a}^{i-1} - x_l^i \right)}{\left(x_{J_a}^i - x_l^i \right)} \right) \left(\prod_{i \in J_a \setminus \{p\}} \frac{q^{-1} t T_{J_a} x_{J_a}^i}{\left(x_{J_a}^i - T_{J_a} x_{J_a}^i \right)} \right) T_{J_a} \right\}. \end{aligned} \quad (1.4)$$

The notation used in this formula is defined in Section 4. Our main result is the following,

Theorem 1.1. *Let $N_\bullet$ be compatible with γ and $\lambda \in \mathbb{Y}_{N, \gamma}$. Then the number of $1 \leq i \leq N$ with $i - \lambda_i - 1 \equiv p \pmod{r}$ is equal to N_p and $P_\lambda X_{N_\bullet}; q, t$ satisfies the eigenfunction equation*

$$D_{p,n} X_{N_\bullet}; q, t P_\lambda X_{N_\bullet}; q, t = e_n \left[\sum_{\substack{i=1 \\ i - \lambda_i - 1 \equiv p \pmod{r}}}^N q^{\lambda_i} t^{-N-i} \right] P_\lambda X_{N_\bullet}; q, t. \quad (1.5)$$

The eigenvalue uses plethystic notation: the N_p variables of the elementary symmetric polynomial are replaced by the summands of the form $q^a t^b$. Our operators (1.4) are much more complicated than the original Macdonald operators (1.1). In the case $r = 1$, we do indeed obtain (1.1) after some simplification. When $r > 1$, the q -shift operator T_{q, x_i} is replaced with what we call a *cyclic-shift operator* T_{J_a} , which cyclically permutes variables of different colors in addition to multiplying by a power of q .

1.1 Integral formulas

Our strategy for deriving the eigenoperators (1.4) and establishing the eigenfunction equation (1.5) uses work of the third author [5]. We study the wreath Macdonald polynomials using the *quantum toroidal algebra* $\check{U} = U_{q, \delta} \check{\mathfrak{sl}}_r$ and its *vertex representation* W . The aforementioned work proves that infinite-variable wreath Macdonald polynomials can be naturally embedded inside W such that they diagonalize a large commutative subalgebra of $\check{U}$ known as the *horizontal Heisenberg subalgebra*. This alone is insufficient for obtaining explicit formulas. We also need work of Neguț [3] realizing $\check{U}$ as a *shuffle algebra*. The shuffle algebra is a space of rational functions endowed with an exotic product structure, and it is isomorphic to a part of $\check{U}$ via a map that is morally (but not precisely) an integration map. Writing its action on W and then specializing from infinite to finite variables, we obtain integral formulas. Finally, to pin down the eigenvalues, we use the twisted isomorphism established by Tsymbaliuk [4] between the vertex representation and the *Fock representation*. We apply this process to the shuffle realizations of well-chosen elements of the horizontal Heisenberg subalgebra which were found in [5].

1.2 Towards duality

In the case $r = 1$, the eigenfunction equation (1.2) is particularly interesting when juxtaposed with the *Pieri rules* [2]. To make this apparent, introduce a continuous extension of the discrete parameters $\lambda = \lambda_1, \dots, \lambda_N : s_i : q^{\lambda_i} t^{N-i}$, $S_N : \{s_1, \dots, s_N\}$. We call the variables X_N the *position variables* and S_N the *spectral variables*. It is natural to interpret the q -shift $T_{q, s_i} P_\lambda(X_N; q, t)$ by adding a box to row i of the partition λ . For a certain renormalization $\tilde{P}_\lambda(X_N; q, t)$ of $P_\lambda(X_N; q, t)$, we can write the Pieri rules as

$$e_n(x_1, \dots, x_N) \tilde{P}_\lambda(X_N; q, t) = t^{\frac{n(n-1)}{2}} \sum_{\substack{I \subset \{1, \dots, N\} \\ |I| = n}} \left(\prod_{\substack{i \in I \\ j \notin I}} \frac{ts_i - s_j}{s_i - s_j} \right) \prod_{i \in I} T_{q, s_i} \tilde{P}_\lambda(X_N; q, t). \quad (1.6)$$

The fact that no shift operator T_{q, s_i} appears more than once enforces the well known support condition of the Pieri rules: the $\tilde{P}_\mu(X_N; q, t)$ that appear on the right hand side of (1.6) are such that $\mu \setminus \lambda$ contains no horizontally adjacent boxes. On the other hand, we

can view the eigenfunction equation (1.2) as describing multiplication by $e_n s_1, \dots, s_N$. The similarity between (1.2) and (1.6) is reflective of a symmetry $X_N \leftrightarrow S_N$. One manifestation of this symmetry is the well-known *evaluation duality* [2]:

$$\tilde{P}_\lambda q^{\mu_1 t^{N-1}, \mu_2 t^{N-2}, \dots, \mu_N} = \tilde{P}_\mu q^{\lambda_1 t^{N-1}, \lambda_2 t^{N-2}, \dots, \lambda_N}.$$

For the wreath case $r > 1$, the spectral variables should also have color. We assign s_l^i to some b such that $b - \lambda_b \equiv i - 1 \pmod r$: $s_l^i : q^{\lambda_b t^{|N_\bullet| - b}}$. Here, we point out a natural motivation for imposing our compatibility condition between the r -core of λ and $N_\bullet$ —it forces there to also be N_i spectral variables of color i . The eigenfunction equation (1.5) then describes multiplication by $e_n s_1^p, \dots, s_{N_p}^p$. Note that adding a box to a row will not only contribute a q -shift but also change the color, and that is precisely what the cyclic-shift operators $T_{\underline{L}_a}$ do. Work of the third author [5] provides one constraint on the support of the wreath Pieri rules. Namely, for a box a, b , if we call the class of $b - a \pmod r$ its color, then $P_\mu X_{N_\bullet}; q, t$ appears as a summand of $e_n x_{p,1}, \dots, x_{p,N_p} P_\lambda X_{N_\bullet}; q, t$ only if $\mu \setminus \lambda$ consists of n boxes of each color such that no boxes of color p and $p - 1$ are horizontally adjacent. One can check that the combinations of $T_{\underline{L}_a}$ appearing in (1.4) enforce this condition after swapping $x_l^i \leftrightarrow s_l^i$. Computer calculations done by the second author also confirm a wreath analogue of evaluation duality. While we are still a long way from establishing a wreath analogue of the $X_N \leftrightarrow S_N$ symmetry, our strange operators seem to go out of their way to say it must be true.

1.3 Outline

Section 2 discusses the core-quotient combinatorics in detail. Section 3 defines wreath Macdonald symmetric functions and their finitizations. Section 4 defines the ingredients of the formula (1.4) of our wreath Macdonald eigenoperators.

2 Abaci, cores, quotients, and root lattice combinatorics

Fix a positive integer r and let $I = \mathbb{Z}/r\mathbb{Z}$.

2.1 Partitions and cores

Let $\mathbb{Y}$ be the set of all integer partitions. The diagram of a partition $\mu = \mu_1, \mu_2, \dots \in \mathbb{Y}$ is by definition the French-style diagram $D_\mu = \{a, b \in \mathbb{Z}_{\geq 0}^2 : 0 \leq a < \mu_{b+1}\}$ which uses 0-based first quadrant Cartesian x, y -coordinates for the elements of D_μ . The *residue* or *color* of $a, b \in \mathbb{Z}^2$ is the element $b - a \in \mathbb{Z}/r\mathbb{Z}$. This is the residue mod r of the *negative* of Macdonald's content of a cell [2, Ex. I.1.3].

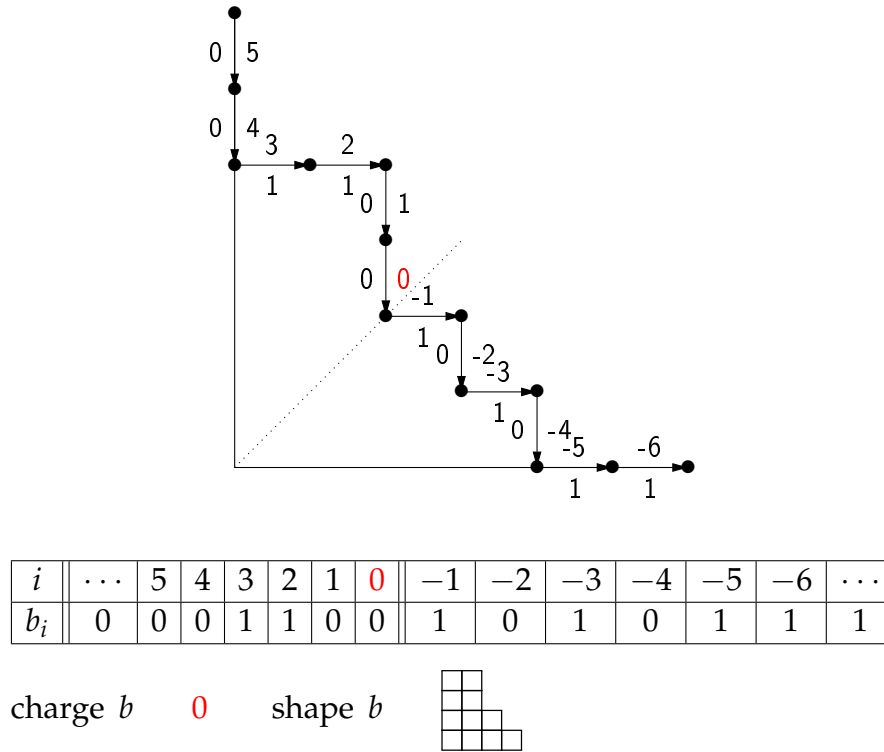


Figure 1: The shape and charge of an edge sequence

For $a, b \in D \mu$ the hook length $h_{\mu} a, b$ is the number of elements $a', b' \in D \mu$ that are in the same row as a, b and weakly to the right (that is, $a', b' \geq a, b$ with $a' \geq a$) or in the same column as a, b and weakly above (that is, $a', b' \geq a, b$ with $b' \geq b$). A partition $\mu \in \mathbb{Y}$ is an r -core if $h_{\mu} a, b \not\equiv r$ for all $a, b \in D \mu$. We denote by $\mathcal{C} = \mathcal{C}_r$ the set of r -cores.

2.2 Edge sequences and partitions

A function $b : \mathbb{Z} \rightarrow \{0, 1\}$ can be viewed as an infinite indexed binary word

$$\dots b_1 b_0 b_{-1} \dots;$$

notice that in writing such a word we index the positions in reverse order. An *inversion* of b is a pair of integers $i > j$ such that $b_i > b_j$, a 1 to the left of a 0. An *edge sequence* (or *Maya diagram*) is a function $b : \mathbb{Z} \rightarrow \{0, 1\}$ such that $b_i = 0$ for $i \gg 0$ and $b_i = 1$ for $i \ll 0$, that is, b has finitely many inversions. Let ES denote the set of edge sequences. The *shape* of $b \in \text{ES}$ is the partition whose French partition diagram has boundary traced out by the values of b from top left to bottom right, where 0 (resp. 1) indicates a vertical downward (resp. horizontal rightward) unit vector. See Figure 1, in which the 0s and 1s

of the edge sequence are written to the left and below their corresponding unit vectors, and the index i of the bit b_i is written to the right or above the unit vector.

The *charge* of $b \in \text{ES}$ is the index of the unit vector that touches the main diagonal from above and left, or equivalently the index of the rightmost 0 in the edge sequence of the form $\cdots 0011 \cdots$ obtained from b by repeatedly swapping adjacent pairs 10 to 01 until none remain. There is a bijection

$$\text{ES} \rightarrow \mathbb{Z} \times \mathbb{Y}, \quad b \mapsto \text{charge } b, \text{shape } b. \quad (2.1)$$

2.3 Cores and quotients

Our goal is to define the bijection

$$\mathbb{Y} \sim \mathcal{C}_r \times \mathbb{Y}^r, \quad \lambda \mapsto \text{core}_r \lambda, \text{quot}_r \lambda \quad (2.2)$$

where core_r is the r -core and quot_r is the r -quotient map.

In the following diagram all horizontal maps are bijections and vertical maps are inclusions.

$$\begin{array}{ccccccc}
 \mathbb{Z} \times \mathbb{Y} & \xleftarrow{\text{charge} \times \text{shape}} & \text{ES} & \xrightarrow{\quad} & \text{ES}^r & \xrightarrow{c^\bullet \times \text{quot}_r} & \mathbb{Z}^r \times \mathbb{Y}^r \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 \{0\} \times \mathbb{Y} & \xleftarrow{\quad} & \text{ES}_0 & \xrightarrow{\quad} & \text{ES}^r_0 & \xrightarrow{\quad} & Q \times \mathbb{Y}^r \xrightarrow{\kappa^{-1} \times \text{id}} \mathcal{C}_r \times \mathbb{Y}^r \\
 \uparrow & & & & & & \uparrow \\
 \{0\} \times \mathcal{C}_r & \xrightarrow{\quad \kappa \quad} & & & & & Q \times \emptyset^r
 \end{array}$$

Elements $b^\bullet = (b^0, b^1, \dots, b^{r-1}) \in \text{ES}^r$ are called *abaci*. We may write them as $\{0, 1, \dots, r-1\} \times \mathbb{Z}$ matrices with entries in $\{0, 1\}$ where a 0 is a bead and a 1 is a hole (position with no bead) and the i -th row represents the edge sequence b^i and is the i -th runner in the abacus.

There is a bijection $\text{ES} \rightarrow \text{ES}^r$ sending b to $(b^0, b^1, \dots, b^{r-1})$ by letting b^i select the bits in b indexed by integers congruent to $i \bmod r$: $b^i_j = b_{rj+i}$ for $0 \leq i < r$ and $j \in \mathbb{Z}$. The inverse map is given by interleaving the sequences $b^0, b^1, \dots, b^{r-1}$. This bijection is charge-additive: $\text{charge } b = \sum_{j=0}^{r-1} \text{charge } b^j$. The r -fold product of the bijection (2.1) yields the bijection $\text{ES}^r \sim \mathbb{Z}^r \times \mathbb{Y}^r$. Denote this by $b^\bullet = (b^0, \dots, b^{r-1}) \mapsto (c_0, \dots, c_{r-1}, \lambda^\bullet)$. We write $\lambda^\bullet = \text{quot}_r b^\bullet$; this is the r -quotient. Call $(c_0, \dots, c_{r-1}) = c^\bullet b^\bullet$ the charge vector. This indicates the position on each runner where the beads end after pushing all beads to the left within each runner. This defines the bijections going across the top row of the diagram.

We now restrict all these bijections. Let $\text{ES}_0 = \{b \in \text{ES} \mid \text{charge } b = 0\}$ and $\text{ES}^r_0 = \{b^\bullet \in \text{ES}^r \mid \sum_{i=0}^{r-1} c_i b^\bullet = 0\}$. Then $c^\bullet b^\bullet$ can be viewed as an element of the $\mathfrak{sl}_r$ root

lattice Q (and belongs to the zero lattice $Q = 0$ when $r = 1$). The second row of the diagram (save the last map) is given by suitable restrictions of the top row of bijections.

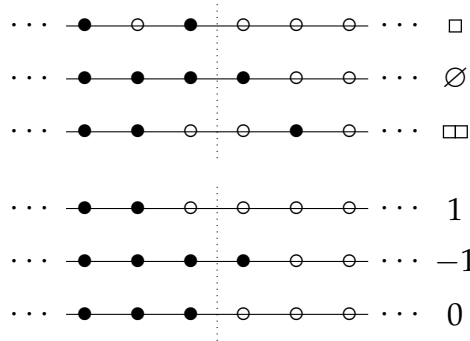
Let $\gamma \in \mathbb{Y}$ and $b \in \text{ES}$ with γ shape b . We see that the following are equivalent: γ is an r -core; $h_\gamma(i, j) \not\equiv r$ for all $i, j \in \gamma$; there is no index k such that $b_k = 1$ and $b_{k+r} = 0$; $\text{quot}_r \gamma$ is the empty multipartition $\emptyset^I$. Therefore the bijection $\{0\} \times \mathbb{Y} \sim Q \times \mathbb{Y}^r$ restricts to the bijection $\{0\} \times \mathcal{C}_r \sim Q \times \emptyset^r$, that is, $\mathcal{C}_r \sim Q$. We call this bijection κ .

Example 2.1. Let $b \in \text{ES}_0$ be as in the previous example. We have λ shape $b = 4, 3, 2, 2$. Set $r = 3$. We map $b \mapsto b^0, b^1, b^2$ which are pictured in the matrix below. Reading up the columns of the $\{0, 1, 2\} \times \mathbb{Z}$ matrix we recover b . Each runner of the abacus is an edge sequence; the corresponding shapes give the 3-quotient of $4, 3, 2, 2$, which is $1, \emptyset, 2$.

To get the 3-core of λ we move all beads to the left in each runner. This produces the second abacus. Reading up columns we obtain the edge sequence $\cdots 0001|1011 \cdots$. Therefore $\text{core}_3 4, 3, 2, 2 = 2$. The charge sequence is $1, -1, 0 \in Q$.

i	$\cdots$	5	4	3	2	1	0	0	-1	-2	-3	-4	-5	-6	$\cdots$
b_i	$\cdots$	0	0	1	1	0	0	0	1	0	1	0	1	1	$\cdots$

	2	1	0	-1	-2	-3
b^0	0	1	0	1	1	1
b^1	0	0	0	0	1	1
b^2	0	0	1	1	0	1



core	i	$\cdots$	5	4	3	2	1	0	-1	-2	-3	-4	-5	-6	$\cdots$
	a_i	0	0	0	0	0	0	1	1	0	1	1	1	1	1

When considering a fixed r , we simply write $\text{core} = \text{core}_r$ and $\text{quot} = \text{quot}_r$.

2.4 Cores to root lattice

Recall that $Q : \{c_0, \dots, c_{r-1} \in \mathbb{Z}^I \mid \sum_{i \in I} c_i = 0\}$ denotes the $\mathfrak{sl}_r$ root lattice (or $Q = 0$ in the case $r = 1$), realized as the zero sum elements in the lattice $\mathbb{Z}^I$. Let $\epsilon_i \in \mathbb{Z}^I$ be the i -th coordinate vector. Then Q is spanned by the elements $\{\alpha_i : \epsilon_{i-1} - \epsilon_i \mid i \in I\}$. We realize the simple roots of $\mathfrak{sl}_r$ as the α_i for $i \neq 0$.

Define the map $\kappa : \mathbb{Y} \rightarrow Q$ by $\kappa \mu = -\sum_{p,q \in \mu} \alpha_{q-p}$. Then the restriction of κ to $\mathcal{C}$ is the same as the bijection $\mathcal{C} \sim Q$ constructed above.

Example 2.2. Let $r = 3$ and consider the 3-core $\gamma = \square\square$. We have $\kappa \square\square = -\alpha_0 - \alpha_2 - \alpha_1$, which agrees with the charge sequence $1, -1, 0 \in Q$ computed above.

We list (in reverse lex order, which refines dominance order) the partitions with core $\gamma = \square\square$ and 3-quotient of size 2.

quot	$\square \cdot \cdot$	$\square \cdot \cdot$	$\square \cdot \square$	$\square \square \cdot$	$\cdot \cdot \square$

quot	$\cdot \square \cdot$	$\cdot \square \square$	$\cdot \cdot \square$	$\cdot \square \cdot$

3 Wreath Macdonald symmetric functions and finitization

3.1 Tensor symmetric functions

Let Λ be the algebra of symmetric functions over $\mathbb{K} = \mathbb{Q}[q, t]$ [2, §I.2]. Denote by $\Lambda^I = \Lambda^{\otimes I}$ the I -fold tensor power of Λ over $\mathbb{K}$. For $f \in \Lambda$, we write $f X^i$ to indicate the element of Λ^I with 1 in tensor factors $j \neq i$ and f in factor i . Λ^I is a polynomial ring over $\mathbb{K}$ with polynomial generators given by the power sums $p_k X^i$ for $i \in I$ and $k > 0$. We write $X^\bullet$ for the I -tuple of alphabets $X^0, \dots, X^{r-1}$ and often denote by $f X^\bullet$ a generic element of Λ^I . For an I -tuple of partitions $\lambda^\bullet = \lambda^0, \lambda^1, \dots, \lambda^{r-1} \in \mathbb{Y}^I$, define the tensor Schur function $s_{\lambda^\bullet} = \bigotimes_{i \in I} s_{\lambda^i} = \prod_{i \in I} s_{\lambda^i} X^i$. The tensor Hall pairing on Λ^I is defined by the orthonormality of the tensor Schur basis: $\langle s_{\lambda^\bullet}, s_{\mu^\bullet} \rangle = \delta_{\lambda^\bullet, \mu^\bullet}$ for $\lambda^\bullet, \mu^\bullet \in \mathbb{Y}^I$.

For any $a = a(q, t) \in \mathbb{K}$, define the $\mathbb{K}$ -algebra automorphism $\mathcal{P}_{\text{id}-a\chi^{-1}}$ of Λ^I by $\mathcal{P}_{\text{id}-a\chi^{-1}} p_k X^i = p_k X^i - a q^k t^k p_k X^{i-1}$ for all $i \in I$ and $k > 0$.

3.2 Wreath Macdonald H and P functions

For $\lambda \in \mathbb{Y}$ let $H_\lambda X^\bullet; q, t$ be the wreath Macdonald H function [1], as defined in [5]. These are characterized by the conditions

$$\mathcal{P}_{\text{id}-q\chi^{-1}} H_\lambda X^\bullet; q, t \in \mathbb{K}^{\times s_{\text{quot } \lambda}} \bigoplus_{\substack{\nu > \lambda \\ \kappa \nu \prec \kappa \lambda}} \mathbb{K} s_{\text{quot } \nu} \quad (3.1)$$

$$\mathcal{P}_{\text{id}-t^{-1}\chi^{-1}} H_\lambda X^\bullet; q, t \in \mathbb{K}^{\times s_{\text{quot } \lambda}} \bigoplus_{\substack{\nu < \lambda \\ \kappa \nu \prec \kappa \lambda}} \mathbb{K} s_{\text{quot } \nu} \quad (3.2)$$

$$\langle s_n X^0, H_\lambda X^\bullet; q, t \rangle = 1. \quad (3.3)$$

where $n = |\text{quot } \lambda|$ and $<$ is the (strict) dominance order on partitions [2, §I.1].

For any $\lambda \in \mathbb{Y}$ the wreath Macdonald P -function $P_\lambda X^\bullet; q, t$ is defined so that $P_\lambda X^\bullet; q, t^{-1}$ is the scalar multiple of $\mathcal{P}_{\text{id}-t^{-1}\chi^{-1}} H_\lambda X^\bullet; q, t$ in which the coefficient of $s_{\text{quot } \lambda}$ is 1. In particular, $P_\lambda X^\bullet; q, t$ satisfies the unitriangularity

$$P_\lambda X^\bullet; q, t \in s_{\text{quot } \lambda} \bigoplus_{\substack{\nu < \lambda \\ \kappa \nu \prec \kappa \lambda}} \mathbb{K} s_{\text{quot } \nu}$$

For any fixed $\alpha \in Q$, $\{P_\lambda X^\bullet; q, t \mid \lambda \in \mathbb{Y} \text{ such that } \kappa \lambda = \alpha\}$ is a homogeneous basis of Λ^I , with $P_\lambda X^\bullet; q, t$ having degree $|\text{quot } \lambda|$.

Example 3.1. For $r = 1$, $P_\lambda X^\bullet; q, t$ is the usual Macdonald P -function.

Example 3.2. If λ is an r -core then $H_\lambda X^\bullet; q, t = 1$ and $P_\lambda X^\bullet; q, t = 1$.

Example 3.3. Let $r = 2$ and $\lambda = 1, 1$. We have $H_{11} X^\bullet; q, t = s_1 X^0 + ts_1 X^1$. To verify this we consider the partitions $2 \supseteq 1, 1$. We need only check

$$\begin{aligned} \mathcal{P}_{\text{id}-t^{-1}\chi^{-1}} s_1 X^0 + ts_1 X^1 &= s_1 X^0 - t^{-1}s_1 X^1 + t s_1 X^1 - t^{-1}s_1 X^0 \\ &= t - t^{-1} s_1 X^1. \end{aligned}$$

and note that $s_{\text{quot } 1,1} = s_{\emptyset, 1} = s_1 X^1$. It follows that $P_{1,1} X^\bullet; q, t^{-1} = s_1 X^1$.

3.3 Symmetric polynomials

Recall the sets of variables (1.3). There is a restriction map

$$\pi_{N_\bullet} : \Lambda^I \rightarrow \Lambda_{N_\bullet}^I : \bigotimes_{i \in I} \mathbb{K} [x_1^i, \dots, x_{N_i}^i]^{\mathfrak{S}_{N_i}} \quad (3.4)$$

given by the tensor product $\pi_{N_\bullet} = \bigotimes_{i \in I} \pi_{N_i}$, where $\pi_N : \Lambda \rightarrow \mathbb{K} [x_1, \dots, x_N]^{\mathfrak{S}_N}$ is the standard projection to symmetric polynomials. We also write $\pi_{N_\bullet} f = f_{X_{N_\bullet}}$.

Let γ be an r -core and $\kappa \gamma = c_0, c_1, \dots, c_{r-1} \in Q \subset \mathbb{Z}^I$. We say that $N_\bullet \in \mathbb{Z}_{\geq 0}^I$ is compatible with γ if

$$N_i - N_{i-1} = \alpha_i^\vee, \kappa \gamma = c_{i-1} - c_i, \quad \text{for all } i \in I, \quad (3.5)$$

where $\langle -, - \rangle : Q^\vee \times Q \rightarrow \mathbb{Z}$ is the standard pairing between $\mathfrak{sl}_r$ root and coroot lattices. All $N_\bullet$ compatible with a given γ differ from each other by a vector in $\mathbb{Z}^{1^r}$.

Example 3.4. In the setting of [Example 2.2](#), the core is $\gamma = 2$ and $\kappa \gamma = 1, -1, 0$. The smallest $N_\bullet$ compatible with γ is $N_\bullet = 0, 2, 1 = -\kappa \lambda = 1, 1, 1$. To this we can add the vector $1, 1, 1$ any number of times to obtain the other $N_\bullet$ compatible with γ .

Lemma 3.5. *Let γ be an r -core and $N_\bullet \in \mathbb{Z}_{\geq 0}^I$ be compatible with γ . Then*

1. *The quantity $N = |N_\bullet| : \sum_{i \in I} N_i$ is divisible by r .*
2. *For all $\lambda \in \mathbb{Y}_{N, \gamma}$ and all $p \in I$,*

$$N_p = |\{1 \leq i \leq N : i - \lambda_i - 1 \equiv p \pmod{r}\}|.$$

In particular, quotient $\lambda = \lambda^\bullet$ satisfies $\ell(\lambda^i) \leq N_i$ for all $i \in I$.

3. *For any $\lambda^\bullet \in \mathbb{Y}^I$ satisfying $\ell(\lambda^i) \leq N_i$ for all i , the unique partition λ with core γ and quotient $\lambda^\bullet$ satisfies $\lambda \in \mathbb{Y}_{N, \gamma}$.*

An immediate consequence of parts (2) and (3) of [Lemma 3.5](#) is the following:

Proposition 3.6. *Let γ be an r -core and $N_\bullet$ be compatible with γ . Then $\{P_\lambda X_{N_\bullet}; q, t \mid \lambda \in \mathbb{Y}_{N, \gamma}\}$ forms a basis of the space of color symmetric polynomials $\Lambda_{N_\bullet}^I$.*

4 Wreath Macdonald eigenoperators

Fix an r -core γ and compatible $N_\bullet \in \mathbb{Z}_{\geq 0}^I$. As before, let $X_{N_\bullet}$ be the variables as in (1.3) satisfying (3.5). Define a *shift pattern* of $X_{N_\bullet}$ to be a subset of $X_{N_\bullet}$ that contains no more than one variable of each color. A shift pattern *contains the color* $p \in I$ if it contains a variable of color p . Let $Sh_p X_{N_\bullet}$ denote the set of all shift patterns containing p .

For a shift pattern $\underline{J}$, let $J \subset I$ denote the set of colors of the variables in $\underline{J}$. We denote the variables in $\underline{J}$ by $x_{\underline{J}}^i$, so $\underline{J} = \{x_{\underline{J}}^i\}_{i \in J}$. To $\underline{J}$ we associate the following:

1. *Gap labels:* For $i \in I$ let $i^\nabla \in J$ be the first element less than or equal to i in the cyclic order. We stipulate that $0 \leq i - i^\nabla \leq r - 1$. Define

$$x_{\underline{J}^\nabla}^i = q^{i - i^\nabla} x_{\underline{J}}^{i^\nabla}.$$

In particular $x_{\underline{J}^\nabla}^i = x_{\underline{J}}^i$ if $i \in J$. Thus, while $\underline{J}$ gives a list of variables colored by $J \subset I$, we ‘fill in the gaps’ for values $i \in I \setminus J$ with q -shifts of the elements of $\underline{J}$.

2. A cyclic-shift operator: For $i \in J$, let $i^\nabla \in J$ be the first element strictly less than i in the cyclic order. We set $1 \leq i - i^\nabla \leq r$, where r occurs if and only if $|J| = \{i\}$. We then define the operator $T_{\underline{J}}$ on $\mathbb{K}[X_{N_\bullet}]$ as the $\mathbb{K}$ -algebra map induced by

$$T_{\underline{J}} x_l^i = \begin{cases} q^{i-i^\nabla} x_{\underline{J}}^{i^\nabla} & \text{if } i \in J \text{ and } x_l = x_{\underline{J}}^i \\ x_l^i & \text{otherwise.} \end{cases}$$

For an n -tuple $\underline{J} = J_1, \dots, J_n$ of shift patterns and $0 \leq k \leq n$, we denote

$$|\underline{J}| = J_1 \cup \dots \cup J_n \subset X_{N_\bullet}, \quad |\underline{J}|_{\leq k} = J_1 \cup \dots \cup J_k \subset X_{N_\bullet}, \quad |\underline{J}|_{\geq k} = J_k \cup \dots \cup J_n \subset X_{N_\bullet}.$$

If $\underline{J}$ is an n -tuple of shift patterns all containing color p , we say $\underline{J}$ is p -distinct if the p -colored variables $x_{J_k}^p$ are all distinct. Let $Sh_p^n(X_{N_\bullet})$ denote the set of all p -distinct n -tuples of shift patterns containing color p .

Finally, if $a_1, a_2, \dots, a_n$ are operators then define $\prod_{j=1}^n a_j : a_1 \circ a_2 \circ \dots \circ a_n$. This completes the definition of the operators $D_{p,n}$ in (1.4).

Example 4.1. Let $r = 1$, $p = 1$, $N_\bullet = (2, 1, 0)$, and $\lambda = (3, 1, 1)$. In this case, λ is a 3-core and by [Example 3.2](#) $P_\lambda(X_\bullet; q, t) = 1$ so that $P_\lambda(X_{N_\bullet}; q, t) = 1$. There are three shift patterns containing 1: $J_1 = \{x_1^1\}$, $J_2 = \{x_1^0, x_1^1\}$, and $J_3 = \{x_2^0, x_1^1\}$. We have

$$D_{1,1}(X_{N_\bullet}; q, t) = q \left(\frac{qtx_1^1 - x_1^0}{q^2x_1^1 - x_1^0} \right) \left(\frac{qtx_1^1 - x_2^0}{q^2x_1^1 - x_2^0} \right) T_{J_1} \quad (4.1)$$

$$1 - qt^{-1} q \left(\frac{qtx_1^1 - x_2^0}{x_1^0 - x_2^0} \right) \left(\frac{qtx_1^1}{x_1^0 - q^2x_1^1} \right) T_{J_2} \quad (4.2)$$

$$1 - qt^{-1} q \left(\frac{qtx_1^1 - x_1^0}{x_2^0 - x_1^0} \right) \left(\frac{qtx_1^1}{x_2^0 - q^2x_1^1} \right) T_{J_3}. \quad (4.3)$$

The cyclic-shift operators act trivially on $P_\lambda(X_{N_\bullet}; q, t) = 1$. After a miraculous simplification one obtains $D_{1,1}(X_{N_\bullet}; q, t) = P_\lambda(X_{N_\bullet}; q, t) = qP_\lambda(X_{N_\bullet}; q, t)$.

Example 4.2. Let $r = 2$, $p = 0$, $N_\bullet = (1, 1)$, and $\lambda = (1, 1)$. We have $P_{1,1}(X_\bullet; q, t) = s_1(X^1)$ from [Example 3.3](#). Finitizing we obtain $P_\lambda(X_{N_\bullet}; q, t) = x_1^1$. There are two shift patterns containing 0: $J_1 = \{x_1^0\}$, $J_2 = \{x_1^0, x_1^1\}$. We then have

$$D_{0,1}(X_{N_\bullet}; q, t) = q \left(\frac{tx_1^0 - x_1^1}{qx_1^0 - x_1^1} \right) T_{J_1} = 1 - qt^{-1} \frac{x_1^1}{x_1^0} \left(\frac{tx_1^0}{x_1^1 - qx_1^0} \right) T_{J_2}.$$

Observe that $T_{J_1} x_1^1 = x_1^1$ and $T_{J_2} x_1^1 = qx_1^0$. Altogether then,

$$\begin{aligned} D_{0,1} X_{N_\bullet}; q, t P_\lambda X_{N_\bullet}; q, t &= q \left(\frac{tx_1^0 - x_1^1}{qx_1^0 - x_1^1} \right) x_1^1 - 1 - qt^{-1} \frac{x_1^1}{x_1^0} \left(\frac{tx_1^0}{x_1^1 - qx_1^0} \right) qx_1^0 \\ &\quad - qx_1^1 \left(\frac{tx_1^0 - x_1^1 - t - q x_1^0}{qx_1^0 - x_1^1} \right) = qt P_\lambda X_{N_\bullet}; q, t. \end{aligned}$$

Example 4.3. Let $r = 2$, $p = 1$, $N_\bullet = 0, 2$, and $\lambda = 1$. Because λ is a 2-core, $P_\lambda X_{N_\bullet}; q, t = 1$. There are only two shift patterns containing 1: $J_1 = \{x_1^1\}$, $J_2 = \{x_2^1\}$. Note that $T_{J_1} x_1^1 = q^2 x_1^1$, $T_{J_1} x_2^1 = x_1^1$, $T_{J_2} x_1^1 = x_1^1$, $T_{J_2} x_2^1 = q^2 x_2^1$. Therefore,

$$\begin{aligned} D_{1,1} X_{N_\bullet}; q, t P_\lambda X_\bullet; q, t &= \frac{-1 - 1 - qt^{-1}}{1 - q^2 t^{-2}} \left\{ \frac{qtx_2^1 - q^2 x_1^1}{x_1^1 - x_2^1} - \frac{qtx_1^1 - q^2 x_2^1}{x_2^1 - x_1^1} \right\} \\ &\quad - \frac{-1 - 1 - qt^{-1} - qt - q^2}{1 - q^2 t^{-2}} = qt P_\lambda X_{N_\bullet}; q, t. \end{aligned}$$

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The isomorphism problem for cominuscule Schubert varieties

Edward Richmond^{*1}, Mihail Tarigradschi⁺², and Weihong Xu^{‡3}

¹Department of Mathematics, Oklahoma State University, Stillwater, OK

²Department of Mathematics, Rutgers University, Piscataway, NJ

³Department of Mathematics, Virginia Tech, Blacksburg, VA

Abstract. Cominuscule flag varieties generalize Grassmannians to other Lie types. Schubert varieties in cominuscule flag varieties are indexed by posets of roots labeled long/short. These labeled posets generalize Young diagrams. We prove that Schubert varieties in potentially different cominuscule flag varieties are isomorphic as varieties if and only if their corresponding labeled posets are isomorphic, generalizing the classification of Grassmannian Schubert varieties using Young diagrams by the last two authors. Our proof is type-independent.

Keywords: Schubert variety, flag variety, cominuscule

1 Introduction

Cominuscule flag varieties correspond to algebraic varieties that admit the structure of a compact Hermitian symmetric space and have been studied extensively due their shared properties with Grassmannians [1, 3, 14, 22, 8, 19, 4, 20, 5, 18, 6]. These varieties come in four infinite families and two exceptional types and are determined by a pair $(\mathcal{D}, \gamma)$ of a Dynkin diagram $\mathcal{D}$ of a reductive Lie group and a cominuscule simple root γ . See Table 1 for a classification of cominuscule flag varieties. Let X denote the cominuscule flag variety corresponding to $(\mathcal{D}, \gamma)$ and R denote the root system of the Dynkin diagram $\mathcal{D}$. Set $\mathcal{P}_X = \{\alpha \in R : \alpha \geq \gamma\}$ with the partial order $\alpha \leq \beta$ if $\beta - \alpha$ is a non-negative sum of simple roots, and give $\mathcal{P}_X$ a labeling of long/short roots. By [7, Theorem 2.4], Schubert varieties in X are indexed by lower order ideals in $\mathcal{P}_X$, generalizing the fact that Schubert varieties in a Grassmannian are indexed by Young diagrams.

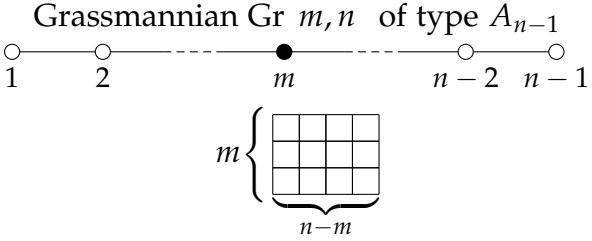
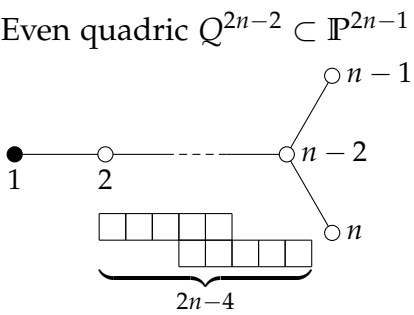
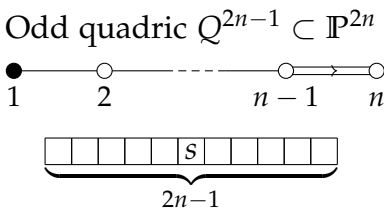
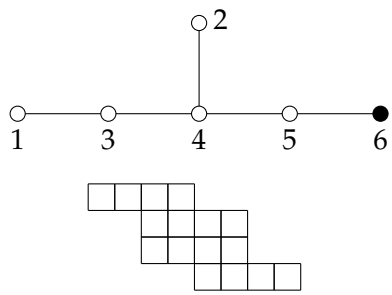
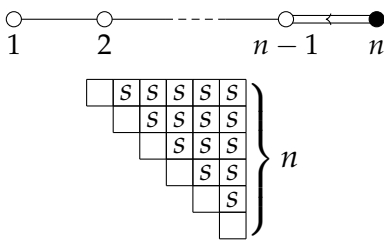
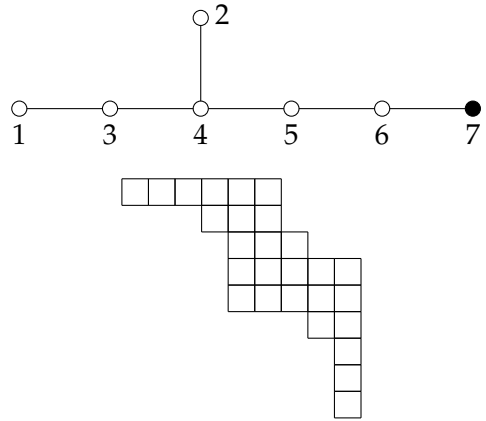
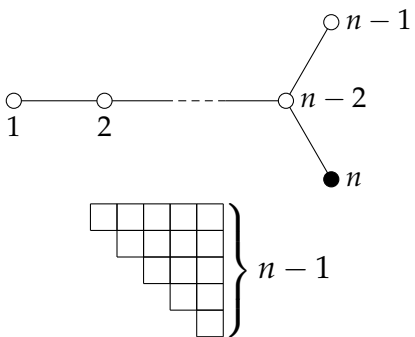
Our main result Theorem 1 is a combinatorial criterion for distinguishing isomorphism classes of Schubert varieties coming from cominuscule flag varieties.

^{*}edward.richmond@okstate.edu, supported by a grant from the Simons Foundation 941273

[†]mt994@math.rutgers.edu, supported by NSF grant MS-2152316

[‡]weihong@vt.edu, supported by NSF grant MS-2152316

Table 1: The labeled poset $\mathcal{P}_X$ for a cominuscle flag variety X . Each element in $\mathcal{P}_X$ is drawn as a box, and boxes decorated with an “s” correspond to short roots. The partial order on boxes is given by $\alpha \leq \beta$ if and only if α is weakly north-west of β . This table is a modification of [6, Table 1].

Grassmannian $\text{Gr } m, n$ of type A_{n-1} 	Even quadric $Q^{2n-2} \subset \mathbb{P}^{2n-1}$ 
Odd quadric $Q^{2n-1} \subset \mathbb{P}^{2n}$ 	Cayley Plane E_6/P_6 
Lagrangian Grassmannian $\text{LG } n, 2n$ 	Freudenthal variety E_7/P_7 
Max. orthog. Grassmannian $\text{OG } n, 2n$ 	

Theorem 1. *Let $X_\lambda \subseteq X$ and $Y_\mu \subseteq Y$ be cominuscule Schubert varieties indexed by lower order ideals $\lambda \subseteq \mathcal{P}_X$ and $\mu \subseteq \mathcal{P}_Y$, respectively. Then X_λ and Y_μ are algebraically isomorphic if and only if λ and μ are isomorphic as labeled posets.*

For illustrative examples of [Theorem 1](#), see [Section 2](#).

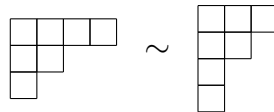
Since Grassmannians are cominuscule flag varieties, [Theorem 1](#) extends the work of Tarigradschi and Xu in [\[21\]](#), where they prove two Grassmannian Schubert varieties are isomorphic if and only if their Young diagrams are the same or the transpose of each other. Other related works include Richmond and Slofstra’s characterization of the isomorphism classes of Schubert varieties coming from complete flag varieties in [\[17\]](#) using Cartan equivalence. However, they also note that Cartan equivalence is neither necessary nor sufficient to distinguish Schubert varieties in partial flag varieties. A class of smooth Schubert varieties in type A partial flag varieties are classified by Develin, Martin, and Reiner in [\[9\]](#). Yet many Schubert varieties are singular, with the first example being the Schubert divisor in the Grassmannian $\text{Gr } 2, 4$.

We discuss preliminaries in [Section 3](#), and then in [Section 4](#), we prove [Theorem 1](#) and illustrate it with examples. Our proof is type-independent and employs several new techniques.

2 Examples of [Theorem 1](#)

For the following examples, recall that cominuscule Schubert varieties are indexed by lower order ideals in $\mathcal{P}_X$. Examples of $\mathcal{P}_X$ are illustrated in [Table 1](#), where each element in $\mathcal{P}_X$ is drawn as a box, and boxes decorated with an “s” correspond to short roots. The partial order on boxes is given by $\alpha \leq \beta$ if and only if α is weakly north-west of β . In particular, lower order ideals are given by subsets of boxes that are closed under moving to the north and west.

Example 2. As illustrated below, transposing a Young diagram does not change the poset structure. Therefore, two Grassmannian Schubert varieties are isomorphic if their indexing Young diagrams are the transpose of each other. Geometrically, this is related to the isomorphism $\text{Gr } m, m-k \sim \text{Gr } k, m-k$.



Example 3. Using [Table 1](#), it is not hard to see that if a Grassmannian Schubert variety is isomorphic to a non-type A cominuscule Schubert variety, then they are both isomorphic to a projective space. Indeed, in order to fit inside a $\mathcal{P}_X$ of another type, the lower order ideal is forced to be a chain.

Example 4. The Schubert divisor in Q^3 is not isomorphic to $\mathbb{P}^2$, because the labeling of their posets does not match:

$$\boxed{} \boxed{S} \neq \boxed{} \boxed{} .$$

Example 5. The quadric Q^3 embeds in LG $n, 2n$ ($n \geq 3$) as a Schubert variety, as illustrated by

	S	S	S
		S	S
			S

Let G be a complex reductive linear algebraic group. We fix subgroups $T \subset B \subset G$, where T is a maximal torus and B is a Borel subgroup. With this setup, $T \subset G$ determines

a root system R of G , with corresponding Weyl group $W : N(T)/T$, and B determines a set of simple roots $\Delta \subseteq R$. The set of roots decomposes into positive and negative roots $R = R^+ \sqcup R^-$, with R^+ being non-negative sums of simple roots. The Weyl group W is generated by the set of simple reflections

$$S : \{s_\alpha : \alpha \in \Delta\}.$$

To each subset $I \subseteq S$ one can associate a Weyl subgroup $W_I : \langle s : s \in I \rangle \subseteq W$, a parabolic subgroup $P_I = BW_I B \subseteq G$ and the corresponding (partial) flag variety $X = G/P_I$. Schubert varieties in X are indexed by W^I , the set of minimal length coset representatives of W/W_I . Explicitly, for $w \in W^I$, the Schubert variety

$$X_w : \overline{BwP_I/P_I}$$

has dimension the Coxeter length of w , denoted $\ell(w)$. Moreover, for any $u \in W^I$, we have $X_u \subseteq X_w$ if and only if $u \leq w$ in Bruhat order.

From now on, X is a cominuscule flag variety. In other words, $I = S \setminus \{s_\gamma\}$, where γ is a cominuscule simple root, i.e., γ appears with coefficient 1 in the highest root of R . Cominuscule roots are illustrated by filled-in circles in Table 1.

Recall that

$$\mathcal{P}_X : \{\alpha \in R : \alpha \geq \gamma\}$$

inherits the usual partial order on roots, i.e., $\alpha \leq \beta$ if $\beta - \alpha$ is a non-negative sum of simple roots, and in addition, we give $\mathcal{P}_X$ a labeling of long/short roots.

In [15], Proctor proves that W^I is a distributive lattice under the induced Bruhat partial order from W . Birkhoff's representation theorem implies there is a bijection between W^I and the set of lower order ideals in $\mathcal{P}_X$. In particular, the join-irreducible elements of W^I are identified with principal lower order ideals of $\mathcal{P}_X$ and hence with $\mathcal{P}_X$ itself. Explicitly, to each $w \in W^I$ we associate its inversion set

$$\lambda(w) : \{\alpha \in R^+ : w.\alpha < 0\}, \quad (3.1)$$

viewed as a sub-poset of $\mathcal{P}_X$. It is well known that $\ell(w) = |\lambda(w)|$. Moreover, the following proposition was proved in [7, Theorem 2.4 and Corollary 2.6]:

Proposition 9 (Buch–Samuel). *For any $w \in W^I$, the inversion set $\lambda(w)$ is a lower order ideal in $\mathcal{P}_X$. Moreover:*

1. *The map $w \mapsto \lambda(w)$ is a bijection between W^I and the set of lower order ideals in $\mathcal{P}_X$.*
2. *For any $u \in W^I$, we have $u \leq w$ in Bruhat order if and only if $\lambda(u) \subseteq \lambda(w)$.*
3. *If $\alpha \in \lambda(w)$ and $\lambda(w) \setminus \{\alpha\}$ is a lower order ideal, then $ws_\alpha \in W^I$ and $\lambda(ws_\alpha) = \lambda(w) \setminus \{\alpha\}$, where $s_\alpha \in W$ is the reflection corresponding to α .*

Notation 10. Given a lower order ideal $\lambda \subseteq \mathcal{P}_X$, we will write w_λ for the element of W^I corresponding to λ in [Proposition 9](#). We also use $X_\lambda : X_{w_\lambda}$ to denote the corresponding Schubert variety.

In [Section 4.2](#), we will use a map $\delta : \mathcal{P}_X \rightarrow \Delta$ defined in [\[6\]](#) as follows.

Definition 11. For $\alpha \in \mathcal{P}_X$, let λ_α be the principal lower order ideal generated by α . Let $s_{\delta \alpha} = -w_{\lambda_\alpha} \cdot \alpha \in R$. Then $s_{\delta \alpha} = w_{\lambda_\alpha} s_\alpha w_{\lambda_\alpha}^{-1}$ has length 1 [\[6, Section 4.1\]](#). Therefore, $\delta \alpha \in \Delta$.

The following lemma is a restatement of [\[7, Corollary 2.10\]](#). See also [\[6, Section 4.1\]](#).

Lemma 12 (Buch–Samuel). *Let $\lambda \subseteq \mathcal{P}_X$ be a lower order ideal and $\beta_1, \beta_2, \dots, \beta_\ell$ be the boxes it contains written in an increasing order. Then $s_{\delta \beta_\ell} \cdots s_{\delta \beta_2} s_{\delta \beta_1}$ is a reduced decomposition of w_λ . Moreover, every reduced decomposition of w_λ can be obtained in this way.*

4 Proof of [Theorem 1](#)

In this section, we prove each direction of [Theorem 1](#) separately.

4.1 Forward direction: the isomorphism class of X_λ determines the labeled poset λ

Let $i_\lambda : X_\lambda \hookrightarrow X$ denote the embedding of a Schubert subvariety into a cominuscle flag variety $X = G/P_I$. The definition of the labeled poset λ depends on the root system of the reductive group G and hence on the embedding $i_\lambda : X_\lambda \hookrightarrow X$. The goal of this section is to show that λ (as a labeled poset) can be constructed using only the variety structure of X_λ and is therefore intrinsic to the isomorphism class of X_λ . We prove the following proposition which states the “forward” direction of [Theorem 1](#).

Proposition 13. *Let $X_\lambda \subseteq X$ and $Y_\mu \subseteq Y$ be cominuscle Schubert varieties indexed by lower order ideals $\lambda \subseteq \mathcal{P}_X$ and $\mu \subseteq \mathcal{P}_Y$, respectively. If X_λ and Y_μ are algebraically isomorphic, then λ and μ are isomorphic as labeled posets.*

Our primary tools come from the intersection theory of algebraic varieties (see [\[10\]](#) for more details). Let $\text{Pic } X_\lambda$ and $A_* X_\lambda$ denote the Picard and Chow groups of X_λ . It is well known that these groups are algebraic invariants of X_λ . Recall that the k -th Chow group $A_k Z$ of a scheme Z is the free abelian group on the k -dimensional subvarieties of Z modulo rational equivalence. When Z is a normal variety, the Picard group $\text{Pic } Z$ can be identified with the subgroup of $A_{\dim Z - 1} Z$ generated by classes of locally principal

divisors (note that all Schubert varieties are normal). Our aim is to construct the labeled poset λ from the intersection class map or intersection product [10, Definition 2.3]:

$$\text{Pic } X_\lambda \times A_* X_\lambda \rightarrow A_* X_\lambda .$$

If $\sigma, \tau \in \text{Pic } X_\lambda \times A_* X_\lambda$, we denote the image of the intersection product by $\sigma \cdot \tau$. Next, we consider the effective cone of a scheme:

Definition 14. Let Z be a scheme. The *effective cone* in the Chow group $A_* Z$ is the semigroup in $A_* Z$ generated by the classes of closed subvarieties of Z .

Since the flag variety X is cominuscule, there is a unique Schubert variety of codimension 1 in X , called the Schubert divisor. Its class, denoted D , is the unique effective generator of the Picard group $\text{Pic } X \subseteq A_* X$. Recall that $i_\lambda : X_\lambda \hookrightarrow X$ is a closed embedding of varieties. Lemma 15 below follows from [13, Proposition 6] (see also [2, Proposition 2.2.8 part (ii)]).

Lemma 15. For any non-empty lower order ideal $\lambda \subseteq \mathcal{P}_X$, the map $i_\lambda^* : \text{Pic } X \rightarrow \text{Pic } X_\lambda$ is an isomorphism.

Since D is effective and generates $\text{Pic } X$, Lemma 15 implies that $i_\lambda^* D$ the unique effective generator of $\text{Pic } X_\lambda$. Recall from Proposition 9 that we have lower order ideals $\mu \subseteq \lambda$ if and only if $w_\mu \leq w_\lambda$ in Bruhat order. Hence, we have $\mu \subseteq \lambda$ if and only if $X_\mu \subseteq X_\lambda$. We write X_μ for the class of X_μ in $A_* X_\lambda$ and in $A_* X$, in an abuse of notation. It is well known that the classes $\{X_\mu\}_{\mu \subseteq \lambda}$ form an integral basis of $A_* X_\lambda$. Lemma 16 below is a special case of [11, Corollary of Theorem 1] and allows us to identify Schubert classes (the effective cone) in $A_* X_\lambda$ without referring to the embedding $i_\lambda : X_\lambda \hookrightarrow X$ or using the B -variety structure of X_λ .

Lemma 16 (Fulton–MacPherson–Sottile–Sturmfels). The Schubert classes X_μ such that $\mu \subseteq \lambda$ are exactly the minimal elements in the extremal rays of the effective cone in $A_* X_\lambda$.

We shall see later that the poset structure of λ can be recovered from the intersection products $i_\lambda^* D \cdot X_\mu$. Let $j : X_\mu \hookrightarrow X_\lambda$ be an inclusion of Schubert varieties, where $\mu \subseteq \lambda$. By the definition of intersection product,

$$i_\lambda^* D \cdot X_\mu = j^* i_\lambda^* D = i_\lambda \circ j^* D = D \cdot X_\mu .$$

This implies that the product $i_\lambda^* D \cdot X_\mu$ on X_λ can be computed on X . By [10, Example 19.1.11], $A_* X$ can be identified with the homology group $H_* X$. By [10, Proposition 19.1.2] we have that the intersection product $D \cdot X_\mu$ can be identified with a cap product. Since X is a smooth complex variety, the Poincaré duality further identifies the intersection product with the cup product of cohomology classes corresponding to D and X_μ . This cup product is given by the Chevalley formula [12, Lemma 8.1]. Using these identifications, we restate the Chevalley formula for cominuscule flag varieties (and hence Schubert varieties):

Lemma 17 (Fulton–Woodward). *Let X be a cominuscule flag variety with corresponding cominuscule simple root γ . For any lower order ideals $\mu \subseteq \lambda \subseteq \mathcal{P}_X$, we have*

$$i_\lambda^* D \cdot X_\mu = \sum_{\alpha} \frac{\gamma, \gamma}{\alpha, \alpha} X_{\mu \setminus \{\alpha\}},$$

the sum over all positive roots α such that $\mu \setminus \{\alpha\}$ is a lower order ideal in $\mathcal{P}_X$. Here $\cdot, \cdot$ denotes the usual inner product.

Observe that [Lemma 17](#) reinterprets the Chevalley formula as a degree lowering operator since intersection product with divisors is a map from $A_k X_\lambda$ to $A_{k-1} X_\lambda$. This is opposite to the standard presentation of the Chevalley formula as a degree raising operator in cohomology.

Example 18. By [Lemma 17](#), the following calculations hold for $X = \text{LG}(3, 6)$. We refer to [Table 1](#) for the poset $\mathcal{P}_X$.

$$D \cdot \left[X_{\begin{smallmatrix} \square & \square & \square \\ & \square & \square \end{smallmatrix}} \right] = 2 \left[X_{\begin{smallmatrix} \square & \square \\ & \square \end{smallmatrix}} \right] - \left[X_{\begin{smallmatrix} \square & \square \\ & \square \end{smallmatrix}} \right], \quad D \cdot \left[X_{\begin{smallmatrix} \square & \square \\ & \square \end{smallmatrix}} \right] = \left[X_{\begin{smallmatrix} \square & \square \\ & \square \end{smallmatrix}} \right], \quad D \cdot \left[X_{\begin{smallmatrix} \square & \square \\ & \square \end{smallmatrix}} \right] = 2 \left[X_{\begin{smallmatrix} \square & \square \\ & \square \end{smallmatrix}} \right].$$

Note that a coefficient 2 occurs whenever the removed box (root) is short.

Proof of Proposition 13. [Lemma 15](#) allows us to identify the unique effective generator $i_\lambda^* D$ of $\text{Pic } X_\lambda$, and [Lemma 16](#) allows us to identify Schubert classes in $A^* X_\lambda$, or equivalently, their indexing set: the set of lower order ideals in $\mathcal{P}_X$ that are contained in λ . The Bruhat order on this set is recovered from [Lemma 17](#), and its sub-poset of join-irreducible elements recovers the poset λ as discussed in [Section 3](#). Finally, note that if μ is a join-irreducible diagram contained in λ with (unique) maximal box α , then

$$i_\lambda^* D \cdot X_\mu = \frac{\gamma, \gamma}{\alpha, \alpha} X_{\mu \setminus \{\alpha\}}.$$

Since γ is always long, the coefficient $\gamma, \gamma / \alpha, \alpha$ is not 1 if and only if α is short. Hence, the labeling of the boxes in λ is also determined. $\square$

4.2 Converse direction: the labeled poset λ determines the isomorphism class of X_λ

Let $X = G/P_I$ be a cominuscule flag variety and $\lambda \subseteq \mathcal{P}_X$ be a lower order ideal. In this section, we prove that the poset λ and its labeling of long/short roots determine the isomorphism class of X_λ . More precisely, we prove the following proposition, which states the “converse” direction of [Theorem 1](#).

Proposition 19. *Let $X_\lambda \subseteq X$ and $Y_\mu \subseteq Y$ be cominuscule Schubert varieties indexed by lower order ideals $\lambda \subseteq \mathcal{P}_X$ and $\mu \subseteq \mathcal{P}_Y$, respectively. If λ and μ are isomorphic as labeled posets, then X_λ and Y_μ are algebraically isomorphic.*

Our strategy is to embed X_λ in a “minimal” flag variety X' determined by the labeled poset λ .

Recall that S is the set of simple reflections defined in [Section 3](#).

Definition 20. The *support* of λ is defined as

$$S_\lambda : \{s \in S : s \leq w_\lambda\}.$$

Equivalently, S_λ is the set of simple reflections appearing in any reduced decomposition of w_λ .

Every reduced decomposition of w_λ , and in particular, S_λ , can be read out from the poset λ [[6](#), Section 4]. The variety X' is constructed using S_λ as follows. Let G' be the reductive subgroup of P_{S_λ} with Weyl group $W' : W_{S_\lambda}$ and $P' : G' \cap P_I$ be the reductive subgroup of G' corresponding to $I' : I \cap S_\lambda$. Set $X' : G'/P'$. Note that $w_\lambda \in W'^{I'}$.

[Lemma 21](#) below is a restatement of [[16](#), Lemma 4.8].

Lemma 21 (Richmond–Slofstra). *The inclusion $X' \hookrightarrow X$ induces an isomorphism $X'_{w_\lambda} \rightarrow X_\lambda$.*

Let Y be another cominuscule flag variety and $\mu \subseteq \mathcal{P}_Y$ be a lower order ideal. Next, we show that a labeled poset isomorphism between λ and μ induces an isomorphism between X' and Y' , which restricts to an isomorphism between X_λ and Y_μ . We shall see that X' and Y' are cominuscule and that this isomorphism is given by an isomorphism of their Dynkin diagrams.

In the following, let $\mathcal{D}_X$ be the Dynkin diagram of X with vertex set Δ_X .

Definition 22. The diagram $\mathcal{D}_X^\lambda$ is defined to be the full subgraph of $\mathcal{D}_X$ with vertex set

$$\Delta_X^\lambda : \{\alpha \in \Delta_X : s_\alpha \in S_\lambda\}.$$

Definition 23. Let a *Dynkin chain* in $\mathcal{P}_X$ be a chain $\pi \subseteq \mathcal{P}_X$ such that:

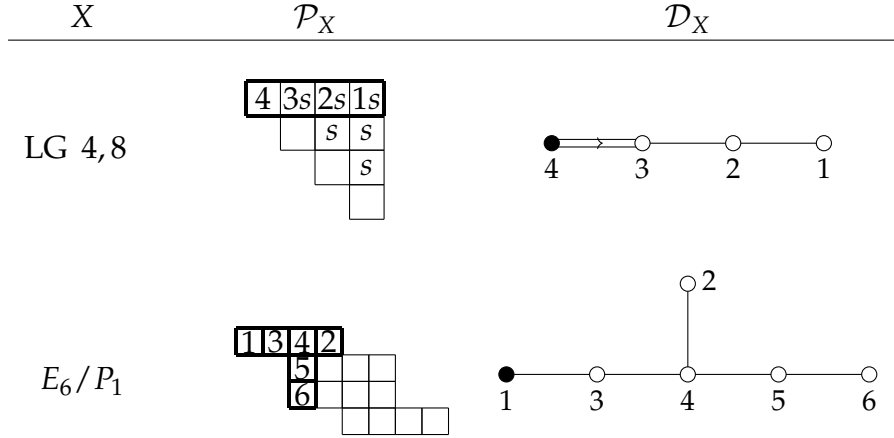
1. the set π is a lower order ideal;
2. the lengths of roots in π are weakly decreasing.

The lower order ideal $\mathcal{P}_X^\Delta \subseteq \mathcal{P}_X$ is defined to be the union of all Dynkin chains in $\mathcal{P}_X$.

In fact, Dynkin chains in $\mathcal{P}_X$ correspond to paths in $\mathcal{D}_X$ starting from the cominuscule root γ . Examples of $\mathcal{P}_X^\Delta$ are illustrated in [Figure 1](#).

The proof of [Lemma 24](#) and [Corollary 25](#) below are left to the reader.

Figure 1: We highlight $\mathcal{P}_X^\Delta \subset \mathcal{P}_X$ with a bold border and label its boxes by the images of δ .



Lemma 24. *The restriction $\delta : \mathcal{P}_X^\Delta \rightarrow \Delta_X$ is a bijection.*

Corollary 25. *Let $\lambda \subseteq \mathcal{P}_X$ be a lower order ideal. Then $\delta \lambda \cap \mathcal{P}_X^\Delta = \Delta_X^\lambda$, and $\mathcal{D}_X^\lambda$ is a connected Dynkin diagram.*

Remark 26. Corollary 25 implies that X' is the cominuscule flag variety given by $\mathcal{D}_X^\lambda, \gamma$.

The last ingredient is Proposition 27, a purely combinatorial result. Geometrically, it implies that the “minimal” cominuscule flag varieties for Schubert varieties with isomorphic labeled posets are isomorphic.

Proposition 27. *Let $\lambda \subseteq \mathcal{P}_X$ and $\mu \subseteq \mathcal{P}_Y$ be lower order ideals. Then every labeled poset isomorphism between λ and μ induces a graph isomorphism between $\mathcal{D}_X^\lambda$ and $\mathcal{D}_Y^\mu$ that identifies reduced decompositions of w_λ and w_μ .*

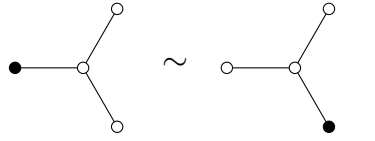
Proof of Proposition 19. Let γ_X and γ_Y denote the cominuscule simple roots corresponding to the cominuscule flag varieties X and Y . Let X' and Y' denote the cominuscule flag varieties given by the pairs $\mathcal{D}_X^\lambda, \gamma_X$ and $\mathcal{D}_Y^\mu, \gamma_Y$. Proposition 27 implies the cominuscule flag varieties X' and Y' are isomorphic. By Lemma 21, this isomorphism restricts to an isomorphism between X_λ and Y_μ , upon identifying them with Schubert varieties in X' and Y' , respectively. $\square$

We illustrate the above process with Example 28 and Example 29 below.

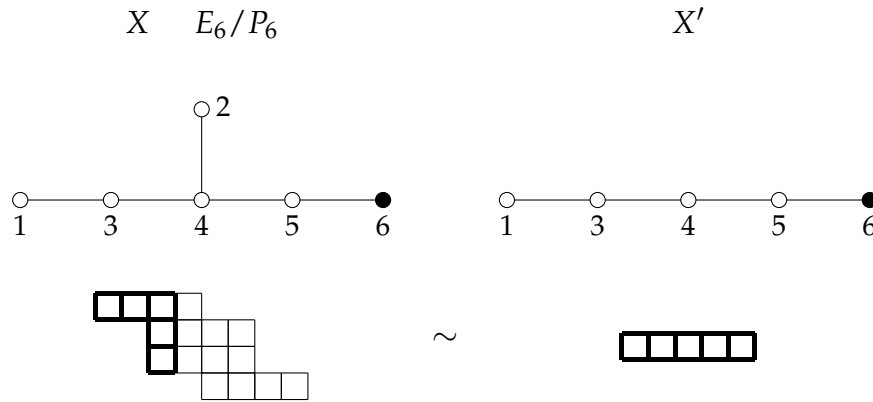
Example 28.

$$\mathcal{P}_{Q^6} \quad \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \sim \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \quad \mathcal{P}_{OG\ 4,8},$$

therefore, $Q^6 \sim OG\ 4,8$. This isomorphism comes from a symmetry of the D_4 Dynkin diagram:



Example 29. Let $X = E_6/P_6$ and λ be the lower order ideal depicted on the left below. Then $S_\lambda = \{s_1, s_3, s_4, s_5, s_6\}$, where s_i is the simple reflection corresponding to the simple root labeled by i . Therefore, the pair $(\mathcal{D}_X^\lambda, \gamma)$ is as depicted on the right, isomorphic to that of $\mathbb{P}^5$, showing $X_\lambda \sim X' \sim \mathbb{P}^5$.



Acknowledgements

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0-Hecke Modules, Quasisymmetric Functions, and Peak Functions in Type B

Colin Defant^{*1} and Dominic Searles^{†2}

¹*Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA*

²*Department of Mathematics and Statistics, University of Otago, Dunedin 9016, New Zealand*

Abstract. We extend the recently-introduced *ascent-compatibility framework* to arbitrary Coxeter systems, yielding a uniform method for constructing modules of 0-Hecke algebras. In type B, we apply this method to produce 0-Hecke modules whose type B quasisymmetric characteristics are notable type B quasisymmetric functions. We also construct a 0-Hecke module on the set of standard domino tableaux of a given shape; the type B quasisymmetric characteristic of this module is a certain type B analogue of a Schur function called a *domino function*. We then introduce an analogous function defined on shifted domino tableaux and prove that this function expands positively in the type B analogue of the peak functions. Finally, we introduce a type B variant of the 0-Hecke–Clifford algebra and consider the modules of this algebra induced from the simple type B 0-Hecke modules. We characterize the isomorphism classes of these induced modules in terms of type B peak sets and prove that the type B quasisymmetric characteristics of the restrictions of these induced modules are precisely the type B peak functions.

1 Introduction

The 0-Hecke algebra H_W^0 of a Coxeter system (W, S) is a certain deformation of the group algebra of W . In type A, the Grothendieck group of finite-dimensional 0-Hecke modules is isomorphic to the Hopf algebra QSym of quasisymmetric functions via the *quasisymmetric characteristic map* [7]. There has been a great deal of recent work devoted to finding 0-Hecke modules whose quasisymmetric characteristics are certain notable quasisymmetric functions, thereby giving representation-theoretic interpretations for those quasisymmetric functions, and studying their properties, e.g., [1], [2], [9], [11], [13], [15]. Recently, a framework for producing such modules that relies on a simple *ascent-compatibility condition* was introduced in [14]; as far as we are aware, all of

^{*}colindefant@gmail.com. Colin Defant was supported by the National Science Foundation under Award No. 2201907 and by a Benjamin Peirce Fellowship at Harvard University.

[†]dominic.searles@otago.ac.nz. Dominic Searles was supported by the Marsden Fund, administered by the Royal Society of New Zealand Te Apārangi.

the notable families of quasisymmetric functions for which 0-Hecke modules have been constructed can be obtained via the ascent-compatibility framework.

In this extended abstract, we summarize results from [6]. Our first main result extends the ascent-compatibility condition from type A to all Coxeter systems, thus providing a framework for constructing 0-Hecke modules for any Coxeter system. Most of our attention is directed toward hyperoctahedral groups, where previous researchers have already laid the groundwork to extend much of the story from type A. Indeed, Chow [5] introduced the ring QSym^B of type B quasisymmetric functions, and Huang [8] recently defined a type B quasisymmetric characteristic map from the Grothendieck group of finite-dimensional type B 0-Hecke modules to QSym^B . We provide several applications in which the ascent-compatibility framework readily produces type B 0-Hecke modules whose quasisymmetric characteristics are interesting type B quasisymmetric functions, thus providing a representation-theoretic interpretation of these functions.

Next, we define an action of the type B 0-Hecke algebra on *standard domino tableaux* of partition shape, thus obtaining a family of type B 0-Hecke modules whose quasisymmetric characteristics are the type B analogues of Schur functions defined in [10]. We also consider the shifted domino tableaux that were introduced and used to provide an expansion formula for products of Schur Q -functions in [4]. We define variants of these tableaux whose associated generating functions are in QSym^B , and we prove that these type B analogues of the Schur Q -functions expand positively in Petersen's type B peak functions [12], with the expansion indexed by type B peak sets of standard shifted domino tableaux.

Finally, we consider the *0-Hecke–Clifford algebra* obtained by combining the 0-Hecke algebra and the Clifford algebra. The representation theory of this algebra is governed by the peak subalgebra of QSym [3], and type A ascent-compatibility was applied in [14] to construct 0-Hecke–Clifford modules for important families of functions in the peak algebra. We define a type B analogue of the 0-Hecke–Clifford algebra and show that the isomorphism classes of the modules of this algebra induced from the simple type B 0-Hecke modules are indexed by the type B peak sets. We further show that the quasisymmetric characteristics of the restrictions of these modules to the type B 0-Hecke algebra are precisely the type B peak functions [12], thus providing a representation-theoretic interpretation of these functions.

2 Background

2.1 Quasisymmetric functions

A *composition* $\alpha = \alpha_1, \alpha_2, \dots, \alpha_k$ is a finite sequence of positive integers. The integers α_i are called the *parts* of α . If the parts of α sum to n , then we say α is a composition of n . There is a natural bijection between compositions of n and subsets of $[n-1]$: given a

composition $\alpha = \alpha_1, \alpha_2, \dots, \alpha_k$ of n , its *descent set* $\text{Des } \alpha$ is the set $\{\alpha_1, \alpha_1 - \alpha_2, \dots, \alpha_1 - \alpha_2 - \dots - \alpha_{k-1}\}$.

For $I \subseteq [n-1]$, the *fundamental quasisymmetric function* F_I is defined by

$$F_I = \sum_{\substack{1 \leq i_1 \leq i_2 \leq \dots \leq i_n \\ j \in I \Rightarrow i_j < i_{j+1}}} x_{i_1} x_{i_2} \dots x_{i_n}.$$

As α ranges over all compositions, the fundamental quasisymmetric functions $F_{\text{Des } \alpha}$ form a basis for the algebra QSym of quasisymmetric functions.

Example 2.1. If $n = 4$, then $F_{\{2,3\}} = \sum_{1 \leq i_1 \leq i_2 < i_3 < i_4} x_{i_1} x_{i_2} x_{i_3} x_{i_4}$.

A type B analogue of the fundamental quasisymmetric functions was introduced by Chow [5]. These involve an additional variable x_0 . For $I \subseteq [0, n-1]$, the *type B fundamental quasisymmetric function* F_I^B is defined by

$$F_I^B = \sum_{\substack{0 \leq i_0 \leq i_1 \leq \dots \leq i_n \\ j \in I \Rightarrow i_j < i_{j+1}}} x_{i_1} x_{i_2} \dots x_{i_n}.$$

Note that the variable x_0 appears in F_I^B if and only if $0 \notin I$.

Example 2.2. If $n = 4$, then

$$F_{\{2,3\}}^B = \sum_{0 \leq i_1 \leq i_2 < i_3 < i_4} x_{i_1} x_{i_2} x_{i_3} x_{i_4} \quad \text{and} \quad F_{\{0,3\}}^B = \sum_{0 < i_1 \leq i_2 \leq i_3 < i_4} x_{i_1} x_{i_2} x_{i_3} x_{i_4}.$$

A subset $P \subseteq [n-1]$ is called a *peak set* if $1 \notin P$ and P contains no two consecutive integers. The compositions whose descent sets are peak sets are called *peak compositions*; these are precisely the compositions whose parts are all greater than 1, except possibly the last part. Given a peak set $P \subseteq [n-1]$, the *peak function* K_P is defined as

$$K_P = 2^{|P|-1} \sum_{J \subseteq [n-1] : P \subseteq J \Delta J-1} F_J,$$

where $J-1 = \{j-1 : j \in J\}$ and Δ denotes symmetric difference.

Petersen introduced type B analogues of the peak functions in [12]. A subset $P \subseteq [n-1]$ is called a *type B peak set* if P contains no two consecutive integers; note that a type B peak set is permitted to contain 1. For each type B peak set P , there are two type B peak functions $K_{0,P}$ and $K_{1,P}$ defined by

$$K_{0,P} = 2^{|P|} \sum_{J \subseteq [0, n-1] : P \subseteq J \Delta J-1} F_J^B \quad \text{and} \quad K_{1,P} = 2^{|P|-1} \sum_{0 \in J \subseteq [0, n-1] : P \subseteq J \Delta J-1} F_J^B,$$

with the proviso that $K_{1,P}$ is defined only if $1 \notin P$. For $I \subseteq [0, n-1]$, let $\text{Peak}^B I = \{p \in I : p \geq 1 \text{ and } p-1 \notin I\}$ and define

$$\Delta^B I = \begin{cases} K_{0, \text{Peak}^B I} & \text{if } 0 \notin I \\ K_{1, \text{Peak}^B I} & \text{if } 0 \in I. \end{cases}$$

2.2 0-Hecke algebras and modules

Given symbols γ, δ and a nonnegative integer r , we write $\gamma \mid \delta_r$ for the word $\gamma\delta \cdots$ of length r that starts with γ and alternates between γ and δ . For example, $\gamma \mid \delta_5 = \gamma\delta\gamma\delta\gamma$.

Let W, S be a Coxeter system. Thus, W is generated by S , and each element of S is an involution. For any distinct $s, t \in S$, we have the relation $s \mid t_{m_{st}} = t \mid s_{m_{st}}$, where $m_{st} \in \{2, 3, \dots\} \cup \{\infty\}$ (the relation is not present if $m_{st} = \infty$). The *0-Hecke algebra* H_W^0 of the Coxeter system W, S is a certain deformation of the group algebra of W ; it has a generating set $\{\pi_s : s \in S\}$ satisfying the relations

$$\pi_s^2 = -\pi_s \quad \text{and} \quad \pi_s \mid \pi_t_{m_{st}} = \pi_t \mid \pi_s_{m_{st}}.$$

The simple H_W^0 -modules are all one-dimensional and indexed by subsets I of S . The structure of a simple H_W^0 -module S_I is given by

$$\pi_j v = \begin{cases} -v & \text{if } j \in I \\ 0 & \text{if } j \notin I, \end{cases}$$

where v is a basis element of S_I .

We will mainly consider the 0-Hecke algebras where W is the symmetric group S_n or the hyperoctahedral group B_n . These are referred to as, respectively, the type A 0-Hecke algebra H_n^0 and the type B 0-Hecke algebra H_n^B . The type A 0-Hecke algebra H_n^0 is generated by $\{\pi_1, \dots, \pi_{n-1}\}$ with relations

$$\begin{aligned} \pi_i^2 &= -\pi_i; \\ \pi_i \pi_j &= \pi_j \pi_i \quad \text{if } |i - j| \geq 2; \\ \pi_i \pi_{i-1} \pi_i &= \pi_{i-1} \pi_i \pi_{i-1}, \end{aligned}$$

while the type B 0-Hecke algebra H_n^B is generated by $\{\pi_0, \pi_1, \dots, \pi_{n-1}\}$ with the same relations as above for $1 \leq i \leq n-1$ and the additional relations

$$\begin{aligned} \pi_0^2 &= -\pi_0; \\ \pi_0 \pi_i &= \pi_i \pi_0; \quad \text{if } i \geq 2 \\ \pi_0 \pi_1 \pi_0 \pi_1 &= \pi_1 \pi_0 \pi_1 \pi_0. \end{aligned}$$

For $I \subseteq \{1, \dots, n-1\}$, let us (slightly abusing notation) write S_I for the simple H_n^0 -module corresponding to the set $\{s_i : i \in I\}$ of simple generators of A_{n-1} . Similarly, for $I \subseteq \{0, 1, \dots, n-1\}$, write S_I^B for the simple H_n^B -module corresponding to the set $\{s_i : i \in I\}$ of simple generators of B_n .

The *quasisymmetric characteristic map* is the Hopf algebra isomorphism ch from the Grothendieck group of finite-dimensional type A 0-Hecke modules to QSym defined by $\text{ch}(S_I) = F_I$ [7]. Huang [8] defined a type B quasisymmetric characteristic map ch^B from the Grothendieck group of finite-dimensional type B 0-Hecke modules to QSym^B by $\text{ch}^B(S_I^B) = F_I^B$.

3 Ascent-compatibility and applications in type B

Let (W, S) be a Coxeter system. A simple generator $s \in S$ is an *ascent* of an element $w \in W$ if $\ell(sw) > \ell(w)$; otherwise, s is a *descent* of w . The set of descents of w is denoted $\text{Des } w$. Fix $X \subseteq W$, and let $\mathbf{N}_X$ be the complex vector space with basis X . For $s \in S$, define a linear operator $\pi_s : \mathbf{N}_X \rightarrow \mathbf{N}_X$ by

$$\pi_s x = \begin{cases} -x & \text{if } s \in \text{Des } x \\ 0 & \text{if } s \notin \text{Des } x \text{ and } sx \notin X \\ sx & \text{if } s \notin \text{Des } x \text{ and } sx \in X \end{cases}$$

for all $x \in X$. We say X is *ascent-compatible* if for all $v, w \in X$ and all $s, t \in S$ such that s is an ascent of v , t is an ascent of w , and $v^{-1}sv = w^{-1}tw$, we have that $sv \in X$ if and only if $tw \in X$.

Theorem 3.1. *Let (W, S) be a Coxeter system. If X is an ascent-compatible subset of W , then the operators $\{\pi_s : s \in S\}$ define an action of the 0-Hecke algebra of W on $\mathbf{N}_X$.*

The type A version of Theorem 3.1 was proved in [14].

Example 3.2. Consider the type A Coxeter group S_3 , which has two simple generators s_1, s_2 . Let

$$X = \{1, 2, 3, 2, 1, 3, 2, 3, 1, 1, 3, 2\} \subseteq S_3$$

where the elements of X are written in one-line notation. This set X is not ascent-compatible, because $1, 2, 3 \xrightarrow{-1s_1} 1, 2, 3$ and $2, 3, 1 \xrightarrow{-1s_2} 2, 3, 1$ are equal, but $s_1 1, 2, 3 = 2, 1, 3 \in X$ whereas $s_2 2, 3, 1 = 3, 2, 1 \notin X$.

On the other hand, the set

$$X' = \{2, 3, 1, 3, 1, 2, 1, 3, 2\} \subseteq S_3$$

is ascent-compatible.

We now provide a family of ascent-compatible sets across all Coxeter groups. Given a Coxeter system (W, S) , the *left weak order* $\leq_L$ is the partial order on W defined by $x \leq_L y$ if there is a reduced word for y that contains a reduced word for x as a suffix. If $x \leq_L y$, then the *interval* between x and y in the left weak order is the set $x, y \llbracket = \{z \in W : x \leq_L z \leq_L y\}$. A set $X \subseteq W$ is called *convex* (in the left weak order) if for all $x, y \in X$ with $x \leq_L y$, we have $x, y \llbracket \subseteq X$.

Theorem 3.3. *Let (W, S) be a Coxeter system, and let $X \subseteq W$. If X is convex in the left weak order and has a unique maximal element under the left weak order, then X is ascent-compatible.*

In type A, Theorem 3.3 generalizes the result that intervals in left weak order on S_n are ascent-compatible [14]. The corresponding 0-Hecke modules were studied in [9] and shown to be very useful for interpreting and relating many previously-studied families of 0-Hecke modules in type A.

For 0-Hecke algebras for which a quasisymmetric characteristic map is defined, one can obtain a formula for the characteristic of $\mathbf{N}_X$. In particular, in types A and B, we have the following (the type A version of this theorem appeared in [14]).

Theorem 3.4. *Let X be an ascent-compatible subset of S_n and Y an ascent-compatible subset of B_n . Then*

$$\text{ch } \mathbf{N}_X = \sum_{x \in X} F_{\text{Des } x} \quad \text{and} \quad \text{ch}^B \mathbf{N}_Y = \sum_{y \in Y} F_{\text{Des } y}^B.$$

Example 3.5. Let $\mathbf{N}_{X'}$ be the 0-Hecke module defined on the set X' from Example 3.2. Then

$$\text{ch } \mathbf{N}_{X'} = F_{\{1\}} + 2F_{\{2\}}.$$

Given a function expressed as sums of type A or type B fundamental quasisymmetric functions associated to descents of Coxeter group elements, Theorems 3.1 and 3.4 yield a method for finding a 0-Hecke module whose quasisymmetric characteristic is that function. Specifically, it suffices to determine that the set of Coxeter group elements indexing the expansion is ascent-compatible. There has been significant interest in finding 0-Hecke modules for many such functions in type A, and the type A version of these theorems was used in [14] to give a common interpretation of these modules.

Motivated by a type B analogue of Schur positivity, Mayorova and Vassilieva [10] studied several interesting subsets of B_n and functions associated to them. For $X \subseteq B_n$, let

$$\mathcal{Q} X = \sum_{x \in X} F_{\text{Des } x^{-1}}^B.$$

Note that realizing $\mathcal{Q} X$ as the characteristic of an $H_n^B(0)$ -module via Theorems 3.1 and 3.4 follows from ascent-compatibility of the set $X^{-1} = \{x^{-1} : x \in X\}$.

We view B_n as the group of permutations σ of $\{\pm 1, \dots, \pm n\}$ satisfying $\sigma(-i) = -\sigma(i)$ for all i . Two sets $X \subseteq B_n$ whose corresponding functions $\mathcal{Q} X$ were studied in [10] are the *left descent classes* and the *left-unimodal* permutations. The left descent class associated to $I \subseteq [0, n-1]$ is the set $D_I^B = \{\sigma \in B_n : \text{Des } \pi \subseteq I\}$, and the left-unimodal permutations are the elements of $L^B = \bigcup_{i=1}^n L_i^B$, where L_i^B is the set of all $\sigma \in B_n$ such that $\sigma^{-1}(1) > \dots > \sigma^{-1}(i) < \dots < \sigma^{-1}(n)$.

Theorem 3.6. *The sets $D_I^B(0)^{-1}$ and $L_i^B(0)^{-1}$ are ascent-compatible. Therefore, both $\mathbb{C}[D_I^B(0)^{-1}]$ and $\mathbb{C}[L_i^B(0)^{-1}]$ are $H_n^B(0)$ -modules, and we have*

$$\text{ch}^B \mathbb{C}[D_I^B(0)^{-1}] = \mathcal{Q} D_I^B \quad \text{and} \quad \text{ch}^B \mathbb{C}[L_i^B(0)^{-1}] = \mathcal{Q} L_i^B;$$

the latter of these identities implies that $\text{ch}^B \bigoplus_{i=1}^n \mathbb{C}[L_i^B(0)^{-1}] = \mathcal{Q} L^B$.

The sets D_I^B and L_i^B are also ascent-compatible, so we additionally obtain $\mathcal{Q} = D_I^B - 1$ and $\mathcal{Q} = L_i^B - 1$ as characteristics of $H_n^B(0)$ -modules. Let us remark that each L_i^B is a subset of a left descent class; more generally, any subset of a left descent class is ascent-compatible.

Another example considered in [10] is the *type B Knuth class* C_T^B associated to a standard domino tableau T : this is the set of all $\sigma \in B_n$ that insert to T under a certain type B analogue of the Robinson–Schensted correspondence (see Section 4 for the definition of a standard domino tableau). The set $C_T^B - 1$ is not in general ascent-compatible, but C_T^B itself is a subset of a left descent class, thus ascent-compatible. Therefore the functions $\mathcal{Q} = C_T^B - 1$ are characteristics of $H_n^B(0)$ -modules.

The final example considered in [10] is the *signed arc permutations* $\mathcal{A}_n^B$. These are the elements of B_n whose one-line notation is a shuffle of a word $a_1 \cdots a_p$ over the alphabet $[n]$ and a word $b_1 \cdots b_{n-p}$ over the alphabet $[-n]$ satisfying the following conditions:

- $a_{i-1} \equiv a_i - 1 \pmod{n}$ for all $1 \leq i \leq p-1$;
- $b_{j-1} \equiv b_j - 1 \pmod{n}$ for all $1 \leq j \leq n-p-1$;
- $a_1 \equiv -b_1 - 1 \pmod{n}$ if p and $n-p$ are both nonzero.

The set of inverses of signed arc permutations is not ascent-compatible, but the set of signed arc permutations themselves is. Accordingly, we obtain the following theorem.

Theorem 3.7. *The set $\mathcal{A}_n^B$ is ascent-compatible. Therefore, $\mathbb{C}\mathcal{A}_n^B$ is an $H_n^B(0)$ -module, and*

$$\text{ch}^B \mathbb{C}\mathcal{A}_n^B = \mathcal{Q} = \mathcal{A}_n^B - 1.$$

4 Domino tableaux and type B peak functions

A central motivation for the families of functions considered in Section 3 is their positive expansions into a type B analogue of Schur functions known as domino functions, which have a formula in terms of domino tableaux [10]. In this section, we define an action of the type B 0-Hecke algebra on standard domino tableaux, thus realizing the domino functions as quasisymmetric characteristics of 0-Hecke modules. We then consider a shifted analogue of domino tableaux introduced by Chemli [4], modify these to allow 0 entries, and introduce standard shifted domino tableaux. We show that the generating functions of the (modified) shifted domino tableaux expand positively in the type B peak functions, indexed by standard shifted domino tableaux.

Let $\lambda \vdash 2n$. A *domino tiling* of λ is a tiling of the Young diagram of shape λ by 1×2 and 2×1 rectangles called *dominoes*. A *standard domino tableau* of shape λ is a bijective filling of a domino tiling of λ with entries from $[n]$ such that entries increase from left to

1	2	5
	4	6
3		7

Figure 1: A standard domino tableau of shape $(5,4,4,1) \vdash 14$.

right along rows and from top to bottom in columns (see Figure 1). Let $\text{SDT } \lambda$ denote the set of all standard domino tableaux of shape λ .

Suppose $T \in \text{SDT } \lambda$. Let $\text{dom}_i T$ denote the domino in T filled with the entry i . A number $i \in [0, n-1]$ is a *descent* of T if $i = 0$ and $\text{dom}_1 T$ is vertical or if $i > 0$ and $\text{dom}_{i-1} T$ is strictly lower than $\text{dom}_i T$. Let $\text{Des } T$ be the set of descents of T . For example, the descent set of the domino tableau in Figure 1 is $\{0, 2, 5, 6\}$. The *domino function* G_λ [10] is defined by

$$G_\lambda = \sum_{T \in \text{SDT } \lambda} F_{\text{Des } T}^B.$$

For $1 \leq i \leq n-1$, let $s_i T$ be the tableau obtained by exchanging the entries i and $i+1$. If $\text{dom}_1 T$ and $\text{dom}_2 T$ tile a square in T , then we say these dominoes are *aligned* and let $s_0 T$ be the tableau obtained by switching the orientation of both these dominoes (i.e., from horizontal to vertical or vice versa); otherwise, we say $s_0 T$ is not defined (in particular, we would say $s_0 T \notin \text{SDT } \lambda$ in this case). We define operators $\pi_0, \pi_1, \dots, \pi_{n-1}$ on standard domino tableaux by

$$\pi_i T = \begin{cases} -T & \text{if } i \in \text{Des } T \\ 0 & \text{if } i \notin \text{Des } T \text{ and } s_i T \notin \text{SDT } \lambda \\ s_i T & \text{if } i \notin \text{Des } T \text{ and } s_i T \in \text{SDT } \lambda. \end{cases}$$

Example 4.1. Let $\lambda = (4, 3, 3)$, and suppose

$$T = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & \\ \hline \end{array} \in \text{SDT } \lambda.$$

Then

$$\pi_0 T = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 4 & 5 \\ \hline \end{array} \quad \text{and} \quad \pi_2 T = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 & \\ \hline \end{array}.$$

We also have $\pi_1 T = \pi_3 T = -T$ and $\pi_4 T = 0$.

Theorem 4.2. *The operators $\pi_0, \pi_1, \dots, \pi_{n-1}$ define an $H_n^B 0$ -action on the space spanned by the standard domino tableaux of shape λ . Moreover, the type B quasisymmetric characteristic of the resulting $H_n^B 0$ -module is precisely the domino function G_λ .*

We now consider the shifted domino tableaux [4]. The definition of these tableaux requires the notion of the *2-quotient* of a partition $\lambda = \lambda_1, \dots, \lambda_k$, which is the pair of partitions μ, ν obtained as follows. Let λ^* be the partition obtained by adding $k - i$ to the part λ_i for each $1 \leq i \leq k$. Now let w be obtained from λ^* by replacing the odd parts of λ^* with $1, 3, 5, \dots$ and the even parts of λ^* with $0, 2, 4, \dots$, from right to left. Then μ (respectively, ν) is the partition found by subtracting the even (respectively, odd) entries of w from the even (respectively, odd) entries of λ^* , dividing the result by 2, and removing any 0 entries.

Example 4.3. If $\lambda = 7, 7, 6, 5, 1$, then $\lambda^* = 11, 10, 8, 6, 1$, $w = 3, 4, 2, 0, 1$, $\mu = 3, 3, 3$, and $\nu = 4$.

Let λ be a partition whose 2-quotient $\mu = \mu_1, \dots, \mu_p, \nu = \nu_1, \dots, \nu_q$ satisfies $\mu_p \geq p$ and $\nu_q \geq q$. A domino tiling of λ is a *shifted tiling* of λ if there is no vertical domino d lying on the main diagonal such that all dominoes left of d and adjacent to d are strictly below the main diagonal. Given a shifted tiling of λ , Chemli [4] defined a *semistandard shifted domino tableau* of shape λ to be a filling of the dominoes in the tiling lying weakly above the main diagonal with entries from $1' < 1 < 2' < 2 < \dots$ such that

1. entries weakly increase from left to right along rows and from top to bottom in columns
2. each row contains at most one i' , and each column contains at most one i .

Under this definition, the weight generating function of semistandard shifted domino tableaux of shape λ is equal to the product of the Schur Q -functions indexed by μ and ν [4].

We modify Chemli's definition of semistandard shifted domino tableaux to introduce *zeroed semistandard shifted domino tableaux*. In these tableaux, we allow 0 entries (but not $0'$), where $0 < 1' < 1 < \dots$; we also require that all dominoes with a 0 entry are horizontal. See the left of Figure 2. Let $\text{SSShDT } \lambda$ denote the set of zeroed semistandard shifted domino tableaux. For $T \in \text{SSShDT } \lambda$, let the *weight* $\text{wt } T$ of T be defined by $\text{wt } T = \text{wt}_0 T, \text{wt}_1 T, \dots$, where $\text{wt}_i T$ is the number of entries in T equal to i or i' (so $\text{wt}_0 T$ is just the number of 0 entries in T). Let $x^{\text{wt } T} = \prod_{i \geq 0} x_i^{\text{wt}_i T}$. We define the *shifted domino function* H_λ by

$$H_\lambda = \sum_{T \in \text{SSShDT } \lambda} x^{\text{wt } T}.$$



Figure 2: A semistandard shifted domino tableau $T \in \text{SSShDT } \lambda$ (left) and a standard shifted domino tableau $U \in \text{SShDT } \lambda$ (right), where $\lambda = 7, 7, 6, 5, 1$. We have $\text{wt } T = 1, 4, 0, 1, 2, 2$ and $\text{Des } U = \{1, 5, 7, 8\}$.

The zeroed variant of semistandard shifted domino tableaux is analogous to a zeroed variant of a semistandard version of (nonshifted) domino tableaux introduced in [10], whose weight generating function is G_λ . In this way, the shifted domino functions may be regarded as a type B analogue of the Schur Q -functions, similarly to how domino functions provide a type B analogue of Schur functions.

Just as the Schur Q -functions expand positively in the peak functions, we claim the shifted domino function expands positively in the type B peak functions. To this end, we define a *standard shifted domino tableau* of shape λ to be a bijective filling of the dominoes weakly above the main diagonal in a shifted tiling of λ with entries in $[m]$ (where m is the number of dominoes weakly above the main diagonal) such that entries strictly increase from left to right along rows and from top to bottom in columns. See the right of Figure 2. Let $\text{SShDT } \lambda$ denote the set of standard shifted domino tableaux of shape λ .

Let $U \in \text{SShDT } \lambda$, and suppose U uses the entries $1, \dots, m$. Write $\text{dom}_i U$ for the domino filled with the entry i in U . An integer $i \in [0, m-1]$ is a *descent* of U if $i = 0$ and $\text{dom}_1 U$ is vertical or if $i > 0$ and $\text{dom}_{i-1} U$ is strictly lower than $\text{dom}_i U$. Let $\text{Des } U$ be the set of descents of U . Recall the definition of the type B peak function $\Delta^B I$ from Section 2.1.

Theorem 4.4. *We have*

$$H_\lambda = \sum_{U \in \text{SShDT } \lambda} \Delta^B \text{Des } U.$$

We remark that the same operators used in Theorem 4.2 also define a $H_n^B \otimes \mathbb{C}$ -action on the space spanned by $\text{SShDT } \lambda$.

5 A type B analogue of the 0-Hecke–Clifford algebra

The *0-Hecke–Clifford algebra* $HCl_n \otimes \mathbb{C}$ is the algebra with generators $\pi_1, \dots, \pi_{n-1}$ and $c_1, \dots, c_n$ such that the π_i generate $H_n \otimes \mathbb{C}$, the c_j satisfy the Clifford relations $c_j^2 = -1$

and $c_i c_j = c_j c_i$ for $i \neq j$, and the π_i and c_j satisfy the additional relations

$$\begin{aligned} \pi_i c_j &= c_j \pi_i \quad \text{for } j \neq i, i \neq 1; \\ \pi_i c_i &= c_{i-1} \pi_i; \\ \pi_i &= 1 - c_{i-1} - c_i \pi_i - 1. \end{aligned}$$

We construct an analogue $HCl_n^B 0$ of this algebra using the type B 0-Hecke algebra $H_n^B 0$. This algebra has generators $\pi_0, \pi_1, \dots, \pi_{n-1}$ and $c_1, \dots, c_n$ such that the π_i generate $H_n^B 0$, the c_j satisfy the same Clifford relations $c_j^2 = -1$ and $c_i c_j = c_j c_i$ for $i \neq j$, and the π_i and c_j satisfy the additional relations

$$\begin{aligned} \pi_i c_j &= c_j \pi_i \quad \text{for } j \neq i, i \neq 1; \\ \pi_i c_i &= c_{i-1} \pi_i \quad \text{for } 1 \leq i \leq n-1; \\ \pi_i &= 1 - c_i - c_{i-1} \pi_i - 1; \\ \pi_0 c_1 &= \sqrt{-1} \pi_0. \end{aligned}$$

Recall that we write S_I^B for the simple $H_n^B 0$ -module corresponding to the set $\{s_i : i \in I\}$ of simple generators of B_n . Let M_I^B denote the $HCl_n^B 0$ -module induced from S_I^B , and consider the restriction $\text{Res}_{H_n^B 0} M_I^B$ of M_I^B to $H_n^B 0$. The following theorem, whose type A analogue was proven in [3], provides a representation-theoretic interpretation of Petersen's type B peak functions $\Delta^B I$ [12].

Theorem 5.1. *Let $I \subseteq [0, n-1]$ and let $I^c = [0, n-1] \setminus I$. Then*

$$\text{ch}^B \text{Res}_{H_n^B 0} M_{I^c}^B = \Delta^B I.$$

Moreover, two induced modules M_I^B and M_J^B are isomorphic if and only if

$$\text{Peak}^B I^c = \text{Peak}^B J^c \quad \text{and} \quad 0 \notin I \triangle J.$$

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Modules of the 0-Hecke algebras for genomic Schur functions

Young-Hun Kim^{*1} and Semin Yoo⁺²

¹Center for quantum structures in modules and spaces, Seoul National University, Seoul 08826, Republic of Korea

²School of Computational Sciences, Korea Institute for Advanced Study, 85 Hoegiro Dongdaemun-gu, Seoul 02455, Republic of Korea

Abstract. We construct an $H_m(0)$ -module $\mathbf{G}_{\lambda,m}$ whose image under the quasisymmetric characteristic is the m th degree homogeneous component of the genomic Schur function U_λ by defining an $H_m(0)$ -action on increasing gapless tableaux. We then provide a direct sum decomposition of $\mathbf{G}_{\lambda,m}$ and show that each summand of this decomposition is isomorphic to a weak Bruhat interval module.

Keywords: 0-Hecke algebra, weak Bruhat order, genomic Schur function, quasisymmetric characteristic

1 Introduction

Let $X = \text{Gr}_k(\mathbb{C}^n)$ be the Grassmannian of k -dimensional subspaces of $\mathbb{C}^n$. Since the early 2000s, several combinatorial interpretations for the *K-theoretic Littlewood-Richardson rule* have been introduced. For instance, see [3, 11, 15, 16]. In particular, Pechenik and Yong [11] gave a combinatorial interpretation by using *genomic tableaux*. Therein, they defined a symmetric function U_λ , called the *genomic Schur function*, as a generating function for genomic tableaux of shape λ for all partition λ . Further, they proved that $\{U_\lambda \mid \lambda \text{ is a partition}\}$ is a basis for the ring of symmetric functions and pointed out that genomic Schur functions are not Schur-positive in general. As an alternative positivity, Pechenik [10] showed that genomic Schur functions are fundamental positive. Specifically, for any partition λ ,

$$U_\lambda = \sum_{T \in \text{IGLT}(\lambda)} F_{\text{comp}(T)},$$

^{*}ykim.math@gmail.com. Young-Hun Kim was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government (NRF-2020R1A5A1016126 and NRF-2022R1A2C1004045).

⁺syoo19@kias.re.kr. Semin Yoo was supported by the KIAS Individual Grant (CG082701) at Korea Institute for Advanced Study.

where $\text{IGLT}(\lambda)$ is the set of *increasing gapless tableaux* of shape λ , $\text{comp}(T)$ is the composition associated to T , and $F_{\text{comp}(T)}$ is the *fundamental quasisymmetric function* associated to $\text{comp}(T)$. For the precise definitions, see Subsection 2.4.

The 0-Hecke algebra $H_n(0)$ is the $\mathbb{C}$ -algebra obtained from the Hecke algebra $H_n(q)$ by specializing q to 0. Duchamp, Krob, Leclerc, and Thibon [6] introduced a ring isomorphism, called *quasisymmetric characteristic*,

$$\text{ch} : \mathcal{G}_0(H_\bullet(0)) \rightarrow \text{QSym}, \quad [\mathbf{F}_\alpha] \mapsto F_\alpha.$$

Here, $\mathcal{G}_0(H_\bullet(0))$ is the Grothendieck ring associated to 0-Hecke algebras and QSym is the ring of quasisymmetric functions. In view of this correspondence, there have been considerable attempts to provide a representation theoretic interpretation of noteworthy quasisymmetric functions by constructing appropriate 0-Hecke modules. For instance, see [1, 2, 4, 12, 13, 14]. Recently, Jung, Kim, Lee, and Oh [7] introduced the *weak Bruhat interval module* $B(\sigma, \rho)$ to provide a unified method to study the $H_n(0)$ -modules in the papers mentioned above. Here, σ and ρ are permutations in the symmetric group $\mathfrak{S}_n$.

The purpose of this paper is to provide a nice representation theoretic interpretation of genomic Schur functions. First, for each $1 \leq m \leq n$, we construct an $H_m(0)$ -module $\mathbf{G}_{\lambda;m}$ by defining an $H_m(0)$ -action on the $\mathbb{C}$ -span of the set $\text{IGLT}(\lambda)_m$ of increasing gapless tableaux of shape λ with maximum entry m . And, we see that the image of $\mathbf{G}_{\lambda;m}$ under the quasisymmetric characteristic is the m th homogeneous component of U_λ . Next, we define an equivalence relation $\sim_{\lambda;m}$ on $\text{IGLT}(\lambda)_m$ and show that the $\mathbb{C}$ -span of each equivalence class is closed under the $H_m(0)$ -action. Thus, we obtain a direct sum decomposition

$$\mathbf{G}_{\lambda;m} = \bigoplus_{E \in \mathcal{E}_{\lambda;m}} \mathbf{G}_E,$$

where $\mathcal{E}_{\lambda;m}$ is the set of all equivalence classes with respect to $\sim_{\lambda;m}$ and $\mathbf{G}_E$ is the submodule of $\mathbf{G}_{\lambda;m}$ whose underlying space is the $\mathbb{C}$ -span of E . Finally, we show that $\mathbf{G}_E$ is isomorphic to a weak Bruhat interval module for $E \in \mathcal{E}_{\lambda;m}$. To do this, we prove that there exist unique source tableau T_E and sink tableau T'_E in E . In addition, we assign a permutation $\text{read}(T)$, called *standardized reading word*, to each $T \in E$. With these preparations, we show that

$$\mathbf{G}_E \cong B(\text{read}(T_E), \text{read}(T'_E)).$$

We end with providing an avenue for future research. This paper is an extended abstract of our paper [8].

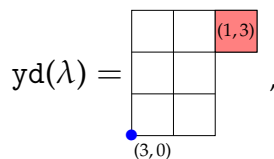
2 Preliminaries

Given any integers m and n , define $[m, n]$ to be the set $\{k \in \mathbb{Z} \mid m \leq k \leq n\}$ if $m \leq n$ or the empty set otherwise. Throughout this section, n denotes a nonnegative integer.

2.1 Compositions and Diagrams

A *composition* α of a nonnegative integer n , denoted by $\alpha \models n$, is a finite ordered list of positive integers $(\alpha_1, \alpha_2, \dots, \alpha_k)$ satisfying $\sum_{i=1}^k \alpha_i = n$. We call $k =: \ell(\alpha)$ the *length* of α and $n =: |\alpha|$ the *size* of α . Given $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{\ell(\alpha)}) \models n$, we define $\text{set}(\alpha) := \{\alpha_1, \alpha_1 + \alpha_2, \dots, \sum_{i=1}^{\ell(\alpha)-1} \alpha_i\} \subseteq [1, n-1]$.

If $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{\ell(\lambda)}) \models n$ satisfies $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{\ell(\lambda)}$, then we say that λ is a *partition* of n and denote it by $\lambda \vdash n$. For $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{\ell(\lambda)}) \vdash n$, we define the *Young diagram* $\text{yd}(\lambda)$ of λ by a left-justified array of n boxes where the i th row from the top has λ_i boxes for $1 \leq i \leq \ell(\lambda)$. We say that a box in $\text{yd}(\lambda)$ is *in the i th row* if it is in the i th row from the top and *in the j th column* if it is in the j th column from the left. We denote by (i, j) the box in the i th row and j th column. Denoting $(i, j) \in \text{yd}(\lambda)$ means that $1 \leq i \leq \ell(\lambda)$ and $1 \leq j \leq \lambda_i$. We also say that a lattice point on $\text{yd}(\lambda)$ is *in the i th row* if it is in the $(i+1)$ st horizontal line from the top and *in the j th column* if it is in the $(j+1)$ st vertical line from the left. We denote by $(\underline{i}, \underline{j})$ the lattice point in the i th row and j th column. For example, if $\lambda = (3, 2, 2)$, then



the box $(1, 3)$ is the box filled with red, and the lattice point $(\underline{3}, \underline{0})$ is the point marked by the blue dot. A *filling* of $\text{yd}(\lambda)$ is a function $T : \text{yd}(\lambda) \rightarrow \mathbb{Z}_{>0}$. Throughout this paper, we assume that

$$\begin{aligned} T((i, j)) &= \infty && \text{if } (i, j) \in (\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}) \setminus \text{yd}(\lambda) \quad \text{and} \\ T((i, j)) &= -\infty && \text{if } (i, j) \in (\mathbb{Z} \times \mathbb{Z}) \setminus (\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}). \end{aligned}$$

For any filling T of $\text{yd}(\lambda)$, let $\max(T) := \max\{T((i, j)) \mid (i, j) \in \text{yd}(\lambda)\}$.

2.2 The 0-Hecke algebra and the quasisymmetric characteristic

To begin with, we recall that the symmetric group $\mathfrak{S}_n$ is generated by simple transpositions $s_i := (i, i+1)$ with $1 \leq i \leq n-1$. An expression for $\sigma \in \mathfrak{S}_n$ of the form $s_{i_1} s_{i_2} \cdots s_{i_p}$ that uses the minimal number of simple transpositions is called a *reduced expression* for σ . The number of simple transpositions in any reduced expression for σ , denoted by $\ell(\sigma)$, is called the *length* of σ .

The 0-Hecke algebra $H_n(0)$ is the $\mathbb{C}$ -algebra generated by $\pi_1, \pi_2, \dots, \pi_{n-1}$ subject to the following three relations: (1) $\pi_i^2 = \pi_i$ for $1 \leq i \leq n-1$, (2) $\pi_i \pi_{i+1} \pi_i = \pi_{i+1} \pi_i \pi_{i+1}$ for $1 \leq i \leq n-2$, and (3) $\pi_i \pi_j = \pi_j \pi_i$ if $|i-j| \geq 2$. Pick up any reduced expression $s_{i_1} s_{i_2} \cdots s_{i_p}$ for a permutation $\sigma \in \mathfrak{S}_n$. We define the element π_σ of $H_n(0)$ by $\pi_\sigma := \pi_{i_1} \pi_{i_2} \cdots \pi_{i_p}$. It is well known that π_σ is independent of the choice of reduced expressions, and $\{\pi_\sigma \mid \sigma \in \mathfrak{S}_n\}$ is a basis for $H_n(0)$.

In [9], Norton classified all irreducible modules of the 0-Hecke algebras. It was shown that there are 2^{n-1} distinct irreducible $H_n(0)$ -modules which are naturally parametrized by compositions of n . For each $\alpha \models n$, the irreducible module $\mathbf{F}_\alpha$ corresponding to α is the 1-dimensional $H_n(0)$ -module spanned by a vector v_α whose $H_n(0)$ -action is given by

$$\pi_i \cdot v_\alpha = \begin{cases} 0 & i \in \text{set}(\alpha), \\ v_\alpha & i \notin \text{set}(\alpha), \end{cases} \quad (1 \leq i \leq n-1).$$

Let $\mathcal{R}(H_n(0))$ denote the $\mathbb{Z}$ -span of the isomorphism classes of finite dimensional $H_n(0)$ -modules. The isomorphism class corresponding to an $H_n(0)$ -module M will be denoted by $[M]$. The *Grothendieck group* $\mathcal{G}_0(H_n(0))$ is the quotient of $\mathcal{R}(H_n(0))$ modulo the relations $[M] = [M'] + [M'']$ whenever there exists a short exact sequence $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$. The irreducible $H_n(0)$ -modules form a free $\mathbb{Z}$ -basis for $\mathcal{G}_0(H_n(0))$. Let $\mathcal{G}_0(H_\bullet(0)) := \bigoplus_{n \geq 0} \mathcal{G}_0(H_n(0))$ be the ring equipped with the induction product. In [6], Duchamp, Krob, Leclerc, and Thibon revealed a deep connection between $\mathcal{G}_0(H_\bullet(0))$ and the ring QSym of quasisymmetric functions by providing a ring isomorphism

$$\text{ch} : \mathcal{G}_0(H_\bullet(0)) \rightarrow \text{QSym}, \quad [\mathbf{F}_\alpha] \mapsto F_\alpha,$$

called *quasisymmetric characteristic*. Here, F_α is the *fundamental quasisymmetric function*.

2.3 Weak Bruhat interval modules of the 0-Hecke algebra

Given $\sigma \in \mathfrak{S}_n$, let $\text{Des}_L(\sigma) := \{i \in [1, n-1] \mid \ell(s_i\sigma) < \ell(\sigma)\}$. The *left weak Bruhat order* $\preceq_L$ on $\mathfrak{S}_n$ is the partial order on $\mathfrak{S}_n$ whose covering relation $\preceq_L^c$ is defined as follows: $\sigma \preceq_L^c s_i\sigma$ if and only if $i \notin \text{Des}_L(\sigma)$. Given $\sigma, \rho \in \mathfrak{S}_n$, the *left weak Bruhat interval from σ to ρ* , denoted by $[\sigma, \rho]_L$, is the closed interval $\{\gamma \in \mathfrak{S}_n \mid \sigma \preceq_L \gamma \preceq_L \rho\}$.

Definition 2.1. ([7]) Let $\sigma, \rho \in \mathfrak{S}_n$. The *weak Bruhat interval module associated to $[\sigma, \rho]_L$* , denoted by $B(\sigma, \rho)$, is the $H_n(0)$ -module with the underlying space $\mathbb{C}[\sigma, \rho]_L$ and with the $H_n(0)$ -action defined by

$$\pi_i \cdot \gamma := \begin{cases} \gamma & \text{if } i \in \text{Des}_L(\gamma), \\ 0 & \text{if } i \notin \text{Des}_L(\gamma) \text{ and } s_i\gamma \notin [\sigma, \rho]_L, \\ s_i\gamma & \text{if } i \notin \text{Des}_L(\gamma) \text{ and } s_i\gamma \in [\sigma, \rho]_L. \end{cases}$$

2.4 Genomic Schur functions

Given $\lambda \vdash n$, an *increasing gapless tableau* of shape λ is a filling of $\text{yd}(\lambda)$ such that the entries in each row strictly increase from left to right, the entries in each column strictly increase from top to bottom, and the set $T^{-1}(i)$ is nonempty for all $1 \leq i \leq \max(T)$. Let $\text{IGLT}(\lambda)$ be the set of all increasing gapless tableaux of shape λ . Given $T \in \text{IGLT}(\lambda)$ and

$1 \leq i \leq \max(T)$, let $\text{Top}_i(T)$ (resp. $\text{Bot}_i(T)$) be the highest (resp. lowest) box in T having the entry i . Let

$$(r_b^{(i)}(T), c_b^{(i)}(T)) := \text{Bot}_i(T) \quad \text{and} \quad (r_t^{(i)}(T), c_t^{(i)}(T)) := \text{Top}_i(T).$$

If T is clear in the context, we simply write $r_b^{(i)}, c_b^{(i)}, r_t^{(i)}$, and $c_t^{(i)}$ instead of $r_b^{(i)}(T), c_b^{(i)}(T), r_t^{(i)}(T)$, and $c_t^{(i)}(T)$, respectively. We call an index $i \in [1, \max(T) - 1]$ a *descent* of T if there is some instance of i strictly above some instance of $i + 1$ in T . Let $\text{Des}(T)$ be the set of all descents of T and let $\text{comp}(T) := \text{comp}(\text{Des}(T))$.

Definition 2.2. ([10, 11]) For $\lambda \vdash n$, the *genomic Schur function* U_λ is defined by

$$U_\lambda := \sum_{T \in \text{IGLT}(\lambda)} F_{\text{comp}(T)}.$$

Given $1 \leq m \leq n$, we define

$$\text{IGLT}(\lambda)_m := \{T \in \text{IGLT}(\lambda) \mid \max(T) = m\} \quad \text{and} \quad U_{\lambda;m} := \sum_{T \in \text{IGLT}(\lambda)_m} F_{\text{comp}(T)}.$$

From the definition, it immediately follows that $U_{\lambda;m}$ is the m th degree homogeneous component of U_λ .

Hereafter, we assume that n is a positive integer, m is a positive integer less than or equal to n , and λ is a partition of n , unless otherwise stated.

3 0-Hecke modules from increasing gapless tableaux

3.1 An $H_m(0)$ -module for $U_{\lambda;m}$

We start by introducing the necessary definitions.

Definition 3.1. Given $T \in \text{IGLT}(\lambda)$ and $1 \leq i \leq \max(T) - 1$, we say that i is an *attacking descent* if $i \in \text{Des}(T)$, and either

- (a) there exists $(j, k) \in \text{yd}(\lambda)$ such that $T((j, k)) = i$ and $T((j + 1, k)) = i + 1$, or
- (b) there exists a box $B \in T^{-1}(i + 1)$ placed weakly above $\text{Bot}_i(T)$.

Take any $1 \leq m \leq n$. For each $1 \leq i \leq m - 1$, we define a linear operator $\pi_i : \mathbb{C} \text{IGLT}(\lambda)_m \rightarrow \mathbb{C} \text{IGLT}(\lambda)_m$ by

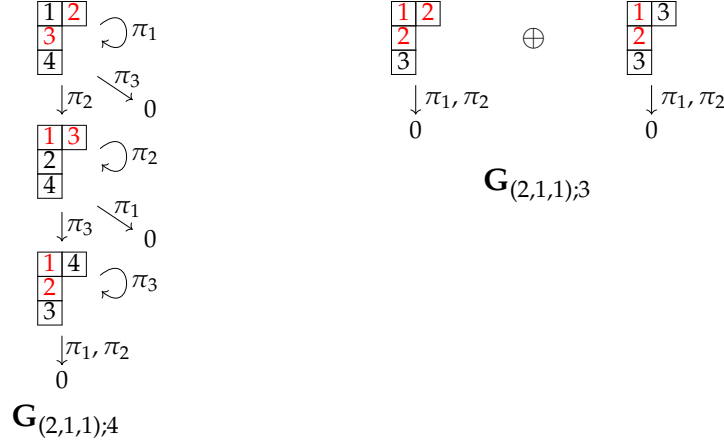
$$\pi_i(T) := \begin{cases} T & \text{if } i \text{ is not a descent of } T, \\ 0 & \text{if } i \text{ is an attacking descent of } T, \\ s_i \cdot T & \text{if } i \text{ is a non-attacking descent of } T \end{cases} \quad (3.1)$$

for $T \in \text{IGLT}(\lambda)_m$ and extending it by linearity. Here, $s_i \cdot T$ is the tableau obtained from T by replacing i and $i + 1$ with $i + 1$ and i , respectively. By proving $\pi_i^2 = \pi_i$ ($1 \leq i \leq m - 1$), $\pi_i \pi_{i+1} \pi_i = \pi_{i+1} \pi_i \pi_{i+1}$ ($1 \leq i \leq m - 2$), and $\pi_i \pi_j = \pi_j \pi_i$ ($1 \leq i, j \leq m - 1$ with $|i - j| > 1$), we obtain the following theorem. For details, see [8, Subsection 3.1].

Theorem 3.2. For any $1 \leq m \leq n$, the operators $\pi_1, \pi_2, \dots, \pi_{m-1}$ satisfy the same relations as the generators $\pi_1, \pi_2, \dots, \pi_{m-1}$ for $H_m(0)$. In other words, $\pi_1, \pi_2, \dots, \pi_{m-1}$ define an $H_m(0)$ -action on $\mathbb{C} \text{IGLT}(\lambda)_m$.

Example 3.3. (1) When $T = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 6 \\ \hline 2 & 3 & 5 & 7 \\ \hline 4 & 6 & & \\ \hline \end{array}$, we have $\pi_3(T) = s_3 \cdot T$, $\pi_4(T) = T$, and $\pi_i(T) = 0$ for $i = 1, 2, 5, 6$. Here, the indices in red are used to indicate the descents of the tableau.

(2) Note that $\text{IGLT}((2, 1, 1)) = \left\{ \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & 3 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & 3 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 3 \\ \hline \end{array} \right\}$. One can see that $U_{(2,1,1)} = (F_{(2,1,1)} + F_{(1,2,1)} + F_{(1,1,2)}) + 2F_{(1,1,1)}$. The following figures illustrate the $H_m(0)$ -action on $\mathbf{G}_{(2,1,1);m}$ for $m = 3, 4$:



The following proposition follows immediately from (3.1) and Theorem 3.2.

Proposition 3.4. For any $\lambda \vdash n$ and $1 \leq m \leq n$, $\text{ch}([\mathbf{G}_{\lambda;m}]) = U_{\lambda;m}$. Consequently, $\sum_{1 \leq m \leq n} \text{ch}([\mathbf{G}_{\lambda;m}]) = U_{\lambda}$.

Remark 3.5. In [5, Theorem 4.7], Duchamp, Hivert, and Thibon described the *Ext-quiver* of $H_m(0)$. According to their result, for any $\alpha \models m$, we have $\text{Ext}_{H_m(0)}^1(\mathbf{F}_{\alpha}, \mathbf{F}_{\alpha}) = 0$, equivalently, there is no indecomposable $H_m(0)$ -module M such that $\text{ch}([M]) = 2F_{\alpha}$. On the other hand, in Example 3.3, we see that $U_{(2,1,1);3} = 2F_{(1,1,1)}$. Thus, we conclude that there is no indecomposable $H_3(0)$ -module M satisfying $\text{ch}([M]) = U_{(2,1,1);3}$.

3.2 A direct sum decomposition of $\mathbf{G}_{\lambda;m}$ into $H_m(0)$ -submodules

Let us start with necessary definitions and notation. Given $T \in \text{IGLT}(\lambda)_m$, let $\mathcal{I}(T) := \{i \in [1, m] \mid |T^{-1}(i)| > 1\}$. Given $i \in \mathcal{I}(T)$, let $\Gamma_i(T)$ be the lattice path from $(\underline{r_b^{(i)}}, \underline{c_b^{(i)}} - 1)$ to $(\underline{r_t^{(i)}} - 1, \underline{c_t^{(i)}})$ satisfying the following two conditions:

- (i) if the path passes through two boxes horizontally, then the entry at the above box is smaller than i and the entry at the below box is greater than i , and

- (ii) if the path passes through two boxes vertically, then the entry at the left box is smaller than i and the entry at the right box is greater than i .

Example 3.6. Let

$$T = \begin{array}{cccccccc} 1 & 6 & 10 & 14 & 22 & 24 & 26 & 27 \\ 2 & 7 & 11 & 15 & 23 & 25 & 29 & \\ 3 & 8 & 12 & 16 & 28 & 29 & & \\ 4 & 9 & 13 & 17 & & & & \\ 5 & 17 & 27 & & & & & \\ 18 & 20 & & & & & & \\ 19 & 21 & & & & & & \\ 21 & & & & & & & \end{array}.$$

Note that $\mathcal{I}(T) = \{17, 21, 27, 29\}$. By following the way of defining lattice paths, we obtain the lattice paths $\Gamma_{17}(T), \Gamma_{21}(T), \Gamma_{27}(T)$, and $\Gamma_{29}(T)$ as follows:

$$\begin{array}{cccc} \begin{array}{cccccccc} 1 & 6 & 10 & 14 & 22 & 24 & 26 & 27 \\ 2 & 7 & 11 & 15 & 23 & 25 & 29 & \\ 3 & 8 & 12 & 16 & 28 & 29 & & \\ 4 & 9 & 13 & 17 & & & & \\ 5 & 17 & 27 & & & & & \\ 18 & 20 & & & & & & \\ 19 & 21 & & & & & & \\ 21 & & & & & & & \end{array} & \begin{array}{cccccccc} 1 & 6 & 10 & 14 & 22 & 24 & 26 & 27 \\ 2 & 7 & 11 & 15 & 23 & 25 & 29 & \\ 3 & 8 & 12 & 16 & 28 & 29 & & \\ 4 & 9 & 13 & 17 & & & & \\ 5 & 17 & 27 & & & & & \\ 18 & 20 & & & & & & \\ 19 & 21 & & & & & & \\ 21 & & & & & & & \end{array} & \begin{array}{cccccccc} 1 & 6 & 10 & 14 & 22 & 24 & 26 & 27 \\ 2 & 7 & 11 & 15 & 23 & 25 & 29 & \\ 3 & 8 & 12 & 16 & 28 & 29 & & \\ 4 & 9 & 13 & 17 & & & & \\ 5 & 17 & 27 & & & & & \\ 18 & 20 & & & & & & \\ 19 & 21 & & & & & & \\ 21 & & & & & & & \end{array} & \begin{array}{cccccccc} 1 & 6 & 10 & 14 & 22 & 24 & 26 & 27 \\ 2 & 7 & 11 & 15 & 23 & 25 & 29 & \\ 3 & 8 & 12 & 16 & 28 & 29 & & \\ 4 & 9 & 13 & 17 & & & & \\ 5 & 17 & 27 & & & & & \\ 18 & 20 & & & & & & \\ 19 & 21 & & & & & & \\ 21 & & & & & & & \end{array} \\ \Gamma_{17}(T) & \Gamma_{21}(T) & \Gamma_{27}(T) & \Gamma_{29}(T) \end{array}$$

Given a lattice path Γ , let $V(\Gamma)$ be the set of lattice points through which Γ passes. For two lattice paths Γ and Γ' , we write $\Gamma = \Gamma'$ if $V(\Gamma) = V(\Gamma')$.

Definition 3.7. Let $\lambda \vdash n$ and $T_1, T_2 \in \text{IGLT}(\lambda)_m$. The equivalence relation $\sim_{\lambda; m}$ on $\text{IGLT}(\lambda)_m$ is defined by $T_1 \sim_{\lambda; m} T_2$ if and only if

$$\left\{ \left(\Gamma_i(T_1), T_1^{-1}(i) \right) \mid i \in \mathcal{I}(T_1) \right\} = \left\{ \left(\Gamma_i(T_2), T_2^{-1}(i) \right) \mid i \in \mathcal{I}(T_2) \right\}.$$

Example 3.8. Let $T_1 = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 5 \\ \hline 2 & 4 & 5 & 6 \\ \hline 3 & 5 & 7 & \\ \hline \end{array}$, $T_2 = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 5 \\ \hline 2 & 4 & 5 & 7 \\ \hline 3 & 5 & 6 & \\ \hline \end{array}$, $T_3 = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 2 & 3 & 5 & 6 \\ \hline 3 & 4 & 7 & \\ \hline \end{array}$, and $T_4 = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 5 \\ \hline 2 & 3 & 5 & 6 \\ \hline 4 & 5 & 7 & \\ \hline \end{array}$.

Then, $T_1 \sim_{(4,3,2);5} T_2$, but $T_1 \not\sim_{(4,3,2);5} T_k$ for $k = 3, 4$.

Let $\mathcal{E}_{\lambda; m}$ be the set of equivalence classes of $\text{IGLT}(\lambda)_m$ with respect to $\sim_{\lambda; m}$.

Theorem 3.9. Let m and n be positive integers with $m \leq n$ and let $\lambda \vdash n$. For any $1 \leq i \leq m-1$ and $E \in \mathcal{E}_{\lambda; m}$, $\pi_i \cdot \mathbb{C}E \subseteq \mathbb{C}E$.

For each $E \in \mathcal{E}_{\lambda; m}$, let $\mathbf{G}_E$ be the $H_m(0)$ -submodule of $\mathbf{G}_{\lambda; m}$ whose underlying space is the $\mathbb{C}$ -span of E . Then, we have the following direct sum decomposition

$$\mathbf{G}_{\lambda; m} = \bigoplus_{E \in \mathcal{E}_{\lambda; m}} \mathbf{G}_E.$$

Hereafter, E denotes an equivalence class of $\text{IGLT}(\lambda)_m$ with respect to $\sim_{\lambda; m}$ and T denotes a tableau contained in $\text{IGLT}(\lambda)_m$ unless otherwise stated.

4 Source and sink tableaux

The goal of this section is to show that there are two distinguished tableaux, called *source* and *sink tableaux*, in each equivalence class $E \in \mathcal{E}_{\lambda;m}$. To achieve our goal, we first construct two tableaux $\text{source}(T)$ and $\text{sink}(T)$ for each $T \in E$. Then, we prove that $\text{source}(T)$ (resp. $\text{sink}(T)$) is the unique source tableau (resp. sink tableau) in E , where T is an arbitrary chosen element in E .

To begin with, we give definitions for source tableaux and sink tableaux in $\text{IGLT}(\lambda)_m$.

Definition 4.1. Let $T \in \text{IGLT}(\lambda)_m$.

- (1) T is said to be a *source tableau* if there do not exist $T' \in \text{IGLT}(\lambda)_m$ and $1 \leq i \leq m-1$ such that $\pi_i \cdot T' = T$ and $T' \neq T$.
- (2) T is said to be a *sink tableau* if there do not exist $T' \in \text{IGLT}(\lambda)_m$ and $1 \leq i \leq m-1$ such that $\pi_i \cdot T = T'$ and $T' \neq T$.

To construct the desired tableau $\text{source}(T)$, we need the following preparation. Given two lattice points P and P' in the same row, we denote the horizontal line from P to P' by $\text{HL}(P, P')$. For each $i \in \mathcal{I}(T)$, we define a new lattice path $\tilde{\Gamma}_i(T)$ by extending $\Gamma_i(T)$ with the following algorithm.

Algorithm 4.2. Fix $i \in \mathcal{I}(T)$ and let $\lambda_0 := \lambda_1$.

Step 1. For each $j \in \mathcal{I}(T)$, let Γ'_j be the lattice path obtained by connecting three lattice

$$\text{paths } \text{HL}\left(\underline{(r_b^{(j)}, 0)}, \underline{(r_b^{(j)}, c_b^{(j)} - 1)}\right), \Gamma_j(T), \text{ and } \text{HL}\left(\underline{(r_t^{(j)} - 1, c_t^{(j)})}, \underline{(r_t^{(j)} - 1, \lambda_{r_t^{(j)} - 1}^{(j)})}\right).$$

Step 2. Let $r_t = \min\{r \mid (r, c) \in V(\Gamma'_i)\}$ and $c_t = \min\{c \mid (r_t, c) \in V(\Gamma'_i)\}$.

Step 3. If there exists $j \in \mathcal{I}(T)$ such that

$$r' < r_t < r'' \quad \text{and} \quad c', c'' > c_t \quad \text{for some } (r', c'), (r'', c'') \in V(\Gamma'_j), \quad (4.1)$$

then go to *Step 4*. Otherwise, go to *Step 5*.

Step 4. Let $j_0 = \min\{j \mid \Gamma'_j \text{ satisfies (4.1)}\}$ and $c_0 = \min\{c \mid (r_t, c) \in V(\Gamma'_{j_0})\}$. Then, let Γ be the lattice path satisfying that

$$V(\Gamma) = V(\Gamma'_i) \setminus \{(r_t, c) \mid c \geq c_0\} \cup \{(r, c) \in V(\Gamma'_{j_0}) \mid r \leq r_t \text{ and } c \geq c_0\}.$$

Set $\Gamma'_i := \Gamma$. Go to *Step 2*.

Step 5. Return $\tilde{\Gamma}_i(T) := \Gamma'_i$ and terminate the algorithm.

If T is clear in the context, we simply write the lattice path $\tilde{\Gamma}_i(T)$ by $\tilde{\Gamma}_i$ for $i \in \mathcal{I}(T)$.

Example 4.3. Let us revisit Example 3.6. By applying Algorithm 4.2 to each $i \in \mathcal{I}(T)$, we obtain $\tilde{\Gamma}_{17}$, $\tilde{\Gamma}_{21}$, $\tilde{\Gamma}_{27}$, and $\tilde{\Gamma}_{29}$ as follows:

1	6	10	14	22	24	26	27
2	7	11	15	23	25	29	
3	8	12	16	28	29		
4	9	13	17				
5	17	27					
18	20						
19	21						
21							

$\tilde{\Gamma}_{17}(T)$

1	6	10	14	22	24	26	27
2	7	11	15	23	25	29	
3	8	12	16	28	29		
4	9	13	17				
5	17	27					
18	20						
19	21						
21							

$\tilde{\Gamma}_{21}(T)$

1	6	10	14	22	24	26	27
2	7	11	15	23	25	29	
3	8	12	16	28	29		
4	9	13	17				
5	17	27					
18	20						
19	21						
21							

$\tilde{\Gamma}_{27}(T)$

1	6	10	14	22	24	26	27
2	7	11	15	23	25	29	
3	8	12	16	28	29		
4	9	13	17				
5	17	27					
18	20						
19	21						
21							

$\tilde{\Gamma}_{29}(T)$

For $i \in \mathcal{I}(T)$, we set $p'_i \in \{1, 2, \dots, |\mathcal{I}(T)|\}$ satisfying the following: Let $i, j \in \mathcal{I}(T)$.

- C1.** If $r_b^{(i)} < r_b^{(j)}$, then $p'_i < p'_j$. And, if $r_b^{(i)} > r_b^{(j)}$, then $p'_i > p'_j$.
- C2.** When $r_b^{(i)} = r_b^{(j)}$, consider the lowest lattice point $p \in V(\tilde{\Gamma}_i) \cap V(\tilde{\Gamma}_j)$ such that neither $p + (-1, 0)$ nor $p + (0, 1)$ are contained in $V(\tilde{\Gamma}_i) \cap V(\tilde{\Gamma}_j)$. If $p + (-1, 0) \in V(\tilde{\Gamma}_i)$, then $p'_i < p'_j$. Otherwise, $p'_i > p'_j$.

Given $i, j \in \mathcal{I}(T)$, if there exist $(r', c'), (r'', c'') \in V(\tilde{\Gamma}_j)$ such that $r' < r_b^{(i)} < r''$ and $c', c'' < c_b^{(i)}$, then we say that $\tilde{\Gamma}_j$ crosses the bottom path of $\tilde{\Gamma}_i$. By rearranging p'_i 's with the following algorithm, we define a bijection $p_T : \mathcal{I}(T) \rightarrow \{1, 2, \dots, |\mathcal{I}(T)|\}$.

Algorithm 4.4. For each $i \in \mathcal{I}(T)$, let $p_i := p'_i$, where p'_i is the index defined above.

Step 1. Let $k = 1$.

Step 2. Take i_k and i_{k+1} in $\mathcal{I}(T)$ such that $p_{i_k} = k$ and $p_{i_{k+1}} = k + 1$.

Step 3. If $\tilde{\Gamma}_{i_{k+1}}$ crosses the bottom path of $\tilde{\Gamma}_{i_k}$, then set $p_{i_k} := k + 1$ and $p_{i_{k+1}} := k$ and go to *Step 1*. Otherwise, go to *Step 4*.

Step 4. If $k < |\mathcal{I}(T)| - 1$, then set $k = k + 1$ and go to *Step 2*. Otherwise, set $p_T(i) := p_i$ for each $i \in \mathcal{I}(T)$ and go to *Step 5*.

Step 5. Return $(p_T(i))_{i \in \mathcal{I}(T)}$ and terminate the algorithm.

Given $u \in [1, |\mathcal{I}(T)|]$, let A_u be the subdiagram of $\text{yd}(\lambda)$ consisting of the boxes located above $\tilde{\Gamma}^{(u)}$. Let

$$D_u^{(1)}(T) := A_u \setminus \left(\bigcup_{1 \leq v < u} (A_v \cup T^{-1}(p_T^{-1}(v))) \right) \quad \text{and} \quad D_u^{(2)}(T) := T^{-1}(p_T^{-1}(u)).$$

Now, we construct the desired tableau $\text{source}(T)$ with the following algorithm.

Algorithm 4.5. Let $T \in \text{IGLT}(\lambda)_m$. Set $e_0 = 0$ and $M_0 = 0$. For $1 \leq u \leq |\mathcal{I}(T)|$, let $e_u := |D_u^{(1)}(T)| + 1$ and $M_u = \sum_{v=0}^u e_v$.

Step 1. Set $v := 1$.

Step 2. Fill the boxes in $D_v^{(1)}(T)$ by $M_{v-1} + 1, M_{v-1} + 2, \dots, M_{v-1} + e_v - 1$ from left to right starting from the top.

Step 3. Fill the boxes in $D_v^{(2)}(T)$ by M_v .

Step 4. If $v < |\mathcal{I}(T)|$, then set $v := v + 1$ and go to *Step 2*. Otherwise, fill the remaining boxes by $M_{|\mathcal{I}(T)|} + 1, M_{|\mathcal{I}(T)|} + 2, \dots, m$ from left to right starting from the top. Set $\text{source}(T)$ to be the resulting filling. Return $\text{source}(T)$ and terminate the algorithm.

Example 4.6. Let us revisit [Example 4.3](#). One can easily see that $p'_{17} = 2, p'_{21} = 4, p'_{27} = 3$, and $p'_{29} = 1$. By applying [Algorithm 4.4](#), we have

$$p_T(17) = 1, \quad p_T(21) = 4, \quad p_T(27) = 2, \quad \text{and} \quad p_T(29) = 3.$$

In addition, by applying [Algorithm 4.5](#), we have

$$T = \begin{array}{cccccccc} 1 & 6 & 10 & 14 & 22 & 24 & 26 & 27 \\ 2 & 7 & 11 & 15 & 23 & 25 & 29 & \\ 3 & 8 & 12 & 16 & 28 & 29 & & \\ 4 & 9 & 13 & 17 & & & & \\ 5 & 17 & 27 & & & & & \\ 18 & 20 & & & & & & \\ 19 & 21 & & & & & & \\ 21 & & & & & & & \end{array} \xrightarrow{\text{Algorithm 4.5}} \text{source}(T) = \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 23 \\ 8 & 9 & 10 & 11 & 12 & 13 & 25 & \\ 14 & 15 & 16 & 17 & 24 & 25 & & \\ 18 & 19 & 20 & 22 & & & & \\ 21 & 22 & 23 & & & & & \\ 26 & 27 & & & & & & \\ 28 & 29 & & & & & & \\ 29 & & & & & & & \end{array}.$$

Theorem 4.7. For any $T \in E$, $\text{source}(T)$ is the unique source tableau in E .

Similarly, we construct $\text{sink}(T)$ for each $T \in E$ and prove the following theorem.

Theorem 4.8. For any $T \in E$, $\text{sink}(T)$ is the unique sink tableau in E .

We denote by T_E and T'_E the unique source tableau and sink tableau in E , respectively.

5 A weak Bruhat interval module description of G_E

Throughout this section, we let $\text{Des}(T_E) = \{d_1 < d_2 < \dots < d_k\}$, $d_0 := 0$, and $d_{k+1} := m$. For each $1 \leq j \leq k+1$, let $H_j := T_E^{-1}([d_{j-1} + 1, d_j])$.

Example 5.1. When

$$T_E = \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 11 & 12 & 14 & 15 \\ 6 & 7 & 8 & 10 & 11 & 14 & & & \\ 9 & 10 & & & & & & & \\ 13 & 14 & & & & & & & \end{array},$$

we have that $\text{Des}(T_E) = \{5, 8, 12\}$. In this case, H_j ($1 \leq j \leq 4$) are given as follows:

$$\begin{array}{cccccc} H_1 & H_1 & H_1 & H_1 & H_1 & H_3 & H_3 & H_4 & H_4 \\ H_2 & H_2 & H_2 & H_3 & H_3 & H_4 & & & \\ H_3 & H_3 & & & & & & & \\ H_4 & H_4 & & & & & & & \end{array}$$

For each $1 \leq j \leq k$, let $w^{(j)}(T)$ be the word obtained by reading the entries of T contained in H_j from right to left. Note that if an integer i appears multiple times in $w^{(j)}(T)$, then the integer i 's are placed consecutively. We define $\bar{w}^{(j)}(T)$ as the word obtained from $w^{(j)}(T)$ by erasing all i 's except one i for each i that appears in $w^{(j)}(T)$.

Definition 5.2. For $T \in E$, the *standardized reading word* $\text{read}(T)$ of T is defined to be the word $\bar{w}^{(1)}(T)\bar{w}^{(2)}(T) \cdots \bar{w}^{(k+1)}(T)$ obtained by concatenating $\bar{w}^{(j)}(T)$ for $1 \leq j \leq k+1$.

Example 5.3. We revisit [Example 5.1](#). One can see that

$$\text{read}(T_E) = 5\ 4\ 3\ 2\ 1\ 8\ 7\ 6\ 12\ 11\ 10\ 9\ 15\ 14\ 13 \in \mathfrak{S}_{15}.$$

Theorem 5.4. For each $E \in \mathcal{E}_{\lambda;m}$, $\mathbf{G}_E \cong \mathbf{B}(\text{read}(T_E), \text{read}(T'_E))$ as $H_m(0)$ -modules.

6 Further avenue

In [10], Pechenik proved that for all $\lambda = (\lambda_1, \lambda_2) \vdash n$,

$$U_\lambda = \sum_{l_\lambda \leq m \leq n} \sum_{\mu \in \text{Par}(\lambda;m)} s_\mu, \quad (6.1)$$

where $l_\lambda := \max\{\lambda_1, \lambda_2 + 1\}$, $\text{Par}(\lambda; n) := \{(\lambda_1, \lambda_2)\}$, and

$$\text{Par}(\lambda; m) := \begin{cases} \{(\lambda_1 - k_m, \lambda_1 - k_m, 1^{k_m})\} & \text{if } \lambda_1 = \lambda_2, \\ \{(\lambda_1 - k_m, \lambda_2 - k_m, 1^{k_m}), (\lambda_1 - k_m, \lambda_2 - k_m + 1, 1^{k_m-1})\} & \text{if } \lambda_1 > \lambda_2 \end{cases}$$

for all $l \leq m < n$. Here, $k_m := n - m$ and $s_\mu := 0$ if μ is not a partition. And, for each $\lambda \vdash m$, Searles [12] introduced the $H_m(0)$ -module X_λ such that $\text{ch}([X_\lambda]) = s_\lambda$. The study of representation theoretic interpretation for (6.1) will be pursued in the near future by using $\mathbf{G}_{\lambda;m}$ and X_μ . In this direction, we leave the following conjecture.

Conjecture 6.1. Let $\lambda = (\lambda_1, \lambda_2) \vdash n$. For each $l_\lambda \leq m \leq n$, there exists a partition $\{\mathcal{E}_\mu \mid \mu \in \text{Par}(\lambda; m)\}$ of $\mathcal{E}_{\lambda;m}$ satisfying the following: For each $\mu \in \text{Par}(\lambda; m)$,

- (1) $\sum_{E \in \mathcal{E}_\mu} \text{ch}([\mathbf{G}_E]) = s_\mu$,
- (2) there exist a total order $\prec_\mu$ on $\mathcal{E}_\mu = \{E_1 \prec_\mu \cdots \prec_\mu E_{|\mathcal{E}_\mu|}\}$ and a filtration $M_0 = \{0\} \subseteq M_1 \subseteq \cdots \subseteq M_{|\mathcal{E}_\mu|} = X_\mu$ such that $\mathbf{G}_{E_i} \cong M_i/M_{i-1}$ for all $1 \leq i \leq |\mathcal{E}_\mu|$.

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Minimal skew semistandard Young tableaux and the Hillman–Grassl correspondence

Alejandro H. Morales^{*1}, Greta Panova^{†2}, and GaYee Park^{‡3}

¹Department of Mathematics and Statistics, UMass, Amherst MA, USA

²Department of Mathematics, University of Southern California, Los Angeles CA, USA

³LaCIM, Université du Québec à Montréal, Montréal QC, Canada

Abstract. Standard tableaux of skew shape are fundamental objects in enumerative and algebraic combinatorics and no product formula for the number is known. In 2014, Naruse gave a formula (NHLF) as a positive sum over excited diagrams of products of hook-lengths. Subsequently, Morales, Pak, and Panova gave several proofs and generalized it to two q -analogues. They also showed, partly algebraically, that the Hillman–Grassl map restricted to skew shapes must be a bijection. We study the problem of circumventing the algebraic part and proving the bijection completely combinatorially. For a skew shape, we define a new set of semi-standard Young tableaux, called the *minimal SSYT*, that are equinumerous with excited diagrams via a new description of the Hillman–Grassl bijection and a version of excited moves. Lastly, we relate the minimal skew SSYT with the terms of the Okounkov–Olshanski formula (OOF) for counting SYTs of skew shape. Our construction immediately implies that the summands in the NHLF are less than the summands in the OOF.

Keywords: Standard Young tableaux, Hillman–Grassl correspondence

1 Introduction

Given a partition $\lambda \vdash n$, the number f^λ of standard Young tableaux (SYT) of shape λ is an important quantity in enumerative, algebraic, and probabilistic combinatorics and is given by the following *hook-length formula* of Frame–Robinson–Thrall [2] 1954.

Theorem 1.1 (Frame–Robinson–Thrall [2]). *For λ a partition of n we have*

$$f^\lambda = \frac{n!}{\prod_{i,j \in \lambda} h_{i,j}}, \quad (1.1)$$

where λ is the Young diagram of λ and $h_{i,j} = \lambda_i - i + \lambda'_j - j + 1$.

^{*}amoralesborr@umass.edu. Morales is partially supported by NSF grant DMS-22030407.

[†]gpanova@usc.edu. Panova is partially supported by NSF grant CCF-2007652.

[‡]park.ga_yee@uqam.ca.

Unlike for straight shapes, there is no such product formula for the number $f^{\lambda/\mu}$ of SYT of skew shape λ/μ . There is a determinantal formula for $f^{\lambda/\mu}$ like the *Jacobi–Trudi identity*, which is efficient but inherently non-positive. There is a classical positive formula for $f^{\lambda/\mu}$ involving the *Littlewood–Richardson coefficients* $c_{\mu,\nu}^\lambda$, which are themselves hard to compute.

Central to this paper are two other positive formulas for $f^{\lambda/\mu}$ coming from equivariant Schubert calculus or more explicitly from evaluations of *factorial Schur functions*: the Okounkov–Olshanski formula [14] from 1998 and the Naruse hook-length formula [13] from 2014. We start with the latter since it resembles (1.1).

Theorem 1.2 (Naruse [13], [10]). *For a skew shape λ/μ of size n we have*

$$f^{\lambda/\mu} = n! \sum_{S \in \mathcal{E} \lambda/\mu} \prod_{u \in \lambda \setminus S} \frac{1}{h_u}, \quad (\text{NHLF})$$

where $\mathcal{E} \lambda/\mu$ is the set of excited diagrams of λ/μ .

The excited diagrams of shape λ/μ , denoted by $\mathcal{E} \lambda/\mu$, are certain subsets of size $|\mu|$ of the Young diagram of λ obtained from the Young diagram of μ by recursively doing a local move β [5]. Excited diagrams are in correspondence with certain SSYT of shape μ that are *flagged*, i.e. with certain bounds on the entries in each row. The Naruse formula has been actively studied [1, 6, 9, 10].

The other positive formula by Okounkov–Olshanski is also a sum over certain SSYT of shape μ with entries at most ℓ_λ called *Okounkov–Olshanski tableaux* $\text{OOT} \lambda/\mu$.

Theorem 1.3 (Okounkov–Olshanskii [14]). *For a skew shape λ/μ of size n we have*

$$f^{\lambda/\mu} = \frac{n!}{\prod_{u \in \lambda} h_u} \sum_{T \in \text{OOT} \lambda/\mu} \prod_{i,j \in \mu} \lambda_{\ell_i - 1 - T_{i,j} - i - j}. \quad (\text{OOF})$$

In [12], Morales–Zhu did a similar study of (OOF) as Morales–Pak–Panova did in [9, 10] for (NHLF). In particular, they gave in [12, Cor. 5.7] a reformulation of (OOF) in terms of the following flagged tableaux: SSYT of shape λ/μ with entries in row i at most $i - 1$, whose set we denote by $\mathcal{SF} \lambda/\mu$.

A new formulation of Naruse’s formula: A natural question, which is the start of our investigation, is to also find a reformulation of (NHLF) in terms of some SSYT of skew shape λ/μ . We define the set $\text{SSYT}_{\min} \lambda/\mu$ of *minimal skew SSYT* obtained from the *minimum SSYT*, T_0 of shape λ/μ with entries $\{0, 1, \dots, \lambda'_i - \mu'_i - 1\}$ by recursively doing local moves δ . Our first result is a bijection Φ between excited diagrams and tableaux $\text{SSYT}_{\min} \lambda/\mu$ that intertwines with the respective local moves β and δ , respectively.

Theorem 1.4. *For a skew shape λ/μ the map $\Phi : \mathcal{E} \lambda/\mu \rightarrow \text{SSYT}_{\min} \lambda/\mu$ is a bijection that intertwines with the respective local moves, that is $\Phi \circ \beta = \delta \circ \Phi$.*

As a corollary, we obtain a new reformulation of (NHLF). Let $\theta_1, \dots, \theta_k$ be the Lascoux–Pragacz decomposition of the shape λ/μ into border strips.

Theorem 1.5 (minimal SSYT version of (NHLF)). *For a skew shape λ/μ of size n we have*

$$f^{\lambda/\mu} = \frac{n!}{\prod_{u \in \lambda} h^u} \sum_{T \in \text{SSYT}_{\min} \lambda/\mu} \prod_{i,j \in \mu} h^{i - \alpha_j} \alpha_j, \quad (1.2)$$

where α is the number strips θ_k such that $\bar{T}(\theta_k, j) > \mu'_j - i$ and $\bar{T} = T - T_0$.

We give an explicit non-recursive description of the tableaux in $\text{SSYT}_{\min} \lambda/\mu$ (Theorem 2.6) that immediately shows they are a subset of the skew flagged tableaux $\mathcal{SF} \lambda/\mu$ (see Lemma 3.4). As a corollary, we obtain that the Okounkov–Olshanski formula has at least as many terms as the Naruse formula and characterize the skew shapes where equality is attained. We denote the number of terms of each formula by $\text{ED} \lambda$ and $\text{OOT} \lambda/\mu$, respectively.

Theorem 1.6. *For a connected skew shape λ/μ with $d = \ell \lambda$ and $r = \max\{i \mid \mu_1 = \mu_i\}$, we have that $\text{ED} \lambda/\mu \leq \text{OOT} \lambda/\mu$ with equality if and only if $\lambda_d \geq \mu_r - d - r$.*

Relation with the Hillman–Grassl correspondence: The hook-length formula (1.1) has a q -analogue by Littlewood that is a special case of Stanley’s hook-content formula for the generating series of SSYT of shape λ .

Theorem 1.7 (Littlewood). *For a partition λ we have*

$$s_\lambda(1, q, q^2, \dots) = q^{\sum_i i-1 \lambda_i} \cdot \prod_{u \in \lambda} \frac{1}{1 - q^{h^u}}. \quad (1.3)$$

This q -analogue has a bijective proof by Hillman–Grassl via the correspondence $HG \cdot$ between reverse plane partitions of shape λ and arrays of nonnegative integers of shape λ . This correspondence is related to the famous RSK correspondence (see [3, 10]) and has recent connections to *quiver representations* [4].

In [10], Morales–Pak–Panova gave a q -analogue of (NHLF) for the generating functions of SSYT of skew shape λ/μ .

Theorem 1.8 (Morales–Pak–Panova [10]). *For a skew shape λ/μ we have*

$$s_{\lambda/\mu}(1, q, q^2, \dots) = \sum_{D \in \mathcal{E} \lambda/\mu} \prod_{i,j \in D} \frac{q^{\lambda'_j - i}}{1 - q^{h^{i,j}}}. \quad (1.4)$$

Interestingly, this identity corresponds to restricting the Hillman–Grassl bijection to SSYT of shape λ/μ [10]. The resulting arrays $\mathcal{A}^* \lambda/\mu$ have support on the complement of excited diagrams and with certain forced nonzero entries on *broken diagonals*. The proof of this connection with the Hillman–Grassl map $HG \cdot$ is partly algebraic and remained mysterious. Our next results elucidate on this.

First, we show that the bijection between minimal tableaux and excited diagrams coincides with the Hillman–Grassl map.

Theorem 1.9. *The map Φ is equivalent to the inverse Hillman–Grassl map HG^{-1} on the arrays $\mathcal{A}^* \lambda/\mu$. That is for A_D in $\mathcal{A}^* \lambda/\mu$ we have $HG^{-1} A_D = \Phi D$.*

As a corollary, we obtain a bijective proof of part of (1.4) by taking the *leading terms* of each summand on the right-hand-side.

Corollary 1.10. *For a shape λ/μ we have $\sum_{T \in \text{SSYT}_{\min} \lambda/\mu} q^{|T|} = \sum_{D \in \mathcal{E} \lambda/\mu} q^{\sum_{i,j \in D} \lambda'_j - i}$.*

We also give a fully bijective proof of (1.4) for the case of *border strips*. The details of this last result and full version of this abstract will appear in [11].

2 Minimal semistandard tableaux of shape λ/μ

2.1 Excited diagram and broken diagonals

Given a subset D of λ , a cell (i, j) of D is *active* if the cells $(i-1, j)$, $(i, j-1)$, $(i-1, j-1)$ are not in D but are in λ . Given an active cell (i, j) of D , let $\beta_{i,j} : D \rightarrow D'$ be the map that replaces cell (i, j) in D by $(i-1, j-1)$. We call such $\beta_{i,j}$ an *excited move*. An *excited diagram* of λ/μ is a set of $|\mu|$ cells obtained from $\mu \subseteq \lambda$ by applying excited moves. We let $\mathcal{E} \lambda/\mu$ be the set of excited diagrams of λ/μ (see Figure 2a). These diagrams were introduced by Ikeda–Naruse in [5].

Each excited diagram D is associated with the *broken diagonals* $\text{Br } D$, which are obtained from the sub-diagonals of λ/μ by applying excited moves. For $\beta_{i,j} : D \rightarrow D'$, then $\text{Br } D' = \text{Br } D \setminus \{(i-1, j-1)\} \cup \{(i, j)\}$ (see Figure 2a).

2.2 Lascoux–Pragacz and Kreiman decomposition of λ/μ

We review two different decompositions of a Young diagram of shape λ/μ into *strips* (connected skew shapes with no 2×2 box), see Figure 1a.

The *Lascoux–Pragacz decomposition* of λ/μ is a tuple of non-intersecting lattice paths $\theta_1, \dots, \theta_k$, where θ_1 is the outer border strip of λ , which starts at the southwest corner of λ/μ and ends at the northeast corner of λ/μ [8]. Then θ_2 is the outer border strip of $\lambda \setminus \theta_1$ and each θ_i is the outer border strip of $\lambda \setminus \theta_1 \setminus \dots \setminus \theta_{i-1}$. We assume each strip

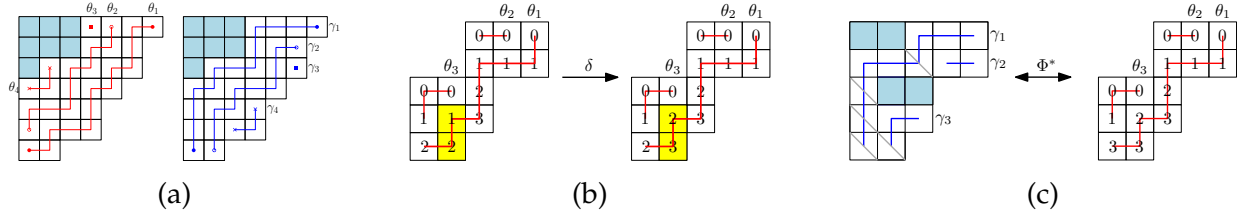


Figure 1: (a) Examples of the Lascoux–Pragacz decomposition and Kreiman decomposition of the skew shape λ/μ . (b) Excited move δ on $\text{SSYT}_{\min}$ (c) Excited diagram to $\text{SSYT}_{\min}$

θ_i starts at the left or bottom of λ/μ and ends at the right or top of λ/μ . Given a skew shape λ/μ , and its Lascoux–Pragacz decompositions $\theta_1, \dots, \theta_k$, denote f_{θ_r} the row number of the top most element of θ_r . We define the θ_r -height of row i , denoted as $ht_{\theta_r} i$ to be $i - f_{\theta_r}$. For example, in Figure 1a, the height of the element $7, 2$ in θ_1 is 6 whereas the height of $4, 1$ in θ_4 is 1.

The Kreiman decomposition of λ/μ is a tuple $\gamma_1, \dots, \gamma_k$ of non-intersecting lattice paths with support λ/μ , where each γ_i begins at the southwest corner and ends at northeast corner [7] (see Figure 1a). We denote ϵ_i the distance between γ_1 and γ_i . The support of such paths are the complement of excited diagrams [7, §5, §6]. If an excited diagram D corresponds to the tuple $\gamma'_1, \dots, \gamma'_k$, we denote by $\gamma_i D$ γ'_i , the path obtained from γ_i after applying the corresponding ladder moves to the excited moves where μ is obtained from D (see Figure 2a).

Lemma 2.1. Let $\gamma_1, \dots, \gamma_k$ and $\theta_1, \dots, \theta_k$ be the Kreiman and Lascoux–Pragacz decomposition of shape λ/μ . For each $i \in \{1, \dots, k\}$, let $\gamma_i : f_i, g_i \rightarrow c_i, d_i$ and $\theta_i : s_i, t_i \rightarrow u_i, v_i$. Then $g_i - t_i = \epsilon_i$, i.e. the column difference of the starting point of γ_i and θ_i is ϵ_i .

2.3 Minimal skew Semistandard Young Tableaux

The minimal skew SSYT are obtained from the minimum SSYT T_0 by applying a sequence of excited moves δ , defined as follows (see Figure 1b). Let λ/μ be a skew partition and $\theta_1, \dots, \theta_k$ be its Lascoux–Pragacz decomposition. Given a strip θ_k and column j , let $\theta_k j$ be the j th column segment of θ_k . Let i, j be the top-most cell and i', j be the bottom-most cell of $\theta_k j$ (which may agree if the column has size one).

Definition 2.2 (excited move δ). The column $\theta_k j$ of T in $\text{SSYT } \lambda/\mu$ is active if

- (i) $T_{i,j} < ht_{\theta_k} i$,
- (ii) $T_{i,j} < T_{i,j-1}$ and $T_{i',j} < T_{i'-1,j-1}$.

Given an active column $\theta_k j$ of T , the excited move δ adds one to each entry in the column segment $\theta_k j$ and $T' : \delta_{k,j} T$ be the tableau obtained from T by the move δ .

We define a bijection between $\mathcal{E} \lambda/\mu$ and $\text{SSYT}_{\min} \lambda/\mu$ denoted $\Phi : \mathcal{E} \lambda/\mu \rightarrow \text{SSYT} \lambda/\mu$, which commutes with the excited moves in $\mathcal{E} \lambda/\mu$ and $\text{SSYT}_{\min} \lambda/\mu$.

Definition 2.3. Given $D \in \mathcal{E} \lambda/\mu$, let $\gamma_1 D, \dots, \gamma_k D$ and $\theta_1, \dots, \theta_k$ be its Kreiman and Lascoux-Pragacz decomposition into paths and $\text{Br } D$ be its set of broken diagonals. Let b_j be the number of broken diagonals on the j th column of $\gamma_i D$. For each $\gamma_i D$ and θ_i , do the following procedure starting from the left most column of each path to obtain $T : \Phi D$:

1. Denote the bottom-most element of the j th column of θ_i as s_j, t_j . Let

$$T_{s_j, t_j} = \begin{cases} b_j & \text{if } j = 1, \\ T_{s_j, t_j - 1} - b_j & \text{if } j > 1. \end{cases}$$

2. Fill the rest of the j th column of θ_i so that each entry differs by 1.

The proof that Φ is well defined can be found in the full version [11].

Example 2.4. Let $D \in \mathcal{E} \lambda/\mu$ be the excited diagram shown in Figure 1c. We apply the correspondence to obtain $T = \Phi D$. The first column of $\gamma_1 D$ has three broken diagonals, so $T_{5,1} = 3$. The second column of $\gamma_1 D$ has no broken diagonals, so $T_{5,2} = 3$, then the rest of the column of θ_1 is filled with each entry differing by 1 to maintain column strictness. We continue the algorithm to obtain the final tableau T of skew shape λ/μ .

For the Lascoux-Pragacz decomposition of λ/μ and the Kreiman paths $\gamma_1, \dots, \gamma_k$ corresponding to an excited diagram D , we denote by $\text{col}_{\lambda/\mu} \theta_i, n$ and $\text{col}_{\lambda/\mu} \gamma_i, n$ the column on λ/μ where the n th column of θ_i and γ_i is, respectively.

Lemma 2.5. Given an excited diagram D in $\mathcal{E} \lambda/\mu$ and an active cell $u = a, b$ of D we have

$$\Phi \beta_u D = \delta_{f, u} \Phi D,$$

where $f = a, b = i; \mu = a, b$, i is the index of the Kreiman path $\gamma_i D$ modified by $\beta_{a, b}$, and $\mu = a, b$ is the column of the original cell in μ where a, b came from.

Proof. The excited move $\beta_{a, b}$ corresponds to a ladder move on $\gamma_i D \rightarrow \gamma_i D'$, for some i and a broken diagonal shift from $a-1, b-1$ to $a-1, b$. Let g_i and t_i be the starting column of $\gamma_i D$ and θ_i , respectively. Suppose $a-1, b$ is in the j th column of the path $\gamma_i D$. Then the excited move $\beta_{a, b}$ shifts a broken diagonal to the j -th column of $\gamma_i D'$, where $j = b - g_i - 1$. The j -th column of $\gamma_i D$ corresponds to the j -th column of θ_i , which corresponds to column $t_i - j + 1$ of shape λ/μ . By Lemma 2.1 we know that $g_i - t_i = \epsilon_i$. Thus we have that

$$\text{col}_{\lambda/\mu} \theta_i, j = t_i - j + 1 = g_i - \epsilon_i = b - g_i - 1 - 1 = b - \epsilon_i.$$

We know that ϵ_i is also the number of times a, b has been excited, which is $b - \mu = a, b$. Therefore we have that $\text{col}_{\lambda/\mu} \theta_i, j = \mu = a, b$. This shows that $\Phi \beta_u D$ has increments of 1 on θ_i at column $\mu = a, b$ compared to ΦD . This is equivalent to what the move $\delta_{i; \mu = a, b}$ does on ΦD . Thus $\Phi \beta_u D = \delta_{i; \mu = a, b} \Phi D$, as desired. $\square$

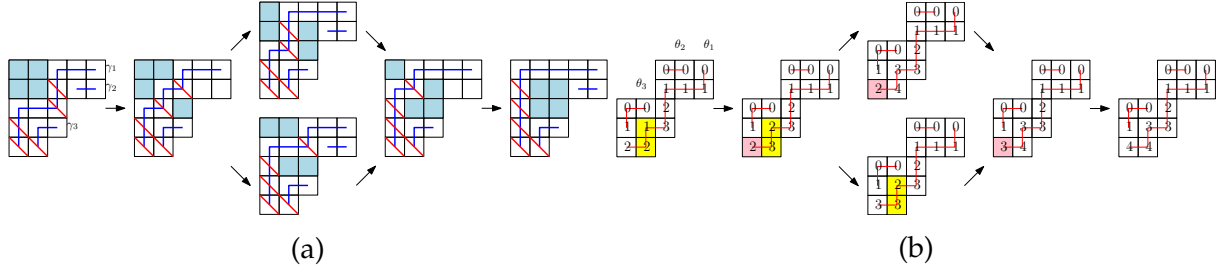


Figure 2: (a) The $\mathcal{E}$ $55332/22$, their corresponding $\gamma_i D$ (in blue) and broken diagonals (in red). (b) The $\text{SSYT}_{\min}$ $55332/22$ with the active columns colored in yellow and red.

Proof sketch of Theorem 1.4. By Lemma 2.5, Φ intertwines the excited moves β and δ . One can check that Φ is injective. Note that $\Phi \mu = T_0$. Given any excited diagram D in $\mathcal{E} \lambda/\mu$, there exists a sequence $\beta_1, \dots, \beta_m$ of excited moves such that $D = \beta_m \circ \dots \circ \beta_1 \mu$. Iterating Lemma 2.5 gives that $\Phi D = \delta_m \circ \dots \circ \delta_1 T_0$.

Thus ΦD is in $\text{SSYT}_{\min} \lambda/\mu$ and so $\Phi \mathcal{E} \lambda/\mu \subseteq \text{SSYT}_{\min} \lambda/\mu$. It remains to show that ΦD is surjective. Given T in $\text{SSYT}_{\min} \lambda/\mu$, there exists a sequence of $\delta_1, \dots, \delta_k$ of excited moves such that $\delta_k \circ \dots \circ \delta_1 T_0 = T$. Again, iterating Lemma 2.5 we have that for $D' = \beta_k \circ \dots \circ \beta_1 \mu$, one obtains $\Phi D' = T$, as desired. $\square$

There is the following non-recursive description of the minimal SSYT. This new characterization will be related to the Okounkov–Olshanski tableaux in the next section. We omit the details of the proof.

Theorem 2.6. A SSYT T of shape λ/μ is in $\text{SSYT}_{\min} \lambda/\mu$ if and only if

- (i) For each i_r, j_r in θ_r , $T(i_r, j_r) \leq ht_{\theta_r}(i_r)$
- (ii) For any i_r, j_r and $i_r + 1, j_r$ in θ_r , $T(i_r + 1, j_r) - T(i_r, j_r) = 1$.

In other words, the set $\text{SSYT}_{\min} \lambda/\mu$ consists of SSYT of shape λ/μ where the values along each path θ_r are bounded by the height ht_{θ_r} and the values along entries in the same column of a strip θ_r differ by one.

3 Reformulation of (NHLF) and comparison with (OOF)

In this section we use the bijection Φ between minimal SSYT and excited diagrams of shape λ/μ to reformulate the Naruse hook formula (NHLF) in terms of minimal SSYT. We need the following description of the inverse of Φ .

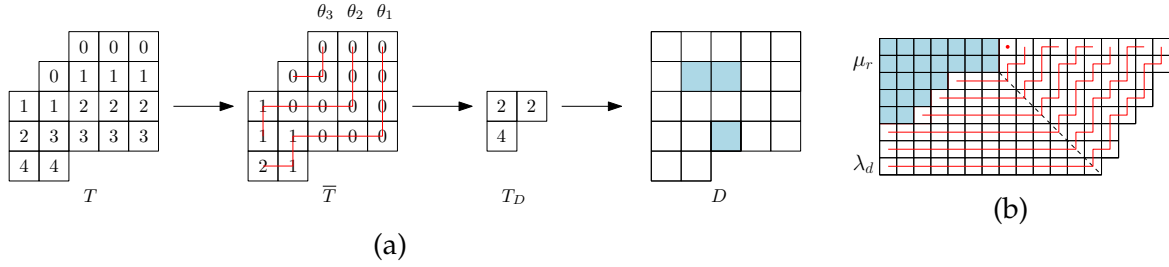


Figure 3: (a) Example of bijection Φ^{-1} between a minimal SSYT and excited diagrams for the shape $\lambda/\mu = 55552/21$. (b) Illustration of skew shapes λ/μ and their Lascoux–Pragacz decomposition with $\lambda_d \geq \mu_r$ $d - r$ where ED $\lambda/\mu =$ OOT λ/μ .

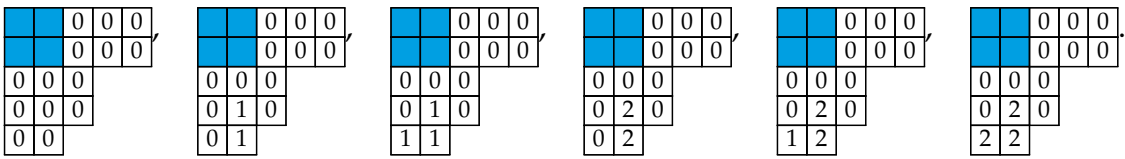
Lemma 3.1 (Inverse of Φ). *Let $D \in \mathcal{E}_{\lambda/\mu}$ and recall $\bar{T} = T - T_0$, where T_0 is the minimum skew SSYT and $T = \Phi(D)$. Then*

$$D = \{ i \rightarrow j \mid i, j \in \mu \},$$

where α is the number of strips θ_k such that $\overline{T} \theta_k j > \mu'_j - i$.

Proof of Theorem 1.5. The result follows by combining the excited diagram formulation (NHLF) of Naruse’s formula and the description of inverse of Φ from Lemma 3.1. $\square$

Example 3.2. For the shape $\lambda/\mu = 55332/22$, from each of its minimal tableaux in Figure 2b we find $\bar{T}$:



Next, for each $\bar{T}$ and i, j in $\mu \setminus 22$, we find α . Putting everything together using (1.2) gives

$$f^{\lambda/\mu} \frac{14!}{9 \cdot 8^2 \cdot 7 \cdot 6 \cdot 5^2 \cdot 4^2 \cdot 3^2 \cdot 2^4} (h \ 1,1 \ h \ 1,2 \ h \ 2,1 \ h \ 2,2 \quad h \ 1,1 \ h \ 1,2 \ h \ 2,1 \ h \ 2 \ \textcolor{red}{1},2 \ \textcolor{red}{1} \\ h \ 1,1 \ h \ 1 \ \textcolor{red}{1},2 \ \textcolor{red}{1} \ h \ 2,1 \ h \ 2 \ \textcolor{red}{1},2 \ \textcolor{red}{1} \quad h \ 1,1 \ h \ 1,2 \ h \ 2 \ \textcolor{red}{1},1 \ \textcolor{red}{1} \ h \ 2 \ \textcolor{red}{1},2 \ \textcolor{red}{1} \\ h \ 1,1 \ h \ 1 \ \textcolor{red}{1},2 \ \textcolor{red}{1} \ h \ 2 \ \textcolor{red}{1},1 \ \textcolor{red}{1} \ h \ 2 \ \textcolor{red}{1},2 \ \textcolor{red}{1} \quad h \ 1 \ \textcolor{red}{1},1 \ \textcolor{red}{1} \ h \ 1 \ \textcolor{red}{1},2 \ \textcolor{red}{1} \ h \ 2 \ \textcolor{red}{1},2 \ \textcolor{red}{1} \ \textcolor{red}{1}).$$

In the rest of this section we use the explicit non-recursive description of the skew minimal tableaux in $\text{SSYT}_{\min} \lambda/\mu$ of [Theorem 2.6](#) to relate the number $\text{ED } \lambda/\mu$ of terms of [\(NHLF\)](#) and the number $\text{OOT } \lambda/\mu$ of terms of [\(OOF\)](#).

Theorem 3.3. *For a skew shape λ/μ we have that $\text{ED } \lambda/\mu \leq \text{OOT } \lambda/\mu$.*

The proof follows immediately from the following lemma.

Lemma 3.4. *For a skew shape λ/μ we have that $\text{SSYT}_{\min} \lambda/\mu \subseteq \mathcal{SF} \lambda/\mu$.*

Proof. Let T be a tableau in $\text{SSYT}_{\min} \lambda/\mu$. By Condition (i) in [Theorem 2.6](#), we have that for i, j in θ_r , $T(i, j) \leq ht_{\theta_r}(i)$. By definition of the height, we have that $ht_{\theta_r}(i) \leq i - 1$. Thus T is a flagged tableau in $\mathcal{SF} \lambda/\mu$. $\square$

[Theorem 1.6](#) characterizes the skew shapes where the Naruse hook-length formula and the Okounkov–Olshanski formula have the same number of terms.

Proof sketch of Theorem 1.6. The inequality $\text{ED} \lambda/\mu \leq \text{OOT} \lambda/\mu$ follows from [Theorem 3.3](#). By [Lemma 3.4](#) we have that equality occurs if and only if all the flagged tableaux in $\mathcal{SF} \lambda/\mu$ are minimal. First, define the maximal tableau $T \in \mathcal{SF} \lambda/\mu$ as $T(i, j) = i - 1$ for $(i, j) \in \lambda/\mu$, and let $\theta_1, \dots, \theta_k$ be the Lascoux–Pragacz decomposition of λ/μ .

By the height condition in the definition of minimal tableaux (Condition (i) in [Theorem 2.6](#)), it follows that all border strips θ_k reach the top row. The border strips θ_k are entirely horizontal in the first μ_1 columns of λ/μ and we have $d - r$ of them starting in the first μ_1 columns. Thus $\lambda_d \geq \mu_1$. Also, since these border strips reach the top row, they must all cross the diagonal starting at (r, μ_r) and thus this diagonal extends to the last row of the shape, so $\lambda_d \geq \mu_r - d - r$, which proves the forward direction. See [Figure 3b](#).

For the other direction, if $\lambda_d \geq \mu_r - d - r$, then it is easy to see that all border strips θ_i reach the top row and they are horizontal in the first μ_1 columns. The tableaux in $\mathcal{SF} \lambda/\mu$ consisting of the columns past μ_1 are of straight shape, and one can show the entries in row i are $i - 1$. For the first μ_1 columns, since all border strips are horizontal, there are no further restrictions coming from the minimal tableaux. Thus $\text{SSYT}_{\min} \lambda/\mu = \mathcal{SF} \lambda/\mu$, as desired. $\square$

For shapes where $\text{ED} \lambda/\mu = \text{OOT} \lambda/\mu$, this number is given by the hook-content formula. We omit the proof.

Corollary 3.5. *For a connected skew shape λ/μ satisfying $\lambda_d \geq \mu_r - d - r$ where $d = \ell(\lambda)$ and $r = \max\{i \mid \mu_1 \leq \mu_i\}$, we have that $\text{ED} \lambda/\mu = \text{OOT} \lambda/\mu = s_\mu 1^{d-r}$.*

4 The Hillman–Grassl correspondence and the map Φ

In this section, we sketch the proof of [Theorem 1.9](#).

4.1 Background on Hillman–Grassl

We denote by $HG : \cdot$ the *Hillman–Grassl bijection* between reverse plane partitions in $\text{RPP } \lambda$ ranked by size and integer arrays of shape λ ranked by hook weight. That is, if $HG : \pi \mapsto A$ then $|\pi| = \sum_{u \in \lambda} h(u) \cdot A_u$. We follow the definition of $HG : \cdot$ in [15, §7.22]. This bijection also implies (1.3) as a corollary.

Definition 4.1 (Hillman–Grassl correspondence). *The correspondence is obtained via a sequence of pairs $(\pi_0, A_0) : (\pi, \mathbf{0})$, $(\pi_1, A_1) \dots, (\pi_k, A_k) : (\mathbf{0}, HG(\pi))$ where each π_i is a reverse plane partition and A_i is an array of nonnegative integers of shape λ obtained recursively as follows:*

- (i) start at the SW most nonzero cell of π_i ,
- (ii) traverse a NE path: from cell (a, b) , if $\pi_i(a, b) > \pi_i(a-1, b)$ then the path moves North, otherwise the path moves East.
- (iii) Terminate when this is no longer possible.

Let $u_i = (c, d)$ where d is the column where the path starts and c is the row where it ends. Note that $|\pi_i| - |\pi_{i-1}| = h(u_i)$. We obtain the array A_{i-1} from A_i by adding one to cell u_i . We continue until π_k as only zero entries. We let $A_k = HG(\pi)$.

We extend HG to $\text{SSYT } \lambda/\mu$ by viewing each skew $\text{SSYT } T$ as a plane partition of shape λ with zero entries in μ .

For any excited diagram $D \in \mathcal{E}(\lambda/\mu)$, denote by A_D the 0-1 array of shape λ with support on the broken diagonals $\text{Br } D$ of D and let $\mathcal{A}^*(\lambda/\mu) = \{A_D \mid D \in \mathcal{E}(\lambda/\mu)\}$. Let $\mathcal{A}_D^*$ be the set of arrays A of nonnegative integers of shape λ with support contained in $\lambda \setminus D$ and positive entries $A_{i,j} > 0$ if $A_D(i,j) = 1$.

Theorem 4.2 ([10, Thm. 7.7]). *The (restricted) Hillman–Grassl map HG is a bijection*

$$HG : \text{SSYT } \lambda/\mu \rightarrow \bigcup_{D \in \mathcal{E}(\lambda/\mu)} \mathcal{A}_D^*.$$

This property combined with the correspondence between RPPs and SSYTs of straight shape yields (1.4) as a corollary (see [10, Sec. 7.1]). The proof of this result is partly algebraic and the authors of [10] asked for a fully bijective proof. We make progress on this question by (i) giving a combinatorial proof for border strips λ/μ , and (ii) characterizing what is the image of $\text{SSYT}_{\min} \lambda/\mu$. We sketch the latter in the next section and the former is in [11].

4.2 Hillman–Grassl on minimal skew SSYT

In this section we give the necessary lemmas to prove Theorem 1.9. Fix a skew shape λ/μ with Lascoux–Pragacz decomposition $\theta_1, \dots, \theta_k$.

Let $A_{\gamma_i D}$ be the a subarray of A_D of shape $\lambda \setminus \cup_{j=1}^{i-1} \gamma_j D$ with support on the broken diagonal of $\gamma_i D$. We define the sum $A_{\gamma_i D} + A_{\gamma_{i-1} D}$ as an array of shape $A_{\gamma_i D}$ with supports on the broken diagonals of $\gamma_i D$ and $\gamma_{i-1} D$. In other words, it is the entry-wise sum of the arrays where alignment of $A_{\gamma_i D}$ and $A_{\gamma_{i-1} D}$ corresponds to the original A_D (see Figure 4a).

Similarly, let T_{θ_i} be a SSYT of shape $\lambda \setminus \cup_{j=1}^{i-1} \theta_j D$ with tableau entries in θ_i . We define the $T_{\theta_i} \cup T_{\theta_{i-1}}$ to be the tableaux of shape T_{θ_i} with entries of both θ_i and θ_{i-1} .

The first lemma is a variation of Theorem 1.9 restricted to the individual γ_i and θ_i (see Figure 4a).

Lemma 4.3. *Let $D \in \mathcal{E} \lambda/\mu$ and $T = \Phi D \in \text{SSYT}_{\min} \lambda/\mu$. Then for $i = 1, \dots, k$ we have that $HG T_{\theta_i} = A_{\gamma_i D}$.*

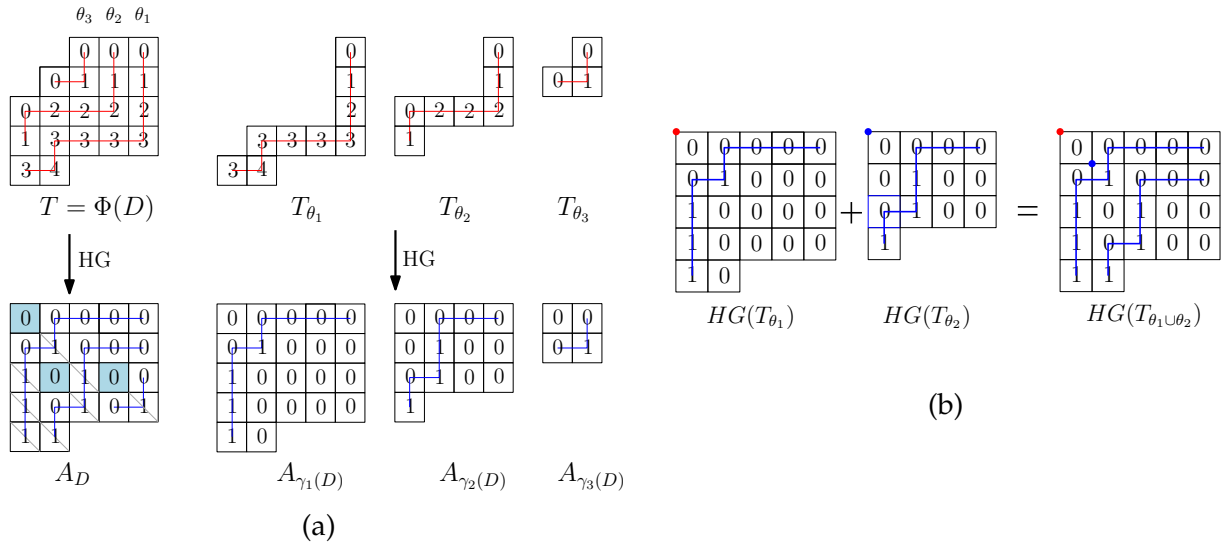


Figure 4: (a) The Hillman–Grassl map of T_{θ_1} , T_{θ_2} , and T_{θ_3} separately (b) Additivity of the Hillman–Grassl correspondence on T_{θ_i} .

In order to show Theorem 1.9, we need the following additivity results (see Section 4.2). We define the sum of arrays $HG T_{\theta_i}$ and $HG T_{\theta_{i-1}}$ similarly to the sum of $A_{\gamma_i D}$ and $A_{\gamma_{i-1} D}$. We omit the technical proof by induction.

Lemma 4.4. *We have that $\sum_{i=1}^k HG T_{\theta_i} = HG T_{\theta_1 \cup \dots \cup \theta_k}$.*

Proof sketch of Theorem 1.9. We have that $T_{\theta_1 \cup \dots \cup \theta_k} = \Phi D$. We obtain the desired identity $HG \Phi D = \sum_{i=1}^k A_{\gamma_i D} = A_D$ by Lemma 4.3 and Lemma 4.4. $\square$

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Stable-Limit Non-symmetric Macdonald Functions in Type A

Milo James Bechtloff Weising^{*1}

¹Department of Mathematics, University of California, Davis

Abstract. We construct and study an explicit simultaneous $\mathcal{Y}$ eigenbasis of Ion and Wu's standard representation of the stable-limit double affine Hecke algebra for the limit Cherednik operators $\mathcal{Y}_i$. This basis arises as a generalization of Cherednik's non-symmetric Macdonald polynomials of type GL_n . We utilize links between stable-limit double affine Hecke algebra theory of Ion and Wu and the double Dyck path algebra of Carlsson and Mellit that arose in their proof of the Shuffle Conjecture. As a consequence, the spectral theory for the limit Cherednik operators is understood.

Keywords: stable-limit, Macdonald polynomials, double affine Hecke algebra, double Dyck path algebra, Cherednik operators

1 Introduction

The Shuffle Conjecture, now the Shuffle Theorem [2], is a combinatorial statement regarding the Frobenius character, $\mathcal{F}_{R_n}$, of the diagonal coinvariant algebra R_n which generalizes the coinvariant algebra arising from the geometry of flag varieties. The following explicit formula is due to Haiman [5]:

$$\mathcal{F}_{R_n}(X; q, t) = (-1)^n \nabla e_n(X)$$

where the operator ∇ is an eigenoperator on symmetric functions prescribed by its action on the modified Macdonald symmetric functions as

$$\nabla \tilde{H}_\mu = \tilde{H}_\mu - (-1)^{l(\mu)} \tilde{H}_\mu.$$

The original conjecture of Haglund, Haiman, Loehr, Remmel, and Ulyanov states the following:

Theorem 1 (Shuffle Theorem). [5]

$$(-1)^n \nabla e_n(X) = \sum_{\pi} \sum_{w \in WP_\pi} t^{\text{area } \pi} q^{\text{dinv } \pi, w} x_w.$$

^{*}mjbechtloffweising@ucdavis.edu.

In the above, π ranges over the set of Dyck paths of length n and WP_π is the set of word parking functions corresponding to π . The values $area\ \pi$ and $dinv\ \pi, w$ are certain statistics corresponding to π and $w \in WP_\pi$.

In [2], Carlsson and Mellit prove the Compositional Shuffle Conjecture, a generalization of the original Shuffle Conjecture. The authors construct and investigate a quiver path algebra called the Double Dyck Path algebra $\mathbb{A}_{q,t}$. They construct a representation of $\mathbb{A}_{q,t}$, called the standard representation, built on certain mixed symmetric and non-symmetric polynomial algebras with actions from Demazure-Lusztig operators, Hall-Littlewood creation operators, and plethysms. The Compositional Shuffle Conjecture falls out after a rich understanding of the standard representation is developed. Later analysis done by Gorsky, Carlsson, and Mellit [1] showed that in fact $\mathbb{A}_{q,t}$ occurs naturally in the context of equivariant cohomology of Hilbert schemes.

Recent work by Ion and Wu [7] has made progress in linking the work of Carlsson and Mellit on $\mathbb{A}_{q,t}$ to the representation theory of double affine Hecke algebras. Ion and Wu introduce the stable-limit double affine Hecke algebra $\mathcal{H}$ along with a representation $\mathcal{P}_{as}$ of $\mathcal{H}$ from which one can recover the standard $\mathbb{A}_{q,t}$ representation. The main obstruction in making a stable-limit theory for the double affine Hecke algebras is the lack of an inverse system of the double affine Hecke algebras in the traditional sense. Ion and Wu get around this obstruction by introducing a new notion of convergence (Definition 6) for sequences of polynomials with increasing numbers of variables along with limit versions of the standard Cherednik operators defined by this convergence.

Central to the study of the standard Cherednik operators are the non-symmetric Macdonald polynomials. The non-symmetric Macdonald polynomials in full generality were introduced first by Cherednik [3] in the context of proving the Macdonald constant-term conjecture. The introduction of the double affine Hecke algebra, along with the non-symmetric Macdonald polynomials by Cherednik, constituted a significant development in representation theory. They serve as a non-symmetric counterpart to the symmetric Macdonald polynomials introduced by Macdonald as a q,t -analog of Schur functions. Further, they give an orthogonal basis of the polynomial representation consisting of weight vectors for the Cherednik operators. In particular, the correct choice of symmetrization applied to a non-symmetric Macdonald polynomial will yield their symmetric counterpart. The type A symmetric Macdonald polynomials are a remarkable basis for symmetric polynomials simultaneously generalizing many other well studied bases which can be recovered by appropriate specializations of values for q and t . The aforementioned modified Macdonald functions $\tilde{H}_\mu$ can be obtained via a plethystic transformation from the symmetric Macdonald polynomials in sufficiently many variables. The spectral theory of non-symmetric Macdonald polynomials is well understood using the combinatorics of affine Weyl groups.

It is natural to seek an asymptotic extension for the non-symmetric Macdonald polynomials following the methods of Ion and Wu. In particular, does the standard $\mathcal{H}$

representation $\mathcal{P}_{as}$ have a basis of weight vectors for the limit Cherednik operators $\mathcal{Y}_i$? The main theorem of this paper answers this question in the affirmative.

The strategy for finding a basis of weight vectors for the limit Cherednik operators $\mathcal{Y}_i$ is the following. First, we show that the non-symmetric Macdonald polynomials have stable-limits in the sense that if we start with a composition μ and consider the compositions $\mu * 0^m$ for $m \geq 0$ then the corresponding sequence of non-symmetric Macdonald polynomials $E_{\mu * 0^m}$ converges to an element $\tilde{E}_\mu$ of $\mathcal{P}_{as}$. Next, we show that these limits of non-symmetric Macdonald polynomials are $\mathcal{Y}$ -weight vectors. Importantly, the newly constructed set of $\tilde{E}_\mu$ do *not* span $\mathcal{P}_{as}$. To fill in these gaps, the lowering operators d_- from $\mathbb{A}_{q,t}$ are used to create enough $\mathcal{Y}$ weight vectors to span $\mathcal{P}_{as}$. Finally, a symmetrization operator is used to show that the spanning set obtained from this process is actually a basis in [Theorem 7](#).

[Lemma 1](#), [Theorem 5](#), and [Lemma 5](#) together give a description of the weights across all weight vectors in $\mathcal{P}_{as}$. The author would like to thank the FPSAC referees who alerted the author to an unpublished work of Ion and Wu [\[6\]](#) which independently determines the same explicit description of these eigenvalues.

2 Definitions and Notation

2.1 Double Affine Hecke Algebras in Type GL

Definition 1. Define the *double affine Hecke algebra* $\mathcal{H}_n$ to be the $\mathbb{Q}[q, t]$ -algebra generated by $T_1, \dots, T_{n-1}$, $X_1^{\pm 1}, \dots, X_n^{\pm 1}$, and $Y_1^{\pm 1}, \dots, Y_n^{\pm 1}$ with the following relations:

(i) $T_i - 1 - T_i - t = 0$, $T_i T_{i-1} T_i = T_{i-1} T_i T_{i-1}$, $T_i T_j = T_j T_i, i - j > 1$, (ii) $T_i^{-1} X_i T_i^{-1} = t^{-1} X_{i-1}$, $T_i X_j = X_j T_i, i \notin \{j, j-1\}$, $X_i X_j = X_j X_i$,	(iii) $T_i Y_i T_i = t Y_{i-1}$, $T_i Y_j = Y_j T_i, i \notin \{j, j-1\}$, $Y_i Y_j = Y_j Y_i$, (iv) $Y_1 T_1 X_1 = X_2 Y_1 T_1$, (v) $Y_1 X_1 \cdots X_n = q X_1 \cdots X_n Y_1$

Further, define the special element ω_n by

$$\omega_n : T_{n-1}^{-1} \cdots T_1^{-1} Y_1^{-1}$$

2.1.1 Standard DAHA representation

Definition 2. Let $\mathcal{P}_n = \mathbb{Q}[q, t, x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. The *standard representation of $\mathcal{H}_n$* is given by the following action on $\mathcal{P}_n$:

- $T_i f(x_1, \dots, x_n) = s_i f(x_1, \dots, x_n) = (1 - t x_i \frac{1-s_i}{x_i - x_{i-1}}) f(x_1, \dots, x_n)$
- $X_i f(x_1, \dots, x_n) = x_i f(x_1, \dots, x_n)$
- $\omega_n f(x_1, \dots, x_n) = f(q^{-1} x_n, x_1, \dots, x_{n-1})$

Here s_i denotes the operator that swaps the variables x_i and x_{i-1} . Under this action the T_i operators are known as the *Demazure-Lusztig operators*. For q, t generic $\mathcal{P}_n$ is known to be a faithful representation of $\mathcal{H}_n$. The action of the elements $Y_1, \dots, Y_n \in \mathcal{H}_n$ are called *Cherednik operators*.

Set $\mathcal{H}_n$ to be the positive part of $\mathcal{H}_n$ i.e. the subalgebra generated by $T_1, \dots, T_{n-1}$, $X_1, \dots, X_n$, and $Y_1, \dots, Y_n$ without allowing for inverses in the X and Y elements and set $\mathcal{P}_n = \mathbb{Q}[q, t, x_1, \dots, x_n]$. Importantly, $\mathcal{P}_n$ is a $\mathcal{H}_n$ submodule of $\mathcal{P}_n$.

2.1.2 Non-symmetric Macdonald Polynomials and Symmetric Functions

Definition 3. The *non-symmetric Macdonald polynomials* (for GL_n) are a family of Laurent polynomials $E_\mu \in \mathcal{P}_n$ for $\mu \in \mathbb{Z}^n$ uniquely determined by the following:

- **Triangularity:** Each E_μ has a monomial expansion of the form $E_\mu = \sum_{\lambda < \mu} a_\lambda x^\lambda$ where " $<$ " denotes the Bruhat order for $\mathbb{Z}^n$
- **Weight Vector:** Each E_μ is a weight vector for the operators $Y_1, \dots, Y_n \in \mathcal{H}_n$.

The non-symmetric Macdonald polynomials are a Y weight basis for the $\mathcal{H}_n$ standard representation $\mathcal{P}_n$. For $\mu \in \mathbb{Z}^n$, E_μ is homogeneous with degree $\mu_1 + \dots + \mu_n$. Further, the set of E_μ corresponding to $\mu \in \mathbb{Z}_{\geq 0}^n$ gives a basis for $\mathcal{P}_n$.

Definition 4. In this paper, a *composition* will refer to a finite tuple $\mu = \mu_1, \dots, \mu_n$ of non-negative integers. We allow for the empty composition $\emptyset$ with no parts. The length of a composition $\mu = \mu_1, \dots, \mu_n$ is $\ell(\mu) = n$ and the size of the composition is $|\mu| = \mu_1 + \dots + \mu_n$. Given two compositions $\mu = \mu_1, \dots, \mu_n$ and $\beta = \beta_1, \dots, \beta_m$, define $\mu * \beta = \mu_1, \dots, \mu_n, \beta_1, \dots, \beta_m$. A *partition* is a composition $\lambda = \lambda_1, \dots, \lambda_n$ with $\lambda_1 \geq \dots \geq \lambda_n \geq 1$. We denote $\text{sort } \mu$ to be the partition obtained by ordering the nonzero elements of μ in weakly decreasing order. Define the *ring of symmetric functions* Λ to be the subalgebra of bounded x -degree elements of the inverse limit of the symmetric polynomial rings $\mathbb{Q}[q, t, x_1, \dots, x_n]^{S_n}$ with respect to the quotient maps sending $x_n \rightarrow 0$.

In this paper we use plethystic notation. For a complete introduction and explanation of plethysm we refer the reader to [8]. For example, if $F \in \Lambda$ and $\{t_1, t_2, \dots\}$ is a set of independent variables, then we write $F t_1 t_2 \dots$ for the symmetric function given by F with variables in the set $\{t_1, t_2, \dots\}$. We will in a few instances use the notation $\mathbb{1} p$ to denote the value 1 if the statement p is true and 0 otherwise.

2.2 Stable-Limit DAHA of Ion and Wu

Definition 5. The *stable-limit double affine Hecke algebra* of Ion and Wu, $\mathcal{H}$, is the algebra generated over $\mathbb{Q}[q, t]$ by the elements T_i, X_i, Y_i for $i \in \mathbb{N}$ satisfying the following relations:

- The generators T_i, X_i for $i \in \mathbb{N}$ satisfy (i) and (ii) of Definition 1.
- The generators T_i, Y_i for $i \in \mathbb{N}$ satisfy (i) and (iii) of Definition 1.
- $Y_1 T_1 X_1 = X_2 Y_1 T_1$

We include Ion and Wu's full definition of convergence in Definition 6 for the sake of completeness. A full understanding of convergence is not required to follow the rest of this paper.

Definition 6. [7] Let $\mathcal{P}_k : \mathbb{Q}[q, t][x_1, \dots, x_k] \otimes \Lambda[x_{k+1}, x_{k+2}, \dots]$. Define the *ring of almost symmetric functions* $\mathcal{P}_{as} : \bigcup_{k \geq 0} \mathcal{P}_k$. Further, let $\mathcal{P}_\infty$ denote the inverse limit of the rings $\mathcal{P}_k$ with respect to the homomorphisms which send x_{k+1} to 0 at each step. Note $\mathcal{P}_{as} \subset \mathcal{P}_\infty$. Define $\rho : \mathcal{P}_{as} \rightarrow x_1 \mathcal{P}_{as}$ to be the linear map defined by $\rho(x_1^{a_1} \dots x_n^{a_n} F x_m \dots x_{m+1} \dots) = \mathbb{1} a_1 > 0 x_1^{a_1-1} \dots x_n^{a_n} F x_m \dots x_{m+1} \dots$ for $F \in \Lambda$. Let f_k be a sequence of polynomials with $f_k \in \mathcal{P}_k$. Then the sequence f_k is *convergent* if there exist some N and auxiliary sequences h_k , g_k^i for $k \geq 1$, and a_k^i for $1 \leq i \leq N$ with $h_k, g_k^i \in \mathcal{P}_k$, $a_k^i \in \mathbb{Q}[q, t]$ with the following properties:

- For all k , $f_k = h_k + \sum_{i=1}^N a_k^i g_k^i$.
- The sequences h_k , g_k^i for $k \geq 1$ converge in $\mathcal{P}_\infty$ with limits h, g^i respectively. Further, $g^i \in \mathcal{P}_{as}$.
- The sequences a_k^i for $1 \leq i \leq N$ converge with respect to the t -adic topology on $\mathbb{Q}[q, t]$ with limits a^i which are required to be in $\mathbb{Q}[q, t]$.

The sequence is said to have a limit given by $\lim_k f_k = h + \sum_{i=1}^N a^i g^i$.

Ion and Wu use their definition of convergence to define asymptotic versions of the Cherednik operators.

Theorem 2. [7] Consider the sequence of operators $\tilde{Y}_1^n : t^n \rho \circ Y_1^n$ where Y_1^n is the operator coming from the action of $Y_1 \in \mathcal{H}_n$ on $\mathcal{P}_n$. Let $\pi_n : \mathcal{P}_{as} \rightarrow \mathcal{P}_n$ be the canonical projection and let $f \in \mathcal{P}_{as}$. Then the sequence $\tilde{Y}_1^n \circ \pi_n f$ $n \geq 1$ is convergent with limit which is also almost symmetric. This yields a well-defined operator $\mathcal{Y}_1 : \mathcal{P}_{as} \rightarrow \mathcal{P}_{as}$ given by $\mathcal{Y}_1 f : \lim_n \tilde{Y}_1^n \circ \pi_n f$. Further, the operator $\mathcal{Y}_1$ along with the Demazure-Lusztig action of the T_i 's and multiplication by the X_i 's generate an $\mathcal{H}$ action on $\mathcal{P}_{as}$.

3 Stable-Limits of Non-symmetric Macdonald Polynomials

Given a composition μ , consider the compositions $\mu * 0^m$ for $m \geq 0$ and the corresponding sequence of non-symmetric Macdonald polynomials $E_{\mu * 0^m}$ $m \geq 0$. In order to prove the convergence of these sequences we use the following result of [4] giving an explicit combinatorial formula for the non-symmetric Macdonald polynomials. Note that the q, t conventions in [4] differ from those appearing in this paper. In the below theorem the appropriate translation $q \rightarrow q^{-1}$ has been made.

Theorem 3. [4] For a composition μ with $\ell \mu = n$ the following holds:

$$E_\mu = \sum_{\substack{\sigma: \mu \rightarrow n \\ \text{non-attacking}}} X^\sigma q^{-\text{maj } \hat{\sigma}} t^{\text{coinv } \hat{\sigma}} \prod_{\substack{u \in dg' \mu \\ \hat{\sigma} u / \hat{\sigma} d u}} \left(\frac{1-t}{1-q^{-\ell u-1} t^{a u-1}} \right)$$

The combinatorial description of non-symmetric Macdonald polynomials in the Haiman-Haglund-Loehr formula relies on the combinatorics of *non-attacking labellings* of certain box diagrams corresponding to compositions. In the interest of space we refer the reader to [4] for all the notation used above such as $\hat{\sigma}$, d , a , ℓ , maj , and coinv .

We now show the convergence for the sequence $E_{\mu * 0^m}$ $m \geq 0$. The method used shows convergence and gives an explicit combinatorial formula for the limit functions.

Theorem 4. For a composition μ with $\ell \mu = n$ the sequence $E_{\mu * 0^m}$ $m \geq 0$ is convergent with limit $\tilde{E}_\mu$ in $\mathcal{P}_{as}$ given by

$$\tilde{E}_\mu : \sum_{\substack{\lambda \text{ partition} \\ |\lambda| \leq |\mu|}} m_\lambda x_{n-1} \cdots \sum_{\substack{\sigma: \mu * 0^\ell \lambda \rightarrow \{1, \dots, n-\ell, \lambda\} \\ \text{non-attacking} \\ |\sigma^{-1} n-i| = \lambda_i}} x_1^{|\sigma^{-1} 1|} \cdots x_n^{|\sigma^{-1} n|} q^{-\text{maj } \hat{\sigma}} t^{\text{coinv } \hat{\sigma}} \tilde{\Gamma} \hat{\sigma}$$

where

$$\tilde{\Gamma} \hat{\sigma} = \prod_{\substack{u \in dg' \mu * 0^\ell \lambda \\ \hat{\sigma} u / \hat{\sigma} d u \\ u \text{ not in row 1}}} \left(\frac{1-t}{1-q^{-\ell u-1} t^{a u-1}} \right) \prod_{\substack{u \in dg' \mu * 0^\ell \lambda \\ \hat{\sigma} u / \hat{\sigma} d u \\ u \text{ in row 1}}} 1-t$$

Proof Sketch. Start by using the HHL formula to expand $E_{\mu*0^m}$ for $m \geq 1$. Because $E_{\mu*0^m}$ is symmetric in $x_{n-1}, \dots, x_{n-m}$ we can expand relative to the monomial symmetric functions $m_\lambda x_{n-1} \dots x_{n-m}$. This is made explicit using the combinatorics of non-attacking labellings as per HHL. For sufficiently large $m \geq |\mu|$ the $\mathbb{Q}[q, t]$ coefficients of the $m_\lambda x_{n-1} \dots x_{n-m}$ stabilize to polynomials with coefficients that converge t -adically. $\square$

Remark. Note importantly, that for any composition μ and $m \geq 0$, by definition $\tilde{E}_{\mu*0^m} = \tilde{E}_\mu$.

3.0.1 Example

Here we list a few simple examples.

- $\tilde{E}_1 = x_1$
- $\tilde{E}_{2,0} = x_1^2 - \frac{q^{-1}1-t}{1-q^{-1}t} x_1 m_1 x_2 - x_3 - \dots$
- $\tilde{E}_{0,2} = x_2^2 - (1-t)x_1^2 - \frac{1-q^{-1}t}{1-q^{-1}t} \frac{q^{-1}}{1-t} x_1 x_2 - \left(\frac{q^{-1}1-t}{1-q^{-1}t} x_2 - \frac{q^{-1}1-t^2}{1-q^{-1}t} x_1 \right) m_1 x_3 - \dots$
- $\tilde{E}_{2,2} = x_1^2 x_2^2 - \frac{q^{-1}1-t}{1-q^{-1}t} x_1^2 x_2 - x_1 x_2^2 m_1 x_3 - x_4 - \dots - \left(\frac{q^{-2}1-t^2}{q^{-2}t^3 - q^{-1}t^2 - q^{-1}t - 1} \right) x_1 x_2 m_{1,1} x_3 - x_4$
 $\dots$

4 $\mathcal{Y}$ Weight Basis of $\mathcal{P}_{as}$

Given a family of commuting operators $\{y_i : i \in I\}$ and a weight vector v we denote its weight by the function $\alpha : I \rightarrow \mathbb{Q}[q, t]$ such that $y_i v = \alpha(i) v$. We sometimes denote α as $\alpha_1, \alpha_2, \dots$.

4.1 The $\tilde{E}_\mu$ are $\mathcal{Y}$ weight vectors

In what follows, the classical spectral theory for non-symmetric Macdonald polynomials is used to demonstrate that the limit functions $\tilde{E}_\mu$ are $\mathcal{Y}$ weight vectors. The below lemma is a simple application of this classical theory and of basic properties of the t -adic topology on $\mathbb{Q}[q, t]$.

Lemma 1. For a composition μ with $\ell(\mu) = n$ define α_μ^m to be the weight of $E_{\mu*0^m}$. Then in the t -adic topology on $\mathbb{Q}[q, t]$ the sequence $t^{n-m} \alpha_\mu^m$ converges in m to some $\tilde{\alpha}_\mu \in \mathbb{Q}[q, t]$. In particular, $\tilde{\alpha}_\mu(i) = 0$ for $i > n$ and for $1 \leq i \leq n$ we have that $\tilde{\alpha}_\mu(i) = 0$ exactly when $\mu_i = 0$.

Proof: Take $\mu = \mu_1, \dots, \mu_n$. From classic double affine Hecke algebra theory we have $\alpha_\mu^0 i = q^{\mu_i} t^{1-\beta_\mu i}$ where

$$\beta_\mu i : \#\{j : 1 \leq j \leq i, \mu_j \leq \mu_i\} - \#\{j : i < j \leq n, \mu_i > \mu_j\}.$$

It follows then that

$$t^{n-m} \alpha_\mu^m i = \begin{cases} q^{\mu_i} t^{n-m-1-\beta_\mu i} m \mathbb{1}_{\mu_i/0} & t^n \alpha_\mu^0 i & i \leq n, \mu_i \neq 0 \\ q^{\mu_i} t^{n-m-1-\beta_\mu i} m \mathbb{1}_{\mu_i/0} & t^{n-m} \alpha_\mu^0 i & i \leq n, \mu_i = 0 \\ t^{n-m-1-\#\mu_j=0} i-n & t^{\#\mu_j/0} t^{m-1-i-n} & i > n \end{cases}$$

Lastly, by limiting $m \rightarrow \infty$ we get the result. $\square$

For a composition μ define the list of scalars $\tilde{\alpha}_\mu$ using the formula in [Lemma 1](#) for $\tilde{\alpha}_\mu i$ for $i \in \mathbb{N}$. We use [Lemma 1](#) to show that certain denominators that occur in the proof of [Lemma 2](#) below do not vanish in the limit as $m \rightarrow \infty$.

Lemma 2. For $\mu = \mu_1, \dots, \mu_n$ with $\mu_i \neq 0$ for $1 \leq i \leq n$, $\tilde{E}_\mu$ is a $\mathcal{Y}$ -weight vector with weight $\tilde{\alpha}_\mu$.

Proof. We spare the reader the direct calculation which uses the limit definition of the $\mathcal{Y}_r$ operators and Prop. 6.21 from [7] which leads to

$$\mathcal{Y}_r \tilde{E}_\mu = \tilde{\alpha}_\mu r T_{r-1} \cdots T_1 \rho T_1^{-1} \cdots T_{r-1}^{-1} \tilde{E}_\mu. \quad (4.1)$$

We will show that the right side of (4.1) is $\tilde{\alpha}_\mu r \tilde{E}_\mu$. As $\tilde{\alpha}_\mu r = 0$ for $r > n$ by [Lemma 1](#), the lemma holds for $r \leq n$. Now let us consider some fixed $r \leq n$. Below we show that $x_1 | T_1^{-1} \cdots T_{r-1}^{-1} \tilde{E}_\mu$ from which it follows that

$$\rho T_1^{-1} \cdots T_{r-1}^{-1} \tilde{E}_\mu = T_1^{-1} \cdots T_{r-1}^{-1} \tilde{E}_\mu$$

implying

$$\begin{aligned} \mathcal{Y}_r \tilde{E}_\mu &= \tilde{\alpha}_\mu r T_{r-1} \cdots T_1 \rho T_1^{-1} \cdots T_{r-1}^{-1} \tilde{E}_\mu \\ &= \tilde{\alpha}_\mu r T_{r-1} \cdots T_1 T_1^{-1} \cdots T_{r-1}^{-1} \tilde{E}_\mu \\ &= \tilde{\alpha}_\mu r \tilde{E}_\mu \end{aligned}$$

as desired. To show that $x_1 | T_1^{-1} \cdots T_{r-1}^{-1} \tilde{E}_\mu$ it suffices to show that for all $m \geq 0$, $x_1 | T_1^{-1} \cdots T_{r-1}^{-1} E_{\mu*0^m}$. To this end fix $m \geq 0$. We have that

$$\begin{aligned} \alpha_\mu^m r E_{\mu*0^m} &= Y_r^{n-m} E_{\mu*0^m} \\ &= t^{-r-1} T_{r-1} \cdots T_1 \omega_n^{-1} T_n^{-1} \cdots T_r^{-1} E_{\mu*0^m}. \end{aligned}$$

Since $\alpha_\mu^m \neq 0$ we can have $\frac{1}{\alpha_\mu^m} T_1^{-1} \cdots T_{r-1}^{-1}$ act on both sides to get

$$T_1^{-1} \cdots T_{r-1}^{-1} E_{\mu * 0^m} = \frac{t^{-r-1}}{\alpha_\mu^m} \omega_{n-m}^{-1} T_{n-m-1}^{-1} \cdots T_r^{-1} E_{\mu * 0^m}.$$

By HHL any non-attacking labelling of $\mu * 0^m$ will have row 1 diagram labels given by $\{1, 2, \dots, n\}$ so $x_1 \cdots x_n$ divides $E_{\mu * 0^m}$ so in particular x_r divides $E_{\mu * 0^m}$ for all $m \geq 0$. Lastly,

$$\begin{aligned} \omega_{n-m}^{-1} T_{n-m-1}^{-1} \cdots T_r^{-1} X_r &= \omega_{n-m}^{-1} t^{-n-m-r} X_{n-m} T_{n-m-1} \cdots T_r \\ &= q t^{-n-m-r} X_1 \omega_{n-m}^{-1} T_{n-m-1} \cdots T_r \end{aligned}$$

Thus x_1 divides $T_1^{-1} \cdots T_{r-1}^{-1} E_{\mu * 0^m}$ for all $m \geq 0$ showing the result. $\square$

Now we consider the general situation where the composition μ can have some parts which are 0. We can extend the above result, [Lemma 2](#), by a straight-forward argument using intertwiner theory from the study of affine Hecke algebras.

Theorem 5. For all compositions μ , $\tilde{E}_\mu$ is a $\mathcal{Y}$ -weight vector with weight $\tilde{\alpha}_\mu$.

Proof Sketch: [Lemma 2](#) shows that this statement holds for any composition with all parts nonzero. Further, every composition μ can be written as a permutation of a composition of the form $\nu * 0^m$ for a partition ν and some $m \geq 0$. Hence, it suffices to show that for any composition μ , if $\tilde{E}_\mu$ satisfies the theorem then so will $\tilde{E}_{s_i \mu}$. This process is made rigorous by using induction on Bruhat order. Using the intertwiner operators from standard affine Hecke algebra theory, given by $\varphi_i = T_i \mathcal{Y}_i - \mathcal{Y}_i T_i$, we only need to show that for any μ with $s_i \mu > \mu$ in Bruhat order,

$$\varphi_i \tilde{E}_\mu = \tilde{\alpha}_\mu(i) - \tilde{\alpha}_\mu(i) - 1 \tilde{E}_{s_i \mu}.$$

Suppose the theorem holds for some μ with $\ell \mu = n$ and let $1 \leq i \leq n$ such that $s_i \mu > \mu$. Then we have the following:

$$\begin{aligned} \varphi_i \tilde{E}_\mu &= T_i \mathcal{Y}_i - \mathcal{Y}_i T_i = (1 - t \mathcal{Y}_i) \tilde{E}_\mu \\ &= \tilde{\alpha}_\mu(i) - \tilde{\alpha}_\mu(i) - 1 T_i \tilde{E}_\mu = (1 - t \tilde{\alpha}_\mu(i) - 1) \tilde{E}_\mu \\ \lim_m t^{n-m} \alpha_\mu^m(i) - t^{n-m} \alpha_\mu^m(i) - 1 T_i E_{\mu * 0^m} &= (1 - t t^{n-m} \alpha_\mu^m(i) - 1) E_{\mu * 0^m} \\ \lim_m t^{n-m} \alpha_\mu^m(i) - t^{n-m} \alpha_\mu^m(i) - 1 E_{s_i \mu * 0^m} &= \\ \tilde{\alpha}_\mu(i) - \tilde{\alpha}_\mu(i) - 1 \tilde{E}_{s_i \mu} &. \end{aligned}$$

$\square$

We have shown in [Theorem 5](#) there is an explicit collection of $\mathcal{Y}$ -weight vectors $\tilde{E}_\mu$ in $\mathcal{P}_{as}$ arising as the limits of non-symmetric Macdonald polynomials $E_{\mu*0^m}$. Unfortunately, these $\tilde{E}_\mu$ do not span $\mathcal{P}_{as}$. To see this note that one cannot write a non-constant symmetric function as a linear combination of the $\tilde{E}_\mu$. However, in the below work we build a full $\mathcal{Y}$ weight basis.

4.2 Constructing the Weight Basis

To complete our construction of a full weight basis of $\mathcal{P}_{as}$ one needs the ∂_-^k operators from Ion and Wu. These operators are, up to a change of variables and plethysm, the d_- operators from Carlsson and Mellit's standard $\mathbb{A}_{q,t}$ representation.

Definition 7. [7] Define the operator $\partial_-^k : \mathcal{P}^k \rightarrow \mathcal{P}^{k-1}$ to be the $\mathcal{P}_{k-1}$ -linear map which acts on elements of the form $x_k^n F x_{k-1} x_{k-2} \cdots$ for $F \in \Lambda$ and $n \geq 0$ as

$$\partial_-^k x_k^n F x_{k-1} x_{k-2} \cdots = \mathcal{B}_n F x_k x_{k-1} \cdots.$$

Here the $\mathcal{B}_n$ are the Jing operators which serve as creation operators for the Hall-Littlewood symmetric functions $\mathcal{P}_\lambda$ given explicitly by the following plethystic formula:

$$\mathcal{B}_n F[X] = \langle z^n \rangle F[X - z^{-1} \text{Exp } 1 - t zX].$$

We refer the reader to [7] for a discussion on the Jing operators. Importantly, the ∂_-^k operators do not come from the $\mathcal{H}$ action itself. Note that the ∂_-^k operators are homogeneous by construction.

We require the following lemma.

Lemma 3. [7] The map $\partial_-^n : \mathcal{P}^n \rightarrow \mathcal{P}^{n-1}$ is a projection onto $\mathcal{P}^{n-1}$ i.e. for $f \in \mathcal{P}^{n-1} \subset \mathcal{P}^n$ we have that $\partial_-^n f = f$.

[Lemma 3](#) shows that the following operator is well defined.

Definition 8. For $f \in \mathcal{P}^n \subset \mathcal{P}_{as}$ define $\tilde{\sigma} f := \partial_-^1 \cdots \partial_-^n f$. Then $\tilde{\sigma}$ defines an operator $\mathcal{P}_{as} \rightarrow \Lambda$ which we call the *stable-limit symmetrization operator*. For a partition λ define $\mathcal{A}_\lambda = \tilde{\sigma} \tilde{E}_\lambda \in \Lambda$.

The $\mathcal{A}_\lambda$ symmetric functions have many useful properties including, but not limited to, the following.

Theorem 6. The set $\{\mathcal{A}_\lambda : \lambda \text{ is a partition}\}$ is a basis of Λ .

Proof Sketch. The result follows after proving the stronger property that each $\mathcal{A}_\lambda$ has a unitriangular expansion with respect to dominance order into the Hall-Littlewood symmetric function basis. $\square$

Stable-limit symmetrization behaves well with respect to permuting the defining composition μ of each $\tilde{E}_\mu$.

Lemma 4. For any composition μ there is some nonzero scalar $\gamma_\mu \in \mathbb{Q}[q, t]$ such that

$$\tilde{\sigma} \tilde{E}_\mu = \gamma_\mu \mathcal{A}_{\text{sort } \mu}$$

where $\gamma_\mu = 1$ when μ is a partition.

We can now construct a full $\mathcal{Y}$ -weight basis of $\mathcal{P}_{as}$. We parameterize this basis by pairs $\mu|\lambda$ for μ a composition with $\mu_\ell \neq 0$ and λ a partition.

Definition 9. For μ a composition with $\mu_\ell \neq 0$ and λ a partition define the *stable-limit non-symmetric Macdonald function* corresponding to $\mu|\lambda$ as

$$\tilde{E}_{\mu|\lambda} := \partial_-^{\ell(\mu)-1} \dots \partial_-^{\ell(\mu)-\ell(\lambda)} \tilde{E}_{\mu*\lambda}.$$

Remark. Note importantly $\tilde{E}_{\mu|\lambda} \in \mathcal{P}^{\ell(\mu)}$, $\tilde{\sigma} \tilde{E}_{\mu|\lambda} = \tilde{\sigma} \tilde{E}_{\mu*\lambda}$, and $\tilde{E}_{\mu|\lambda}$ is homogeneous of degree $|\mu| - |\lambda|$. Further, for any composition μ and partition λ we have $\tilde{E}_{\mu|\emptyset} = \tilde{E}_\mu$ and $\tilde{E}_{\emptyset|\lambda} = \mathcal{A}_\lambda$. We are only considering compositions μ with $\mu_\ell \neq 0$ to avoid redundancies.

The following simple lemma shows that the stable-limit non-symmetric Macdonald functions $\tilde{E}_{\mu|\lambda}$ are $\mathcal{Y}$ -weight vectors.

Lemma 5. Suppose $f \in \mathcal{P}^k$ is a $\mathcal{Y}$ -weight vector with weight $(\alpha_1, \dots, \alpha_k, 0, 0, \dots)$. Then $\partial_-^k f \in \mathcal{P}^{k-1}$ is a $\mathcal{Y}$ -weight vector with weight $(\alpha_1, \dots, \alpha_{k-1}, 0, 0, \dots)$.

Here we give a few basic examples of stable-limit non-symmetric Macdonald functions expanded in the Hall-Littlewood basis $\mathcal{P}_\lambda$ and their corresponding weights:

- $\tilde{E}_{\emptyset|2} = \mathcal{P}_2(x_1) \dots = \frac{q^{-1}}{1-q^{-1}t} \mathcal{P}_{1,1}(x_1) \dots$ and has weight $(0, 0, \dots)$
- $\tilde{E}_{2|\emptyset} = x_1^2 - \frac{q^{-1}}{1-q^{-1}t} x_1 \mathcal{P}_1(x_2) \dots$ and has weight $(q^2t, 0, \dots)$
- $\tilde{E}_{1|1,1} = x_1 \mathcal{P}_{1,1}(x_2) \dots$ and has weight $(qt^3, 0, \dots)$

Finally, we prove that the stable-limit non-symmetric Macdonald functions are a basis for $\mathcal{P}_{as}$.

Theorem 7. (Main Theorem) The $\tilde{E}_{\mu|\lambda}$ are a $\mathcal{Y}$ -weight basis for $\mathcal{P}_{as}$.

Proof Sketch. As there are sufficiently many $\tilde{E}_{\mu|\lambda}$ in each graded component of every $\mathcal{P}^k$ it suffices to show that these functions are linearly independent. Obviously weight vectors in distinct weight spaces are linearly independent. Using [Lemmas 2](#) and [5](#), we deduce that if $\tilde{E}_{\mu_1|\lambda_1}$ and $\tilde{E}_{\mu_2|\lambda_2}$ have the same weight then necessarily $\mu_1 = \mu_2$. Hence, we can restrict to the case where we have a dependence relation

$$c_1 \tilde{E}_{\mu|\lambda^1} + \cdots + c_N \tilde{E}_{\mu|\lambda^N} = 0$$

for $\lambda^1, \dots, \lambda^N$ distinct partitions. By applying the stable-limit symmetrization operator we see that

$$\tilde{\sigma} c_1 \tilde{E}_{\mu|\lambda^1} + \cdots + c_N \tilde{E}_{\mu|\lambda^N} = \tilde{\sigma} c_1 \tilde{E}_{\mu*\lambda^1} + \cdots + c_N \tilde{E}_{\mu*\lambda^N} = 0.$$

Now by [Lemma 4](#), $\tilde{\sigma} \tilde{E}_{\mu*\lambda^i} = \gamma_{\mu*\lambda^i} \mathcal{A}_{\text{sort } \mu*\lambda^i}$ with nonzero scalars $\gamma_{\mu*\lambda^i}$ so

$$0 = c'_1 \mathcal{A}_{\text{sort } \mu*\lambda^1} + \cdots + c'_N \mathcal{A}_{\text{sort } \mu*\lambda^N}.$$

The partitions λ^i are distinct so we know that the partitions $\text{sort } \mu*\lambda^i$ are distinct as well. By [Theorem 6](#) the symmetric functions $\mathcal{A}_{\text{sort } \mu*\lambda^i}$ are linearly independent. Thus $c'_i = 0$ implying $c_i = 0$ for all $1 \leq i \leq N$ as desired. $\square$

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Divisors, orbit harmonics, and DT invariants

M. Reineke^{*1}, B. Rhoades^{†2}, and V. Tewari^{‡3}

¹*Faculty of Mathematics, Ruhr University Bochum, Bochum, Germany*

²*Department of Mathematics, University of California San Diego, La Jolla, CA 92093, USA*

³*Department of Mathematics, University of Hawaii at Manoa, Honolulu HI, 96822, USA*

Abstract. We give a combinatorial interpretation to Donaldson–Thomas invariants of symmetric quivers by using the method of orbit harmonics. Along the way we connect Young subgroup actions on break divisors of graphs, quotients of the polynomial ring modulo power ideals of Postnikov–Ardila, and a construction of Efimov in the context of Cohomological Hall algebras.

Keywords: break divisors, Donaldson–Thomas invariants, orbit harmonics, power ideals

1 Introduction

Donaldson–Thomas (DT) invariants of quivers with potential were introduced in the seminal work of Kontsevich–Soibelman [10] as a mathematical definition of the string-theoretic concept of BPS state count. We give a high-level introduction to DT invariants with the goal of emphasizing that the algebraic formalism hides an interesting combinatorial question, which is what we address.

In [10], (quantum) DT invariants were introduced as follows: Attached to a quiver Q on k vertices is a *cohomological Hall algebra* $\mathcal{H}$ which carries a $\mathbb{Z}_{\geq 0}^k$ -grading. If Q is *symmetric* then $\mathcal{H}$ is $\mathbb{Z}_{\geq 0}^k \times \mathbb{Z}$ -graded. Kontsevich and Soibelman conjectured [10, Conjecture 1] that in this case $\mathcal{H}$ is a free (super-)commutative algebra generated by a $\mathbb{Z}_{\geq 0}^k \times \mathbb{Z}$ -graded vector space V of the form $V^{\text{prim}} \otimes \mathbb{Q}[x]$. Furthermore that, given a fixed *dimension vector* $\gamma \in \mathbb{Z}_{\geq 0}^k$, only finitely many $c_{\gamma,k} : \dim V_{\gamma,k}^{\text{prim}}$ are nonzero.

Efimov [6] proved this conjecture via an ingenious but mysterious construction for the spaces V_{γ}^{prim} . The *quantum DT invariant* $\Omega_{Q,\gamma} q$ is the Laurent *polynomial* tracking the $c_{\gamma,k}$ as k varies over $\mathbb{Z}$. Naturally its coefficients lie in $\mathbb{Z}_{\geq 0}$. The *numerical DT-invariant* $\text{DT}_{Q,\gamma}$ is the evaluation of $\Omega_{Q,\gamma} q$ at $q = 1$. We note that these numbers are

^{*}markus.reineke@rub.de. M.R. was supported by DFG CRC-TRR 191 “Symplectic structures in geometry, algebra and dynamics”.

[†]bprhoades@math.ucsd.edu. B.R. was supported by NSF Grant DMS-1953781.

[‡]vtewari@math.hawaii.edu. V.T. was supported by Simons Collaboration Grant #855592.

called by various names in (mathematical physics) literature, e.g. LMOV invariants and BPS invariants.

At the level of Poincaré series, [10, Conjecture 1] implies a ‘sum=product’ flavored equality common in combinatorics and number theory. For instance, the numerical DT-invariants $\text{DT}_{Q,n}$ for a single vertex $m-1$ -loop quiver Q with dimension vector n for $n \in \mathbb{Z}_{\geq 1}$ may be defined by considering an *Euler product* factorization:

$$\sum_{n \geq 0} \frac{1}{mn-1} \binom{m-1}{n} t^n = \prod_{n \geq 1} (1 - (-1)^{mn} t^n)^{-(-1)^{mn} \text{DT}_{Q,n}}. \quad (1.1)$$

On the left-hand side is the generating function for Fuss–Catalan numbers well known to count $m-1$ -ary trees. It is not immediate from Equation (1.1) that the exponents $\text{DT}_{Q,n}$ are nonnegative and integral.

In what follows we give a combinatorial interpretation for $\text{DT}_{Q,\gamma}$ for a symmetric quiver Q with at least one loop at each vertex and dimension vector γ . Our route is winding and hinges on realizing Efimov’s construction as an instance of a construction of Dahmen–Micchelli from the 1980s.

Let X be a finite collection of nonzero vectors in $\mathbb{R}^n$. Motivated by problems in approximation theory, Dahmen–Micchelli associated a pair of vector spaces $\mathcal{P}(X), \mathcal{D}(X)$ to X . Subsequent work [2, 9, 15] cast more light on these spaces. When the aforementioned collection of vectors is obtained from a finite connected graph G , work of Postnikov–Shapiro [15] lends a fresh perspective involving divisors on graphs. This will be our point of entry, with $\mathcal{P}(X)$ becoming the Postnikov–Shapiro slim subgraph space.

Throughout, graphs are finite connected graphs with multiple edges allowed but not self loops. Given such a graph G , we let $\mathcal{V}(G)$ (respectively $\mathcal{E}(G)$) denote its vertex set (respectively multiset of edges). An assignment $D : \mathcal{V}(G) \rightarrow \mathbb{Z}$ is called a *divisor*. Baker–Norine [3] studied this notion in depth and shed light on rich combinatorics underlying it. Another place where divisors play a role is tropical geometry, thanks to work of Mikhalkin–Zharkov [14] who introduced break divisors in this context; see [1] for more on their combinatorics. We care particularly about this class of divisors.

The *genus* of G is $g(G) = |\mathcal{V}(G)| - |\mathcal{E}(G)| + 1$. A divisor D is *effective* if $D(v) \geq 0$ for all $v \in \mathcal{V}(G)$. The *degree* of D is equal to $\sum_{v \in \mathcal{V}(G)} D(v)$. We say that D is a *break divisor* if its degree is $g(G)$ and furthermore, for every connected subgraph H of G the degree of D restricted to H weakly exceeds $g(H)$. This description implies that the set $\text{Break}(G)$ of break divisors on G may be identified with integer lattice points of a polytope. It also implies that the automorphism group $\text{Aut}(G)$ of G acts by permutations on $\text{Break}(G)$.

With this elementary setup, one may ask an elementary question: *Does the $\text{Aut}(G)$ module $\text{Break}(G)$ carry interesting information?* We hope to convince the reader that the answer is in the affirmative by giving an enumerative application to DT theory. Given a symmetric quiver Q and dimension vector γ , we associate a connected graph $G_{Q,\gamma}$ that we call the *covering graph*. Let S_γ denote the Young subgroup determined by γ .

Unraveling Efimov's construction using $G_{Q,\gamma}$ allows us to cast his results in the context of the space $\mathcal{P} G_{Q,\gamma}$. Here is our chief result.

Theorem 1. *The dimensions of the graded pieces of $\mathcal{P} G_{Q,\gamma}^{S_\gamma}$ are the coefficients of the quantum DT invariant $\tilde{\Omega}_\gamma q$. In particular, the numerical DT invariant $\text{DT}_{Q,\gamma} : \tilde{\Omega}_\gamma 1$ equals the number of S_γ -orbits on $\text{Break } G$.*

[Theorem 1](#) gives, to the best of our knowledge, the first manifestly nonnegative combinatorial interpretation for $\text{DT}_{Q,\gamma}$ by describing it as an orbit count. Our interpretation comes from invoking the theory of orbit harmonics (see [Theorem 3](#)). In fact, this theory will give an interpretation of quantum DT invariant as the graded dimension of a space of S_γ -invariants in a certain polynomial ring quotient (see [Theorem 7](#)).

The full version of this extended abstract with proofs appears on the arXiv [\[16\]](#).

2 The Postnikov–Shapiro slim subgraph space

For reasons of space, we discuss the objects in the title solely in the context they matter to us, referring the reader to [\[5\]](#) and references therein for a detailed treatment. Some of the jargon is matroid-theoretic, and for good reason.

Let G be a connected graph as before. We typically identify the vertex set with $n : \{1, \dots, n\}$ where $n = |\mathcal{V} G|$. Given a nonempty proper subset $S \subset \mathcal{V} G$, we denote by G_S the graph induced on vertices in S . We define the *edge cut* ∂S of G associated to S to be the set of edges with one endpoint in S and the other in $\bar{S} = \mathcal{V} G \setminus S$. We let $d S : |\partial S|$. We denote the set of edge cuts by $\mathcal{C} G$. A minimal non-empty edge cut is called a *bond*. We denote the set of bonds by $\mathcal{B} G$.

Suppose $H \subset \mathcal{C} G$ is such that the subgraph of G given by $G' : \mathcal{V} G, \mathcal{E} G \setminus H$ is connected. We say that H (or the subgraph in G determined by it) is *slim*. Such subgraphs play a key role in two works that serve as our motivation, namely [\[4, 15\]](#). We attach certain polynomials to subgraphs of G and subsequently discuss the special role played by slim subgraphs.

For each edge $e \in \mathcal{E} G$ joining vertices i and j where $i < j$, consider the linear form $p_e : x_i - x_j$. Given $H \subseteq \mathcal{C} G$, define the polynomial

$$p_H = \prod_{e \in H} p_e. \quad (2.1)$$

Consider the polynomial ring $\mathbb{Q}[p_e \mid e \in \mathcal{E} G]$ which must equal $\mathbb{Q}[x_i - x_j \mid i < j]$ as G is connected. In this ring, the *cocircuit ideal* $I G$ is defined as

$$I G : \langle p_H \mid H \in \mathcal{B} G \rangle. \quad (2.2)$$

Note that we could have defined $I G$ as the ideal generated by *all* p_H where $H \in \mathcal{C} G$. Indeed $p_{H'}$ for H' a cut is a multiple of p_H for H some bond.

Now define the (*Postnikov–Shapiro*) *slim subgraph space* (also known as the *central P-space*) as the $\mathbb{Q}$ -vector space associated to G as follows:

$$\mathcal{P} G : \mathbb{Q} \{p_H \mid H \text{ slim subgraph of } G\}. \quad (2.3)$$

Example 2. Say G is the graph on three vertices (identified with $\{3\}$) with two edges between vertices 1 and 2, and two between 2 and 3. Then it is easily checked that

$$\begin{aligned} I G &= \langle x_1 - x_2^2, x_2 - x_3^2 \rangle, \\ \mathcal{P} G &= \mathbb{Q} \{1, x_1 - x_2, x_2 - x_3, x_1 - x_2 - x_2 - x_3\}. \end{aligned}$$

Observe that $\mathbb{Q} x_1 - x_2, x_1 - x_3, x_2 - x_3 = I G \oplus \mathcal{P} G$, and that $\dim \mathcal{P} G = 4$. This is no coincidence, as we explain next.

Let $\text{ST } G$ denote the set of *spanning trees* of G , which are in bijection with the set of *bases* in the collection of vectors $\{\mathbf{e}_i - \mathbf{e}_j \mid \{i < j\} \in \mathcal{E} G\}$. The vectors $\mathbf{e}_1$ through $\mathbf{e}_n$ denote the standard basis vectors for $\mathbb{R}^n$. The following fundamental results then hold (see [9, Theorem 3.8] or [5, Corollary 11.23]):

1. $\mathbb{Q} x_i - x_j \mid i < j = \mathcal{P} G \oplus I G$,
2. $\dim \mathcal{P} G = |\text{ST } G|$.

Let $\mathbb{Q} \mathbf{x}_n : \mathbb{Q} x_1, \dots, x_n$. By throwing in the linear form $x_1 + \dots + x_n$ into the mix, the first of these may be written as $\mathbb{Q} \mathbf{x}_n = \mathcal{P} G \oplus \hat{I} G$, where $\hat{I} G$ is now the ideal in $\mathbb{Q} \mathbf{x}_n$ with the same generators as $I G$ as well as $x_1 + \dots + x_n$. Observe that the central P -space $\mathcal{P} G$ remains unchanged in both direct sum decompositions. We will henceforth abuse notation and treat $\hat{I} G$ and $I G$ as the same, with the ambient ring serving to distinguish them.

An alternative perspective on $\mathcal{P} G$ than the one just offered will eventually allow us to describe it by the method of orbit harmonics. $\mathcal{P} G$ is also the *Macaulay-inverse* to a *power ideal* determined by G . We proceed to make this precise. Like before the set of bonds $\mathcal{B} G$ will be important. Given nonempty $S \subseteq [n]$, let $x_S : \sum_{i \in S} x_i$. Recall that for S nonempty and proper, we denote by $d S$ the size of the edge cut ∂S . Consider the power ideal $\mathcal{I} G$ in $\mathbb{Q} \mathbf{x}_n$ defined by

$$\mathcal{I} G = \langle x_1 + \dots + x_n, x_S^{d S} \text{ where } \emptyset \neq S \subsetneq [n] \text{ such that } \partial S \in \mathcal{B} G \rangle$$

Several nice properties of such ideals (and corresponding quotients) defined in this manner were brought forth by Ardila–Postnikov [2]. The quotient $\mathbb{Q} \mathbf{x}_n / \mathcal{I} G$ is known as the *central zonotopal algebra* attached to G .¹

¹We note that the ideals $\mathcal{I} G$ also appear in [15], but their set of generators has many redundant elements, and Postnikov–Shapiro eliminate x_n and work in a quotient of $\mathbb{Q} \mathbf{x}_{n-1}$.

How does $\mathcal{I}(G)$ relate to $\mathcal{P}(G)$? Let $\partial_i := \frac{\partial}{\partial x_i}$. We have ([9, Theorem 3.8 (5)], or [5, Theorem 11.25]):

$$\mathcal{P}(G) = \{f \in \mathbb{Q}[x_n] \mid p \cdot \partial_1, \dots, \partial_n f = 0 \text{ for all } p \in \mathcal{I}(G)\}. \quad (2.4)$$

Going back to [Example 2](#), we see for instance that the differential operator $\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}^2$ corresponding to $x_1 - x_2^2 \in \mathcal{I}(G)$ annihilates the four polynomials that span $\mathcal{P}(G)$: three for degree reasons, and the fourth by a routine computation. Furthermore $\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3}$ annihilates any polynomial in the differences $x_i - x_j$ for $1 \leq i < j \leq 3$.

3 Orbit harmonics and break divisors

Recall from the introduction that we denote the set of break divisors on G by $\text{Break}(G)$. To motivate the connection of these divisors with the results in the preceding section, we note here the important equality $|\text{Break}(G)| = |\text{ST}(G)|$ [1, Theorem 4.25].

Under our usual identification of $\mathcal{V}(G)$ with n , elements of $\text{Break}(G)$ become certain lattice points in $\mathbb{R}^n$. In fact, they are exactly the set of lattice points in a close cousin of a polytope attached to G . Consider the *graphical zonotope* $\mathcal{Z}_G \in \mathbb{R}^n$ determined from G by taking the Minkowski sum of line segments $\mathbf{e}_i, \mathbf{e}_j$ for every edge $\{i, j\} \in \mathcal{E}(G)$. Let $\Delta_{n-1, n}$ denotes the $(n-1)$ -th hypersimplex obtained as the convex hull of the S_n -orbit of the point $(1^{n-1}, 0) \in \mathbb{R}^n$. This given we have [12, Proposition 2.1]:

$$\text{Break}(G) = \mathcal{Z}_G - \Delta_{n-1, n} \cap \mathbb{Z}^n,$$

where $\mathcal{Z}_G - \Delta_{n-1, n}$ denotes the *trimmed graphical zonotope* obtained as a Minkowski difference. As mentioned in the introduction, $\text{Aut}(G)$ acts on $\text{Break}(G)$ by permutations. Thus we have a point set on which a linear group acts, and such a setup is ideal for the method of orbit harmonics.

The method of *orbit harmonics* gives a technique for turning an ungraded permutation representation of some linear group $\mathcal{G}$ acting on a finite point locus Y into a graded $\mathcal{G}$ -module. Orbit harmonics was introduced by Kostant [13] and has seen subsequent application (for example) by Garsia–Procesi [7] in the context of Springer fibers and by Haglund–Rhoades–Shimozono [8] in the context of Macdonald-theoretic delta operators.

Consider a finite point set $Y \subset \mathbb{Q}^n$ and let $I(Y) \subseteq \mathbb{Q}[x_n]$ be the ideal of polynomials which vanish on Y . The quotient of $\mathbb{Q}[x_n]$ by the ideal $I(Y)$ has vector space dimension $\dim_{\mathbb{Q}} \mathbb{Q}[x_n] / I(Y) = |Y|$. If the set Y is $\mathcal{G}$ -stable for some group $\mathcal{G} \subset \text{GL}_n(\mathbb{Q})$, then we have an isomorphism of ungraded $\mathcal{G}$ -modules

$$\mathbb{Q}[x_n] / I(Y) \sim_{\mathcal{G}} \mathbb{Q}[Y].$$

Orbit harmonics produces a quotient that will afford the structure of a *graded* $\mathcal{G}$ -module. Given a nonzero polynomial $f \in \mathbb{Q}[\mathbf{x}_n]$, let τf be the top degree homogeneous component of f . Define an ideal $T(Y) \subseteq \mathbb{Q}[\mathbf{x}_n]$ by

$$T(Y) = \langle \tau f : f \in I(Y), f \neq 0 \rangle.$$

The $\mathcal{G}$ -module isomorphism given above extends to a chain

$$\mathbb{Q}[\mathbf{x}_n] / T(Y) \sim_{\mathcal{G}} \mathbb{Q}[\mathbf{x}_n] / I(Y) \sim_{\mathcal{G}} \mathbb{Q}[Y],$$

with the added feature that the left hand side is a graded $\mathcal{G}$ -module.

3.1 Applying orbit harmonics to Break G

We will apply orbit harmonics to the locus $Y = \text{Break } G$ with $\mathcal{G} = \text{Aut } G$. Remarkably, the ideal $T(Y)$ which so arises will coincide with the power ideal $\mathcal{I}(G)$. For brevity we refer to the ideals $I(\text{Break } G)$ and $T(\text{Break } G)$ by $I(G)$ and $T(G)$ respectively. We then have the following result.

Theorem 3. *We have the equality of ideals $T(G) = \mathcal{I}(G)$. Consequently we have the following isomorphisms and equalities of ungraded $\text{Aut } G$ -modules:*

$$\mathbb{Q}[\text{Break } G] \sim \mathbb{Q}[\mathbf{x}_n] / T(G) = \mathbb{Q}[\mathbf{x}_n] / \mathcal{I}(G) \sim \mathcal{P}(G).$$

where the middle equality and right isomorphism are in the category of graded $\text{Aut } G$ -modules.

Sketch of proof. We begin by identifying a family of polynomials indexed by $\mathcal{B}(G)$ that vanish on $\text{Break } G$. Subsequently we consider the ideal $\tilde{T}(G)$ generated by top degree homogeneous summands of the polynomials in the aforementioned family. The inclusion $\tilde{T}(G) \subset T(G)$ on the one hand implies that

$$\dim(\mathbb{Q}[\mathbf{x}_n] / \tilde{T}(G)) \geq \dim(\mathbb{Q}[\mathbf{x}_n] / T(G)).$$

On the other hand, it transpires that $\tilde{T}(G)$ equals the power ideal $\mathcal{I}(G)$ and thus the left hand side equals $|\text{ST}(G)|$. Additionally, the right hand side equals $|\text{Break } G|$. Given that $|\text{ST}(G)| = |\text{Break } G|$ we conclude that $\tilde{T}(G) = T(G)$. $\square$

We demonstrate several key notions of this section in the following example.

Example 4. Let G be the complete bipartite graph $K_{2,3}$. Let us identify the vertex set with 5 with $\{1,2\}$ and $\{3,4,5\}$ being the two partite sets. Elements of $\text{Break } G$ are certain tuples $(d_1, d_2, d_3, d_4, d_5) \in \mathbb{Z}_{\geq 0}^5$ where $\sum_{1 \leq i \leq 5} d_i = g(K_{2,3}) = 2$. Any subgraph that is a tree gives a trivial constraint. From the $S_2 \times S_3$ symmetry in G it follows that given any break divisor we can permute d_1 and d_2 (respectively d_3, d_4 , and d_5) to get

another break divisor. Thus, up to $S_2 \times S_3$ symmetry, the only relevant inequality is $d_1 = d_2 = d_3 = d_4 \geq 1$. In summary we obtain the following four orbit representatives for $S_2 \times S_3$ -action on Break G : $(2,0,0,0,0)$, $(1,1,0,0,0)$, $(1,0,1,0,0)$, $(0,0,1,1,0)$. We thus obtain $|\text{Break } G| = 12$, which agrees with the number of spanning trees of G .

Consider the bond ∂S determined by choosing $S = \{1\}$. This determines the linear form x_1 . The smallest (respectively largest) value that x_1 attains on Break G is 0 (respectively 2). Thus the polynomial $x_1(x_1 - 1)(x_1 - 2)$ vanishes on Break G . This polynomial has x_1^3 as the top degree homogeneous summand, and this polynomial is indeed in $\mathcal{I}(K_{2,3})$ as well as $\mathcal{T}(K_{2,3})$. We leave it to the reader to verify that

$$\mathcal{T}(K_{2,3}) = \langle x_1 \cdots x_5, x_i^3, x_j^2, x_i x_j^3 \text{ where } i \in \{1,2\}, j \in \{3,4,5\} \rangle.$$

The quotient $\mathbb{Q}[x_5] / \mathcal{T}(K_{2,3})$ has a monomial basis

$$\{1, x_1, x_2, x_3, x_4, x_1^2, x_2^2, x_1 x_2, x_1 x_3, x_2 x_3, x_1 x_4, x_2 x_4\}.$$

It can be shown that the elements $\{1, x_1 - x_2, x_1^2 - x_2^2, x_1 x_2\}$ span the space of $S_2 \times S_3$ -invariants. We thus get

$$\text{Hilb } \mathbb{Q}[x_5] / \mathcal{T}(K_{2,3})^{S_2 \times S_3} = 1 + q + 2q^2.$$

As we shall soon see, this last polynomial is essentially a quantum DT invariant; see [Example 8](#). This shall follow once we make a connection to Efimov's work.

We conclude this section with one final remark. In general, it follows from [\[15\]](#) that the quotient $\mathbb{Q}[x_n] / \mathcal{I}(G)$ has a monomial basis indexed by G -parking functions. Thus, [Theorem 3](#) provides an algebraic perspective on the fact that the set of q -reduced divisors on G (essentially G -parking functions) has the same cardinality as Break G . Indeed, this same quotient arises when applying orbit harmonics to Break G .

4 A construction of Efimov for quantum DT invariants

Let $A = (a_{ij})_{i,j \in k}$ be a symmetric matrix with nonnegative integer entries. We assume $a_{ii} \geq 1$ throughout. The matrix A determines a quiver Q on k vertices labeled 1 through k , with a_{ij} arrows from i to j for $i, j \in k$. Note that $a_{ii} \geq 1$ ensures at least one loop at each vertex. We further assume A is such that Q is connected. Let $\gamma = (\gamma_1, \dots, \gamma_k) \in \mathbb{Z}_{\geq 0}^k$ be a dimension vector. We begin by describing the essential construction in [\[6\]](#).

Consider variables $x_{i,\alpha}$ for $1 \leq i \leq k$ and $1 \leq \alpha \leq \gamma_i$, and let σ_γ be their sum. Define the polynomial ring

$$A_\gamma = \mathbb{Q}[x_{i,\alpha} \mid 1 \leq i \leq k, 1 \leq \alpha \leq \gamma_i]. \quad (4.1)$$

Furthermore, let

$$A_\gamma^{\text{prim}} = \mathbb{Q}[x_{j,\alpha_2} - x_{i,\alpha_1} \mid 1 \leq i, j \leq k, 1 \leq \alpha_1 \leq \gamma_i, 1 \leq \alpha_2 \leq \gamma_j]. \quad (4.2)$$

Then we have $A_\gamma = A_\gamma^{\text{prim}} \otimes \mathbb{Q}[\sigma_\gamma]$. Define $S_\gamma : S_{\gamma_1} \times \cdots \times S_{\gamma_k}$.

Define J_γ to be the smallest S_γ -stable A_γ^{prim} -submodule of A_γ^{prim} such that the following holds: for any decomposition of $\gamma = \delta + \bar{\delta}$ where both δ and $\bar{\delta}$ are nonzero we have that

$$f_{\delta, \bar{\delta}} = \prod_{i/j \in k} \prod_{\alpha_1=1}^{\delta_i} \prod_{\alpha_2=1}^{\gamma_j} (x_{j,\alpha_2} - x_{i,\alpha_1})^{a_{ij}} \prod_{i \in k} \prod_{\alpha_1=1}^{\delta_i} \prod_{\alpha_2=1}^{\gamma_i} (x_{i,\alpha_2} - x_{i,\alpha_1})^{a_{ii}-1} \in J_\gamma. \quad (4.3)$$

We shall later reinterpret $f_{\delta, \bar{\delta}}$ in terms of cuts (as the indexing hints) in a graph constructed from (A, γ) .

Letting $\mathcal{H}_\gamma^{\text{prim}} : A_\gamma^{\text{prim}} \rightarrow S_\gamma$, consider the decomposition [6, p. 1139]

$$\mathcal{H}_\gamma^{\text{prim}} = V_\gamma^{\text{prim}} \oplus J_\gamma^{S_\gamma}. \quad (4.4)$$

The quantum DT invariants of the quiver Q with dimension vector γ (assuming trivial potential and stability) arise as dimensions of the graded pieces of the graded vector space V_γ^{prim} as explained in [6, Section 4]. We describe the $\mathbb{Z}$ -grading employed.

Given dimension vectors γ and δ , the *Euler form* $\chi_Q(\gamma, \delta)$ is defined as

$$\chi_Q(\gamma, \delta) := \sum_{1 \leq i \leq k} \gamma_i \delta_i - \sum_{1 \leq i, j \leq k} a_{ij} \gamma_i \delta_j. \quad (4.5)$$

Homogeneous polynomials $f \in \mathcal{H}_\gamma$ of degree d get assigned the grading $2d - \chi_Q(\gamma, \gamma)$. We let $V_{\gamma, m}^{\text{prim}}$ be the space of elements in V_γ^{prim} with grading m . Following [6, Section 4], and as in the introduction, set $c_{\gamma, m} := \dim V_{\gamma, m}^{\text{prim}}$, and consider the polynomial in $\mathbb{Z}_{\geq 0}[q^{\pm \frac{1}{2}}]$:

$$\Omega_\gamma(q) = \sum_{m \in \mathbb{Z}} c_{\gamma, m} q^{m/2}. \quad (4.6)$$

These $\Omega_\gamma(q)$ are the *quantum DT-invariants* of the quiver Q .

We get a necessary condition for when $V_{\gamma, m}^{\text{prim}}$ is nonzero in [6, Theorem 1.2] (also presented as [6, Theorem 3.10]). It involves a statistic $N_\gamma(Q)$ [6, Section 1] whose precise definition is unnecessary for the purposes of this abstract.

Theorem 5 ([6, Theorem 3.10]). *If $V_{\gamma, m}^{\text{prim}} \neq 0$, then $\gamma \neq 0$, $m \equiv \chi_Q(\gamma, \gamma) \pmod{2}$, and $\chi_Q(\gamma, \gamma) \leq m < \chi_Q(\gamma, \gamma) + 2N_\gamma(Q)$.*

In view of [Theorem 5](#) we may rewrite $\Omega_\gamma q$ as

$$\Omega_\gamma q = q^{\frac{1}{2}\chi_Q(\gamma,\gamma)} \sum_{0 \leq m \leq N_\gamma(Q)-1} c_{\gamma,2m} \chi_Q(\gamma,\gamma) q^m. \quad (4.7)$$

Denote the sum on the right by $\tilde{\Omega}_\gamma q$. It lies in $\mathbb{Z}_{\geq 0} q$ and its degree is bounded by $N_\gamma(Q) - 1$. We abuse notation and refer to $\tilde{\Omega}_\gamma q$ as the quantum DT invariant as well.²

4.1 The covering graph construction

The reader might find the preceding construction both ingenious and mysterious. It transpires that the space $V_{\gamma,m}^{\text{prim}}$ is the space of S_γ -invariants for a slim subgraph space determined naturally from the quiver and the dimension vector. We proceed to describe this construction.

Our point of departure from Efimov is to consider an analogue of (4.4) where we do not take S_γ -invariants. We will construct an S_γ -stable space W_γ^{prim} so that

$$A_\gamma^{\text{prim}} = W_\gamma^{\text{prim}} \oplus J_\gamma. \quad (4.8)$$

Our next construction is crucial to this end.

Given Q as before, we construct an undirected graph $G_{Q,\gamma}$: Consider a set of vertices $v_{i,\alpha}$ for $1 \leq i \leq k$ and $1 \leq \alpha \leq \gamma_i$. For $i \in k$, the restriction of $G_{Q,\gamma}$ to the vertices $v_{i,1}, \dots, v_{i,\gamma_i}$ is the clique on γ_i vertices with $a_{ii} - 1$ edges between any two distinct vertices. In particular, if $a_{ii} = 1$, then we have a collection of γ_i totally disconnected vertices. For $i \neq j \in k$ we draw a_{ij} edges between any vertex v_{i,α_1} and v_{j,α_2} for $1 \leq \alpha_1 \leq \gamma_i$ and $1 \leq \alpha_2 \leq \gamma_j$. This determines $G_{Q,\gamma}$. Note that S_γ is a subgroup of $\text{Aut } G_{Q,\gamma}$.

We will assume throughout that $G_{Q,\gamma}$ is connected. For the remainder of this section, we fix our symmetric quiver Q and dimension vector γ , and we will drop them from notation. In particular, unless otherwise noted, let $G = G_{Q,\gamma}$.

We reinterpret the important element $f_{\delta,\bar{\delta}} \in J_\gamma$ from Equation (4.3) as the polynomial p_H for some cut in G .

Lemma 6. *Consider a decomposition $\gamma = \delta + \bar{\delta}$ where $\delta, \bar{\delta} \vdash \gamma$. The following hold.*

1. $f_{\delta,\bar{\delta}} = p_H$ for some $H \in \mathcal{C}(G)$.
2. For $\emptyset \neq S \subsetneq \mathcal{V}(G)$ such that $\partial S \in \mathcal{B}(G)$, we have that $p_{\partial S} \in J_\gamma$. More specifically, there exists a decomposition $\gamma = \delta + \bar{\delta}$ such that $\sigma f_{\delta,\bar{\delta}} = p_{\partial S}$ for some $\sigma \in S_\gamma$.

The preceding lemma simply states that Efimov's ideal J_γ is precisely the ideal generated by p_H for $H \in \mathcal{B}(G)$, i.e. $J_\gamma = I(G)$ where the latter is the cocircuit ideal from Equation (2.2). We thus obtain the following theorem that ‘combinatorializes’ Efimov's construction and describes the DT invariants in terms of $\mathcal{P}(G)$.

²As the careful reader may have realized, we already do so in the introduction.

Theorem 7. *The following decomposition holds: $A_\gamma^{\text{prim}} = \mathcal{P}(G) \oplus J_\gamma$. Consequently we have the equality*

$$\text{Hilb } \mathcal{P}(G)^{S_\gamma} = \tilde{\Omega}_\gamma(q).$$

From the chain of isomorphisms in Theorem 3 we have $\text{DT}_{Q,\gamma} = \tilde{\Omega}_\gamma(1) = \dim \mathbb{Q} \text{Break } G^{S_\gamma}$, i.e. the number of S_γ -orbits on $\text{Break } G$.

To demonstrate our main result, we revisit Example 4, this time starting with a quiver and a dimension vector.

Example 8. Consider the quiver with two vertices, each with a self-loop, and two edges between the vertices, one in each direction. Pick $\gamma = (2, 3)$. The covering graph construction then yields $G_{Q,\gamma}$ as the bipartite graph $K_{2,3}$. We reindex our variables $x_1 = x_{11}$, $x_2 = x_{12}$, $x_3 = x_{21}$, $x_4 = x_{22}$, and $x_5 = x_{32}$. We let $S_2 \times S_3$ act on $\mathbb{Q}[x_1, \dots, x_5]$ by letting S_2 (respectively S_3) act on x_1, x_2 (respectively x_3, x_4, x_5).

$\mathcal{P}(G)$ is spanned by p_H for slim subgraphs H which may be obtained as the S_γ -orbit of elements in $\{1, x_1 - x_3, x_1 - x_3 - x_1 - x_4, x_1 - x_3 - x_2 - x_4\}$. The space $\mathcal{P}(G)^{S_\gamma}$ has basis elements

$$\left\{ 1, \sum_{\sigma \in S_\gamma} \sigma \cdot (x_1 - x_3), \sum_{\sigma \in S_\gamma} \sigma \cdot (x_1 - x_3 - x_1 - x_4), \sum_{\sigma \in S_\gamma} \sigma \cdot (x_1 - x_3 - x_2 - x_4) \right\}.$$

Explicitly, other than the constant polynomial 1, up to normalization these equal

$$\begin{aligned} & 3x_1x_2 - 2x_3x_4x_5 \\ & 3x_1^2x_2^2 - 2x_1x_2x_3x_4x_5 - 2x_3x_4x_5 \\ & 3x_1x_2 - x_1x_2x_3x_4x_5 - x_3x_4x_5. \end{aligned}$$

We thus infer that

$$\tilde{\Omega}_\gamma(q) = 1 + q + 2q^2,$$

and therefore that $\tilde{\Omega}_\gamma(1) = 4$. Going back to Example 4, this agrees with the number of $S_2 \times S_3$ -orbits on $\text{Break } K_{2,3}$. We also encountered the polynomial $1 + q + 2q^2$ in that same example. Back then we obtained it as $\text{Hilb } \mathbb{Q}[x_5] / \mathcal{I}_{K_{2,3}}^{S_2 \times S_3}$.

5 Two applications

We close with two representation-theoretic results of independent interest, with motivation coming from [4]. Like before, we set $G := G_{Q,\gamma}$.

The top degree of $\mathcal{P}(G)$ as an S_γ -module. The graph G (with $\mathcal{V}(G) = n$) determines a hyperplane arrangement $\mathcal{A}_G$ in $\mathbb{C}^n$ by considering the union of hyperplanes $x_i - x_j$

0 for edges $\{i, j\} \in E(G)$. The de Rham cohomology ring $H^*(\mathbb{C}^n \setminus \mathcal{A}_G)$ over $\mathbb{C}$ of the complement $\mathbb{C}^n \setminus \mathcal{A}_G$ is the exterior algebra over the generators $d \log \beta = \frac{d\beta}{\beta}$ where β ranges over the linear forms that cut out $\mathcal{A}_G$. Consider the collection of vectors $X_G = \{\mathbf{e}_i - \mathbf{e}_j \mid \{i, j\} \in \mathcal{E}(G)\}$, and endow it with a total order. The top degree graded piece of $H^*(\mathbb{C}^n \setminus \mathcal{A}_G)$, denoted by $H^*(\mathbb{C}^n \setminus \mathcal{A}_G)_{\text{top}}$, has a basis indexed by so-called *unbroken bases*. Such bases also form a basis for the top degree of $\mathcal{P}(G)$, denoted by $\mathcal{P}(G)_{\text{top}}$. This equality in dimension of the two spaces is a shadow of an S_γ -isomorphism between two vector spaces. The following result generalizes [4, Theorem 5].

Proposition 9. *We have the following S_γ -isomorphism:*

$$\mathcal{P}(G)_{\text{top}} \sim H^*(\mathbb{C}^n \setminus \mathcal{A}_G)_{\text{top}} \otimes \varepsilon_{\gamma_1}^{a_{11}-1} \otimes \cdots \otimes \varepsilon_{\gamma_k}^{a_{kk}-1}.$$

Here $a_{ii} - 1$ is one less than the number of loops at vertex i in Q , and ε_{γ_i} denotes the sign representation of S_{γ_i} .

The $m-1$ -loop quiver. Let m be a positive integer. We revisit the case of the $m-1$ -loop quiver Q with dimension vector $\gamma = n$ where $n \geq 1$. The covering graph $G : G_{Q,\gamma}$ is the complete multipartite graph K_n^m , i.e. the graph on n vertices with m edges between any two distinct vertices. $\text{Aut}(G)$ is clearly the symmetric group S_n . Theorem 3 then gives us (isomorphic) spaces with dimension $m^{n-1}n^{n-2}$.

Berget–Rhoades [4] consider the S_n -modules $\mathcal{P}(K_n^m)$ and establish several interesting results. While the graded Frobenius characteristics of these modules remains elusive, we are able to shed light on related matters. In fact these modules encode a large class of DT invariants; see [16, Section 5.4] for more.

Theorem 3 allows us to connect the space $\mathcal{P}(K_n^m)$ to $\text{Q-Break}(K_n^m)$ and, among other things, resolve [11, Conjecture 3.3] and recover one of the main results in [4].

Corollary 10. *The following hold.*

1. *We have the isomorphism $\mathcal{P}(K_n^m) \sim_{S_n} \text{Q-Break}(K_n^m)$ of ungraded representations. In particular, the Frobenius characteristic $\text{Frob}(\mathcal{P}(K_n^m))$ is h -positive.*
2. *The graded multiplicity of the trivial representation of S_n in $\mathcal{P}(K_n^m)$ is given by the quantum DT invariant $\tilde{\Omega}_{n,q}$. At $q = 1$ we obtain*

$$\dim \mathcal{P}(K_n^m)^{S_n} = \text{DT}_{Q,n} = \frac{1}{mn^2} \sum_{d|n} (-1)^{mn - \frac{mn}{d}} \mu(d) \left(\frac{m-1}{d} \frac{n}{d} - 1 \right).$$

3. *On the level of graded Frobenius characteristics, we have*

$$\text{grFrob}(\mathcal{P}(K_n^m)) \downarrow_{S_{n-1}}^{S_n} \text{grFrob}(\text{PF}_{n-1}),$$

where PF_{n-1} is the parking function representation of S_{n-1} , with basis indexed by parking functions and grading given by the sum of entries in a parking function.

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Acyclic pipe dreams, subword complexes and lattice quotients of weak order intervals

Noémie Cartier¹

¹Laboratoire Interdisciplinaire des Sciences du Numérique – UMR CNRS 9015, Université Paris-Saclay, CNRS, Orsay, France

Abstract. The Tamari lattice is a well-known quotient of the weak order on permutations, and can be realized as the increasing flip poset of a family of pipe dreams with a well-chosen exit permutation. We show that for any permutation ω , the increasing flip poset on acyclic pipe dreams with exit permutation ω is a lattice quotient of the interval id, ω of the weak order. We then give similar quotients on acyclic pipe dreams on a family of non-triangular shapes. We finally discuss conjectural generalizations of these results to acyclic facets of subword complexes on any finite Coxeter group.

Résumé. Le treillis de Tamari est un quotient bien connu de l'ordre faible sur les permutations, et il peut être réalisé comme l'ordre des flips croissants sur une famille d'arrangements de tuyaux de permutation de sortie bien choisie. Nous montrons que pour toute permutation ω , l'ordre des flips croissants sur les arrangements de tuyaux acycliques de permutation de sortie ω est un quotient de treillis de l'intervalle id, ω de l'ordre faible. Nous donnons ensuite des quotients similaires sur des arrangements de tuyaux acycliques sur une famille de formes non triangulaires. Nous discutons finalement des généralisations conjecturales de ces résultats aux facettes acycliques de complexes de sous-mots sur des groupes de Coxeter finis quelconques.

Keywords: weak order, Tamari lattice, lattice quotient, pipe dream, subword complex

1 Introduction

A *triangular pipe dream* is a filling of a triangular shape with crosses $+$ and contacts $\nearrow$ so that all pipes entering on the left side exit on the top side, as illustrated in Figure 1. We will only consider *reduced* pipe dreams, where two pipes have at most one crossing. The pipes are labeled from 1 to n on the left size from top to bottom. For a pipe dream P with n pipes, its *exit permutation* is the order in which the pipes exit the shape at the top, read from left to right. For a fixed permutation $\omega \in \mathfrak{S}_n$, we denote by $\Pi \omega$ the set of reduced pipe dreams with exit permutation ω .

A contact c on a pipe dream P is *flippable* if the two pipes passing through c have a crossing x . The *flip* on c exchange the contact c with the crossing x to obtain a pipe dream with the same exit permutation as P , as illustrated in Figure 1. The flip is *increasing* if c is north-east of x and *decreasing* otherwise. The *increasing flip graph* is the graph on $\Pi \omega$

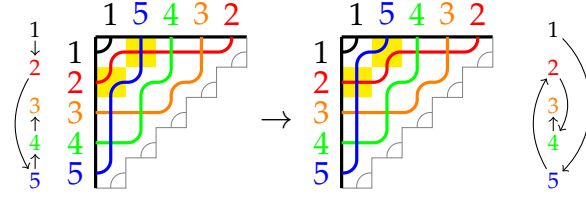


Figure 1: An increasing flip between two triangular pipe dreams with exit permutation 15432 and their contact graphs.

with an arc from P to P' if we can obtain P' by doing an increasing flip on P . The *increasing flip order* is the reflexive and transitive closure of the increasing flip graph.

Several authors [14, 9, 13] have observed that the set $\Pi 1n \ n-1 \ \dots \ 32$ is counted by the Catalan number C_{n-1} . Moreover, the increasing flip graph on this set is isomorphic to the Hasse diagram of the Tamari lattice. Since there exists a simple lattice isomorphism from binary trees with $n-1$ nodes to those pipe dreams and a surjective lattice morphism from the weak order on $\mathfrak{S}_{n-1}$ to binary trees, we can combine them to obtain a lattice morphism from $\mathfrak{S}_{n-1}$ to $\Pi 1n \ n-1 \ \dots \ 32$. The increasing flip order in this particular case is therefore a quotient of the weak order on permutations. In this paper, we discuss several extensions of this result obtained by changing the exit permutation, the shapes of pipe dreams, and even the group on which they are defined.

In Section 2, we will first study the triangular pipe dreams with any fixed exit permutation. We will prove in Theorem 2.8 that for any permutation ω , the increasing flip graph on acyclic pipe dreams with exit permutation ω is a lattice quotient of the weak order interval id, ω . We will give a lattice morphism from this interval to the acyclic pipe dreams and several ways to compute the image of a permutation.

Then in Section 3, we will consider pipe dreams on non-triangular shapes and discuss what hypothesis need to be modified to obtain a similar theorem, given in Theorem 3.3. We will see that the quotient may sometimes be realized on a subset of acyclic pipe dreams and may not use all of the increasing flip graph, but that it is linked to the brick polyhedron, an object introduced on spherical subword complexes in [11] and generalized to any subword complex in [5].

Finally, as pipe dreams can be interpreted as the facets of some subword complexes in type A Coxeter groups, we will discuss in Section 4 how those results can be generalized to subword complexes in any finite Coxeter group and give a conjecture that has been extensively tested in this context in Conjecture 4.3.

The details and proofs omitted in this extended abstract will soon be available in two articles in preparation [1, 2]. I am grateful to Nantel Bergeron, Cesar Ceballos and Vincent Pilaud for allowing me to use the results of [1] here.

2 Triangular pipe dreams

We defined triangular pipe dreams and their flips in the introduction. The *contact graph* of a pipe dream P is the directed graph $P^\#$ that has a vertex for each pipe of P and contains the edges a, b if and only if a contact $a \curvearrowright b$ appears somewhere in P (see two examples in Figure 1). We say that P is *acyclic* if and only if its contact graph is acyclic, and in that case we denote by $\text{lin } P$ the linear extensions of its contact graph $P^\#$. For any permutation $\omega \in \mathfrak{S}_n$ we denote by $\Sigma \omega$ the set of acyclic pipe dreams with exit permutation ω . Our first result deals with these sets of linear extensions of pipe dreams.

Theorem 2.1. *The set $\{\text{lin } P \mid P \in \Sigma \omega\}$ is a partition of the weak order interval id, ω .*

This partition defines an equivalence relation $\equiv_\omega$ on id, ω with an equivalence class $\text{lin } P$ for each $P \in \Sigma \omega$.

For instance, consider the exit permutation $\omega = 1 \ n \ n-1 \ \dots \ 3 \ 2$. Then the interval id, ω , i.e the set of all permutations in $\mathfrak{S}_n$ starting with 1, is isomorphic for the weak order to $\mathfrak{S}_{n-1}$ via the map $\phi : \sigma \in \mathfrak{S}_{n-1} \mapsto 1 \ \sigma \ 1 \ \dots \ \sigma \ n-1 \ 1 \in \text{id}, \omega$. Moreover, the contact graph of a pipe dream of $\Pi \omega$ is a binary search tree on $\{2, \dots, n\}$ and with an additional node 1 pointing to the root. Hence, the relation $\equiv_\omega$ is the image by ϕ of the sylvester congruence defined in [4], whose classes are linear extensions of standard binary search trees. This congruence is also the transitive closure of the rewriting rule $UijV \equiv UjiV$ if and only if U contains a letter k such that $i < k < j$, with U and V (possibly empty) words on n and $i < j$ integers, and the following theorem gives a similar alternate definition of $\equiv_\omega$ for any permutation.

Theorem 2.2. *The equivalence relation $\equiv_\omega$ is the transitive closure of the rewriting rule saying that for any weak order cover $UijV \lessdot UjiV$ in id, ω , then $UijV \equiv_\omega UjiV$ if and only if*

$$|\{k < i \mid \omega^{-1} k < \omega^{-1} j\}| \leq |\{k \in U \mid i < k < j \text{ and } \omega^{-1} j < \omega^{-1} k < \omega^{-1} i\}|$$

A *congruence* of a lattice $L, \leq, \wedge, \vee$ is an equivalence relation $\equiv$ on L which respects meets and joins: if $x \equiv x'$ and $y \equiv y'$, then $x \wedge y \equiv x' \wedge y'$ and $x \vee y \equiv x' \vee y'$. In that case, the *lattice quotient* $L / \equiv$ is the lattice on the classes of $\equiv$ where for any two classes X and Y , we have $X \leq Y$ if and only if there exist $x \in X$ and $y \in Y$ such that $x \leq y$ [12]. As the sylvester congruence is a lattice congruence, and the Tamari lattice is the associated lattice quotient of the weak order, it is natural to study the same properties on $\equiv_\omega$ for any permutation ω .

Theorem 2.3. *For any permutation ω , the equivalence relation $\equiv_\omega$ is a lattice congruence of the weak order interval id, ω .*

Theorem 2.1 also allows us to define a map $\text{ins}_\omega : \text{id}, \omega \mapsto \Sigma \omega$ such that for any $\pi \in \text{id}, \omega$, the permutation π is a linear extension of the contact graph of $\text{ins}_\omega \pi$.

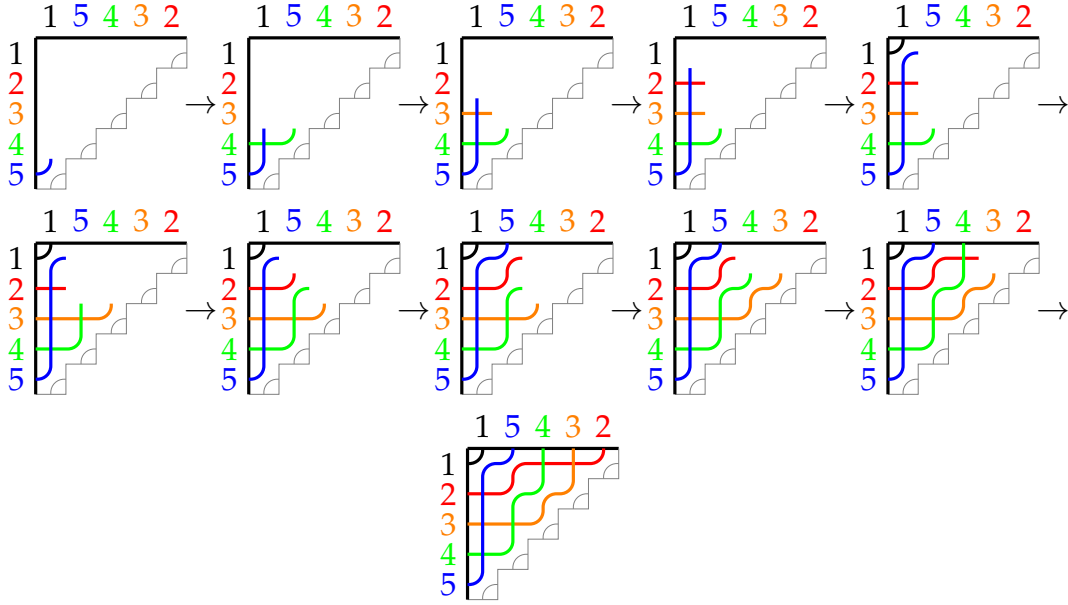


Figure 2: The sweeping algorithm with $\omega = 15432$ and $\pi = 15243$.

We note that $\text{ins}_\omega \text{id}$ is the unique source of the increasing flip graph on $\Pi \omega$, and $\text{ins}_\omega \omega$ is its unique sink. The fibers of the map ins_ω are the equivalence classes of the congruence $\equiv_\omega$.

When $w = 1n \ n-1 \ \dots \ 32$, the image $\text{ins}_\omega \pi$ can be computed by iterated insertions in a binary search tree as described in [4]. For general w , there are two known algorithms to compute $\text{ins}_\omega \pi$ for any $\pi \leq \omega$: the sweeping algorithm, which fill the cells of the triangular shape one by one, and the insertion algorithm, which adds the pipes to the pipe dream one by one.

Algorithm 2.4. The sweeping algorithm on triangular pipe dreams, illustrated in Figure 2, computes $\text{ins}_\omega \pi$ by filling the cells of the triangular shape from south-west to north-east with contacts and crosses as follows. We denote by i the pipe entering the current cell from the west side and by j the one entering from the south side, and we fill the cell with a cross if and only if

- $i < j$ and $\omega^{-1} i > \omega^{-1} j$ and
- $\pi^{-1} i > \pi^{-1} j$ or the current cell lies in column $\omega^{-1} j$.

Algorithm 2.5. The insertion algorithm on triangular pipe dreams, illustrated in Figure 3, computes $\text{ins}_\omega \pi$ by drawing the pipes of the pipe dream one by one in order of π . At step t , we draw pipe πt starting in row πt , ending in column $\omega^{-1} \pi t$ and covering with its south-to-east elbows $\nearrow$ the west-to-north free elbows $\searrow$ in the rectangle between those two points.

We know from Theorem 2.3 that the fibers of ins_ω respect the lattice properties of the weak order on id, ω . The following theorem shows that this application also send the

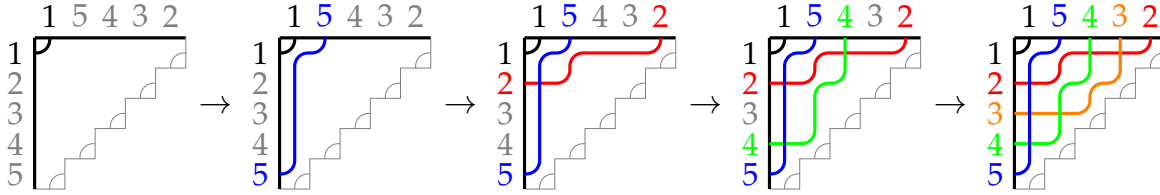


Figure 3: The insertion algorithm with $\omega = 15432$ and $\pi = 15243$.

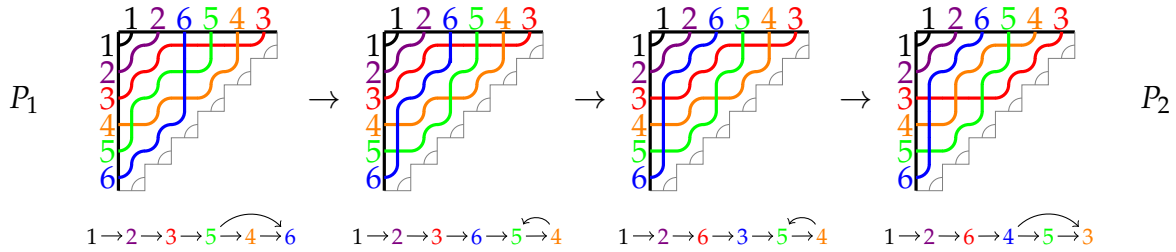


Figure 4: Two pipe dreams comparable in the flip order but not in the weak order.

weak order to the increasing flip graph on $\Sigma \omega$.

Theorem 2.6. *For any two pipe dreams $P, P' \in \Sigma \omega$, there is a path from P to P' in the increasing flip graph on acyclic pipe dreams if and only if there exist linear extensions π of P and π' of P' such that π is below π' in the weak order.*

Remark 2.7. The image of the weak order by ins_ω is not always the restriction of the increasing flip order on $\Pi \omega$ to acyclic pipe dreams. An example is given in Figure 4: a sequence of ascending flips from P_1 to P_2 is represented, therefore $P_1 < P_2$ in the flip order. Moreover, we see that $\text{lin } P_1 = \{123546\}$ and $\text{lin } P_2 = \{126453\}$ so both P_1 and P_2 are acyclic. However, the only linear extension of P_1 is not comparable to the only linear extension of P_2 in the weak order, and thus $P_1 \not\prec P_2$ in the image of the weak order; this is possible because the two middle pipe dreams are not acyclic, so the flips are not part of the increasing flip graph on $\Sigma 126543$.

Combining Theorem 2.6 and Theorem 2.3 tells us that we can quotient id, ω by $\equiv_\omega$ to obtain a lattice quotient whose elements are the fibers of ins_ω , and we thus obtain the following result.

Theorem 2.8. *The increasing flip graph on $\Sigma \omega$ is isomorphic to the Hasse diagram of the lattice quotient $\text{id}, \omega / \equiv_\omega$. Moreover, the map ins_ω is a lattice morphism from the weak order on id, ω to the reflexive and transitive closure of the increasing flip graph on $\Sigma \omega$.*

3 Generalized pipe dreams

The previous section proved that increasing flip graphs on acyclic triangular pipe dreams were lattice quotients of weak order intervals. A similar result was given in [8], where we observe that cambrian lattices – a generalization of the Tamari lattice introduced in [12] – can be realized with pipe dreams in some well-chosen shapes on the two dimensional cartesian grid. Here we consider even more general shapes inspired by this work.

To describe the boundaries of our shapes on this grid, we will describe *paths* by their origin in $\mathbb{Z}^2$ and a sequence of steps of length one given by their direction: the letters N , S , E and W correspond respectively to northward, southward, eastward and westward step. Moreover, for a path $\mathcal{P}$, the notation $|\mathcal{P}|_x$ denotes the numbers of steps in $\mathcal{P}$ with direction x .

In general, we will define an n -shape F for $n \geq 2$ as a collection of cells on the two-dimensional cartesian grid delimited by four paths as follows:

- a *starting path* $\mathcal{S}_F$ from $0,0$ to $|\mathcal{S}_F|_E, -|\mathcal{S}_F|_S$ with n steps S or E ;
- a *NW stair path* from $0,0$ to t_F, t_F of the form NE^{t_F} for some $t_F \geq 0$;
- a *SE stair path* from $|\mathcal{S}_F|_E, -|\mathcal{S}_F|_S$ to $|\mathcal{S}_F|_E + b_F, -|\mathcal{S}_F|_S + b_F$ of the form EN^{b_F} for some $b_F \geq 0$;
- an *ending path* $\mathcal{E}_F$ from t_F, t_F to $|\mathcal{S}_F|_E + b_F, -|\mathcal{S}_F|_S + b_F$ with n steps S or E .

An example is given in Figure 5: the 4-shape is drawn in black, with the starting and ending path bolded and the stair paths drawn with finer lines.

A *pipe dream* on the n -shape F is then a filling of this shape with crosses $+$ and contacts $\curvearrowright$ such that the figure contains n pipes starting on a step of the starting path and ending on a step of the ending path. The pipes are numbered in the natural order from north-west to south-east on the starting path, and the order in which they appear on the ending path from north-west to south-east is the *exit permutation* of the pipe dream. As in the triangular case, we only consider *reduced* pipe dreams, i.e those where any two pipes cross at most once. For an n -shape F and a permutation $\omega \in \mathfrak{S}_n$, we denote by $\Pi_F \omega$ the set of reduced pipe dreams on F with exit permutation ω .

We note that the triangular shape used to create triangular pipe dreams is an n -shape:

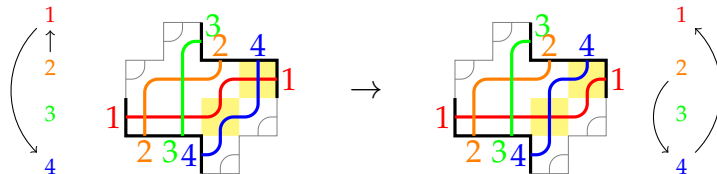


Figure 5: An ascending flip between two reduced pipe dreams on a 4-shape with exit permutation 3241 and their contact graphs.

the starting path only has S steps, the ending path only has E steps, the NW stair path has length 0 and the SE stair path has length $2n$. Cambrian shapes in [12] are another particular case of these shapes, where the starting and ending paths are opposite.

Flips, increasing and decreasing flips, the increasing flip order and the contact graph of a pipe dream are defined the same way as in the triangular case, as illustrated in Figure 5. We denote by $\Sigma_F \omega$ the set of acyclic reduced pipe dreams on the n -shape F for any permutation $\omega \in \mathfrak{S}_n$.

Until now, the generalization of triangular pipe dreams to n -shapes seems to be very simple, but some limitations appear very quickly. The first is that n -shapes can be "too small", in the sense that one can find an n -shape F and a permutation $\omega \in \mathfrak{S}_n$ such that $\Pi \omega$ is empty because there is not enough space in F for the pipes of a pipe dream to cross each other in a way to exit in order ω . In general, we will say that a permutation $\omega \in \mathfrak{S}_n$ is *sortable* on an n -shape F if $\Pi \omega$ contains at least one pipe dream. In that case, it turns out that $\Sigma \omega$ is also nonempty. Indeed, one can always find a reduced acyclic pipe dream from any pipe dream: first by removing pairs of crossings of the same two pipes, and then by doing descending flips on the result until no such flip is possible; the resulting pipe dream has $\text{id}_n \in \mathfrak{S}_n$ as a linear extension.

Moreover, even when considering a permutation sortable on a shape, the linear extensions of acyclic pipe dreams behave significantly less nicely than in the triangular case. The statement given by Theorem 2.1 does not translate as is to general pipe dreams: some linear extensions lie outside of the weak order interval below the exit permutation. The closest true statement is the following.

Theorem 3.1. *Let F be an n -shape and $\omega \in \mathfrak{S}_n$ be a permutation sortable on F , we denote by $\text{lin}_F \omega$ the set of all linear extensions of pipe dreams in $\Sigma_F \omega$. Then:*

- *the set $\{\text{lin } P \mid P \in \Sigma_F \omega\}$ is a partition of $\text{lin}_F \omega$;*
- *the set $\text{lin}_F \omega$ is a lower ideal of the weak order on $\mathfrak{S}_n$ that contains the interval id, ω .*

While this theorem still allows for the definition of an equivalence relation and a map similar to $\equiv_\omega$ and ins_ω in the triangular case, their relationship with the lattice properties of the weak order are limited by two aspects: one is that the maximal possible domain of this function, which is $\text{lin}_F \omega$, is not always an interval and as such is not always a lattice, and the second one is that if we only define our function on the interval id, ω , some acyclic pipe dreams may not have any preimage, as is the case in the example in Figure 6: the pipe dream has only one linear extension 15234 and it is not below its exit permutation 31524 in the weak order. We thus have to make a choice between lattice properties of the map and surjectivity on acyclic facets.

Since our goal is to study lattice quotients of weak order intervals in pipe dreams, we chose to limit ourselves to the interval id, ω . We thus define *strongly acyclic* pipe dreams as pipe dreams with at least one linear extension below their exit permutation in the weak order. The example in Figure 6 is an acyclic pipe dream that is not strongly

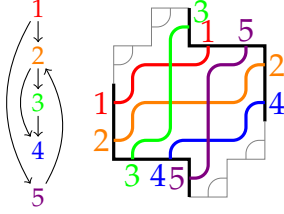


Figure 6: A reduced acyclic pipe dream that is not strongly acyclic.

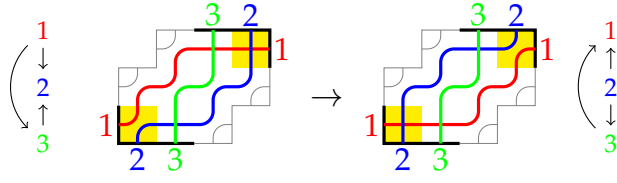


Figure 7: An increasing flip between two non-comparable acyclic pipe dreams.

acyclic. We denote by $\Sigma'_F \omega$ the set of strongly acyclic pipe dreams on the n -shape F with exit permutation ω . We note that when F is a triangular shape like in the previous section, since $\text{lin}_F \omega$ is id, ω for any permutation $\omega \in \mathfrak{S}_n$, any acyclic pipe dream is also strongly acyclic and $\Sigma'_F \omega = \Sigma_F \omega$.

We can now define the equivalence relation $\equiv_{F, \omega}$ on id, ω for any n -shape F and any permutation $\omega \in \mathfrak{S}_n$ as the one defined on id, ω whose equivalence classes are the sets of $\{\text{lin } P \cap \text{id}, \omega \mid P \in \Sigma'_F \omega\}$. However, while the Cambrian congruences that define Cambrian lattices as weak order quotients can be defined by rules similar as the one in [Theorem 2.2](#) (see section 6 of [\[12\]](#)), there is no known characterization of $\equiv_{F, \omega}$ outside of triangular shapes, even on Cambrian shapes. We can nevertheless still study the properties of $\equiv_{F, \omega}$ to obtain the following theorem.

Theorem 3.2. *For any n -shape F and any permutation $\omega \in \mathfrak{S}_n$ sortable on F , the relation $\equiv_{F, \omega}$ is a lattice congruence of the weak order interval id, ω .*

Therefore we can quotient id, ω by $\equiv_{F, \omega}$ to obtain a lattice quotient whose elements are the fibers of $\text{ins}_{F, \omega}$.

We can also define the map $\text{ins}_{F, \omega} : \text{id}, \omega \mapsto \Sigma'_F \omega$ which associate to any permutation in id, ω the pipe dream on F with exit permutation ω that has this permutation as a linear extension of its contact graph. As illustrated in [Figures 8 and 9](#), the sweeping algorithm and the insertion algorithm can be adapted to work on n -shapes.

The application $\text{ins}_{F, \omega}$ still shows a link between the weak order and increasing flips in $\Sigma'_F \omega$ in the sense that for any two permutations π, π' in id, ω , if $\pi \leq \pi'$ then there is a path from $\text{ins}_{F, \omega} \pi$ to $\text{ins}_{F, \omega} \pi'$ in the increasing flip graph. However, the converse is not true and [Theorem 2.6](#) does not hold for any n -shape. This is illustrated on [Figure 7](#), where the only linear extension 132 of the left pipe dream is not comparable to the only linear extension 231 of the right pipe dream, while the two pipe dreams are connected by the flip on the yellow cells. Therefore, the image of the weak order by $\text{ins}_{F, \omega}$ in general is a suborder of the increasing flip order that cannot be determined by simply looking

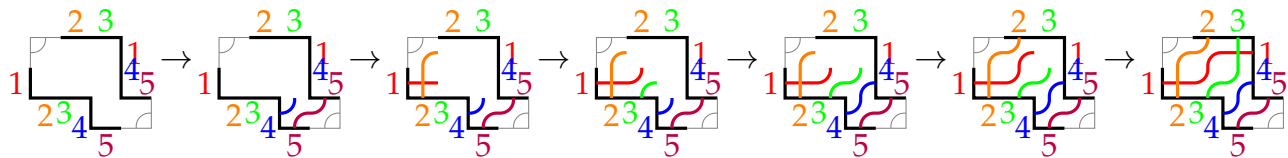


Figure 8: The sweeping algorithm on a 5-shape with $\omega = 23145$ and $\pi = 21345$.

at the increasing flip graph of $\Pi_F \omega$; we call this order the *acyclic flip order* on $\Sigma'_F \omega$. Combined with [Theorem 3.2](#), we get the following result.

Theorem 3.3. *The acyclic flip order on $\Sigma'_F \omega$ is isomorphic to the lattice quotient $\text{id}, \omega / \equiv_{F, \omega}$. Moreover, the map $\text{ins}_{F, \omega}$ is a lattice morphism from the weak order on id, ω to the acyclic flip order on $\Sigma'_F \omega$.*

While this acyclic flip order seems very artificial and ad-hoc, we can actually find its covers in an object introduced first in [10] and extended to any subword complex in [5]: the brick polyhedron of a subword complex. This object was behind the first introduction of acyclic pipe dreams: the brick polytope is defined on some subword complexes as the convex hull of the brick vectors associated with facets, and acyclic facets are the one associated to vertices of this polytope. The brick polyhedron is a generalization to non-spherical subword complexes. In both case, a flip between two strongly acyclic pipe dreams is associated to a cover of the acyclic flip order if and only if the brick vectors of those pipe dreams are linked by an edge of the brick polyhedron.

We can thus say that the quotient of the weak order interval obtained with $\equiv_{F, \omega}$ is isomorphic to the restriction of an orientation of the skeleton of the brick polyhedron to strongly acyclic facets. However, the removal of acyclic facets that are not strongly acyclic makes this result weaker, leading to us trying to find some conditions under which all acyclic facets are also strongly acyclic, like in the triangular case. This lead to the following theorem.

Theorem 3.4. *Let F be an n -shape such that the maximal permutation of size n is sortable on F , then for any permutation $\omega \in \mathfrak{S}_n$ we have $\text{lin}_F \omega = \text{id}, \omega$ and all acyclic pipe dreams on F are strongly acyclic.*

Therefore, if the n -shape is big enough, we can remove the distinction between acyclic and strongly acyclic pipe dreams and consider the full brick polyhedron. This condition on F is necessary to obtain that for any $\omega \in \mathfrak{S}_n$, the linear extensions of pipe dreams in $\Sigma_F \omega$ are exactly id, ω .

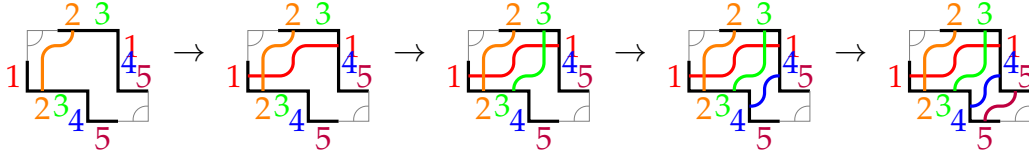


Figure 9: The insertion algorithm on a 5-shape with $\omega = 23145$ and $\pi = 21345$.

4 Extension to Coxeter groups

Let (W, S) be a finite Coxeter system and let (Φ, Δ) be a root system for (W, S) . In the following section, for any simple reflection $s \in S$ we will denote by α_s the simple root associated to s . For any $w \in W$, we define the *inversion set* of w as $\text{inv } w = w \Phi^- \cap \Phi^+$, and the *noninversion set* of w as $\text{ninv } w = \Phi^+ \setminus \text{inv } w$. The *right weak order* on W is the order defined by inclusion on the inversion sets and it is a lattice.

Let Q be any word on the alphabet S and denote by m its length. For $w \in W$, we denote by $\text{SC } Q, w$ the *subword complex* defined by w on Q , i.e the simplicial complex with ground set $[m]$ and whose faces are the sets of indices of Q whose complement contains a reduced expression of w . This object was introduced in [7] and developed in [6] to study combinatorial properties of Schubert polynomials. The facets of $\text{SC } Q, w$ are the complements of subwords of Q that are reduced expressions of w . Two distinct facets I and J of the same subword complex are linked by a *flip* if and only if there exist $i \in I$ and $j \in J$ such that $I = J \setminus \{j\} \cup \{i\}$, and this flip is *increasing from I to J* if $j < i$ and decreasing otherwise. The *increasing flip graph* of $\text{SC } Q, w$ has the facets as vertices and an edge from I to J if and only if there is an increasing flip from I to J , and the transitive closure of this graph defines the *increasing flip order* on facets of $\text{SC } Q, w$.

Let Q be a word on S and w be an element of W with $\text{SC } Q, w$ nonempty. The *root function* $r_I, \cdot : [m] \rightarrow \Phi$ on a facet I of this subword complex, introduced in [3], is defined by

$$r_{I,k} = \left(\prod_{i \in \{1, \dots, k-1\} \setminus I} Q_i \right) \alpha_{Q_k}$$

i.e it is the application of the prefix of w written by facet I up to index $k-1$ on Q to the root associated with the k^{th} letter of Q . We note that for any facet I of $\text{SC } Q, w$, the values of the root function of the indices not in I are the inversions of w , each appearing exactly once. Moreover, for $i \in I$, there exists a flip from I to a facet J with $i \notin J$ if and only if $r_{I,i} \in \pm \text{inv } w$, and this flip is increasing if $r_{I,i} \in \Phi^+$ and decreasing otherwise.

The *root configuration* of a facet I , denoted by R_I , is the set $\{r_{I,k} \mid k \in [m] \setminus I\}$ of the values of the root function on the other indices of Q . An element π of W is a

linear extension of I if $R(I) \subseteq \pi \Phi^-$. A facet is *acyclic* if the cone generated by its root configuration is pointed, or, equivalently, if it has at least one linear extension, and *strongly acyclic* if it has a linear extension in e, w . We note that the contact graph of pipe dreams defined in parts 2 and 3 give a representation of the root configuration of the facet of the associated subword complex, and therefore the following theorem is a direct generalization of [Theorem 2.1](#) and [Theorem 3.1](#).

Theorem 4.1. *Let Q be a word on S and $w \in W$ be such that $\text{SC}(Q, w)$ is nonempty. Let $\text{lin}_Q(w)$ be the set of linear extensions of facets of $\text{SC}(Q, w)$. Then:*

- *the set $\{\text{lin}(I) \mid I \in \text{SC}(Q, w)\}$ is a partition of $\text{lin}_Q(w)$;*
- *the set $\text{lin}_Q(w)$ is a lower ideal of the right weak order on W and contains e, w .*

As in [Section 2](#) and [Section 3](#), we can therefore define an equivalence relation $\equiv_{Q, w}$ whose equivalence classes are the $\text{lin}(I)$ and a map $\text{ins}_{Q, w} : e, w \mapsto \text{SC}(Q, w)$ such that $\pi \in \text{lin}(\text{ins}_{Q, w}(\pi))$ for any $\pi \in e, w$. Moreover, we can prove that if $\pi < \pi'$ are two elements of e, w , then $\text{ins}_{Q, w}(\pi) \leq \text{ins}_{Q, w}(\pi')$ in the increasing flip order. However, the restrictions on the properties of $\text{ins}_{F, \omega}$ defined in [Section 3](#) are also true for $\text{ins}_{Q, \omega}$: it is not always surjective on the acyclic facets of $\text{SC}(Q, w)$ and the image of the weak order is included in the increasing flip order but can be strictly weaker. We can however still prove that the flips that are covers of the image of the weak order are linked to the edges of the brick polyhedron defined on $\text{SC}(Q, w)$ in [\[5\]](#), and give a generalization of the sweeping algorithm in type A that computes $\text{ins}_{Q, w}(\pi)$ when it exists.

Algorithm 4.2. *The sweeping algorithm on subword complexes computes $\text{ins}_{Q, w}$ by considering the indices of Q from first to last and adding t or not to the facet according to the following rules:*

1. *if $r(\text{ins}_{Q, w}(\pi), t) \in w \Phi^-$ then t is added;*
2. *else if $r(\text{ins}_{Q, w}(\pi), t) \in \pi \Phi^-$ then t is not added;*
3. *else t if and only if the result can be completed into a facet of $\text{SC}(Q, w)$.*

We have now defined the objects we need to conjecture a generalization to [Theorem 3.2](#) and [Theorem 3.3](#). However, even in type A, the quotient of e, w by $\equiv_{Q, w}$ is not always a lattice quotient, as illustrated in Figure 9 of [\[11\]](#). We thus need to restrict the conjecture to a specific category of words. The generalization of n -shapes to general Coxeter groups are *alternating words*, i.e words such that for any $s \in S$, if $Q_i = Q_j = s$ and $i < j$, then for any $t \in S$ such that $st \neq ts$, the letter t appears in Q between indices i and j . In type A, the alternating words are equivalent to the n -shapes. This leads us to the following conjecture, tested extensively on various finite Coxeter groups.

Conjecture 4.3. *Let Q be an alternating word on S and $w \in W$ be such that $\text{SC}(Q, w)$ is nonempty, then the relation $\equiv_{Q, w}$ is a lattice congruence of the right weak order interval e, w . The quotient of e, w is isomorphic to the strongly acyclic part of the brick polyhedron of $\text{SC}(Q, w)$ oriented by ascending flips, and $\text{ins}_{Q, w}$ is a lattice morphism from e, w to this.*

Lastly, a generalization of [Theorem 3.4](#) was proven in [\[5\]](#), where an algorithm very similar to [Algorithm 4.2](#) was also defined.

Theorem 4.4 ([\[5\]](#)). *Let Q be a word on S and w be any element of W . If Q contains a reduced word of w_0 the longest element of W , then for any facet of $\text{SC } Q, w$, the noninversions of w are all in the cone generated by the root configuration of the facet, and thus any linear extension of the facet is in e, w .*

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A combinatorial interpretation of the noncommutative inverse Kostka matrix

Edward E Allen¹ and Sarah Mason¹

¹*Department of Mathematics, Wake Forest University, Winston-Salem, NC 27109*

Abstract. We provide a combinatorial formula for the expansion of immaculate noncommutative symmetric functions into complete homogeneous noncommutative symmetric functions. To do this, we introduce generalizations of Ferrers diagrams which we call GBPR diagrams. We define tunnel hooks, which play a role similar to that of the special rim hooks appearing in the Eğecioğlu-Remmel formula for the symmetric inverse Kostka matrix. We extend this interpretation to skew shapes and fully generalize to define immaculate functions indexed by integer sequences skewed by integer sequences. Finally, as an application of our combinatorial formula, we extend Campbell's results on ribbon decompositions of immaculate functions to a larger class of shapes.

Keywords: immaculate functions, inverse Kostka, NSym

1 Background and Introduction

The ring Sym of symmetric functions on a set of commuting variables consists of all polynomials invariant under the action of the symmetric group. Symmetric functions play an important role in representation theory, combinatorics, and other areas of mathematics and the physical and natural sciences. The Schur function basis for Sym , which can be defined combinatorially as the generating function for semi-standard Young tableaux or algebraically through *Bernstein creation operators*, corresponds to irreducible representations of the symmetric group. Schur function multiplication corresponds to the cohomology of the Grassmannian [6, 10, 11].

The inverse Kostka matrix is the transition matrix from the Schur basis $\{s_\lambda\}_\lambda$ of Sym to the *complete homogeneous basis* $\{h_\lambda\}_\lambda$. The *Jacobi-Trudi Formula* [10] is a determinantal formula $s_{\lambda/\nu} = \det(h_{\lambda_i - i - (\nu_j - j)})$ for decomposing Schur functions $s_{\lambda/\nu}$ into complete homogeneous symmetric functions h_λ . Eğecioğlu and Remmel's combinatorial interpretation of the inverse Kostka matrix [5] provides a method for writing Schur functions in terms of complete homogeneous symmetric functions using decompositions of the indexing shapes into *special rim hooks*.

The Hopf algebra NSym of noncommutative symmetric functions can be thought of as the free associative algebra $\mathbf{K}\langle H_1, H_2, \dots \rangle$ generated by algebraically independent,

noncommuting complete homogeneous symmetric functions H_i over a fixed commutative field $\mathbf{K}$ of characteristic zero. Set $H_a := 0$ if a is a negative integer and $H_0 := 1$. The function $H_{(\alpha_1, \alpha_2, \dots, \alpha_\ell)} = H_{\alpha_1} H_{\alpha_2} \cdots H_{\alpha_\ell}$ maps to the complete homogeneous symmetric function h_α under the “forgetful map” from NSym to Sym.

The creation operator construction of the Schur functions can be extended to NSym to produce a new basis for NSym called the *immaculate basis* [2], corresponding to indecomposable modules of the 0-Hecke algebra [3]. We use the equivalent Jacobi-Trudi style formula [2] to define the immaculate basis and we also extend this construction to introduce the *skew immaculate functions*.

Definition 1.1. Let $\mu, \nu \in \mathbb{Z}^k$ be sequences of integers. Define $(M_{\mu/\nu})_{i,j} = H_{(\mu_i - i) - (\nu_j - j)}$. Then $\mathfrak{S}_{\mu/\nu} = \det(M_{\mu/\nu})$, where the noncommutative determinant in NSym is obtained by performing the Laplace expansion row by row, starting from the top row.

Example 1.2. With $\mu = (4, 5, 2)$ and $\nu = (3, 1, 1)$, then

$$\begin{aligned} \mathfrak{S}_{\mu/\nu} &= \det(M_{\mu/\nu}) = \det \begin{pmatrix} H_1 & H_4 & H_5 \\ H_1 & H_4 & H_5 \\ H_{-3} & H_0 & H_1 \end{pmatrix} \\ &= H_1 H_4 H_1 - H_1 H_5 H_0 - H_4 H_1 H_1 + H_4 H_5 H_{-3} + H_5 H_1 H_0 - H_5 H_4 H_{-3} \\ &= H_{1,4,1} - H_{1,5} - H_{4,1,1} + H_{5,1} \end{aligned}$$

since $H_0 = 1$ and $H_{-1} = 0$.

In this extended abstract, we provide a combinatorial formula for the expansion of the immaculate basis into the complete homogeneous basis for NSym using ribbon-like objects we call *tunnel hooks*, generalizing the combinatorial inverse Kostka formula to NSym. A complete discussion of these results including detailed proofs and additional examples can be found in [1].

Theorem 1.3. The decomposition of the skew immaculate noncommutative symmetric functions $\mathfrak{S}_{\mu/\nu}$ (with $\mu \in \mathbb{Z}^k$ and ν a partition with at most k parts, possibly the empty partition) into the complete homogeneous noncommutative symmetric functions is given by the following formula.

$$\mathfrak{S}_{\mu/\nu} = \sum_{\gamma \in \text{THC}_{\mu/\nu}} \prod_{r=1}^k \epsilon(\mathfrak{h}(r, \tau_r)) H_{\Delta(\mathfrak{h}(r, \tau_r))}, \quad (1.1)$$

where $\text{THC}_{\mu/\nu}$ denotes the tunnel hook coverings of a diagram of shape μ/ν , and a sign $\epsilon(\mathfrak{h}(r, \tau_r))$ and integer value $\Delta(\mathfrak{h}(r, \tau_r))$ are assigned to each tunnel hook $\mathfrak{h}(r, \tau_r)$ in each THC γ .

Note that the product $\prod_{r=1}^k \epsilon(\mathfrak{h}(r, \tau_r)) H_{\Delta(\mathfrak{h}(r, \tau_r))}$ in Eq. (1.1) is taken in order from $r = 1$ to k since the functions $H_{\Delta(\mathfrak{h}(r, \tau_r))}$ do not commute.

Loehr and Niese recently published a combinatorial interpretation of the immaculate inverse Kostka matrix [9]. Their approach uses transitive tournaments and recursively defined sums, which is quite different from our diagrammatic approach. Loehr and Niese also provide a diagrammatic method for computing the decomposition of an immaculate function into the complete homogeneous basis when the indexing shape is a partition. Our diagrammatic approach works for all indexing shapes, including all sequences of integers. Our decomposition can be determined directly by looking at the diagram and recording the values associated to the tunnel hooks.

2 GBPR diagrams and tunnel cells

Let $\mu = (\mu_1, \dots, \mu_k)$ be a sequence and $\nu = (\nu_1, \dots, \nu_k)$ be a partition. The *Grey-Blue-Purple-Red (GBPR) diagram* $D_{\mu/\nu}$ for the skew shape μ/ν is obtained as follows.

1. Place ν_i grey cells in row i of the diagram (for $1 \leq i \leq k$), working from bottom to top (to stay consistent with French notation).
2. For each $1 \leq i \leq k$, divide into cases to place red and blue cells into row i as follows.
 - (a) If $\mu_i > 0$ and $\nu_i \leq \mu_i$, place $\mu_i - \nu_i$ blue cells in row i immediately to the right of the grey cells.
 - (b) If $\mu_i > 0$ and $\mu_i < \nu_i$, place $\nu_i - \mu_i$ red cells in row i immediately to the right of the grey cells.
 - (c) If $\mu_i \leq 0$, place $|\mu_i| + \nu_i$ red cells in row i immediately to the right of the grey cells.
3. Any cell in the first quadrant not colored grey, red, or blue is *purple*, but we do not typically color these in illustrations since there are infinitely many purple cells.

See [Example 2.8](#) for an example of a GBPR diagram. Although the GBPR diagram does not necessarily contain exactly μ_i cells in row i , the value of μ_i can be determined from the number of grey, blue, and red cells. Let a_i be the number of grey cells, b_i be the number of blue cells, and c_i be the number of red cells in row i of a GBPR diagram. Then

$$a_i + b_i - c_i = \mu_i.$$

Definition 2.1. Cells (p, q) and (r, s) are adjacent if and only if $|p - r| + |q - s| = 1$. A collection C of cells is connected if for any cells $c, d \in C$, there is a sequence $c = \tau_1, \tau_2, \dots, \tau_j = d$ where $\tau_i \in C$ and τ_i and τ_{i+1} are adjacent for $1 \leq i \leq j - 1$. Cells (p, q) and (r, s) are diagonally adjoining if both $r = p + 1$ and $s = q + 1$ or both $p = r + 1$ and $q = s + 1$.

At times, it will be convenient to construct diagrams for shapes obtained via prefix removal. Note that any diagram can be considered as a partial diagram by setting $r = 0$.

Definition 2.2. Let $r \in \mathbb{Z}_{\geq 0}$, $\mu \in \mathbb{Z}^k$, and $v \in \mathbb{Z}_{\geq 0}^k$ such that $v_{r+1} \geq v_{r+2} \geq \dots \geq v_k$. The partial diagram $D_{\mu/v}^{(r)}$ is obtained by constructing the GBPR diagram $D_{(\mu_{r+1}, \dots, \mu_k) / (v_{r+1}, \dots, v_k)}$ and shifting the resulting diagram up by r rows, so that the first nonempty row is in row $r + 1$.

This construction allows us to compute the GBPR diagram for row $r + 1$ through row k even when the first r parts of v are not necessarily weakly decreasing.

Example 2.3. Let $\mu = (-3, 1, -1, 0, 3, -2)$ and $v = (2, 4, 1, 0, 0)$. Then $D_{\mu/v}^{(2)}$ is given by

6, 1	6, 2	
5, 1	5, 2	5, 3
4, 1		
3, 1	3, 2	3, 3

We now define *boundary cells*, which intuitively are the cells which lie on the boundary of v . Boundary cells will be used to construct hooks that will play a role similar to that of *rim hooks* in the combinatorial interpretation of the symmetric inverse Kostka matrix [5].

Definition 2.4. Let $\mu \in \mathbb{Z}^k$ and let r be a positive integer such that $1 \leq r \leq k$. Let $v \in \mathbb{Z}_{\geq 0}^k$ such that $(v_r, v_{r+1}, \dots, v_k)$ is a partition and $r \leq p \leq k$. A cell in location (p, q) is a boundary cell of $D_{\mu/v}^{(r-1)}$ iff

$$v_p + 1 \leq q \leq v_{p-1} + 1,$$

for $p > r$ and

$$v_r + 1 \leq q \leq \max\{v_r + 1, a_r + b_r + c_r\}$$

if $p = r$. A tunnel cell (p, q) of $D_{\mu/v}^{(r-1)}$ is a boundary cell such that

$$q = v_p + 1.$$

The inequality $v_r + 1 \leq q \leq \max\{v_r + 1, a_r + b_r + c_r\}$ for row r forces the cell $(r, v_r + 1)$, as well as all red or blue cells in row r , to be boundary cells. Also note that a cell (p, q) with $p < r$ cannot be a boundary cell of $D_{\mu/v}^{(r)}$. Let $\mathcal{B}_{\mu/v}^{(r)}$ and $\mathcal{T}_{\mu/v}^{(r)}$ respectively denote the collections of boundary cells and tunnel cells of $D_{\mu/v}^{(r)}$. Set $\mathcal{N}_{\mu/v}^{(r)} = \mathcal{B}_{\mu/v}^{(r)} - \mathcal{T}_{\mu/v}^{(r)}$.

Definition 2.5. Let $\tau_r = (p, q) \in \mathcal{T}_{\mu/v}^{(r-1)}$ be a tunnel cell in $D_{\mu/v}^{(r-1)}$. The tunnel hook determined by τ_r is the collection $\mathfrak{h}(r, \tau_r)$ consisting of all boundary cells in rows r through p . The cell τ_r (the farthest northwest cell of $\mathfrak{h}(r, \tau_r)$) is called the terminal cell of $\mathfrak{h}(r, \tau_r)$. The sign of the tunnel hook $\mathfrak{h}(r, \tau_r)$, denoted by $\epsilon(\mathfrak{h}(r, \tau_r))$, equals $(-1)^{p-r}$. If cell $\hat{c} \in \mathfrak{h}(r, \tau)$ then we say that $\mathfrak{h}(r, \tau)$ covers $\hat{c}$.

There are many similar objects in the literature such as skew hooks, ribbons, and border strips [10], and the rim hooks [5] appearing in the combinatorial interpretation of the inverse Kostka matrix in Sym. To generalize these combinatorial objects to the NSym setting, tunnel hooks need to "tunnel" into the diagram instead of remaining on the rim.

With $\mathfrak{h}(r, \tau_r)$ a tunnel hook in the partial diagram $D_{\mu/\nu}^{(r)}$ terminating at cell $\tau = (p, q)$, set

$$\Delta(\mathfrak{h}(r, \tau_r)) = b_r - c_r + (\nu_r + 1 - q) + (p - r). \quad (2.1)$$

Note that in Equation (2.1), since ν is a partition (whose parts might be 0), the cell $(p, q) \in \mathcal{T}_{\mu/\nu}^{(r)}$ will be weakly west of $(r, \nu_r + 1)$.

The proof of the following lemma is immediate since there is only one tunnel cell in each row and moving up a row strictly increases the value of $\Delta(\mathfrak{h}(r, \tau_r))$.

Lemma 2.6. *Given a partial diagram $D_{\mu/\nu}^{(r)}$ with μ a sequence and ν a partition, for any fixed $j \in \mathbb{Z}$, there is at most one tunnel cell τ such that $j = \Delta(\mathfrak{h}(r, \tau_r))$.*

The following iterative procedure provides a method for constructing a *tunnel hook covering* (THC) of the diagram $D_{\mu/\nu}$.

Procedure 2.7. Consider $\mu = (\mu_1, \mu_2, \dots, \mu_k) \in \mathbb{Z}^k$ and a partition $\nu = (\nu_1, \nu_2, \dots, \nu_k)$.

1. Construct the partial GBPR diagram $D_{\mu/\nu}^{(0)}$ of shape μ/ν . Set $\nu^{(0)} = \nu$.
2. Repeat the following steps, once for each value of r from 1 to k .
 - (a) Choose a tunnel hook $\mathfrak{h}(r, \tau_r)$ in $D_{\mu/\nu^{(r-1)}}^{(r-1)}$ and set $\alpha_r := \Delta(\mathfrak{h}(r, \tau_r))$.
 - (b) For each $1 \leq i \leq k$, let $\eta_i^{(r)}$ be the number of cells in row i of $\mathfrak{h}(r, \tau_r)$ and let $\nu^{(r)}$ be the partition defined by $\nu_i^{(r)} = \nu_i^{(r-1)} + \eta_i^{(r)}$ for $1 \leq i \leq k$.
 - (c) Construct the partial GBPR diagram $D_{\mu/\nu^{(r)}}^{(r)}$.
3. Set $\alpha = (\alpha_1, \dots, \alpha_k)$. This will become a subscript appearing in the H-expansion of $\mathfrak{S}_\mu$.

It is not hard to show that if $\nu^{(r)}$ is the sequence of nonnegative integers produced during Step 2b of Procedure 2.7, then $(\nu_{r+1}^{(r)}, \nu_{r+2}^{(r)}, \dots, \nu_k^{(r)})$ is a partition.

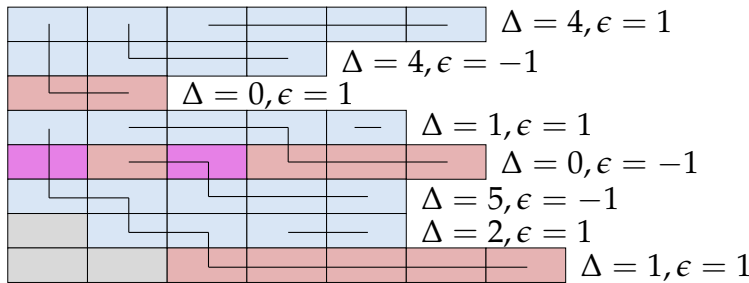
Example 2.8. Letting $\mu/\nu = (-3, 5, 5, 0, 5, -2, 4, 6)/(2, 1)$, we construct a tunnel hook covering of $D_{\mu/\nu}$. First, we give the GBPR diagram of $D_{\mu/\nu}^{(0)}$ with cells labelled by (row, column). We then use a table to provide details for a particular tunnel hook covering. Finally, we illustrate this THC on the GBPR diagram itself.

8,1	8,2	8,3	8,4	8,5	8,6	
7,1	7,2	7,3	7,4			
6,1	6,2					
5,1	5,2	5,3	5,4	5,5		
4,1						
3,1	3,2	3,3	3,4	3,5		
2,1	2,2	2,3	2,4	2,5		
1,1	1,2	1,3	1,4	1,5	1,6	1,7

The following table records the process of decomposing the GBPR diagram $D_{\mu/\nu}$ into tunnel hooks. Each row r indicates the situation before the r^{th} tunnel hook is placed. Here τ_r is the tunnel cell at which the tunnel hook beginning in the r^{th} row of the partial diagram terminates.

r	$(\mu_r, \mu_{r+1}, \dots, \mu_k)$	$\nu^{(r-1)}$	τ_r	$\Delta(\mathfrak{h}(r, \tau_r))$
1	$(-3, 5, 5, 0, 5, -2, 4, 6)$	$(2, 1, 0, 0, 0, 0, 0, 0)$	$(5, 1)$	$-5 + 6 = 1$
2	$(5, 5, 0, 5, -2, 4, 6)$	$(7, 3, 2, 1, 1, 0, 0, 0)$	$(2, 4)$	$2 + 0 = 2$
3	$(5, 0, 5, -2, 4, 6)$	$(7, 5, 2, 1, 1, 0, 0, 0)$	$(4, 2)$	$3 + 2 = 5$
4	$(0, 5, -2, 4, 6)$	$(7, 5, 5, 3, 1, 0, 0, 0)$	$(5, 2)$	$-3 + 3 = 0$
5	$(5, -2, 4, 6)$	$(7, 5, 5, 6, 4, 0, 0, 0)$	$(5, 5)$	$1 + 0 = 1$
6	$(-2, 4, 6)$	$(7, 5, 5, 6, 5, 0, 0, 0)$	$(8, 1)$	$-2 + 2 = 0$
7	$(4, 6)$	$(7, 5, 5, 6, 5, 2, 1, 1)$	$(8, 2)$	$3 + 1 = 4$
8	(6)	$(7, 5, 5, 6, 5, 2, 4, 2)$	$(8, 3)$	$4 + 0 = 4$

The following illustration of the THC of $D_{\mu/\nu}^{(0)}$ depicts all the tunnel hooks at once and omits the step of converting the colors of the cells to grey as they are covered by tunnel hooks.



3 Proof of Theorem 1.3 for non-skewed sequences

In this section, we give an overview of the proof Theorem 1.3 when the indexing shape is not a skew shape. Recall Definition 1.1 [2] states that

$$\mathfrak{S}_\mu = \det(M_\mu) = \sum_{\sigma \in S_k} \epsilon(\sigma) (M_\mu)_{i, \sigma_i}.$$

Theorem 1.3 provides a combinatorial interpretation of the decomposition of immaculate functions into complete homogeneous functions using tunnel hook coverings. The main idea behind our proof is a bijection between tunnel hook coverings and permutations (**Proposition 3.5**). Furthermore, each tunnel hook is associated to a number equal to the subscript of the corresponding complete homogeneous function appearing in the matrix M_μ (**Lemma 3.6**). Finally, we show that the product of the signs of the tunnel hooks in a tunnel hook covering is equal to the sign of the corresponding permutation.

The first lemma needed for the proof of the non-skew case of **Theorem 1.3** provides foundational insight into how tunnel cells are related to the diagonals parallel to the line $y = x$. Let

$$\mathcal{L}_j := \{(p, q) : p - q + 1 = j\} = \{(j + m, 1 + m) | m \in \mathbb{Z}^{\geq 0}\} \quad (3.1)$$

be the collection of cells in the j^{th} diagonal of the first quadrant of the plane, for $1 \leq j \leq k$. These diagonals (whose properties are described in the following lemma) will correspond to the entries in the permutations used when computing the determinant of the matrix M_μ .

Lemma 3.1. *Let $D_{\mu/\nu^{(r-1)}}^{(r-1)}$ be a partial GBPR diagram for $\mu \in \mathbb{Z}^k$ and $\nu^{(r-1)} \in \mathbb{Z}_{\geq 0}^k$ such that $(\nu_r^{(r-1)}, \nu_{r+1}^{(r-1)}, \dots, \nu_k^{(r-1)})$ is a partition. Suppose $\tau_r, t \in \mathcal{T}_{\mu/\nu^{(r-1)}}^{(r-1)}$, $\tau_r \neq t$, and $\xi \in \mathcal{N}_{\mu/\nu^{(r-1)}}^{(r-1)}$. Furthermore, suppose $\tau_r = (p_1, q_1)$, $t = (p_2, q_2)$, and $\xi = (p_3, q_3)$. Finally, let $\mathfrak{h}(r, \tau_r)$ be a tunnel hook in the diagram $D_{\mu/\nu^{(r-1)}}^{(r-1)}$. Then*

- A. $(p_1 + 1, q_1 + 1) \in \mathcal{N}_{\mu/\nu^{(r)}}^{(r)}$.
- B. If $\mathfrak{h}(r, \tau_r)$ does not cover t then $t \in \mathcal{T}_{\mu/\nu^{(r)}}^{(r)}$.
- C. If $\mathfrak{h}(r, \tau_r)$ covers t then $(p_2 + 1, q_2 + 1) \in \mathcal{T}_{\mu/\nu^{(r)}}^{(r)}$.
- D. If $\mathfrak{h}(r, \tau_r)$ does not cover ξ then $\xi \in \mathcal{N}_{\mu/\nu^{(r)}}^{(r)}$.
- E. If $\mathfrak{h}(r, \tau_r)$ covers ξ then $(p_3 + 1, q_3 + 1) \in \mathcal{N}_{\mu/\nu^{(r)}}^{(r)}$.

Lemma 3.1 implies an algorithm for identifying the cells in $\mathcal{T}_{\mu/\nu^{(r-1)}}^{(r-1)}$ available to become terminal cells at each step r in the construction of a THC. Along the way, we uncover a permutation associated with each THC. This algorithm to identify tunnel cells and produce the associated permutation is described below.

Procedure 3.2. *The following algorithm constructs a sequence of cells (which we will see are terminal cells for a tunnel hook covering) and also produces a permutation associated to each choice of cells.*

1. Let $\mathbb{T}_{(0)} = \{(1, 1), (2, 1), \dots, (k, 1)\}$.
2. Select a cell $\tau_1 = (p, 1)$ from $\mathbb{T}_{(0)}$ and set

$$\mathbb{T}_{(1)}^{\{\tau_1\}} = \{(2, 2), (3, 2), \dots, (p, 2), (p+1, 1), \dots, (k, 1)\};$$

$\mathbb{T}_{(1)}^{\{\tau_1\}}$ is the set constructed from $\mathbb{T}_{(0)}$ by removing $(p, 1)$ and adding $(1, 1)$ to each of the cells from rows lower than row p .

3. Let $\sigma_1 = p - 1 + 1 = p$. Note that $\mathcal{L}_{\sigma_1}$ is the diagonal containing $(p, 1)$.
4. Repeat the following steps, once for each value of r from 2 to k .
 - (a) Select a cell $\tau_r = (p_r, q_r)$ from $\mathbb{T}_{(r-1)}^{\{\tau_1, \dots, \tau_{r-1}\}}$.
 - (b) Construct $\mathbb{T}_{(r)}^{\{\tau_1, \dots, \tau_r\}}$ from $\mathbb{T}_{(r-1)}^{\{\tau_1, \dots, \tau_{r-1}\}}$ by removing τ_r and adding $(1, 1)$ to each of the cells from rows lower than row p_r .
 - (c) Set $\sigma_r = p_r - q_r + 1$. Note that $\mathcal{L}_{\sigma_r}$ is the diagonal containing (p_r, q_r) .
5. Set $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_k)$.

The following lemma shows that [Procedure 3.2](#) provides a method for selecting the terminal cells of a tunnel hook covering of any non-skew shape μ . Although the cells chosen do not depend on the diagram μ (provided μ is non-skew), the value Δ of each tunnel hook selected will vary based on the diagram μ .

Lemma 3.3. *Let μ be an arbitrary integer sequence with k parts and v be the empty partition. Let $\mu/v^{(r-1)}$ be the shape obtained after $r - 1$ iterations of the second step in [Procedure 2.7](#) with terminal cells $\{\tau_1, \tau_2, \dots, \tau_{r-1}\}$. Then the cells contained in $\mathbb{T}_{(r-1)}^{\{\tau_1, \dots, \tau_{r-1}\}}$ are precisely the tunnel cells in $\mathcal{T}_{\mu/v^{(r-1)}}^{(r-1)}$ for $1 \leq r - 1 \leq k$. Furthermore, the sequence $\sigma = (\sigma_1, \dots, \sigma_k)$ produced in step (5) of the procedure is a permutation in S_k .*

Lemma 3.3 proves that [Procedure 3.2](#) is equivalent to the THC construction procedure, since selecting a tunnel hook starting in row r can be done by simply selecting its terminal cell. Every possible terminal cell for a tunnel hook starting in row r (for the diagram $D_{\mu/v^{(r-1)}}^{(r-1)}$) is included in $\mathbb{T}_{(r-1)}^{\{\tau_1, \dots, \tau_{r-1}\}}$.

Example 3.4. *Let σ be a permutation of length $k = 10$ and consider step $i = 6$ in a tunnel hook covering construction of μ . Assume the first 5 tunnel hooks have been constructed with terminal cells $\tau_1, \dots, \tau_5$ and $\sigma_i \in \{2, 3, 6, 7, 9\}$ for $1 \leq i \leq 5$. Then the tunnel cells in $\mathbb{T}_{(5)}^{\{\tau_1, \dots, \tau_5\}}$ are in diagonals $\mathcal{L}_1, \mathcal{L}_4, \mathcal{L}_5, \mathcal{L}_8$, and $\mathcal{L}_{10}$, and, moreover,*

$$\mathbb{T}_{(5)}^{\{\tau_1, \dots, \tau_5\}} = \{(6, 6), (7, 4), (8, 4), (9, 2), (10, 1)\}.$$

If, for example, the cell $(8, 4)$ is selected as the terminal cell τ_6 for the next tunnel hook, then the new collection of tunnel cells will become

$$\mathbb{T}_{(6)}^{\{\tau_1, \dots, \tau_6\}} = \{(7, 7), (8, 5), (9, 2), (10, 1)\},$$

since $(1, 1)$ is added to the first two entries of $\mathbb{T}_{(5)}^{\{\tau_1, \dots, \tau_5\}}$ and $(8, 4)$ is removed.

The following proposition is a direct consequence of [Procedure 3.2](#) and [Lemma 3.3](#).

Proposition 3.5. *Let $\mu = (\mu_1, \dots, \mu_k)$ be a sequence. There is a bijection between tunnel hook coverings of the GBPR diagram for μ and permutations $\sigma \in S_k$.*

Additionally, given a permutation $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_k)$ in one-line notation, consider the subscripts of the entries of the submatrix of $(M_\mu)_{i,j} = H_{\mu_i - i + j}$ obtained by removing row i and column σ_i for $1 \leq i \leq r - 1$. These subscripts equal the Δ values of the tunnel hooks in the partial diagram obtained after $r - 1$ tunnel hooks (corresponding to $\sigma_1, \sigma_2, \dots, \sigma_{r-1}$) have been constructed.

Lemma 3.6. *Let μ be a sequence of length k and let M_μ be defined as above. Assume the first $r - 1$ tunnel hooks have been constructed so that their terminal cells do not lie in $\mathcal{L}_j$. Then $(M_\mu)_{r,j} = H_{\Delta(\mathfrak{h}(r, \tau_r))}$, where τ_r is the unique cell in the diagonal $\mathcal{L}_j$ that is also in $\mathbb{T}_{(r-1)}^{\{\tau_1, \dots, \tau_{r-1}\}}$.*

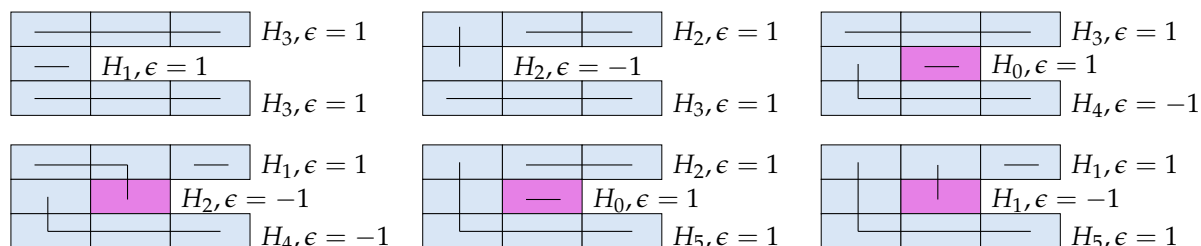
We now have all the pieces we need to complete the proof of [Theorem 1.3](#).

Sketch of Theorem 1.3 proof (non-skew case). Recall that $\mathfrak{S}_\mu = \det(M_\mu) = \sum_{\sigma \in S_k} \epsilon(\sigma) (M_\mu)_{i, \sigma_i}$.

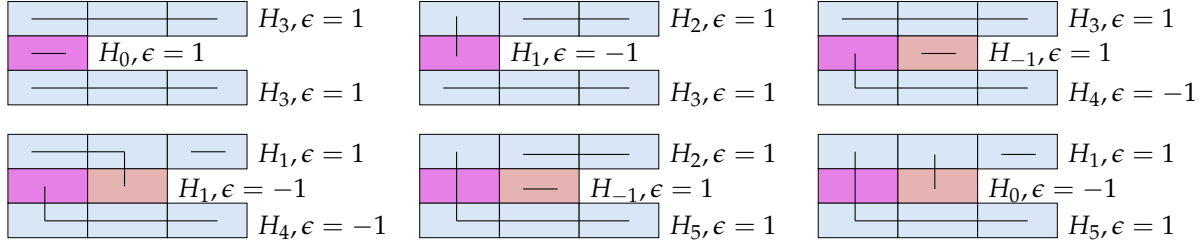
[Proposition 3.5](#) gives a bijection between permutations and tunnel hook coverings. In [Lemma 3.6](#) we show that the subscripts on the entries of the matrix M_μ equal the Δ values for the corresponding tunnel hooks. All that remains is to show that $\epsilon(\sigma)$ equals the product of the signs of the tunnel hooks. This is done using a connection between the Lehmer code [\[8\]](#), the sign of a permutation, and the number of rows covered by a tunnel hook. $\square$

The following three examples demonstrate how to construct the tunnel hook coverings for three different indexing shapes. Notice that the 6 configurations of terminal cells are the same in all three examples, but the shapes of the actual tunnel hooks (as well as the resulting Δ values) differ based on the initial diagram μ .

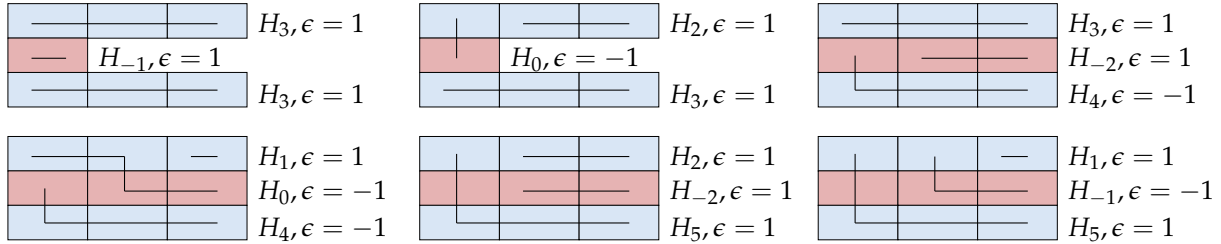
Example 3.7. $\mathfrak{S}_{(3,1,3)} = H_{(3,1,3)} - H_{(3,2,2)} - H_{(4,0,3)} + H_{(4,2,1)} + H_{(5,0,2)} - H_{(5,1,1)}$



Example 3.8. $\mathfrak{S}_{(3,0,3)} = H_{(3,0,3)} - H_{(3,1,2)} + H_{(4,1,1)} - H_{(5,0,1)}$



Example 3.9. $\mathfrak{S}_{(3,-1,3)} = -H_{(3,0,2)} + H_{(4,0,1)}$.



We now briefly discuss what happens when an immaculate is indexed by a sequence skewed by partition. First note that the partial GBPR diagrams produced during the construction of a tunnel hook covering are GBPR diagrams of skew shapes. Therefore finding the tunnel hook coverings of a shape skewed by a partition amounts to choosing an appropriate initial sequence and collection of initial tunnel hooks and then completing the remaining tunnel hook coverings. See [1] for details.

4 Immaculate functions indexed by sequences skewed by non-partition shapes

We now extend [Theorem 1.3](#) to find the H -decomposition of $\mathfrak{S}_{\mu/\nu}$ for any pair of integer sequences $\mu, \nu \in \mathbb{Z}^k$. If all parts of ν are nonnegative, set $\hat{\nu} = (v_1, v_2, \dots, v_{p-1}, v_{p+1} - 1, v_p + 1, v_{p+2}, \dots, v_k)$ and $(M_{\mu/\hat{\nu}})_{i,j} = H_{(\mu_i - i) - (\hat{\nu}_j - j)}$, so that $(M_{\mu/\nu})_{i,j} = (M_{\mu/\hat{\nu}})_{i,j}$ for $j \notin \{p, p+1\}$. Furthermore, since $\hat{\nu}_{p+1} = v_p + 1$ and $\hat{\nu}_p = v_{p+1} - 1$, we have $\mathfrak{S}_{\mu/\nu} = -\mathfrak{S}_{\mu/\hat{\nu}}$, since we obtain $M_{\mu/\hat{\nu}}$ by swapping columns p and $p+1$ of $M_{\mu/\nu}$. If the entries of ν are not weakly decreasing, apply the straightening operator $\hat{\nu}$ (adjusting the sign each time) until the result is a partition (in which case [Theorem 1.3](#) applies) or a sequence containing a one-step increase (in which case the skew immaculate function is zero). If any terms of ν are negative, let ν_j be the smallest part of ν . Add $-\nu_j$ to every part of μ and every part of ν ; call the resulting sequences $\text{aug}_{\nu_j}(\mu)$ and $\text{aug}_{\nu_j}(\nu)$ respectively. Then $M_{\text{aug}_{\nu_j}(\mu)/\text{aug}_{\nu_j}(\nu)} = M_{\mu/\nu}$, so $\mathfrak{S}_{\mu/\nu} = \mathfrak{S}_{\text{aug}_{\nu_j}(\mu)/\text{aug}_{\nu_j}(\nu)}$ and we can apply the above

techniques to $\mathfrak{S}_{\text{aug}_{v_j}(\mu)/\text{aug}_{v_j}(\nu)}$ to find the decomposition of $\mathfrak{S}_{\mu/\nu}$ into the complete homogeneous noncommutative symmetric functions.

Example 4.1. Let $\mu = (2, -5, 0, 1)$ and $\nu = (2, -3, 1, 6)$. To find $\mathfrak{S}_{\mu/\nu}$, first add 3 to every part of ν and every part of μ to get $\text{aug}_3(\mu)/\text{aug}_3(\nu) = (5, -2, 3, 4)/(5, 0, 4, 9)$. Next, since $\text{aug}_3(\nu)$ is not a partition, apply the straightening operator to get

$$\mathfrak{S}_{\mu/\nu} = -\mathfrak{S}_{(5,-2,3,4)/(5,0,8,5)} = \mathfrak{S}_{(5,-2,3,4)/(5,7,1,5)} = -\mathfrak{S}_{(5,-2,3,4)/(6,6,1,5)} = \mathfrak{S}_{(5,-2,3,4)/(6,6,4,2)}.$$

Corollary 4.2. For $1 \leq m \leq k$, we have

$$\mathfrak{S}_{\mu} = \sum_{\pi \in A_{k,m}} \epsilon(\pi) H_{\mu_1-1+\pi_1} \cdots H_{\mu_{m-1}-(m-1)+\pi_{m-1}} \mathfrak{S}_{\mu/\nu^{(m)}},$$

where $\nu^{(m)}$ is the partition obtained from the construction of the tunnel hooks corresponding to π in the diagram D_{μ} and $A_{k,m}$ is the set of all ordered m -element subsets of a k -element set.

For example, the skew immaculate $\mathfrak{S}_{(5,-1,3,4)/(3,1,0,0)}$ can be obtained by selecting the terms appearing in $\mathfrak{S}_{(3,3,3,5,-1,3,4)}$ whose first three terms are $H_7 H_4 H_2$. Applying the forgetful map to this expansion produces the expansion of a Schur function in terms of skew Schur functions with complete homogeneous symmetric functions as coefficients.

5 Ribbon decompositions of immaculate functions

Ribbons are a Schur-like basis for NSym related to Schur functions by the forgetful map; there are a number of excellent sources for background on ribbons in Sym and NSym [7, 10]. The ribbon basis expands positively into the immaculate basis via standard immaculate tableaux [2], but the expansion of the immaculate basis into the ribbon basis is only known for certain special cases. Campbell [4] provides formulas for the ribbon expansion of immaculate functions indexed by rectangles and immaculate functions indexed by products of two rectangles satisfying certain size conditions as follows.

Theorem 5.1. [4] The ribbon expansion of an immaculate function indexed by either a rectangle $\alpha = m^{\ell(\alpha)}$ or the product $\alpha = (a^b, c^d)$ of rectangles satisfying $b \leq c$ and $b \leq a$ is given by

$$\mathfrak{S}_{\alpha} = \sum_{\sigma \in S_{\ell(\alpha)}} \epsilon(\sigma) R_{(\alpha_1-1+\sigma_1, \alpha_2-2+\sigma_2, \dots, \alpha_{\ell(\alpha)}-\ell(\alpha)+\sigma_{\ell(\alpha)})}.$$

It is not true in general that the ribbon decomposition has the same indexing set as the homogeneous decomposition. One open question is to classify the compositions for which the indexing compositions are the same. We use tunnel hooks to extend Campbell's results to a much larger class of compositions.

Theorem 5.2. Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ be a composition for which there exists j with $1 \leq j \leq k$ such that $\alpha_i \geq i$ for $1 \leq i \leq j$, $\alpha_{j+1} \geq j$, and $\alpha_\ell = \alpha_{j+1}$ for $j+2 \leq \ell \leq k$. Then

$$\mathfrak{S}_\alpha = \sum_{\sigma \in S_k} \epsilon(\sigma) R_{(\alpha_1-1+\sigma_1, \alpha_2-2+\sigma_2, \dots, \alpha_k-k+\sigma_k)},$$

with the convention that R_α vanishes if α contains any nonpositive parts.

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Polyhedral and Tropical Geometry of Flag Positroids

Jonathan Boretsky^{*1}, Christopher Eur^{†1}, and Lauren Williams^{‡1}

¹Department of Mathematics, Harvard University, Cambridge, MA, 02138

Abstract. A *flag positroid* of ranks $\mathbf{r} : r_1 < \dots < r_k$ on n is a flag matroid that can be realized by a real $r_k \times n$ matrix A such that the $r_i \times r_i$ minors of A involving rows $1, 2, \dots, r_i$ are nonnegative for all $1 \leq i \leq k$. In this abstract, we explore the polyhedral and tropical geometry of flag positroids, particularly when $\mathbf{r} : a, a-1, \dots, b$ is a sequence of consecutive numbers. In this case we show that the nonnegative tropical flag variety $\text{TrFl}_{\mathbf{r},n}^{\geq 0}$ equals the nonnegative flag Dressian $\text{FIDr}_{\mathbf{r},n}^{\geq 0}$, and that the points $\mu = \mu_a, \dots, \mu_b$ of $\text{TrFl}_{\mathbf{r},n}^{\geq 0} = \text{FIDr}_{\mathbf{r},n}^{\geq 0}$ give rise to coherent subdivisions of flag positroid polytopes into (smaller) flag positroid polytopes. Our results have applications to Bruhat interval polytopes. For example, we show that a complete flag matroid polytope is a Bruhat interval polytope if and only if its ≤ 2 -dimensional faces are Bruhat interval polytopes. Our results also have applications to realizability questions. We define a *positively oriented flag matroid* to be a sequence of positively oriented matroids $\chi_1, \dots, \chi_k$ which is also an oriented flag matroid. We then prove that every positively oriented flag matroid of ranks $\mathbf{r} : a, a-1, \dots, b$ is realizable.

Keywords: Flag varieties, Tropical varieties, Positroids, Polytopal subdivisions, Matroid polytopes, Oriented flag matroids

1 Introduction

In recent years there has been a great deal of interest in the tropical Grassmannian [30, 19, 20, 14, 11], and subdivisions of matroid polytopes [33, 1, 15]. Further, there has been research into “positive” [29, 31, 2, 24, 25, 32, 4] and “flag” [35, 13, 12, 21, 22, 8] versions of the above objects. The aim of this paper is to illustrate the beautiful relationships between the nonnegative tropical flag variety, the nonnegative flag Dressian, and subdivisions of flag positroid polytopes, unifying and generalizing some of the existing results. We will

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[‡]williams@math.harvard.edu. L.W. is partially supported by the National Science Foundation (DMS-1854512 and DMS-2152991).

particularly focus on the case of flag varieties (respectively, flag positroids) consisting of subspaces (respectively, positroids) of *consecutive* ranks. This includes both Grassmannians and complete flag varieties.

For positive integers n and d with $d < n$, we let $[n]$ denote the set $\{1, \dots, n\}$ and we let $\binom{[n]}{d}$ denote the collection of all d -element subsets of $[n]$. Given a subset $S \subseteq \binom{[n]}{d}$, we let $\mathbf{e}_S$ denote the sum of standard basis vectors $\sum_{i \in S} \mathbf{e}_i$. For a collection $\mathcal{B} \subset \binom{[n]}{d}$, we let $P(\mathcal{B})$ be the convex hull of $\{\mathbf{e}_B : B \in \mathcal{B}\}$ in $\mathbb{R}^n$. The collection $\mathcal{B}$ is said to define a *matroid* M of rank d on $[n]$ if every edge of the polytope $P(\mathcal{B})$ is parallel to $\mathbf{e}_i - \mathbf{e}_j$ for some $i \neq j \in [n]$. In this case, we call $\mathcal{B}$ the set of *bases* of M , and define the *matroid polytope* $P(M)$ of M to be the polytope $P(\mathcal{B})$. When $\mathcal{B}$ indexes the nonvanishing Plücker coordinates of an element A of the Grassmannian $\text{Gr}_{d,n}(\mathbb{C})$, we say that A *realizes* M , and it is well-known that $P(\mathcal{B})$ is the moment map image of the closure of the torus orbit of A in the Grassmannian [17]. We assume familiarity with the fundamentals of matroid theory as in [28] and [10].

The above definition of a matroid in terms of its polytope is due to [17]. Flag matroids are natural generalizations of matroids that admit the following polytopal definition.

Definition 1.1. [10, Corollary 1.13.5 and Theorem 1.13.6] Let $\mathbf{r} = r_1, \dots, r_k$ be a sequence of increasing integers in $[n]$. A *flag matroid* of ranks $\mathbf{r}$ on $[n]$ is a sequence $M = M_1, \dots, M_k$ of matroids of ranks $r_1, \dots, r_k$ on $[n]$ such that all vertices of the polytope

$$P(M) = P(M_1) + \dots + P(M_k), \text{ the Minkowski sum of matroid polytopes,}$$

are equidistant from the origin. This polytope is called the *flag matroid polytope* of M .

Flag matroids are exactly the type A objects in the theory of Coxeter matroids [18, 10]. Just as a realization of a matroid is a point in a Grassmannian, a realization of a flag matroid is a point in a flag variety. More concretely, a *realization* of a flag matroid of ranks $r_1, \dots, r_k$ over a field is an $r_k \times n$ matrix A over that field such that for each $1 \leq i \leq k$, the first r_i rows of A is a realization of M_i . For an equivalent definition of flag matroids in terms of Plücker relations on partial flag varieties, see [21, Proposition A]. There are natural “positive” analogues of matroids, flag matroids, and their polytopes.

Definition 1.2. Let $\mathbf{r} = r_1, \dots, r_k$ be a sequence of increasing integers in $[n]$. We say that a flag matroid $M = M_1, \dots, M_k$ of ranks $\mathbf{r}$ on $[n]$ is a *flag positroid* if it has a realization by a real matrix A such that the $r_i \times n$ submatrix of A formed by the first r_i rows of A has all nonnegative minors for each $1 \leq i \leq k$.

We refer to the flag matroid polytope of a flag positroid as a *flag positroid polytope*. It follows from our definition above that flag positroids are realizable.

Setting $k = 1$ in Definition 1.2 gives the well-studied notion of *positroids* and *positroid polytopes* [29, 2]. Therefore each flag positroid is a sequence of positroids.

In recent years it has been gradually understood that the tropical geometry of the Grassmannian and flag variety, and in particular, the *Dressian* and *flag Dressian*, are intimately connected to (flag) matroid polytopes and their subdivisions [33, 19, 13, 27].

The tropical geometry of the *positive* Grassmannian and of the flag variety are particularly nice: the positive tropical Grassmannian equals the positive Dressian, whose cones in turn parameterize subdivisions of the hypersimplex into positroid polytopes [31, 32, 25, 4], while the positive tropical complete flag variety equals the positive complete flag Dressian, whose cones parameterize subdivisions of the permutohedron into *Bruhat interval polytopes* [8, 22].

Our main theorem, which follows, unifies and generalizes the above results. In particular, [Theorem 1.3](#) applies to the *nonnegative* flag variety of rank $\mathbf{r}$, where $\mathbf{r}$ consists of any consecutive integers. This includes both the Grassmannian case, when $\mathbf{r} = d$ and the complete flag case, when $\mathbf{r} = 1, 2, \dots, n$. Much of the terminology appearing in this theorem statement will be defined in [Section 2](#).

Theorem 1.3. Suppose $\mathbf{r}$ is a sequence of consecutive integers $a, \dots, b$ for some $1 \leq a \leq b \leq n$. Then, for $\underline{\mu} = \mu_a, \dots, \mu_b \in \prod_{i=a}^b \mathbb{P}(\mathbb{T}^n_i)$, the following are equivalent:

- (a) $\underline{\mu} \in \text{TrFl}_{\mathbf{r},n}^{\geq 0}$, the nonnegative tropicalization of the flag variety, i.e. the closure of the coordinate-wise valuation of points in $\text{Fl}_{\mathbf{r},n} \subset \mathbb{C}_{\geq 0}^n$.
- (b) $\underline{\mu} \in \text{FlDr}_{\mathbf{r},n}^{\geq 0}$, the nonnegative flag Dressian, i.e. the “solutions” to the positive-tropical Grassmann-Plücker and incidence-Plücker relations.
- (c) Every face in the coherent subdivision $\mathcal{D}_{\underline{\mu}}$ of the polytope $P_{\underline{\mu}} = P_{\underline{\mu}_1} \cdots P_{\underline{\mu}_k}$ induced by $\underline{\mu}$ is a flag positroid polytope (of rank $\mathbf{r}$), where $\underline{\mu}_i$ is the support of $\underline{\mu}_i$ for $1 \leq i \leq k$.
- (d) Every face of dimension at most 2 in the subdivision $\mathcal{D}_{\underline{\mu}}$ of $P_{\underline{\mu}}$ is a flag positroid polytope (of rank $\mathbf{r}$).
- (e) The support $\underline{\mu}$ of $\underline{\mu}$ is a flag matroid, and $\underline{\mu}$ satisfies every three-term positive-tropical incidence relation (respectively, Grassmann-Plücker relation) when $a < b$ (respectively, $a = b$).

We present here three useful corollaries of this result.

Corollary 1.4. Suppose $\mathbf{r}$ is a sequence of consecutive integers and let $\underline{\mu} \in \text{FlDr}_{\mathbf{r},n}^{\geq 0}$. Then $P_{\underline{\mu}}$ is a flag positroid polytope.

We note that in the Grassmannian case, that is, the case that $\mathbf{r} = d$ is a single integer, [Theorem 1.3](#) and [Corollary 1.4](#) describe the relationship between the nonnegative tropical Grassmannian, the nonnegative Dressian, and subdivisions of positroid polytopes (e.g. the hypersimplex, if $\underline{\mu}$ has no coordinates equal to ∞) into positroid polytopes. Moreover, when $\mathbf{r} = 1, 2, \dots, n$, [Theorem 1.3](#) and [Corollary 1.4](#) describe the relationship between the nonnegative tropical complete flag variety, the nonnegative complete flag Dressian,

and subdivisions of Bruhat interval polytopes (e.g. the permutohedron, if μ has no coordinates equal to ∞) into Bruhat interval polytopes. We illustrate this relationship in the case where μ has no coordinates equal to ∞ in Figure 1.

We now present an application of Theorem 1.3 to flag matroid polytopes.

Corollary 1.5. For a flag matroid $M = M_a, M_{a+1}, \dots, M_b$ of consecutive ranks $\mathbf{r}$, its flag matroid polytope $P(M)$ is a flag positroid polytope if and only if its ≤ 2 -dimensional faces are flag positroid polytopes (of rank $\mathbf{r}$).

In the Grassmannian case, the flag positroid polytopes of rank $\mathbf{r} = d$ are precisely the positroid polytopes and Corollary 1.5 appeared as [25, Theorem 3.9].

It has been shown that all *positively oriented matroids* are positroids, which is to say, realizable by a matrix with nonnegative maximal minors [3][32]. It is natural to ask when a sequence of positroids is a flag positroid, which is to say, realizable as in Definition 1.2. Note however that questions of realizability for flag matroids are rather subtle, as demonstrated in Example 2.14. By working with *oriented* flag matroids, we give an answer to this realizability question for flag positroids, in the case of consecutive ranks.

Corollary 1.6. Suppose $M_1, \dots, M_k$ is a sequence of positroids on n of consecutive ranks. Then, considered as a sequence of positively oriented matroids, $M_1, \dots, M_k$ is a flag positroid if and only if it is an oriented flag matroid.

We define a *positively oriented flag matroid* to be a sequence of positively oriented matroids $\chi_1, \dots, \chi_k$ which is also an oriented flag matroid. Corollary 1.6 then says that every positively oriented flag matroid of consecutive ranks is realizable. We do not know whether Corollary 1.6 holds if $\mathbf{r}$ fails to satisfy the consecutive rank condition.

The structure of this abstract is as follows. In Section 2, we define the terms and concepts appearing in Theorem 1.3. In Section 3, we present further results related to Theorem 1.3, including an application to generalized Bruhat interval polytopes and comments on what can be said about general $\mathbf{r}$. This is an extended abstract of the full article [9].

2 Background and Definitions

2.1 Nonnegative Flag Varieties

Let $n \in \mathbb{Z}$ and let $\mathbf{r} = \{r_1 < \dots < r_k\} \subseteq n$. For a field $\mathbb{k}$, let $G = \mathrm{GL}_n(\mathbb{k})$, and let $P_{\mathbf{r};n}(\mathbb{k})$ denote the parabolic subgroup of G of block upper-triangular matrices with diagonal blocks of sizes $r_1, r_2 - r_1, \dots, r_k - r_{k-1}, n - r_k$. We define the *partial flag variety*

$$\mathrm{Fl}_{\mathbf{r};n}(\mathbb{k}) := \mathrm{GL}_n(\mathbb{k}) / P_{\mathbf{r};n}(\mathbb{k}).$$

As usual, we identify $\mathrm{Fl}_{\mathbf{r};n}(\mathbb{k})$ with the variety of partial flags of subspaces in $\mathbb{k}^n$:

$$\mathrm{Fl}_{\mathbf{r};n}(\mathbb{k}) = \{V_1 \subset \dots \subset V_k : V_i \text{ a linear subspace of } \mathbb{k}^n \text{ of dimension } r_i \text{ for } i = 1, \dots, k\}$$

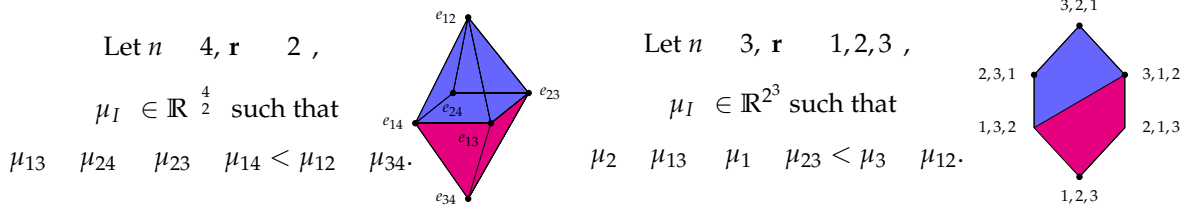


Figure 1: Coherent subdivision of the $n = 4, r = 2$ hypersimplex into positroid polytopes induced by a point $\mu \in \text{FIDr}_{2,4}^{\geq 0}$ (left) and coherent subdivision of the $n = 3$ permutohedron into flag positroid polytopes induced by a point $\mu \in \text{FIDr}_{1,2,3,3}^{\geq 0}$ (right).

We write $\text{Fl}_n \mathbb{k}$ for the complete flag variety $\text{Fl}_{1,2,\dots,n;n} \mathbb{k}$. Note that $\text{Fl}_n \mathbb{k}$ can be identified with $\text{GL}_n \mathbb{k} / B \mathbb{k}$, where $B \mathbb{k}$ is the subgroup of upper-triangular matrices. There is a natural projection π from $\text{Fl}_n \mathbb{k}$ to any partial flag variety by simply forgetting some of the subspaces.

If A is an $r_k \times n$ matrix such that V_{r_i} is the span of the first r_i rows, we say that A is a *realization* of V : $V_1 \subset \dots \subset V_k \in \text{Fl}_{\mathbf{r},n}$. Given any realization A of V and any $1 \leq i \leq k$, we have the *Plücker coordinates* or *flag minors* $p_I A$ where $I \in \binom{n}{r_i}$; concretely, $p_I A$ is the determinant of the submatrix of A occupying the first r_i rows and columns I . This gives the Plücker embedding of $\text{Fl}_{\mathbf{r},n} \mathbb{k}$ into $\mathbb{P}^{r_1-1} \times \dots \times \mathbb{P}^{r_k-1}$ taking V to $\left((p_I A)_{I \in \binom{n}{r_1}}, \dots, (p_I A)_{I \in \binom{n}{r_k}} \right)$.

Definition 2.1. For integers $0 < r \leq s < n$, the (single-exchange) Plücker relations of type $r, s; n$ are polynomials in variables $\{x_I : I \in \binom{n}{r} \cup \binom{n}{s}\}$ defined as

$$\mathcal{P}_{r,s;n} = \left\{ \sum_{j \in J \setminus I} \text{sign } j, I, J x_{I \cup j} x_{J \setminus j} \mid I \in \binom{n}{r-1}, J \in \binom{n}{s+1} \right\},$$

where $\text{sign } j, I, J = (-1)^{|\{k \in J \mid k < j\}| - |\{i \in I \mid j < i\}|}$. When $r = s$, the elements of $\mathcal{P}_{r,r;n}$ are called the *Grassmann-Plücker relations* (of type $r; n$), and when $r < s$, the elements of $\mathcal{P}_{r,s;n}$ are called the *incidence-Plücker relations* (of type $r, s; n$).

As in the introduction, let $\mathbf{r} = r_1 < \dots < r_k$ be a sequence of increasing integers in n . We let $\mathcal{P}_{\mathbf{r},n} = \bigcup_{r,s \in \mathbf{r}} \mathcal{P}_{r,s;n}$, and let $\langle \mathcal{P}_{\mathbf{r},n} \rangle$ be the ideal generated by $\mathcal{P}_{\mathbf{r},n}$. It is well-known that for any field $\mathbb{k}$ the ideal $\langle \mathcal{P}_{\mathbf{r},n} \rangle$ set-theoretically carves out the partial flag variety $\text{Fl}_{\mathbf{r},n} \mathbb{k}$ embedded in $\prod_{i=1}^k \mathbb{P} \left(\mathbb{k}^{\binom{n}{r_i}} \right)$ via the Plücker embedding [16, §9].

We refer to the subset of relations in $\mathcal{P}_{\mathbf{r};n}$ which are polynomial relations containing precisely three terms as the *three-term Plücker relations*. We will now often drop the $\mathbb{k}$ from our notation, with the understanding that $\mathbb{k} = \mathbb{R}$.

Definition 2.2. We say that a real matrix is *totally positive* if all of its minors are positive. We let $\mathrm{GL}_n^{>0}$ denote the subset of GL_n of totally positive matrices.

There are two natural ways to define positivity for partial flag varieties. The first notion comes from work of Lusztig [26]. The second notion uses Plücker coordinates, and was initiated in work of Postnikov [29].

Definition 2.3. We define the (*Lusztig*) *positive part* of $\mathrm{Fl}_{\mathbf{r};n}$, denoted by $\mathrm{Fl}_{\mathbf{r};n}^{>0}$, as the image of $\mathrm{GL}_n^{>0}$ inside $\mathrm{Fl}_{\mathbf{r};n} = \mathrm{GL}_n / \mathrm{P}_{\mathbf{r};n}$. We define the (*Lusztig*) *nonnegative part* of $\mathrm{Fl}_{\mathbf{r};n}$, denoted by $\mathrm{Fl}_{\mathbf{r};n}^{\geq 0}$, as the closure of $\mathrm{Fl}_{\mathbf{r};n}^{>0}$ in the Euclidean topology.

We define the *Plücker positive part* (respectively, *Plücker nonnegative part*) of $\mathrm{Fl}_{\mathbf{r};n}$ to be the subset of $\mathrm{Fl}_{\mathbf{r};n}$ where all Plücker coordinates are positive (respectively, nonnegative).¹

It is well-known that the Lusztig positive part of $\mathrm{Fl}_{\mathbf{r};n}$ is a subset of the Plücker positive part of $\mathrm{Fl}_{\mathbf{r};n}$, and that the two notions agree in the case of the Grassmannian [34, Corollary 1.2]. The two notions also agree in the case of the complete flag variety [8, Theorem 5.21]. More generally, we have the following.

Theorem 2.4. [7, Theorem 1.1] The Lusztig positive (respectively, Lusztig nonnegative) part of $\mathrm{Fl}_{\mathbf{r};n}$ equals the Plücker positive (respectively, Plücker nonnegative) part of $\mathrm{Fl}_{\mathbf{r};n}$ if and only if the set $\mathbf{r}$ consists of consecutive integers.

See [7, Section 1.4] for more references and a nice discussion of the history. Since in this abstract we will be mainly studying the case where $\mathbf{r}$ consists of consecutive integers, we will use the two notions interchangeably when there is no ambiguity.

2.2 Tropical Geometry

Before discussing tropical varieties, we introduce the tropical hyperfield and what it means to find a solution of a tropical polynomial. For more details, see [27] and [31].

Definition 2.5. Let $\mathbb{T} = \mathbb{R} \cup \{\infty\}$ be the set underlying the tropical hyperfield, endowed with the topology such that $-\log : \mathbb{R}_{\geq 0} \rightarrow \mathbb{T}$ is a homeomorphism. Given a point $w \in \mathbb{T}^{\binom{n}{r}}$, we define the *support* of w to be $\underline{w} = \{S \in \binom{[n]}{r} : w_S \neq \infty\}$. When $\underline{w}$ is the set of bases of a matroid, we identify $\underline{w}$ with that matroid. Let $\mathbb{P}\left(\mathbb{T}^{\binom{n}{r}}\right)$ be the tropical projective space of $\mathbb{T}^{\binom{n}{r}}$, which is defined as $(\mathbb{T}^{\binom{n}{r}} \setminus \{\infty, \dots, \infty\}) / \sim$, where $w \sim w'$ if $w = w' \cdot c, \dots, c$ for some $c \in \mathbb{R}$.

¹The reader who is concerned about the fact that we are working with projective coordinates can replace “all Plücker coordinates are nonnegative” by “all nonzero Plücker coordinates have the same sign”.

For a point $w = w_1, \dots, w_m \in \mathbb{T}^m \setminus \{\infty, \dots, \infty\}$, we write $\bar{w}$ for its image in $\mathbb{P}^{\mathbb{T}^m}$. For $a = a_1, \dots, a_m \in \mathbb{Z}^m$, write $a \bullet w = a_1 w_1 \cdots a_m w_m$.

Definition 2.6. For a real homogeneous polynomial

$$f = \sum_{a \in \mathcal{A}} c_a x^a \in \mathbb{R}[x_1, \dots, x_m], \quad \text{where } \mathcal{A} \subset \mathbb{Z}_{\geq 0}^m \text{ is finite and } c_a \in \mathbb{R} \setminus \{0\} \text{ for all } a \in \mathcal{A},$$

the *extended tropical hypersurface* $V_{\text{trop}} f$ and the *nonnegative tropical hypersurface* $V_{\text{trop}}^{\geq 0} f$ are subsets of the tropical projective space $\mathbb{P}^{\mathbb{T}^m}$ defined by

$$V_{\text{trop}} f = \left\{ \bar{w} \in \mathbb{P}^{\mathbb{T}^m} \mid \begin{array}{l} \text{the minimum in } \min_{a \in \mathcal{A}} a \bullet w, \text{ if finite, occurs at least twice} \end{array} \right\},$$

and

$$V_{\text{trop}}^{\geq 0} f = \left\{ \bar{w} \in \mathbb{P}^{\mathbb{T}^m} \mid \begin{array}{l} \text{the minimum in } \min_{a \in \mathcal{A}} a \bullet w, \text{ if finite, occurs at least twice,} \\ \text{at some } a, a' \in \mathcal{A} \text{ such that } c_a \text{ and } c_{a'} \text{ have opposite signs} \end{array} \right\}.$$

We say that a point *satisfies the tropical relation* of f if it is in $V_{\text{trop}} f$, and that it *satisfies the positive-tropical relation* of f if it is in $V_{\text{trop}}^{\geq 0} f$.

When f is a multihomogeneous real polynomial, we define $V_{\text{trop}} f$ and $V_{\text{trop}}^{\geq 0} f$ similarly as subsets of a product of tropical projective spaces. We will consider tropical hypersurfaces of Plücker relations, yielding tropical analogues of partial flag varieties.

Definition 2.7. The *tropicalization* $\text{TrFl}_{\mathbf{r},n}$ of $\text{Fl}_{\mathbf{r},n}$, the *nonnegative tropicalization* $\text{TrFl}_{\mathbf{r},n}^{\geq 0}$ of $\text{Fl}_{\mathbf{r},n}$, the *flag Dressian* $\text{FIDr}_{\mathbf{r},n}$, and the *nonnegative flag Dressian* $\text{FIDr}_{\mathbf{r},n}^{\geq 0}$ are subsets of $\prod_{i=1}^k \mathbb{P}^{\mathbb{T}^{\binom{n}{r_i}}}$ defined as

$$\begin{aligned} \text{TrFl}_{\mathbf{r},n} &= \bigcap_{f \in \langle \mathcal{P}_{\mathbf{r},n} \rangle} V_{\text{trop}} f \quad \text{and} \quad \text{TrFl}_{\mathbf{r},n}^{\geq 0} = \bigcap_{f \in \langle \mathcal{P}_{\mathbf{r},n} \rangle} V_{\text{trop}}^{\geq 0} f, \\ \text{FIDr}_{\mathbf{r},n} &= \bigcap_{f \in \mathcal{P}_{\mathbf{r},n}} V_{\text{trop}} f \quad \text{and} \quad \text{FIDr}_{\mathbf{r},n}^{\geq 0} = \bigcap_{f \in \mathcal{P}_{\mathbf{r},n}} V_{\text{trop}}^{\geq 0} f. \end{aligned}$$

When $\mathbf{r} = d$ consists of a single integer, one obtains the (nonnegative) tropicalization of the Grassmannian and the (nonnegative) Dressian studied in [30, 31, 32, 4].

Remark 2.8. We record a useful equivalent description of the (nonnegative) tropicalization of a partial flag variety using Puiseux series.

Definition 2.9. Let $\mathcal{C} = \mathbb{C}\{\{t\}\}$ be the field of Puiseux series with coefficients in $\mathbb{C}$, with the usual valuation map $\text{val} : \mathcal{C} \rightarrow \mathbb{T}$. Concretely, for $f \neq 0$, $\text{val } f$ is the exponent of the initial term of f , and $\text{val } 0 = \infty$. Let

$$\mathcal{C}_{>0} = \{f \in \mathcal{C} : \text{the initial coefficient of } f \text{ is real and positive}\} \text{ and } \mathcal{C}_{\geq 0} = \mathcal{C}_{>0} \cup \{0\}.$$

The valuation val can also be applied coordinate wise, yielding a map $\text{val} : \mathcal{C}^n \rightarrow \mathbb{T}^n$,

Proposition 2.10. [27, Theorems 3.2.3 & 6.2.15][31, Proposition 2.2] We have

$\text{TrFl}_{\mathbf{r};n}$ the closure of $\{\text{val } p : \{0, \dots, 0\} / p \in \text{Fl}_{\mathbf{r};n} \mathcal{C}\}$ in $\prod_{i=1}^k \mathbb{P}\left(\mathbb{T}^{\binom{n}{r_i}}\right)$ and
 $\text{TrFl}_{\mathbf{r};n}^{\geq 0}$ the closure of $\{\text{val } p : \{0, \dots, 0\} / p \in \text{Fl}_{\mathbf{r};n} \mathcal{C}_{\geq 0}\}$ in $\prod_{i=1}^k \mathbb{P}\left(\mathbb{T}^{\binom{n}{r_i}}\right)$.

2.3 Coherent Subdivisions

Consider a point $\underline{\mu} = \mu_1, \dots, \mu_k \in \prod_{i=1}^k \mathbb{P}\left(\mathbb{T}^{\binom{n}{r_i}}\right)$ such that its support $\underline{\mu}$ is a flag matroid. By construction, the vertices of the flag matroid polytope $P_{\underline{\mu}}$ have the form $\mathbf{e}_{B_1} \cdots \mathbf{e}_{B_k}$ where B_i is a basis of the matroid $\underline{\mu}_i$ for each $i = 1, \dots, k$.

Definition 2.11. We define $\mathcal{D}_{\underline{\mu}}$ to be the coherent subdivision of $P_{\underline{\mu}}$ induced by assigning each vertex $\mathbf{e}_{B_1} \cdots \mathbf{e}_{B_k}$ of $P_{\underline{\mu}}$ the weight $\mu_1 |B_1| \cdots \mu_k |B_k|$. That is, the faces of $\mathcal{D}_{\underline{\mu}}$ correspond to the faces of the lower convex hull of the set of points

$$\{\mathbf{e}_{B_1} \cdots \mathbf{e}_{B_k}, \mu_1 |B_1| \cdots \mu_k |B_k| \in \mathbb{R}^n \times \mathbb{R} : \mathbf{e}_{B_1} \cdots \mathbf{e}_{B_k} \text{ a vertex of } P_{\underline{\mu}}\}.$$

In [13, Theorem A], it is shown that a point $\underline{\mu} \in \prod_{i=1}^k \mathbb{P}\left(\mathbb{T}^{\binom{n}{r_i}}\right)$ is in the flag Dressian $\text{FlDr}_{\mathbf{r};n}$ if and only if the all faces of the subdivision $\mathcal{D}_{\underline{\mu}}$ are flag matroid polytopes.

2.4 Oriented Flag Matroids

Let $\mathbb{S} = \{-1, 0, 1\}$ be the *hyperfield of signs*, introduced in [5]. For a polynomial $f = \sum_{a \in \mathcal{A}} c_a x^a \in \mathbb{R}[x_1, \dots, x_m]$, we say that an element $\chi \in \mathbb{S}^m$ is *in the null set* of f if the set $\{\text{sign } c_a \chi^a\}_{a \in \mathcal{A}}$ is either $\{0\}$ or contains $\{-1, 1\}$.

Definition 2.12. An *oriented matroid* of rank r on $[n]$ is a point $\chi \in \mathbb{S}^{\binom{n}{r}}$, called a *chirotope*, which is in the null set of each $f \in \mathcal{P}_{r;n}$. Similarly, an *oriented flag matroid* of ranks $\mathbf{r}$ is a point $\chi = \chi_1, \dots, \chi_k \in \prod_{i=1}^k \mathbb{S}^{\binom{n}{r_i}}$ such that χ is in the null set of each $f \in \mathcal{P}_{\mathbf{r};n}$.

While these definitions may seem different from those in the standard reference [6] on oriented matroids, one can show that Definition 2.12 is equivalent to [6, Definition 3.5.3].

Definition 2.13. A *positively oriented matroid* is an oriented matroid χ such that χ only takes values 0 or 1. Similarly, we define a *positively oriented flag matroid* to be an oriented flag matroid χ such that χ only takes values 0 or 1.

A positroid M defines a positively oriented matroid $\chi = \chi_M$ where χ takes value 1 on its bases and 0 otherwise. Every positively oriented matroid χ is realizable, i.e. has the form χ_M for some positroid M [3]. Thus, each positively oriented flag matroid is a sequence of positroids which is also an oriented flag matroid. However, we remark that imposing the oriented flag matroid condition, as in [Corollary 1.6](#), is stronger than imposing that we have a realizable flag matroid whose consistent matroids are positroids.

Example 2.14. We give an example of a realizable flag matroid that has positroids as its constituent matroids but is not a flag positroid. This example also appeared in [22, Example 5] and [7, Example 6]. Let M, M' be matroids of ranks 1 and 2 on 3 whose sets of bases are $\{1, 3\}$ and $\{12, 13, 23\}$, respectively. Both are positroids. We can realize M, M' as a flag matroid using the matrix

$$\begin{bmatrix} a & 0 & b \\ c & d & e \end{bmatrix},$$

where the minors $a, b, ad, -bd, ae - bc$ are nonzero. In order to realize M, M' as a flag positroid, we need all these minors to be strictly positive, which is impossible. Since M, M' does *not* form an oriented flag matroid, this is consistent with [Corollary 1.6](#).

3 Further Results

We begin by presenting two results that are crucial to the proof [Theorem 1.3](#), and which are also interesting in their own rights.

Theorem 3.1. Suppose $\underline{\mu} = (\mu_1, \mu_2) \in \mathbb{P} \left(\mathbb{T}^{\binom{n}{r}} \right) \times \mathbb{P} \left(\mathbb{T}^{\binom{n}{r-1}} \right)$ satisfies every three-term positive-tropical incidence relation. If the support $\text{supp } \underline{\mu}$ is a flag matroid, then $\underline{\mu} \in \text{FIDr}_{r,r-1;n}^{\geq 0}$.

We also expect that our proof of [Theorem 3.1](#) in [9] adapts well to arbitrary perfect hyperfields, in the language of [5]. The following result gives a characterization of positively oriented flag matroids (and equivalently, by [Corollary 1.6](#), of flag positroids).

Theorem 3.2. The set of positively oriented flag matroids of ranks $\mathbf{r}$ can be identified with the set of points of the nonnegative flag Dressian $\text{FIDr}_{\mathbf{r};n}^{\geq 0}$ whose coordinates are all either 0 or ∞ .

We continue with a further corollary of [Theorem 1.3](#). A *generalized Bruhat interval polytope* [35, Definition 7.8 and Lemma 7.9] can be defined as the moment map image of the closure of the torus orbit of a point A in the nonnegative part $G/P^{\geq 0}$ (in the sense of Lusztig) of a flag variety G/P . When $\mathbf{r}$ is a sequence of consecutive integers, it then follows from [7] that generalized Bruhat interval polytopes for $\text{Fl}_{\mathbf{r};n}^{\geq 0}$ are precisely the flag

positroid polytopes of ranks $\mathbf{r}$. In the complete flag case, a generalized Bruhat interval polytope is just a *Bruhat interval polytope* [23], that is, the convex hull of the permutation vectors $z_1, \dots, z_n$ for all permutations z lying in some Bruhat interval u, v . Thus, we can now restate [Corollary 1.5](#) as follows.

Corollary 3.3. For a flag matroid on n of consecutive ranks $\mathbf{r}$, its flag matroid polytope is a generalized Bruhat interval polytope if and only if its ≤ 2 -dimensional faces are generalized Bruhat interval polytopes. In particular, for a complete flag matroid on n , its flag matroid polytope is a Bruhat interval polytope if and only if its ≤ 2 -dimensional faces are Bruhat interval polytopes.

As a final point, we comment on what happens when $\mathbf{r}$ does not consist of consecutive integers. A key fact we use about flag positroids of consecutive ranks is that they can be extended to complete flag positroids. This is no longer necessarily true if $\mathbf{r}$ does not consist of consecutive integers.

Example 3.4. Let the sets of bases of M and M' be $\{1, 2, 3, 4\}$ and $\{123, 234\}$, respectively. These form a flag positroid (M, M') on 4 of ranks $(1, 3)$. After row reduction, a realization of this flag matroid can be written as

$$\begin{bmatrix} 1 & a & b & c \\ 0 & x & y & 0 \\ 0 & z & w & 0 \end{bmatrix}$$

where $a, b, c > 0$ and $xw - yz > 0$. The minors of the matrix formed by the first two rows include $x, y, -cx, -cy$, which cannot be all nonnegative since $c > 0$ and not both of x and y are zero. Thus, there is no flag positroid (M, M_2, M') with rank of M_2 equal to 2.

Passing through complete flag positroids is a key step in our proof of [Theorem 1.3](#) and, in fact, most of [Theorem 1.3](#) no longer holds if $\mathbf{r}$ does not consist of consecutive integers. However, we still have the following:

Lemma 3.5. For any $\mathbf{r}$, the modification of (c) of [Theorem 1.3](#) to:

- (c') Every face in the coherent subdivision $\mathcal{D}_\mu$ of $P_{\underline{\mu}}$ is the flag matroid polytope of a positively oriented flag matroid.

is equivalent to (b) of [Theorem 1.3](#).

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A new perspective on positivity in (consecutive) permutation patterns

Natasha Blitvić^{*1}, Mohammed Slim Kammoun^{†2},
and Einar Steingrímsson^{‡3}

¹*School of Mathematical Sciences, Queen Mary University of London, London UK*

²*Department of Mathematics and Statistics, Lancaster University, UK, and Institut de Mathématiques de Toulouse, Toulouse France*

³*Department of Mathematics and Statistics, University of Strathclyde, Glasgow UK*

Abstract. We present a point of view on consecutive permutation patterns that interprets these in terms of (1) natural generalizations of the descent set of a permutation, (2) paths of a k -dependent point process, (3) refined clusters in the cluster method, and, surprisingly, (4) as conjectured moments of probability measures on the real line. At the heart of this paper is a recursive enumeration formula that allows us to get a grip on the aforementioned quantities and further enables us to formulate and numerically verify the conjecture (4), which provides a new unifying perspective on moment sequences arising from the study of permutation patterns.

Keywords: permutation patterns, descent set, point processes, moment sequences

1 Introduction

There are times in mathematics, when the conditions are right and a certain kind of threshold has been reached, that an idea crystalizes in plain sight. In the past several years, different groups of authors [4, 6, 7, 13, 18] have independently observed that several classes of integer sequences enumerating combinatorial objects, in particular permutations avoiding fixed permutation patterns, are moment sequences of positive Borel measures on the real line.

The Catalan numbers offer a canonical example. For concreteness, fix a permutation $p \in S_3$. (Avoidance of permutation patterns of length three is well understood, in contrast to patterns of length four or greater.) We recall that for $\sigma \in S_n$ ($n \in \mathbb{N}$), σ *avoids the classical permutation pattern p* if there are no indices $1 \leq i < j < k \leq n$ such that

^{*}n.blitvic@qmul.ac.uk. Partially supported by grant EP/V048902/1 from The Engineering and Physical Sciences Research Council and RPG-2020-103 from the Leverhulme Trust.

[†]medslimkammoun@gmail.com. Partially supported by grant RPG-2020-103 from the Leverhulme Trust.

[‡]einar@alum.mit.edu.

the image of i, j, k under σ is order-equivalent to p . (For example, the permutation 243165, in the one-line notation, avoids the classical permutation pattern $p = 312$; in contrast, 352614 fails to do so, containing an occurrence of p on indices (2,3,6), among others.) Combining the well-known enumeration result [17] for patterns of length three with the famous integral representation of the Catalan numbers as the (even) moments of Wigner's semicircle law, we obtain that

$$\#\{\sigma \in S_n \mid \sigma \text{ avoids } p\} = \int_{-2}^2 x^{2n} \frac{\sqrt{4-x^2}}{2\pi} dx.$$

Although no general conjecture has formally been published, it is fair to say there is belief among several aforementioned authors (including us), further reinforced by numerical evidence [6, 7, 13], that the numbers of permutations avoiding any given *classical* permutation pattern give a moment sequences. If true, this conjecture would pave the way to more efficient asymptotics [13] and place the study of permutation patterns within a broader probabilistic and combinatorial context [4].

But what of the consecutive patterns?

Even for consecutive patterns of length three, that is the patterns whose occurrence is predicated on the existence of a triple i, j, k of now *consecutive* indices, i.e. where $i = j - 1 = k - 2$, the picture is significantly less clear. In particular, a straightforward computation of determinants (see [4, p. 5486]) shows that the sequence enumerating permutations avoiding the consecutive pattern 132 is not a moment sequence. This is in contrast to the consecutive pattern 123, which gives rise to moment sequences even in its distributional form, whenever the variable carrying the number of occurrences is specialized to a real number. ([4, Corollary 4])

And yet, there is a general way of looking at consecutive permutation patterns that does seem to uniformly give rise to moment sequences. **Presenting this point of view**, as well as providing the **enumerative formulas** that supply the numerical evidence to support this **new conjecture**, are the main goals of this paper.

1.1 A different look at consecutive permutation patterns

In the following, we regard permutations as words $a_1 a_2 \dots a_n$ on distinct integers, usually on the set $[n] = \{1, 2, \dots, n\}$. We recall that a *descent* in a permutation is an i such that $a_i > a_{i+1}$. The *descent set* of a permutation σ is the set $\{i \mid a_i > a_{i+1}\}$, that is, the set of indices at which σ has descents.

In a seminal paper [14] Gessel and Viennot gave a formula, in terms of determinants, for the number of permutations of length n with any given descent set (a special case of the celebrated Lindström-âĂŞGessel-âĂŞViennot Lemma). The descent set was later interpreted probabilistically by Borodin, Diaconis, and Fulman, as a sample path of a

stationary one-dependent determinantal point process [5], providing a canonical example in a broader theory of such processes. An obvious question that doesn't seem to have received much attention is what type of combinatorial or probabilistic structure is achievable when, instead of looking at descents, we count permutations according to each starting point of an occurrence of a given *consecutive pattern*.

In this paper we present a recursive formula for the number of permutations of length n that are *covered* by an arbitrary given consecutive pattern p of length k with *overlaps* m (which we now define), and provide explicit closed formulas for several families of such patterns.

Definition 1. We say that a consecutive pattern p *covers* a permutation π with *overlaps* m if there is a sequence of occurrences of p in π containing between them all letters of π and such that two successive occurrences in the sequence share exactly m letters of π . We say that the occurrences in this sequence are *prescribed by the covering by p with overlaps m* .

As an example, the permutation 14263758 is covered by 1324 with overlaps 2, since 1426, 2637 and 3758 are all occurrences of 1324. Note that permutations covered by a pattern form a *cluster*, in the sense of the cluster method of Goulden and Jackson, although the method as normally used in relation to consecutive permutation patterns (see e.g. [3, 11, 12]) does not keep track of the positions of the occurrences of a pattern.

It is important to highlight that we count permutations that have occurrences of a pattern p in prescribed positions, but disregard whether p might occur in more places. (In other words, we count permutations with occurrences in "at least" the prescribed positions.) This is the natural approach from the point of view of point processes. Indeed, our enumerative formula gives the *correlation functions for a stationary k -dependent point process on positive integers* that encodes the occurrences of consecutive patterns occurring in uniformly distributed permutations. Of course, whether it is possible to have additional occurrences of a pattern depends on the length of the pattern p , the length of the overlaps and the *auto-correlation* of p , that is, the maximum possible overlap of two occurrences of p in a permutation. In order for a pattern p of length k to possibly occur in more places in a permutation than those prescribed by overlaps $m \geq 1$, p must have auto-correlation at least $k - m / 2$.

Furthermore, our formula in [Theorem 2](#) requires every letter of a permutation to belong to an occurrence of the pattern and that two successive occurrences overlap in the permutation. However, allowing gaps between occurrences, or two occurrences being adjacent without overlap, simply adds trivially computed multiplicative factors to the formula (see remark following the main theorem). The formula can also be extended to count permutations with the pattern occurring in any set of positions, rather than at regular intervals.

The proof of [Theorem 2](#) is based on a generalization of the 'positive approach' of Baxter, Nakamura and Zeilberger [2]. They keep track only of the number of occur-

rences of a pattern, but not their positions (which are essential here). Also worth noting is a similar argument preceding Equation (1) in Section 2.1 in [3]. Implicit in our argument, in the special case where the overlap is 1, is what Elizalde conjectured [10, p. 123] and was proved independently by Duane and Remmel [9, p. 31] and by Dotsenko and Khoroshkin [8, Theorem 28], that the number of permutations of a given length with a given number of occurrences with overlap 1 depends only on the first and last letter of the pattern. To be precise, they assume the auto-correlation of the pattern is 1, which implies that this counts all occurrences of the pattern, since two occurrences can overlap in at most one letter.

In addition to our main result we give as corollaries several results for some infinite families of patterns, and in some of these cases also bijective connections to other combinatorial structures. There are some recurring themes here, such as the “stretched factorials”

$$n!_k : \prod_{0 \leq i < n/k} n - ik$$

for different k . Some of these results are listed in Table 1. Among these sequences, there is a surprising number of well-known moment sequences, including the Catalan numbers (moments of the semicircle law), Fuss-Catalan numbers (corresponding to the so-called free Bessel laws), stretched factorials (associated with Gamma distributions), and others. The recursive formula of Theorem 2 in turn allows to efficiently compute the Hankel determinants [15] of the lesser-known sequences arising from Theorem 2. What we observe appears to strongly support the following conjecture:

Conjecture 1. *Fix a consecutive permutation pattern p of length k . Let $a_j^{p,m}$ be the number of permutations of length $k + jk - m$ covered by p with overlaps m ($m \leq k$). Then, $a_j^{p,m}_{j \geq 0}$ is the moment sequence of some probability measure on the real line.*

We began investigating the positivity of these sequences based on intuition from non-commutative probability that suggested a notion of multiplication based on the ‘concatenation’ of patterns. The subsequent numerical evidence afforded by Theorem 2 greatly strengthens this intuition. Ultimately, Conjecture 1 provides a consistent perspective on the positivity of sequences arising from the study of consecutive permutation patterns, mirroring — in a non-obvious way — the behaviors observed for classical permutation patterns. Namely, whereas it is the *avoidance* of *classical* permutation patterns that appears to give rise to familiar sequences such as the Catalan numbers and a host of other sequences all of which are conjectured to be moment sequences, it is the *packing* of the occurrences of *consecutive* permutation patterns that gives rise to the analogous positivity phenomena.

This is an extended abstract of a paper in preparation in which we also study the universality properties and other probabilistic aspects of the point processes defined by the positions of occurrences of a pattern in a random permutation.

2 Main results

We call a prefix (resp. suffix) of length m an m -prefix (resp. m -suffix). Note that a pattern covering with overlaps m must have m -suffix order isomorphic to its m -prefix, that is, the letters of the m -suffix must appear in the same order of size as those of the m -prefix.

Lemma 1. *The number of permutations of length n covered by a pattern p with overlaps m depends only on the m -prefix and m -suffix of p .*

Proof. When $m \geq 2k$, where k is the length of p , the result is obvious because p is then determined by its m -prefix and m -suffix. Suppose now that $m < 2k$. Given a permutation σ covered by p with overlaps m , let A_i be the segment of σ between the m -prefix and m -suffix of the i -th occurrence of p prescribed by the covering with overlaps m . Any dependencies between the possible letters in A_i and A_{i-1} are determined solely by the m -prefix and m -suffix of p , and the ordering of the letters in each A_i is uniquely determined by the segment M of p between its m -prefix and m -suffix. Let p' be a pattern of the same length as p , with the same m -prefix and m -suffix as p , but with M' as its segment between these. A bijection taking a permutation σ covered by p with overlaps m to a permutation σ' covered by p' with overlaps m simply permutes the letters of each A_i by the permutation that takes M to M' . $\square$

We now prove the main theorem of this paper, giving a recursive formula for computing the number of permutations of a given length that are covered by a given pattern with overlaps m . For simplicity we assume the m -prefix (and thus the m -suffix) of the pattern in question is increasing, but this restriction will be removed by a following lemma. We also explain, in a following remark, that the theorem and proof can easily be extended to the case where the overlaps are allowed to vary in length.

Theorem 2. *Let p be a pattern of length $K \geq 2m$ with prefix $\mathbf{p} = p_1 p_2 \dots p_m$ and suffix $\mathbf{s} = p_{m-1} p_{m-2} \dots p_{2m}$, where $p_1 < p_2 < \dots < p_m$ and $p_{m-1} < p_{m-2} < \dots < p_{2m}$.*

Let π_i be the place of the i -th smallest letter in the concatenated word $\mathbf{ps} = p_1 p_2 \dots p_{2m}$, and let $\pi_0 = 0$ and $\pi_{2m-1} = 2m - 1$.

Let $\ell_0 = 0, \ell_{2m-1} = n - 1$, where $n = K - j, j = K - m$.

Let $p_0 = 0$ and $p_{2m-1} = K - 1$.

Let $L = \ell_{m-1} \ell_{m-2} \dots \ell_{2m}$.

Let $g_j L$ be the number of permutations of length $K - j, j = K - m$ with p -overlap m and ending with L .

Then $g_0 \mathbf{s} = 1, g_0 L = 0$ if $L \neq \mathbf{s}$, and for $j \geq 0$ we have

$$g_{j+1} L = \sum_{p_1 \leq \ell_1 < \ell_2 < \dots < \ell_m} g_j \ell_1 - p_1 - 1, \ell_2 - p_2 - 2, \dots, \ell_m - p_m - m \prod_{i=0}^{2m} \left(\frac{\ell_{\pi_i-1} - \ell_{\pi_i} - 1}{p_{\pi_i-1} - p_{\pi_i} - 1} \right). \quad (2.1)$$

Consequently, letting h_j be the number of permutations of length $n = K - j + K - m$ covered by p with overlaps m , we have

$$h_j = \sum_L g_j(L), \quad (2.2)$$

where L runs over all sequences $\ell_1, \ell_2, \dots, \ell_m$ with $1 \leq \ell_1 < \ell_2 < \dots < \ell_m \leq K - j + K - m$.

Proof. We prove this by induction on j , the base case, when $j = 0$, being $g_0(L) = 1$ if and only if $L = s$, since there is precisely one permutation of length K with an occurrence of the pattern p , namely the permutation consisting of p itself.

In order to construct a permutation of length $K - j + 1 + K - m$ covered by p with overlaps m and ending with $L = \ell_{m-1}\ell_{m-2} \dots \ell_{2m}$, let $\mathcal{O} = \mathcal{O}_F \mathcal{O}_M \mathcal{O}_L$ be the word consisting of the last occurrence of p in such a permutation σ , and such that both $\mathcal{O}_F$ and $\mathcal{O}_L$ have length m . We first analyze the possible choices of letters in σ that constitute $\mathcal{O}_M$.

Let $a < b$ be two letters in $\mathcal{O}$ that belong to the union of $\mathcal{O}_F$ and $\mathcal{O}_L$ and such that no letter c in $\mathcal{O}_F$ or $\mathcal{O}_L$ satisfies $a < c < b$. Thus, $a = \ell_{\pi_i}$ and $b = \ell_{\pi_{i-1}}$ for some i , where π is the permutation defined in the statement of the theorem. However, there may be letters in $\mathcal{O}_M$ that lie between a and b in size. In fact, there must be precisely $p_{\pi_{i-1}} - p_{\pi_i} - 1$ such letters in $\mathcal{O}_M$, in order for $\mathcal{O}$ to be an occurrence of p . Moreover, these letters can be any of those that lie strictly between ℓ_{π_i} and $\ell_{\pi_{i-1}}$ in size. In addition, there may be letters in $\mathcal{O}_M$ that are smaller or greater than all letters in $\mathcal{O}_F$ and $\mathcal{O}_L$, which is taken care of by the definitions of π_0, π_{2m-1} etc. There are therefore $\prod_{i=1}^{2m} \binom{\ell_{\pi_{i-1}} - \ell_{\pi_i} - 1}{p_{\pi_{i-1}} - p_{\pi_i} - 1}$ possible choices for the letters in $\mathcal{O}_M$. Having thus chosen the letters to form $\mathcal{O}_M$ there is only one way to order them so that they will form the part of an occurrence of p that lies between the m -prefix and the m -suffix of that occurrence.

In order to complete the construction of the permutation σ we must construct its prefix $\sigma' = \sigma_1 \sigma_2 \dots \sigma_{K-j+K-m}$, that is, σ with $\mathcal{O}_M$ and $\mathcal{O}_L$ removed. This σ' has p -overlaps m , and must end in an increasing sequence $L : \ell_1 < \ell_2 < \dots < \ell_m$. If we standardize σ to consist of the letters $\{1, 2, \dots, K - j + K - m\}$ then the sequence L becomes $\ell_1 - p_1 - 1, \ell_2 - p_2 - 2, \dots, \ell_m - p_m - m$, since there are precisely $p_i - i$ letters in $\mathcal{O}_M$ and $\mathcal{O}_L$ that are smaller than ℓ_i . By the inductive hypothesis, there are precisely $g_j(\ell_1 - p_1 - 1, \ell_2 - p_2 - 2, \dots, \ell_m - p_m - m)$ such σ' , which establishes identity (2.1), and equation (2.2) now follows readily. $\square$

Thanks to the following lemma (which has a straightforward bijective proof, and can also be proved by showing the isomorphism of the corresponding *cluster posets* introduced in [12]), Theorem 2 can be applied to any pattern p , regardless of the order of letters in its overlap prefix/suffix, since sorting the prefix and suffix, respectively, to increasing order doesn't change the number of permutations covered by the pattern in question with overlaps m .

Lemma 2. Let $p = \mathbf{pws}$ be a pattern where $\mathbf{p} = p_1p_2 \dots p_m$ and $\mathbf{s} = s_1s_2 \dots s_m$ are order isomorphic and $\mathbf{w}$ is any permutation of the remaining letters. Let $\mathbf{p}' = p'_1p'_2 \dots p'_m$ be a permutation of $p_1p_2 \dots p_m$, let $\mathbf{s}' = s'_1s'_2 \dots s'_m$ be $s_1s_2 \dots s_m$ permuted in the same way and let $p' = \mathbf{p}'\mathbf{w}\mathbf{s}'$. Then the number of n -permutations covered by p with overlaps m equals the number of n -permutations covered by p' with overlaps m .

Remark 1. Although [Theorem 2](#) and its proof are stated in terms of fixed size overlaps, it is easy to see how these can be extended to the enumeration of permutations with occurrences of the pattern p in arbitrary prescribed positions. Namely, for each step in the recursion we can simply specify the length of the overlap and adjust the length of the sequence $\ell_1, \ell_2, \dots, \ell_m$ accordingly. Also, if two consecutive occurrences of the pattern aren't required to overlap then we can arbitrarily choose the set of letters to form the part of the permutation up to the end of the first of these occurrences (and recursively if there are more such non-overlaps), and we can choose and order arbitrarily letters to fill positions that are not required to belong to an occurrence. (Recall that we count occurrences in "at least" the prescribed positions, i.e. we allow for more occurrences than those specified.) In such cases, where not all pairs of successive occurrences are required to overlap, this breaks the problem into smaller pieces of overlapping occurrences, to which our formula can be applied, and segments of letters in a permutation that are not required to belong to an occurrence. We then need to multiply by the appropriate multinomial coefficients, corresponding to choices of letters for each of these pieces, and the appropriate factorials for those segments that are not required to belong to an occurrence, since the letters of those segments can be ordered arbitrarily.

Finally, note that the usual symmetries when enumerating consecutive permutation patterns — reverse, complement and their composition — apply here, in addition to [Lemma 2](#) and the following:

Lemma 3. Let $p = \mathbf{pws}$ be a pattern whose prefix $\mathbf{p}$ and suffix $\mathbf{s}$ have the same length m . The number of permutations of length n covered by p with overlaps m equals the number of permutations of length n covered by $p' = \mathbf{swp}$ with overlaps m .

Proof. Given a permutation σ covered by p with overlaps m , let $\sigma = p_1w_1s_1 \dots p_jw_js_j$, where, for each i , $p_iw_is_i$ and $s_iw_{i-1}p_{i-1}$ are occurrences of p , such that each p_i and each s_i is an m -prefix and/or m -suffix of an occurrence of p . Then $\sigma' = s_jw_jp_j \dots s_1w_1p_1$ is a permutation covered by p' with overlaps m and this process is clearly invertible. $\square$

3 Special cases

We now highlight some of the interesting sequences that can be obtained as specializations of [Theorem 2](#). Some of these have been obtained previously by other authors (or are easily derived from such results).

Proposition 1. Let $p = p_1 p_2 \dots p_k$ be a pattern of length k where $p_1 = 1$ and let $d = k - p_k$. The number of permutations of length $n = k + j(k - 1)$ covered by p with overlaps 1 is

$$\prod_{i=0}^j \binom{i(k-1) + d}{d}.$$

(See also Proposition 26 in [8] and the poset in Figure 1 in [12], with $a = 1$.)

Corollary 1. With j , n and p as in Proposition 1, if $p_k = k - 1$ then the number of permutations of length n in question is $(n - k + 1)!_{k-1}$.

In the special case of Proposition 1 where $p_k = 2$ the number sequence obtained equals that for partitions of a set of size $k - 1 + j - 1$ into $j - 1$ blocks of size $k - 1$ each, and we present a simple bijection (for $k = 1$ instead of k). First, it is easy to see that a permutation of this kind must begin with the letter 1, and that the letter in place $k - 1$ (common to the first and second occurrence of the pattern) must be 2. A straightforward argument shows that the letters in places $ik + 1$, for $0 \leq i \leq j - 1$ are the right-to-left minima of such a permutation, that is, each is smaller than all the letters to its right. Delete the first letter, 1, and split the resulting tail of the permutation into blocks of size k , so that the last letter of each block is smaller than any letter to its right in the permutation. Because each block is an occurrence of the tail of the pattern p , each block has its letters in a prescribed order. Sorting each block increasingly and keeping the order of blocks (and decrementing each letter by 1) gives a standard representation of a partition of the set $n - 1$ into $j - 1$ blocks of size k each. Conversely, any such partition is transformed into a permutation of this kind by reversing the above construction. As an example for $k = 3$, $j = 2$ (so $n = 10$) and the pattern 1342, the above bijection gives:

$$1\ 8\ 10\ 2\ 4\ 5\ 3\ 7\ 9\ 6 \leftrightarrow 8\ 10\ 2 - 4\ 5\ 3 - 7\ 9\ 6 \leftrightarrow 791 - 342 - 685 \leftrightarrow 179 - 234 - 568$$

Proposition 2. Let p be the pattern $1\ k - 1\ 2\ k - 2\ \dots\ k\ 2k$, where $k \geq 2$. The number of permutations of length $n = 2j + 2k$ covered by p with overlaps $2k - 2$ is the Catalan number C_{j-1} .

This also holds for the pattern $q = k - 1\ 1\ k - 2\ 2\ \dots\ 2k\ k$. The same is true when $2k - 1$ is appended to q , with overlaps $2k - 1$, and also when 1 is prepended to p and each letter of p incremented by 1, with overlaps $2k - 1$.

Proposition 3. (See [12, Thm. 4.2]) The number of permutations of length $m = j(m - 1)$ covered by $134\dots m - 2\ 2\ m - 3\ m - 4\ \dots\ 2m - 1$ with overlaps m equals $\frac{1}{mj+1} \binom{m}{j} j!$.

Proposition 4. The number of permutations of length $5 + 3j$ covered by 14523 with overlaps 2 is $2j - 1!_2$.

So far, the special cases considered involve regular overlaps. As previously mentioned, Theorem 2 straightforwardly adapts to arbitrary overlap vectors, but for some

classes of patterns, we observe a further decoupling that allows us to more directly ‘upgrade’ the results obtained for regular overlaps to arbitrary ones. In particular, let p be a pattern of length k with auto-correlation at least 2. Tracking only the occurrences of p arising from overlaps of sizes 1 and 2, we can express the vector of successive overlaps in terms of runs of 1s and 2s as

$$\underline{1}_{r_1}, \underline{2}_{q_1}, \dots, \underline{1}_{r_a}, \underline{2}_{q_a}$$

where $a, q_1, r_2, q_2, \dots, r_a \in \mathbb{N}$, $r_1, q_a \in \mathbb{N}_0$ and $\underline{x}_n$ is the word $x, \dots, x$ of length n (or the empty word when $n = 0$).

Proposition 5. *Let p be a pattern of length k of the form $1\sigma k$ with auto-correlation at least 2. Let $\underline{1}_{r_1}, \underline{2}_{q_1}, \dots, \underline{1}_{r_a}, \underline{2}_{q_a}$ give the successive overlaps of p in a permutation, counting only the overlaps of sizes 1 and 2. Then the number of such permutations is given by the product*

$$N_{q_1} \cdot N_{q_2} \cdot \dots \cdot N_{q_a},$$

where N_q is the number of permutations covered by p with q regularly-spaced overlaps of size 2.

Proof. Given a permutation σ of this form we claim that the letter ℓ of the leftmost overlap of size 1 in σ is greater than all letters preceding ℓ . Now, ℓ is clearly greater than all the letters in the occurrence of p of which ℓ is the last letter. Therefore, if this occurrence overlaps in two letters with a preceding occurrence, ℓ is larger than both letters in that overlap. In particular, it is larger than the last letter of that overlap, which in turn is greater than all other letters in that preceding occurrence. An inductive argument now shows that ℓ must be greater than all the letters preceding it in σ . An analogous argument shows that ℓ is smaller than all letters following it in σ .

This determines which subset of letters of σ belongs to each of the segments between successive 1-overlaps in σ . A segment between two successive 1-overlaps is thus a permutation covered by p with overlaps 2, whose number is N_i where i is the number of 2-overlaps in that segment. Since the ordering of letters in such a segment is independent of the ordering in any other such the proof follows. $\square$

Therefore, for $p = 1324$, for example, which can have overlaps of sizes 1 and 2 only, we obtain the full analogue of the descent set where we count permutations with occurrences of p in arbitrary prescribed positions, using [Proposition 2](#). This corollary can also be deduced from arguments in the proof of Thm. 4.1 in [\[12\]](#):

Corollary 2. *The number of permutations σ with occurrences of the consecutive pattern 1324 prescribed by the overlaps $\underline{1}_{r_1}, \underline{2}_{q_1}, \dots, \underline{1}_{r_a}, \underline{2}_{q_a}$ equals the following product of Catalan numbers:*

$$C_{q_1-1} \cdot C_{q_2-1} \cdot \dots \cdot C_{q_a-1}.$$

(For non-overlapping occurrences, the picture is even simpler; see [Remark 1](#).)

4 Moment Sequences

Several of the families of sequences outlined in the previous sections are moment sequences of known probability measures. Among this company, we again find the Catalan numbers (of course), as well as the following:

Proposition 6. Fix $k \in \mathbb{N}$. The sequence $a_n = k^{-1} n!_{k-1}$, for $n = k, 2k, \dots$ (see [Corollary 1](#)), is the moment sequence of the probability measure on the real line with density $f(x) = x^{1/k-1} e^{-x/k} k^{-1/k} \Gamma(1/k)^{-1}$.

Proposition 7. The number of permutations of length $m + j - m - 1$ covered by $134\dots m$ with overlaps m (see [Proposition 3](#)) is the moment sequence of the free multiplicative convolution of m free-Poisson random variables (members of a broader family of the so-called free Bessel laws [\[1\]](#)).

While the aforementioned are fundamental laws from classical and free probability, specializations of [Theorem 2](#) also include a great number of sequences that are more difficult to characterize. Nevertheless, by allowing us to compute these sequences efficiently to a very high number of terms, [Theorem 2](#) enables us to at least numerically verify the positivity, up to a point, of the lesser-known sequences arising in this context by calculating their Hankel determinants. The result is our [Conjecture 1](#) (see [Section 1](#)).

Finally, note that while [Theorem 2](#) provides an enumeration formula for overlapping patterns, one can (more easily) derive results for the cases where patterns are not overlapping, but are still required to occur at regular intervals. Concretely, consider permutations of length $n = k + j - k - m$ such that a fixed pattern p of length k occurs with overlaps $m \leq 0$, that is, at indices $1, k - 1 - m, 2k - 1 - m, \dots$. Recall that we count permutations with patterns occurring "at least" at specified positions; that is, we do not prohibit further occurrences of p , beyond the prescribed ones. As a result, regardless of the choice of the pattern, the number of permutations is simply $\frac{k + j - k - m}{k!^{j-1}}$, which can be expressed as the integral

$$\int_0^\infty y^j C y^{\frac{m-1}{k-m}} e^{-ky^{\frac{1}{k-m}}} dy,$$

where $C > 0$. This is clearly a moment sequence. In fact, for $m = 0$, we recognize here the moments of the random variable $Y = \frac{X^k}{k!}$ with X a rate-one exponential.

We do not yet understand the apparent positivity of sequences arising from the various ways of counting permutation patterns (avoiders of classical permutation patterns, packings of consecutive permutation patterns, and perhaps other formulations still to be discovered). Yet the fundamental nature of the measures we observe and consistency of the numerical evidence strongly suggest that there is interesting mathematics at play.

pattern	overlaps 1	overlaps 2	overlaps 3
312	$n - 2 !_2$		
1243	$n - 3 !_3$ Coro. 1	—	
1324	1	C_{j-1} Prop. 2	
1342	$\frac{3j!}{j! 3! j}$	—	
1423	$n - 3 !_3$ Coro. 1	1	
2143	1, 9, 234, 12204, 1067040, ...	1	
2413	1, 9, 234, 12204, 1067040, ...	C_{j-1} , Prop. 2	
12354	$n - 4 !_4$ Coro. 1	—	—
12435	1	1	—
12453	$\prod_{i=0}^j \frac{4i-2}{2}$ Prop. 1	—	—
12534	$n - 4 !_4$ Coro. 1	$n - 4 !_3$	—
13254	$n - 4 !_4$ Coro. 1	—	1
13425	1	$\frac{1}{2j-3} \binom{3j-3}{j-1}$ Prop. 3	—
13452	$\frac{n-1!}{j-1! 4! j-1}$	—	—
13524	$n - 4 !_4$ Coro. 1	1, 6, 81, 1806, 57447, ...	—
14253	$\prod_{i=0}^j \frac{4i-2}{2}$ Prop. 1	—	C_{j-1} Prop. 2
14325	1	A274644 [16]	—
14523	$\prod_{i=0}^j \frac{4i-2}{2}$ Prop. 1	$2j-1 !_2$ Prop. 4	—
15234	$n - 4 !_4$ Coro. 1	1	—
15243	$\prod_{i=0}^j \frac{4i-2}{2}$ Prop. 1	—	1
15324	$n - 4 !_4$ Coro. 1	$n - 4 !_3$	—
15423	$\prod_{i=0}^j \frac{4i-2}{2}$ Prop. 1	1	—
21354	1, 16, 816, 86528, 15661440, ...	1	—
21453	1, 30, 4440, 1867920, ...	1	—
21534	1, 16, 816, 86528, 15661440, ...	—	—
21543	1, 30, 4440, 1867920, ...	$n - 4 !_3$	—
23514	1, 16, 816, 86528, 15661440, ...	$2j-1 !_2$ Prop. 4	—
24153	1, 30, 4440, 1867920, ...	—	—
24513	1, 30, 4440, 1867920, ...	1, 6, 81, 1806, 57447, ...	—
25314	1, 16, 816, 86528, 15661440, ...	A274644 [16]	—
25413	1, 30, 4440, 1867920, ...	$\frac{1}{2j-3} \binom{3j-3}{j-1}$ Prop. 3	—

Table 1: Enumerating sequences: equivalence classes of patterns of length ≤ 5 .

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Fine polyhedral adjunction theory

Sofía Garzón^{*1}, and Christian Haase⁺¹

¹*Department of Mathematics, Freie Universität Berlin, Germany*

Abstract. Originally introduced by Fine and Reid in the study of plurigenera of toric hypersurfaces (Fine 1983, Reid et al. 1985), the Fine interior of a lattice polytope got recently into the focus of research: it has been used for constructing canonical models in the sense of Mori Theory (Batyrev 2020). Based on the Fine interior, we propose here a modification of the original adjoint polytopes by defining the Fine adjoint polytope $P^{F(s)}$ of P as consisting of the points in P that have lattice distance at least s to all valid inequalities for P . We obtain a Fine polyhedral adjunction theory that is, in many respects, better behaved than its original analogue. Many existing results in polyhedral adjunction theory carry over, some with stronger conclusions, such as decomposing polytopes into Cayley sums, and most with simpler, more natural proofs as in the case of the finiteness of the Fine spectrum.

Keywords: Polyhedral adjunction theory, Cayley polytopes, Fine interior

1 Introduction

Let $P \subset \mathbb{R}^d$ be a rational polytope, i.e., the convex hull of a finite set of points in $\mathbb{Q}^n$. For $s > 0$ we define the Fine adjoint polytope $P^{F(s)}$ as the set of points satisfying $\langle a, x \rangle \geq b + s$ whenever $\langle a, x \rangle \geq b$ is valid for all points in P . The Fine interior of a polytope, namely the Fine adjoint polytope for $s = 1$, was first introduced by Jonathan Fine in [5], where it was referred to as the heart of a polytope. Along the lines of this idea, we take here the Fine adjoint polytope as consisting of the points of P with lattice distance at least s to *all valid inequalities* for P (cf. [Definition 2.2](#)). In this paper we argue that the resulting Fine polyhedral adjunction theory is a more natural version of the polyhedral adjunction theory introduced in [3], where only facet defining inequalities were considered. Various of the results obtained from the original polyhedral adjunction theory carry over to the Fine case. Some of them give us stronger conclusions, such as the Decomposition [Theorem 4.3](#). Moreover, we are often able to provide simpler, more elegant proofs, as in the case of the finiteness of the Fine spectrum as mentioned in [Theorem 5.4](#).

^{*}sofiag96@zedat.fu-berlin.de. Supported by the German Science Foundation (DFG), Graduiertenkolleg “Facets of Complexity” (GRK 2434).

⁺haase@math.fu-berlin.de. Supported by the German Science Foundation (DFG) under grant HA 4383/8-1.

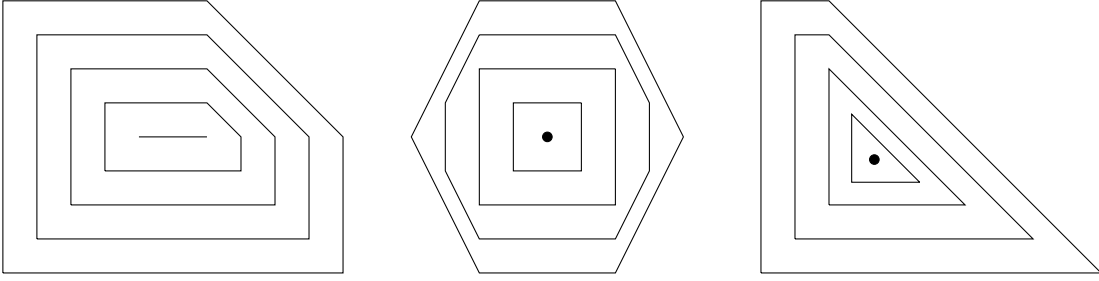


Figure 1: Examples of polytopes with their Fine adjoint polytopes.

Of particular importance is the largest value s_0 of s so that $P^{F(s)}$ is non-empty. For historical reasons, we record its reciprocal $\mu^F := 1/s_0$. A first indication that the Fine theory is better behaved is the monotonicity of this parameter:

$$P \subseteq Q \quad \implies \quad \mu^F(P) \geq \mu^F(Q),$$

which does not hold for the analogous parameter μ , as defined in [3].

One of the main results of polyhedral adjunction theory is the decomposition theorem (cf. [3, 4, 7]), which gives lower bounds on the codegree of a polytope for it to have a Cayley sum structure. Here, we refer to a *Cayley sum* $P_0 \star \cdots \star P_t$ of $t + 1$ polytopes as being a polytope which is constructed by positioning the $t + 1$ polytopes along the vertices of a t -dimensional standard simplex and taking its convex hull (as in Definition 4.1). It was conjectured by Batyrev and Nill that there is a function $f(n)$ giving such lower bounds on the codegree of a polytope which decomposes as a Cayley sum. In particular, a conjecture posed by Dickenstein and Nill [4, Conj. 1.2] for n -dimensional polytopes where $f(n) = \frac{n+3}{2}$ has been disproven by Higashitani [8]. Instead a weaker version was proposed [3, Conj. 1.3] which states the following.

Conjecture 1.1. *If an n -dimensional lattice polytope P satisfies $\mu(P) > \frac{n+1}{2}$, then P decomposes as a Cayley sum of lattice polytopes of dimension at most $\lfloor 2(n + 1 - \mu(P)) \rfloor$.*

This conjecture is still open but a slightly weaker version was proven in [3, Thm. 3.4] in which $\mu(P) > \frac{n+1}{2}$ is replaced with $\mu(P) \geq \frac{n+2}{2}$. Because of the way the Fine adjoint polytopes are defined, we obtain the relation of the Q-codegree and Fine Q-codegree of a rational polytope P , $\mu^F(P)$, to be such that

$$\mu(P) \leq \mu^F(P).$$

It is due to this that in Theorem 4.3 we prove a Fine version of this decomposition theorem where essentially the same proof yields a stronger result.

Our second main result is related to Fujita's Spectrum Conjecture, of great interest in the context of polarized varieties, which can be translated to the case of toric varieties.

Conjecture 1.2 (Spectrum Conjecture, Fujita [6]). *For any $n \in \mathbb{Z}_{\geq 1}$, let S_n be the set of unnormalized spectral values of a smooth polarized n -fold. Then, for any $\varepsilon > 0$, the set $\{\mu \in S_n \mid \mu > \varepsilon\}$ is a finite set of rational numbers.*

A polyhedral version of Fujita's conjecture was proven by Paffenholz [10, Thm 3.1] even allowing certain, α -canonical, singularities (cf. Lemma 5.1 below). In this paper, we show that the analogous set $\{\mu^F \in S_n^F \mid \mu^F > \varepsilon\}$ of Fine spectral values is finite without any assumption on the singularities (cf. Theorem 5.4). As a result, our proof is simpler than Paffenholz', and it should allow for classification results in the future.

2 Redefining polyhedral adjunction theory

In what follows, unless stated otherwise, we consider $P \subseteq \mathbb{R}^n$ to be an n -dimensional rational polytope, which is described in a unique minimal way by inequalities as $P = \{x \in \mathbb{R}^n \mid \langle a_i, x \rangle \geq b_i, i = 1, \dots, m\}$, where $b_i \in \mathbb{Q}$ and $a_i \in (\mathbb{Z}^n)^*$ are the primitive rows of a matrix A , i.e., they are not the multiple of another lattice vector, and $b \in \mathbb{Q}^m$. We will refer to P being a *rational polytope* when its vertices lie in $\mathbb{Q}^n$ and we will say that P is a *lattice polytope* when its vertices lie in $\mathbb{Z}^n$. We introduce our first definitions.

Definition 2.1. Let f be the affine functional $f(x) = \langle a, x \rangle - b$ for some $b \in \mathbb{Q}$ and $a \in (\mathbb{Z}^n)^*$. Such a functional is said to be *valid* for a polytope P if for the halfspace $\mathcal{H}_+ := \{x \in \mathbb{R}^n \mid f(x) \geq 0\}$, we have that $P \subseteq \mathcal{H}_+$. Moreover, if there is some $p \in P$ with $f(p) = 0$, i.e., at least one point of P lies in $\mathcal{H}$, the hyperplane generated by f , we say f is a *tight valid inequality* for P .

Since P is a polyhedron, note that it can be described by a finite subset of all tight valid inequalities for P of which there is an infinite number, namely at least one for each primitive $a \in (\mathbb{Z}^n)^* \setminus \{0\}$.

Definition 2.2. Let $\alpha \in (\mathbb{R}^n)^*$ we define the *distance function* associated with P as

$$d_P^F : (\mathbb{R}^n)^* \rightarrow \mathbb{R}, \quad \alpha \mapsto \min_{x \in P} \langle \alpha, x \rangle.$$

In terms of this function, for some real number $s > 0$, we may define the *Fine adjoint polytope*, which is a rational polytope, as

$$P^{F(s)} := \{x \in \mathbb{R}^n \mid \langle a, x \rangle \geq d_P^F(a) + s, \text{ for all } a \in (\mathbb{Z}^n)^* \setminus \{0\}\}.$$

We will refer to the study of such Fine adjoint polytopes as *Fine polyhedral adjunction theory*.

As previously mentioned, the Fine adjoint polytopes we have introduced are a variant of the adjoint polytopes as defined in [3, Definition 1.1]. In order to compare between these definitions, we recall the original one here.

Definition 2.3. Let P be a rational polytope of dimension n given by the inequalities $\langle a_i, \cdot \rangle \geq b_i$ for $i = 1, \dots, m$ that define facets $F_1, \dots, F_m$ in a minimal way. Then for $x \in \mathbb{R}^n$, the *lattice distance* from the facet F_i is given by $d_{F_i} := \langle a_i, x \rangle - b_i$ and the *lattice distance with respect to the boundary* ∂P of P is $d_P := \min_{i=1, \dots, m} d_{F_i}(x)$. For $s > 0$, the *adjoint polytope* is defined as

$$P^{(s)} := \{x \in \mathbb{R}^n \mid d_P(x) \geq s\}.$$

Remark 2.4. In some cases, taking Fine adjoint polytopes of a polytope P is equivalent to considering the original adjoint polytopes, as is the case of the rightmost and leftmost examples in [Figure 1](#).

In what follows, we will prove a crucial result, namely that only finitely many tight valid inequalities $f_1, \dots, f_t$ will be relevant when computing the Fine adjoint polytopes. Moreover, from its proof we will obtain a characterization for when exactly an inequality will be relevant when computing Fine adjoint polytopes. We make this notion of a relevant inequality more precise.

Definition 2.5. Let $\mathcal{F}$ be the set of all valid inequalities for P , where an element $f \in \mathcal{F}$ is of the form $\langle a_f, x \rangle \geq b_f$. A valid inequality $f \in \mathcal{F}$ is said to be *relevant* for P if for some $s > 0$, it holds that

$$\{x \in \mathbb{R}^n \mid \langle a_f, x \rangle \geq d_P^F(a_f) + s, \forall f \in \mathcal{F}\} \neq \{x \in \mathbb{R}^n \mid \langle a_f, x \rangle \geq d_P^F(a_f) + s, \forall f \in \mathcal{F} \setminus \{f\}\}.$$

The valid inequality f is said to be *irrelevant* if it is not relevant.

The following proposition will be very useful for our results and computations below.

Proposition 2.6 ([1, Proposition 3.11]). *Let P be a rational polytope of dimension n . Then there exists a finite set $\mathcal{S} \subset \mathcal{F}$ of valid inequalities for P such that the set $\mathcal{S}$ contains all relevant valid inequalities for P .*

From [Proposition 2.6](#) we obtain a useful description of the relevant valid inequalities which we state as a Corollary.

Corollary 2.7. *Let P be a rational polytope of dimension n . All $a \in (\mathbb{Z}^n)^*$ which yield valid relevant inequalities for P of the form $\langle a, \cdot \rangle \geq d_P^F(a)$ belong to $\text{conv}(a_1, \dots, a_m)$ where the a_i for $1 \leq i \leq m$ are the primitive inward pointing facet normals of P .*

We will now consider the polytope P to be defined as $P = \{x \in \mathbb{R}^n \mid \langle a_i, x \rangle \geq b_i, i = 1, \dots, m\}$, where $b_i \in \mathbb{Q}$ and $a_i \in (\mathbb{Z}^n)^*$ are the primitive rows of a matrix A including all relevant valid inequalities for P .

Remark 2.8. Note that in the Fine case, we have that taking Fine adjoint polytopes satisfies monotonicity with respect to inclusion of polytopes, i.e., if P and P' are two polytopes, such that $P \subseteq P'$, then we have that for any $s \geq 0$, $P^{F(s)} \subseteq P'^{F(s)}$. This holds since for any $a \in (\mathbb{Z}^n)^*$ we have that $d_P^F(a) \geq d_{P'}^F(a)$, but this does not necessarily hold in the classical polyhedral adjunction theory case.

Now, using the Fine adjoint polytopes, we may reformulate the concept of classical $\mathbb{Q}$ -codegree.

Definition 2.9. The *Fine $\mathbb{Q}$ -codegree* of a rational polytope P is

$$\mu^F(P) := (\sup\{s > 0 \mid P^{F(s)} \neq \emptyset\})^{-1},$$

and the *Fine core* of P is

$$\text{core}^F(P) := P^{F(1/\mu^F(P))}.$$

For completeness, recall that the concepts of $\mu(P)$ and $\text{core}(P)$ were defined in the context of the original polyhedral adjunction theory as above, where simply the Fine adjoints $P^{F(s)}$ are the original adjoints $P^{(s)}$.

Example 2.10. In general, the core and the Fine core of a given polytope can differ and may even have different dimensions, as in the case of the polytopes in [Figure 2](#).

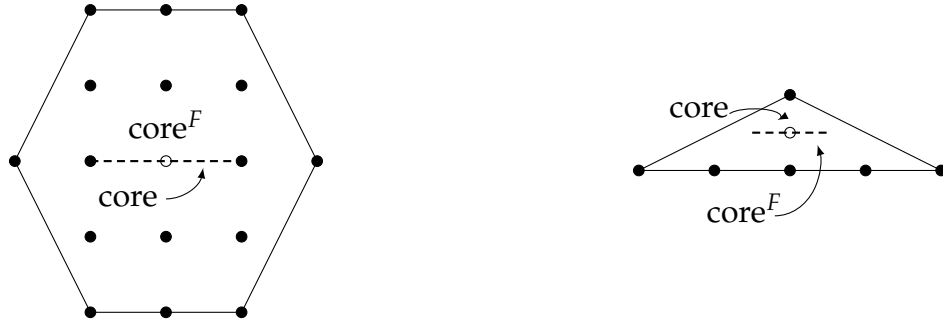


Figure 2: Examples of polytopes whose Fine and classical cores differ.

Example 2.11. Consider the polytope given as in [\[3, Figure 6\]](#) for the case $h = 10$ by

$$P = \text{conv} \begin{bmatrix} 0 & 2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 & 10 & 10 \end{bmatrix}.$$

We may compute the core of P and the Fine core of P to be

$$\text{core}(P) = \text{conv} \begin{bmatrix} 4/3 & 4/3 \\ 4/3 & 4/3 \\ 4/3 & 2 \end{bmatrix}, \quad \text{core}^F(P) = \text{conv} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 3 & 1 & 3 \end{bmatrix}.$$

Thus, in this case, as seen in [Figure 3](#), we have that $\text{core}^F(P) \not\subseteq \text{core}(P)$. The core and the Fine core are even disjoint for this example.

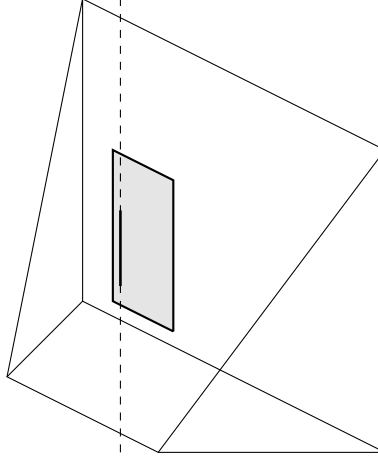


Figure 3: Polytope whose Fine core is not contained in its classical core.

We will now introduce a definition which we will need in later sections.

Definition 2.12. If we consider an element y of a k -dimensional rational polyhedral cone σ with generators $a_1, \dots, a_k$, then the *height function* associated with the cone σ is the piecewise linear function given by

$$\text{height}_\sigma(y) := \max \left\{ \sum_{i=1}^k \lambda_i \mid y = \sum_{i=1}^k \lambda_i a_i, \text{ and } \lambda_i \geq 0 \text{ for } 1 \leq i \leq k \right\}.$$

The cone σ is α -canonical for some $\alpha > 0$ if $\text{height}_\sigma(y) \geq \alpha$ for any integral point $y \in \sigma \cap (\mathbb{Z}^n)^*$. A complete rational polyhedral fan Σ is α -canonical if every cone in Σ is α -canonical. Furthermore, a cone or fan is *canonical* if it is α -canonical for $\alpha = 1$.

3 Natural Projections in the Fine case

We first want to study the behaviour of the Fine $\mathbb{Q}$ -codegree under projections, so we introduce the following definition.

Definition 3.1. Let $K(P)$ be the linear space parallel to $\text{aff}(\text{core}^F(P))$. Then the projection $\pi_P : \mathbb{R}^n \rightarrow \mathbb{R}^n / K(P)$ is called the natural projection associated with P .

We now have the following Lemma from [3] which holds with the same proof in the case of Fine Adjunction Theory.

Lemma 3.2. Let $x \in \text{relint}(\text{core}^F(P))$. Let us denote by $f_1, \dots, f_t$ the relevant valid inequalities for P with $d_{f_i}(x) := \langle a_i, y \rangle - b_i = \mu^F(P)^{-1}$. Then their primitive inner normals $a_1, \dots, a_t$

positively span the linear subspace $K(P)^\perp$. Moreover, if $\text{core}^F(P) = \{x\}$, then

$$\{y \in \mathbb{R}^n \mid d_{f_i}(y) \geq 0 \text{ for all } i = 1, \dots, t\}$$

is a rational polytope containing P .

However, we can prove the following stronger result, which does not hold in the classical polyhedral adjunction theory case. We include here the proof of the direction that was previously not valid.

Proposition 3.3. *The image $Q := \pi_P(P)$ of the natural projection of P is a rational polytope satisfying $\mu^F(Q) = \mu^F(P)$. Moreover $\text{core}^F(Q)$ is the point $\pi_P(\text{core}^F(P))$.*

Proof. To prove that $\mu^F(P)^{-1} \leq \mu(Q)^{-1}$, let g be a valid inequality for Q and let $p \in P$ with $\pi_P(p) = q \in \text{core}^F(Q)$. Then, for $\pi_P^*g = f$, we have

$$\mu^F(Q)^{-1} = g(q) - \min_{\tilde{q} \in Q} g(\tilde{q}) = \pi_P^*g_P(p) - \min_{\tilde{p} \in P} \pi_P^*g(\tilde{p}) \geq \mu^F(P)^{-1}. \quad \square$$

Remark 3.4. Note that we have described the behaviour of the Q -codegree under the natural projection of P . However, under any projection π' of P , we have that $\mu^F(\pi'(P)) \leq \mu^F(P)$.

4 Cayley Decompositions and the Fine structure theorem

We let $P \subseteq \mathbb{R}^n$ be an n -dimensional lattice polytope and we recall the following definition.

Definition 4.1. Given lattice polytopes $P_0, \dots, P_t \subseteq \mathbb{R}^k$, then the *Cayley sum* $P_0 * \dots * P_t$ is the convex hull of

$$(P_0 \times 0) \cup (P_1 \times e_1) \cup \dots \cup (P_t \times e_t) \subseteq \mathbb{R}^k \times \mathbb{R}^t$$

for $e_1, \dots, e_t$ the standard basis of $\mathbb{R}^t$.

As a means of comparison, we will now define the notion of codegree which comes up in the context of Ehrhart Theory of lattice polytopes [2].

Definition 4.2. Let P be a rational polytope. The *codegree* of P is given by

$$\text{cd}(P) := \min\{k \in \mathbb{N}_{\geq 1} \mid \text{int}(kP) \cap \mathbb{Z}^n \neq \emptyset\}.$$

Let us define the value

$$d^F(P) := \begin{cases} 2(n - \lfloor \mu^F(P) \rfloor), & \text{if } \mu^F(P) \notin \mathbb{N} \\ 2(n - \mu^F(P)) + 1, & \text{if } \mu^F(P) \in \mathbb{N} \end{cases}$$

We have that $P \cong \Delta_n$ if and only if $\text{cd}(P) = n + 1$. Since $\mu(P) \leq \mu^F(P) \leq \text{cd}(P) \leq n + 1$ and $\mu(\Delta_n) = n + 1$, then it holds that $P \cong \Delta_n$ if and only if $\mu(P) = n + 1$, in which case also $\mu^F(P) = n + 1$.

We now come to the following strengthening of the Decomposition Theorem for Cayley sums [3, Theorem 3.4] whose proof follows the one presented in [3] slightly adapted to the Fine case.

Theorem 4.3. *Let P be an n -dimensional lattice polytope with $P \not\cong \Delta_n$. If $n > d^F(P)$, then P is a Cayley sum of lattice polytopes in $\mathbb{R}^m$ with $m \leq d^F(P)$.*

Let us consider the following example where we compute the codegree in our three settings.

Example 4.4. Let $\Delta_n(a) := \text{conv}(0, ae_1, e_2, \dots, e_n)$ for positive integers $a \in \mathbb{Z}_{>0}$, where for $1 \leq i \leq n$ the e_i form the standard basis of $\mathbb{R}^n$. Note that in the case where $a = 1$ this yields the standard simplex and it has been argued before that it satisfies

$$\text{cd}(\Delta_n(1)) = \mu(\Delta_n(1)) = \mu^F(\Delta_n(1)) = n + 1.$$

Thus, let us consider the case where $a > 1$ and $n \geq 2$. It is easy to check that $\text{cd}(\Delta_n(a)) = n$. Moreover, it has been computed in [11] that in this case the Q-codegree is given by

$$\mu(\Delta_n(a)) = n - 1 + \frac{2}{a}.$$

Finally, since in the Fine case the inequality defined by $\sum_{i=2}^n x_i \leq 1$ is relevant, it can be computed that

$$\mu^F(\Delta_n(a)) = n.$$

Thus, we obtain that $\mu(\Delta_n(a)) < \mu^F(\Delta_n(a)) = \text{cd}(\Delta_n(a))$.

From this example we see that the Q-codegree and the Fine Q-codegree can take different values on the same polytope P .

5 Finiteness of the Fine Q-codegree spectrum

It has been proven already that when bounded from below by some $\varepsilon > 0$, the set of values that the Q-codegree can take under certain conditions is finite. We will shortly review these conditions for the case of classical polyhedral adjunction theory.

Let $P \subseteq \mathbb{R}^n$ be a lattice polytope of dimension n , which we assume to be full-dimensional. We let $\mathcal{N}(P)$ denote the normal fan of P and define the following sets as in [11],

$$\mathcal{S}(n, \varepsilon) := \{P \mid P \text{ is an } n\text{-dimensional lattice polytope, } \mu(P) \geq \varepsilon\},$$

$$\mathcal{S}_\alpha^{\text{can}}(n, \varepsilon) := \{P \mid P \in \mathcal{S}(n, \varepsilon) \text{ and } \mathcal{N}(P) \text{ is } \alpha\text{-canonical}\}.$$

The theorem proven in [11] is stated as follows.

Lemma 5.1 (Paffenholz, [11, Theorem 3.1]). *Let $n \in \mathbb{N}$ and $\alpha, \varepsilon > 0$ be given. Then*

$$\{\mu(P) \mid P \in \mathcal{S}_\alpha^{\text{can}}(n, \varepsilon)\}$$

is finite.

Note that in this result the α -canonical assumption on the polytopes was necessary.

Example 5.2. A natural example to consider in order to see the importance of this assumption is the family of polytopes

$$\Delta_n(a) = \text{conv}(0, ae_1, \dots, e_n)$$

where the $e_1, \dots, e_n$ are the standard basis vectors and $a \in \mathbb{Z}_{>0}$, which was previously studied in Example 4.4. For these polytopes, we have that their normal fan is $\mathbb{Q}$ -Gorenstein with index a and if $a > 1$, so that

$$\mu(\Delta_n(a)) = n - 1 + \frac{2}{a},$$

but the polytopes $\Delta_n(a)$ are not α -canonical for any $\alpha > 0$ and their $\mathbb{Q}$ -codegree can take an infinite number of values.

In what follows we will study a generalization of this theorem to the case of Fine adjunction theory. We will follow the proof presented in [11] and adapt it to the Fine polyhedral adjunction theory case where the remarkable difference will be the fact that we will not be assuming that the polytopes are α -canonical, hence in this new setting the theorem holds in much greater generality and with much weaker assumptions.

Definition 5.3. A vector a_i is a *Fine core normal* if for all $y \in \text{core}^F(P)$,

$$\langle a_i, y \rangle = d_P^F(a_i) + \mu^F(P)^{-1}.$$

The set of *Fine core normals* will be denoted by $\mathcal{N}_{\text{core}}^F(P)$. We also define the set

$$\mathcal{S}^F(n, \varepsilon) := \{P \mid P \text{ is an } n\text{-dimensional lattice polytope, } \mu^F(P) \geq \varepsilon\}.$$

We can now state our main result.

Theorem 5.4. *Let $n \in \mathbb{N}$ and $\varepsilon > 0$ be given. Then*

$$\{\mu^F(P) \mid P \in \mathcal{S}^F(n, \varepsilon)\}$$

is finite.

The proof here will also consist of two main parts. First, we will show that up to lattice equivalence, for a fixed $n \in \mathbb{Z}_{>0}$, there are only finitely many sets of core normals for n -dimensional lattice polytopes. Then we will show that each such configuration of core normals gives rise to finitely many values for the Fine Q-codegree above any positive threshold. Thus, if we let P be described by all relevant inequalities as

$$P = \{x \in \mathbb{R}^n \mid \langle a_i, x \rangle \geq b_i, i = 1, \dots, m\},$$

note that, up to relabelling, we can assume that $a_1, \dots, a_k$ for some $k \leq m$ is the set of valid inequalities defining the affine hull of the Fine core of P , $\text{aff}(\text{core}^F(P))$. That is,

$$\text{aff}^F(\text{core}(P)) = \{x \mid \langle a_i, x \rangle = b_i + \mu^F(P)^{-1}, 1 \leq i \leq k\}.$$

Definition 5.5. Define the set A_{core}^F to be

$$A_{\text{core}}^F := \text{conv}(a_1, \dots, a_k) \subseteq (\mathbb{R}^n)^*$$

as the convex hull of the Fine core normals.

For P as defined above, the following lemmas will show that all the a_i are vertices of A_{core}^F and that the origin is a relative interior point.

Lemma 5.6 ([3, Lemma 2.2]). *The origin is in the relative interior of A_{core}^F .*

Moreover, the following result proven in [11] gives us precisely the vertices of A_{core}^F .

Lemma 5.7 ([11, Lemma 3.6]). *The vertices of A_{core}^F are $a_1, \dots, a_k$.*

We now want to show that, independently of the polytope being α -canonical or not, the origin is the only lattice point in the relative interior of A_{core}^F .

Lemma 5.8. *For A_{core}^F as above, we have that $\text{relint}(A_{\text{core}}^F) \cap (\mathbb{Z}^n)^* = \{0\}$.*

Proof. We prove this by contradiction. Hence, assume that there is some vector $a \in (\mathbb{Z}^n)^* \setminus \{0\}$ contained in the relative interior of A_{core}^F . As $0 \in \text{relint}(A_{\text{core}}^F)$, the point a is contained in the cone spanned by the vertices of some facet F of A_{core}^F , and defines the valid inequality $\langle a, x \rangle \geq b$ for P . If we let $a_1, \dots, a_k$ be the vertices of A_{core}^F , we can find $\lambda_1, \dots, \lambda_k \geq 0$ with $\lambda_i = 0$ if $a_i \notin F$ such that $a = \sum_{i=1}^k \lambda_i a_i$ and satisfying

$\sum_{i=1}^k \lambda_i < 1$. This last inequality follows from the fact that a is in the relative interior of A_{core}^F . Let $x_{\text{core}} \in \text{relint}(\text{core}^F(P))$. By definition, we have that $\langle a, x_{\text{core}} \rangle - b \geq (\mu^F(P))^{-1}$. Now, considering the sum over all valid inequalities associated to the core normals a_i for $1 \leq i \leq k$, for such y we obtain

$$\langle a, x_{\text{core}} \rangle - b = \sum_{i=1}^k \lambda_i (\langle a_i, x_{\text{core}} \rangle - b_i) = \sum_{i=1}^k \lambda_i (\mu^F(P))^{-1} < (\mu^F(P))^{-1}$$

where the last inequality follows from the fact that $\sum_{i=1}^k \lambda_i < 1$, but this contradicts the previous relation. $\square$

The last result we need in this first part of the proof is the following one by Lagarias and Ziegler. We say two lattice polytopes P and Q are *lattice equivalent* if there is an affine lattice isomorphism mapping P onto Q .

Theorem 5.9 (Lagarias, Ziegler [9, Theorem 1]). *Let integers $n, l \geq 1$ be given. There are, up to lattice equivalence, only finitely many different lattice polytopes of dimension d with exactly l interior points in the lattice $\mathbb{Z}^n$.*

Since we have proven that for A_{core}^F the only lattice point in its relative interior is $\{0\}$, combining this result with Theorem 5.9 we have shown that for a fixed $n \in \mathbb{Z}_{>0}$, only finitely many sets define the Fine core normals of an n -dimensional lattice polytope P . We record this as the following result.

Lemma 5.10. *Let $n \in \mathbb{Z}_{>0}$ be fixed and let P be an n -dimensional lattice polytope. Then only finitely many sets define the Fine core normals of P .*

In what follows, we will continue with the second step of the proof. We make use of the following lemma proven in [11] where we do not require the α -canonicity of P .

Lemma 5.11 ([11, Lemma 3.10]). *Fix some $\varepsilon > 0$ and some $n \in \mathbb{Z}_{>0}$. Let P be an n -dimensional lattice polytope with set of Fine core normals $\mathcal{A}$. Then the set*

$$\{\mu^F(P) \mid P \text{ is } n\text{-dimensional with set of Fine core normals } \mathcal{A}, \mu^F(P) \geq \varepsilon\}$$

is finite.

Proof of Theorem 5.4. Combining this last lemma together with the previous one we obtain the following. First of all, by Lemma 5.10 we have that up to lattice equivalence, there are only finitely many sets of Fine core normals for some n -dimensional lattice polytope. Finally, by Lemma 5.11, the set of values μ^F is finite for n -dimensional lattice polytopes with a fixed set $\mathcal{A}$ of Fine core normals. $\square$

In the Fine case, a stronger version of the theorem regarding the finiteness of the $\mathbb{Q}$ -codegree spectrum holds, dropping the α -canonicity assumption. Hence, considering all valid inequalities is a condition that highly restricts the shape and properties of the polytope A_{core}^F , i.e., the convex hull of the Fine core normals of a polytope P , since all such polytopes contain just one interior lattice point, namely the origin. Due to this we are able to prove the result in greater generality for the Fine $\mathbb{Q}$ -codegree spectrum.

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Generic curves and non-coprime Catalans

Eugene Gorsky^{*1}, Mikhail Mazin^{†2}, and Alexei Oblomkov^{‡3}

¹Department of Mathematics, UC Davis, Davis, California

²Department of Mathematics, Kansas State University, Manhattan, Kansas

³Department of Mathematics, UMass Amherst, Amherst, Massachusetts

Abstract. We compute the Poincaré polynomials of the compactified Jacobians for plane curve singularities with Puiseux exponents $nd, md, md-1$, and relate them to the combinatorics of q, t -Catalan numbers in the non-coprime case. We also confirm a conjecture of Cherednik and Danilenko for such curves.

Keywords: Compactified Jacobians, Rational Catalan Combinatorics, Dyck Paths

1 Introduction

In this paper, we study the topology of compactified Jacobians of plane curve singularities. We focus on the case where the curve is reduced and locally irreducible (or *unibranched*), and it is known [1, 2] that in this case compactified Jacobian is irreducible as well.

Compactified Jacobians play an important role in modern geometric representation theory. First, they are closely related to Hilbert schemes of points on singular curves, singular fibers in the Hitchin fibration and affine Springer fibers. In particular, counting points in the compactified Jacobians over finite fields is related to certain orbital integrals [21, 20, 30]. Recent works [8, 19, 11] relate them to the representation theory of Coulomb branch algebras, defined by Braverman, Finkelberg and Nakajima [4]. Second, a set of conjectures of the third author, Rasmussen and Shende [26, 25] relates the homology of compactified Jacobians to the *Khovanov-Rozansky homology* of the corresponding knots and links. In particular, the conjectures imply that the homology of the compactified Jacobian is expected to be determined by the topology of the link or, in the unibranched case, by the collection of Puiseux pairs of the singularity.

The progress in explicit computations of the homology of compactified Jacobians has been quite slow. For the quasi-homogeneous curves $C = \{x^m - y^n\}$, $\text{GCD}(m, n) = 1$, the homology was computed in many sources, starting with Lusztig and Smelt [22]. The key observation is that in this case $\overline{JC}$ admits a paving by affine cells. These cells and their dimensions have been given a number of combinatorial interpretations in [12, 13, 15, 18], where they were related to q, t -Catalan combinatorics. In [27, 28] the third author and Yun determined the ring structure on the homology in this case.

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[†]mmazin@gmail.com. Mikhail Mazin was partially supported by the Simons Collaboration grant 524324

[‡]Alexei Oblomkov was supported by the NSF grant DMS-1760373

In a different direction, Piontkowski [29] have computed the homology of compactified Jacobians for curves with one Puiseux pair defined by the parameterization $x, y = t^n, t^m \dots$, $\text{GCD}(m, n) = 1$. He showed that $\overline{JC}$ again admits an affine paving, and the combinatorics of cells depends only on m, n and hence agrees with the quasi-homogeneous case $x, y = t^n, t^m$. Moreover, Piontkowski computed the cell decompositions for some curves with two Puiseux exponents, where the combinatorics becomes rather subtle. In our main theorem, we vastly generalize the results of Piontkowski and prove the following.

Theorem 1.1. *Suppose $\text{GCD}(m, n) = 1$ and $d \geq 1$, consider the plane curve singularity C defined by the parameterization*

$$x = t, y = t^{nd}, t^{md} - \lambda t^{md-1} \dots, \lambda \neq 0 \quad (1.1)$$

Then:

- a) *The compactified Jacobian $\overline{JC}$ admits an affine paving where the cells are in bijection with Dyck paths D in an $nd \times md$ rectangle.*
- b) *The Poincaré polynomial of $\overline{JC}$ is given by*

$$P_{\overline{JC}}(t) = \sum_{D \in \text{Dyck}(nd, md)} t^{2(\delta - \text{dinv } D)}$$

where dinv is a certain statistics on Dyck paths defined in Section 4, and δ is the number of boxes weakly under the diagonal in an $nd \times md$ rectangle.

- c) *In particular, the Poincaré polynomial does not depend on λ (as long as it is nonzero) or on the higher order terms in (1.1).*

We call the curves (1.1) *generic curves*, since a generic curve with the first Puiseux pair nd, md has this form. The affine paving in the statement of Theorem 1.1 is obtained by intersecting the compactified Jacobian with the Schubert cell paving of the affine Grassmannian. Thus one can define a partial order on the strata such that the boundary of an affine cell lies in the union of the cells with smaller indices in this order. Hence we conclude:

Corollary 1.2. *For generic curves, the cohomology $H^*(\overline{JC})$ is supported in even degrees and the weight filtration on $H^*(\overline{JC})$ is pure in the sense of [10, 9].*

Very recently, Kivinen and Tsai [20] used completely different methods (p -adic harmonic analysis) to count points in arbitrary compactified Jacobians over finite fields $\mathbb{F}_q$. They proved that the result is always a polynomial in q and hence recovers the weight polynomial of $\overline{JC}$. Given the above purity result, for generic curves the Poincaré polynomial of $\overline{JC}$ agrees with the weight polynomial and our result agrees with theirs.

As a corollary of Theorem 1.1, we get that the Euler characteristic of $\overline{JC}$ is given by the number of Dyck paths in $nd \times md$ rectangle. For example, for the curve

$C = t^4, t^6 - t^7$ the Euler characteristic $\chi(\overline{JC}) = 23$ is equal to the number of Dyck paths in a 4×6 rectangle, in agreement with [29].

Next, we address the conjectures of Cherednik, Danilenko and Philipp [6, 7], which proposed an expression for the Poincaré polynomials of compactified Jacobians in terms of certain matrix elements of certain operators in the double affine Hecke algebra, see Section 5 for more details. We are able to prove this conjecture for generic curves:

Theorem 1.3. *Consider the two-variable polynomial*

$$C_{nd,md}(q, t) = \sum_{D \in \text{Dyck}(n, m)} q^{\text{area } D} t^{\text{dinv } D} \quad (1.2)$$

where $\text{area } D$ is the number of full boxes between a Dyck path D and the diagonal. It satisfies the following properties:

- a) It is symmetric in q and t : $C_{nd,md}(q, t) = C_{nd,md}(t, q)$
- b) At $q = 1$ it specializes to the Poincaré polynomial of $\overline{JC}$ (up to a linear change of the variable).
- c) It is given by the matrix element $\langle \gamma_{n,m} e_d - 1, e_{nd} \rangle$ of the elliptic Hall algebra operator $\gamma_{n,m} e_d$.
- d) It agrees with the Cherednik-Danilenko conjecture (Conjecture 5.1).

Compositional Rational Shuffle Theorem [23] implies a different, manifestly symmetric in q, t , explicit formula for $C_{nd,md}(q, t)$, which implies part (a). Part (c) also follows from the Compositional Rational Shuffle Theorem, part (d) follows from (c). Part (b) is a rephrasing of Theorem 1.1. Theorem 1.3(a) allows us to give a simple formula for the Poincaré polynomial of the compactified Jacobian:

Corollary 1.4. *The Poincaré polynomial of $\overline{JC}$ equals*

$$t^{2\delta} C_{nd,md}(1, t^{-2}) = t^{2\delta} C_{nd,md}(t^{-2}, 1) = \sum_{D \in \text{Dyck}(n, m)} t^{2(\delta - \text{area } D)}.$$

A more detailed version of the material presented in this extended abstract can be found in [14].

2 Background

2.1 Compactified Jacobians and semigroups

Let C be an irreducible (and reduced) plane curve singularity at $(0, 0)$. We can parametrize C as $x = t, y = t^k$, and write the local ring of functions on C as $\mathcal{O}_C = \mathbb{C}[x = t, y = t^k] \subset \mathbb{C}[[t]]$. Given a function $f(t) \in \mathcal{O}_C$, we can write

$$f(t) = \alpha_k t^k + \alpha_{k+1} t^{k+1} + \dots, \alpha_k \neq 0$$

and define the order of $f \in t^k$ by $\text{Ord } f \in t^k$. The compactified Jacobian $\overline{JC}$ is defined as the moduli space of rank one torsion free sheaves on C or, equivalently, $\mathcal{O}_C$ -submodules $M \subset \mathbb{C}[t]$. Note that M is an $\mathcal{O}_C$ -submodule if and only if $x \in tM \subset M$, $y \in tM \subset M$. Given such a subspace M , we define

$$\Delta_M = \{\text{Ord } f \in t^k : f \in M\} \subset \mathbb{Z}_{\geq 0}.$$

In particular, for $M = \mathcal{O}_C$ we obtain the **semigroup** of C :

$$\Gamma = \Delta_{\mathcal{O}_C} = \{\text{Ord } f \in t^k : f \in \mathcal{O}_C\}.$$

If M is an $\mathcal{O}_C$ -submodule then Δ_M is a Γ -module: $\Delta_M + \Gamma \subset \Delta_M$. This motivates the following

Definition 2.1. Given a subset $\Delta \subset \mathbb{Z}_{\geq 0}$, we define the stratum in the compactified Jacobian:

$$J_\Delta = \{M \subset \mathbb{C}[t] : \mathcal{O}_C M \subset M, \Delta_M = \Delta\}.$$

Clearly, J_Δ give a subdivision of $\overline{JC}$ and J_Δ is empty if Δ is not Γ -invariant. As a warning to the reader, J_Δ could be empty even if Δ is Γ -invariant.

2.2 Generic curves

It is well known that any curve C can be parametrized using Puiseux expansion:

$$x = t^{nd}, y = t^{md} + \lambda t^{md-1} + \dots, \text{GCD}(m, n) = 1.$$

If $d = 1$ then C has one Puiseux pair (n, m) and its link is the (n, m) torus knot. In this paper, we will be mostly interested in the case $d > 1$.

Definition 2.2. Assume $d > 1$. A curve C is called generic if $\lambda \neq 0$.

It is easy to see that the definition of a generic curve is symmetric in n and m . Indeed, we can choose the new parameter

$$\tilde{t} = \sqrt[m]{y} = t^{md} \sqrt[m]{1 + \lambda t^{md-1} + \dots} = t \left(1 + \frac{\lambda}{md} t^{md-1} + \dots \right),$$

then

$$y = \tilde{t}^{md}, x = \tilde{t}^{nd} - \frac{n\lambda}{md} \tilde{t}^{nd-1} + \dots$$

If $n = 1$, then the curve has one Puiseux pair $(d, md) = 1$. Otherwise, it has two Puiseux pairs (nd, md) and $(d, md) = 1$, which completely determine the topological type of the corresponding knot as a $(d, mnd) = 1$ -cable of the (n, m) torus knot.

2.3 Invariant subsets

. A subset $\Delta \subset \mathbb{Z}_{\geq 0}$ is called 0-normalized if $0 \in \Delta$. We will mostly consider 0-normalized subsets, as any subset of $\mathbb{Z}_{\geq 0}$ can be shifted to a unique 0-normalized one. We call Δ cofinite if $\mathbb{Z}_{\geq 0} \setminus \Delta$ is finite. For a cofinite subset Δ and $x \geq 0$ we write

$$\text{Gaps } x = \text{Gaps}_{\Delta} x : x, \infty \setminus \Delta.$$

Definition 2.3. We call Δ an nd -invariant subset if $nd \in \Delta \subset \Delta$. A number a is called an nd -generator of Δ if $a \in \Delta$ but $a - nd \notin \Delta$.

It is clear that for a cofinite nd -invariant subset Δ there is exactly one nd -generator in each remainder modulo nd . We will group them according to their remainders modulo d , so that the generators $a_{j,i}$, $i = 0, \dots, n-1$ all have remainder j modulo d . We will further reorder $a_{j,i}$ so that $a_{j,i} \equiv m \pmod{n}$. We write

$$\Delta_j = \bigcup_{i=0, \dots, n-1} a_{j,i} + nd\mathbb{Z}_{\geq 0}, \quad \Delta = \bigcup_{j=0, \dots, d} \Delta_j.$$

We will call the integers $a_{j,i} \equiv m \pmod{n}$ the combinatorial syzygys of Δ . It is clear that in each remainder modulo d there are n such syzygys.

3 Topology of compactified Jacobians

Throughout this section we fix a generic curve C with parametrization

$$x = t, y = t^{nd}, t^{md} + \lambda t^{md-1} + \dots$$

with $\lambda \neq 0$. We will use the notation $\mathcal{O}_C = \mathbb{C}[x = t, y = t^{nd}, t^{md} + \lambda t^{md-1} + \dots]$ for the ring of functions on C .

We also fix a cofinite nd, md -invariant subset $\Delta \subset \mathbb{Z}_{\geq 0}$ with nd -generators $a_{j,i}$ as in [Section 2.3](#). We denote by $A = \{a_{j,i}\}$ the set of all nd -generators. The main goal of this section is to describe the stratum J_{Δ} in the compactified Jacobian $\overline{JC}$.

3.1 Equations for J_{Δ}

Consider an $\mathcal{O}_C$ -module $M \in J_{\Delta}$.

Lemma 3.1. *Then for all $k \in \Delta$ there exist a unique canonical representative*

$$f_k = t^k + \sum_{l \in \text{Gaps } k} f_{k;l} t^l \in M.$$

The canonical generators form a topological basis in M .

Vice versa, consider a collection of polynomials

$$\left\{ g_a - t^a \sum_{l \in \text{Gaps } a} g_{a;l-a} t^l \mid a \in A \right\}.$$

It will be convenient to consider the coefficients $g_{a;l-a}$ for $l \in \text{Gaps } a$ as parameters. For $l \notin \text{Gaps } a$ we use conventions

$$g_{a;0} = 1, \quad g_{a;l-a} = 0 \text{ if } l \in \Delta \setminus \{a\}.$$

Let N be the $\mathcal{O}_C$ -submodule generated by the collection, and let $\tilde{N}$ be the $\mathbb{C}[t^{dn}]$ submodule generated by the same collection. Then $N \in J_\Delta$ iff $N \subset \tilde{N}$, or, equivalently,

$$y - t g_a \in \tilde{N} \quad \forall a \in A.$$

For every $a \in A$ let $s_a - dm$ be a polynomial in t whose coefficients are polynomials in $g_{a';r}$, $a' \in A$, $r \in \mathbb{Z}_{>0}$ such that

$$y - t g_a - s_a - dm \in \tilde{N}$$

and

$$s_a - dm = \sum_{l \in \text{Gaps } a - dm} s_{a - dm;l - dm - a} t^l.$$

The condition $y - t g_a \in \tilde{N}$ is equivalent to $s_a - dm = 0$ for all $a \in A$. Thus J_Δ is a subset of the affine space $\text{Gen} = \mathbb{C}^{G \setminus \Delta}$ defined by $E \setminus \Delta$ equations:

$$G \setminus \Delta = \sum_{i,j} |\text{Gaps } a_{j,i}|, \quad E \setminus \Delta = \sum_{i,j} |\text{Gaps } a_{j,i} - dm|.$$

In particular, the coefficients $g_{a_{j,i};x}$ for $x \in \text{Gaps } a_{j,i}$ are natural coordinates on Gen . The equations are

$$s_{a_{j,i} - dm;x} = 0, \quad a_{j,i} - dm - x \notin \Delta.$$

3.2 Admissibility

Definition 3.2. We call $x > 0$ j -suspicious if $a_{j,i} - md - x \notin \Delta$ for all i , and suspicious if it is j -suspicious for at least one remainder j .

Lemma 3.3. A number x is j -suspicious if and only if $\Delta_j - x \subset \Delta_j - md - nd - x$.

Definition 3.4. We call Δ admissible, if 1 is not suspicious for Δ .

Lemma 3.5. Suppose that $J_\Delta \neq \emptyset$. Then Δ is admissible.

Proof. Suppose Δ is not admissible, then there exists some j such that $a_{j,i} - md - 1 \notin \Delta$ for all i . This implies $a_{j,i} - 1 \notin \Delta$ for all i , so we have canonical generators $g_{a_{j,i}}$ $t^{a_{j,i}} = g_{a_{j,i};1} t^{a_{j,i}-1} \dots$. If $a_{j,i} - md - a_{j,i} - 1$ and then we get

$$\begin{aligned} t^{a_{j,i} - md} g_{a_{j,i} - 1} - y - t g_{a_{j,i}} &= t^{a_{j,i} - md} g_{a_{j,i} - 1} - t^{md} \lambda t^{a_{j,i} - md - 1} \dots g_{a_{j,i}} \\ &= g_{a_{j,i} - 1;1} - g_{a_{j,i};1} - \lambda t^{a_{j,i} - md - 1} \dots \end{aligned}$$

hence $g_{a_{j,i} - 1;1} - g_{a_{j,i};1} - \lambda = 0$ for all i . By adding these for all i we get $n\lambda = 0$, contradiction. $\square$

3.3 Paving by affine spaces

Recall that Gen has coordinates $g_{a_{j,i};x}$ where $a_{j,i} \neq x \notin \Delta$. Sometimes we will use the notation $g_{a_{j,i};x}$ for all x , assuming

$$g_{a_{j,i};x} = 0 \quad \text{if } a_{j,i} = x \in \Delta. \quad (3.1)$$

It is also convenient to consider Gen as a graded vector space $\text{Gen} = \bigoplus_{x=1}^{\infty} \text{Gen}_x$, where Gen_x is spanned by $g_{a_{j,i};x}$ with a fixed x . There are coordinates on Gen that are most suitable for study of equations s_k in the case Δ is admissible. Let us define

$$g_{j,i;x}^- = g_{a_{j,i};x} - g_{a_{j,i-1};x}, \quad i = 0, \dots, n-1.$$

These functions are linearly dependent, for example $\sum_i g_{j,i;x}^- = 0$. We choose a subset of these as follows:

- (1) If x is j -suspicious, we define $I_{j;x} = \{0, \dots, n-2\}$.
- (2) If x is not j -suspicious, we can define $I_{j;x}$ to be a set of i such that $a_{j,i} \neq dm$ $x \notin \Delta$.

Lemma 3.6. *The functions $g_{j,i;x}^-$, $i \in I_{j;x}$ are linearly independent.*

Finally, if there exists at least one integer i such that $a_{j,i} \neq x \notin \Delta$, then we set

$$g_{j;x} = \sum_{i: a_{j,i} \neq x \notin \Delta} g_{a_{j,i};x}.$$

Thus if $I_{j;x} \neq \emptyset$ then $\{g_{j,i;x}^-, g_{j;x}\}$, $i \in I_{j;x}$ are linearly independent linear coordinates on the space of generators Gen_x . For each j, x such that $I_{j;x} \neq \emptyset$ let us fix a subset $\bar{I}_{j;x}$ such that $\{g_{j,i;x}^-, g_{j;x}, g_{a_{j,i'};x}\}$ $i \in I_{j;x}$, $i' \in \bar{I}_{j;x}$ is a basis of linear coordinates on Gen_x . For $I_{j;x} = \emptyset$ set $\bar{I}_{j;x} = \{0, 1, \dots, n-1\}$. As we will see later the coordinates $g_{a_{j,i'};x}$, $i' \in \bar{I}_{j;x}$ are not constrained by the equations for J_Δ , so we call them *free variables*. On the rest of the coordinates $g_{*,*}^*$ we introduce a partial order generated as follows. Allowing $*$ to take any independent values,

$$g_{*,*}^- < g_{*,x} < g_{*,*,x-1}^-,$$

coordinates $g_{*,*}^-$ are ordered in any arbitrary way, and

$$g_{j-1;x} < g_{j;x}$$

(cyclic notation modulo d) for $j \neq d - x - 1$.

Lemma 3.7. *For x such that $a_{j,i} \neq dm$ $x \notin \Delta$ we have*

$$s_{a_{j,i} \neq dm;x} g_{j,i;x}^- \text{ polynomial in free variables and variables } < g_{j,i;x}^- \quad (3.2)$$

and if x is j -suspicious then

$$\sum_{i=0}^{n-1} s_{a_{j,i} \neq dm;x} \lambda g_{j,x-1} \text{ polynomial in free variables and variables } < g_{j,x-1} \quad (3.3)$$

Proposition 3.8. *If Δ is admissible then*

$$J_\Delta = \mathbb{C}^{\dim \Delta}, \quad \dim \Delta = \dim G \Delta - E \Delta.$$

Proof. Recall that J_Δ is defined by the equations $s_{a_{j,i} \text{ } dm; x} = 0$ for j, i, x such that $a_{j,i} \text{ } dm \text{ } x \notin \Delta$. We can modify this system of equations as follows. Whenever x is j -suspicious, replace $s_{a_{j,n-1} \text{ } dm; x} = 0$ by $\sum_{i=0}^{n-1} s_{a_{j,i} \text{ } dm; x} = 0$. Clearly, the new system of equations is equivalent to the old one. Furthermore, according to Lemma 3.7, the new system of equations expresses some of the elements of a basis of the space Gen in terms of the smaller variables with respect to the order $<$. Therefore, one can use the equations to eliminate these variables one by one. Since $\dim \text{Gen} = \dim G \Delta$ and there are $E \Delta$ equations, we obtain the required result. $\square$

4 Combinatorics

4.1 More on invariant subsets

Let Δ be an nd, md -invariant subset. We call b an md -cogenerator for Δ if $b \notin \Delta$ but $b \text{ } md \in \Delta$.

Lemma 4.1. *Let Δ be an admissible cofinite nd, md -invariant subset. The dimension of J_Δ equals to the number of pairs a, b such that a is an nd -generator of Δ , b is an md -cogenerator and $a < b$.*

Let Θ be an n, m -invariant subset in $\mathbb{Z}$. As before, n, m are relatively prime.

Definition 4.2. *The skeleton S of Θ is the union of the n -generators and m -cogenerators of Θ .*

4.2 Equivalence classes of dn, dm -invariant subsets

Let us remind the definitions of the equivalence classes of invariant subsets from [16].

Let $\Theta = \{\Theta_0^0, \dots, \Theta_{d-1}^0\}$ be a collection of 0-normalized n, m -invariant subsets. For every $x_1, \dots, x_{d-1} \in \mathbb{R}_{\geq 0}^{d-1}$ consider

$$\Delta = \Delta_{x_1, \dots, x_{d-1}} : \bigcup_{k=0}^{d-1} d\Theta_k^0 = x_k,$$

where $x_0 = 0$. If all shift parameters $x_0, \dots, x_{d-1}$ are integers with different remainders modulo d , then Δ is a cofinite 0-normalized nd, md -invariant subset.

For each Θ_k^0 let S_k^0 be its skeleton. Consider the space $\mathbb{R}_{\geq 0}^l$ of all possible shifts $x_1, \dots, x_{d-1}$. Consider the subset $\Sigma_\Theta \subset \mathbb{R}_{\geq 0}^{d-1}$ consisting of all shifts $x_1, \dots, x_{d-1}$ for which there exists i and j such that

$$dS_i^0 = x_i \cap dS_j^0 = x_j \neq \emptyset.$$

Clearly, Σ_Θ is a hyperplane arrangement. We say that two nd, md -invariant subsets are equivalent if the corresponding shifts belong to the same connected component of the complement to Σ_Θ . One can show that every connected component of the complement to Σ_Θ contains at least one point corresponding to an dn, dm -invariant subset. To summarize, in order to get all the equivalence classes of the cofinite dn, dm -invariant subsets, one should consider all possible d -tuples of 0-normalized n, m -invariant subsets $\Theta_0^0, \dots, \Theta_{d-1}^0$, and for each such d -tuple consider the set of connected components of the complement to Σ_Θ in the space of shifts. Furthermore, one should consider these connected components up to symmetry: if two of the subsets are equal $\Theta_i^0 = \Theta_j^0$ then switching the corresponding shift coordinates x_i and x_j interchanges the connected components corresponding to the same equivalence class of dn, dm -invariant subsets.

4.3 Admissible representatives

Let Δ be a cofinite 0-normalized dn, dm -invariant subset. In the spirit of the above notations, let $\Delta = \bigcup_k d\Theta_k^0 - x_k - k$, where $x_1, \dots, x_{d-1} \in d\mathbb{Z}_{\geq 0}^{d-1} \subset \mathbb{R}_{\geq 0}^{d-1}$, $x_0 = 0$, and $\Theta_0^0, \dots, \Theta_{d-1}^0$ are 0-normalized n, m -invariant subsets. Denote $\Theta_k : \Theta_k^0 - \frac{x_k}{d}$ for all $k \in \{0, \dots, d-1\}$.

In these notations we get that Δ is admissible if for every $k \in \{0, \dots, d-1\}$ one has

$$d\Theta_{k-1} - k - 1 \not\subseteq d\Theta_k - k - dn - dm - 1,$$

or, equivalently,

$$\Theta_{k-1} \not\subseteq \Theta_k - n - m.$$

The following is our key combinatorial result:

Theorem 4.3. *Every equivalence class contains a unique admissible representative.*

Let $\text{Inv } dm, dn$ be the set of cofinite 0-normalized dm, dn -invariant subsets. The following Theorem is the main result of [16]:

Theorem 4.4 ([16]). *There exists a bijection $\mathcal{D} : \text{Inv } dm, dn / \sim \rightarrow \text{Dyck } dm, dn$, where $\sim$ is the equivalence relation defined above, and $\text{Dyck } dm, dn$ is the set of dm, dn -Dyck paths. Furthermore, $\dim \Delta = \delta - \text{dinv } \mathcal{D}(\Delta)$ for all $\Delta \in \text{Inv } nd, md$.*

4.4 Proof of Theorem 1.1

We combine all of the above results to prove Theorem 1.1. The compactified Jacobian $\overline{JC}$ is stratified into locally closed subsets J_Δ and by Proposition 3.8 they are isomorphic to affine spaces of dimension $\dim \Delta$ if Δ is admissible and empty otherwise. Therefore the Poincaré polynomial has the form

$$P(t) = \sum_{\Delta \text{ admissible}} t^{2 \dim \Delta}.$$

Next, we consider the infinite set $\text{Inv } nd, md$ of all nd, md -invariant subsets in $\mathbb{Z}_{\geq 0}$ and the equivalence relation on it. By [Theorem 4.3](#) in each equivalence class there is a unique admissible representative. Furthermore, by [Lemma 4.1](#) the “combinatorial dimension” $\dim \Delta$ depends only on the order of generators and cogenerators, and hence is constant on each equivalence class. Therefore we can write

$$P(t) = \sum_{\Delta \in \text{Inv } nd, md / \sim} t^{2 \dim \Delta}.$$

Finally, by [Theorem 4.4](#) there is a bijection between the equivalence classes and Dyck paths in $nd \times md$ rectangle, and the statistic $\dim \Delta$ on the former corresponds to the statistic $\text{codinv} = \delta - \text{dinv}$ on the latter, so

$$P(t) = \sum_{D \in \text{Dyck } nd, md} t^{2(\delta - \text{dinv } D)}.$$

5 Rational Shuffle Theorem and generic curves

5.1 Elliptic Hall algebra

We briefly recall some notations for the elliptic Hall algebra [\[5\]](#), and refer the reader to [\[3, 17, 23, 24\]](#) for more precise statements and details.

The elliptic Hall algebra $\mathcal{E}$ is generated by elements $P_{kn, km}$ for all possible kn, km . The universal cover of the group $\text{SL}(2, \mathbb{Z})$ acts on $\mathcal{E}$ by automorphisms. For coprime m and n we denote by $\gamma_{n, m}$ an element of $\text{SL}(2, \mathbb{Z})$ such that $\gamma_{n, m} \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} = \begin{pmatrix} m & 1 \\ 0 & 1 \end{pmatrix}$. Then one gets $P_{kn, km} = \gamma_{n, m} P_{k, 0}$.

Let Λ be the ring of symmetric functions in infinitely many variables. The elliptic Hall algebra $\mathcal{E}$ acts on Λ , and the multiplication operators by power sum symmetric functions p_k correspond (up to a scalar) to the generators $P_{k, 0}$ of $\mathcal{E}$. Furthermore, $\mathcal{E}$ is graded, and the grading is compatible with the grading on Λ . The generator $P_{kn, km}$ has degree kn .

5.2 Cherednik-Danilenko conjecture

Consider a Puiseux expansion of plane curve singularity:

$$y = b_1 x^{\frac{m_1}{r_1}} + b_2 x^{\frac{m_2}{r_1 r_2}} + b_3 x^{\frac{m_3}{r_1 r_2 r_3}} + \dots, \quad b_i \neq 0.$$

Here we assume $\text{GCD}(m_i, r_1 \cdots r_i) = 1$. The exponents are related to characteristic pairs (r_i, s_i) by the equations $m_1 = s_1, m_i = s_i - r_i m_{i-1}$ $i > 1$. Given a sequence of characteristic pairs $(r_1, s_1), \dots, (r_\ell, s_\ell)$, the authors of [\[6\]](#) define a sequence of symmetric functions $f_{\ell-1}, f_\ell, \dots, f_1$ by setting $f_{\ell-1} = p_1$ and $f_k = \gamma_{r_\ell, s_\ell} f_{k-1}^{-1}$, where f_{k-1} is viewed as a multiplication operator on Λ , and thus an element of $\mathcal{E}$.

Conjecture 5.1 ([\[6\]](#)). *Let f_1 be the symmetric function of degree $r_1 \cdots r_\ell$ obtained by the above procedure. The specialization of $f_1, e_{r_1 \cdots r_\ell}$ at $t = 1$ agrees with the Poincaré polynomial of the compactified Jacobian of an algebraic curve with characteristic pairs $(r_1, s_1), \dots, (r_\ell, s_\ell)$.*

In the case of generic curves (1.1) we have

$$y = x^{\frac{m}{n}} \lambda x^{\frac{md-1}{nd}} \dots, m_1 = m, r_1 = n, m_2 = md-1, r_2 = d.$$

This means that $s_1 = m$ and $s_2 = 1$. To follow the above procedure, we first need to compute the operator $\gamma_{r_2, s_2} p_1$ and the corresponding symmetric function f_2 . By [17, Corollary 6.5] we have $f_2 = \gamma_{d,1} p_1 = 1 = P_{d,1} = e_d$. Therefore the next symmetric function is $f_1 = \gamma_{n,m} e_d = 1$. Then $f_1, e_{nd} = C_{nd,md} q, t$ follows from Rational Shuffle Theorem [23]. We conclude that for generic curves Conjecture 5.1 is true and follows from Theorem 1.1.

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Partial Permutohedra

Roger E. Behrend^{*1}, Federico Castillo^{†2}, Anastasia Chavez^{‡3}, Alexander Diaz-Lopez^{§4}, Laura Escobar^{¶5}, Pamela Harris^{||6}, and Erik Insko^{**7}

¹*School of Mathematics, Cardiff University, Cardiff, UK*

²*Departamento de Matemáticas, Pontificia Universidad Católica de Chile, Santiago, Chile*

³*Department of Mathematics and Computer Science, Saint Mary's College of California, Moraga, CA, USA*

⁴*Department of Mathematics and Statistics, Villanova University, Villanova, PA, USA*

⁵*Department of Mathematics and Statistics, Washington University in St. Louis, St. Louis, MO, USA*

⁶*Department of Mathematical Sciences, University of Wisconsin Milwaukee, Milwaukee, WI, USA & Department of Mathematics and Statistics, Williams College, Williamstown, MA, USA*

⁷*Department of Mathematics, Florida Gulf Coast University, Fort Myers, FL, USA*

Abstract. Partial permutohedra are lattice polytopes which were recently introduced and studied by Heuer and Striker [arXiv:2012.09901]. For positive integers m and n , the partial permutohedron $\mathcal{P}_{m,n}$ is the convex hull of all vectors in $\{0, 1, \dots, n\}^m$ whose nonzero entries are distinct. We study the face lattice, volume and Ehrhart polynomial of $\mathcal{P}_{m,n}$, and our methods and results include the following. For any m and n , we obtain a bijection between the nonempty faces of $\mathcal{P}_{m,n}$ and certain chains of subsets of $\{1, \dots, m\}$, thereby confirming a conjecture of Heuer and Striker. We use this characterization of faces to obtain a closed expression for the h -polynomial of $\mathcal{P}_{m,n}$. For any m and n with $n \geq m - 1$, we use a pyramidal subdivision of $\mathcal{P}_{m,n}$ to establish a recursive formula for the normalized volume of $\mathcal{P}_{m,n}$, from which we then obtain closed expressions for this volume. We also use a sculpting process (in which $\mathcal{P}_{m,n}$ is reached by successively removing certain pieces from a simplex or hypercube) to obtain closed expressions for the Ehrhart polynomial of $\mathcal{P}_{m,n}$ with arbitrary m and fixed $n \leq 3$, the volume of $\mathcal{P}_{m,4}$ with arbitrary m , and the Ehrhart polynomial of $\mathcal{P}_{m,n}$ with fixed $m \leq 4$ and arbitrary $n \geq m - 1$.

^{*}behrendr@cardiff.ac.uk. Roger E. Behrend was partially supported by Leverhulme Trust Grant RPG-2019-083.

[†]federico.castillo@mat.uc.cl. Federico Castillo was partially supported by FONDECYT Grant 1221133.

[‡]amc59@stmarys-ca.edu.

[§]alexander.diaz-lopez@villanova.edu.

[¶]laurae@wustl.edu. Laura Escobar was partially supported by NSF Grant DMS-1855598 and NSF CAREER Grant DMS-2142656.

^{||}peharris@uwm.edu. Pamela Harris was partially supported by a Karen Uhlenbeck EDGE Fellowship.

^{**}einsko@fgcu.edu.

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1 Introduction

Computing the volume of a polytope is hard, even when the complete face structure is known [8]. In fact, few exact volume formulas have been discovered in much generality. Stanley gave a notable volume formula for the regular permutohedron $\Pi 1, 2, \dots, m$, specifically that its normalized volume is m^{m-2} [13, Example 3.1]. More generally, Postnikov [11] studied the permutohedron $\Pi z_1, \dots, z_m$ (which is the convex hull of all vectors obtained by permuting the entries of an arbitrary vector $z_1, \dots, z_m$ in $\mathbb{R}^m$), as well as a certain class of generalized permutohedra, and obtained three distinct formulas for the volume of $\Pi z_1, \dots, z_m$ [11, Theorems 3.1, 5.1 and 17.1], each one subtle in its own way.

In this paper, we study a related family of polytopes called *partial permutohedra*, which were introduced recently by Heuer and Striker [7]. For positive integers m and n , the partial permutohedron $\mathcal{P}_{m,n}$ is the convex hull of all vectors in $\{0, 1, \dots, n\}^m$ whose nonzero entries are distinct. These lattice polytopes are anti-blocking versions of certain permutohedra. We also note that for any $n \geq m$, $\mathcal{P}_{m,n}$ is combinatorially equivalent to the *m-stellohedron* which, for example, has been studied in [10, Section 10.4] and has appeared recently in connection with matroid theory [4].

We expand on the work of Heuer and Striker [7] by obtaining, in [Theorem 2.4](#), a bijection between the nonempty faces of $\mathcal{P}_{m,n}$ and certain chains of subsets of $\{1, \dots, m\}$, for any m and n , thus proving Conjecture 5.25 of [7]. This conjecture has also, very recently, been independently proved by Black and Sanyal [3, Theorem 6.5]. We then use this characterization of the faces of $\mathcal{P}_{m,n}$ to obtain, in [Theorem 2.5](#), a closed expression for the h -polynomial of $\mathcal{P}_{m,n}$ with any m and n .

Our results for the volumes of partial permutohedra include the following. In [Theorem 3.1](#), we use a technique, in which $\mathcal{P}_{m,n}$ is subdivided into certain pyramids, to establish a recursive formula for the normalized volume of $\mathcal{P}_{m,n}$, for any m and n with $n \geq m - 1$. Using this recursion and Stong's computation for a particular case [14], we are then able to prove, in [Theorem 3.3](#), a closed expression for the normalized volume of $\mathcal{P}_{m,n}$ for the case $n \geq m - 1$. For the remaining case $m - 1 > n$, the combinatorial type of $\mathcal{P}_{m,n}$ depends on both m and n (whereas for $n \geq m - 1$ it depends only on m), which suggests that finding a completely general closed formula for the volume in this case is unlikely.

We also compute the Ehrhart polynomial of $\mathcal{P}_{m,n}$ when m or n is small. One of our main techniques is based on the idea, exploited by algebraists in the days of yore, of completing the (hyper)cube. We start with a lattice polytope for which we know

the volume or Ehrhart polynomial, and then carefully remove pieces until we reach the polytope of interest. This idea makes itself apparent after analyzing certain expressions, as given in [Example 3.2](#), for the normalized volume of $\mathcal{P}_{m,n}$ for small fixed m and any $n \geq m - 1$. These expressions are polynomials in n , with all coefficients negative, except in the leading term which is $m! n^m$, the normalized volume of a hypercube in $\mathbb{R}^m$ of side-length n . This suggests that we start with a hypercube from which we can sculpt a partial permutohedron, and this is precisely what we do in [Section 4](#) for the case $n \geq m - 1$. This sculpting approach has been used recently to compute the Ehrhart polynomials of matroid polytopes starting from the hypersimplex. See [\[5\]](#) for sparse paving matroids, and [\[6\]](#) for paving matroids.

In [Theorem 4.2](#), we provide results for the normalized volume and Ehrhart polynomial of $\mathcal{P}_{m,n}$ with arbitrary m and fixed $n \leq 4$, where these are obtained by sculpting $\mathcal{P}_{m,n}$ from a $\binom{n-1}{2}$ -dilated standard m -simplex $\Delta_m = \{\mathbf{x} \in \mathbb{R}^m \mid x_i \geq 0 \text{ for all } i, \sum_{i=1}^m x_i \leq 1\}$. For example, we find that the normalized volume of $\mathcal{P}_{m,2}$ is $3^m - 3$, thereby confirming Conjecture 5.30 of [\[7\]](#), we obtain explicit expressions for the Ehrhart polynomials of $\mathcal{P}_{m,1}$, $\mathcal{P}_{m,2}$ and $\mathcal{P}_{m,3}$ with arbitrary m , and we obtain an explicit expression for the normalized volume of $\mathcal{P}_{m,4}$ with arbitrary m .

In [Theorem 4.4](#), we provide explicit expressions for the Ehrhart polynomials of $\mathcal{P}_{m,n}$ with fixed $m \leq 4$ and arbitrary $n \geq m - 1$, where these are obtained by sculpting $\mathcal{P}_{m,n}$ from a hypercube $[0, n]^m$.

We end, in [Conjecture 5.1](#), by conjecturing a closed expression for the Ehrhart polynomial of $\mathcal{P}_{m,n}$ with $n \geq m - 1$ which generalizes our volume and Ehrhart polynomial results for this case.

2 Faces of partial permutohedra

2.1 Description of the partial permutohedron $\mathcal{P}_{m,n}$

We introduce the partial permutohedron $\mathcal{P}_{m,n}$, for any positive integers m and n , using a similar approach to that used by Heuer and Striker [\[7, Section 5\]](#).

Definition 2.1. Let the *partial permutohedron* $\mathcal{P}_{m,n}$ be the polytope given by the convex hull of all vectors in $\{0, 1, \dots, n\}^m$ whose nonzero entries are distinct.

It follows from the definition that $\mathcal{P}_{m,n}$ is a lattice polytope. As noted in [\[7, Remark 5.5\]](#), it has dimension m . From [\[7, Proposition 5.7\]](#), its vertices are the vectors in $\mathbb{R}^m$ with entries of zero in any $m - k$ positions, and with the other k entries being $n, n - 1, \dots, n - k + 1$ in any order, where k ranges from 0 to $\min\{m, n\}$. Its facet description is given in

Theorems 5.10 and 5.11 in [7] as

$$\mathcal{P}_{m,n} = \left\{ \mathbf{x} \in \mathbb{R}^m \mid \begin{array}{ll} 0 \leq x_i, & \text{for all } i \in m, \\ \sum_{i \in S} x_i \leq \binom{n-1}{2} - \binom{n-1-|S|}{2}, & \text{for all nonempty } S \subseteq m \\ & \text{with } |S| \leq n-1 \text{ or } |S| = m \end{array} \right\}, \quad (2.1)$$

where the inequalities correspond to distinct facets, and where $\binom{n-1-|S|}{2}$ is taken to be 0 if $n-1-|S| \leq 1$ (which occurs if $|S| = m \geq n$). Also, m denotes the set $\{1, \dots, m\}$.

In [7, Theorem 5.27], it is shown that $\mathcal{P}_{m,n}$ is a projection of the partial permutation polytope $\text{PPerm}_{m,n}$ (or polytope of $m \times n$ doubly substochastic matrices), which can be defined as the convex hull in $\mathbb{R}^{mn}$ of all $m \times n$ matrices in which each entry is in $\{0, 1\}$ and each row and column contains at most one 1. We remark that $\text{PPerm}_{m,n}$ can be regarded as the matching polytope of the complete bipartite graph $K_{m,n}$, and it is thus of interest in the context of combinatorial optimization (see, for example, [12, Chapters 18 and 25]). The polytope $\text{PPerm}_{m,m}$ was studied by Kohl, Olsen and Sanyal [9, Section 5].

2.2 The faces of $\mathcal{P}_{m,n}$

The whole face structure of $\mathcal{P}_{m,n}$ was conjectured in [7, Conjecture 5.25], and we prove this conjecture in Theorem 2.4. Recently, this was independently proved by Black and Sanyal [3, Theorem 6.5] in the context of monotone path polytopes of polymatroids. To characterize the faces of $\mathcal{P}_{m,n}$, we need some further definitions, as follows.

Definition 2.2. The Boolean lattice $\mathcal{B}_m$ is the poset consisting of subsets $A \subseteq m$, ordered by inclusion, where $A \in \mathcal{B}_m$ has rank $|A|$, the cardinality of A . A chain C in $\mathcal{B}_m$ is a nonempty ordered collection $C = A_1 \subsetneq A_2 \subsetneq \dots \subsetneq A_\ell$ of subsets $A_i \in \mathcal{B}_m$. We say that a rank i is *missing* from a chain C in $\mathcal{B}_m$ if there is no subset of rank i in C and there is a subset of rank greater than i in C .

Definition 2.3. Let $\mathcal{C}_{m,n}$ denote the set of all chains $A_1 \subsetneq \dots \subsetneq A_\ell$ in $\mathcal{B}_m$ which satisfy the following:

- If $A_1 \neq \emptyset$, then $|A_\ell \setminus A_1| \leq n-1$.
- If $A_1 = \emptyset$ and $\ell \geq 2$, then $|A_\ell \setminus A_2| \leq n-1$.

Our theorem for the face structure of $\mathcal{P}_{m,n}$, with any m and n , is as follows.

Theorem 2.4. Given a chain $C = A_1 \subsetneq \dots \subsetneq A_\ell$ in $\mathcal{C}_{m,n}$, let $\mathcal{F}_C$ be the intersection of $\mathcal{P}_{m,n}$ with the following hyperplanes:

- (i) $\{\mathbf{x} \in \mathbb{R}^m \mid x_i = 0\}$, for all $i \in m \setminus A_\ell$.
- (ii) $\left\{ \mathbf{x} \in \mathbb{R}^m \mid \sum_{i \in A_\ell \setminus A_j} x_i = \binom{n-1}{2} - \binom{n-1-|A_\ell \setminus A_j|}{2} \right\}$, for all $2 \leq j \leq \ell-1$, and also for $j=1$ unless $A_1 = \emptyset$ and $|A_\ell| \geq n$.
- (iii) $\left\{ \mathbf{x} \in \mathbb{R}^m \mid \sum_{i=1}^m x_i = \binom{n-1}{2} \right\}$, if $A_1 = \emptyset$ and $|A_\ell| \geq n$.

Then the following is a bijection:

$$\begin{aligned} \mathcal{C}_{m,n} &\longrightarrow \{\text{nonempty faces of } \mathcal{P}_{m,n}\}, \\ C &\longmapsto \mathcal{F}_C. \end{aligned}$$

Moreover, this bijection maps chains with k missing ranks to faces of dimension k , for each $k = 0, \dots, m$.

Our proof of [Theorem 2.4](#) involves verifying directly that the stated mapping is well-defined, injective and surjective.

2.3 The h -polynomial of $\mathcal{P}_{m,n}$

We now consider the h -polynomial of $\mathcal{P}_{m,n}$. Given a d -dimensional polytope P and $0 \leq i \leq d$, let $f_i(P)$ denote the number of i -dimensional faces of P . The f -polynomial of P is then defined as $f_P(t) = \sum_{i=0}^d f_i(P) t^i$, and the h -polynomial of P is defined as $h_P(t) = f_P(t-1)$.

Since Eulerian polynomials will play a role in the h -polynomial of $\mathcal{P}_{m,n}$, we proceed to introduce them. For a positive integer m , let $A_m(t)$ denote the Eulerian polynomial for $\mathfrak{S}_m$, i.e., $A_m(t) = \sum_{i=0}^{m-1} A(m,i) t^i$, where the Eulerian number $A(m,i)$ is the number of permutations in $\mathfrak{S}_m$ with exactly i descents. Also, let $A(0,i) = \delta_{0,i}$ and $A_0(t) = 1$.

Our theorem for the h -polynomial of $\mathcal{P}_{m,n}$, with any m and n , is as follows.

Theorem 2.5. *The h -polynomial of $\mathcal{P}_{m,n}$ is*

$$h_{\mathcal{P}_{m,n}}(t) = 1 + \sum_{i=0}^{n-1} \sum_{j=1}^{m-i} \binom{m}{i} A_i(t) t^j. \quad (2.2)$$

This is equivalent to the recurrence relation and initial condition

$$h_{\mathcal{P}_{m,n-1}}(t) = h_{\mathcal{P}_{m,n}}(t) - \binom{m}{n} A_n(t) \sum_{i=1}^{m-n} t^i, \quad h_{\mathcal{P}_{m,1}}(t) = \sum_{i=0}^m t^i. \quad (2.3)$$

Our proof of [Theorem 2.5](#) uses the characterization of faces provided by [Theorem 2.4](#) to verify (2.3).

3 Volume of $\mathcal{P}_{m,n}$ with $n \geq m - 1$

Let $v_{m,n}$ denote the normalized volume of $\mathcal{P}_{m,n}$. Heuer and Striker [7, Figure 6] used SageMath to compute $v_{m,n}$ for $m, n \leq 7$. In this section, we consider $v_{m,n}$ for $n \geq m - 1$. The first result is a recursive formula, and is proved by subdividing $\mathcal{P}_{m,n}$ into pyramids, for which the apex is the origin and the base is a facet that does not contain the origin. The same method was used to obtain Theorem 3.1 for the case $n = m - 1$ in [1, Theorem 4.1] and [14, Problem 12191].

Theorem 3.1. *For any m and n with $n \geq m - 1$, the normalized volume of $\mathcal{P}_{m,n}$ is given recursively by*

$$v_{m,n} = \frac{(m-1)!}{m-k!} \sum_{k=1}^m k^{k-2} \frac{v_{m-k,n-k}}{m-k!} \left(kn - \binom{k}{2} \right) \binom{m}{k}, \quad (3.1)$$

with the initial condition $v_{0,n} = 1$.

Example 3.2. For $1 \leq m \leq 7$ and $n \geq m - 1$, Theorem 3.1 gives the following expressions for $v_{m,n}$:

$$\begin{array}{llllllll} v_{1,n} & & & & & & & n, \\ v_{2,n} & -1 & & & & & & 2n^2, \\ v_{3,n} & -6 & -9n & & & & & 6n^3, \\ v_{4,n} & -54 & -96n & -72n^2 & & & & 24n^4, \\ v_{5,n} & -840 & -1350n & -1200n^2 & -600n^3 & & & 120n^5, \\ v_{6,n} & -21150 & -30240n & -24300n^2 & -14400n^3 & -5400n^4 & & 720n^6, \\ v_{7,n} & -782460 & -1036350n & -740880n^2 & -396900n^3 & -176400n^4 & -52920n^5 & 5040n^7. \end{array}$$

Note that the expression $v_{2,n} = 2n^2 - 1$ was obtained in [7, Theorem 5.29].

Using the recursion of (3.1), and building on ideas from [1] and [14, Problem 12191], we are able to obtain the following explicit expressions for $v_{m,n}$.

Theorem 3.3. *For any m and n with $n \geq m - 1$, the normalized volume of $\mathcal{P}_{m,n}$ is explicitly*

$$v_{m,n} = -\frac{m!}{2^m} \sum_{0 \leq i \leq k \leq m} \binom{m}{k} \binom{k}{i} (2i-3)!! \, 2n^{m-k}, \quad (3.2)$$

or equivalently

$$v_{m,n} = -\frac{m!}{2^m} \sum_{i=0}^m \binom{m}{i} (2i-3)!! \, 2n^{m-i}. \quad (3.3)$$

Remark 3.4. We note that, by regarding n as fixed and defining $V_{m,n}$ to be the RHS of (3.2) or (3.3) for any nonnegative integer m (with the restriction $n \geq m - 1$ no longer applying) we have the generating function expression

$$\sum_{m=0}^{\infty} \frac{V_{m,n}}{m!} z^m = \sqrt{1-z} e^{n-1/2} z. \quad (3.4)$$

Furthermore, by defining $W_{m,n} = V_{m,n} / m!$, we have the simple recurrence relation

$$W_{m,n} = m - n - 1 W_{m-1,n} - m - 1 - n - 1/2 W_{m-2,n}, \quad (3.5)$$

with initial conditions $W_{0,m} = 1$ and $W_{1,m} = n$.

4 The Ehrhart polynomial of some partial permutohedra

We now shift our focus to the computation of the Ehrhart polynomial of $\mathcal{P}_{m,n}$ for small fixed values of m or n .

4.1 Ehrhart polynomials

We begin by recalling some basic facts about Ehrhart polynomials. For a lattice polytope $\mathcal{P} \subseteq \mathbb{R}^m$, the function $|t\mathcal{P} \cap \mathbb{Z}^m|$ of a positive integer variable t (where $t\mathcal{P}$ denotes the t -th dilate $\{t\mathbf{x} \mid \mathbf{x} \in \mathcal{P}\}$ of $\mathcal{P}$) is known to agree with a polynomial $\text{Ehr } \mathcal{P} \in \mathbb{Q}[t]$ of degree $\dim \mathcal{P}$, called the *Ehrhart polynomial* of $\mathcal{P}$. Furthermore, the coefficient of the leading term of $\text{Ehr } \mathcal{P}$ is the relative volume of $\mathcal{P}$, where this is a normalized volume if $\mathcal{P}$ is non-full-dimensional, see [2, Corollary 3.20]. For the polytopes $\mathcal{P}_{m,n}$ considered here, the normalized volume is $m!$ times the relative volume.

4.2 The sculpting strategy

All of our remaining proofs follow the same sculpting strategy. We start with a well-known polytope and remove other known polytopes by adding inequalities, until we obtain the desired polytope $\mathcal{P}_{m,n}$. More precisely, to compute the volume or Ehrhart polynomial of $\mathcal{P}_{m,n}$, we create a sequence of lattice polytopes $\mathcal{P}_1, \dots, \mathcal{P}_k \subseteq \mathcal{P}_{m,n}$, where $\mathcal{P}_1$ is either the $\binom{n-1}{2}$ -dilated standard m -simplex $\binom{n-1}{2} \Delta_m$ (in Theorem 4.2) or the m -cube $[0, n]^m$ of side-length n (in Theorem 4.4). We then obtain $\mathcal{P}_{i+1}$ from $\mathcal{P}_i$ by adding inequalities to $\mathcal{P}_i$, i.e., by taking an intersection of $\mathcal{P}_i$ with closed halfspaces, and thus removing some pieces from $\mathcal{P}_i$.

A simple example which illustrates this idea is as follows.

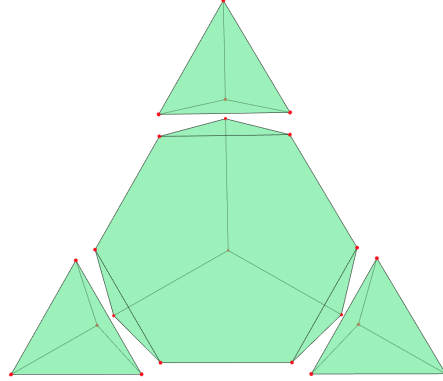


Figure 1: Illustration of $\mathcal{P}_{3,2}$ as $3\Delta_3$, from which three copies of Δ_3 have been removed.

Example 4.1. Figure 1 shows the partial permutohedron $\mathcal{P}_{3,2} = \{\mathbf{x} \in \mathbb{R}^3 \mid 0 \leq x_i \leq 2 \text{ for all } i \in [3], x_1 + x_2 + x_3 \leq 3\}$ as the 3-dilated standard 3-simplex $3\Delta_3$, from which three copies of the standard 3-simplex Δ_3 have been removed.

Further illustrations of the sculpting process are given in Figure 2 (for $\mathcal{P}_{3,3}$) and Figure 3 (for $\mathcal{P}_{3,n}$ with $n \geq 2$).

4.3 New results

We have the following formulas for $\mathcal{P}_{m,n}$ with arbitrary m and $n \leq 4$. The expression for $v_{m,2}$ was conjectured in [7, Conjecture 5.30]. Our sculpting methods become harder as n grows, and for $\mathcal{P}_{m,4}$ we have only computed the volume.

Theorem 4.2. *The following statements are true.*

1. For any m , $\mathcal{P}_{m,1}$ is the standard m -simplex Δ_m , and so $\mathcal{P}_{m,1}$ has Ehrhart polynomial $\text{Ehr } \mathcal{P}_{m,1} = \binom{t+m}{m}$ and normalized volume $v_{m,1} = 1$.
2. For any m , the Ehrhart polynomial of $\mathcal{P}_{m,2}$ is

$$\text{Ehr } \mathcal{P}_{m,2} = \binom{3t+m}{m} - m \binom{t+m-1}{m}, \quad (4.1)$$

and thus, taking $m!$ times the coefficient of t^m in $\text{Ehr } \mathcal{P}_{m,2}$, the normalized volume of $\mathcal{P}_{m,2}$ is

$$v_{m,2} = 3^m - m.$$

3. For any m , the Ehrhart polynomial of $\mathcal{P}_{m,3}$ is

$$\begin{aligned} \text{Ehr } \mathcal{P}_{m,3} &= \binom{6t+m}{m} - m \binom{3t+m-1}{m} \\ &\quad - \binom{m}{2} \left(\binom{t+m-1}{m} + m-2 \binom{t+m-2}{m} \right), \end{aligned} \quad (4.2)$$

and thus, taking $m!$ times the coefficient of t^m in $\text{Ehr } \mathcal{P}_{m,3}$, the normalized volume of $\mathcal{P}_{m,3}$ is

$$v_{m,3} = 6^m - m 3^m - (m-1) \binom{m}{2}.$$

4. For any m , the normalized volume of $\mathcal{P}_{m,4}$ is

$$v_{m,4} = 10^m - m 6^m - \frac{m(m-1)(m-3)}{6} 3^m - 3m^2 - 6m - 1 \binom{m}{3}. \quad (4.3)$$

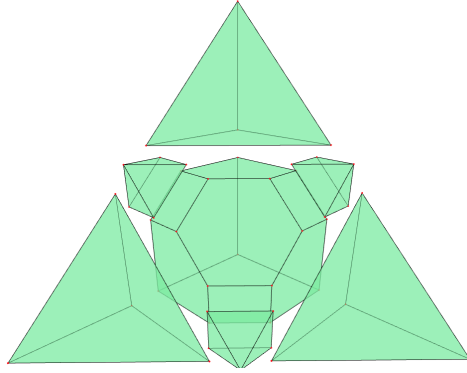


Figure 2: The sculpting of $\mathcal{P}_{3,3}$ (in the center), as used in our proof of (4.2) with $m=3$.

Having calculated $v_{m,n}$ for $n \leq 4$, an obvious open problem remains.

Open Problem 4.3. Find $v_{m,n}$ for all m, n with $n > 4$.

We now return to $\mathcal{P}_{m,n}$ in the case $n \geq m-1$. It follows from Postnikov's results on generalized permutohedra [11], and an interpretation of partial permutohedra as certain generalized permutohedra, that in this case $\text{Ehr } \mathcal{P}_{m,n}$ is a polynomial in n of degree m and that $\mathcal{P}_{m,n}$ is Ehrhart positive (i.e., that each coefficient of $\text{Ehr } \mathcal{P}_{m,n}$, as a polynomial in t , is positive). We obtain explicit expressions for the Ehrhart polynomial of $\mathcal{P}_{m,n}$ for fixed $m \leq 4$ and arbitrary $n \geq m-1$, from which this Ehrhart positivity can also be easily verified.

Theorem 4.4. *The following statements are true.*

1. For $m = 1$ and any n , $\mathcal{P}_{1,n}$ is the line segment $[0, n]$, and so the Ehrhart polynomial of $\mathcal{P}_{1,n}$ is

$$\text{Ehr } \mathcal{P}_{1,n} = nt + 1. \quad (4.4)$$

2. For $m = 2$ and any n , the Ehrhart polynomial of $\mathcal{P}_{2,n}$ is

$$\text{Ehr } \mathcal{P}_{2,n} = n^2 - 1/2 t^2 + 2n - 1/2 t + 1, \quad (4.5)$$

where this can be obtained, as a simple application of the sculpting process, by constructing $\mathcal{P}_{2,n}$ as a square $[0, n]^2$, from which a half-open triangle

$$\text{ConvexHull} \{ [n-1, n], [n, n], [n, n-1] \} \setminus \text{ConvexHull} \{ [n-1, n], [n, n-1] \}$$

has been removed.

3. For any $n \geq 2$, the Ehrhart polynomial of $\mathcal{P}_{3,n}$ is

$$\begin{aligned} \text{Ehr } \mathcal{P}_{3,n} \\ (n^3 - 3n/2 - 1)t^3 + (3n^2 - 3n/2 - 3/2)t^2 + (3n - 3/2)t + 1. \end{aligned} \quad (4.6)$$

4. For any $n \geq 3$, the Ehrhart polynomial of $\mathcal{P}_{4,n}$ is

$$\begin{aligned} \text{Ehr } \mathcal{P}_{4,n} \\ (n^4 - 3n^2 - 4n - 9/4)t^4 + (4n^3 - 3n^2 - 6n - 5/2)t^3 \\ + (6n^2 - 6n - 9/4)t^2 + (4n - 3)t + 1. \end{aligned} \quad (4.7)$$

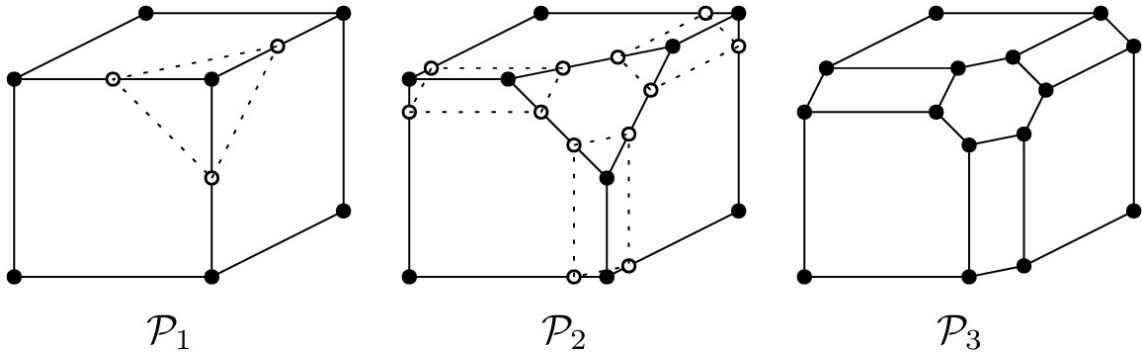


Figure 3: The sculpting of $\mathcal{P}_{3,n}$, as used in our proof of (4.6).

5 Further directions

The expressions in (3.3) and (4.4)–(4.7) lead us to the following conjecture of a completely explicit formula for $\text{Ehr } \mathcal{P}_{m,n}$ with any $n \geq m - 1$.

Conjecture 5.1. *We conjecture that, for any m and n with $n \geq m - 1$, the Ehrhart polynomial of $\mathcal{P}_{m,n}$ is explicitly*

$$\text{Ehr } \mathcal{P}_{m,n} = \frac{1}{2^m} \sum_{i=0}^{\lfloor m/2 \rfloor} \sum_{j=2i}^m (-1)^{i-1} \binom{m}{m-j, j-2i, i} i! (2j-4i-3)!! t^{j-i} 2^{nt} t^{\frac{1}{2}m-j}. \quad (5.1)$$

In terms of generating functions, we have

$$\sum_{m=0}^{\infty} \frac{E_{m,n,t}}{m!} 2z^{\frac{m}{2}} = \sqrt{1-2tz} e^{2nt} t^{\frac{1}{2}z-tz^2}, \quad (5.2)$$

where $E_{m,n,t}$ is the RHS of (5.1) for any nonnegative integer m , with n and t fixed.

Conjecture 5.1 can be seen to generalize Theorem 3.3. Specifically, since $v_{m,n}/m!$ is the leading coefficient of $\text{Ehr } \mathcal{P}_{m,n}$, as a polynomial in t , and since the degree of this polynomial is m (as $\dim \mathcal{P}_{m,n} = m$), it follows that $v_{m,n}/m!$ is given by $\lim_{T \rightarrow 0} (T^m \text{Ehr } \mathcal{P}_{m,n} |_{t=1/T})$. Now (5.2) gives

$$\sum_{m=0}^{\infty} \frac{E_{m,n,1/t}}{m!} 2tz^{\frac{m}{2}} = \sqrt{1-2z} e^{2n-1-2tz-tz^2},$$

and taking $t \rightarrow 0$ on both sides then gives (3.4).

It can also easily be checked that (5.1) reproduces the expressions (4.4)–(4.7) for $\text{Ehr } \mathcal{P}_{m,n}$ with $m \leq 4$ and $n \geq m - 1$.

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A cocharge formula for the Δ -Springer modules

Maria Gillespie^{*1} and Sean T. Griffin^{†2}

¹Department of Mathematics, Colorado State University, Fort Collins, CO, USA

²Department of Mathematics, University of California, Davis, Davis, CA, USA

Abstract. We conjecture a simple combinatorial formula for the Schur expansion of the Frobenius series of the S_n -modules $R_{n,\lambda,s}$, which appear as the cohomology rings of the “ Δ -Springer” varieties. These modules interpolate between the Garsia-Procesi modules R_μ (which are the type A Springer fiber cohomology rings) and the rings $R_{n,k}$ defined by Haglund, Rhoades, and Shimozono in the context of the Delta Conjecture.

Our formula directly generalizes the known cocharge formula for Garsia-Procesi modules and gives a new cocharge formula for the Delta Conjecture at $t = 0$, by introducing *battery-powered tableaux* that “store” extra charge in their battery. Our conjecture has been verified by computer for all $n \leq 10$ and $s \leq \ell(\lambda) - 2$, as well as for $n \leq 8$ and $s \leq \ell(\lambda) - 7$. We prove it holds for several infinite families of n, λ, s .

Keywords: Cocharge, Springer fiber, Hall-Littlewood polynomials, Delta conjecture

1 Introduction and results

The rings $R_{n,\lambda,s}$, for integers n, s and a partition λ with $|\lambda| = k \leq n$ and $s \geq \ell(\lambda)$, were first introduced in [8]. They are graded rings with an S_n -action that interpolate between the well-known Garsia-Procesi modules R_μ [6] and the rings $R_{n,k}$ introduced by Haglund, Rhoades, and Shimozono [12]. The rings R_μ , which coincide with $R_{n,\lambda,s}$ for $\lambda = \mu$, $n = |\mu|$, are isomorphic to the cohomology rings of the Springer fibers and have a natural S_n action, being a symmetric quotient of the polynomial ring $\mathbb{Q}[x_1, \dots, x_n]$. The rings $R_{n,k}$, which coincide with $R_{n,\lambda,s}$ for $\lambda = 1^k$ and $s = k$, arise naturally in the $t = 0$ case of the famous *Delta conjecture* [11] (part of which was proven in [2, 4]) and have two geometric interpretations [10, 14]. The common generalization $R_{n,\lambda,s}$ have been shown to have a geometric interpretation as the cohomology rings of the *Δ -Springer varieties* [10], and we therefore refer to them here as the *Δ -Springer modules*.

The decomposition of a graded S_n -module $R = \bigoplus_d R_d$ into irreducibles can be described by its *graded Frobenius character*

$$\text{grFrob } R; q : \sum_d \text{Frob } R_d q^d$$

^{*}Maria.Gillespie@colostate.edu. Partially supported by NSF DMS award number 2054391.

[†]stgriffin@ucdavis.edu. Partially supported by NSF Grant DMS-1439786 while in residence at ICERM during the Spring 2021 semester.

where R_d is the d -th graded piece and Frob is the additive map on representations that sends the irreducible S_n -module V_ν to the Schur function s_ν .

For the Garsia-Procesi modules R_μ , it is known (see [6]) that $\text{grFrob } R_\mu; q = \tilde{H}_\mu(x; q)$ where $\tilde{H}_\mu(x; q)$ are the (transformed) symmetric *Hall-Littlewood polynomials*. Lascoux and Schutzenberger [13] provided the following explicit combinatorial formula for their Schur expansions. For a partition μ , define $\text{SSYT } \mu$ to be the set of all (straight shape) **semistandard Young tableaux** of **content** μ , meaning that the tableau entries consist of μ_i copies of i for each i , and the entries are weakly increasing across rows and strictly increasing up columns in French notation (as in the “device” part of the tableau at left in Figure 1). Then

$$\text{grFrob } R_\mu; q = \tilde{H}_\mu(x; q) = \sum_{T \in \text{SSYT } \mu} q^{\text{cc } T} s_{\text{sh } T}$$

where $\text{sh } T$ is the **shape** of the tableau T , that is, the partition whose i -th part is the length of the i -th row of T , and $s_{\text{sh } T}$ is the corresponding Schur function. Here cc is the *cocharge* statistic as defined in Section 2 below.

In this paper we provide the following full conjectural formula for $\text{grFrob } R_{n,\lambda,s}; q$ that directly generalizes the above.

Definition 1.1. For a fixed n, λ, s with $k = |\lambda| \leq s$, define $\Lambda_{n,\lambda,s}$ to be the partition formed by adding an $s \times (n - k)$ rectangle at the left of the diagram of λ . In other words $\Lambda_{n,\lambda,s} = (n - k, n - k, \dots, n - k, \lambda_1, n - k, \lambda_2, \dots, n - k, \lambda_r, n - k, \dots, n - k)$ where there are s parts in total. As an example, for $n = 8, \lambda = (2, 1, 1), s = 4$, we have $\Lambda_{n,\lambda,s} = (6, 5, 5, 4)$.

Definition 1.2. A **battery-powered tableau** of parameters n, λ, s consists of a pair (D, B) of semistandard Young tableaux, where B is rectangular of shape $(s - 1) \times (n - k)$, and the total content of D and B is $\Lambda_{n,\lambda,s}$. We call D the **device** of T and B the **battery**. We define the **shape** of T to be the shape of its device, that is, $\text{sh } T = \text{sh } D$.

We write $\mathcal{T}_{n,\lambda,s}$ to denote the set of all battery-powered tableaux of parameters n, λ, s . For $T \in \mathcal{T}_{n,\lambda,s}$, we write $\text{cc } T$ and $\text{ch } T$, respectively to denote the cocharge and charge of the word formed by concatenating the reading words of D and B in that order (see Section 2).

Remark 1.3. We will usually draw the battery down-and-right from the device, as in Figure 1, so that the device and the battery together form a **skew tableau** (that is, a tableau of shape θ/ρ , where θ/ρ is formed by deleting the diagram of a partition ρ from a larger partition θ). We write this tableau as $D \cdot B$ so that $\text{cc } T = \text{cc } D \cdot B$.

Conjecture 1.4. We have

$$\tilde{H}_{n,\lambda,s}(x; q) = \text{grFrob } R_{n,\lambda,s}; q = \frac{1}{q^{\frac{s-1}{2}(n-k)}} \sum_{T \in \mathcal{T}_{n,\lambda,s}} q^{\text{cc } T} s_{\text{sh } T}.$$

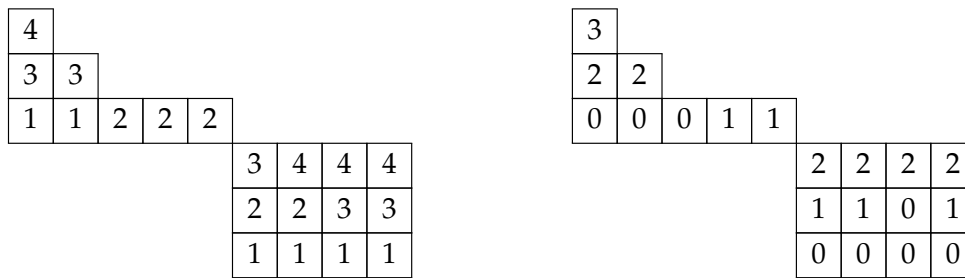


Figure 1: At left, a battery-powered tableau T for $n = 8$, $\lambda = 2, 1, 1$, and $s = 4$. Its shape is $5, 2, 1$. The cocharge labels are shown at right, giving $\text{cc}(T) = 20$.

We think of the battery as storing extra charge for the device. The q -exponent $\binom{s-1}{2} n - k$ is the largest amount of cocharge that may be stored in the battery.

Example 1.5. Suppose $n = 8$, $\lambda = (2, 1, 1)$, and $s = 4$. Then $\Lambda_{n,\lambda,s} = (6, 5, 5, 4)$ and an example of a battery-powered tableau is shown in Figure 1. Its cocharge is 20 and shape is $(5, 2, 1)$, and the normalization factor in Conjecture 1.4 is $q^{-\frac{3}{2} \cdot 4} = q^{-12}$, so one of the terms of the summation above is $q^{-12} \cdot q^{20s}_{5,2,1} = q^{8s}_{5,2,1}$.

By applying rev_q (reversing the coefficients of the q polynomial by setting $q \rightarrow q^{-1}$ and multiplying by q^d where d is the degree) to [Conjecture 1.4](#), we can obtain the following alternative simpler expansion in terms of the generalized charge statistic.

Conjecture 1.6. *We have*

$$\mathrm{rev}_q \left(\tilde{H}_{n,\lambda,s} \right) = \mathrm{rev}_q \circ \mathrm{grFrob} \circ R_{n,\lambda,s} \sum_{T \in \mathcal{T}_{n,\lambda,s}} q^{\mathrm{ch} \ T} s_{\mathrm{sh} \ T}.$$

Using Sage [15], we have tested the above conjectures for all n, λ, s such that $n \leq 10$ and $s \leq \ell(\lambda) - 2$ (where $\ell(\lambda)$ is the number of parts of λ), as well as for $n \leq 8$ and $s \leq \ell(\lambda) - 7$. The following theorem summarizes our progress towards a proof.

Theorem 1. *Conjecture 1.4 holds for $n - k \geq 1$ (and any n, s), for $s \geq 2$ (and any n, λ), and in the $R_{n,k}$ case for the coefficient of s^n . Furthermore, for any n, λ, s , the highest degree terms match, which are both of q -degree $n - \lambda \leq n - k - s - 1$.*

Note that the conjecture also agrees with the Lascoux-Schutzenberger formula for R_u , since there is no battery when $n - k = 0$.

Below, we provide more background and terminology in [Section 2](#). In [Section 3](#) we prove the conjecture for $n - k = 1$. We prove that [Conjectures 1.4](#) and [1.6](#) are equivalent in [Section 4](#) and also show that the highest degree terms match. In [Section 5](#) we give a brief sketch of our proof for $s = 2$. Finally in [Section 6](#) we explore the connections to the *minimaj* formula [[1](#), [11](#)] for $\text{grFrob } R_{n,k}; q$. Full versions of the proofs sketched here will be provided in [[7](#)].

2 Background and definitions

2.1 Cocharge and insertion

The **reading word** of a tableau is the word formed by concatenating the rows from top to bottom. For instance, the reading word of the battery-powered tableau in [Figure 1](#) is

$$4\ 3\ 3\ 1\ 1\ 2\ 2\ 2\ 3\ 4\ 4\ 4\ 2\ 2\ 3\ 3\ 1\ 1\ 1\ 1\ .$$

The **first cocharge subword** is formed by searching right to left in the reading word for a 1, then continuing from that position to search for a 2 (wrapping around the end cyclically if necessary), and so on until we have reached the largest letter of the word:

$$4\ 3\ 3\ 1\ 1\ 2\ 2\ 2\ 3\ 4\ 4\ 4\ 2\ 2\ 3\ 3\ 1\ 1\ 1\ 1\ .$$

The **cocharge labeling** of a permutation is computed by searching right to left cyclically as before, labeling the entries $1, 2, 3, \dots$ in order, and starting by labeling the 1 with a 0 and incrementing the label if and only if the next entry is to the left of the previous:

$$4_3\ 3\ 3\ 1\ 1\ 2\ 2\ 2\ 3_2\ 4\ 4\ 4\ 2\ 2_1\ 3\ 3\ 1\ 1\ 1\ 1_0.$$

We then similarly find and label the *second cocharge subword* among the unlabeled letters:

$$4_3\ 3\ 3_2\ 1\ 1\ 2\ 2\ 2\ 3_2\ 4\ 4\ 4_2\ 2_1\ 2_1\ 3\ 3\ 1\ 1\ 1_0\ 1_0.$$

We continue to iterate this process on the unlabeled letters until all have been labeled:

$$4_3\ 3_2\ 3_2\ 1_0\ 1_0\ 2_0\ 2_1\ 2_1\ 3_2\ 4_2\ 4_2\ 4_2\ 2_1\ 2_1\ 3_0\ 3_1\ 1_0\ 1_0\ 1_0.$$

In [Figure 1](#), the cocharge labels on the reading word elements are shown in the corresponding squares at right. The **charge labels** are placed in the same order as cocharge labels except we increment when the next element is to the **right** of the previous.

The **cocharge** (resp. **charge**) of T , written $cc\ T$ and $ch\ T$ respectively, is the sum of the cocharge (resp. charge) labels of its reading word. Therefore, the cocharge of the word above is $3\ 2\ 2\ 1\ 1\ 2\ 2\ 2\ 2\ 1\ 1\ 1\ 20$.

The **RSK insertion** or **bumping** of a letter i into a tableau T is the tableau T' formed by row inserting i into the bottom row of T , where it is placed at the end if i is greater than or equal to every element of the row and otherwise it replaces the leftmost entry m greater than i . Then m is inserted into the second row in the same manner, and so on until the process is complete and a new entry is added. It is well known [5] that this RSK insertion algorithm is bijective, and we call the reverse process **unbumping**. We also say the **RSK insertion** of a tableau B into a tableau D (such as in the case of a battery B and device D) is the tableau T' formed by inserting the letters of the reading word of B one at a time into D .

Cocharge and charge are invariant under bumping: we have $\text{ch } D \cdot B = \text{cc } T'$ and $\text{ch } D \cdot B = \text{ch } T'$ where T' is the insertion of B into D . This is because RSK insertion preserves the *Knuth equivalence class* of the reading word [5], and cocharge and charge are invariant under Knuth equivalence [13].

2.2 S_μ -invariants

A **symmetric function** is a formal power series in infinitely variables $x_1, x_2, \dots$ with coefficients in $\mathbb{Q}[q]$. We refer to [5] for background on symmetric functions and graded Frobenius characteristic (see Section 2 for the definition), including the definition of Schur functions. We will need the following fact about the Frobenius characteristic.

Let V be an S_K -module, and let $S_\mu \subseteq S_K$ be a Young subgroup. Then the S_μ -**anti-invariants** (or **alternants**) of V are

$$V^{S_\mu\text{-anti}} : \{v \in V : \sigma \cdot v = \text{sgn } \sigma \cdot v \text{ for all } \sigma \in S_\mu\}.$$

Suppose μ is a refinement of the composition $1^n, K - n$. Then $V^{S_\mu\text{-anti}}$ is an S_n -module, and it is well known (as in [12]) that

$$\text{Frob } V^{S_\mu\text{-anti}} = e_{\mu_n}^\perp e_{\mu_{n-2}}^\perp \cdots e_{\mu_\ell}^\perp \text{Frob } V.$$

2.3 Spaltenstein and Δ -Springer varieties

Given K an integer and μ a composition of K with nonzero parts, let $\mathcal{B}_\mu$ be the partial flag variety of partial flags in $\mathbb{C}^K$ with dimension jumps recorded by the parts of μ . Given a $K \times K$ nilpotent matrix X of Jordan type ν , let $\mathcal{B}_\mu^\nu$ be the **Spaltenstein variety**,

$$\mathcal{B}_\mu^\nu : \{V_\bullet \in \mathcal{B}_\mu : XV_i \subseteq V_{i-1} \text{ for all } i \leq \ell(\mu)\}.$$

When $\mu = 1^K$, then the Spaltenstein variety is the Springer fiber indexed by Jordan type ν . Letting $H^* =$ denote singular cohomology with $\mathbb{C}$ coefficients, Borho and Macpherson [3] proved the following isomorphism of graded $\mathbb{C}$ -vector spaces,

$$H^* \mathcal{B}_\mu^\nu \sim H^* \mathcal{B}^\nu S_\mu\text{-anti } 2d_\mu \quad (2.1)$$

where $d_\mu = \dim \mathcal{B} - \dim \mathcal{B}_\mu = \sum_i \frac{\mu_i}{2}$ and the $2d_\mu$ denotes a shift in degree.

The Spaltenstein variety has the following connection with Δ -Springer varieties. Let $\nu = \Lambda$, let $K = |\Lambda|$, and let $\mu = 1^n, s-1, n-k$, which is the composition consisting of n many 1s followed by $n-k$ many $s-1$'s. Letting $\pi_n : \mathcal{B}_\mu \rightarrow \mathcal{B}_{1^n, K-n}$, be the projection map remembering only the first n many parts of the flag (and the ambient space), then the Δ -**Springer variety** $Y_{n,\lambda,s}$ [10] is

$$Y_{n,\lambda,s} = \pi_n \mathcal{B}_\mu^\Lambda.$$

The map π_n gives us a map on cohomology

$$\pi_n^* : H^* Y_{n,\lambda,s} \hookrightarrow H^* \mathcal{B}_\mu^\Lambda,$$

that is injective by [10], and the map is the one induced by sending x_i to x_i for all i . Letting S_n act on $x_1, \dots, x_n$ by permutation of the indices, then $H^* \mathcal{B}_\mu^\Lambda$ is an S_n -module, and in fact $H^* Y_{n,\lambda,s}$ is an S_n -submodule of $H^* \mathcal{B}_\mu^\Lambda \sim H^* \mathcal{B}^{\Lambda, S_\mu\text{-anti}} 2d_\mu$.

3 Spaltenstein motivation and a special case algebraically

In this subsection, we use the geometry of Δ -Springer varieties to prove [Conjecture 1.4](#) in the case when $n - k = 1$. We then explain how this case helped us find the general case of the conjecture.

Fix $n - k = 1$. Then the projection π_n from [Section 2](#) is the identity map, and $Y_{n,\lambda,s} = \mathcal{B}_\mu^\Lambda$, where $\mu = 1^n, s = 1$. Therefore, in this case we have that $H^* Y_{n,\lambda,s} \sim H^* \mathcal{B}^{\Lambda, S_\mu\text{-anti}} 2d_\mu$ by (2.1). Since $\text{grFrob } H^* \mathcal{B}^\Lambda; q = \text{grFrob } R_\Lambda; q = \tilde{H}_\Lambda; q$, then

$$\tilde{H}_{n,\lambda,s} = q^{-n} \mu^T e_{s-1}^\perp \tilde{H}_\Lambda.$$

For the proof below, recall that a **j -vertical strip** in a partition ν is a skew shape ν/μ of size j with no two cells in the same row. Also recall a **j -horizontal strip** is a skew shape ν/μ of size j with no two cells in the same column.

Proof of [Conjecture 1.4](#) when $n - k = 1$. When $n - k = 1$, then $n = \mu^T = \frac{s-1}{2}$, so it suffices to show

$$q^{\frac{s-1}{2}} \langle s_\mu, \tilde{H}_{n,\lambda,s} \rangle = \langle s_\mu, e_{s-1}^\perp \tilde{H}_\Lambda \rangle = \sum_{\substack{T', C \in \mathcal{T}_{n,\lambda,s} \\ \text{sh } T' = \mu}} q^{\text{cc } S} \quad (3.1)$$

for all $\mu \vdash |\Lambda|$. Letting $N = |\Lambda|$, we have

$$\langle s_\mu, e_{s-1}^\perp \tilde{H}_\Lambda \rangle = \langle e_{s-1} s_\mu, \tilde{H}_\Lambda \rangle = \sum_{\substack{\nu \vdash N \\ \nu/\mu \text{ a } s-1\text{-vertical strip}}} \sum_{T \in \text{SSYT } \nu, \Lambda} q^{\text{cc } T} \quad (3.2)$$

which follows from the Pieri rule for $e_{s-1} s_\mu$ and the cocharge formula for $\tilde{H}_\Lambda$.

We now give a bijection between the tableaux T that we are summing over above and the battery-powered tableaux $T = B, D \in \mathcal{T}_{n,\lambda,s}$ with $\text{sh } B = \mu$ such that, if $T \mapsto T', C$, we have $\text{cc } T = \text{cc } T', C$. This will imply that (3.2) is equal to

$$\sum_{\substack{T', C \in \mathcal{T}_{n,\lambda,s} \\ \text{sh } T' = \mu}} q^{\text{cc } S}$$

and the result will follow.

The bijection is as follows: consider the entries of T in the squares of ν/μ , and unbump them in order from top to bottom, recording the entries that get bumped out of the tableau in a column C from bottom to top. Notice that each unbumping path P_i is weakly to the right of the previous path P_{i-1} (see [5] for proofs of these facts about unbumping). Moreover, the previous path P_{i-1} bumped out an entry w_{i-1} from the bottom row and replaced it with a larger entry (because it is not the last bumping path of the vertical strip and so we are not starting from the bottom row). It follows that the next path P_i bumps out an entry w_i that is strictly larger than w_{i-1} . This means C is increasing from bottom to top, and so we get a pair $(T', C) \in \mathcal{T}_{n, \lambda, s}$.

This process is a bijection because given a (T', C) , inserting the column C into T' adds a vertical strip to the shape μ to form a tableau T of content Λ . Finally, cocharge is constant on Knuth equivalence classes and hence is unchanged under bumping.

Thus, (3.1) holds and the result holds when $n - k = 1$. $\square$

We now explain the motivation behind [Conjecture 1.4](#). Recall that in general the cohomology ring $H^* Y_{n, \lambda, s}$ is a graded S_n -submodule of $H^* \mathcal{B}^\Lambda S_{\mu\text{-anti}} 2d_\mu$, whose graded Frobenius characteristic is the symmetric function $q^{-\frac{s-1}{2}n-k} e^\perp_{s-1, n-k} \tilde{H}_\Lambda$. By a similar argument as above, the s_μ coefficient of $e^\perp_{s-1, n-k} \tilde{H}_\Lambda$ is

$$\sum_{S = T' \cdot C_1 \cdot C_2 \cdots C_{n-k}} q^{\text{cc } S},$$

where the sum is over all $T' \in \text{SSYT } \mu$, and $C_i \in \text{SSYT } 1^{s-1}$ for all i . Therefore,

$$q^{\frac{s-1}{2}n-k} \langle s_\mu, \tilde{H}_{n, \lambda, s} \rangle \leq \sum_{S = T' \cdot C_1 \cdot C_2 \cdots C_{n-k}} q^{\text{cc } S} \quad (3.3)$$

where the inequality is coefficient-wise for each power of q . Thus, [Conjecture 1.4](#) can be interpreted as stating that the terms of the right-hand side of (3.3) that correspond to $\tilde{H}_{n, \lambda, s}$ are those such that $C_1 \cdot C_2 \cdots C_{n-k}$ rectifies to an element of $\text{SSYT } n - k^{s-1}$.

4 Equivalence of conjectures and highest degree terms

For any partition ν , define the statistic $n(\nu) = \sum_i i - 1 \nu_i$.

Proposition 4.1. *The maximum value of $\text{cc } T$ for $T \in \mathcal{T}_{n, \lambda, s}$ is*

$$n(\lambda) + \binom{s}{2} (n - k).$$

Moreover, there is precisely one tableau T with this value of cc for each device shape ν with $\ell(\nu) \leq s$ and where ν/λ is a horizontal strip.

Proof. The maximal cocharge among all words of a given content Λ occurs when each cocharge subword has its letters appearing in order from right to left, and in that case the cocharge is $n - \Lambda$. For this to occur, the battery columns must be filled with $1, 2, \dots, s-1$ from bottom to top, for otherwise some entry of the battery B would be to the right of the previous element in its cocharge subword. The subwords starting at the 1's in the bottom of B will then contain $1, 2, \dots, s$ from right to left, with the s being in the device.

For the cocharge subwords starting at 1's in the device D to be in right to left order, D must contain the unique tableau D' of content λ and shape λ (with λ_i entries i in the i -th row for all i). So, D is formed by adding a horizontal strip of length $n - k$ labeled by s to D' such that the result is semistandard. Thus there is one tableau of maximal cocharge for each shape of height $\leq s$ formed by adding a horizontal strip to λ .

For such pairs D, B , we have $\text{cc}(D, B) = n - \Lambda = n - \lambda + \frac{s-1}{2}(n-k)$, as desired. $\square$

Dividing out by the factor $q^{\frac{s-1}{2}(n-k)}$, we obtain the following corollary.

Corollary 4.2. *The top q -degree of the polynomial on the right hand side of [Conjecture 1.4](#) is $d = n - \lambda + \frac{s-1}{2}(n-k)$, and the coefficient of q^d is $\sum_{s_\nu} 1$ where the sum ranges over all partitions ν of n with $\ell(\nu) \leq s$ and ν/λ a horizontal strip.*

The value d matches with the formula given for the top degree of $\text{grFrob}_q R_{n,\lambda,s}$ in [8]. In [10], it was shown that the coefficient of q^d is the skew Schur function $s_{\Lambda/\lambda} = s_{\nu/\lambda}$. A straightforward application of the Littlewood-Richardson rule shows that this agrees with our formula in [Corollary 4.2](#), and we refer to [7] for details.

Finally, we show that [Conjectures 1.4](#) and [1.6](#) are equivalent. Taking the q -reversal of both sides of [Conjecture 1.4](#), we have

$$\text{rev}_q \left(\tilde{H}_{n,\lambda,s} \right) = \sum_{T \in \mathcal{T}_{n,\lambda,s}} q^{n - \lambda + \frac{s-1}{2}(n-k) - \text{cc}(T)} s_{\text{sh}(T)}.$$

Then the exponent $n - \lambda + \frac{s-1}{2}(n-k) - \text{cc}(T)$ is equal to $n - \Lambda - \text{cc}(T)$, which is simply $\text{ch}(T)$ by the definition of charge. This gives [Conjecture 1.6](#).

5 The case $s = 2$

To prove [Conjecture 1.6](#) in the case $s = 2$, we first outline a general strategy for a proof in all cases. Consider the expansion of $\tilde{H}_{n,\lambda,s}$ into Hall-Littlewood polynomials [9]:

$$\text{rev}_q \left(\tilde{H}_{n,\lambda,s}(X; q) \right) = \sum_{\substack{\alpha \vdash n, \\ \alpha \supset \lambda}} q^{n - \alpha/\lambda - \text{coinv}(\alpha)} H_{\text{sort}(\alpha)}(x; q) \quad (5.1)$$

Above, α is a weak composition of n of length s , and $\text{sort } \alpha$ is the partition formed by sorting the parts of α in nonincreasing order. The quantity $n \alpha / \lambda$ is the sum $\sum \frac{c_i}{2}$ where c_i is the number of boxes in the i -th column of the diagram of $\text{sort } \alpha / \lambda$.

The polynomials $H_{\text{sort } \alpha}$, written without the tilde, denote the *charge* version of Hall-Littlewood polynomials, which expand as $H_{\mu}(x; q) = \sum_{T \in \text{SSYT } \mu} q^{\text{ch } T} s_{\text{sh } T}$. Substituting into (5.1) yields

$$\text{rev}_q \left(\tilde{H}_{n, \lambda, s}(X; q) \right) = \sum_{\substack{\alpha = (\alpha_1, \dots, \alpha_s) \\ \alpha \supset \lambda}} \sum_{n, T \in \text{SSYT } \text{sort } \alpha} q^{n \alpha / \lambda - \text{coinv } \alpha - \text{ch } T} s_{\text{sh } T} \quad (5.2)$$

Thus, to prove [Conjecture 1.6](#), removing both rev_q 's, it suffices to show that

$$\sum_{T \in \mathcal{T}_{n, \lambda, s}} q^{\text{ch } T} s_{\text{sh } T} = \sum_{\substack{\alpha = (\alpha_1, \dots, \alpha_s) \\ \alpha \supset \lambda}} \sum_{n, U \in \text{SSYT } \text{sort } \alpha} q^{n \alpha / \lambda - \text{coinv } \alpha - \text{ch } U} s_{\text{sh } U}.$$

To prove this, we need a shape-preserving bijection between $\mathcal{T}_{n, \lambda, s}$ and

$$\mathcal{A}_{n, \lambda, s} : \{ (\alpha, U) \mid \alpha = (\alpha_1, \dots, \alpha_s) \mid n, \alpha \supset \lambda, U \in \text{SSYT } \text{sort } \alpha \}$$

such that, if T corresponds to (α, U) , then $\text{ch } T = \text{ch } U - n \alpha / \lambda - \text{coinv } \alpha$.

We now give such a bijection in the case $s = 2$. To do so, we begin with the following observation about the compositions α .

Definition 5.1. Let $\lambda = (\lambda_1, \lambda_2)$ be a partition with $\lambda_1 \geq \lambda_2 \geq 0$, and let $\alpha = (\alpha_1, \alpha_2)$ be a composition that contains λ . Then define $\varphi \alpha$ to be the composition formed by moving $n \alpha / \lambda - \text{coinv } \alpha$ boxes from the bottom row of $\text{sort } \alpha$ to the top row.

As a running example, let $n = 11$, $\lambda = (3, 1)$, $s = 2$, and $\alpha = (5, 6)$. Then $n \alpha / \lambda - \text{coinv } \alpha = 2 - 1 = 3$. Since $\text{sort } \alpha = (6, 5)$, then $\varphi \alpha = (3, 8)$.

Now, let $\lambda = (\lambda_1, \lambda_2)$ with $\lambda_1 - \lambda_2 = k \leq n$ and $\lambda_1 \geq \lambda_2 \geq 0$. We construct a bijection from $\mathcal{A}_{n, \lambda, 2}$ to $\mathcal{T}_{n, \lambda, 2}$ as follows.

Definition 5.2. Let $(\alpha, U) \in \mathcal{A}_{n, \lambda, 2}$. Define ψU to be the tableau formed by changing 1's to 2's in the bottom row of U , starting with the rightmost 1 and moving leftwards, until we obtain a tableau of content $\varphi \alpha$.

Continuing our running example with $\alpha = (5, 6)$, letting U be the following tableau with $\text{ch } U = 2$, then ψU is as below:

$$U = \begin{array}{|c|c|c|c|c|c|c|c|} \hline 2 & 2 & 2 & & & & & \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 \\ \hline \end{array}, \quad \psi U = \begin{array}{|c|c|c|c|c|c|c|c|} \hline 2 & 2 & 2 & & & & & \\ \hline 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 \\ \hline \end{array}$$

Remark 5.3. The tableau ψU is not necessarily semistandard; it may have columns containing two 2's.

Definition 5.4. Let $\alpha, U \in \mathcal{A}_{n,\lambda,2}$. Define $\Phi \alpha, U$ as follows. First, compute ψU , and add 1's to the left of the bottom row and 2's to the left of the top row until the resulting tableau S has content $\Lambda_{n,\lambda,2}$ (and then left-justifying). Then, unbump a horizontal strip of size $n - k$ from S from right to left to form a tableau D of the same shape as U , and an unbumped row B of length $n - k$. We set $\Phi \alpha, U = D, B$.

For our running example, we have $\Lambda_{n,\lambda,s} = 10, 8$ and

$$\Phi \alpha, U = \begin{array}{|c|c|c|} \hline 2 & 2 & 2 \\ \hline 1 & 1 & 1 \\ \hline \end{array} \begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline \end{array} \begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 \\ \hline \end{array}$$

so that $\text{ch } \Phi \alpha, U = 5 = \text{ch } U = n - \alpha/\lambda = \text{coinv } \alpha$. We have the following result, whose proof we omit and refer to [7].

Theorem 2. The map Φ is a bijection from $\mathcal{A}_{n,\lambda,2}$ to $\mathcal{T}_{n,\lambda,s}$. Moreover, if $T = \Phi \alpha, U$ then $\text{ch } T = \text{ch } U = n - \alpha/\lambda = \text{coinv } \alpha$.

6 Bijection for coefficient of s_n for $R_{n,k}$ case

We now consider the setting in which $\lambda = 1^k$ and $s = k$, so that $R_{n,\lambda,s} = R_{n,k}$. We recall the positive Schur expansion of $\text{grFrob}_q R_{n,k}$ given in [1]. An **ordered set partition**, or OSP, of n is a partition of $\{1, 2, \dots, n\}$ into a disjoint union of subsets called **blocks**, along with an ordering of the blocks from left to right. For instance, $45|367|28|19$ denotes an OSP of 9.

A **descent** of a permutation π is an index d such that $\pi_d > \pi_{d+1}$, and the **major index** of π is the sum of its descents. The **minimaj** of an OSP is the major index of the *minimaj word* formed by ordering each block's entries from least to greatest and then reading the letters in the OSP from left to right. For instance, the associated word to $45|367|28|19$ is 453672819, and it has descents in positions 2, 5, 7, so the minimaj is $2 + 5 + 7 = 14$.

The **reading word** $\text{rw } P$ of an OSP P (different from its minimaj word) is formed by reading the smallest entry of each block from right to left, and then the remaining entries from left to right. For instance, the reading word of $45|367|28|19$ is 123456789.

It follows from the general Schur expansion of $\text{grFrob } R_{n,k}$ in [1] that the coefficient of the Schur function s_n is

$$\sum_{\substack{P \in \text{OSP } n \\ \text{rw } P = 123 \dots n}} q^{\text{minimaj } P}. \quad (6.1)$$

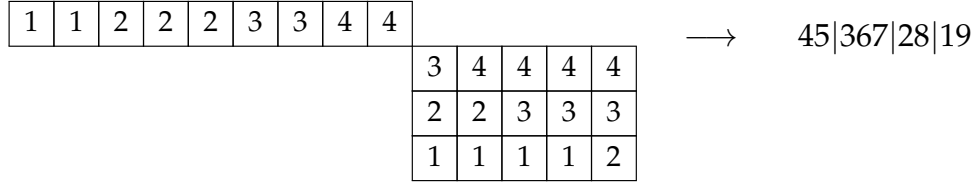


Figure 2: A battery-powered tableau T of shape 9 for $\lambda = 1^4$ and $s = 4$, and the corresponding ordered set partition P . We have $\text{ch } T = \text{minimaj } P = 14$.

On the other hand, the coefficient of s_n in the ch formula of [Conjecture 1.6](#) is

$$\sum_{\substack{T \in \mathcal{T}_{n, 1^k, k} \\ \text{sh } T = n}} q^{\text{ch } T} \quad (6.2)$$

Define a bijection f from the set of T tableaux in (6.2) to the OPSs in (6.1) as follows.

Definition 6.1. Given $T \in \mathcal{T}_{n, 1^k, k}$ with shape n , define $f(T)$ to be the OSP P constructed as follows. Let P have exactly k blocks $B_1, \dots, B_k$ in that order, which initially contain $k, k-1, k-2, \dots, 1$ respectively. Then let m_i be the number of i 's in the device of T , and place the numbers $k-1, k-2, \dots, n$ into the blocks from left to right in the unique way so that each block B_i has size m_i for all i . The resulting OSP is P .

The map f is depicted in [Figure 2](#). To show it is well-defined, we have $\Lambda_{n, 1^k, k} = n - k - 1^k$, and so $\mathcal{T}$ has exactly $n - k - 1$ copies of each letter from 1 through k . Thus we have $m_i \geq 1$ for all i , so P is a well-defined OSP. By its construction, the reading word of P is $123 \cdots n$, and the process is reversible since there is a unique way to fill the one-row device and the battery for any sequence of block sizes m_i . Thus f is a bijection.

To see that f is weight-preserving, sending ch to minimaj , we refer the reader to [\[7\]](#) for details. As a sketch proof, the charge labels of the battery in this case are always either 0 or 1, with the 1 labels being precisely on the entries of the battery that are larger than their row index. In the device, all of the charge labels are 0 except for those of the final charge subword, which is $123 \cdots k$ in order, which has charge $\frac{k}{2}$. This is the minimaj value formed by placing $k, k-1, \dots, 1$ in the blocks from left to right, and then placing the remaining letters in the blocks increases the minimaj by precisely the amount of charge stored in the battery.

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Shifted key polynomials

Eric Marberg^{*1} and Travis Scrimshaw⁺²

¹Department of Mathematics, Hong Kong University of Science and Technology

²Department of Mathematics, Hokkaido University

Abstract. We introduce shifted analogues of key (resp. atom) polynomials that we call P - and Q -key (resp. atom) polynomials. These families are defined in terms of isobaric divided difference operators applied to dominant symplectic and orthogonal Schubert polynomials. We establish a number of fundamental properties of these functions, formally similar to classical results on key polynomials. For example, we show that our shifted key polynomials are partial versions of Schur P - and Q -functions in a precise sense. We conjecture that symplectic/orthogonal Schubert polynomials expand positively in terms of P/Q -key polynomials. As evidence for this conjecture, we also show that shifted key polynomials are the characters of certain shifted analogues of Demazure crystals.

Keywords: Schur P/Q -function, key polynomial, Lie superalgebra, Demazure crystal

1 Introduction

Fix a positive integer n . Set $G = \mathrm{GL}_n(\mathbb{C})$ and write $B \subseteq G$ for the Borel subgroup of upper triangular matrices. Let K be either the orthogonal group $O_n(\mathbb{C})$ or symplectic group $\mathrm{Sp}_n(\mathbb{C})$ (when n is even). For brevity, we omit the rank and field. The *(complete) flag variety* G/B decomposes into finitely many B -orbits indexed by the symmetric group S_n . The closures of these orbits give rise to the *Schubert classes* in $H^*(G/B)$. The cohomology ring $H^*(G/B)$ is naturally a quotient of $\mathbb{Z}[x_1, x_2, \dots]$ and the well-known *Schubert polynomials* $\mathfrak{S}_w \in \mathbb{Z}[x]$ for $w \in S_n$ provide representatives for the Schubert classes.

The flag variety G/B also decomposes into finitely many K -orbits, indexed by involutions in S_n when $K = O$ and by fixed-point-free involutions in S_n when $K = \mathrm{Sp}$ and n is even. The closures of the K -orbits give rise to cohomology classes in $H^*(G/B)$ that are positive sums of Schubert classes. Polynomial representatives for these classes are provided by the *orthogonal* and *symplectic Schubert polynomials* (which we abbreviate as *K-Schubert polynomials*) characterized in [22]. A precise expansion of K -Schubert polynomials into usual Schubert polynomials was given in [3]; see also [5, 8].

It is known [15, Ex. 2.2.2] that $\mathfrak{S}_w = x^\lambda$ whenever $w \in S_n$ has *Rothe diagram* equal to the Young diagram of a partition $\lambda \vdash w$. Such w are called *dominant* and correspond

^{*}emarberg@ust.hk. Partially supported by Hong Kong RGC grant GRF 16306120.

⁺tscrimsg@gmail.com. Partially supported by Grant-in-Aid for JSPS Fellows 21F51028.

to GL dominant weights. There exists a stable limit F_w of $\mathfrak{S}_w$ known as a *Stanley symmetric function* [21], and when w is dominant, it holds that $F_w = s_{\lambda(w)}$ is a Schur function. Combining the *Billey–Jockusch–Stanley (BJS) formula* [2] for $\mathfrak{S}_w$ with the stable limit shows that if $w \in S_n$ is dominant, then $s_{\lambda(w)}$ is the weight-generating function for the set $\text{RF } w$ of factorizations of reduced words for w into decreasing subwords. (We refer to elements of $\text{RF } w$ as *reduced factorizations*.) Morse and Schilling showed this directly in [19] by constructing a (*Kashiwara*) *crystal* [12] on the set $\text{RF } w$.

The BJS formula for $\mathfrak{S}_w$ is a sum over a certain class of *bounded reduced factorizations* $\text{BRF } w \subset \text{RF } w$, and so we can restrict the crystal structure on $\text{RF } w$ to this subset and consider the resulting connected components. These connected components were shown in [1] to be the crystals for B -representations called *Demazure modules* that are constructed as “partial” versions of highest weight GL-representations. The characters of these $\mathfrak{gl}_n$ -*Demazure crystals* are the so-called *key polynomials* $\kappa_{u\lambda}$; here $u \in S_n$ and λ is a partition with at most n parts.

The precise definition of a key polynomial is $\kappa_{u\lambda} = \pi_u \mathfrak{S}_w$ where π_u is an *isobaric divided difference operator* and w is a dominant permutation with $\lambda(w) = \lambda$. For each choice of $K \in \{O, \text{Sp}\}$, there is an analogous notion of a *K-dominant* involution z . These elements index the K -Schubert polynomials that are products of binomials $x_i - x_j$ indexed by positions in the associated Rothe diagram. By considering all expressions of the form $\pi_u \mathfrak{S}_z^K$ where z is a K -dominant involution in S_n , we obtain a new family of objects that we refer to as *P- and Q-key polynomials*, or collectively as *shifted key polynomials*.

Using the fact that each $\mathfrak{S}_z^K$ is an $\mathbb{N}$ -linear combination of Schubert polynomials, we can show that each shifted key polynomial is an $\mathbb{N}$ -linear combination of key polynomials (see Theorem 3.4). This suggests that shifted key polynomials may form a combinatorially interesting family. Key polynomials are partial versions of Schur functions, since if $w_0 \in S_n$ denotes the reverse permutation then $s_{\lambda}(x_1, x_2, \dots, x_n) = \kappa_{w_0\lambda}$. Similarly, we show that P - and Q -key polynomials are partial versions of the Schur P - and Q -functions related to the projective representation theory of S_n (see Theorem 3.5).

Classical key polynomials form a $\mathbb{Z}$ -basis for all polynomials, are uniquely indexed by *weak compositions*, and decompose every Schubert polynomial with positive coefficients. Shifted key polynomials are not so well-behaved: they are not linearly independent over $\mathbb{Z}$, nor is it clear how to index them uniquely. In spite of this, we conjecture (Conjecture 3.7) that K -Schubert polynomials also expand positively for some choice of shifted key polynomials.

In order to classify a good set of linear independent shifted key polynomials, we consider a certain “truncated” crystal structure on a set of bounded reduced factorizations associated to an involution z , analogous to constructions in [1, 19]. However, in our case, the crystal will not be for $\mathfrak{gl}_n$ but for the queer Lie superalgebra $\mathfrak{q}_n$ and its extended version $\mathfrak{q}_n$ recently introduced in [18]. The full set of reduced factorizations relevant to K -Schubert polynomials were given a $\mathfrak{q}_n/\mathfrak{q}_n$ crystal structure in [16, 18]. We show in

Theorem 3.8 that when this crystal structure is restricted to its bounded elements for a K -dominant involution, we obtain a connected object whose character is a shifted key polynomial. We conclude by describing a crystal-theoretic generalization of our conjecture that K -Schubert polynomials expand positively into shifted key polynomials.

This extended abstract is organized as follows. [Section 2](#) gives some background on key polynomials. [Section 3](#) contains our main results on shifted key polynomials. We have omitted all proofs to save space. Complete arguments can be found in two full-length articles associated to this abstract, this first of which is available as [\[17\]](#).

2 Key polynomials, Schubert calculus, and crystals

Throughout, n is a positive integer, $n = \{1, 2, \dots, n\}$, $\mathbb{N} = \{0, 1, 2, \dots\}$, and $\mathbb{P} = \{1, 2, 3, \dots\}$. Let $\mathbf{x} = x_1, x_2, \dots$ be commuting indeterminates.

Define $S_\infty = \langle s_1, s_2, s_3, \dots \rangle$ to be the group permutations of $\mathbb{P}$ fixing all but finitely many elements, with $s_i = (i \ i+1)$ denoting a simple transposition. Set $S_n = \langle s_i : i \in n-1 \rangle \subset S_\infty$. A *reduced word* for $w \in S_\infty$ a minimal length sequence $i_1 i_2 \dots i_\ell$ such that $w = s_{i_1} s_{i_2} \dots s_{i_\ell}$, where $\ell = \ell(w)$ is the length of w . The group S_∞ acts on the polynomial ring $\mathbb{Z}[\mathbf{x}]$ by permuting variables. The *Rothe diagram* of $w \in S_\infty$ is $D(w) = \{ (i, w(j)) : i < j \text{ and } w(i) > w(j) \} \subset \mathbb{P} \times \mathbb{P}$.

A *word* is a possibly empty sequence of positive integers. For $w \in S_\infty$, let $\text{RF}(w)$ denote the set of sequences $a = a^1, a^2, a^3, \dots$ where each a^i is a strictly decreasing word such that the concatenation $a^1 a^2 a^3 \dots$ is a reduced word for w . We refer to elements of this set as *reduced factorizations* and define $\text{RF}_n(w)$ to be the set of such a with a^i empty for all $i > n$. In examples we express elements of $\text{RF}_n(w)$ as n -tuples rather than as infinite sequences. Let $\text{BRF}_n(w)$ denote the set of reduced factorizations in $\text{RF}_n(w)$ that are *bounded* in the sense that $i \leq \min a^i$ for all nonempty a^i . Set $\text{BRF}(w) = \bigsqcup_{n=1}^\infty \text{BRF}_n(w)$.

A *weak composition* is a nonnegative integer sequence $\alpha = (\alpha_i)_{i=1}^\infty$ with finite sum $|\alpha| = \sum_{i=1}^\infty \alpha_i$, and a *partition* is a weakly decreasing weak composition. We frequently omit the trailing 0's when writing weak compositions in examples. Given a weak composition α , let $\lambda(\alpha)$ be the partition sorting α and define $\mathbf{x}^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots$. There is a unique $u(\alpha) \in S_\infty$ such that $u(\alpha)\alpha$ is a partition $\lambda(\alpha)$, where the permutation acts on α by permuting indices; e.g., if $\alpha = 1021$, then $\lambda(\alpha) = 2110$ and $u(\alpha) = 3142 = s_2 s_1 s_3$.

For $i \in \mathbb{P}$, let ∂_i be the *divided difference operator* on $f \in \mathbb{Z}[\mathbf{x}]$ defined by $\partial_i f = (f - s_i f) / (x_i - x_{i+1})$. The *isobaric divided difference operators* are then given by $\pi_i f = \partial_i x_i f$ and $\bar{\pi}_i = \pi_i - 1$. For $w \in S_\infty$ with reduced word $i_1 \dots i_\ell$, define $\pi_w = \pi_{i_1} \dots \pi_{i_\ell}$ and $\bar{\pi}_w = \bar{\pi}_{i_1} \dots \bar{\pi}_{i_\ell}$; these formulas do not depend on the choice of reduced word. The *key polynomial* of a weak composition α is then $\kappa_\alpha = \pi_{u(\alpha)} \mathbf{x}^{\lambda(\alpha)}$ while the *atom polynomial* of α is $\bar{\kappa}_\alpha = \bar{\pi}_{u(\alpha)} \mathbf{x}^{\lambda(\alpha)}$. It is well-known that $\{\kappa_\alpha : \text{weak compositions } \alpha\}$ is a basis for $\mathbb{Z}[\mathbf{x}]$ and that key polynomials are unitriangular with $\kappa_\alpha = \mathbf{x}^\alpha + \text{lower order terms}$

with respect to lexicographic order [20, Cor. 7]. Key polynomials are related to atom polynomials by the identity $\kappa_\alpha = \sum_{\beta \leq \alpha} \bar{\kappa}_\beta$, where we write $\alpha \leq \beta$ if $\lambda_\alpha \leq \lambda_\beta$ and $u_\alpha \leq u_\beta$ in Bruhat order. For more background on key polynomials, see [20].

A permutation $w \in S_\infty$ is *dominant* if its Rothe diagram D_w is the Young diagram $D_\lambda = \{i, j \in \mathbb{P} \times \mathbb{P} : j \leq \lambda_i\}$ of a partition $\lambda = \lambda_w$. This occurs precisely when w is 132-avoiding [15, Ex. 2.2.2]. The *Schubert polynomial* of $w \in S_\infty$ is defined recursively by setting $\mathfrak{S}_w = \mathbf{x}^{\lambda_w}$ when w is dominant and requiring that $\mathfrak{S}_{ws_i} = \partial_i \mathfrak{S}_w$ for $w \cdot i > w \cdot i^{-1}$ [13]. For any $w \in S_\infty$ the Billey–Jockusch–Stanley formula [2] asserts that

$$\mathfrak{S}_w = \sum_{a \in \text{BRF}_n w} \mathbf{x}^{\text{wt } a}, \quad \text{where } \text{wt } a : \ell a^1, \ell a^2, \dots. \quad (2.1)$$

Key polynomials can be defined in terms of Schubert polynomials, since if $w \in S_\infty$ is dominant of shape $\lambda = \lambda_w$ then $\kappa_\alpha = \pi_{u_\alpha} \mathfrak{S}_w$. On the other hand, every Schubert polynomial expands as a positive linear combination of key polynomials with an explicit combinatorial description [20, Thm. 4].

The BJS formula (2.1) can be interpreted as a character formula for certain *Demazure crystals* which we describe below. A *crystal* [12] for $\mathfrak{gl}_n$ is a set $\mathcal{B}$ with *crystal operators* $e_i, f_i : \mathcal{B} \rightarrow \mathcal{B} \sqcup \{0\}$ for $i \in [n-1]$ and a *weight function* $\text{wt} : \mathcal{B} \rightarrow \mathbb{Z}^n$ that satisfy certain conditions. We can encode this data as a weighted directed graph called a *crystal graph* with vertices $\mathcal{B}$ and edges $b \xrightarrow{i} f_i b$ whenever $b \in \mathcal{B}$ and $f_i b \neq 0$. For each $w \in S_\infty$, the set $\text{RF}_n w$ already has a weight function as used in (2.1). Morse and Schilling [19] identified a natural $\mathfrak{gl}_n$ -crystal structure on $\text{RF}_n w$, using a certain bracketing rule to describe the crystal operators. See [4, §10] for more information on these crystals.

Suppose $w \in S_\infty$ is dominant of shape $\lambda = \lambda_w$. Assume λ has at most n nonzero parts. Then $\text{RF}_n w$ contains a single bounded reduced factorization $b_\lambda \in \text{BRF}_n w$. This element has weight λ and is *highest weight* in the sense that $e_i b_\lambda = 0$ for all $i \in [n-1]$. If α is a weak composition with $\lambda = \lambda_\alpha$ and $u_\alpha \in S_n$ then we define

$$\text{Dem}_n \alpha : \left\{ a \in \text{RF}_n w : \begin{array}{l} e_{i_1}^{m_1} e_{i_2}^{m_2} \cdots e_{i_\ell}^{m_\ell} a = b_\lambda \text{ for some reduced word} \\ i_1 i_2 \cdots i_\ell \text{ of } u_\alpha \text{ and some } m_1, m_2, \dots, m_\ell \in \mathbb{N} \end{array} \right\}. \quad (2.2)$$

We refer to this subset as a *$\mathfrak{gl}_n$ -Demazure crystal*. We identify it with the (connected) subgraph induced from the crystal graph of $\text{RF}_n w$. See Figure 1 for an example. The *character* of any finite subset $\mathcal{X}$ of a crystal $\mathcal{B}$ is $\text{ch } \mathcal{X} : \sum_{b \in \mathcal{X}} \mathbf{x}^{\text{wt } b} \in \mathbb{Z}[\mathbf{x}]$.

Theorem 2.1 (See [4]). *If α is a weak composition with $u_\alpha \in S_n$ such that λ_α has at most n parts then $\text{ch } \text{Dem}_n \alpha = \kappa_\alpha$.*

On the other hand, Assaf and Schilling [1] have shown the following:

Theorem 2.2 ([1]). *For any $w \in S_\infty$, the set $\text{BRF}_n w$ is a disjoint union of $\mathfrak{gl}_n$ -Demazure crystals, in the sense that there is a weight-preserving isomorphism from each connected component of the subgraph of the crystal graph of $\text{RF}_n w$ induced on $\text{BRF}_n w$ to $\text{Dem}_n \alpha$ for some α .*

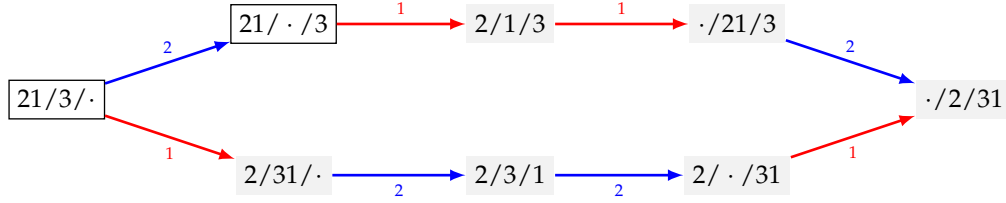


Figure 1: For the dominant $w = 3142 = s_2 s_1 s_3 \in S_\infty$ of shape $\lambda = w = 2, 1, 0$, the $\mathfrak{gl}_3$ -crystal $\text{RF}_3(w)$. The unique reduced factorization is $b_\lambda w = 21, 3, \emptyset$. The elements in the $\mathfrak{gl}_n$ -Demazure crystal $\text{Dem}_3(\alpha)$ for $\alpha = 2, 0, 1$, which are all in $\text{BRF}(w)$, are boxed.

Since $\text{ch } \text{BRF}_n(w) = \mathfrak{S}_w$ if n is sufficiently large, taking characters in this theorem recovers the nontrivial fact noted above that every Schubert polynomial expands as a positive linear combination of key polynomials [20, Thm. 4].

3 Shifted key polynomials

In this section, we introduce two shifted analogues of key and atom polynomials. We then present our main results about these polynomials and state a number of conjectures.

A partition λ is *strict* if its nonzero parts are all distinct; alternatively, if $\lambda = \lambda_1 > \dots > \lambda_\ell > 0$. We say λ is *symmetric* if $\lambda^\top = \lambda$, where $\lambda^\top$ is the conjugate shape. A partition λ is *skew-symmetric* if $\lambda^\top = \lambda$ and if i maximal such that $i, i \in D_\lambda$, then we cannot add or remove the box $i, i - 1$ from D_λ and still have the diagram of a partition. When λ is symmetric, we define its *shifted diagram* to be $S(\lambda) = \{i, j \in D_\lambda : i \leq j\}$ and *strict shifted diagram* to be $\hat{S}(\lambda) = \{i, j \in D_\lambda : i < j\}$.

Let $H(\lambda)$ (resp. $\hat{H}(\lambda)$) be the (*strict*) *half diagram* formed by sliding all boxes to the left of $S(\lambda)$ (resp. $\hat{S}(\lambda)$). This is the diagram of the strict partition λ^H (resp. $\lambda^{\hat{H}}$) whose parts count the number of boxes in the distinct rows of $S(\lambda)$ (resp. $\hat{S}(\lambda)$). The map $\lambda \mapsto \lambda^H$ (resp. $\lambda \mapsto \lambda^{\hat{H}}$) is a bijection from symmetric (resp. skew-symmetric) partitions to strict partitions. We say a weak composition α is (*skew*-)*symmetric* if $\lambda = \alpha$ is (skew-)symmetric.

Definition 3.1. Let α be a symmetric weak composition, and set $\lambda = \lambda(\alpha)$. Define

$$\kappa_\alpha^Q = \pi_{u(\alpha)} \left(\prod_{i,j \in S(\lambda)} x_i - x_j \right) \quad \text{and} \quad \bar{\kappa}_\alpha^Q = \bar{\pi}_{u(\alpha)} \left(\prod_{i,j \in \hat{S}(\lambda)} x_i - x_j \right).$$

We refer to these functions as *Q-key polynomials* and *Q-atom polynomials*. Similarly, when

α is skew-symmetric we define

$$\kappa_\alpha^P = \pi_{u \cdot \alpha} \left(\prod_{i,j \in \widehat{S} \cdot \lambda} x_i \cdot x_j \right) \quad \text{and} \quad \bar{\kappa}_\alpha^P = \bar{\pi}_{u \cdot \alpha} \left(\prod_{i,j \in \widehat{S} \cdot \lambda} x_i \cdot x_j \right).$$

We refer to these functions as *P-key polynomials* and *P-atom polynomials*.

The definitions of κ_α^P and $\bar{\kappa}_\alpha^P$ make sense if α is symmetric but not skew-symmetric, but in this case there is always a skew-symmetric β with $\kappa_\alpha^P = \kappa_\beta^P$ and $\bar{\kappa}_\alpha^P = \bar{\kappa}_\beta^P$.

Example 3.2. If $\alpha = 3143$ then $\lambda \cdot \alpha = 4211$ is skew-symmetric with $\lambda^{\widehat{H}} = 3100$ and $u \cdot \alpha = 3142 = s_2 s_1 s_3$, so we have

$$\begin{aligned} \kappa_{3143}^P &= \pi_2 \pi_1 \pi_3 (x_1 \cdot x_2 \cdot x_1 \cdot x_3 \cdot x_1 \cdot x_4 \cdot x_2 \cdot x_3) = \kappa_{0022} + \kappa_{0031} + \kappa_{0112}, \\ \bar{\kappa}_{3143}^P &= \bar{\pi}_2 \bar{\pi}_1 \bar{\pi}_3 (x_1 \cdot x_2 \cdot x_1 \cdot x_3 \cdot x_1 \cdot x_4 \cdot x_2 \cdot x_3) = \bar{\kappa}_{0022} + \bar{\kappa}_{0031}. \end{aligned}$$

If $\alpha = 2031$ then $\lambda \cdot \alpha = 3210$ is symmetric with $\lambda \cdot \alpha^H = 3100$ and $u \cdot \alpha = s_2 s_1 s_3$, so

$$\begin{aligned} \kappa_{2031}^Q &= \pi_2 \pi_1 \pi_3 (4x_1 x_2 \cdot x_1 \cdot x_2 \cdot x_1 \cdot x_3) = 4\kappa_{103} + 4\kappa_{202} + 4\kappa_{1021}, \\ \bar{\kappa}_{2031}^Q &= \bar{\pi}_2 \bar{\pi}_1 \bar{\pi}_3 (4x_1 x_2 \cdot x_1 \cdot x_2 \cdot x_1 \cdot x_3) = 0. \end{aligned}$$

We refer to κ_α^P and κ_α^Q collectively as *shifted key polynomials*, and to $\bar{\kappa}_\alpha^P$ and $\bar{\kappa}_\alpha^Q$ as *shifted atom polynomials*. Shifted atom polynomials are related to shifted key polynomials via the *Bruhat order* $\leq$ on S_∞ . Recall that $\beta \leq \alpha$ if $\lambda \cdot \beta = \lambda \cdot \alpha$ and $u \cdot \beta \leq u \cdot \alpha$.

Proposition 3.3. *We have $\kappa_\alpha^P = \sum_{\beta \leq \alpha} \bar{\kappa}_\beta^P$ and $\kappa_\alpha^Q = \sum_{\beta \leq \alpha} \bar{\kappa}_\beta^Q$. Moreover, κ_α^Q and $\bar{\kappa}_\alpha^Q$ are divisible by 2^ℓ , where ℓ is the length of $\lambda \cdot \alpha^H$.*

Our first substantial result about shifted key and atom polynomials is the following.

Theorem 3.4. *Let α be a symmetric composition. Then κ_α^P and κ_α^Q (resp. $\bar{\kappa}_\alpha^P$ and $\bar{\kappa}_\alpha^Q$) are linear combinations of key (resp. atom) polynomials with nonnegative integer coefficients. Consequently, the polynomials κ_α^P , $\bar{\kappa}_\alpha^P$, κ_α^Q , and $\bar{\kappa}_\alpha^Q$ are all in $\mathbb{N} \mathbf{x}$.*

Key polynomials are partial Schur functions in the sense that if $\lambda = \lambda_1 \geq \lambda_2 \geq \dots \geq 0$ is a partition with at most n nonzero parts then $\kappa_\alpha = s_\lambda(x_1, x_2, \dots, x_n)$ for $\alpha = \lambda_n, \dots, \lambda_2, \lambda_1$ [20, §2]. Analogously, we can prove that shifted key polynomials are partial *Schur P/Q-functions* (see, [14, §III.8] for background on these functions):

Theorem 3.5. *If $\lambda = \lambda_1 \geq \lambda_2 \geq \dots \geq 0$ is a symmetric partition with $\lambda_1 \leq n$ then*

$$\kappa_\alpha^P = P_\mu(x_1, x_2, \dots, x_n) \quad \text{and} \quad \kappa_\alpha^Q = Q_\nu(x_1, x_2, \dots, x_n)$$

for $\alpha = \lambda_n, \dots, \lambda_2, \lambda_1, 0, 0, \dots$, $\mu = \lambda^{\widehat{H}}$, and $\nu = \lambda^H$.

We have conjectural formulas for the leading terms of κ_α^P and κ_α^Q . Assume α is a symmetric composition with $\lambda = \lambda_\alpha$ and $u = u_\alpha$. Define $D_\alpha = \{u_i, u_j : i, j \in D_\lambda\}$. Let $\rho_\alpha = \rho_1, \rho_2, \dots$ and $\theta_\alpha = \theta_1, \theta_2, \dots$ where $\rho_i = |\{a, b \in D_\alpha : i \leq a \leq b\}|$ and $\theta_i = |\{a, b \in D_\alpha : a \leq b \leq i\}|$. Also define $\tilde{\rho}_\alpha = \tilde{\rho}_1, \tilde{\rho}_2, \dots$ and $\tilde{\theta}_\alpha = \tilde{\theta}_1, \tilde{\theta}_2, \dots$ where $\tilde{\rho}_i = |\{a, b \in D_\alpha : i \leq a < b\}|$ and $\tilde{\theta}_j = |\{a, b \in D_\alpha : a < b \leq j\}|$. These are the row/column counts of D_α below the main diagonal.

Conjecture 3.6. Suppose α and β are symmetric compositions with β skew-symmetric. Then

$$\kappa_\alpha^Q \in 2^{\ell(\lambda_\alpha) - H} \left(\mathbf{x}^{\rho_\alpha} \mathbf{x}^{\theta_\alpha} \sum_{\gamma / \rho_\alpha} \mathbb{N} \mathbf{x}^\gamma \right) \quad \text{and} \quad \kappa_\beta^P \in \mathbf{x}^{\tilde{\rho}_\beta} \mathbf{x}^{\tilde{\theta}_\beta} \sum_{\gamma / \tilde{\rho}_\beta} \mathbb{N} \mathbf{x}^\gamma.$$

Moreover, $\mathbf{x}^{\rho_\alpha}$ and $\mathbf{x}^{\tilde{\rho}_\beta}$ are the leading terms of κ_α^Q and κ_β^P in lexicographic order.

We can prove that if α and β are as above then D_α (hence, also α) is uniquely determined by ρ_α and θ_α , while D_β (hence, also β) is uniquely determined by $\tilde{\rho}_\beta$ and $\tilde{\theta}_\beta$. This does not hold for general symmetric subsets of $\mathbb{P} \times \mathbb{P}$. Shifted key/atom polynomials are not as well-behaved as their classical analogues in a few other ways:

- Shifted atom polynomials are zero for some indices α . They can also coincide for different indices. For example, $\bar{\kappa}_{30023}^Q = \bar{\kappa}_{21014}^Q = 0$ and $\bar{\kappa}_{402402}^P = \bar{\kappa}_{313501}^P = 0$.
- P -key polynomials are not uniquely indexed by skew-symmetric compositions: for example, $\kappa_{4313}^P = \kappa_{4133}^P = 0$.
- However, we have not yet been able to find a pair of distinct symmetric compositions $\alpha \neq \beta$ such that $\kappa_\alpha^Q = \kappa_\beta^Q$. It is possible that the Q -key polynomials are uniquely indexed by symmetric compositions.
- Even if this is the case, the Q -key polynomials are still not linearly independent. For example, we have $\kappa_{123}^Q = \kappa_{0321}^Q = \kappa_{132}^Q = \kappa_{0231}^Q$.

Shifted key polynomials are closely related to certain “orthogonal” and “symplectic” versions of type A Schubert polynomials. Let $I_\infty^O = \{z \in S_\infty : z = z^{-1}\}$ and let I_∞^{Sp} be the S_∞ -conjugacy class of $1 \text{ f p f } 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ \dots$. If λ is a symmetric partition, then the unique dominant element of S_∞ of shape λ already belongs to I_∞^O . If λ is a skew-symmetric partition, then there is a unique $z \in I_\infty^{Sp}$ with $\{i, j \in D_z : i \leq j\} = \{i, j \in D_\lambda : i \leq j\}$, which we call the *dominant* element of I_∞^{Sp} with shape λ .

Let $K \in \{Sp, O\}$. By results in [22], there are unique polynomials $\{\mathfrak{S}_z^K\}_{z \in I_\infty^K}$ with $\mathfrak{S}_\lambda^K = \kappa_\lambda^P$ when $K = Sp$ (resp. $\mathfrak{S}_\lambda^K = \kappa_\lambda^Q$ when $K = O$) and $z \in I_\infty^K$ is dominant of shape λ ,

and which satisfy

$$\partial_i \mathfrak{S}_z^{\text{Sp}} = \begin{cases} 0 & \text{if } z \cdot i < z \cdot i - 1, \\ 0 & \text{if } z \cdot i = i - 1, \\ \mathfrak{S}_{s_i z s_i}^{\text{Sp}} & \text{otherwise,} \end{cases} \quad \text{and} \quad \partial_i \mathfrak{S}_z^{\text{O}} = \begin{cases} 0 & \text{if } z \cdot i < z \cdot i - 1, \\ 2\mathfrak{S}_{z s_i}^{\text{O}} & \text{if } z \cdot i = i - 1, \\ \mathfrak{S}_{s_i z s_i}^{\text{O}} & \text{otherwise,} \end{cases}$$

for all $z \in I_\infty^K$ and $i \in \mathbb{P}$. We refer to the $\mathfrak{S}_z^K$'s as *K-Schubert polynomials*. These elements, called *involution Schubert polynomials* in [9, 10, 11], represent cohomology classes of the closures of the Sp- and O-orbits in the complete flag variety [22]. The following conjecture is one of our primary motivations for studying shifted key polynomials:

Conjecture 3.7. *Each polynomial $\mathfrak{S}_z^{\text{Sp}}$ for $z \in I_\infty^{\text{Sp}}$ (resp. $\mathfrak{S}_z^{\text{O}}$ for $z \in I_\infty^{\text{O}}$) is an $\mathbb{N}$ -linear combination of P-key polynomials (resp. Q-key polynomials).*

This conjecture is supported by many computational examples and closely parallels the classical case. Below, we will outline a shifted analogue of Theorem 2.1 that also provides some heuristic support for the conjecture.

Like ordinary Schubert polynomials, K-Schubert polynomials can be expressed via a BJS-type formula as $\mathfrak{S}_z^K = \sum_{a \in \text{BRF}_z^K} \mathbf{x}^{\text{wt } a}$ for an analogue BRF_z^K of the set BRF_z [9]. For each choice of K, BRF_z^K consists of the bounded elements in a larger set of *K-reduced factorizations* RF_z^K . If $K = \text{Sp}$ then RF_z^K is explicitly given as the disjoint union of RF_w over all minimal length $w \in S_\infty$ with $z = w^{-1} 1_{\text{pf}} w$. The definition of RF_z^{O} for $z \in I_\infty^{\text{O}}$ is more involved: this is formed by taking another disjoint union of sets RF_w for certain $w \in S_\infty$, and then optionally annotating some letters in each reduced factorization by primes; see [18] for the precise details.

Define $\text{BRF}_n^K(z) \subseteq \text{RF}_n^K(z)$ to be the respective subsets of $\text{BRF}_z^K \subseteq \text{RF}_z^K$ consisting of the tuples $a = (a^1, a^2, \dots)$ with a^i empty for $i > n$. Like $\text{RF}_n(w)$, the sets $\text{RF}_n^K(z)$ have natural crystal structures. Results in [16] identify a crystal structure on $\text{RF}_n^{\text{Sp}}(z)$ corresponding to the *queer Lie superalgebra* $\mathfrak{q}_n$ (as described axiomatically in [7]), extending the $\mathfrak{gl}_n$ crystal in [19]. The set $\text{RF}_n^{\text{O}}(z)$ similarly is the prototypical example of what is called an *extended queer supercrystal* or $\mathfrak{q}_n$ -crystal in [18].

Crystals for $\mathfrak{q}_n$ are $\mathfrak{gl}_n$ crystals with *odd crystal operators* $e_{\bar{i}}, f_{\bar{i}}$ for $\bar{i} \in \overline{n-1} : \{\bar{1}, \dots, \overline{n-1}\}$ satisfying certain axioms. It is sufficient to define $e_{\bar{1}}, f_{\bar{1}}$ as the other odd crystal operators come from inductively twisting the root system [6, Lemma 2.2]:

$$e_{\bar{i}} = s_{i-1} s_i e_{\overline{i-1}} s_i s_{i-1} \quad \text{and} \quad f_{\bar{i}} = s_{i-1} s_i f_{\overline{i-1}} s_i s_{i-1}. \quad (3.1)$$

Here s_i acts as the crystal operator that reverses each *i-string*. We remark that $e_{\bar{1}}$ is not defined by a usual bracketing rule but by a weight condition. For $\mathfrak{q}_n$ crystals, there are additional crystal operators e_0, f_0 . Similar to $\mathfrak{gl}_n$ crystals, we encode $\mathfrak{q}_n/\mathfrak{q}_n$ crystals as crystal graphs. In view of (3.1), we omit $\bar{2}, \bar{3}, \dots$ when drawing the crystal graphs.

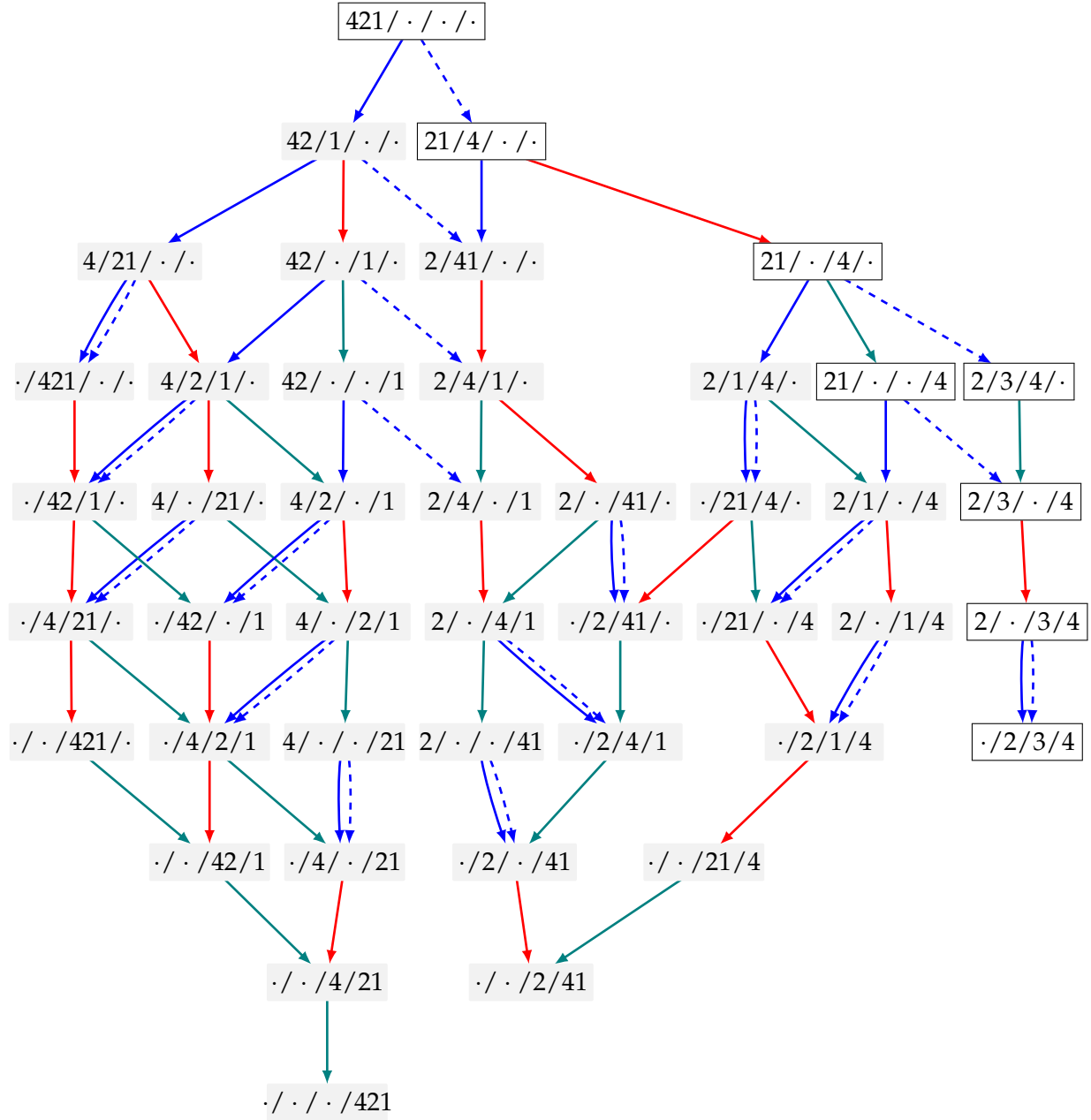


Figure 2: The q_4 -crystal on $\text{RF}^{\text{Sp}} z$ corresponding to κ_λ^P for $z = s_1 s_2 s_4 \cdot 1_{\text{pf}} \cdot s_4 s_2 s_1$ and $\lambda = 4, 1, 1, 1$. The boxed elements are in $\text{BRF}^{\text{Sp}} z$. Solid blue, red, and green arrows indicate 1-, 2-, and 3-edges, respectively, while dashed blue arrows are $\bar{1}$ -edges.

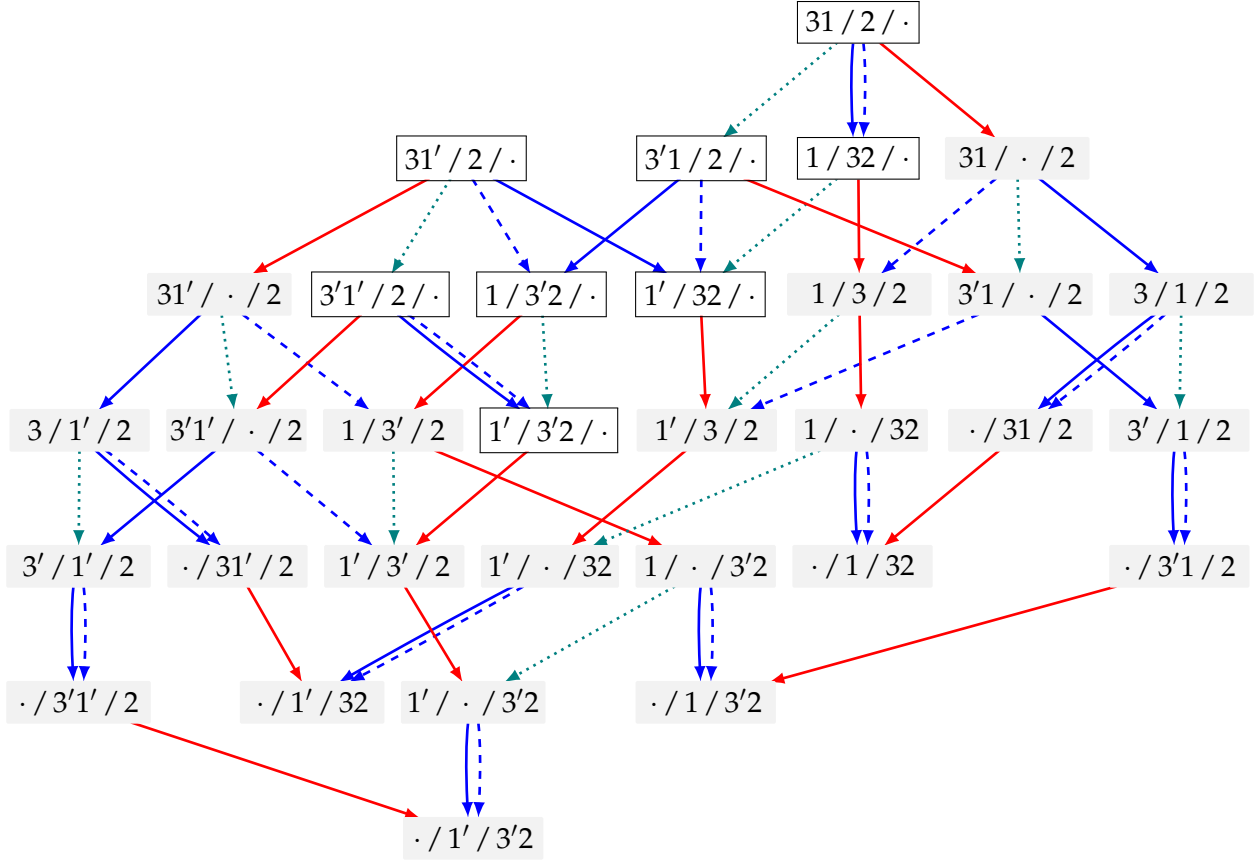


Figure 3: The q_3 -crystal on $\text{RF}^{\text{O}} z$ corresponding to $\kappa_{\lambda}^{\text{Q}}$ for $z = 1\ 3\ 2\ 4$ and $\lambda = 3, 3, 3$. The boxed elements are in $\text{BRF}^{\text{O}} z$. Solid blue, solid red, dotted green, and dashed blue arrows are i -edges for $i = 1, 2, 0, \bar{1}$, respectively.

Now suppose $z \in I_{\infty}^{\text{K}}$ is dominant with (skew-)symmetric shape λ having $\lambda_1 \leq n$. In general, the set $\text{BRF}_n^{\text{K}} z$ contains more than one element in $\text{RF}_n^{\text{K}} z$. Nevertheless, there is an interesting shifted analogue of the $\mathfrak{gl}_n$ -Demazure crystal (2.2).

Fix a (skew-)symmetric composition α with $\lambda = \lambda + \alpha$ and $u_{\alpha} \in S_n$. Then define

$$\text{Dem}_n^{\text{K}} \alpha : \left\{ a \in \text{RF}_n^{\text{K}} z : \begin{array}{l} e_{i_1}^{m_1} e_{i_2}^{m_2} \cdots e_{i_{\ell}}^{m_{\ell}} a \in \text{BRF}_n^{\text{K}} z \text{ for some reduced word} \\ i_1 i_2 \cdots i_{\ell} \text{ for } u_{\alpha} \text{ and some } m_1, m_2, \dots, m_{\ell} \in \mathbb{N} \end{array} \right\}.$$

We refer to this subset as a *K-Demazure crystal*; see Figures 2 and 3 for examples. We identify $\text{Dem}_n^{\text{K}} \alpha$ with the subgraph that it induces in the *extended q_n/q_n -crystal graph* of $\text{RF}_n^{\text{K}} z$ (formed by drawing arrows corresponding to all even and odd crystal operators). This directed subgraph is connected, although this is not at all obvious:

Theorem 3.8. Assume $u, \alpha \in S_n$ and λ, α has at most n parts. Then the K -Demazure crystal $\text{Dem}_n^K \alpha$ is connected with character equal to κ_α^P when $K = \text{Sp}$ and to κ_α^Q when $K = \text{O}$.

Our second main conjecture is the following analogue of [Theorem 2.2](#):

Conjecture 3.9. If $z \in I_\infty^K$ then $\text{BRF}_n^K z$ is a disjoint union of K -Demazure crystals, in the sense that there is a weight-preserving isomorphism from each connected component of the subgraph of the extended crystal graph of $\text{RF}_n^K z$ induced on $\text{BRF}_n^K z$ to $\text{Dem}_n^K \alpha$ for some α .

As $\text{ch } \text{BRF}_n^K z = \mathfrak{S}_z^K$ if $n \gg 0$, [Conjecture 3.7](#) would follow from this conjecture on taking characters. This conjecture is again supported by extensive computer calculations. [Conjectures 3.7](#) and [3.9](#) also have more refined versions involving K -reduced factorizations that are bounded by arbitrary *flags*, which would generalize [\[20, Thm. 21\]](#).

An interesting discrepancy with the classical case is that the subsets $\text{Dem}_n^K \alpha$ are not closed under $e_{\bar{i}}$ for all $\bar{i} \in \overline{n-1}$. Contrast this with the situation for $\mathfrak{gl}_n$ -Demazure crystals: if b_β is the unique element of weight $\beta \leq \alpha$ in a $\mathfrak{gl}_n$ -Demazure crystal, then while we cannot obtain all elements from b_α through applying $\{e_i : i \in \overline{n-1}\}$, we can obtain everything by applying these operators to the closure of $\{b_\beta : \beta \leq \alpha\}$.

Example 3.10. Consider the q_4 -crystal $\text{RF}^{\text{Sp}} z$ in [Figure 2](#). For $b = 2/3/4/\cdot \in \text{BRF}^{\text{Sp}} z$, we have $e_{\bar{2}} 2/3/4/\cdot = s_1 s_2 e_{\bar{1}} s_2 s_1 2/3/4/\cdot = s_1 s_2 e_{\bar{1}} 2/3/4/\cdot = s_1 s_2 21/\cdot/4/\cdot$. $s_1 21/4/\cdot/\cdot = 2/41/\cdot/\cdot \notin \text{BRF}^{\text{Sp}} z$.

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Chainlink Polytopes

Ezgi Kantarcı Oğuz^{*1}, Cem Yalın Özel^{†2}, and Mohan Ravichandran^{‡2}

¹Department of Mathematics, Galatasaray University, İstanbul

²Department of Mathematics, Bogazici University, İstanbul

Abstract. We introduce a class of polytopes that we call chainlink polytopes and which allow us to construct infinite families of pairs of non isomorphic rational polytopes with the same Ehrhart quasi-polynomial. Our construction is inspired by a non-obvious and non-trivial symmetry in the rank sequences of circular fence posets. We show that this symmetry can be lifted to yield an analogous symmetry at the level of polytopes. We show this symmetry property of Chainlink polytopes by introducing the related class of chainlink posets and show that they exhibit the same symmetry properties using linear algebraic techniques. We further prove an outstanding conjecture on the unimodality of circular rank polynomials.

Keywords: polytope, fence poset, symmetry, Ehrhart theory

This paper is about a class of polytopes, that naturally arise in poset theory, specifically in the study of fence posets and related objects. They are easy to describe, pliable of study, possess certain unexpected properties and throw up several puzzles. These polytopes will be indexed by compositions; let $\bar{a} = a_1, \dots, a_s$ be a composition of n and let l be a non-negative integer. The *chainlink polytope* $CL(\bar{a}, l)$ with *chain composition* $\bar{a}$ and *link number* l is defined to be the polytope:

$$CL(\bar{a}, l) = \{x \in \mathbb{R}^s \mid 0 \leq x_i \leq a_i, x_i - x_{i-1 \bmod s} \leq a_i - l, i \in [s]\}.$$

This is a polytope that naturally lies in $\mathbb{R}^s$ and has a maximum of $3s$ facets. When the link number l is equal to zero, the second set of constraints become redundant and the polytope becomes a cuboid, $CL(\bar{a}, 0) = [0, a_1] \times [0, a_2] \times \dots \times [0, a_s]$. When the link number is larger, new facets emerge. For an example, see [Figure 1](#).

We will also work with certain special sections of these chainlink polytopes. For a positive real number t , we define

$$CL^t(\bar{a}, l) = CL(\bar{a}, l) \cap \{x_1 = \dots = x_s = t\}.$$

The polytopes $CL^t(\bar{a}, l)$ are rational, see [Proposition 2.1](#), are non-empty for $t \in [0, n]$, where $n = a_1 + \dots + a_s$. One of the main results in this paper is the following (unexpected) symmetry property of these sections of chainlink polytopes.

^{*}ezgikantarcioguz@gmail.com. EKO was partially supported by Tübitak BİDEP 2218-121C385.

[†]yalim98@gmail.com.

[‡]mohan.ravichandran@gmail.com. MR gratefully acknowledges financial support from the Bogazici Solidarity fund.

Theorem 4.7. Let $\bar{a} = a_1, \dots, a_s$ be a composition of n , let l be a positive integer such that $2l \leq \min\{a_i\}_{i \in s}$ and let t be a positive integer. Then complementary sections of the chainlink polytope have the same volume,

$$|\text{CL}^t \bar{a}, l| = |\text{CL}^{n-t} \bar{a}, l|,$$

where $|P|$ for a polytope denotes the relative volume.

There will be no ambiguity in the definition of the relative volume for us. All our polytopes will lie on hyperplanes of the form $\{x_1 + \dots + x_s = t\}$ and we will work with the volume form that assigns volume 1 to the polytope $\mathcal{P} = \text{conv}\{0, e_1 - e_2, e_1 - e_3, \dots, e_1 - e_s\}$. This theorem is a special case of the following more general theorem. The terms used will be formally defined in the next section.

Theorem 1. Let $\bar{a} = a_1, \dots, a_s$ be a composition of n , let l be a non-negative integer such that $2l \leq \min\{a_i\}_{i \in s}$ and let t be a positive integer. Then complementary sections of the chainlink polytope have the same Ehrhart quasipolynomial,

$$\text{Ehr} \text{CL}^t \bar{a}, l = \text{Ehr} \text{CL}^{n-t} \bar{a}, l.$$

To see why this is unexpected, consider the chainlink polytope $\text{CL}^{6,4,5}, 2$ as above. The sections at $t = 4$ and $t = 11$ have the same volume, but are non-isomorphic. We plot the first one on the triangular lattice, the natural choice given that these lie on the hyperplanes $x + y + z = \text{const}$.

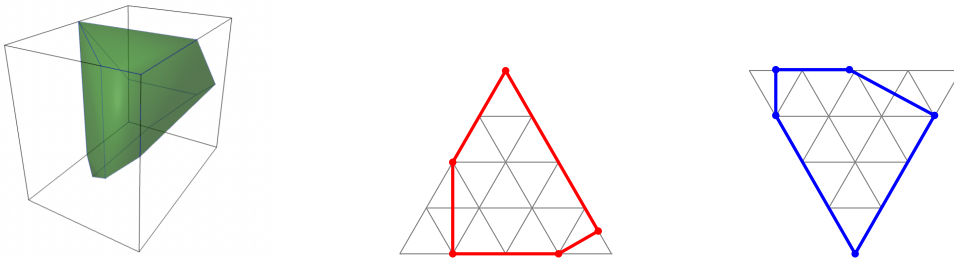


Figure 1: The polytope $\text{CL}^{6,4,5}, l = 2$ with sections $\text{CL}^4 \bar{a}, l$ and $\text{CL}^{11} \bar{a}, l$.

1 Background

Where do these chainlink polytopes come from? At first sight, they (might perhaps) seem unmotivated, if (again perhaps) natural. We were led to these following the paper by the first and third authors [6] on fence posets and in particular, a tricky problem that they had been unable to solve. We first recall the definition of fence posets.

Definition 1.1. Given a composition $\bar{c} = c_1, \dots, c_k$, the **fence poset** is the poset on $n = c_1 + \dots + c_k + 1$ nodes, where $n = c_1 + \dots + c_k + 1$ defined by the cover relations

$$x_1 \prec x_2 \prec \dots \prec x_{c_1+1} \succ x_{c_1+2} \succ \dots \succ x_{c_1+c_2+1} \prec x_{c_1+c_2+2} \prec \dots$$

For an example, see Figure 2. These posets arise in cluster algebras, quiver representation theory and combinatorics. They also appeared in recent work of Morier-Genoud and Ovsienko [8], where they introduced a q -deformation of the rational numbers. In this same paper, the authors conjectured the following, which was proved in [6].

Theorem 1.2. *The rank polynomials of fence posets are unimodal.*

The main step in the proof of this theorem involved the introduction of the ancillary class of *circular fence posets*, and an unexpected property of these posets.

Definition 1.3. Given an even length composition $\bar{c} = c_1, \dots, c_{2s}$, the *circular fence poset* $\bar{F} \bar{c}$ is the poset on $n = c_1 + \dots + c_{2s}$ nodes where $n = c_1 + \dots + c_{2s}$, defined by the cover relations

$$x_1 \prec \dots \prec x_{c_1+1} \succ x_{c_1+2} \succ \dots \succ x_{c_1+c_2+1} \prec x_{c_1+c_2+2} \prec \dots \prec x_{1+\sum_{i=1}^{2s-1} c_i} \succ \dots \succ x_{\sum_{i=1}^{2s} c_i} \succ x_1.$$

In other words, this is what we get by identifying the two end points of a regular fence poset.

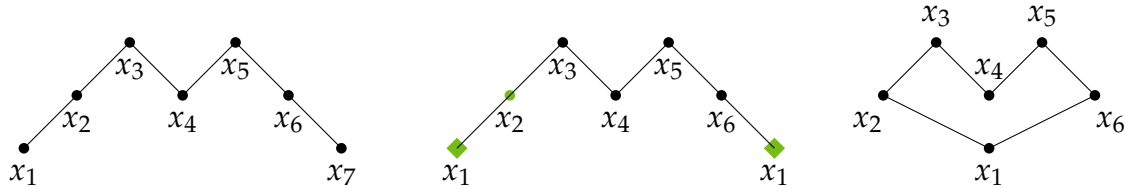


Figure 2: The fence poset $F(2,1,1,2)$ (left) and two depictions of the circular fence poset $\bar{F}(2,1,1,2)$ (center, right). In the middle one, the nodes marked x_1 are identified.

In [6], the authors showed that circular fence posets satisfy an unexpected property.

Theorem 1.4 ([6]). *Rank polynomials of circular fence posets are symmetric.*

Let us make a comment on why this result is unexpected. Given a composition $\bar{c} = c_1, \dots, c_{2s}$, let $\bar{c}_{\text{shift}}$ be the composition that is the cyclical shift of $\bar{c}$, that is $\bar{c}_{\text{shift}} = c_{2s}, c_1, c_2, \dots, c_{2s-1}$. A calculation shows that the symmetry of the rank polynomial of $\bar{F} \bar{c}$ is equivalent to the statement that the posets $\bar{F} \bar{c}$ and $\bar{F} \bar{c}_{\text{shift}}$ have the same rank polynomial. It is also possible to see that this same rank symmetry may also be expressed as saying that the poset of lower ideals (our $\bar{F} \bar{a}$) and the poset of upper

Composition	Fence Poset	Hasse Diagram of Lattice of Lower Ideals
$\bar{c}$ $2, 1, 1, 2$		
$\bar{c}_{\text{shift}}$ $2, 2, 1, 1$		

Table 1: Example showing that lattices of upper and lower ideals can be non-isomorphic (the inclusion in the Hasse diagrams given is in the direction left to right)

ideals of the same fence poset have the same rank polynomial. However, except in very special cases, the two posets *are not isomorphic*. For instance, take $\bar{c} = 2, 1, 1, 2$ as in Figure 2 and compare the Hasse diagrams of $\bar{F} \bar{c}$ and $\bar{F} \bar{c}_{\text{shift}}$. The second fence can be seen as the vertical reflection of the first and we have labeled the elements appropriately.

A second, this time bijective, proof of Theorem 1.4 was given by Elizalde and Sagan in [1]. Both proofs of this result are highly intricate and it is natural to seek a transparent proof of this basic result. We present such a proof in this paper, see Corollary 3.3.

As mentioned above, in [6], the symmetry of the rank polynomials of circular fence posets was used to prove Theorem 1.2, that rank polynomials of (regular) fence posets are unimodal. Generically, rank polynomials of circular fence posets seemed to be unimodal as well, though there are certain exceptions; a calculation shows that

$$\bar{R}(1, 1, 1, 1; q) = 1 + 2q + q^2 + 2q^3 + q^4.$$

Extensive computer calculations however suggested the following conjecture.

Conjecture 1.5 ([6]). *The rank polynomial $\bar{R}(\bar{a}; q)$ of a circular fence poset $\bar{F} \bar{c}$ is unimodal except when $\bar{c} = a, 1, a, 1$ or $1, a, 1, a$ for some positive integer a .*

In this same paper, the authors showed that if $\bar{R}(\bar{a})$ is not unimodal, then the composition $\bar{a}$ may not have any two adjacent entries larger than 1. For the purposes of this paper, we will focus on the more concrete case, where our compositions have form:

$$\bar{a} = a_1, 1, a_2, 1, \dots, a_s, 1. \quad (1.1)$$

Such compositions play an important role in the bijective proof of symmetry of Elizalde and Sagan in [1], where the authors refer to such circular fence posets as *gate posets*.

It turns out that the number of rank k ideals of gate posets coming from a composition $a_1, 1, \dots, a_s, 1$ equal the number of lattice points in $\text{CL}^k \bar{a}, 1$ where $\bar{a} = a_1, a_2, \dots, a_s$. We were naturally led to investigate whether the generalization

$$\text{Ehr CL}^k \bar{a}, 1 = \text{Ehr CL}^{n-k} \bar{a}, 1, \quad (1.2)$$

where $n = a_1 + \dots + a_s$ is true as well. Well, it is! And this is the content of the main theorem in this paper. The equality of Ehrhart quasipolynomials yields as a corollary the equality of volumes of these polytopes, see [Theorem 3.4](#).

Remark 1.6. Given a composition $\bar{c} = a_1, 1, \dots, a_s, 1$ of n , it is also true that the order polytope $\mathcal{O} \bar{F} \bar{c}$ [9] of the composition is such that for each integer,

$$\#\mathcal{O}^k \bar{F} \bar{c} = \#\mathcal{O}^{n-k} \bar{F} \bar{c},$$

where $\mathcal{O}^k \bar{F} \bar{c} = \mathcal{O}^k \bar{F} \bar{c} \cap \{x_1 = \dots = x_k = n\}$. However the Ehrhart polynomials of these polytopes need not be equal, see [7].

Proving [Theorem 3.4](#) needed several new ideas. Denote the Ehrhart polynomial $\text{Ehr CL}^k \bar{a}, 1$ by f_k . This polynomial evaluated at 1 counts the number of ideals of size k in the associated circular fence poset, that corresponding to $\bar{c} = a_1, 1, \dots, a_s, 1$. When evaluated at other integers, say $f_k m$, we will show that the value can again be interpreted as the number of lower ideals of size mk in a certain poset, which we call a *chainlink poset*. These posets share a familial resemblance to circular fence posets; they are introduced in [Section 2](#) (see [Figure 3](#) for an example).

We will show that all chainlink posets have symmetric rank polynomials. The strategies for showing symmetry for circular fence posets in [6, 1] do not carry over and we needed to approach the problem differently. The new ingredient is a linear algebraic approach coming from the theory of oriented posets (see [5]) that we believe yields a transparent proof. We note that this yields a new (third) proof of rank symmetry for circular fences as well (See [Corollary 3.3](#)).

This approach has as its starting point the following basic feature of fence posets: They can be built up by gluing chains in an iterative manner. We review in [Section 2.1](#) how the rank polynomials of fence (and chainlink) posets can be computed by multiplying certain 2×2 matrices: The entries of these matrices are certain polynomials that encode refined order relations.

This approach has another felicitous consequence : We discovered new recurrences, that lead to a proof of [Conjecture 1.5](#). We include a proof of this in [Section 4](#).

2 Chainlink Polytopes and Chainlink Posets

We note here a basic integrality property of chainlink polytopes.

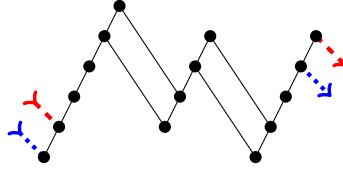


Figure 3: The chainlink poset with $\bar{a} = 6, 4, 5$ and $l = 2$, the two top right nodes are connected to the two bottom left nodes.

Proposition 2.1. Let $\bar{a} = a_1, \dots, a_s$ be a composition of n , let $l, t \in \mathbb{Z}_{\geq 0}$. Then

- The vertices of $\text{CL } \bar{a}, l$ are integral.
- Assume that $2l \leq \min \bar{a}$. Then the vertices of $\text{CL}^t \bar{a}, l$ are either integral or half integral.

Let $\text{CL } \bar{a}, l$ be a chainlink polytope with $2l \leq \min_i a_i$. Consider the integer points that lie inside the polytope. When $l = 1$, these points correspond to ideals of the circular fence poset $F_{a_1 - 1, 1, a_2 - 1, 1, \dots, a_s - 1, 1}$, where the rank of the ideal corresponds to the sum of the coordinates of the point. For general l , the integer points can be interpreted as ideals of a poset $F_l \bar{a}$ formed by adding extra edges to the Hasse diagram of $F_{a_1 - 1, 1, a_2 - 1, 1, \dots, a_s - 1, 1}$ as shown in Figure 3. More precisely, we can define chainlink posets as follows:

Definition 2.2. Let $\bar{a}$ be a composition, with $l = \bar{a}$ parts and l be a positive integer satisfying $2l \leq \min_i a_i$ as in the case for chainlink polytopes. The **chainlink poset** $P_{\text{CL}} \bar{a}, l$ is given by points $x_{i,j}$ for $1 \leq i \leq \ell(\bar{a})$ and $0 \leq j \leq a_i$ with the generating relations $x_{i,0} \prec x_{i,1} \prec \dots \prec x_{i,a_i}$ and $x_{i,a_i-l+1-k} \succ x_{i-1,k}$ for $0 \leq k < l$ and for each i , where $i-1$ is calculated cyclically.

Recall that we use $\text{CL}^t \bar{a}, l$ to denote the slice of the polytope with respect to the hyperplane $x_1 + \dots + x_s = t$. Note that this slice can be non-empty only when $t \in [0, n]$. Furthermore, the number of integer points in $\text{CL}^t \bar{a}, l$ is given by the coefficient of q^t in the rank polynomial of $P_{\text{CL}} \bar{a}, l$.

The connection between integer points of the polytope and rank polynomial of the corresponding poset still holds if we scale the polytope by a number k . This allows us to describe the coefficients of the Ehrhart quasi polynomial of slices of the chainlink polytope in terms of coefficients of rank polynomials of some chainlink posets. That means a general statement about the symmetry of rank polynomials of all chainlink posets can be used to prove the main theorem stated in the introduction, which is precisely what we do in the next few sections.

2.1 Matrix Formulation

An oriented poset $P \nearrow (P, x_L, x_R)$ consists of a poset P with two specialized vertices x_L and x_R which can be thought as the target (left) vertex $\odot$ and the source (right) vertex $\ominus$.

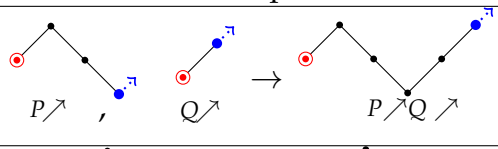
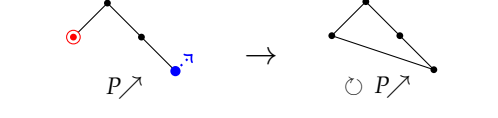
	Formula	Example
$P \nearrow Q$	$\mathfrak{M}_q P \nearrow Q \nearrow \mathfrak{M}_q P \nearrow \cdot \mathfrak{M}_q Q \nearrow$	
$\circlearrowleft P \nearrow$	$\mathfrak{R} \circlearrowleft P \nearrow ; w \quad \text{tr } \mathfrak{M}_q P \nearrow$	

Table 2: The moves are shown through examples

$\rightarrow$. One can think of an oriented poset as a poset with an upwards arrow coming out of the source vertex x_R . One can combine oriented posets by linking the arrow of one poset with target of another via $x_R \preceq y_L$ ($x_R \nearrow y_L$) to get $P \nearrow Q \nearrow^1$.

The effect of this operation on the rank polynomial can be calculated easily by 2×2 matrices. A *rank matrix* of an oriented poset $P \nearrow$ is defined as follows:

$$\mathfrak{M}_q P \nearrow : \begin{bmatrix} \mathfrak{R} P; w \mid_{x_R \in I} & \mathfrak{R} P; w \mid_{x_R \notin I} \\ \mathfrak{R} P; w \mid_{x_R \in I, x_L \notin I} & \mathfrak{R} P; w \mid_{x_R \notin I, x_L \notin I} \end{bmatrix}$$

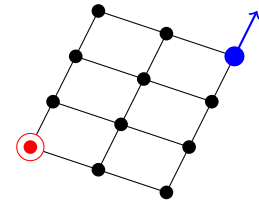
The entries are partial rank polynomials, which correspond to restricting to ideals of the poset P that satisfy the given constraints. We also use the notation $\circlearrowleft P \searrow$ (resp. $\circlearrowleft P \nearrow$) to denote the structure obtained by adding the relation $x_R \succeq x_L$ (resp. $x_R \preceq x_L$). On the rank matrix level, this corresponds to taking the trace. See Table 2 for precise formulas and examples of these operations.

In particular, consider when P is formed of a single node equal to both x_R and x_L . We call this oriented poset an *up step* and denote the corresponding matrix by U .

$$U : \mathfrak{M}_q \bullet \nearrow : \begin{bmatrix} q & 1 \\ 0 & 1 \end{bmatrix}, \quad \mathfrak{M}_q C_k \nearrow U^{k-1}.$$

The second part comes from the fact that combining $k-1$ such posets gives us a chain C_k of length k with x_L corresponding to the minimal element, and x_R to the maximal.

Let $B_{a \times b} \nearrow$ denote the ab -element oriented box poset given by the direct product of two chains C_{a-1} and C_{b-1} with the left vertex given by $a-1, 0$ and the right vertex is given by $0, b-1$. The directed poset $B_{3 \times 4} \nearrow$ is shown in Figure 4.

Figure 4: $B_{3 \times 4} \nearrow$

¹Linking via $x_R \succeq y_L$ is also an option, see [5].

The rank matrix of an oriented box poset is given as follows:

$$\mathfrak{M}_q B_{a \times b} \nearrow \left[\begin{array}{cc|cc} q^{b \ a} & b-1 & a \ b-1 & \\ & b & b-1 & q \\ \hline q^{b \ a} & b-2 & a \ b-2 & \\ & b & b-1 & q \end{array} \right] \left[\begin{array}{cc|cc} a \ b & a \ b-1 & a \ b & q^{b \ a} \ b-1 \\ & b-1 & q & b \\ \hline q^{b \ a} & b-2 & & a \ b-2 \\ & b & q & b-1 \ q \end{array} \right]$$

Next we will see that any given chainlink poset can be realized by combining copies of the up step and the box poset $B_{2 \times l} \nearrow$.

Proposition 2.3. *Consider the chainlink poset $P_{CL} \bar{a}, l$ with $2l \leq \min_i a_i$. Let $\bar{d}$ be the weak composition formed by taking $d_i = a_i - 2l$. The rank polynomial of $P_{CL} \bar{a}, l$ is given by:*

$$\Re P_{\text{CL}}^{\bar{a}, l; q} = \text{tr } U^{d_1} \cdot B \cdot U^{d_2} \cdot B \cdot \dots \cdot U^{d_1} \cdot B \quad (2.1)$$

where B denotes the rank matrix $\mathfrak{M}_q \ B_{2 \times l} \nearrow$.

Example 2.4. The chainlink poset given in Figure 3 with $\bar{a} = 6, 4, 5$ and $l = 2$ can be formed by combining 2×2 boxes with up steps and then taking the closure: $\circlearrowleft B_{2 \times 2} \nearrow \bullet \nearrow \bullet \nearrow \bullet B_{2 \times 2} \nearrow \bullet B_{2 \times 2} \nearrow \bullet \nearrow \bullet$. The corresponding rank polynomial is given by:

$$\text{tr} \left(\begin{bmatrix} q^2 & 3q & 3q \\ q^2 & 2q & \end{bmatrix} \cdot \begin{bmatrix} q & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} q & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} q^2 & 3q & 3q \\ q^2 & 2q & \end{bmatrix} \cdot \begin{bmatrix} q^2 & 3q & 3q \\ q^2 & 2q & \end{bmatrix} \cdot \begin{bmatrix} q & 1 \\ 0 & 1 \end{bmatrix} \right)$$

$$\begin{matrix} 1 & 3q & 6q^2 & 9q^3 & 12q^4 & 14q^5 & 16q^6 & 17q^7 \\ & & & & 17q^8 & 16q^9 & 14q^{10} & 12q^{11} & 9q^{12} & 6q^{13} & 3q^{14} & q^{15}. \end{matrix}$$

Note that the rank polynomial given in this instance is symmetric. Next, we will show that this is always the case.

3 Recurrence Relations and Rank Symmetry

One advantage of building posets via matrices is that the characteristic equations of matrices give us recurrence relations for the rank polynomial. For example, consider the characteristic polynomial of U . Plugging U in the place of x gives us

$$U^2 \quad q \quad 1 \quad U \quad q,$$

Note that the coefficient of U is symmetric around $q^{1/2}$ and q is trivially symmetric around q .

Lemma 3.1. *Let $B \in \mathfrak{M}_q(B_{a \times b})$ for some fixed a, b . The characteristic polynomials of the matrices B and $B \cdot U$ have coefficients that are symmetric polynomials in q . In particular, the trace and determinant of B and $B \cdot U$ are symmetric about $ab/2$, ab , $ab - 1/2$ and $ab - 1$ respectively.*

Theorem 3.2. Let $d_1, d_2, \dots, d_s$ be a weak composition and $B_{a \times b} \nearrow$ be an oriented box poset where B denotes the rank matrix $\mathfrak{M}_q B_{a \times b} \nearrow$. Then the following polynomial is symmetric.

$$\text{tr } U^{d_1} \cdot B \cdot U^{d_2} \cdot B \cdots U^{d_1} \cdot B.$$

Note that when $l = 1$, we recover the rank symmetry of gate posets.

Corollary 3.3. The rank polynomial of any chainlink poset is symmetric.

We use this machinery to prove our main theorem.

Theorem 3.4. Let $\bar{a}$ be a composition of n , let l be a positive integer such that $2l \leq \min\{a_i\}_{i \in s}$ and let t be a non-negative integer. Then complementary sections of the chainlink polytope have the same Ehrhart quasi-polynomial,

$$\text{Ehr CL}^t \bar{a}, l = \text{Ehr CL}^{n-t} \bar{a}, l.$$

Using a scaling argument, we can deduce the following.

Corollary 3.5. Let a_i be positive real numbers such that $a_1 + \dots + a_s = n$, let l be a positive real number such that $2l \leq \min\{a_i\}_{i \in s}$ and let t be a non-negative real number. Then complementary sections of the chainlink polytope have the same volume,

$$|\text{CL}^t \bar{a}, l| = |\text{CL}^{n-t} \bar{a}, l|.$$

4 Unimodality

The recurrence relations deduced from characteristic matrices have other applications as well. In this subsection, we prove the following result.

Theorem 4.1. Rank polynomials of circular fence posets $\bar{F} \bar{c}$ are unimodal except when $\bar{c} = a, 1, a, 1$ or $1, a, 1, a$ for some positive integer a .

We define the matrix for a *down step* denoted by D as follows:

$$D : \begin{bmatrix} 1 & q & -q \\ & 1 & 0 \end{bmatrix}.$$

The following lemma is an easy consequence of the work in [5]. The interested reader is referred there to learn about how down steps fit into the framework of oriented posets.

Lemma 4.2. Let DC_n denote a decreasing chain, an n -element chain poset oriented by taking the maximal vertex as the target and the minimal vertex as the source. Then we have,

$$\mathfrak{M}_q DC_n \nearrow = D^{n-1} \cdot U.$$

That means the above theorem may be restated as follows:

Theorem 4.1'. *For any composition $\bar{a}$ with an even number of parts the following polynomial is unimodal except when $\bar{a} = a, 1, a, 1$ or $1, a, 1, a$ for some positive integer a :*

$$\bar{R}(\bar{a}, q) = \text{tr } D^{a_1} U^{a_2} D^{a_3} U^{a_4} \dots D^{a_{s-1}} U^{a_s} = \text{tr } U^{a_1} D^{a_2} U^{a_3} D^{a_4} \dots U^{a_{s-1}} D^{a_s}.$$

We prove this theorem by using recurrence identities similar to those described in the previous section. A particular one we use is the following matrix identity:

$$DUD - DU - UD - U = D^3 - D^2. \quad (\text{Id } 1)$$

On the rank polynomial level, (Id 1) translates to:

$$\begin{aligned} \bar{R}(a, 1, b, X; q) &= \bar{R}(a-1, 1, b, X; q) - \bar{R}(a, 1, b-1, X; q) \\ &\quad - \bar{R}(a-1, 1, b-1, X; q) + \bar{R}(a-b-1, X; q) \\ &\quad - \bar{R}(a-b, X; q). \end{aligned} \quad (\text{Id } 1')$$

That allows us to prove the theorem inductively, using the following to reduce the problem to compositions formed of 2's and 1's, and resolve those case by case.

Proposition 4.3. *For an odd-length sequence $X = x_1, x_2, \dots, x_k$ of positive integers suppose that $a, b \geq 1$ and $\bar{R}(a-1, 1, b-1, X)$ is unimodal. If $a > 1$ or $\ell(X) > 1$ with $x_1 > 1$ or $b \geq x_2$ then $\bar{R}(a, 1, b, X)$ is also unimodal.*

5 Properties of Chainlink polytopes

In this section, we investigate properties of our chainlink polytopes.

Lemma 5.1. *Let $\bar{a} \in \mathbb{N}^s$ be a composition of n and let $l \in \mathbb{R}$. The chainlink polytope $\text{CL}(\bar{a}, l)$ is full dimensional when $l < \min \bar{a}$.*

Determining exactly when these polytopes are non-empty is a tricky problem and does not seem to have a nice solution. We note though that a routine application of LP duality shows that the condition $l \leq (a_1 + \dots + a_s)/s$ is necessary.

Lemma 5.2. *Let $\bar{a} \in \mathbb{R}_{>0}^s$ and $l \in \mathbb{R}_{\geq 0}$. Suppose that $0 < l < \min_{i \in [s]} a_i$. Then the polytope $\text{CL}(\bar{a}, l)$ has exactly $3s$ facets, defined by the equalities $x_i = 0$, $x_i = a_i$ and $x_i - x_{i-1} = a_i - l$.*

Proposition 5.3. *If we have the strict inequality $2l < \min_{i \in [s]} a_i$, then the polytope $\text{CL}(\bar{a}, l)$ is simple and the combinatorial structure is independent of $\bar{a}$ and l .*

Proposition 5.4. Let $\bar{a} \in \mathbb{R}_{>0}^s$ and $l \in \mathbb{R}_{\geq 0}$. Suppose that $2l \leq \min_{i \in s} a_i$. The number of vertices of the chainlink polytope $\text{CL } \bar{a}, l$ is given by

$$\text{Vert CL } \bar{a}, l = \text{tr } A_1 \cdots A_s$$

where each $A_i = A$ if $a_i > 2l$ or $A_i = B$ if $a_i = 2l$, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Corollary 5.5. In particular if $2l < \min_{i \in s} a_i$, then

$$\text{Vert CL } \bar{a}, l = \text{tr } A^s$$

which satisfies the linear recurrence $\text{tr } A^s = 2 \text{tr } A^{s-1} - \text{tr } A^{s-2}$ for $s \geq 3$. The number of vertices is the "Companion Pell Numbers" in OEIS A002203.

The calculation of the volume of a chainlink polytope has quite a straightforward formula in the case $2l \leq \min_{i \in s} a_i$.

Proposition 5.6. Let $\bar{a} \in \mathbb{R}_{>0}^s$ and $l \in \mathbb{R}_{\geq 0}$. Suppose that $2l \leq \min_{i \in s} a_i$. The volume of the chainlink polytope $\text{CL } \bar{a}, l$ is given by the following trace formula.

$$\text{Vol CL } \bar{a}, l = \text{tr} \left(\begin{bmatrix} a_1 & -\frac{l^2}{2} \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} a_2 & -\frac{l^2}{2} \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} a_3 & -\frac{l^2}{2} \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_s & -\frac{l^2}{2} \\ 1 & 0 \end{bmatrix} \right).$$

6 Remarks and further work

There are several questions about these chainlink polytopes that naturally arise.

- **Ehrhart-Equivalence:** Two rational polytopes $P, Q \in \mathbb{R}^d$ are said to be Ehrhart-Equivalent if they have the same Ehrhart quasi-polynomial. They are said to be GL equidecomposable if we may partition $P = U_1 \cup \dots \cup U_n$ and $Q = V_1 \cup \dots \cup V_n$ into relatively open simplices such that for each i , U_i and V_i are $GL_d \mathbb{Z}$ equivalent. In [4], it was conjectured that Ehrhart-equivalent polytopes are GL equidecomposable. This is known to be true in dimensions 2 [3] and 3 [2]. Sections of chainlink polytopes provide us with a large class of examples to test this conjecture.
- **Multimodality:** Theorem 4.1 can be expressed in the following way: Let $\bar{a}$ be a composition of n . Then the function from $\{0, \dots, n\}$ to $\mathbb{N}$ given by $k \rightarrow \#CL^k \bar{a}, 1$ is unimodal save when $\bar{a} = a, 1, a, 1$ or $1, a, 1, a$ and is bimodal in these cases. If we instead fix a positive integer l such that $2l \leq \min\{a_i\}$ and look at $k \rightarrow \#CL^k \bar{a}, l$, the function may be multimodal. Indeed, we have that when $\bar{a} = 2k, 2k$ and $l = k$, we seem to have $k + 1$ peaks. Can one describe the maximal number of modes that may arise for fixed l and when these are attained?

- **Polytopal models for general circular fences:** In this paper, we developed a polytopal model for *gates*, which are circular fences coming from compositions of the form $a_1, 1, \dots, a_s, 1$, i.e. where all the down steps have size 1. What about general circular fences, those coming from compositions of the form $a_1, b_1, \dots, a_s, b_s$ with differing lengths of down steps $b_1, \dots, b_s$?

A natural proposal is as follows. The polytope [7] will consist of all real tuples $x_1, y_1, \dots, x_s, y_s$ such that

$$0 \leq x_i \leq a_i - 1, \quad 0 \leq y_i \leq b_i - 1, \quad b_i - 1 - x_i - a_i \leq y_i, \quad y_i \leq b_i - 1 - x_i - 1.$$

where the indices are taken cyclically. Unfortunately, these polytopes do not have the symmetry that chainlink polytopes have. Is there a way of defining polyhedral models for general circular fence posets so that this symmetry does hold?

- **General Chainlink Polytopes:** In the case $2l > \min_{i \in s} a_i$, the polytopes $\text{CL } \bar{a}, l$ begin to exhibit wild behaviour. Propositions 2.1 and 5.6, and Corollary 5.5 no longer hold. Given that the chainlink polytope $\text{CL } \bar{a}, l$ is full-dimensional when $l < \min_{i \in s} a_i$, this leaves a lot to be investigated, both combinatorially and geometrically.

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Combinatorial interpretations of the Macdonald identities for affine root systems

David Wahiche¹

¹Univ. Lyon, Université Claude Bernard Lyon 1, UMR 5208, Institut Camille Jordan, France

Abstract. We explore some connections between vectors of integers and integer partitions seen as bi-infinite words. This methodology enables us to give a combinatorial interpretation of the Macdonald identities for affine root systems of the seven infinite families in terms of symplectic and special orthogonal Schur functions. From these results, we are able to derive q -Nekrasov–Okounkov formulas associated to each family. Nevertheless we only give results for types $\tilde{C}$ and $\tilde{C}^\vee$, and give a sketch of the proof for type $\tilde{C}$.

Keywords: integer partitions, hook length, Macdonald identities for affine root systems, Littlewood decomposition, q -Nekrasov–Okounkov formula.

1 Introduction and notations

Formulas involving hook length abound in combinatorics and representation theory. Between 2006 and 2008, using various methods coming from representation theory [19], gauge theory [12] and combinatorics [5], several authors proved the so-called Nekrasov–Okounkov formula which can be stated as follows:

$$\sum_{\lambda \in \mathcal{P}} T^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \left(1 - \frac{z}{h^2}\right) = \prod_{k \geq 1} (1 - T^k)^{z-1}. \quad (1.1)$$

Here T is a formal variable, $z \in \mathbb{C}$, $\mathcal{P}$ is the set of integer partitions and $\mathcal{H}(\lambda)$ is the multiset of hook lengths of the partition λ .

This formula does not only cover the generating series for $\mathcal{P}$ obtained by setting $z = 0$ in (1.1): it actually gives a connection between powers of the Dedekind η function and integer partitions. Among generalizations of (1.1) that can be found in the literature, a (q, t) -extension was proved by Rains–Warnaar [14], by using refined skew Cauchy-type identities for Macdonald polynomials. This result was also obtained independently by Carlsson–Rodriguez-Villegas [1] by means of vertex operators and the plethystic exponential. As mentioned in [14], the special case $q = t$ is a reformulation of a result due to Dehaye–Han [2] and Iqbal–Nazir–Raza–Salem [7] which reads as follows:

$$\sum_{\lambda \in \mathcal{P}} T^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \frac{(1 - uq^h)(1 - u^{-1}q^h)}{(1 - q^h)^2} = \prod_{k, r \geq 1} \frac{(1 - uq^r T^k)^r (1 - u^{-1}q^r T^k)^r}{(1 - q^{r-1} T^k)^r (1 - q^{r+1} T^k)^r}. \quad (1.2)$$

Here T and q are formal variables and $u \in \mathbb{C}$. Note that taking $u = q^z$ and letting $q \rightarrow 1$ in (1.2) yields (1.1), although it is not immediate for the product side.

Methods used by Han to prove (1.1) and Dehaye–Han for (1.2) both start from a specialization of the Macdonald formula for affine root systems specialized in type $\tilde{A}$. However, one needs an unspecialized Macdonald identity for type $\tilde{A}$ to get (1.2). This can be found for instance in [15], where Rosengren and Schlosser give a proof of Macdonald identities for the seven infinite affine root systems with elliptic determinantal evaluation (see also Stanton’s reformulation in [16]). The next step in [2] to prove (1.2) uses new combinatorial notions such as exploded tableaux and a V_t -coding adapted from Garvan–Kim–Stanton [4]. These techniques are close to the methodology presented here. However their extensions to other types seem complicated.

A *partition* λ of a positive integer n is a nonincreasing sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ such that $|\lambda| := \lambda_1 + \dots + \lambda_\ell = n$. The λ_i ’s are the *parts* of λ , the number ℓ of parts being the *length* of λ , denoted by $\ell(\lambda)$. Each partition can be represented by its Ferrers diagram, which consists in a finite collection of boxes arranged in left-justified rows, with the row lengths in non-increasing order. The *Durfee square* of λ is the maximal square fitting in the Ferrers diagram. Its diagonal Δ will be called the main diagonal of λ . It is of size $d = d_\lambda := \max(s | \lambda_s \geq s)$. Let us introduce a signed statistic ε already appearing in [13]. For a box s of λ of coordinates (i, j) , ε_s is defined as -1 if s is strictly below the main diagonal of the Ferrers diagram and as 1 otherwise, as depicted in Figure 1. The partition $\lambda' = (\lambda'_1, \lambda'_2, \dots, \lambda'_{\lambda_1})$ is the *conjugate* of λ , where λ'_j denotes the number of boxes in the column j .

Recall that the *hook length* of s , denoted by h_s , is the number of boxes v such that either $s = v$, or v lies strictly below (respectively to the right) of s in the same column (respectively row). For any $t \in \mathbb{N}^*$, the multiset of all hook lengths that are congruent to $0 \pmod{t}$ is denoted by $\mathcal{H}_t(\lambda)$. Note that $\mathcal{H}(\lambda) = \mathcal{H}_1(\lambda)$. A partition ω is a t -*core* if $\mathcal{H}_t(\omega) = \emptyset$. For any $\mathcal{A} \subset \mathcal{P}$, let $\mathcal{A}_{(t)}$ be the subset of elements of $\mathcal{A}$ that are t -cores. For example, the only 2-cores are the “staircase” partitions $(k, k-1, \dots, 1)$ for any $k \in \mathbb{N}^*$.

An integer partition λ is *self-conjugate* if its Ferrers diagram is symmetric along the main diagonal. Let SC be the set of self-conjugate partitions. The set of doubled distinct partitions, denoted by DD , is that of all partitions λ with Durfee square of size d_λ such that $\lambda_i = \lambda'_i + 1$ for all $i \in \{1, \dots, d\}$. In Figure 1a for instance, $\lambda = (5, 3, 3, 1, 1) \in SC$ has its main diagonal Δ shaded in green while the main diagonal of $\lambda = (6, 4, 4, 1, 1) \in DD$ is shaded in green in Figure 1b. The strip shaded in yellow corresponds to the boxes added to a self-conjugate partition to obtain a DD partition. These subsets of partitions arise when one expresses the Weyl denominator formula for types B, C and D [11, p.79] and have been of particular interest within the work of Pétr  olle [13] where two Nekrasov–Okounkov type formulas for SC and DD are derived. For instance, he proves the following DD Nekrasov–Okounkov type formula, coming from the basic specialization of the Macdonald identity specialized for type $\tilde{C}$, which stands that for a

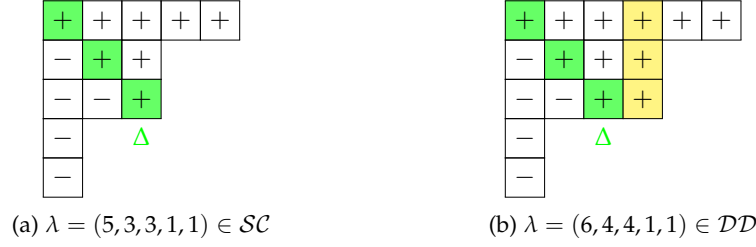


Figure 1: Self-conjugate and doubled distinct partitions filled with ε

formal variable T and any complex number z :

$$\sum_{\lambda \in \mathcal{DD}} (-1)^{d_\lambda} T^{|\lambda|} \prod_{\substack{s \in \lambda \\ h_s \in \mathcal{H}(\lambda)}} \left(1 - \frac{2z + 2}{h_s \varepsilon_s} \right) = \prod_{k \geq 1} (1 - T^k)^{2z^2 + z}. \quad (1.3)$$

Let a and T be formal variables. Recall that the T -Pochhammer symbol is defined as

$$(a; T)_0 = 1, \quad (a; T)_\infty = \prod_{j \geq 0} (1 - aT^j), \quad \text{and} \quad (a_1, \dots, a_n)_\infty = (a_1; T)_\infty \dots (a_n; T)_\infty.$$

We denote by sp (respectively so) the classical symplectic (respectively odd orthogonal) Schur function (see for instance [3]). Let t be a strictly positive integer, set $\mathbf{x} := (x_1, \dots, x_t)$ and let us introduce:

$$K_T(t, \mathbf{x}) = \prod_{1 \leq i < j \leq t} \left(Tx_i x_j, Tx_i^{-1} x_j^{-1}, Tx_i x_j^{-1}, Tx_j x_i^{-1}; T \right)_\infty.$$

The goal of this extended abstract is to investigate combinatorially the connections between all the Macdonald identities and the Nekrasov–Okounkov formulas. According to Corollary 6.2 in [15] (see also [10, 16]), the Macdonald identity for type $\tilde{C}_t$ is as follows:

$$\Delta_C(\mathbf{x}) (T; T)_\infty \prod_{i=1}^t \left(Tx_i^2, Tx_i^{-2}; T \right)_\infty K_T(t, \mathbf{x}) = \sum_{\mathbf{m} \in \mathbb{Z}^t} \sum_{\sigma \in S_t} \text{sgn}(\sigma) \prod_{i=1}^t x_i^{(2t+2)m_i} \times T^{2(t+1)\binom{m_i}{2} + (t+1)m_i} \left((x_i T^{m_i})^{\sigma(i)-t-1} - (x_i T^{m_i})^{t+1-\sigma(i)} \right), \quad (1.4)$$

where $\Delta_C(\mathbf{x}) = \prod_{1 \leq i \leq t} x_i^{-t} (1 - x_i^2) \prod_{1 \leq i < j \leq t} (x_j - x_i)(1 - x_i x_j)$. We will focus on types $\tilde{C}_t$ (denoted $\tilde{C}_t^{(1)}$ in [9]), $\tilde{C}_t^\vee$ (denoted $\tilde{D}_{t+1}^{(2)}$ in [9]) here, but the method is the same for the other types. By computing the Littlewood decomposition to partitions seen as a bi-infinite sequences of “0” and “1” (see Section 2 for precise definitions and properties), the quadratic form which is to the exponent of T in (1.4) can be interpreted as half the weight of a doubled distinct $(2t + 2)$ -core partition using the ideas of Garvan–Kim–Stanton [4]. Introducing the notion of $V_{g,t}$ -coding (see Definition 5) which can be thought of as the last indices (mod g) of letters “0” in the bi-infinite sequences, and $H_+ := \{h_s < g, \varepsilon_s = 1\}$, one can then reinterpret the right-hand side of (1.4) as follows:

Theorem 1. Set $t \in \mathbb{N}^*$. The Macdonald identity for type $\tilde{C}_t$ can be rewritten as follows:

$$\sum_{\omega \in \mathcal{DD}_{(2t+2)}} (-1)^{d_\omega + |H_+|} T^{|\omega|/2} \text{sp}_\mu(\mathbf{x}) = (T)_\infty^t K_T(t, \mathbf{x}) \prod_{i=1}^t \left(T x_i^2, T x_i^{-2}; T \right)_\infty, \quad (1.5)$$

where $\mathbf{v}$ is the $V_{g,t}$ -coding corresponding to ω (see [Definition 5](#)) and $\mu \in \mathcal{P}$ such that $\mu_i := v_i + i - 2t - 2$ for all $1 \leq i \leq t$.

Similarly we get the following:

Theorem 2. Set $t \in \mathbb{N}^*$. The Macdonald identity for type $\tilde{C}_t^\vee$ can be rewritten as follows:

$$\begin{aligned} \sum_{\omega \in \mathcal{SC}_{(2t)}} (-1)^{|H_+| + |H_+ \cap \Delta| + d_\omega} T^{|\omega|/2} \text{so}_\mu(\mathbf{x}) \\ = \left(T^{1/2}; T^{1/2} \right)_\infty (T; T)_\infty^{t-1} K_T(t, \mathbf{x}) \prod_{i=1}^t \left(T^{1/2} x_i, T^{1/2} x_i^{-1}; T^{1/2} \right)_\infty \end{aligned}$$

where $\mathbf{v}$ is the $V_{g,t}$ -coding corresponding to ω (see [Definition 5](#)) and $\mu \in \mathcal{P}$ is such that $\mu_i := v_i + i - 2t$ for all $1 \leq i \leq t$.

As a consequence of our results, we can prove the following q -analogues of Nekrasov–Okounkov type identities.

Theorem 3. For formal variables T, q and any complex number u , we have:

$$\begin{aligned} \sum_{\lambda \in \mathcal{DD}} (-u)^{d_\lambda} T^{|\lambda|/2} \prod_{s \in \lambda} \frac{1 - u^{-2\varepsilon_s} q^{h_s}}{1 - q^{h_s}} \prod_{s \in \Delta} \frac{1 + u q^{h_s/2}}{1 + u^{-1} q^{h_s/2}} \\ = \prod_{m, r \geq 1} \frac{1 + u q^{r-1} T^m}{1 + u^{-1} q^r T^m} \frac{(1 - u^{-2} q^{r+2} T^m)^{r - \lfloor r/2 \rfloor} (1 - u^2 q^{r-1} T^m)^{r - \lfloor r/2 \rfloor}}{(1 - q^r T^m)^{r - \lfloor r/2 \rfloor} (1 - q^{r+1} T^m)^{r - \lfloor r/2 \rfloor}} \end{aligned} \quad (1.6)$$

$$\begin{aligned} \sum_{\lambda \in \mathcal{SC}} (-1)^{d_\lambda} T^{|\lambda|/2} \prod_{s \in \lambda} \frac{1 - u^{-2\varepsilon_s} q^{2h_s}}{1 - q^{2h_s}} \prod_{s \in \Delta} \frac{1 - u q^{h_s}}{1 - u^{-1} q^{h_s}} \\ = \prod_{m, r \geq 1} \frac{(1 - T^{m/2})}{(1 - T^m)} \frac{1 - u^{-1} q^{2r-1} T^{m/2}}{1 - u q^{2r-1} T^{m/2}} \frac{(1 - u^{-2} q^{2(r+1)} T^m)^{r - \lfloor r/2 \rfloor} (1 - u^2 q^{2r} T^m)^{r - \lfloor r/2 \rfloor}}{(1 - q^{2r} T^m)^{r+1 - \lfloor (r+1)/2 \rfloor} (1 - q^{2(r+1)} T^m)^{r - \lfloor r/2 \rfloor}}. \end{aligned} \quad (1.7)$$

Note that taking $u = q^z$ and letting $q \rightarrow 1$ in (1.6) gives (1.3), while in (1.7) it yields a new Nekrasov–Okounkov type formula. Actually all the Macdonald identities in the Appendix of [10] can be derived from specializations and limits of [Theorems 1](#) and [2](#)

and their analogues for other types. In [Section 2](#), we introduce a way of computing the Littlewood decomposition with words, giving an explicit connection between t -cores and vectors of integers. In [Section 3](#), we use this description of the Littlewood decomposition to characterize the product of hook lengths of subsets of t -cores such as $\mathcal{DD}_{(2t+2)}$. We derive [Theorem 6](#) but we only give the lemmas necessary to its proof. In the last section, we sketch the proofs of [Theorems 1](#) and [3](#) for type $\tilde{C}$ only.

2 Combinatorial properties of the Littlewood decomposition on certain subsets of partitions

In this section, we use the formalism of Han–Ji [\[6\]](#). Let $\partial\lambda$ be the border of the Ferrers diagram of λ . Encode the walk along the border from the South-West to the North-East as depicted in [Figure 2](#): take “0” for a vertical step and “1” for a horizontal step. This yields a 0/1 sequence denoted $s(\lambda)$. This resulting word over the $\{0,1\}$ alphabet has infinitely many “0”s at the beginning (respectively “1”s at the end), is indexed by $\mathbb{Z}$, and written $(c_i)_{i \in \mathbb{Z}}$.

This writing as a sequence is not unique (since for any k sequences, $(c_{k+i})_{i \in \mathbb{Z}}$ define the same partition), hence the necessity to set the index 0 uniquely (to ensure this encoding is bijective). To tackle that issue, we set the index 0 when the number of “0”s to the right of that index is equal to the number of “1”s to the left. In other words, the number of horizontal steps along $\partial\lambda$ corresponding to a “1” of negative index in $(c_i)_{i \in \mathbb{Z}}$ must be equal to the number of vertical steps corresponding to “0”s of nonnegative index in $(c_i)_{i \in \mathbb{Z}}$ along $\partial\lambda$. The delimitation between the letter of index -1 and that of index 0 is called the *median* of the word, marked by a $|$ symbol. The size of the Durfee square is then equal to the number of “1”s of negative index. Hence the application s bijectively associates a partition to the word:

$$s(\lambda) = (c_i)_{i \in \mathbb{Z}} = (\dots c_{-2}c_{-1} | c_0c_1c_2 \dots),$$

where $c_i \in \{0,1\}$ for any $i \in \mathbb{Z}$, and such that

$$\#\{i \leq -1, c_i = 1\} = \#\{i \geq 0, c_i = 0\} < \infty.$$

Lemma 1. *This application maps bijectively a box s of hook length h_s of the Ferrers diagram of λ to a pair of indices $(i_s, j_s) \in \mathbb{Z}^2$ of $s(\lambda)$ such that $i_s < j_s$, $c_{i_s} = 1$ and $c_{j_s} = 0$, $j_s - i_s = h_s$.*

[Lemma 2](#) below allows to characterize the position of a box.

Lemma 2. ([\[17, Lemma 2.1\]](#)) *Set $\lambda \in \mathcal{P}$ and $s(\lambda)$ its corresponding word. Let s be a box of the Ferrers diagram of λ . Let $(i_s, j_s) \in \mathbb{Z}^2$ be the indices in $s(\lambda)$ associated with s . Then s is a box strictly above the main diagonal in the Ferrers diagram of λ if and only if $|i_s| \leq |j_s|$.*

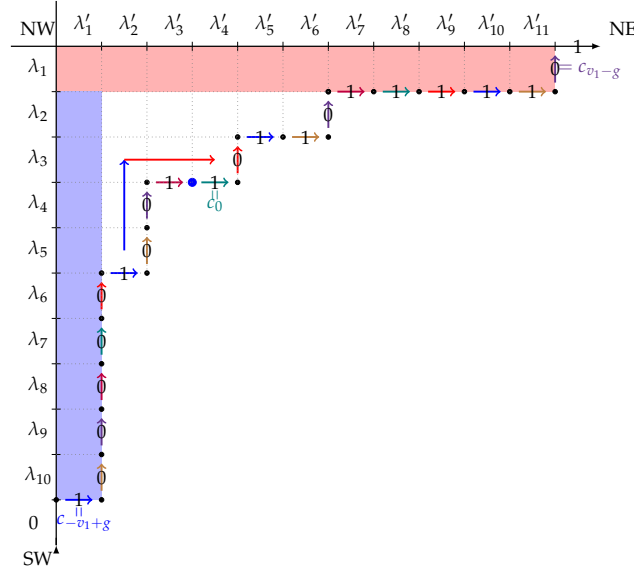


Figure 2: $\omega = (11, 6, 4, 2, 2, 1, 1, 1, 1, 1) \in \mathcal{DD}_{(6)}$ and its binary correspondence

The map below is often called the Littlewood decomposition (see [4, 6] for instance).

Definition 4. Let $t \geq 2$ be an integer and consider:

$$\Phi_t : \begin{cases} \mathcal{P} & \rightarrow \mathcal{P}_{(t)} \times \mathcal{P}^t \\ \lambda & \mapsto (\omega, \nu^{(0)}, \dots, \nu^{(t-1)}), \end{cases}$$

where if $s(\lambda) = (c_i)_{i \in \mathbb{Z}}$, then for all $k \in \{0, \dots, t-1\}$ one has $\nu^{(k)} := s^{-1}((c_{ti+k})_{i \in \mathbb{Z}})$. The tuple $\underline{\nu} = (\nu^{(0)}, \dots, \nu^{(t-1)})$ is the t -quotient of λ , denoted by $quot_t(\lambda)$, while ω is the t -core of λ , denoted by $core_t(\lambda)$.

Obtaining the t -quotient is straightforward from $s(\lambda) = (c_i)_{i \in \mathbb{Z}}$: we just look at subwords with indices congruent to the same values modulo t . The sequence 10 within these subwords are replaced iteratively by 01 until the subwords are all the infinite sequence of “0”s before the infinite sequence of “1”s (in fact it consists in removing all rim hooks in λ of length congruent to 0 (mod t)). Then ω is the partition corresponding to the word which has the subwords (mod t) obtained after the removal of the 10 sequences.

For example, if we take $\lambda = (4, 4, 3, 2)$ and $t = 3$, then $s(\lambda) = \dots 001101|010011\dots$

$$\begin{array}{l} s(\nu^{(0)}) = \dots 001|001\dots \\ s(\nu^{(1)}) = \dots 000|111\dots \\ s(\nu^{(2)}) = \dots 011|011\dots \end{array} \mapsto \begin{array}{l} s(w_0) = \dots 000|011\dots, \\ s(w_1) = \dots 000|111\dots, \\ s(w_2) = \dots 001|111\dots \end{array}$$

Thus $s(\omega) = \dots 000001|011111\dots$, and

$$\text{quot}_3(\lambda) = \left(\nu^{(0)}, \nu^{(1)}, \nu^{(2)} \right) = ((1, 1), \emptyset, (2)), \text{core}_3(\lambda) = \omega = (1)$$

One might therefore see t -core partitions as partitions whose t -quotient in the Littlewood decomposition is empty. Let ω be a partition, it is a t -core if and only if $\text{quot}_t(\omega) = (\emptyset, \dots, \emptyset)$. This is equivalent to say that all subwords $(\bmod t)$ are of the form $\dots 0011\dots$, which is, an infinite sequence of “0”s and then an infinite sequence of “1”s. For any $i \in \{0, \dots, t-1\}$ let us define $n_i := \min\{k \in \mathbb{Z} \mid c_{i+kt} = 1\}$. Each n_i corresponds to the index of the first “1” in the subword of $s(\omega)$ whose index is congruent to $i \pmod t$. Recall that the word $s(\omega)$ has as many “1”s of negative index as “0”s of positive index. This is equivalent to require $\sum_{i=0}^{t-1} n_i = 0$, hence there is a natural bijective map between $\omega \in \mathcal{P}_{(t)}$ and $(n_i)_{i \in \{0, \dots, t-1\}} \in \mathbb{Z}^t$ such that $\sum_{i=0}^{t-1} n_i = 0$.

For example, if we take $\omega = (4, 2)$ and $t = 3$, then

$$s(\omega) = \dots 000011|01101111\dots \longmapsto \begin{cases} s(w_0) = \dots 000 & | \underbrace{001}_n 1\dots, \\ & n_0=2 \\ s(w_1) = \dots 00 & \underbrace{1}_{n_1=-1} & | 111\dots, \\ s(w_2) = \dots 00 & \underbrace{1}_{n_2=-1} & | 111\dots \end{cases}$$

In [8], Johnson uses the fermionic viewpoint of partitions (which is the same as the one described above) to prove that this bijection is actually the one used by Garvan–Kim–Stanton in [4]. We reformulate here what is written in [8, Section 2] in terms of index of words: let λ be a partition and t be a strictly positive integer. Abaci correspond exactly to t -subwords of $s(\lambda)$ with fixed residue $(\bmod t)$ while the n_i ’s as defined in Bijection 2 in [4] correspond to the charge of the i -th runner on the abaci. This implies in particular that if ω is a t -core partition with corresponding word $s(\omega) = (c_i)_{i \in \mathbb{Z}}$ and if we set $\phi(\omega) := (n_0, \dots, n_{t-1})$, then $n_i = \min\{k \in \mathbb{Z} \mid c_{kt+i} = 1\}$. Moreover, we have:

$$|\omega| = \frac{t}{2} \sum_{i=0}^{t-1} n_i^2 + \sum_{i=0}^{t-1} i n_i. \quad (2.1)$$

3 Hook length product of t -core partitions

The aim of this section is to introduce the material required to prove [Theorem 6](#) which is an enumerative result on hook length products. The latter is crucial to get [Theorem 3](#), as explained in the introduction (see also [Section 4](#) for more details). This extended

abstract only focuses on results for $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{C}}^\vee$ that are stated independently in order to avoid characteristic functions (that complicate their statement). However note that analogous statements exist for all seven infinite families. As for technical results allowing to prove [Theorem 6](#), they are stated only in the $\mathcal{DD}_{(2t+2)}$ -case, once again to avoid characteristic functions and technicality they bring. The cases $\mathcal{SC}_{(2t)}$, $\mathcal{SC}_{(2t+1)}$ and $\mathcal{DD}_{(2t+1)}$ are obtained on the same way, uniform formulations and proofs can be found in [\[18\]](#). [Theorem 6](#) involves $V_{g,t}$ -codings: these are a particular kind of vector of integers (see [Definition 5](#)) that can be associated to any $\omega \in \mathcal{DD}_{(2t+2)}$ (see [Proposition 7](#)).

Let $t \in \mathbb{N}^*$. The end of [Section 2](#) associates a vector of integers $\phi(\omega)$ to any t -core ω . Following [\[1\]](#), this vector can be ordered “naturally” (meaning that this order is inherited by that of the residue $(\bmod t)$) by setting:

$$n_k := \left\lfloor \frac{\lambda_i - i}{t} \right\rfloor + 1, \quad i = \min\{v \mid \lambda_v \equiv k \pmod{t}\}. \quad (3.1)$$

The Littlewood decomposition, when restricted to $\mathcal{DD}$, also has interesting properties and can be stated as follows ([\[4, 13\]](#) for instance):

$$\lambda \in \mathcal{DD} \mapsto \begin{cases} (\omega, \tilde{\nu}) \in \mathcal{DD}_{(t)} \times \mathcal{DD} \times \mathcal{P}^{(t-1)/2} & \text{if } t \text{ is odd,} \\ (\omega, \tilde{\nu}, \mu) \in \mathcal{DD}_{(t)} \times \mathcal{DD} \times \mathcal{P}^{(t-2)/2} \times \mathcal{SC} & \text{if } t \text{ is even.} \end{cases} \quad (3.2)$$

This symmetrical behaviour of $\mathcal{DD}_{(2t+2)}$ partitions yields some additional conditions on the associated vector of integers. These have already been studied by Garvan–Kim–Stanton [\[4\]](#) but are stated here in a slightly different way. Let $\omega \in \mathcal{DD}_{(2t+2)}$ and $\phi(\omega) = (n_0, n_1, \dots, n_{2t+1})$ be as above. Symmetries of $\mathcal{DD}$ ensure that $n_0 = 0$, and $n_i = -n_{2t+2-i}$ for all $i \in \{1, \dots, 2t+1\}$, which in particular implies that $n_t = 0$. By [\(2.1\)](#), we then have:

$$|\omega| = (t+1) \sum_{i=0}^{2t+1} n_i^2 + \sum_{i=0}^{2t+1} i n_i = 2 \left((t+1) \sum_{i=0}^{t-1} n_i^2 + \sum_{i=0}^{t-1} (i-t-1) n_i \right) \quad (3.3)$$

So with this formalism one can recover the vector of integers given in [\[4\]](#) from the Littlewood decomposition together with the word decomposition. When developing [\(1.4\)](#) with respect of powers of T , half of the previous quadratic form appear to the exponent.

Example 1. [Figure 2](#) above illustrates this for a $\mathcal{DD}$ when $t = 6$. The arrows are sorted in six different colors, each of them corresponding to a fixed residue $(\bmod 6)$ of the index of the corresponding word of ω . The word corresponding to ω writes as follows:

$$s(\omega) = \dots 000001000001001|1011011111011111111 \dots$$

By extracting the subwords of fixed residue $(\bmod 6)$, we obtain:

$$\begin{aligned} s(w_0) &= \dots 000|111 \dots, & s(w_1) &= \dots 000|011 \dots & s(w_2) &= \dots 011|111 \dots \\ s(w_3) &= \dots 000|111 \dots, & s(w_4) &= \dots 000|001 \dots, & s(w_5) &= \dots 001|111 \dots \end{aligned}$$

so that $(n_0, n_1, n_2, n_3, n_4, n_5) = (0, 1, -2, 0, 2, -1) \in \mathbb{Z}^6$.

The properties of symmetry of any $\mathcal{DD}$ partition along its main diagonal yields the following restriction, as mentioned in [4]:

$$\omega \in \mathcal{DD}_{(6)} \longleftrightarrow (1, -2) \in \mathbb{Z}^2.$$

Similarly, the symmetrical behaviour of $\mathcal{SC}_{(2t)}$ partitions yields additional conditions on the associated vector of integers: let $\omega \in \mathcal{SC}_{(2t)}$ and $\phi(\omega) = (n_0, n_1, \dots, n_{2t-1})$, then $n_i = -n_{2t-1-i}$ for all $i \in \{0, \dots, 2t-1\}$, in particular $n_t = 0$. By (2.1), we have:

$$|\omega| = (2t)/2 \sum_{i=0}^{2t-1} n_i^2 + \sum_{i=0}^{2t-1} i n_i = \sum_{i=0}^{t-1} ((2t)n_i^2 + (2(i-t)-1)n_i). \quad (3.4)$$

The ordered vector of integers $\phi(\omega)$ still lacks of some properties to be the combinatorial tool one needs to prove Theorems 1, 3 and 6. These properties are the one satisfied by vectors of integers called $V_{g,t}$ -codings (defined in [18]). The goal here is to attach a $V_{g,t}$ -coding to any $\phi(\omega)$. Before going any further, let us motivate Definition 5 below, with the following observation: making use of properties of q -series together with the Weyl group action on the affine root systems, Stanton provides a new proof of the Macdonald identities [16]. Sums that appear in these identities involve vectors of integers, so that the existence of ϕ , which is a bijection between partitions and vectors of integers, leads us to rephrase Stanton's work. From this perspective, [16, Proposition 3.7] that focuses on type $\tilde{C}_t$ establishes why only partitions in $\mathcal{DD}_{(2t+2)}$ are to be considered on type $\tilde{C}_t$: they are actually the only ones whose associated vector of integers contribute in a non-trivial way to the sum in the Macdonald identity for the type $\tilde{C}_t$.

As this extended abstract only focuses on results for types $\tilde{C}_t$ and $\tilde{C}_t^\vee$, Definition 5 below restricts to these types. Both the general definition (of [18]) and Definition 5 involve a parameter g of the affine root system defined by the equality $M = g\Lambda$ in [10, p.134]. In particular $g = 2t + 2$ if the affine algebra considered is of type $\tilde{C}_t$ and $g = 2t$ if it is of type $\tilde{C}_t^\vee$.

Definition 5. Let $t \in \mathbb{N}^*$ and $g \in \mathbb{N}^*$ such that $t \leq g$, then set $\lambda \in \mathcal{P}$ and let $s(\lambda) = (c_k)_{k \in \mathbb{Z}}$ be its corresponding binary word. For $i \in \{0, \dots, g-1\}$, define $\beta_i := \max\{(k+1)g + i \mid c_{kg+i} = 0\}$. Let $\sigma : \{1, \dots, g\} \rightarrow \{0, \dots, g-1\}$ be the unique permutation such that $\beta_{\sigma(1)} > \dots > \beta_{\sigma(g)}$. The vector $\mathbf{v} := (\beta_{\sigma(1)}, \dots, \beta_{\sigma(t)})$ is called the $V_{g,t}$ -coding corresponding to λ .

Note that a notion of V_t -coding introduced by Dehay-Han in [2] to connect Macdonald identities for type $\tilde{A}_{t-1}$ and t -core partitions corresponds to the $V_{t,t}$ -coding. We can now state Theorem 6 and provide the key technical results required to prove it.

Theorem 6. Set t a positive integer and $g = 2t + 2$. Let $\omega \in \mathcal{DD}_{(2t+2)}$ and $\mathbf{v} \in \mathbb{Z}^t$ its associated $V_{g,t}$ -coding, and set $r_i = v_i - t - 1$ for any $i \in \{1, \dots, t\}$. Then we have

$$|\omega| = \frac{1}{g} \sum_{i=1}^t r_i^2 - \frac{(g/2 - 1)(g - 1)}{12},$$

and setting $\alpha_i(\omega) := \#\{u \in \omega, h_u = g - i, \varepsilon_u = 1\}$, and for any function $\tau : \mathbb{Z} \rightarrow F^\times$, where F is a field, we also have

$$\prod_{s \in \omega} \frac{\tau(h_s - \varepsilon_s g)}{\tau(h_s)} = \prod_{i=1}^{g-1} \left(\frac{\tau(-i)}{\tau(i)} \right)^{\alpha_i(\omega)} \prod_{i=1}^t \frac{\tau(r_i)}{\tau(i)} \prod_{1 \leq i < j \leq t} \frac{\tau(r_i - r_j)}{\tau(j - i)} \frac{\tau(r_i + r_j)}{\tau(g - i - j)}.$$

To show [Theorem 6](#) one needs to associate bijectively a vector of integers to any partition. For technical and conceptual reasons (respectively facilitating inductive proofs and interpreting the determinants as Schur functions) the involved vectors need to be ordered. The $V_{g,t}$ -codings introduced so far are natural candidate, but one still need to show that they correspond bijectively to partitions. This is the purpose of [Proposition 7](#).

Proposition 7. *Let t be a positive integer. Any λ in one of the sets $\mathcal{DD}_{(2t+2)}$ and $\mathcal{SC}_{(2t)}$ is in bijective correspondence with its $V_{g,t}$ -coding, where g is the index of the corresponding set.*

Let $s(\omega)$ be the word corresponding to ω via the word decomposition. The $V_{g,t}$ -coding associated to ω by [Proposition 7](#) is immediately given when reading $s(\omega)$ from right to left: for instance v_1 is the index of the first “1” in the subword $(\bmod g)$ of $s(\omega)$ that contains the last “0”. [Lemma 3](#) allows to prove [Theorem 6](#) by induction on the length of the Durfee square of the partition. Following the same philosophy as in [Lemmas 1](#) and [2](#), it enumerates the boxes in the largest hook by means of their pair of indices. Ultimately, and as illustrated in [Figure 2](#) in the red and blue shaded areas, let s be a box of λ in the first hook. Then $\varepsilon_s = 1$, respectively $\varepsilon_s = -1$, implies $j_s = v_1 - g$, respectively $i_s = -v_1 + g$ and $j_s < v_1 - g$.

We define the g -intervals as $\mathcal{I}_{m,M}^{g,+} := \{k \in \mathbb{Z} \mid m \leq k < M, k \equiv m \pmod{g}\}$ and $\mathcal{I}_{m,M}^{g,-} := \{l \in \mathbb{Z} \mid m < l \leq M, l \equiv M \pmod{g}\}$. This notion is of particular interest in our case ever since we are trying to enumerate hook lengths with a fixed residue $(\bmod g)$. The key ingredient of the proof of [Theorem 6](#) is that g -intervals allow us to transform products involving generic functions τ into telescopic factors.

Lemma 3. *Set $\omega \in \mathcal{DD}_{(2t+2)}$ so that $g = 2t + 2$ and let $(v_1, \dots, v_t) \in \mathbb{Z}^t$ be its associated $V_{g,t}$ -coding. Then the biggest hook of ω , denoted by H_1 , corresponds to the collection of boxes $\mathcal{H}_{1,+} \cup \mathcal{H}_{1,-}$ where $\mathcal{H}_{1,+}$ is the set of indices of boxes s in the first hook such that $\varepsilon_s = 1$:*

$$\mathcal{H}_{1,+} = \mathcal{I}_{-v_1+g, v_1-g}^{g,+} \cup \mathcal{I}_{0, v_1-g}^{g,+} \cup \mathcal{I}_{g/2, v_1-g}^{g,+} \bigcup_{i=2}^t \left(\mathcal{I}_{v_i, v_1-g}^{g,+} \cup \mathcal{I}_{-v_i+g, v_1-g}^{g,+} \right),$$

and $\mathcal{H}_{1,-}$ is the set of indices of boxes s in the first hook such that $\varepsilon_s = -1$:

$$\mathcal{H}_{1,-} = \mathcal{I}_{-v_1+g, v_1-2g}^{g,-} \cup \mathcal{I}_{-v_1+g, -g}^{g,-} \cup \mathcal{I}_{-v_1+g, -g/2}^{g,-} \bigcup_{i=2}^t \left(\mathcal{I}_{-v_1+g, v_i-g}^{g,-} \cup \mathcal{I}_{v_1-g, -v_i}^{g,-} \right).$$

Finally we will also need the following lemma to apply [Theorem 6](#) to Macdonald identities specializations.

Lemma 4. *Let $\omega \in \mathcal{DD}_{(2t+2)}$ and $s(\omega) = (c_k)_{k \in \mathbb{Z}}$. Let $u_i = (2t+2)n_i + i$ the index of the first letter “1” in the subword $(c_{i+(2t+2)k})_{k \in \mathbb{Z}}$, where $(n_i) = \phi^{-1}(\omega)$. Let $(v_i)_{i \in \{1, \dots, t\}}$ be a $V_{g,t}$ -coding. Then $\sigma : \{1, \dots, t\} \rightarrow \{0, \dots, 2t+1\}$ of [Definition 5](#) is such that that $v_i = u_{\sigma(i)}$ for $i \in \{1, \dots, t\}$. We have that:*

$$|H_+| = \#\{s \in \omega, h_s < 2t+2, \varepsilon_s = 1\} \equiv d_\omega + \text{sgn}(\sigma) \pmod{2}.$$

4 Sketch of proof of [Theorems 1 and 3](#)

We first expand the right-hand side of (1.4) and we extract the terms in T . The power of T corresponds to (3.3). By changes of variables and [Lemma 4](#), we are able to take $\omega = \phi^{-1}(\mathbf{m})$ and derive [Theorem 1](#).

We now derive [Theorem 3](#) for $\mathcal{DD}$. In order to do so, we start by proving the equality setting $u = q^t$ for any strictly positive integer t on both sides. By technical manipulations on products, the right hand side of (1.6) is exactly that of [Theorem 1](#) when $x_i = q^i$ which completes the first step of the proof.

The left-hand side of (1.6) can be obtained by setting $\tau(x) = 1 - q^x$ in [Theorem 6](#) and multiplying the resulting expression by the product of hook lengths on the main diagonal Δ . We prove that:

$$\text{sp}_\mu(q, q^2, \dots, q^t) = (-1)^{|H_+|} q^{(t+1)d_\lambda} \prod_{s \in \lambda} \frac{1 - q^{h_s - (2t+2)\varepsilon_s}}{1 - q^{h_s}} \prod_{s \in \Delta} \frac{1 + q^{t+1+h_s/2}}{1 + q^{-t-1+h_s/2}}. \quad (4.1)$$

The proof of the equality above is done by induction on $r_1 = \max_{1 \leq i \leq t} (r_i)$ as defined in [Theorem 6](#). First remark that the product over elements of Δ on the right-hand side of (4.1) is a telescopic product equal to $\prod_{i=1}^t (1 + q^{r_i}) / (1 + q^i)$. On the left-hand side of (4.1), the quotient of determinants $\det(q^{ir_j} - q^{-ir_j}) / \det(q^{i(t+1-j)} - q^{-i(t+1-j)})$ can be computed by noting that it is equal to $\det((q^{r_j})^i - (q^{r_j})^{-i}) / \det((q^{t+1-j})^i - (q^{t+1-j})^{-i})$. Finally we conclude by checking that both sides of the equation verify the same initial value and the same induction property.

The last step of the proof is to check that both sides of (1.6) are Laurent polynomials in the variable u . An argument of polynomiality that (1.6) holds for any u then allows one to conclude.

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Alternating Sign Matrices and Descending Plane Partitions: a linear number of equivalent statistics

Ilse Fischer^{*1}, Hans Höngesberg^{†1}, and Florian Schreier-Aigner^{‡1}

¹University of Vienna, Austria

Abstract. There is the same number of alternating sign matrices as there is of cyclically symmetric lozenge tilings of a hexagon with a central triangular hole of size 2, but finding an explicit bijection has been an open problem for about 40 years now. This is even more surprising in the view of the fact that, when restricting to vertically symmetric alternating sign matrices, their number equals the number of such lozenge tilings that also exhibit an additional reflective symmetry. To approach such bijections, we present generalizations of these results that involve a linear number of equidistributed statistics. Prior to this work, only a constant number of such statistics were known.

Keywords: alternating sign matrices, plane partitions, non-intersecting lattice paths, bijective proofs

1 Introduction

An *alternating sign matrix* (ASM) is a square matrix with entries in $\{0, \pm 1\}$ such that, in each row and each column, 1's and -1 's alternate and sum to 1. All 3×3 ASMs are given next.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

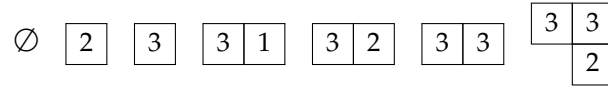
ASMs had been introduced by Robbins and Rumsey in the 1980s [15] and they conjectured that the number of $n \times n$ ASMs is $\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$, which was proven after considerable effort by Zeilberger [16] (in fact, he showed a more general result that includes an additional parameter). Soon after that, Kuperberg used methods from statistical physics to give another, shorter proof [10] (of the special case).

^{*}ilse.fischer@univie.ac.at. Ack. supp. from the Austrian Science Foundation FWF, P34931

[†]hans.hoengesberg@univie.ac.at. Ack. supp. from the Austrian Science Foundation FWF, P34931

[‡]florian.schreier-aigner@univie.ac.at. Ack. supp. from the Austrian Science Foundation FWF, P34931 and J4387

Further observations caused considerable excitement in the combinatorics community, in particular $\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$ appears also as the enumeration formula for *descending plane partitions* (DPPs) with parts no greater than n [3, 12], which are defined next. A *strict partition* is a sequence $\lambda = (\lambda_1, \dots, \lambda_n)$ of positive integers with $\lambda_1 > \dots > \lambda_n$, and the *shifted Young diagram* of shape λ has λ_i cells in row i , where each row is indented by one cell to the right with respect to the previous row. A DPP is a filling of a shifted Young diagram with positive integers such that rows decrease weakly and columns decrease strictly, and the first part of each row is greater than the length of its row and less than or equal to the length of the previous row. The DPPs with all entries less than or equal to 3 are given next.



There has been much effort to construct explicit bijections between ASMs and DPPs, but until recently without much progress. Very recently, a bijective proof [6] of an identity that implies the equinumerosity of ASMs and DPPs was established. The bijection is quite involved, and uses signed sets and a generalization of the involution principle by Garsia and Milne [7]. It seems fair to say that a perfect understanding of these relations despite the many efforts is still missing.

The main purpose of this extended abstract is to shed light on the mysterious relation between ASMs and DPPs by introducing a pair of $n + 3$ equidistributed statistics, one set of $n + 3$ statistics on ASMs and the other set on DPPs. Finding such statistics is a natural approach to construct bijections. Prior to this work, a constant number of such statistics were known. The extended abstract is a summary of our two preprints [1, 5] and we refer the reader to these preprints for all the details.

The introduction of our statistics is at the cost of replacing both ASMs and DPPs by somewhat modified objects, see Sections 2 and 3, respectively, however the relation of these modified objects to ASMs and DPPs is easily established by combinatorial means. We are also able to obtain a result that is similar in vein for vertically symmetric ASMs and the corresponding DPP-objects (Section 4). As the results are also phrased in terms of *non-intersecting lattice paths* on the DPP-side, we spend the rest of the introduction by presenting the lozenge tilings that correspond to DPPs which in turn are in easy bijective correspondence with the lattice paths.

DPPs with parts no greater than n are known [9] to be in easy bijective correspondence with cyclically symmetric lozenge tilings of a hexagon with side lengths $n, n + 2, n, n + 2, n, n + 2$ that have a central triangular hole of size 2, see Figure 1. Such tilings can be encoded as families of non-intersecting lattice paths (*n-DPP paths*), see Figure 1. More precisely, the *n-DPP paths* have starting points $A_i = (0, i - 2)$ and end points $E_i = (i, 0)$ for $i \in S$ where $S \subseteq \{2, \dots, n\}$ and use steps in $(1, 0)$ -direction and $(0, -1)$ -direction (note that our convention differs from the one more frequently used

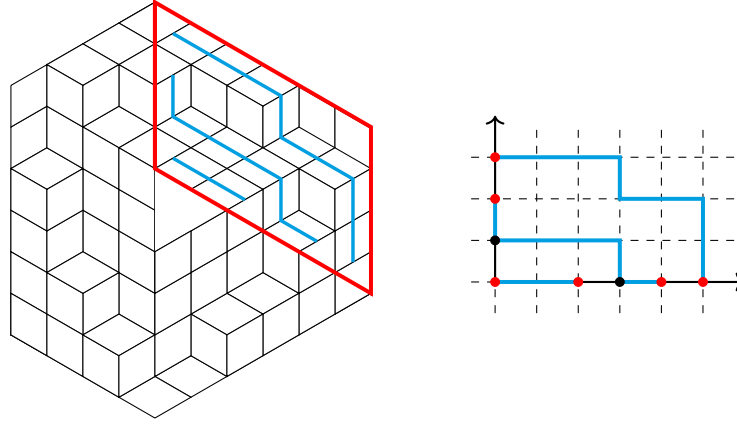


Figure 1: A cyclically symmetric lozenge tiling of a hexagon with the fundamental domain framed in red (left) and the corresponding 5-DPP paths.

by the reflection along the line $y = x$. In Section 4, we consider *vertically symmetric alternating sign matrices* (VSASMs). Due to the vertical symmetry, the central column of a VSASM has to exist and is of the form $(1, -1, 1, \dots, -1, 1)^\top$; hence, VSASMs only exist for an odd order. Amongst the different objects that are known to be equinumerous to ASMs, cyclically symmetric lozenge tilings of a hexagon with a central triangular hole are particularly well-suited for imposing an additional condition that corresponds to the vertical symmetry of VSASMs. In fact, the axial symmetry can be literally transferred to lozenge tilings: it has been shown [14, 11] that $(2n + 1) \times (2n + 1)$ VSASMs are equinumerous with cyclically symmetric lozenge tilings of a hexagon with sides of alternating lengths $2n + 2$ and $2n$ and with a central triangular hole of side length 2 that exhibit an additional reflective symmetry. Figure 2 illustrates an example of such a tiling.

2 Arrowed monotone triangles

A *monotone triangle* is a triangular array $(m_{i,j})_{1 \leq j \leq i \leq n}$ of integers of the form

$$\begin{array}{c}
 m_{1,1} \\
 m_{2,1} \quad m_{2,2} \\
 \vdots \quad \vdots \\
 m_{n-1,1} \quad \dots \quad m_{n-1,n-1} \\
 m_{n,1} \quad m_{n,2} \quad \dots \quad m_{n,n}
 \end{array}$$

with weakly increasing entries along northeast- and southeast-diagonals, i.e., $m_{i+1,j} \leq m_{i,j} \leq m_{i+1,j+1}$, and strictly increasing rows, i.e., $m_{i,j} < m_{i,j+1}$. It is well known that

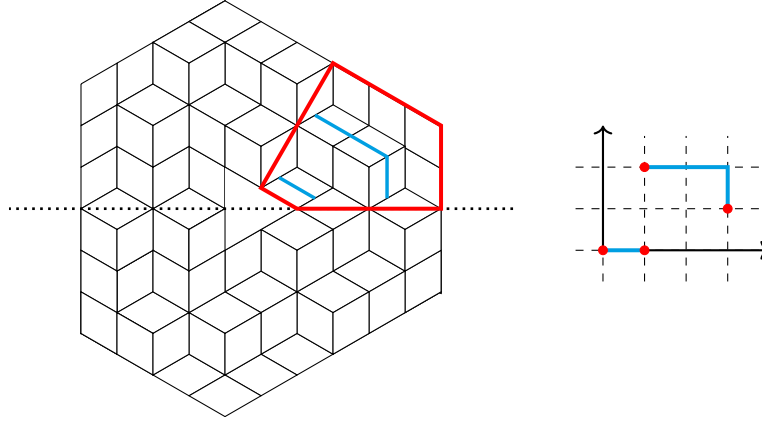


Figure 2: Cyclically symmetric lozenge tiling with an additional reflective symmetry of a hexagon with alternating side lengths $2n + 2$ and $2n$ for $n = 2$. The fundamental domain framed in red corresponds to a sixth of the hexagon. The axis of reflection is indicated with a dotted line. The tiling corresponds to non-intersecting lattice paths from $(i - 1, 2i - 2)$ to $(2i - 1, i - 1)$ for $1 \leq i \leq n$ with step set $\{(1, 0), (0, -1)\}$.

$n \times n$ ASMs are in bijection with monotone triangles with bottom row $(1, 2, \dots, n)$. As we see in Section 4, monotone triangles with bottom row $(0, 2, \dots, 2n - 2)$ are in bijection with $(2n + 1) \times (2n + 1)$ VSASMs in a similar way. Our modified ASM-objects are the following decorated monotone triangles.

An *arrowed monotone triangle* (AMT) is a monotone triangle $(m_{i,j})_{1 \leq j \leq i \leq n}$ together with a decoration of its entries by one of the symbols $\nwarrow, \nearrow, \nwarrow \nearrow$ such that the decoration of $m_{i+1,j}$ has to be $\nwarrow$ if $m_{i+1,j} = m_{i,j}$, while the decoration of $m_{i+1,j+1}$ has to be $\nearrow$ if $m_{i+1,j+1} = m_{i,j}$. In other words, an arrow indicates that the decorated entry has to be different from the entry it is pointing to. We write $\nwarrow m_{i,j}, m_{i,j} \nearrow, \nwarrow m_{i,j} \nearrow$ if the entry $m_{i,j}$ is

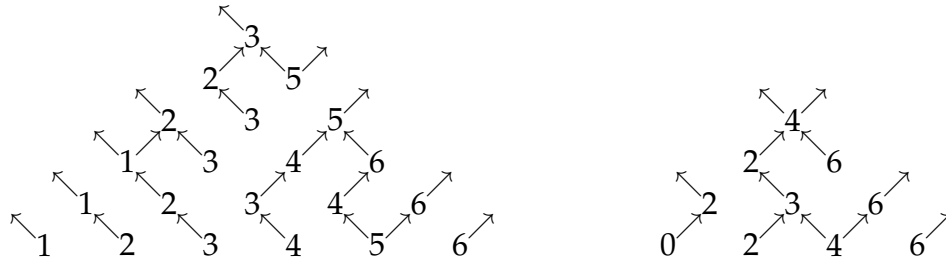


Figure 3: An arrowed monotone triangle with bottom row $(1, 2, 3, 4, 5, 6)$ on the left and with bottom row $(0, 2, 4, 6)$ on the right.

decorated with $\nwarrow, \nearrow$, or $\nwarrow \nearrow$, respectively. See Figure 3 for an example. We assign to an

AMT $M = (m_{i,j})_{1 \leq j \leq i \leq n}$ the weight

$$W_M(u, v, w; \mathbf{x}) = u^{\# \nearrow} v^{\# \nwarrow} w^{\# \nearrow \nwarrow} \prod_{i=1}^n x_i^{\sum_{j=1}^i m_{i,j} - \sum_{j=1}^{i-1} m_{i-1,j} + \#(\nearrow \text{ in row } i) - \#(\nwarrow \text{ in row } i)}.$$

The AMT in Figure 3 (left) has weight $u^7 v^{11} w^3 x_1^2 x_2^5 x_3^2 x_4^3 x_5^3 x_6^2$ and the AMT in Figure 3 (right) has weight $u^5 v^3 w^2 x_1^4 x_2^4 x_3^2 x_4^4$. The exponents of the $n + 3$ variables are the $n + 3$ statistics on the ASM-side of our story. Note that there is a very satisfying analogy to the weight of Gelfand-Tsetlin patterns that corresponds to the Schur weight on semistandard Young tableaux under the classical bijection as this weight is simply $\prod_{i=1}^n x_i^{\sum_{j=1}^i a_{i,j} - \sum_{j=1}^{i-1} a_{i-1,j}}$ for a given Gelfand-Tsetlin pattern $A = (a_{i,j})_{1 \leq j \leq i \leq n}$.

AMTs and monotone triangles are related as follows. Fix a monotone triangle M and consider the set of all AMTs which are mapped to M by forgetting the decorations. We claim that the sum of weights of these AMTs is equal to 1 when setting

$$u = v = 1, \quad w = -1, \quad x_1 = \dots = x_n = 1. \quad (2.1)$$

Indeed, this follows from the observation that each entry of M that is different from its northwest-neighbor and its northeast-neighbor can be decorated with any of $\nwarrow, \nearrow, \nearrow \nwarrow$, while all the other entries can be decorated with either $\nwarrow$ or $\nearrow$. (In [1, Proposition 3.3] we also identify a weight on monotone triangles directly that is up to a simple factor the sum of weights of all arrowed monotone triangles that are obtained by decorating the monotone triangle.) Therefore, we may consider AMTs with bottom row $(1, 2, \dots, n)$ as our modified objects on the ASM-side.

For a sequence $L = (L_1, \dots, L_n)$ of integers (not necessarily decreasing), we define the *extended Schur polynomial* $s_L(\mathbf{x})$ as $s_L(\mathbf{x}) = \frac{\det_{1 \leq i, j \leq n} (x_i^{L_j + n - j})}{\prod_{1 \leq i < j \leq n} (x_i - x_j)}$. For an equivalent definition see [1, Equation (3.6), (5.1)]. We are now able to state our first main result.

Theorem 1 ([1, Theorem 2.2]). *The generating function of arrowed monotone triangles with bottom row $k_1 < \dots < k_n$ is*

$$\prod_{i=1}^n (uX_i + vX_i^{-1} + w) \prod_{1 \leq p < q \leq n} (uE_{k_p} + vE_{k_q}^{-1} + wE_{k_p}E_{k_q}^{-1}) s_{(k_n, k_{n-1}, \dots, k_1)}(x_1, \dots, x_n), \quad (2.2)$$

where E_k denotes the shift operator, defined as $E_k p(k) = p(k + 1)$.

This is a multivariate extension of the operator formula of number of monotone triangles with bottom row [4] and one of the main ingredient in the proofs of our other results.

3 On the relation between ASMs and DPPs

We present the modified objects replacing DPPs along with the multivariate weight with $n + 3$ variables that give rise to the $n + 3$ statistics on the DPP-side.

Extended n -DPP paths are families of non-intersecting lattice paths with starting points $A'_i = (-i, i - 2)$ and end points $E'_i = (i, -2)$ for $i \in S$ and a given subset $S \subseteq \{1, \dots, n\}$ which stay weakly below the line $y = n - 2$ and have the following step sets and weights:

- (i) In the region $\{(x, y) | x < 0\}$, the step set is $\{(1, 0), (0, 1)^2\}$ with weights $x_{p+q+2}v^{-1}$ for a $(1, 0)$ step ending in (p, q) , and where $(0, 1)^2$ indicates that there are two types of $(0, 1)$ steps, one of which has weight $x_{p+q+2}v^{-1}w$ when ending in (p, q) and the other has weight 1.
- (ii) In the region $\{(x, y) | x \geq 0, y \geq -1\}$, the step set is $\{(1, 0), (0, -1)\}$, with weights ux_{q+2} for a $(1, 0)$ step at height q and weight 1 for a $(0, -1)$ step.
- (iii) If we go below the line $y = -1$, the step set is $\{(1, -1), (0, -1)\}$, with weights w for a $(1, -1)$ step and 1 for a $(0, -1)$ step.

The weight of extended n -DPP paths is $v^{\binom{n+1}{2}}$ times the product of the weights of all steps. See Figure 4 for an example. Note that in [1, Section 4] we consider a (very minor) modification of these paths. Analogously to the situation for AMTs and ASMs, we argue

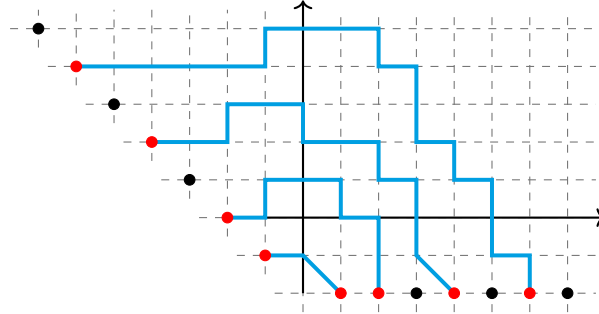


Figure 4: A family of extended 5-DPP paths with $S = \{1, 2, 4, 6\}$ and weight $u^{11}v^5w^5x_1^5x_2^4x_3^6x_4^5x_5^2x_6^2x_7^3$ where all $(0, 1)$ steps left of the y -axis are of first type.

next that the generating function of extended n -DPP paths is the number of n -DPP paths under the specialization (2.1). Indeed, strictly left of the y -axis $(0, 1)$ steps come in pairs with weights of opposite sign; hence, left of the y -axis, we can assume to have only $(1, 0)$ steps. The following is a sign-reversing involution between all such extended n -DPP paths for which not all paths end by two $(0, -1)$ steps (and hence cancels these paths): pick the right-most path ending by either a diagonal $(1, -1)$ step or by a $(1, 0)$ step followed by an $(0, -1)$ step and replace in the second case the two steps by a diagonal

step and vice-versa. By deleting all steps left of the y -axis and below the x -axis of the remaining extended DPP paths, we obtain the above defined DPP paths.

Our second main result is a consequence of [1, Theorem 2.6] using insights from Section 4 in that paper.

Theorem 2. *The generating function of AMTs with bottom row $(1, 2, \dots, n)$ is equal to the generating function of extended n -DPP paths.*

It is an open problem to construct a weight-preserving bijection between AMTs with bottom row $(1, 2, \dots, n)$ and extended n -DPP paths. One surprising observation we made in our work is the following: Suppose that one is able to find a weight preserving bijection between AMTs with bottom row $(1, 2, \dots, n)$ and extended n -DPP paths, which is further compatible with the sign reversing involution described above and the one hinted at in the paragraph around (2.1). Then, as a direct consequence, one would obtain a longed for bijection between ASMs and DPPs. However, we show that such a compatible weight preserving bijection does not exist for $n \geq 3$.

Proposition 1. *Let $n \geq 3$. Then there exists no weight-preserving bijection between AMTs with bottom row $(1, \dots, n)$ and extended n -DPP paths which induces a bijection between monotone triangles with bottom row $(1, \dots, n)$ and n -DPP paths by restricting to n -DPP paths as described above and by ignoring the arrows in the corresponding AMT.*

Proof. We extend n -DPP paths to extended n -DPP paths by adding an appropriate number of $(1, 0)$ steps at the beginning of each path and by adding two $(0, -1)$ steps at the end of each path. The exponent of x_1 thereby counts the number of paths in a given family of n -DPP paths. Hence the maximum exponent of x_1 is $n - 1$ and obtained by the unique family of n -DPP paths consisting of $n - 1$ paths. For AMTs, the exponent of x_1 is $n - 1$ exactly if the top entry is n . The number of monotone triangles with top entry n is equal to the number of monotone triangles with bottom row $(1, \dots, n - 1)$ since the top entry n forces all entries in the right-most northwest-diagonal to be equal to n . For $n \geq 3$, there are at least 2 of such monotone triangles. Hence not all of them can be reached through a weight-preserving bijection. $\square$

There exists also a version of Theorem 2 in terms of certain plane partitions, which we discuss next. Let λ be a partition and $(a_1, \dots, a_l | b_1, \dots, b_l)$ its Frobenius notation, i.e., $a_i = \lambda_i - i$ and $b_i = \lambda'_i - i$, where $l = \max_i \{\lambda_i \geq i\}$. We say that λ is *near-balanced* if, for all i , either $a_i = b_i$ or $a_i = b_i + 1$. We assign the weight $W(\lambda) = w^{l + \sum_{i=1}^l (b_i - a_i)}$. Now, a *set-valued near-balanced column strict plane partition* (SBCSPP) D of shape λ and order n is a filling of a near-balanced partition λ with non-empty subsets of $\{1, 2, \dots, n\}$ such that strictly above the diagonal the subsets are singletons, and (i) rows decrease weakly in the sense that the maxima of the sets form a weakly decreasing sequence if read from left to right, and (ii) columns decrease strictly in the sense that for two adjacent cells in

a column, all elements in the top cell are strictly greater than all elements in the bottom cell. The weight of D is as follows

$$W(D) = W(\lambda) \cdot u^{\# \text{ cells strictly above the main diagonal}} \cdot v^{\binom{n+1}{2} - \# \text{ entries on and below the main diagonal}} \cdot w^{\# \text{ entries} - \# \text{ cells}} \cdot \prod_{i=1}^n x_i^{\# i \text{ in } D}.$$

Next is an example of an SBCSPP of order 9 and weight $u^{16}v^{26}w^3x_1^5x_2^5x_3^4x_4^5x_5^4x_6^4x_7^5x_8^3$.

8	8	8	7	7	6	4	1
7	7	7	6	5	5		
6	6	5	4	4	4		
5	4	3	3,2	3	2		
3	2	2,1	1				
2	1						
1							

Theorem 3 ([1, Theorem 2.6]). *The generating function of arrowed monotone triangles with bottom row $1, 2, \dots, n$ is equal to the generating function of set-valued near-balanced column strict plane partitions with parts in $\{1, 2, \dots, n\}$.*

Concerning the proof of [Theorem 3](#) (and also of [Theorem 4](#)), the rough idea is to use [Theorem 1](#) to obtain a bialternant formula for the generating function of AMTs. We then transform this generating function into a Jacobi-Trudi-type determinant, which we then interpret combinatorially using the Lindström-Gessel-Viennot theorem.

4 Reflective symmetry

We have indicated in [Section 2](#) that there is a one-to-one correspondence between monotone triangles with bottom row $(1, 2, \dots, n)$ and $n \times n$ ASMs. By applying similar ideas, we see that $(2n+1) \times (2n+1)$ VSASMs are in bijection with monotone triangles with bottom row $(0, 2, \dots, 2n-2)$: First we rotate the given matrix by 90° and then map it to a monotone triangle with $2n+1$ rows of which we only need to consider the top n rows due to the axial symmetry of the original VSASM. Since the central column of the VSASM was $(1, -1, 1, -1, \dots, -1, 1)^\top$, the top n rows constitute a monotone triangle with bottom row $(2, 4, \dots, 2n)$. Finally, we subtract 2 from each entry.

Thanks to the correspondence between VSASMs and these monotone triangles, [Theorem 1](#) yields an enumerational refinement of $(2n+1) \times (2n+1)$ VSASMs by $n+3$ parameters. An obvious question is whether there is a corresponding family of non-intersecting lattice paths — that might be interpreted as some kind of plane partitions

or lozenge tilings afterwards — with equally distributed statistics. There are indeed several such objects but just like we needed to broaden the notion of monotone triangles in favour of AMTs in order to incorporate additional statistics, we need to consider here more intricate families of non-intersecting lattice paths than those we get from lozenge tilings; see Figure 2. However, the weights of these lattice paths are signed. In Theorem 4, we provide one such family of non-intersecting lattice paths, an example of which is depicted in Figure 5. By eliminating the sign, these paths can then be further transformed into pairs of certain plane partitions in Theorem 5.

Theorem 4 ([5, Theorem 2.2]). *The generating function of AMTs with bottom row $(0, 2, \dots, 2n - 2)$ is equal to the signed generating function of n lattice paths with starting points $(-1, 1), (-2, 2), \dots, (-n, n)$ and end points $(0, 1), (1, 0), \dots, (n - 1, -n + 2)$ such that:*

- (i) *In the region $\{(x, y) | x \leq 0\}$, the step set is $\{(1, 1), (1, 0)\}$, and steps of type $(1, 0)$ are equipped with the weight w .*
- (ii) *In the region $\{(x, y) | x \geq 0, y \geq 1\}$, the step set is $\{(1, -1), (0, -2)\}$, and steps of type $(0, -2)$ are equipped the weight $-uv$.*
- (iii) *In the region $\{(x, y) | x \geq 0, y \leq 1\}$, the step set is $\{(-1, 0)^2, (0, -1)\}$, and one type of horizontal steps with distance d from the line $y = 2$ is equipped with the weight ux_d and the second type is equipped with the weight vx_d^{-1} .*

The paths are non-intersecting in the first and in the third region. In the second region, we distinguish between even and odd paths depending on whether they contain only even or only odd lattice points, respectively. Lattice paths of the same type are not intersecting each other, but an odd path may have an intersection with an even path.

The weight of a family of lattice paths is $\prod_{i=1}^n x_i^{n-1}$ multiplied by the product of the weights of all its steps where the weight of a step is 1 if it has not been specified. Let σ be the permutation so that the i -th starting point is connected to the $\sigma(i)$ -th end point, then the sign of the family is $\text{sgn } \sigma$.

In Section 5 of [5], we have related the lattice paths of Theorem 4 to non-intersecting lattice paths from $\{(i - 1, 2i - 2) | 1 \leq i \leq n\}$ to $\{(2i - 1, i - 1) | 1 \leq i \leq n\}$ with step set $\{(1, 0), (0, -1)\}$ by purely combinatorial means. The latter are in one-to-one correspondence with cyclically and vertically symmetric lozenge tilings of a hexagon with side lengths $2n + 2, 2n, 2n + 2, 2n, 2n + 2, 2n$ and a central triangular hole of size 2 as described in Figure 2. For this purpose, we have to consider the unrefined case by setting $u = v = 1, w = -1$ and $x_1 = \dots = x_n = 1$; compare with (2.1).

In order to present a class of objects with signless weights, we recall the notion of column-strictness and row-strictness regarding plane partitions. A column-strict plane partition is a filling of a Young diagram with positive integers that weakly decrease along rows and strictly decrease down columns, whereas a row-strict plane partition is

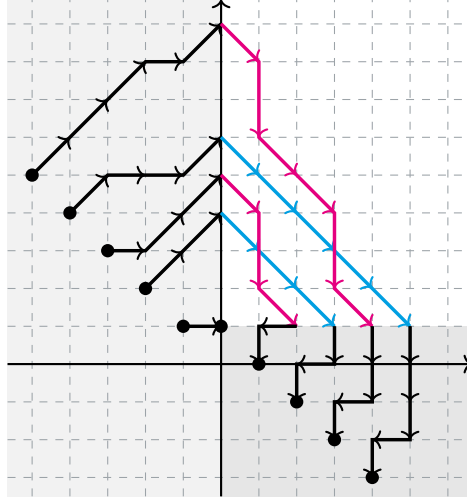


Figure 5: A family of non-intersecting lattice paths as defined in [Theorem 4](#) for $n = 5$. In the second region, we draw even paths in red and odd paths in blue. The associated permutation is $\sigma = (1\ 3\ 2\ 5\ 4)$, and the weight is $(-uv)^3 u^4 w^4 x_1^5 x_2^5 x_3^5 x_4^5 x_5^4$ when always choosing the first type for horizontal steps in the third region.

a filling of a Young diagram with positive integers that weakly decrease down columns and strictly along rows. The following described pairs of plane partitions provide a signless generating function that coincides with the $(n + 3)$ -parameter refinement of VSASMs. [Figure 6](#) shows an example of such a pair.

Theorem 5 ([5, Theorem 2.5]). *The generating function of AMTs with bottom row $(0, 2, \dots, 2n - 2)$ is equal to the generating function of pairs (P, Q) of plane partitions of the same shape with n rows (allowing also rows of length zero) such that P is a column-strict plane partition and Q is a row-strict plane partition, and the entries of P in the i -th row from the bottom are no greater than $2i$, while the entries of Q in the i -th row from the bottom are no greater than i . The weight of such a pair is given by the following monomial:*

$$w^{\binom{n+1}{2} - \# \text{ entries in } Q} \prod_{i=1}^n x_i^{n-1} (ux_i)^{\#(2i-1) \text{ in } P} (vx_i^{-1})^{\#(2i) \text{ in } P}.$$

We are able to prove [Theorem 5](#) in two different ways: one proof is rather based on algebraic manipulations, whilst the other one is more combinatorial. In fact, we can apply two sign-reversing involutions on the lattice paths of [Theorem 4](#) to obtain the pairs of plane partitions in [Theorem 5](#). We essentially start with the family of lattice paths in [Theorem 4](#), but where paths might intersect in the second and third region. The first sign-reversing involution eliminates intersections in the second region and shows that the generating function has no negative coefficients although each family of lattice paths

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The multispecies zero range process and modified Macdonald polynomials

Arvind Ayyer^{*1}, Olya Mandelshtam⁺², and James Martin^{‡3}

¹Department of Mathematics, Indian Institute of Science, Bangalore 560012, India

²Department of Combinatorics and Optimization, University of Waterloo, Canada

³Department of Statistics, University of Oxford, UK

Abstract. In a previous part of this work (FPSAC 2021), we gave a new tableau formula for the modified Macdonald polynomials $\tilde{H}_\lambda(X; q, t)$, using a weight on tableaux involving the *queue inversion* (quinv) statistic. In this paper we explicitly describe the connection between these combinatorial objects and a class of multispecies totally asymmetric zero range processes (mTAZRP) on a ring, with site-dependent jump-rates. We construct a Markov chain on the space of tableaux which projects to the mTAZRP, and whose stationary distribution can be expressed in terms of quinv-weighted tableaux. We also obtain interesting symmetry properties of the mTAZRP probabilities under permutation of the jump rates between the sites. Finally, we give explicit formulas for particle densities and correlations of the process purely in terms of modified Macdonald polynomials.

Keywords: zero range process, Markov chain, tableaux, modified Macdonald polynomials

1 Introduction

Over the past several years, there has been a growing body of research devoted to exploring the relationships of particle models and integrable systems with symmetric (and associated nonsymmetric) functions; see [2] for many examples. These connections have led to new combinatorial objects, new positive formulas [6, 7, 10, 1], and deeper understanding of both the symmetric functions and the associated particle processes. One such connection, coming from [5], was the remarkable discovery that the symmetric Macdonald polynomial $P_\lambda(X; q, t)$ specializes to the partition function of the multispecies asymmetric simple exclusion process (ASEP) on a ring.

The *modified Macdonald polynomials* $\tilde{H}_\lambda(X; q, t)$, introduced by Garsia and Haiman [11], are a transformed version of the P_λ 's, obtained via plethysm from their integral

^{*}arvind@iisc.ac.in. Partially supported by CRG/2021/001592 from the Science and Engineering Research Board, DST, Government of India.

[†]omandels@uwaterloo.ca. Partially supported by NSF grant DMS-1953891 and NSERC.

[‡]martin@stats.ox.ac.uk.

form. Following the discovery of the link between $P_\lambda(X; q, t)$ and the partition function of the multispecies ASEP, a natural question was whether there exists a related statistical mechanics model for which some specialization of $\tilde{H}_\lambda(X; q, t)$ is equal to its partition function. In this extended abstract, we describe such a model, which turns out to be a *multispecies totally asymmetric zero range process (mTAZRP)*, in a form which had previously been considered in [14].

The mTAZRP is a multispecies variant of the more general zero range process (ZRP), which is an important example of a non-equilibrium exactly-solvable interacting particle system. It was introduced by Spitzer in an influential paper [13] that initiated the mathematical study of interacting particle systems, along with the related asymmetric simple exclusion process. The ZRP describes particles hopping to adjacent sites on a graph that allows multiple particles per site, such that the rate of hopping from one site to another depends solely on the content at the site of origin. Among the many reasons that the ZRP is of great interest to physicists is that it is an excellent toy model for the physics of phase-separation and condensation; see [8] for a review.

In this work we study the stationary distribution of the mTAZRP by considering fillings of the diagram of the partition λ with entries in $\{1, \dots, n\}$ – see, for example, Figure 2(b). Let $\text{Tab } \lambda, n$ denote the set of such fillings. We define a function $\mathcal{P}$ (see Definition 3.5) from $\text{Tab } \lambda, n$ to the configurations of the mTAZRP. In earlier work [4], we introduced the “queue inversion” statistic, $\text{quinv} : \text{Tab } \lambda, n \mapsto \mathbb{N}$ (see Definition 3.2) and used it to obtain a new formula for the modified Macdonald polynomial $\tilde{H}_\lambda(X; q, t)$.

The first main result in this article is that the stationary distribution of the mTAZRP on n sites with particle types given by a partition λ is the projection under the map $\mathcal{P}$ of the distribution of the quinv statistic on $\text{Tab } \lambda, n$:

Theorem 1.1. *Consider the mTAZRP on n sites with particle content determined by the partition λ . Then the stationary probability of a configuration w in that mTAZRP equals*

$$\frac{1}{\mathcal{Z}_{\lambda, n}} \sum_{\substack{\sigma \in \text{Tab } \lambda, n \\ \mathcal{P}(\sigma) = w}} t^{\text{quinv } \sigma} x^\sigma,$$

where

$$\mathcal{Z}_{\lambda, n} = \sum_{\sigma \in \text{Tab } n, \lambda} t^{\text{quinv } \sigma} x^\sigma = \tilde{H}_\lambda(x_1, \dots, x_n; 1, t).$$

In this sense, the modified Macdonald polynomial $\tilde{H}_\lambda$ specializes at $q = 1$ to the *partition function* for the mTAZRP model: all the stationary probabilities may be written as rational functions in $\mathbb{N}[t; x_1, \dots, x_n]$ with denominator $\tilde{H}_\lambda(x_1, \dots, x_n; 1, t)$.

Our construction is very closely related to the approach involving *multiline queues*, which was first introduced for the TASEP [9] and has subsequently been used to describe the stationary distributions of a variety of multispecies interacting particle systems. In

particular, a generalized version of multiline queues was used in [7] to interpolate between $P_\lambda(X; q, t)$ and probabilities of the multispecies ASEP.

We prove [Theorem 1.1](#) by constructing a Markov chain on the state space $\text{Tab } \lambda, n$, which has two key properties: (i) it projects to the mTAZRP via the function $\mathcal{P}$; (ii) the stationary probability of a filling σ is proportional to the weight $t^{\text{quinv } \sigma} x^\sigma$.

A key point of interest is the presence of the parameters $x_1, \dots, x_n$ representing site-dependent rates in the TAZRP. In contrast, for the ASEP for example, existing results cover only the case where all rates are equal, involving the specialization of the Macdonald polynomials to $x_1 = \dots = x_n = 1$. By exploiting symmetries of fillings, we obtain interesting new symmetry properties for the mTAZRP under permutation of the state-dependent jump rates.

Theorem 1.2. *Consider the stationary distribution of the mTAZRP on n sites with particle content determined by λ , and parameters $x_1, \dots, x_n, t$. The distribution of the configuration restricted to sites $1, \dots, \ell$ is symmetric in the variables $\{x_{\ell-1}, \dots, x_n\}$.*

The article is organized as follows. The mTAZRP is formally defined in [Section 2](#). [Section 3](#) contains the necessary background on fillings of diagrams and tableaux statistics, the definition of the Markov chain on $\text{Tab } \lambda, n$ for strict partitions λ , as well as the sketch of the proof of our main results for that case. (The Markov chain for general partitions is much more technical and is explained in the longer version of this extended abstract [3].) [Section 4](#) is devoted to observables. In [Section 4.1](#), we study the symmetry properties of the mTAZRP and explain the ideas behind [Theorem 1.2](#). Finally, we end with formulas for single-site densities and currents of particles of a given species in [Section 4.2](#).

2 Definition of the mTAZRP

A partition $\lambda = \lambda_1, \dots, \lambda_k$ is a weakly decreasing sequence of positive integers. When all parts of λ are distinct, λ is called a *strict partition*.

We consider a one-dimensional lattice with n sites (labelled $1, 2, \dots, n$) with periodic boundary conditions, containing k particles of types (or species) $\{\lambda_1, \dots, \lambda_k\}$. Our convention is that particles of larger species are stronger. The set of configurations of the model consists of all possible arrangements of the k particles amongst the n sites. Each site may contain an arbitrary number of particles, and particles of the same type are indistinguishable. We may identify configurations as multiset compositions of type λ with n parts; that is, sequences $w = w_1, \dots, w_n$ where for $1 \leq j \leq n$, w_j is a (possibly empty) multiset, such that the union of all the parts $\bigsqcup_{j=1}^n w_j$ is equal to the multiset λ . We denote the set of such configurations by $\mathcal{T}_{\lambda, n}$.

The system evolves as a continuous-time Markov chain with a global parameter t and site-dependent parameters $x_1, \dots, x_n$. Any transition of the system consists of a

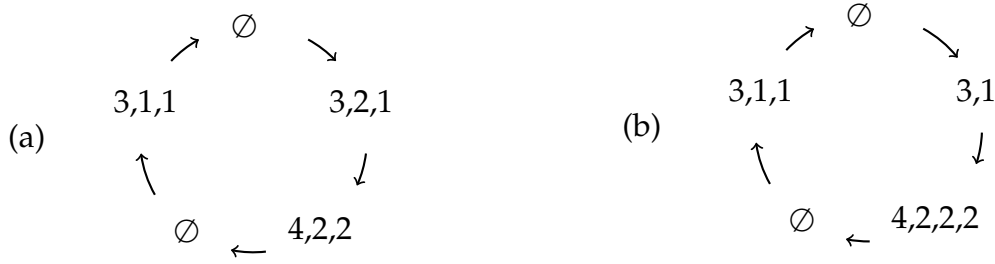


Figure 1: (a) The configuration in this figure is $\cdot|321|422| \cdot|311$ (read clockwise starting from the topmost site), and is an element of $\mathcal{T}_{\lambda,5}$ for $\lambda = 4, 3, 3, 2, 2, 2, 1, 1, 1$. The arrows show the direction of hopping. (b) The configuration is $\cdot|31|4222| \cdot|311$ after a particle of type 2 hops from site 2 to site 3. This hop occurs with rate $x_2^{-1}t$.

single particle jumping from site j to site $j-1$, for some $1 \leq j \leq n$ (sites are considered cyclically mod n). The transition rates are defined as follows. If the number of particles of type a at site j is c_a , and the number of particles of type larger than a at site j is d_a , then a particle of type a from site j jumps with rate $x_j^{-1}t^{d_a} \sum_{i=0}^{c_a-1} t^i$. See Figure 1 for an example.

Example 2.1. $\mathcal{T}_{3,1,1,3}$ consists of the following 18 states:

$$\begin{aligned} &311|\cdot|\cdot, 31|1|\cdot, 31|\cdot|1, 3|11|\cdot, 3|1|1, 3|\cdot|11, 11|3|\cdot, 11|\cdot|3, 1|31|\cdot, \\ &1|1|3, 1|3|1, 1|\cdot|31, \cdot|311|\cdot, \cdot|31|1, \cdot|3|11, \cdot|11|3, \cdot|1|31, \cdot|\cdot|311. \end{aligned}$$

Examples of transitions of the mTAZRP on $\mathcal{T}_{3,1,1,3}$ are:

- The jumps from $\cdot|311|\cdot$ are to $\cdot|11|3$ with rate x_2^{-1} , and to $\cdot|31|1$ with rate $x_2^{-1}t - t^2$;
- The jumps from $\cdot|1|31$ are to $3|1|1$ with rate x_3^{-1} , to $1|1|3$ with rate $x_3^{-1}t$, and to $\cdot|\cdot|311$ with rate x_2^{-1} .

3 Fillings of λ and a Markov chain on these fillings

Let $\lambda = \lambda_1, \dots, \lambda_k$ be a partition. The *diagram of type λ* , which we denote $\text{dg } \lambda$, consists of the cells $\{r, i, 1 \leq i \leq k, 1 \leq r \leq \lambda_i\}$, which we depict using k bottom-justified columns, where the i 'th column from left to right has λ_i boxes. See Figure 2(a) for an illustration of $\text{dg } 3, 2, 1, 1$. The cell (r, i) corresponds to the cell in the i 'th column in the r 'th row of $\text{dg } \lambda$, where rows are labeled from bottom to top.

A *filling* of type λ, n is a function $\sigma : \text{dg } \lambda \rightarrow n$ defined on the cells of $\text{dg } \lambda$ (where $n = \{1, 2, \dots, n\}$). We also refer to a diagram together with a filling of it as a *tableau*. Let $\text{Tab } \lambda, n$ be the set of fillings of type λ, n , and $\text{Tab } \lambda$ the set of fillings $\sigma : \text{dg } \lambda \rightarrow \mathbb{Z}$.

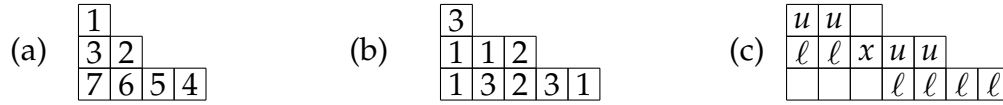


Figure 2: (a) The diagram $\text{dg } 3, 2, 1, 1$ with the cells labeled according to the reading order in Definition 3.1. Our convention is that the cell labelled 5 has coordinates $(1, 3)$. (b) A tableau of type $\lambda = 3, 2, 2, 1, 1$ and $n = 3$. The weight of this filling is $x_1^4 x_2^2 x_3^3 t^7$. (c) The cells in the lower and upper arm of the cell x are labeled ' ℓ ' and ' u ', respectively.

For σ in $\text{Tab } \lambda, n$ (or in $\text{Tab } \lambda$), and a cell $x = r, i$ of $\text{dg } \lambda$, we write $\sigma(x) = \sigma(r, i)$ and call this the *content* of the cell x in the filling σ . Define $\text{South } x$ and $\text{North } x$ to be the cells $(r-1, i)$ and $(r+1, i)$ (respectively), directly below the cell x in the same column, if those cells exist.

Definition 3.1. Define the *reading order* on the cells of a tableau to be along rows from top to bottom and from right to left within each row, as in Figure 2(a).

Definition 3.2. Given a diagram $\text{dg } \lambda$, a *triple* consists of either

- three cells $(r-1, i)$, (r, i) and (r, j) with $i < j$; or
- two cells (r, i) and (r, j) with $i < j$ and $\lambda_i = r$ (in which case the cell $(r-1, i)$ doesn't exist). Such a configuration is called a *degenerate triple*.

Let us write $a = \sigma(r-1, i)$, $b = \sigma(r, i)$, and $c = \sigma(r, j)$, for the contents of the cells of the triple, so that we can depict a triple along with its contents as

$$\begin{array}{|c|} \hline a \\ \hline b \\ \hline \end{array} \dots \begin{array}{|c|} \hline c \\ \hline \end{array}, \quad \text{or} \quad \begin{array}{|c|} \hline \emptyset \\ \hline b \\ \hline \end{array} \dots \begin{array}{|c|} \hline c \\ \hline \end{array} \quad (\text{degenerate}).$$

We say that a triple is a *queue inversion triple*, or a *quinv triple* for short if

$$a \leq b < c \quad \text{or} \quad c < a \leq b \quad \text{or} \quad b < c < a,$$

meaning that its entries are oriented counterclockwise when they are read in increasing order, with ties being broken with respect to reading order. If the triple is degenerate with content b, c , it is a quinv triple if and only if $b < c$.

Definition 3.3. The *weight* of a filling σ is $x^\sigma t^{\text{quinv } \sigma}$, where:

- $x^\sigma = \prod_{u \in \text{dg } \lambda} x_{\sigma(u)}$ is the monomial corresponding to the content of σ ,
- $\text{quinv } \sigma$ is the total number of quinv triples in σ .

See Figure 2(b) for an example of a filling and its weight. In the prequel to this article [4], we used the quinv statistic to obtain a new formula for the modified Macdonald polynomials.

Theorem 3.4 ([4, Theorem 2.6]). *Let λ be a partition. Then*

$$\tilde{H}_\lambda(X; q, t) = \sum_{\sigma \in \text{Tab } \lambda} x^\sigma t^{\text{quinv } \sigma} q^{\text{maj } \sigma}.$$

where $\text{maj } \sigma$ is the major index statistic, defined in [12].

The main focus of this extended abstract will be to obtain formulas for the stationary probabilities and correlations of the mTAZRP on $\mathcal{T}(\lambda, n)$ using weights on the fillings in $\text{Tab } \lambda, n$.

We associate some set of fillings in $\text{Tab } \lambda, n$ to each state of the TAZRP by mapping the bottom row of a filling in $\text{Tab } \lambda, n$ to a state of a TAZRP of type λ on n sites. The correspondence is given as a function $\mathcal{P} : \text{Tab } \lambda, n \rightarrow \mathcal{T}(\lambda, n)$, as follows.

Definition 3.5. Let $\sigma \in \text{Tab } \lambda, n$. Then $\mathcal{P}(\sigma) = w_1, \dots, w_n$, where for $1 \leq j \leq n$,

$$w_j = \{\lambda_i \in \lambda \mid \sigma(1, i) = j\}$$

is the multiset of the heights of the columns of T whose bottom-most entry is j .

Example 3.6. We show all six tableaux corresponding to $(21|1) \in \mathcal{T}(2, 1, 1, 3)$:

$$\begin{array}{|c|c|c|} \hline 1 & & \\ \hline 2 & 3 & 2 \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 2 & & \\ \hline 2 & 3 & 2 \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 3 & & \\ \hline 2 & 3 & 2 \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 1 & & \\ \hline 2 & 2 & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 2 & & \\ \hline 2 & 2 & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 3 & & \\ \hline 2 & 2 & 3 \\ \hline \end{array}$$

For the rest of this section, we shall focus on the case where λ is a strict partition, i.e. has all its parts distinct. This corresponds to the property that all particles of the mTAZRP have distinct types. In particular, fillings of $\text{dg } \lambda$ have no degenerate triples when λ is strict. The case of a general partition λ is much more complicated and involves a technical work-around to treat the degenerate triples, which is given in full detail in the full version of this article [3].

We begin with some definitions to prepare for the description of the Markov chain.

Definition 3.7. Let $u = (r, c)$ be a cell of $\text{dg } \lambda$. The *lower arm* of u , denoted $\text{Larm } u$, consists of all cells to its left in the same row plus all cells to its right in the row below:

$$\text{Larm } u : \{ (r, j) \in \text{dg } \lambda \mid j < c \} \cup \{ (r-1, j) \in \text{dg } \lambda \mid j > c \}.$$

Similarly, we define the *upper arm* of u , denoted $\text{Uarm } u$, to consist of all cells to its left in the row above plus all cells to its right in the same row:

$$\text{Uarm } u : \{ r-1, j \in \text{dg } \lambda \mid j < c \} \cup \{ r, j \in \text{dg } \lambda \mid j > c \}.$$

See Figure 2(c). Note that what we call the lower arm is the statistic ‘arm’ in [12]. Observe that $v \in \text{Larm } u$ if and only if $u \in \text{Uarm } v$. Now we define

$$\text{down } \sigma, u : \# \{ y \in \text{Larm } u : \sigma y = \sigma u \}$$

for a cell u such that $\sigma \text{ South } u = \sigma u$, and its analog

$$\text{up } \sigma, u : \# \{ y \in \text{Uarm } u : \sigma y = \sigma u \}$$

for a cell u such that $\sigma \text{ North } u = \sigma u$.

Let λ be a partition, $u = r, c \in \text{dg } \lambda$ be a cell, and $\sigma \in \text{Tab } \lambda, n$ be a filling of λ . We want to consider the maximal vertical chain of cells upwards from u along which the contents increase by 1 (cyclically mod n) at each step. Let $h_u \sigma$ be the row number of the top of the chain. More precisely, $r \leq h_u \sigma \leq \lambda_c$, and:

- (i) for all i with $r \leq i < h_u \sigma$, we have $\sigma \text{ North } i, c - \sigma i, c \equiv 1 \pmod{n}$;
- (ii) either $h_u \sigma = \lambda_c$, or $\sigma \text{ North } h_u \sigma, c - \sigma h_u \sigma, c \not\equiv 1 \pmod{n}$.

Definition 3.8. Given λ and a cell $u = r, c$, we define a map $R_u : \text{Tab } \lambda, n \rightarrow \text{Tab } \lambda, n$ as follows. If $\sigma \in \text{Tab } \lambda, n$ with $\sigma \text{ South } u = \sigma u$, then let $R_u \sigma = \sigma$. Otherwise, the map R_u will increment by 1 all the contents in the maximal contiguous increasing chain above u . That is, for $v \in \text{dg } \lambda$,

$$R_u \sigma v = \begin{cases} \sigma v + 1 \pmod{n} & v = i, c \text{ for } r \leq i \leq h_u \sigma, \\ \sigma v & \text{otherwise.} \end{cases}$$

One may check from the definition that R_u is invertible. Define $R_y^{-1} : \text{Tab } \lambda, n \rightarrow \text{Tab } \lambda, n$ to be the map such that $R_y^{-1} \circ R_u \sigma = \sigma$, where $y \in \text{dg } \lambda$ that depends on u and σ . See Example 3.10 for examples of both maps.

The following properties of R_y^{-1} are straightforward to check from the definitions:

Lemma 3.9.

- (i) Take $\tau \in \text{Tab } \lambda, n$, and $u = r, j \in \text{dg } \lambda$ such that $\tau u = \tau \text{ South } u$; define $\sigma : R_u \tau$. Then $y = h_u \tau, j$ is the unique cell such that $\tau = R_y^{-1} \sigma$ and this y has the property that $\sigma y = \sigma \text{ North } y$. Moreover,

$$x^\tau = x^\sigma x_{\tau u} x_{\sigma y}^{-1}. \quad (3.1)$$

- (ii) Conversely, take $\sigma \in \text{Tab } \lambda, n$, and $y = r', j \in \text{dg } \lambda$ such that $\sigma \downarrow y \neq \sigma$ North y ; define $\tau := R_y^{-1} \sigma$. Then there is a unique u such that $\sigma = R_u \tau$; moreover, $\tau \downarrow u \neq \tau$ South u and $h_u \tau = r'$.

Example 3.10. Let $\sigma \in \text{Tab } 3, 2, 1, 3$ be the tableau

1		
1	3	
3	2	3

corresponding to the mTAZRP state $\mathcal{P} \sigma = \cdot | 2 | 31$. Note that $R_{3,1} \sigma = R_{2,1}^{-1} \sigma = \sigma$. Below, we show the map R for the cells $1, 1$ and $2, 2$, as well as the inverse map R^{-1} for the cells $3, 1$ and $2, 2$, with the corresponding maximal chain highlighted.

$R_{1,1} \sigma$	$R_{2,2} \sigma$	$R_{3,1}^{-1} \sigma$	$R_{2,2}^{-1} \sigma$																								
<table><tr><td>1</td></tr><tr><td>2</td><td>3</td></tr><tr><td>1</td><td>2</td><td>3</td></tr></table>	1	2	3	1	2	3	<table><tr><td>1</td></tr><tr><td>1</td><td>1</td></tr><tr><td>3</td><td>2</td><td>3</td></tr></table>	1	1	1	3	2	3	<table><tr><td>3</td></tr><tr><td>1</td><td>3</td></tr><tr><td>3</td><td>2</td><td>3</td></tr></table>	3	1	3	3	2	3	<table><tr><td>1</td></tr><tr><td>1</td><td>2</td></tr><tr><td>3</td><td>1</td><td>3</td></tr></table>	1	1	2	3	1	3
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3	1	3																									
rate x_3^{-1}	rate x_2^{-1}	$u = 3, 1$	$u = 1, 2$																								
state $3 2 1$	state $\cdot 2 31$	state $\cdot 2 31$	state $2 \cdot 31$																								

The inverses of the above maps are given by:

$$R_{2,1}^{-1} \circ R_{1,1} \sigma = R_{2,2}^{-1} \circ R_{2,2} \sigma = R_{3,1} \circ R_{3,1}^{-1} \sigma = R_{2,2} \circ R_{2,2}^{-1} \sigma = \sigma.$$

The next couple of lemmas, whose proofs are combinatorial in nature, will be key ingredients in proofs of our theorems. See [3] for details of the proofs.

Lemma 3.11. For all $\sigma \in \text{Tab } \lambda$ and all k , we have

$$\sum_{\substack{u \in \text{dg } \lambda \\ \sigma \downarrow u \neq \sigma \text{ North } u}} t^{\text{up } \sigma, u} = \sum_{\substack{u \in \text{dg } \lambda \\ \sigma \downarrow u \neq \sigma \text{ South } u}} t^{\text{down } \sigma, u}.$$

Lemma 3.12. Fix a strict partition λ and $n \in \mathbb{N}$. For $\tau \in \text{Tab } \lambda, n$, let $u = r, j \in \text{dg } \lambda$ with $\tau \downarrow u \neq \tau$ South u , let $\sigma = R_u \tau$, and let $y = h_u \tau, j$ be the unique cell such that $\tau = R_y^{-1} \sigma$. Then

$$\text{quinv } \tau = \text{quinv } \sigma + \text{up } \sigma, y - \text{down } \tau, u. \quad (3.2)$$

We now define the tableau Markov chain with state space $\text{Tab } \lambda, n$ for the case where λ is a strict partition. Let $\sigma \in \text{Tab } \lambda, n$. We attach an exponential clock with rate $t^{\text{down } \sigma, u} x_{\sigma \downarrow u}^{-1}$ to every cell $u \in \text{dg } \lambda$ such that $\sigma \downarrow u \neq \sigma$ South u . When this clock rings, we make a transition to $R_u \sigma$. We say that such a transition is *triggered by the cell* u . For $\sigma, \tau \in \text{Tab } \lambda, n$, denote by $\text{rate } \sigma, \tau$ the rate of the transition from σ to τ .

Theorem 3.13. *The stationary distribution of the tableau Markov chain defined above is proportional to*

$$\text{wt } \sigma = t^{\text{quinv } \sigma} x^\sigma.$$

Proof. Denote all states reached from σ in one step by $\text{Out } \sigma$, and all states that reach σ in one step by $\text{In } \sigma$:

$$\begin{aligned} \text{Out } \sigma &= \{R_u \sigma \mid u \in \text{dg } \lambda, \sigma \text{ South } u \neq \sigma u\}, \\ \text{In } \sigma &= \{\tau \in \text{Tab } \lambda, n \mid \sigma \in \text{Out } \tau\}. \end{aligned}$$

From Lemma 3.9(ii) we have that $\text{In } \sigma = \{R_y^{-1} \sigma \mid y \in \text{dg } \lambda, \sigma \text{ North } y \neq \sigma y\}$. Then it suffices to show

$$\text{wt } \sigma \sum_{\nu \in \text{Out } \sigma} \text{rate } \sigma, \nu = \sum_{\tau \in \text{In } \sigma} \text{wt } \tau \text{rate } \tau, \sigma, \quad (3.3)$$

which we can expand as

$$x^\sigma t^{\text{quinv } \sigma} \sum_{\substack{u \in \text{dg } \lambda \\ \sigma \text{ South } u \neq \sigma u}} x_{\sigma u}^{-1} t^{\text{down } \sigma, u} = \sum_{\substack{y \in \text{dg } \lambda \\ \sigma \text{ North } y \neq \sigma y}} \text{wt } R_y^{-1} \sigma \text{rate } R_y^{-1} \sigma, \sigma. \quad (3.4)$$

To treat the right-hand side of (3.4), let $\tau = R_y^{-1} \sigma$ for some $y \in \text{dg } \lambda$ such that $\sigma \text{ North } y \neq \sigma y$, and let $u \in \text{dg } \lambda$ be such that $R_u \tau = \sigma$. From (3.1), we have $x^\tau = x^\sigma x_{\sigma y}^{-1} x_{\tau u}$. By Lemma 3.12, we also have

$$\text{wt } \tau \text{rate } \tau, \sigma = x^\tau t^{\text{quinv } \tau} \cdot x_{\tau u}^{-1} t^{\text{down } \tau, u} = x^\sigma x_{\sigma y}^{-1} t^{\text{quinv } \sigma} \text{up } \sigma, y. \quad (3.5)$$

Refining over the coefficients of x -monomials and plugging in (3.5), we get that (3.4) is equivalent to

$$x^\sigma t^{\text{quinv } \sigma} \sum_{k=1}^n x_k^{-1} \sum_{\substack{u \in \text{dg } \lambda \\ \sigma \text{ South } u \neq \sigma u}} t^{\text{down } \sigma, u} = x^\sigma t^{\text{quinv } \sigma} \sum_{k=1}^n x_k^{-1} \sum_{\substack{y \in \text{dg } \lambda \\ \sigma \text{ North } y \neq \sigma y}} t^{\text{up } \sigma, y}. \quad (3.6)$$

Finally, we obtain (3.6) by invoking Lemma 3.11 for each k . $\square$

Recall the projection map $\mathcal{P}$ from $\text{Tab } \lambda, n$ to $\mathcal{T} \lambda, n$ given in Definition 3.5. From the following lemma, we obtain the first main result of this article, Theorem 1.1.

Lemma 3.14. *Let $\sigma \in \text{Tab } \lambda, n$.*

- (i) *For any cell u in the first row of σ , there exists a transition from the TAZRP state $\mathcal{P} \sigma \in \mathcal{T} \lambda, n$ to the TAZRP state $\mathcal{P} R'_u \sigma$.*

- (ii) Let $w = \mathcal{P} \sigma \in \mathcal{T} \lambda, n$ be the corresponding TAZRP state of σ . Let $1 \leq j \leq n$ be a site, $1 \leq r \leq \lambda_1$ be a particle type such that $w_j = r$. Let w' be the state in $\mathcal{T} \lambda, n$ obtained by moving particle r one site to the right. Then

$$\sum_{\substack{u \in \text{dg } \lambda \\ \mathcal{P} R'_u \sigma = w'}} \text{rate } \sigma, R'_u \sigma = \text{rate } w, w'.$$

4 Observables

4.1 Symmetries in probabilities of configurations in the first ℓ sites

Fix a partition λ and a positive integer n , and consider the TAZRP of type λ on n sites. Fix ℓ with $1 \leq \ell \leq n-1$. We will consider the probability of observing some specific configuration w on the sites $1, \dots, \ell$. Formally, w can be seen as a configuration of TAZRP μ, ℓ for some partition μ consisting of a subset of the parts of λ . We write $\bar{w}$ for the set of configurations of TAZRP λ, n whose restriction to $1, \dots, \ell$ is given by w .

Example 4.1. Let $\lambda = 2, 2, 1, 1$, $n = 4$, $\ell = 2$, and $w = 1|2$. (Then $\mu = 2, 1$). The configurations of $\mathcal{T} \lambda, n$ contributing to $\mathbb{P}_{\lambda, n} \bar{w}$ are

$$\{ 1|2|21| \cdot, 1|2|2|1, 1|2|1|2, 1|2| \cdot |21 \}.$$

In this case, the coefficient of t^3 in $P_{\lambda, n} \bar{w}$ is $x_1 x_2 x_2^2 - x_1 x_2 m_2 - x_1 - 2x_2 m_3 - 4x_1 x_2 - 3x_2^2 m_{11} - 3x_1 - 4x_2 m_{21} - m_{31} - 2m_{22} / \tilde{H}_\lambda(x_1, x_2, x_3, x_4; 1, t)$, where $m_\mu = m_\mu(x_3, x_4)$ is the monomial basis evaluated at the variables x_3, x_4 .

The main result we prove in this section is the following.

Proposition 4.2. Fix a partition λ and positive integer n . For any $0 \leq \ell \leq n$ and configuration w on the first ℓ sites of the TAZRP of type λ on n sites, the probability $\mathbb{P}_{\lambda, n} \bar{w}$ is symmetric in the variables $\{x_{\ell-1}, \dots, x_n\}$.

Proposition 4.2 proves **Theorem 1.2**. The proof of this result uses LLT polynomials: fixing sites $1, \dots, \ell$ in the TAZRP is equivalent to fixing a certain set of boxes in the filling to have entries in $\{1, \dots, \ell\}$. The sum over configurations with those sites fixed can be refined as a sum over fillings of subsets of $\text{dg } \lambda$ filled with entries in $\{\ell-1, \dots, n\}$. Such fillings can be described as certain LLT polynomials in the variables $\{\ell-1, \dots, n\}$, which are known to be symmetric in those variables. See [3] for more details.

4.2 Densities and currents

Consider the mTAZRP with particle content given by λ and with n sites. We now assume without loss of generality that λ is *packed*, i.e. $\lambda = \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k = 1$ such that $\lambda_i - \lambda_{i-1} \leq 1$.

Let τ_i^j be the random variable counting the number of particles of type j at site i in a configuration of the mTAZRP. By convention, we will denote the expectation in the stationary distribution by $\langle \cdot \rangle$. The density of a given species (say j) at a given site (say i) is defined to be the expected number of particles of type j at site i in the stationary distribution and is thus given by $\langle \tau_i^j \rangle$.

Note the following cyclic symmetry property, which is immediate from the definition of the process.

Proposition 4.3. *The mTAZRP with partition λ on n sites is invariant under simultaneous translation of the sites $i \rightarrow i + 1$ for $1 \leq i \leq n - 1$ and $n \rightarrow 1$, and relabelling of the site-dependent parameters $x_i \rightarrow x_{i+1}$ for $1 \leq i \leq n - 1$ and $x_n \rightarrow x_1$.*

As an immediate consequence, we can deduce the following result for the densities.

Proposition 4.4. *Suppose $\langle \tau_1^j \rangle = r$ for $x_1, \dots, x_n$. Then for any i , $\langle \tau_i^j \rangle = r$ for $x_i, \dots, x_n, x_1, \dots, x_{i-1}$.*

By Proposition 4.4, it suffices to compute the densities of all species of particles at the first site. The following theorem is proved using a “coloring” argument based on the fact that to a particle of type j , all particles of lower types are invisible.

Theorem 4.5. *For $1 \leq j \leq k$, the density of the j ’th species at the first site is given by*

$$\langle \tau_1^j \rangle = x_1 \partial_{x_1} \log \left(\frac{\tilde{H}_{1^{m_j} \dots m_k} (x_1, \dots, x_n; 1, t)}{\tilde{H}_{1^{m_j-1} \dots m_k} (x_1, \dots, x_n; 1, t)} \right),$$

where the denominator inside the logarithm is 1 if $j = k$.

Currents are important observables in statistical physics because they are a prime indicator of the lack of reversibility in the system. The *current* of a certain species of particle from site i to site j is defined as the number of particles of that species jumping from i to j per unit of time in the large-time limit, or in the stationary state. The current from i to $i + 1$ is the same for all sites i because of particle conservation. Using Proposition 4.3, we obtain:

Theorem 4.6. *Consider the multispecies TAZRP with content λ on n sites. Then, for $1 \leq j \leq k$, the current of particle of species j is given by*

$$m_j \dots m_k t \frac{\tilde{H}_{1^{m_j} \dots m_{k-1}}}{\tilde{H}_{1^{m_j} \dots m_k}} - m_{j-1} \dots m_k t \frac{\tilde{H}_{1^{m_{j-1}} \dots m_{k-1}}}{\tilde{H}_{1^{m_{j-1}} \dots m_k}},$$

where the modified Macdonald polynomials are evaluated at $q = 1$.

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A Generalized RSK for Enumerating Linear Series on n -pointed Curves

Maria Gillespie^{*1} and Andrew Reimer-Berg^{†1}

¹Department of Mathematics, Colorado State University, Fort Collins, CO USA

Abstract. We give a combinatorial proof of a recent geometric result of Farkas and Lian on linear series on curves with prescribed incidence conditions. The result states that the expected number of degree- d morphisms from a general genus- g , n -marked curve C to $\mathbb{P}^r$, sending the marked points on C to specified general points in $\mathbb{P}^r$, is equal to $(r+1)^g$ for sufficiently large d . This computation may be rephrased as an intersection problem on Grassmannians, which has a combinatorial interpretation in terms of Young tableaux by the Littlewood-Richardson rule. We give a bijection, generalizing the RSK correspondence, between the tableaux in question and the $(r+1)$ -ary sequences of length g , and we explore our bijection's combinatorial properties.

We also apply similar methods to give a combinatorial interpretation and proof of the fact that, in the modified setting in which $r=1$ and several marked points map to the same point in $\mathbb{P}^1$, the number of morphisms is still 2^g for sufficiently large d .

Keywords: RSK algorithm, Young tableaux, Schubert calculus, linear series, curves

1 Introduction

In a recent paper [8], Farkas and Lian provide enumerative formulas for the number of maps from a curve C to a complex projective space $\mathbb{P}^r$ with specified incidence conditions. In particular, let C be a general curve of genus g , and let $x_1, \dots, x_n$ be distinct general points on C . Also choose distinct general points $y_1, \dots, y_n$ in $\mathbb{P}^r$. Then we write $L_{g,r,d}$ for the number of degree d morphisms $f : C \rightarrow \mathbb{P}^r$ for which $f(x_i) = y_i$ for all $i = 1, 2, \dots, n$, which is finite precisely when $nr \leq dr \leq d \leq r + rg$.

For sufficiently large d and setting $n = dr - d - r + rg/r$ (assuming r divides d), it was shown in [8] that

$$L_{g,r,d} = (r+1)^g. \quad (1.1)$$

In addition, $L_{g,r,d}$ is equal to a certain intersection of Schubert cycles in the Grassmannian. The latter formula has the following combinatorial interpretation, as we will show in Section 3.

^{*}maria.gillespie@colostate.edu. Partially supported by NSF DMS award number 2054391.

[†]andrew.reimer-berg@colostate.edu. Partially supported by NSF DMS award number 2054391.

Definition 1.1. Define an L -tableau with parameters g, r, d to be a way of filling the boxes of an $(r+1) \times (d-r)$ grid with rg “red” integers and $d-r-r+1-rg$ “blue” integers such that:

- The red integers are left-and-bottom justified, and weakly increase up columns and strictly increase across rows. They consist of the numbers $1, 2, \dots, g$ each occurring exactly r times.
- The blue integers are right-and-top justified, and strictly increase up columns and weakly increase across rows. Their values are from $\{0, 1, \dots, r\}$.

Example 1.2. The following is an L -tableau with parameters $4, 3, 9$. We write the “red” numbers as black font with a red shaded background for clarity.

2	4	1	3	3	3
1	3	4	1	2	2
1	2	3	0	1	1
1	2	3	4	0	0

The work in [8] shows that the number of L -tableaux with parameters g, r, d is equal to $(r+1)^g$ whenever $d \geq rg-r$ (and $r|d$, so that n is an integer) via geometric methods, and asks for a combinatorial proof. Our first main result resolves this open problem by finding a combinatorial proof, of the following stronger result.

Theorem 1.3. *The number of L -tableaux with parameters g, r, d is $(r+1)^g$ when $d \geq g-r$.*

Note that in this purely combinatorial setting we only require $d \geq g-r$ rather than $d \geq rg-r$, and d is not necessarily divisible by r .

Our methods generalize the RSK algorithm (see Section 2.1). In particular, in the case $r=1$, the pair of red and blue tableaux correspond directly under the RSK bijection to the binary sequences of length g (see [10]). For $r > 1$ we introduce an intermediate bijection (see Definition 3.7) to reduce to the RSK correspondence once again.

Remark 1.4. In the case $r=1$, there are several known proofs of the fact that $L_{g,1,d} = 2^g$ for sufficiently large d , including via *scattering amplitudes* for $d \geq g+1$ [15] and by establishing recursions using the boundary geometry of the moduli space of *Hurwitz covers* [2]. Neither of these proofs were combinatorial in nature, though the recursions arising in the latter paper by Cela-Pandharipande-Schmitt [2] are related to Dyck paths and other Catalan objects for small d .

Our second result enumerates a related set of maps. Set $r=1$, fix an integer k with $1 \leq k \leq \min\{n, d\}$, and consider choices of points $y_1, \dots, y_n \in \mathbb{P}^1$ such that

$$y_1 = y_2 = \dots = y_k.$$

Write $L'_{g,d,k}$ for the number, assuming that $n = dr - d - r - rg/r - 2d - 1 - g$, of maps $f : C \rightarrow \mathbb{P}^1$ such that $f(x_i) = y_i$ for all i . We use a similar interpretation in terms of a family of Young tableaux that we call L' -**tableaux**, starting from an intersection theoretic formula in Grassmannians for $L'_{g,d,k}$ given by Farkas and Lian [8], to enumerate these maps for sufficiently high degree curves.

Theorem 1.5. *If $d \geq g - k$, we have $L'_{g,d,k} = 2^g$.*

Notice that the formula 2^g coincides with $r - 1 - g$ at $r = 1$. Indeed, it is remarked in [8] that the simple formula $L'_{g,d,k} = 2^g$ may be explained geometrically in the same way that the formula $L_{g,r,d} = r - 1 - g$ does. Further details and a more general formula for maps with arbitrary ramification profiles were given by Cela and Lian [1]. However, the formula from [8] coming from Grassmannians is stated as a *difference* of two sums of Schubert class intersection products (see Section 4), and there was previously no stated enumerative combinatorial interpretation of the latter formula, or for that matter a direct explanation for why the difference should be positive. We provide such an interpretation in order to prove Theorem 1.5.

In Section 2 below we recall the necessary background results and notation from tableaux theory and intersection theory on Grassmannians. In Section 3 we translate the geometric formulas for $L_{g,r,d}$ into a Young tableaux enumeration problem and give a sketch of the proof of Theorem 1.3. In Section 4 we define L' -tableaux for $L'_{g,d,k}$ and sketch the proof of Theorem 1.5. Full details can be found in our paper [10].

2 Background

We briefly recall several facts about Young tableaux, linear series, and Grassmannians.

2.1 The RSK correspondence on words

The **Young diagram** of a partition λ is the left- and bottom-justified grid of unit squares in the first quadrant (called *boxes*) for which there are λ_i boxes in the i -th row from the bottom for all i . A **semistandard Young tableau**, or **SSYT**, of shape λ is a way of filling the boxes of the Young diagram with nonnegative integers such that the rows are weakly increasing from left to right and the columns are strictly increasing from bottom to top. An SSYT is **standard**, written SYT, if the entries are $1, 2, \dots, n$ each used exactly once. The **shape** of a Young tableau is the underlying Young diagram, and its **content** is the tuple $(m_1, m_2, \dots)$ where m_i is the number of times i appears in the tableau.

The RSK correspondence, in its most general form, is a bijection between two-line arrays and pairs of semistandard Young tableaux of the same shape (see [9] for an excellent overview). In the case that the bottom row of the two-line array consists of the numbers

$1, 2, 3, \dots, n$ in order, the top row may be any sequence, and we obtain the following special case of the RSK correspondence.

Proposition 2.1 (RSK for words). *Let $A_{r,n} = \{0, 1, 2, 3, \dots, r\}^n$ denote the set of all length- n sequences with entries from $\{0, 1, 2, \dots, r\}$. Let $B_{r,n}$ be the set of all pairs (P, Q) such that P is a semistandard Young tableau with letters in $\{0, 1, 2, \dots, r\}$, Q is a standard Young tableau, and P and Q have the same shape of size n . There is an explicit bijection, called the RSK correspondence, from $A_{r,n}$ to $B_{r,n}$ for all r, n .*

We will refer to the length- n sequences of letters from $\{0, 1, \dots, r\}$ as $(r+1)$ -**ary sequences**, generalizing the notion of a **binary sequence** from $\{0, 1\}$. We will use the following property that follows from the definition of semistandard Young tableaux.

Remark 2.2. Since P has letters in $\{0, 1, 2, 3, \dots, r\}$, and columns are strictly increasing, any pair (P, Q) in $B_{r,n}$ can have height at most $r+1$.

Example 2.3. The 4-ary sequence $0, 2, 1, 1, 0, 3, 0, 0, 1$ in $A_{3,9}$ corresponds under RSK to the following pair of tableaux in $B_{3,9}$:

2									5								
1	1	3							3	7	8						
0	0	0	0	1					1	2	4	6	9				

2.2 Linear series and calculations in the Grassmannian

The definitions of $L_{g,r,d}$ and $L'_{g,r,d}$ may be made more rigorous via the theory of linear series on curves. (See [7] for an introduction to this topic.) A **linear series** of type $\mathfrak{g}_d^r$ on a smooth curve C can be thought of as a r -dimensional linear family of sets of d points on C . More formally, this data is encoded by a pair $(\mathcal{L}, V)$ where $\mathcal{L}$ is a line bundle of degree d and $V \subseteq H^0(C, \mathcal{L})$ is a dimension $r+1$ space of sections, which in turn corresponds to a map $\phi_{\mathcal{L},V}$ from C to $\mathbb{P}^r$.

There has been much study of enumeration of linear series with prescribed *ramification* conditions at specified points, including the work of Eisenbud and Harris [6], Osserman [12], Chan and Pflueger [5], Chan, López Martín, Pflueger, and Teixidor i Bigas [3], Larson, Larson, and Vogt [11], and others. Many of these results relied on the combinatorial tools of Young tableaux, symmetric function theory, and other aspects of algebraic combinatorics, and in [4] led to new results in algebraic combinatorics as well.

Here we consider the related problem of enumerating linear series with prescribed *incidence* conditions at specified marked points. Write $G_d^r(C)$ to denote the moduli space of $\mathfrak{g}_d^r$'s on C , and let $x_1, \dots, x_n$ be general marked points on C . Then the values $L_{g,r,d}$ may alternatively be defined as the degree of the evaluation map $\text{ev}_{x_1, \dots, x_n}$ from $G_d^r(C)$ to the moduli space P_r^n of n points in $\mathbb{P}^r$, given by evaluating the maps $\phi_{\mathcal{L},V}$ at the points

$x_1, \dots, x_n$. In [8], a degeneration argument, starting with a reduction to genus 0, is used to reduce the problem of computing the degree of this evaluation map to an intersection problem in Schubert calculus. For genus 0 curves C , $H^0(C, \mathcal{L})$ in general has dimension $d-1$ as a complex vector space for sufficiently high d . Therefore, the $r-1$ -dimensional subspaces $V \subseteq H^0(C, \mathcal{L})$ sweep out a copy of the Grassmannian $\text{Gr}(r-1, d-1)$.

The Grassmannian $\text{Gr}(r-1, d-1)$, defined as the moduli space of $r-1$ -dimensional subspaces of $\mathbb{C}^{d-1}$, has a well-known Schubert decomposition (with respect to a given flag) into Schubert varieties X_λ . (See [9, Ch. 9] for background on Schubert calculus.) Here, λ ranges over all partitions $\lambda = \lambda_1, \lambda_2, \dots, \lambda_k$ whose Young diagram fits inside an $(r-1) \times (d-1)$ grid.

The Schubert varieties give rise to a basis of *Schubert classes* $\sigma_\lambda : X_\lambda$ of its Chow ring $A^\bullet \text{Gr}(r-1, d-1)$. With respect to this basis, it is shown in [8] that whenever either $d \geq rg - r$, $d = r - \frac{rg}{r-1}$, or $r = 1$, we have

$$L_{g,r,d} = \int_{\text{Gr}(r-1, d-1)} \sigma_{1^r}^g \cdot \left[\sum_{\alpha_0 \dots \alpha_r} \sum_{r-1 \leq d-r-rg} \left(\prod_{i=0}^r \sigma_{\alpha_i} \right) \right]. \quad (2.1)$$

Here the notation σ_{1^r} is shorthand for $\sigma_{1,1,1,\dots,1}$ where the tuple $1,1,\dots,1$ has length r , and σ_{α_i} is shorthand for σ_{α_i} . The integral indicates that the sum of products of Schubert cycles in question expands in the Schubert basis as a constant multiple of σ_{d-1-r} : $\sigma_{d-1-r} = \sigma_{d-1,d-1,\dots,d-1}$, and the integral is defined to be this constant coefficient.

The corresponding result from [8] for $L'_{g,d,k}$ is stated in [Section 4](#).

2.3 The iterated Pieri rule

The intersections of Schubert cycles on the Grassmannian may be calculated via symmetric function theory, as products of Schubert classes correspond to products of *Schur functions*. Indeed, let $\{s_\lambda\}$ be the classical Schur function basis of the ring of symmetric functions, where λ ranges over all partitions (see [9] or [14, Ch. 7]). Then the integral in equation (2.1) is equal to the coefficient of $s_{d-1-r-1}$ in the expansion

$$s_{1^r}^g \cdot \left[\sum_{\alpha_0 \dots \alpha_r} \sum_{r-1 \leq d-r-rg} \left(\prod_{i=0}^r s_{\alpha_i} \right) \right] \quad (2.2)$$

The coefficients of products of Schur functions expressed in the Schur basis are called **Littlewood-Richardson coefficients**. In particular we can write

$$s_\lambda \cdot s_\mu = \sum c_{\lambda\mu}^\nu s_\nu$$

where the Littlewood-Richardson coefficients $c_{\lambda\mu}^\nu$ are all nonnegative integers. In our setting we will only need to focus on the cases when one of λ or μ is either a horizontal row

Suppose λ and μ are partitions for which the Young diagram of μ fits inside that of λ . The **skew shape** λ/μ is the set of boxes that are in λ but not in μ . A skew shape is a **horizontal strip** if no two of its boxes are in the same column, and it is a **vertical strip** if no two of its boxes are in the same row.

For $\mu = 1^r$ a single-column partition, the Littlewood-Richardson coefficient $c_{\lambda\mu}^\nu = c_{\lambda, 1^r}^\nu$ is equal to 1 if ν/λ is a vertical strip, and 0 otherwise.

$$S_\lambda \cdot S_\alpha \sum_{\nu/\lambda \in \text{Horz } \alpha} S_\nu \quad S_\lambda \cdot S_{1^r} \sum_{\nu/\lambda \in \text{Vert } r} S_\nu$$

Corollary 2.5 (Iterated Pieri rule). *Let $\mu^1, \mu^2, \dots, \mu^k$ be partitions, each of which is either a horizontal row or a vertical column. Then*

$$s_{\mu^1} \cdot s_{\mu^2} \cdot \dots \cdot s_{\mu^k} \quad \sum c_{\mu^1 \dots \mu^k}^\nu s_\nu$$

In order to keep track of the horizontal and vertical strips, we will label the squares of the strip corresponding to μ^i by i for each i . This results in a tableau-like object that enumerates the generalized Pieri coefficients $c_{\mu^1 \dots \mu^k}^\nu$ above.

In the special case when we have all horizontal strips, we recover the well-known notion of a **semistandard Young tableau**, or **SSYT**: a filling of the boxes of a (possibly skew) Young diagram with numbers such that the rows are weakly increasing from left to right and the columns are strictly increasing from bottom to top. For the case of all vertical strips, we say a **transposed semistandard Young tableau** is a filling of a Young diagram with strictly increasing *rows* and weakly increasing *columns*. We similarly obtain transposed SSYT's in the case of all vertical strips.

3 L -tableaux and enumeration by $r \leq g$

[Corollary 2.5](#), combined with the fact that the integral in equation (2.1) is the coefficient of s_{d-r+1} in the corresponding product (2.2), shows that

$$L_{g,r,d} = \sum_{\alpha_0 \dots \alpha_r} c_{1^r, 1^r, \dots, 1^r, \alpha_0, \dots, \alpha_r}^{d-r+1}$$

where the subscripts include g copies of 1^r . This summation is therefore the number of ways to form a transposed SSYT using each of the numbers $1, 2, \dots, g$ exactly r times, and then extend it to fill the rest of the $(r-1) \times (d-r)$ grid with a semistandard Young tableau using the numbers $0, 1, \dots, r$ in some varying amounts $\alpha_0, \dots, \alpha_r$ each.

This precisely matches the definition of L -tableaux given in [Definition 1.1](#) in the introduction.

Remark 3.1. The preprint [8] mistakenly uses an ordinary (not transposed) SSYT for the red tableau; a correction will appear in a later version of their work and appears in [10].

See [Example 1.2](#). Our discussion thus far, starting from Equation (2.1), has shown:

Proposition 3.2. *The number of L -tableau with parameters (g, r, d) is equal to $L_{g,r,d}$ whenever either $d \geq rg - r$, $d = r - \frac{rg}{r-1}$, or $r = 1$.*

We now note that we can “truncate” by removing some of the right-hand columns of the grid to reduce to a simpler case.

Lemma 3.3 (Truncation). *For any g, r, d with $d \geq g - r$, the number of L -tableaux with parameters (g, r, d) is equal to the number of L -tableaux with parameters $(g, r, g - r)$.*

We omit the proof and refer to [10] for details. The idea is that any column to the right of the g -th column is filled entirely with the blue numbers $0, 1, 2, \dots, r$ in order and therefore is already determined.

[Lemma 3.3](#) tells us that in order to understand $L_{g,r,d}$ for $d \geq g - r$, it suffices to study the case $d = g - r$. We will restrict to this case throughout the remainder of this section.

Remark 3.4. When $d = g - r$, the rectangle containing the L -tableaux is size $(r + 1) \times g$. The red tableau has size rg and so the blue tableau has size g .

We now prove [Theorem 1.3](#). We first define the following sets of tableaux.

Definition 3.5. Let $\text{TrSSYT } g, r$ be the set of all transposed SSYT's of content $(r^g, r, r, \dots, r)$ and height $\leq r + 1$.

Note that $\text{TrSSYT } g, r$ is the set of all possible ‘red’ tableaux in [Definition 1.1](#). We will refer to them as **red tableaux** throughout this section.

Definition 3.6. Define a 180° -rotated SYT to be the result of rotating a standard Young tableaux 180° in the plane. We write $\text{SYT}^{180^\circ} g, r$ for the set of all 180° -rotated SYT of size g and height $\leq r + 1$. We informally call such a tableau a **purple tableau**, as it will be used to relate the red and blue tableaux of [Definition 1.1](#) (see [Example 3.8](#)).

Given a red tableau, note that each number $1, \dots, g$ occurs once in every row except one. Relatedly, a purple tableau in the position of the blue tableau will have each number $1, \dots, g$ in exactly one row. This leads us to define a bijection between the two as follows.

Definition 3.7 (Red to purple bijection). Let $R \in \text{TrSSYT } g, r$ be a red tableau. We define a 180° -rotated tableau φR in the upper right corner of a rectangle by the following iterative process. We add boxes labeled $1, 2, \dots, g$ in order, where on the i th step we place a box labeled i as far to the right as possible in the unique row that does **not** contain an i in R .

Example 3.8. If R is the tableau at left below, its corresponding purple tableau φR is shown at right below.

2	4	5	6							7	3	1
1	3	4	5	7							6	2
1	2	3	5	6	7							4
1	2	3	4	6	7							5

A generalized version of this map was shown to be a bijection in [\[13\]](#). We also provide a more direct proof in this specific case in [\[10\]](#).

Lemma 3.9. The map φ is a bijection from $\text{TrSSYT } g, r$ to $\text{SYT}^{180^\circ} g, r$ for all g, r . Moreover, for any $R \in \text{TrSSYT } g, r$, the shapes of R and φR are complementary in an $(r + 1) \times g$ rectangle.

We now make precise the notion of a ‘blue tableau’ (see [Definition 1.1](#)).

Lemma 3.11. *The pairs of blue and purple tableaux of the same shape correspond to $r - 1$ -ary sequences of length g bijectively, via inverting the entries of the blue tableau (that is, replacing each entry i by $r - i$), rotating both 180° , and applying RSK.*

Example 3.12. Consider the pair of blue and purple tableaux below. The corresponding pair P, Q is shown at right.

Diagram illustrating the transformation of a Young diagram. The initial diagram (left) has rows [0, 2, 3], [1, 2], [1], [0]. The final diagram (right) has rows [3], [2], [1, 2], [0, 1, 3]. An arrow points from the initial to the final diagram.

We finally can produce a bijection between L -tableaux and $(r-1)$ -ary sequences.

Proof. Each such L -tableaux consists of a red tableau and a blue tableau. The bijection follows from combining Lemma 3.9 with Lemma 3.11, which provide bijections between red tableaux with purple tableaux, and between pairs of blue and purple tableaux with $(r-1)$ -ary sequences of length g , respectively. $\square$

2	4	5	6	0	2	3
1	3	4	5	7	1	2
1	2	3	5	6	7	1
1	2	3	5	6	7	0

Theorem 1.3. *The number of L-tableaux with parameters g, r, d is $r - 1$ for all $d \geq r - g$.*

In our full paper [10], we also analyze the case where $r = 1$, which reduces to ordinary RSK, and the case $d = r = \frac{rg}{r-1}$, which recovers a classical theorem of Castelnovo.

4 L' -tableaux and enumeration by 2^g

The result from [8] corresponding to $L'_{g,d,k}$ states that if $k - g \leq 2d - 1$ and $2 \leq k \leq d$:

$$L'_{g,d,k} = \int_{\text{Gr } 2,d-1} \sigma_1^g \sigma_{k-1} \left(\sum_{i+j=2d-g-k-1} \sigma_i \sigma_j \right) - \int_{\text{Gr } 2,d} \sigma_1^g \sigma_{k-2} \left(\sum_{i+j=2d-g-k-2} \sigma_i \sigma_j \right).$$

We first give an interpretation of the left hand integral in the equation above.

Definition 4.1. A **positive L' -tableau with parameters g, d, k** is a way of filling a $2 \times d - 1$ grid with:

- A standard Young tableau of size g in the lower left corner (shaded red),
- A shading of the $k - 1$ rightmost boxes in the top row (gray),
- A skew SSYT in two letters 0, 1 on the remaining squares (blue).

By rearranging so that we think of the σ_{k-1} as last in each product, and applying [Corollary 2.5](#), we see that the positive term in equation (4.1) is equal to the number of positive L' -tableaux. The second term in (4.1), which we are subtracting, is similarly given by a set of smaller tableaux that we call *negative* tableaux.

Definition 4.2. A **negative L' -tableau with parameters g, d, k** is a filling of a $2 \times d - 2$ grid with:

- A standard Young tableau of size g in the lower left corner (shaded red),
- A shading of the $k - 2$ rightmost boxes in the top row (gray),
- A skew SSYT in two letters 0, 1 on the remaining squares (blue).

Example 4.3. Below are a positive and negative L' -tableau with parameters $3, 7, 4$.

3	0	1			
1	2	0	0	1	1

3	0	1		
1	2	0	0	1

Notice that there exist positive L' -tableaux if and only if $k - 1 - g \leq 2d - 1$, which is slightly stronger than the given condition $k - g \leq 2d - 1$. In particular if $k - g \geq 2d$ or $k - g \geq 2d - 1$ we have $L'_{g,d,k} = 0$, so we restrict our attention to the case that $k - g \leq 2d - 1$. We now prove [Theorem 1.5](#).

Definition 4.4. For fixed g, d, k , write L'_+ and L'_- for the set of positive and negative L' tableaux respectively of type g, d, k . Also write $\psi : L'_- \rightarrow L'_+$ for the map that takes a negative tableau T and adds a blue 1 to the end of the bottom row and a gray box to the end of the top row.

Our above analysis shows that $|L'_{g,d,k}| = |L'| - |L'_-|$, and we analyze this difference combinatorially. We note that ψ is a well-defined injective map, and so

$$|L'_{g,d,k}| = |L'| - |\psi(L'_-)|. \quad (4.1)$$

The next proposition characterizes the image $\psi(L'_-)$, and we refer to [10] for our proof.

Proposition 4.5. *A positive tableau T is equal to $\psi(S)$ for some negative tableaux S if and only if the bottom row of T contains a blue 1.*

Applying this proposition and equation (4.1), we obtain the following.

Corollary 4.6. *The quantity $|L'_{g,d,k}|$ is equal to the number of positive L' -tableaux with parameters g, d, k for which the bottom row contains no blue 1.*

For sufficiently large d , we can simplify this characterization even further.

Theorem 1.5. *If $d \geq g + k$, we have $|L'_{g,d,k}| = 2^g$.*

We refer to [10] for the full proof. As a sketch, we may restrict our attention to the columns before any gray squares appear, and then use the RSK bijection on the red and blue tableaux in those columns to obtain a binary sequence.

Example 4.7. The tableau below at left is a positive L' -tableau with parameters $(3, 7, 4)$ that is not the image of a negative one. Since $g = 3$, we restrict our attention to the first three columns (second image below), then consider the associated pair of tableaux of the same shape by rotating the blue tableaux and inverting the labels. Finally, this pair corresponds under RSK to a unique length 3 binary sequence.

$$\begin{array}{|c|c|c|c|c|c|} \hline 3 & 0 & 1 & & & \\ \hline 1 & 2 & 0 & 0 & 0 & 0 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|} \hline 3 & 0 & 1 \\ \hline 1 & 2 & 0 \\ \hline \end{array} \longrightarrow \left(\begin{array}{|c|c|} \hline 3 & \\ \hline 1 & 2 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & \\ \hline 0 & 1 \\ \hline \end{array} \right) \longrightarrow 110$$

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A unipotent realization of the chromatic quasisymmetric function

Lucas Gagnon^{*1}

¹*Department of Mathematics and Statistics, York University, Toronto ON, Canada*

Abstract. The chromatic quasisymmetric function is a t -analogue of Stanley's chromatic symmetric function, and has recently been at the center of a number of exciting developments in algebraic combinatorics. This extended abstract contributes to this trend, describing a novel realization of certain chromatic quasisymmetric functions as characters of the finite general linear group $\mathrm{GL}_n \mathbb{F}_q$. Additional results tie these characters to other aspects of the chromatic quasisymmetric function: point counting in Hessenberg varieties over $\mathbb{F}_q$, realizing the plethystic connection with unicellular LLT polynomials, and re-interpreting positivity conjectures.

Keywords: Chromatic quasisymmetric function; unicellular LLT polynomial; unipotent group; combinatorial Hopf algebra

1 Introduction

The chromatic symmetric function sits at a nexus of disparate areas of mathematics. At face value, this symmetric function encodes the coloring problem of a graph as an analogue of the chromatic polynomial [18]. However, through a well-known equivalence between the ring of symmetric functions and the representation theory of the symmetric groups (see e.g. [14]), some chromatic symmetric functions are also complex characters of the symmetric group [7]. Moreover, by way of a t -analogue known as the chromatic quasisymmetric function, Brosnan and Chow [3] and Guay-Paquet [10] independently proved that the characters corresponding to indifference graphs are afforded by symmetric group modules on the cohomology rings of regular semisimple Hessenberg varieties, as predicted by a conjecture of Shareshian and Wachs [17]. Thus, certain questions about graphs, representation theory, and algebraic geometry coincide in the structure of these symmetric functions, and vice versa.

Based on the paper [6], this extended abstract describes yet another connection to the chromatic quasisymmetric function, this time from the general linear group $\mathrm{GL}_n \mathbb{F}_q$ over the finite field with q elements. The maximal unipotent subgroup $\mathrm{UT}_n \mathbb{F}_q$ of

^{*}lgagnon@yorku.ca.

$GL_n \mathbb{F}_q$ has a family of characters indexed by indifference graphs, and [Theorem 3.1](#) shows that and up to a factor of $q - 1^n$ the induction functor gives a map

$$\left\{ \begin{array}{l} \text{indifference graph} \\ \text{indexed characters} \end{array} \right\} \xrightarrow{\text{Ind}_{UT_n \mathbb{F}_q}^{GL_n \mathbb{F}_q}} \left\{ \begin{array}{l} \text{chromatic quasisymmetric functions for} \\ \text{indifference graphs evaluated at } t = q \end{array} \right\}$$

where unipotently supported class functions of $GL_n \mathbb{F}_q$ are identified with symmetric functions via the Hall algebra. [Section 3](#) describes this result and its proof using a Hopf algebraic interpretation of induction established in a second paper of the author [\[5\]](#).

The remaining sections of this abstract explore implications of [Theorem 3.1](#). While the theorem is reminiscent of the Brosnan–Chow–Guay–Paquet theorem, the underlying association between symmetric functions and $GL_n \mathbb{F}_q$ characters in this paper is quite different and offers a complementary perspective. Moreover, the relevant characters are elementary in nature and may be more accessible from a module theoretic standpoint.

[Section 4](#) gives a geometric interpretation of the characters in [Theorem 3.1](#): their values count the points of Hessenberg varieties associated to an ad-nilpotent ideal over $\mathbb{F}_q$. Even over $\mathbb{C}$, Hessenberg varieties associated to ad-nilpotent ideals are markedly different from the ones in the Brosnan–Chow–Guay–Paquet theorem, and there is no known module structure on their cohomology rings. However, Precup and Sommers [\[16\]](#) have given a geometric connection between the Poincaré polynomials of these Hessenberg varieties and the chromatic quasisymmetric function, and [Corollary 4.4](#) shows how [Theorem 3.1](#) can be seen as a representation theoretic manifestation of this phenomenon.

[Section 5](#) revisits a well-known relationship between chromatic quasisymmetric functions and the family of unicellular LLT polynomials [\[4\]](#) from the perspective of $GL_n \mathbb{F}_q$. The unipotent characters of $GL_n \mathbb{F}_q$ give another realization of symmetric functions, and any character can be projected onto its unipotent summands to produce a symmetric function. For the characters in [Theorem 3.1](#), this association turns out to be a twisted version of the relationship between chromatic quasisymmetric functions and unicellular LLT polynomials, and [Theorem 5.3](#) gives a map

$$\left\{ \begin{array}{l} \text{indifference graph} \\ \text{indexed characters} \end{array} \right\} \xrightarrow{\omega \circ \text{projection} \circ \text{Ind}_{UT_n \mathbb{F}_q}^{GL_n \mathbb{F}_q}} \left\{ \begin{array}{l} \text{unicellular LLT polyno-} \\ \text{mials evaluated at } t = q \end{array} \right\}.$$

No previous connection between LLT polynomials and $GL_n \mathbb{F}_q$ representation theory was known, though similar associations exist for quantum groups [\[13\]](#), affine Hecke algebras [\[8\]](#), and the symmetric groups [\[10, 12\]](#).

Finally, [Section 6](#) describes the $GL_n \mathbb{F}_q$ representation theoretic meaning of two “positivity conjectures” about the chromatic quasisymmetric function and unicellular LLT polynomial. Each conjecture postulates that when the symmetric function in question is expressed in a chosen basis, each coefficient will be nonnegative. For the chromatic quasisymmetric functions, the modified Stanley–Stembridge conjecture [\[17, Conjecture 1.3\]](#) (see also [\[20\]](#)) concerns positivity in the elementary symmetric function basis,

and is entirely open. For the LLT polynomials, positivity in the Schur basis has been established by Grojnowski and Haiman [8], but no positive combinatorial formula for the coefficients is known [11]. While no immediate progress is made on either conjecture in this abstract, the $\mathrm{GL}_n \mathbb{F}_q$ interpretations may be a useful starting point for future work.

The remainder of this abstract is organized as follows. Section 2 gives general background material. Section 3 contains the main result, Theorem 3.1, and Section 4 relates this result to Hessenberg varieties. Section 5 concerns the unicellular LLT polynomial. Finally, Section 6 discusses the aforementioned positivity conjectures.

2 Preliminaries

This section contains preliminary material on Hopf algebras and symmetric functions (Section 2.1), chromatic quasisymmetric functions (Section 2.2), the character theory of $\mathrm{UT}_n \mathbb{F}_q$ (Section 2.3), and its relation to $\mathrm{GL}_n \mathbb{F}_q$ characters (Section 2.4).

2.1 (Quasi-)Symmetric functions and combinatorial Hopf algebras

This section describes the Hopf algebraic techniques required to prove the main result. Throughout, the term *Hopf algebra* will refer to a graded connected $\mathbb{C}$ -vector space

$$H_\bullet = \bigoplus_{n \geq 0} H_n \quad \text{such that} \quad H_0 \sim \mathbb{C},$$

equipped with a $\mathbb{C}$ -bialgebra structure and a compatible antipode map. This extra structure is not used prominently in this work, which instead focuses on Hopf algebra homomorphisms; these can be thought of as a well-behaved subclass of graded linear maps.

Two particularly important examples now follow. A *composition* of $n \in \mathbb{Z}_{\geq 0}$ is a sequence of positive integers $\alpha = \alpha_1, \dots, \alpha_k$ with $\alpha_1 + \dots + \alpha_k = n$. Call each α_i a *part* of α . The *monomial quasisymmetric function* associated to the composition α is

$$M_\alpha = \sum_{i_1 < \dots < i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \dots x_{i_\ell}^{\alpha_\ell} \in \mathbb{C}[\mathbf{x}].$$

where $\mathbf{x} = \{x_1, x_2, \dots\}$ is an infinite, totally ordered set of indeterminates. With grading given by degree, the *Hopf algebra of quasisymmetric functions* is

$$\mathrm{QSym} = \mathbb{C}\text{-span}\{M_\alpha \mid \alpha \text{ is a composition}\}.$$

A *partition* of n is a composition of n with non-increasing parts. Let

$$\mathcal{P} = \bigsqcup_{n \geq 0} \mathcal{P}_n \quad \text{with} \quad \mathcal{P}_n = \{\text{partitions of } n\}.$$

The Hopf algebra of symmetric functions is the subspace

$$\text{Sym} = \mathbb{C}\text{-span}\{m_\lambda \mid \lambda \in \mathcal{P}\} \subseteq \mathcal{Q}\text{Sym} \quad \text{with} \quad m_\lambda = \sum_{\text{sort } \alpha = \lambda} M_\alpha,$$

where $\text{sort } \alpha$ is the partition obtained by listing the parts of α in non-increasing order.

This abstract will use several standard bases of Sym found in [14]: the elementary symmetric functions $\{e_\lambda \mid \lambda \in \mathcal{P}\}$ [14, I.2], the Schur functions $\{s_\lambda \mid \lambda \in \mathcal{P}\}$ [14, I.3], and the degree-shifted Hall-Littlewood symmetric functions $\{\widetilde{P}_\lambda(\mathbf{x}; t) \mid \lambda \in \mathcal{P}\}$ [14, IV.4], which depend on an additional parameter t . We will also use the involution $\omega : \text{Sym} \rightarrow \text{Sym}$, given by $\omega s_\lambda = s_{\lambda'}$, where λ' denotes the transpose of λ .

The paper [1] gives classification of all homomorphisms from an arbitrary Hopf algebra H to $\mathcal{Q}\text{Sym}$ via certain algebra homomorphisms $\zeta : H \rightarrow \mathbb{C}$, called *linear characters* herein in order to avoid confusion with the group characters in this work. The Hopf algebra $\mathcal{Q}\text{Sym}$ has a linear character known as the *first principal specialization*,

$$\begin{aligned} \text{ps}_1 : \mathcal{Q}\text{Sym} &\longrightarrow \mathbb{C} \\ M_\alpha &\longmapsto \begin{cases} 1 & \text{if } \alpha = (1^n) \text{ for } n \geq 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Theorem 2.1 ([1] Theorem 4.1). *Let H be a Hopf algebra. Then the map*

$$\begin{aligned} \{\text{homomorphisms } H \rightarrow \mathcal{Q}\text{Sym}\} &\longrightarrow \{\text{linear characters of } H\} \\ \Psi &\longmapsto \text{ps}_1 \circ \Psi \end{aligned}$$

is a bijection. In particular, for each linear character ζ of H , there is a unique homomorphism $\Psi : H \rightarrow \mathcal{Q}\text{Sym}$ for which $\zeta = \text{ps}_1 \circ \Psi$.

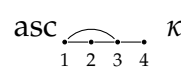
2.2 Indifference graphs and the chromatic quasisymmetric function

This section will describe the chromatic quasisymmetric function of a graph, and go on to define a special class of graphs for which this function is particularly well behaved.

Let γ be a simple, undirected graph with vertex set $[n]$ and edge set $E(\gamma)$. A *coloring* of γ is a function $\kappa : [n] \rightarrow \mathbb{Z}_{>0}$. A coloring κ of γ is *proper* if $\kappa(i) \neq \kappa(j)$ for all $\{i, j\} \in E(\gamma)$. The γ -*ascent number* of a coloring κ is

$$\text{asc}_\gamma(\kappa) = |\{\{i, j\} \in E(\gamma) \mid i < j \text{ and } \kappa(i) < \kappa(j)\}|. \quad (2.2)$$

For example, if $\kappa : [5] \rightarrow \mathbb{Z}_{>0}$ is given by $\kappa(1) = 2, \kappa(2) = 5, \kappa(3) = 1$, and $\kappa(4) = 5$,

$$\text{asc}_\gamma(\kappa) = |\{\{1, 2\}, \{3, 4\}\}| = 2.$$


In this example, κ is a proper coloring of the given graph.

The chromatic quasisymmetric function of γ is

$$X_\gamma \mathbf{x}; t = \sum_{\substack{\kappa: n \rightarrow \mathbb{Z}_{>0} \\ \text{proper}}} t^{\text{asc}_\gamma \kappa} x_{\kappa 1} x_{\kappa 2} \dots x_{\kappa n} \in \mathcal{QSym} t ,$$

so that $X_\gamma \mathbf{x}; t$ is a polynomial in an indeterminate t whose coefficients—by properties of the ascent statistic—are quasisymmetric functions. For example,

$$X_{\begin{smallmatrix} \bullet & \bullet & \bullet \\ 1 & 2 & 3 \end{smallmatrix}} \mathbf{x}; t = t M_{2,1} + t M_{1,2} + t^2 + 4t + 1 M_{1^3} . \quad (2.3)$$

Evaluating the indeterminate t in $X_\gamma \mathbf{x}; t$ at a complex number gives an actual quasisymmetric function. For example, taking $t = 1$ gives the ordinary chromatic symmetric function of γ as defined by Stanley in [18], and the main result of this paper concerns the evaluation of $X_\gamma \mathbf{x}; t$ at $t = q$, the order of the finite field $\mathbb{F}_q$.

The coefficients of $X_\gamma \mathbf{x}; t$ are known to be symmetric under certain conditions. An *indifference graph* of size $n \geq 0$ is a simple, undirected graph γ on the vertex set $[n] = \{1, \dots, n\}$ with edge set E_γ satisfying

$$\text{for each } \{i, l\} \in E_\gamma : \quad \{ \{j, k\} \mid i \leq j < k \leq l \} \subseteq E_\gamma .$$

The empty graph on $\emptyset$ is the unique indifference graph of size zero. Let

$$\mathcal{IG} = \bigsqcup_{n \geq 0} \mathcal{IG}_n \quad \text{with} \quad \mathcal{IG}_n = \{ \text{indifference graphs on } [n] \} .$$

For example,

$$\gamma = \begin{smallmatrix} \bullet & \bullet & \bullet & \bullet \\ 1 & 2 & 3 & 4 \end{smallmatrix} \in \mathcal{IG}_4 \quad \text{but} \quad \sigma = \begin{smallmatrix} \bullet & \bullet & \bullet & \bullet \\ 1 & 2 & 3 & 4 \end{smallmatrix} \notin \mathcal{IG}_4 ,$$

as $\{1, 4\} \in E_\sigma$ but $\{3, 4\} \notin E_\sigma$.

Proposition 2.4 ([17, Theorem 4.5]). *For an indifference graph $\gamma \in \mathcal{IG}_n$, the coefficients of each power of t in $X_\gamma \mathbf{x}; t$ is a symmetric function.*

The indifference graphs of size $n \geq 0$ are enumerated by the n th Catalan number, as seen in the following bijection, found in [19, Sol. 187]. Associate to each $\gamma \in \mathcal{IG}_n$ the southeast lattice path from $(0, 0)$ to $(n, -n)$ which lies directly above the diagonal $x = -y$ and the unit squares centered at $(j - \frac{1}{2}, \frac{1}{2} - i)$ for each $\{i, j\} \in E_\gamma$. For example,

$$\begin{smallmatrix} \bullet & \bullet & \bullet \\ 1 & 2 & 3 \end{smallmatrix} \longleftrightarrow \begin{array}{|c|c|c|} \hline & \textcolor{red}{\{1,2\}} & \\ \hline & & \textcolor{red}{\{2,3\}} \\ \hline & & \\ \hline \end{array} \longleftrightarrow \text{E E S E S S} . \quad (2.5)$$

2.3 Linear algebraic groups and class functions

For a finite group G , the space of complex valued class functions on G is

$$\text{cf } G = \{ \psi : G \longrightarrow \mathbb{C} \mid \psi(g) = \psi(hgh^{-1}) \text{ for all } g, h \in G \}.$$

Under the usual inner product $\langle \cdot, \cdot \rangle : \text{cf } G \otimes \text{cf } G \rightarrow \mathbb{C}$, there are two orthogonal bases of $\text{cf } G$: the irreducible characters of G and the conjugacy class identifier functions,

$$\{ \delta_K \mid K \in \text{Cl} \} \quad \text{with} \quad \delta_K(g) = \begin{cases} 1 & \text{if } g \in K. \\ 0 & \text{otherwise.} \end{cases} \quad (2.6)$$

The remainder of this section will construct a subspace of class functions of a particular family of groups. Fix a prime power q , let $\mathbb{F}_q$ denote the field with q elements, and let $\text{GL}_n = \text{GL}_n(\mathbb{F}_q)$. The *unipotent upper triangular group* is the subgroup

$$\text{UT}_n = \{ g \in \text{GL}_n \mid g - 1_n \text{ is strictly upper triangular} \}$$

where $1_n \in \text{GL}_n$ is the identity matrix. The set $\mathcal{IG}_n$ indexes a family of normal subgroups in UT_n known as *normal pattern subgroups* [15, Lemma 4.1]: for $\gamma \in \mathcal{IG}_n$, let

$$\text{UT}_\gamma = \{ g \in \text{UT}_n \mid g_{i,j} = 0 \text{ if } \{i, j\} \in E_\gamma \},$$

where E_γ denotes the edge set of γ . If π is the Dyck path corresponding to γ , then UT_γ is the subset of elements of UT_n with nonzero entries occurring only on the diagonal or above the path π . For example using the graph and Dyck path from Equation (2.5),

$$\text{UT}_{\begin{array}{c} \text{---} \text{---} \text{---} \\ 1 \quad 2 \quad 3 \end{array}} = \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The characters appearing in the main results below span a subspace of $\text{cf } \text{UT}_n$ which was initially constructed in [2]. For $\gamma \in \mathcal{IG}_n$, let

$$\bar{\chi}^\gamma = \text{Ind}_{\text{UT}_\gamma}^{\text{UT}_n} \mathbb{1},$$

the character of the UT_n -module $\mathbb{C} \text{UT}_n / \text{UT}_\gamma$, so that

$$\bar{\chi}^\gamma = q^{|E_\gamma|} \delta_{\text{UT}_\gamma}. \quad (2.7)$$

The characters $\bar{\chi}^\gamma$ are linearly independent, and span a self-dual subspace

$$\text{scf } \text{UT}_n = \mathbb{C}\text{-span}\{ \bar{\chi}^\gamma \mid \gamma \in \mathcal{IG}_n \},$$

known as the subspace of *superclass functions*.

Remark 2.8. The space $\text{scf } \text{UT}_n$ comes from a supercharacter theory of UT_n [2], and has a number of interesting bases indexed by $\mathcal{IG}_n$. One of these—the “supercharacter basis”—is typically denoted by χ^γ and has the property that $\bar{\chi}^\gamma = \sum_{\sigma \subseteq \gamma} \chi^\sigma$, from which the notation $\bar{\chi}^\gamma$ is derived. More details can be found in the extended version of this abstract [6, Section 2.3].

2.4 Homomorphisms between Hopf algebras of class functions

In [21, III], Zelevinsky defines a graded connected Hopf algebra on the space

$$\text{cf GL}_\bullet = \bigoplus_{n \geq 0} \text{cf GL}_n ,$$

with structure maps coming from the parabolic induction and restriction functors. The paper [5] defines a similar Hopf structure on the spaces

$$\text{scf UT}_\bullet = \bigoplus_{n \geq 0} \text{scf UT}_n , \quad \text{and} \quad \text{cf UT}_\bullet = \bigoplus_{n \geq 0} \text{cf UT}_n ,$$

in which the former is a sub-Hopf algebra of the latter. This section will describe several homomorphisms involving these Hopf algebras.

In [10, Section 6], Guay-Paquet defines a $\mathbb{C} t$ -Hopf algebra on the free $\mathbb{C} t$ -module $\mathbb{C} t \mathcal{IG}$, and specializing $t \mapsto q^{-1}$ gives a Hopf algebra over $\mathbb{C}$; see [5, Section 7].

Theorem 2.9 ([5, Corollary 7.6]). *The map $\gamma \mapsto q^{-|E \gamma|} \chi^\gamma$ is an isomorphism from Guay-Paquet's specialized Hopf algebra to $\text{scf UT}_\bullet$.*

A second map comes from the induction functors $\text{Ind}_{\text{UT}_n}^{\text{GL}_n} : \text{cf UT}_n \rightarrow \text{cf GL}_n$; let

$$\text{Ind}_{\text{UT}}^{\text{GL}} = \bigoplus_{n \geq 0} \text{Ind}_{\text{UT}_n}^{\text{GL}_n} : \text{cf UT}_\bullet \longrightarrow \text{cf GL}_\bullet .$$

Theorem 2.10 ([5, Theorem 6.1]). *The map $\text{Ind}_{\text{UT}}^{\text{GL}}$ is a Hopf algebra homomorphism.*

The image of $\text{Ind}_{\text{UT}}^{\text{GL}}$ is a sub-Hopf algebra, the *class functions with unipotent support*:

$$\text{cf}_{\text{supp}}^{\text{uni}} \text{GL}_\bullet = \text{Ind}_{\text{UT}}^{\text{GL}} \text{cf UT}_\bullet \subseteq \text{cf GL}_\bullet .$$

The space $\text{cf}_{\text{supp}}^{\text{uni}} \text{GL}_\bullet$ can also be constructed more explicitly, as follows. An element $g \in \text{GL}_n$ is *unipotent* if g is conjugate to an element of UT_n . The conjugacy classes of unipotent elements in GL_n are indexed by $\mathcal{P}_n$: the partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ corresponds to the conjugacy class O_λ of the Jordan matrix

$$J_\lambda = J_{\lambda_1} \oplus J_{\lambda_2} \oplus \dots \oplus J_{\lambda_\ell} \quad \text{with} \quad J_{k,i,j} = \begin{cases} 1 & \text{if } j \in \{i, i-1\} \\ 0 & \text{otherwise} \end{cases}$$

Thus, writing $\delta_\lambda = \delta_{O_\lambda}$, standard properties of induction imply that

$$\text{cf}_{\text{supp}}^{\text{uni}} \text{GL}_\bullet = \mathbb{C}\text{-span}\{\delta_\lambda \mid \lambda \in \mathcal{P}\}.$$

Zelevinsky [21, 10.13] (see also [14, IV.4.1]) constructs a Hopf algebra isomorphism

$$\begin{aligned} \mathbf{p}_{\{1\}} : \text{cf}_{\text{supp}}^{\text{uni}} \text{GL}_\bullet &\longrightarrow \text{Sym} \\ \delta_\lambda &\longmapsto \widetilde{P}_\lambda(\mathbf{x}; q) \end{aligned} \tag{2.11}$$

where $\widetilde{P}_\lambda$ is the degree shifted Hall–Littlewood polynomial mentioned in Section 2.1.

3 Realizing $X_\gamma \mathbf{x}; t$ as a GL_n character

This section gives the main result. Recall the definitions of $\mathcal{IG}_n$ and $X_\gamma \mathbf{x}; t$ from [Section 2.2](#), the groups UT_γ from [Section 2.3](#), and the maps $\mathrm{Ind}_{\mathrm{UT}}^{\mathrm{GL}}$ and $\mathbf{p}_{\{1\}}$ from [Section 2.4](#).

Theorem 3.1. *For $n \geq 0$ and $\gamma \in \mathcal{IG}_n$,*

$$\mathrm{Ind}_{\mathrm{UT}_\gamma}^{\mathrm{GL}_n} \mathbb{1} = (q-1)^n \mathbf{p}_{\{1\}}^{-1}(X_\gamma \mathbf{x}; q).$$

A full proof of [Theorem 3.1](#) is given in [6, Section 3], but we will give a short sketch below, after a few preliminaries. [Theorem 2.1](#) states that the homomorphism

$$\mathrm{cano}_{\mathrm{CQS}} : \mathrm{scf} \, \mathrm{UT}_\bullet \xrightarrow{\mathrm{Ind}_{\mathrm{UT}}^{\mathrm{GL}}} \mathrm{cf}_{\mathrm{supp}}^{\mathrm{uni}} \, \mathrm{GL}_\bullet \xrightarrow{\mathbf{p}_{\{1\}}} \mathrm{Sym} \xrightarrow{\mathrm{inclusion}} \mathcal{Q}\mathrm{Sym}$$

is uniquely determined by the linear character $\mathrm{ps}_1 \circ \mathbf{p}_{\{1\}} \circ \mathrm{Ind}_{\mathrm{UT}}^{\mathrm{GL}}$ of $\mathrm{scf} \, \mathrm{UT}_\bullet$. The proof of [Theorem 3.1](#) amounts to showing that this linear character also corresponds to the homomorphism

$$\begin{aligned} \mathrm{scf} \, \mathrm{UT}_\bullet &\longrightarrow \mathcal{Q}\mathrm{Sym} \\ \bar{\chi}^\gamma &\longmapsto (q-1)^n X_\gamma \mathbf{x}; q, \end{aligned} \tag{3.2}$$

where for each $\gamma \in \mathcal{IG}_n$, $\bar{\chi}^\gamma = \mathrm{Ind}_{\mathrm{UT}_\gamma}^{\mathrm{UT}_n} \mathbb{1}$ as in [Section 2.3](#).

A relative of the map in Equation (3.2) has been studied by Guay-Paquet in [10]. Translating through the isomorphism of [Theorem 2.9](#) and a $q \leftrightarrow q^{-1}$ interchange property established in [17, Proposition 2.6], we have the following.

Theorem 3.3 ([10, Theorem 57]). *The map $\bar{\chi}^\gamma \mapsto X_\gamma \mathbf{x}; q$ is a Hopf algebra homomorphism from $\mathrm{scf} \, \mathrm{UT}_\bullet$ to $\mathcal{Q}\mathrm{Sym}$. Writing $\zeta_0 : \mathrm{scf} \, \mathrm{UT}_\bullet \rightarrow \mathbb{C}$ for the corresponding linear character,*

$$\zeta_0 \bar{\chi}^\gamma = \begin{cases} q^{|\mathcal{E}(\gamma)|} & \text{if } \gamma = n, \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

We now relate ζ_0 to the linear character of $\mathrm{cano}_{\mathrm{CQS}}$. Zelevinsky [21, 10.8] has shown that for a unipotently supported class function $\psi \in \mathrm{cf}_{\mathrm{supp}}^{\mathrm{uni}} \, \mathrm{GL}_\bullet$,

$$\mathrm{ps}_1 \circ \mathbf{p}_{\{1\}} \psi = \psi(J_n),$$

where J_n is the unipotent Jordan matrix corresponding to the partition n , as in [Section 2.4](#). Taking $\psi = \mathrm{Ind}_{\mathrm{UT}}^{\mathrm{GL}} \bar{\chi}^\gamma$, standard facts about induction and the conjugacy class of J_n give the following result, proved in [6, Proposition 3.12].

Proposition 3.4. *Let γ be an indifference graph of size $n \geq 0$. Then*

$$\mathrm{ps}_1 \circ \mathbf{p}_{\{1\}} \circ \mathrm{Ind}_{\mathrm{UT}}^{\mathrm{GL}} \bar{\chi}^\gamma = \begin{cases} (q-1)^n q^{|\mathcal{E}(\gamma)|} & \text{if } \gamma = n, \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Proof sketch of Theorem 3.1. Starting from [Theorem 3.3](#), it can be verified directly that the map in Equation (3.2) is a Hopf algebra homomorphism. By [Proposition 3.4](#), this map has the same linear character as $\mathrm{cano}_{\mathrm{CQS}}$, so by [Theorem 2.1](#), they are the same. $\square$

4 Connections to Hessenberg varieties

This section will describe a relationship between the characters in [Theorem 3.1](#) and Hessenberg varieties. Take $n \geq 0$ and let $\mathbb{K}$ be a field. For each element $A \in \text{Mat}_n \mathbb{K}$ and subspace $M \subseteq \text{Mat}_n \mathbb{K}$ which is stable under conjugation by the subgroup of upper triangular matrices $B_n \mathbb{K} \subseteq \text{GL}_n \mathbb{K}$, the *Hessenberg variety* associated to A and M is

$$\mathcal{B}_A^M = \{gB_n \mathbb{K} \in \text{GL}_n \mathbb{K} / B_n \mathbb{K} \mid g^{-1}Ag \in M\}.$$

The following results exclusively concern Hessenberg varieties associated to strictly upper triangular subspaces known as *ad-nilpotent ideals*. For $\gamma \in \mathcal{IG}_n$, let

$$\text{ut}_\gamma \mathbb{K} = \{A \in \text{Mat}_n \mathbb{K} \mid A_{i,j} \neq 0 \text{ only if } i < j \text{ and } \{i, j\} \notin E_\gamma\} \subseteq \text{UT}_\gamma \mathbb{K} = 1_n.$$

Some key examples of Hessenberg varieties of the form $\mathcal{B}_A^{\text{ut}_\gamma \mathbb{K}}$ are well-known, but a specific study of these varieties is quite recent; see [\[16\]](#) and the references therein.

Proposition 4.1. *Let $n \geq 0$ and $\gamma \in \mathcal{IG}_n$. For $A \in \text{Mat}_n \mathbb{F}_q$ with $1_n - A \in \text{GL}_n \mathbb{F}_q$,*

$$\text{Ind}_{\text{UT}_\gamma \mathbb{F}_q}^{\text{GL}_n \mathbb{F}_q} \mathbb{1}_{1_n - A} = (q-1)^n q^{|E_\gamma|} |\mathcal{B}_A^{\text{ut}_\gamma \mathbb{F}_q}|.$$

Proof sketch. The left side is equal to $q^{|E_\gamma|}$ times the number of cosets $h\text{UT}_\gamma \mathbb{F}_q \in \text{GL}_n \mathbb{F}_q / \text{UT}_\gamma \mathbb{F}_q$ with $h^{-1}(1_n - A)h \in \text{UT}_\gamma \mathbb{F}_q$. Each such coset satisfies $hB_n \mathbb{F}_q \in \mathcal{B}_A^{\text{ut}_\gamma \mathbb{F}_q}$, with $q-1$ cosets $h\text{UT}_\gamma \mathbb{F}_q$ for each element of $\mathcal{B}_A^{\text{ut}_\gamma \mathbb{F}_q}$. $\square$

The paper [\[16\]](#) gives a similar result for the analogous Hessenberg varieties over $\mathbb{C}$, which involves the modified Poincaré polynomial

$$\text{Poin}_A^{\text{ut}_\gamma \mathbb{C}}(t) = \sum_{k \geq 0} \beta_{2k} t^k,$$

where β_i is the i th Betti number of $\mathcal{B}_A^{\text{ut}_\gamma \mathbb{C}}$. Recall the Hall–Littlewood symmetric function $\widetilde{P}_\lambda(\mathbf{x}; t)$ from [Section 2.1](#), and define expressions $d_\lambda^\gamma(t)$ for each partition λ of n by

$$X_\gamma(\mathbf{x}; t) = \sum_{\lambda \in \mathcal{P}_n} d_\lambda^\gamma(t) \widetilde{P}_\lambda(\mathbf{x}; t). \quad (4.2)$$

Also note that partitions of n index the similarity classes of nilpotent matrices over any field: $\lambda \in \mathcal{P}_n$ corresponds to the class of $J_\lambda - 1_n$, where J_λ is as defined in [Section 2.4](#).

Theorem 4.3 ([\[16, Equation \(4.7\)\]](#)). *For $n \geq 0$, take $\gamma \in \mathcal{IG}_n$ and $\lambda \in \mathcal{P}_n$. For a nilpotent matrix $A \in \text{Mat}_n \mathbb{C}$ in the similarity class indexed by λ ,* $\text{Poin}_A^{\text{ut}_\gamma \mathbb{C}}(t) = t^{-|E_\gamma|} d_\lambda^\gamma(t)$.

Applying the inverse of the map $\mathbf{p}_{\{1\}}$ defined in [Equation \(2.11\)](#) to [Equation \(4.2\)](#), [Theorem 3.1](#) and [Proposition 4.1](#) give the following.

Corollary 4.4. *For $n \geq 0$, take $\gamma \in \mathcal{IG}_n$ and $\lambda \in \mathcal{P}_n$. Let $A \in \text{Mat}_n \mathbb{F}_q$ and $A' \in \text{Mat}_n \mathbb{C}$ be nilpotent elements belonging to similarity classes indexed by λ . Then $\text{Poin}_{A'}^{\text{ut}_\gamma \mathbb{C}}(q) = |\mathcal{B}_A^{\text{ut}_\gamma \mathbb{F}_q}|$.*

5 Realizing $G_\gamma \mathbf{x}; t$ as a GL_n character

For an indifference graph $\gamma \in \mathcal{IG}$, the *unicellular LLT polynomial* associated to γ is

$$G_\gamma \mathbf{x}; t = \sum_{\kappa: n \rightarrow \mathbb{Z}_{>0}} t^{\mathrm{asc}_\gamma \kappa} x_{\kappa(1)} x_{\kappa(2)} \dots x_{\kappa(n)} \in \mathbb{C}[\mathbf{x}, t],$$

where the sum is over *all* colorings. This is a variation of the polynomials defined by Lascoux, Leclerc, and Thibon in [13], and is due to Carlsson and Mellit [4] (see also [12]). Like $X_\gamma \mathbf{x}; t$, each $G_\gamma \mathbf{x}; t$ is a polynomial in t with symmetric function coefficients.

Theorem 5.1 ([4, Proposition 3.5]). *For $n \geq 0$, let $\gamma \in \mathcal{IG}_n$. Using plethystic notation,*

$$(t-1)^{-n} X_\gamma \mathbf{x}; t \left[\frac{\mathbf{x}}{t-1} \right] = G_\gamma \mathbf{x}; t.$$

Somewhat surprisingly, the plethystic substitution above has a natural meaning for $\mathrm{GL}_n \mathbb{F}_q$. An irreducible character of GL_n is *unipotent* if it is a summand of $\mathrm{Ind}_{B_n}^{\mathrm{GL}_n} \mathbb{1}$, where $B_n \subset \mathrm{GL}_n \mathbb{F}_q$ is as defined in Section 4. These characters are indexed by the partitions of n , and χ^λ will denote the irreducible unipotent character corresponding to $\lambda \in \mathcal{P}_n$, with $\chi^{1^n} = \mathbb{1}$ as in [14]. Zelevinsky [21, 9.4] defines a homomorphism

$$\begin{aligned} \mathbf{p}_\mathbb{1}: \mathrm{cf} \mathrm{GL}_\bullet &\longrightarrow \mathrm{Sym} \\ \psi &\longmapsto \sum_\lambda \langle \psi, \chi^\lambda \rangle s_\lambda, \end{aligned} \tag{5.2}$$

so that for any character ψ of GL_n , the coefficient of s_λ in $\mathbf{p}_\mathbb{1} \psi$ is the multiplicity of χ^λ in ψ . Recalling the map $\mathbf{p}_{\{1\}}$ from Section 2.4, it is known [14, IV.4] that the composite

$$\mathrm{Sym} \xrightarrow{\mathbf{p}_{\{1\}}^{-1}} \mathrm{cf}_{\mathrm{supp}}^{\mathrm{uni}} \mathrm{GL}_\bullet \hookrightarrow \mathrm{cf} \mathrm{GL}_\bullet \xrightarrow{\mathbf{p}_\mathbb{1}} \mathrm{Sym}$$

can be expressed in plethystic notation as $f \mapsto \omega f \left[\frac{\mathbf{x}}{t-1} \right] \big|_{t=q}$. With Theorem 5.1, this implies the following result; see [6, Theorem 5.1] for a complete proof.

Theorem 5.3. *Let γ be an indifference graph. Then $\mathbf{p}_\mathbb{1} \circ \mathrm{Ind}_{\mathrm{UT}_\gamma}^{\mathrm{GL}_n} \mathbb{1} = \omega G_\gamma \mathbf{x}; q$.*

6 Positivity conjectures

Recall the bases of Sym given in Section 2.1. An element $f \in \mathrm{Sym} t$ is respectively *e-positive* or *s-positive* if there are polynomials $a_\lambda t \in \mathbb{Z}_{\geq 0} t$ or $b_\lambda t \in \mathbb{Z}_{\geq 0} t$ for which

$$f = \sum_{\lambda \in \mathcal{P}} a_\lambda t e_\lambda \quad \text{or} \quad f = \sum_{\lambda \in \mathcal{P}} b_\lambda t s_\lambda.$$

For the chromatic quasisymmetric functions in Section 2.2, *e-positivity* generalizes the Stanley–Stembridge conjecture [20, Conjecture 5.5], which by [9] is the $t=1$ case of the statement below.

Conjecture 6.1 ([17, Conjecture 1.3]). For each $\gamma \in \mathcal{IG}$, $X_\gamma \mathbf{x}; t$ is e -positive.

In light of [Theorem 3.1](#), there should be a restatement of [Conjecture 6.1](#) in terms of GL_n . However, the map $\mathbf{p}_{\{1\}}$ defined in [Section 2.4](#) does not associate e_λ to a character of GL_n , so some interpretation is required. One possible restatement uses the Steinberg character χ^n of GL_n defined in [Section 5](#), which agrees with $\mathbf{p}_{\{1\}}^{-1} e_n$ on all unipotent elements. For $\lambda = \lambda_1, \dots, \lambda_\ell \in \mathcal{P}_n$, let St_λ denote the product $\chi^{\lambda_1} \cdots \chi^{\lambda_\ell}$ in $\mathrm{cf} \, \mathrm{GL}_\bullet$.

Conjecture 6.2. Let $n \geq 0$ and $\gamma \in \mathcal{IG}_n$. There are polynomials $a_\lambda^\gamma(t) \in \mathbb{Z}_{\geq 0}[t]$ such that for each prime power q the character $\eta_\gamma = \sum_{\lambda \in \mathcal{P}_n} a_\lambda^\gamma(q) \mathrm{St}_\lambda$ satisfies $q - 1 \mid \eta_\gamma(u)$ for every unipotent element $u \in \mathrm{GL}_n(\mathbb{F}_q)$.

By [\[6, Proposition 6.6\]](#), [Conjectures 6.1](#) and [6.2](#) are equivalent. Ideally, a proof of [Conjecture 6.2](#) would construct a module affording the character η_γ .

For the unicellular LLT polynomials from [Section 5](#), Schur positivity has implications in the study of Macdonald polynomials [\[12\]](#) and is known to hold by [\[8, Corollary 6.9\]](#). Unfortunately, the proof in [\[8\]](#) does not explicitly construct the Schur coefficients.

Open Problem 6.3 ([\[11, Open Problem 6.6\]](#)). Find a (manifestly positive) combinatorial formula for the Schur coefficients $b_\lambda^\gamma(t)$ of $G_\gamma \mathbf{x}; t$.

[Theorem 5.3](#) implies that $b_\lambda^\gamma(q) = \langle \chi^\lambda, \mathrm{Ind}_{\mathrm{UT}_\gamma}^{\mathrm{GL}_n} \mathbb{1} \rangle$ for each prime power q , which is a positive integer, but this does not yet imply Schur positivity as defined above.

Open Problem 6.4. Find a combinatorial formula for $\langle \chi^\lambda, \mathrm{Ind}_{\mathrm{UT}_\gamma}^{\mathrm{GL}_n} \mathbb{1} \rangle$ as a function of q .

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A pipe dream perspective on totally symmetric self-complementary plane partitions

Daoji Huang^{*1} and Jessica Striker⁺²

¹*School of Mathematics, University of Minnesota*

²*Department of Mathematics, North Dakota State University*

Abstract. We characterize totally symmetric self-complementary plane partitions (TSSCPP) as bounded compatible sequences satisfying a Yamanouchi-like condition. As such, they are in bijection with certain pipe dreams. Using this characterization and the recent bijection of [Gao-Huang] between reduced pipe dreams and reduced bumpless pipe dreams, we give a bijection between alternating sign matrices and TSSCPP in the reduced, 1432-avoiding case.

Keywords: Alternating sign matrices, plane partitions, pipe dreams

1 Introduction

Plane partitions are three-dimensional analogues of ordinary partitions. Just as partitions in an $a \times b$ are counted by a lovely formula $\binom{a+b}{a}$, plane partitions in an $a \times b \times c$ box are enumerated by MacMahon's product formula $\prod_{i=1}^a \prod_{j=1}^b \prod_{k=1}^c \frac{i+j+k-1}{i+j+k-2}$ [11]. In a 1986 [13], Stanley considered symmetry operations on plane partitions, namely, reflection (transpose), rotation, and complementation. This yielded 10 symmetry classes of plane partitions consisting of plane partitions invariant under combinations of these operations. The plane partitions invariant under all three operations are called *totally symmetric self-complementary* (TSSCPP). As in the case of all plane partitions, each symmetry class has a nice enumeration. TSSCPP inside a $2n \times 2n \times 2n$ box was shown in 1994 by Andrews [1] to be counted by $\prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$. This was, at the time, the conjectured [12] number of $n \times n$ *alternating sign matrices* (ASM). The 1996 proofs of this conjecture [16, 9] sparked a search for a natural, explicit bijection between TSSCPP and ASM. Partial bijections have been found on small subsets, including the permutation case [14], the case of two monotone triangle diagonals [6, 4], and the 312-avoiding case [2]. This paper interprets TSSCPP

^{*}huan0664@umn.edu. Daoji Huang was supported by NSF DMS-2202900.

[†]jessica.striker@ndsu.edu. Jessica Striker was partially supported by a Simons Foundation/SFARI grant (527204, JS)

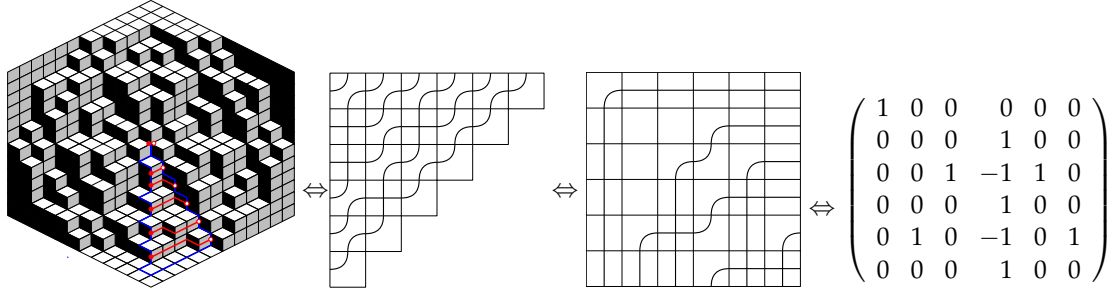


Figure 1: An example of the bijection of this paper. From left to right the objects are: TSSCPP, pipe dream, bumpless pipe dream, ASM. The pipe dream and bumpless pipe dream both have permutation 135264, which avoids 1432.

as *pipe dreams* to extend the bijection of [14] to what appears to be a larger subset than any previous partial bijection. Our main theorem is below; see Figure 1 for an example and Section 2 for definitions. Given $\pi \in S_n$, let $TSSCPP^{red}(\pi)$ denote the set of TSSCPP whose associated pipe dream is reduced and has permutation π , and let $ASM^{red}(\pi)$ denote the set of ASM whose associated *bumpless pipe dream* is reduced and has permutation π . Notice that the reducedness property on TSSCPPs is definitionally inherited from this property on pipe dreams.

Theorem 1.1. *Let $\pi \in S_n$. There is an explicit weight-preserving injection φ from $TSSCPP^{red}(\pi)$ to $ASM^{red}(\pi)$. If π avoids 1432, then φ is a bijection.*

The paper is organized as follows. Section 2 contains background on the relevant objects, including the permutation case TSSCPP bijection of [14] and the bijection of [8] between reduced pipe dreams and reduced bumpless pipe dreams, which are important ingredients in our proof. Section 3 proves Theorem 3.3 characterizing TSSCPP as pipe dreams subject to a Yamanouchi-like condition. Section 4 concerns our main result and its proof. This is an extended abstract only; the full version is posted on the arXiv.

2 Background

In this section, we review relevant definitions and bijections from the literature. Subsections 2.1, 2.2, 2.4, and 2.5 review definitions of ASM, TSSCPP, bumpless pipe dreams, and pipe dreams, respectively. Subsections 2.3 and 2.6 contain less-familiar bijections that are important for our main results.

2.1 Alternating sign matrices

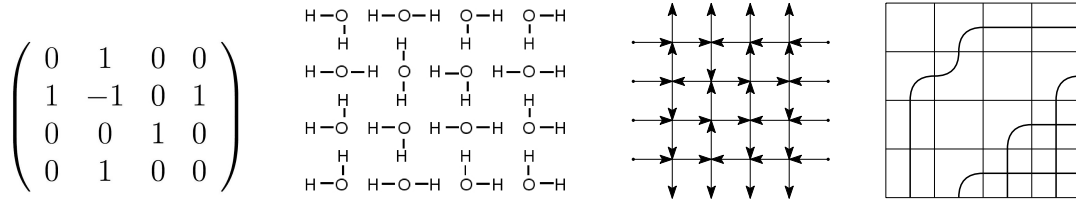
In this subsection, we define alternating sign matrices (see e.g. [12]).

Definition 2.1. An **alternating sign matrix (ASM)** is a square matrix with entries in $\{0, 1, -1\}$ such that the rows and columns each sum to 1 and the nonzero entries alternate in sign across each row and across each column.

Alternating sign matrices are in bijection with configurations of the *six-vertex / square ice model* of statistical physics with *domain wall boundary conditions*; this was an essential element of the enumeration proof of [9]. The 3×3 alternating sign matrices are below.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Below, left and center-left are an alternating sign matrix and its corresponding square ice configuration; the horizontal molecules correspond to $+1$, the vertical molecules correspond to -1 , and all other molecules correspond to 0. Center-right is its six-vertex configuration, where the six molecule configurations are replaced by directed edges. Below, right is the corresponding *bumpless pipe dream*, which will be discussed shortly.



2.2 Totally symmetric self-complementary plane partitions

In this subsection, we define plane partitions and their symmetry classes (see e.g. [13]).

Definition 2.2. A **plane partition** t is a rectangular array of nonnegative integers $(t_{i,j})_{i,j \geq 1}$ such that $t_{i,j} \geq t_{i',j'}$ if $i \leq i', j \leq j'$. We say t is in an $a \times b \times c$ **bounding box** if $t_{i,j} = 0$ whenever $i > a$ or $j > b$ and $t_{i,j} \leq c$ for all i, j . Let $PP(a \times b \times c)$ denote the set of plane partitions in an $a \times b \times c$ bounding box.

Remark 2.3. We can also view $t \in PP(a \times b \times c)$ as a finite set of positive integer lattice points (i, j, k) with $1 \leq i \leq a$, $1 \leq j \leq b$, and $1 \leq k \leq c$ such that if $(i, j, k) \in t$ and $1 \leq i' \leq i$, $1 \leq j' \leq j$, $1 \leq k' \leq k$ then $(i', j', k') \in t$. This well-known bijection is given as $(i, j, k) \in t$ if and only if $t_{i,j} \geq k$. We will use both of these characterizations in the next definition.

Definition 2.4. A plane partition t is **symmetric** if $t_{i,j} = t_{j,i}$ for all i, j . t is **cyclically symmetric** if whenever $(i, j, k) \in t$ then $(j, k, i) \in t$ as well. t is **totally symmetric** if it is both symmetric and cyclically symmetric, so that whenever $(i, j, k) \in t$ then all six permutations of (i, j, k) are also in t . The **complement** t^C of t inside a given bounding box $a \times b \times c$ is defined as $t_{i,j}^C = c - t_{a-i+1, b-j+1}$ for all $1 \leq i \leq a, 1 \leq j \leq b$. That is, $t_{i,j}^C$ equals the number of empty cubes above $t_{a-i+1, b-j+1}$ in the bounding box. A plane partition t is **self-complementary** inside a given bounding box if $t = t^C$. A **totally symmetric self-complementary plane partition (TSSCPP)** is a plane partition which is both totally symmetric and self-complementary.

Note that for there to exist a self-complementary plane partition in an $a \times b \times c$ bounding box, the volume abc of the box must be an even number. In addition, cyclic symmetry requires $a = b = c$, therefore, we need $a = b = c = 2n$ for there to exist a TSSCPP inside an $a \times b \times c$ bounding box.

Definition 2.5. Let $\text{TSSCPP}(n)$ denote the set of TSSCPP inside a $2n \times 2n \times 2n$ box.

2.3 TSSCPP boolean triangles and the permutation case bijection

In this subsection, we review the characterization from [14] of TSSCPP as boolean triangles and the bijection of the same paper between permutation matrices and TSSCPP boolean triangles whose entries weakly decrease along rows.

Definition 2.6 (Def 2.12 of [14]). A **TSSCPP boolean triangle** of order n is a triangular integer array $b = \{b_{i,j}\}$ for $1 \leq i \leq n-1, n-i \leq j \leq n-1$ with entries in $\{0, 1\}$ such that the diagonal partial sums satisfy the following inequality for all $1 \leq j < i \leq n-1$:

$$1 + \sum_{k=j+1}^i b_{k, n-j-1} \geq \sum_{k=j}^i b_{k, n-j}. \quad (2.1)$$

Call this the (i, j) -inequality, in which $n-j$ and $n-j-1$ are the diagonals being compared and i indicates the row index of where the sums stop.

We give below the indexing of a generic TSSCPP boolean triangle.

$$\begin{array}{ccccccc}
 & & & & b_{1,n-1} & & \\
 & & & & & & \\
 & & & b_{2,n-2} & & b_{2,n-1} & \\
 & & b_{3,n-3} & & b_{3,n-2} & & b_{3,n-1} \\
 & & & & \vdots & & \\
 & & & & & & \\
 b_{n-1,1} & & b_{n-1,2} & \cdots & b_{n-1,n-2} & & b_{n-1,n-1}
 \end{array}$$

Below are a non-example and an example of a TSSCPP boolean triangle.

$$\begin{array}{cccccc}
 & & & 1 & & \\
 & & 1 & & 1 & \\
 & 1 & & 0 & & 0 \\
 1 & & 0 & & 0 & 1 \\
 0 & 0 & 0 & 1 & 0 &
 \end{array}
 \qquad
 \begin{array}{cccccc}
 & & & 1 & & \\
 & & 1 & & 1 & \\
 & 1 & & 0 & & 0 \\
 1 & & 0 & & 1 & 1 \\
 0 & 0 & 0 & 0 & 0 & 0
 \end{array}$$

In the left triangle, the $(4,1)$ -inequality is not satisfied, since $\sum_{k=1}^4 b_{k,n-1} = 3$ while $\sum_{k=2}^4 b_{k,n-2} = 1$. In the triangle on the right, all (i,j) -inequalities are satisfied.

Proposition 2.7 (Prop 2.13 of [14]). *TSSCPP boolean triangles of order n are in bijection with $\text{TSSCPP}(n)$.*

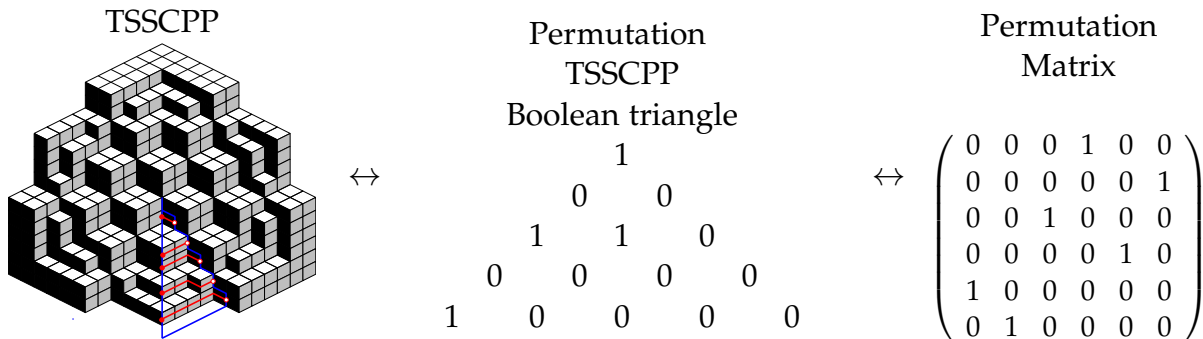
The bijection proceeds by taking the fundamental domain of the TSSCPP, transforming it into a nest of non-intersecting lattice paths, and then recording the two different types of steps in each path as 0 and 1. The diagonal partial sum condition (2.1) is equivalent to the requirement that the paths do not intersect. See [14] for details.

We now review the characterization of a certain subset of TSSCPP boolean triangles.

Definition 2.8 (Def 3.1 of [14]). A **permutation TSSCPP boolean triangle** is a TSSCPP boolean triangle with weakly decreasing rows.

That is, the entries equal to one in a permutation TSSCPP boolean triangle are all left-justified. The terminology ‘permutation’ in the above definition is justified by the statistic-preserving bijection in the theorem below.

Theorem 2.9 (Theorem 3.5 of [14]). *There is a natural, statistic-preserving bijection Φ between $n \times n$ permutation matrices with inversion number p and permutation TSSCPP boolean triangles of order n with p zeros.*



The injection φ in Theorem 1.1 extends this bijection under the mild transformation of flipping the resulting matrix vertically (or reversing the one-line notation of the permutation). Thus, in Theorem 1.1 we instead map the TSSCPP above, to the vertical reflection of the matrix shown above.

2.4 Bumpless pipe dreams

In this subsection, we define bumpless pipe dreams and describe the bijection with alternating sign matrices.

Definition 2.10. A **bumpless pipe dream** [10] of size n is a tiling of an $n \times n$ grid of squares by the following six types of tiles: $\begin{smallmatrix} \oplus \\ \ominus \end{smallmatrix}$, $\begin{smallmatrix} \ominus \\ \oplus \end{smallmatrix}$, $\begin{smallmatrix} \oplus \\ \oplus \end{smallmatrix}$, $\begin{smallmatrix} \ominus \\ \ominus \end{smallmatrix}$, $\begin{smallmatrix} \oplus \\ \ominus \end{smallmatrix}$, $\begin{smallmatrix} \ominus \\ \oplus \end{smallmatrix}$, such that n pipes traveling from the south border to the east border are formed. We denote the set of bumpless pipe dreams of size n as $\text{BPD}(n)$. We say a bumpless pipe dream is **reduced** if no two pipes cross twice. We associate a permutation to each reduced bumpless pipe dream by labeling the pipes $1, \dots, n$ from left to right on the south border and read off the pipe labels from top to bottom on the east border. Let $\text{BPD}^{\text{red}}(\pi)$ denote the set of all reduced bumpless pipe dreams with permutation π .

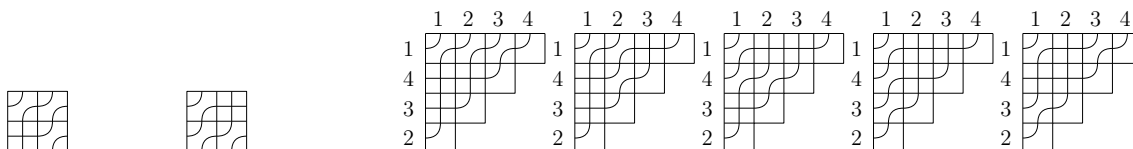
There is a natural bijection between $\text{BPD}(n)$ and $\text{ASM}(n)$, as described in [15]. To obtain an ASM from a BPD we simply replace each $\begin{smallmatrix} \oplus \\ \ominus \end{smallmatrix}$ with a 1, each $\begin{smallmatrix} \ominus \\ \oplus \end{smallmatrix}$ with a -1 , and the rest with 0s. For the inverse map it is not difficult to see the positions of $\begin{smallmatrix} \oplus \\ \oplus \end{smallmatrix}$ and $\begin{smallmatrix} \ominus \\ \ominus \end{smallmatrix}$ uniquely determine a bumpless pipe dream. See p.3 for an example.

2.5 Pipe dreams

In this subsection, we define pipe dreams and bounded compatible sequences and describe the bijection between them.

Definition 2.11. A **pipe dream** [3] of size n is a tiling of a square $n \times n$ grid of squares with two kinds of tiles, the cross-tile $\begin{smallmatrix} \oplus \\ \oplus \end{smallmatrix}$ and elbow-tile $\begin{smallmatrix} \oplus \\ \ominus \end{smallmatrix}$, such that the positions on or below the main (anti)diagonal only consist of elbow-tiles. We think of a pipe dream as n pipes, labelled $1, \dots, n$ traveling from the north border and exiting from the west border. We denote the set of pipe dreams of size n as $\text{PD}(n)$. We say a pipe dream is **reduced** if no two pipes cross twice. We associate a permutation to each reduced pipe dream by reading from top to bottom the labels of each pipe on the west border of the pipe dream. Let $\text{PD}^{\text{red}}(\pi)$ denote the set of reduced pipe dreams with permutation π .

The set of pipe dreams for a fixed permutations are connected by *chute* and *ladder* moves. For precise definitions see [3]. When a ladder (or chute) move is bounded by a 2×2 square, we call this move a **simple slide**, as shown below, left.



Above, right shows the set of pipe dreams $\text{PD}^{\text{red}}(1432)$. The first four pipe dreams are connected by simple slides; the fifth is not.

Definition 2.12. A **bounded compatible sequence** [5] is a pair $(\mathbf{a}, \mathbf{r})$ where $\mathbf{a} = (a_1, \dots, a_\ell)$ and $\mathbf{r} = (r_1, \dots, r_\ell)$ are words of positive integers, satisfying the following conditions:

- (a) $r_1 \geq r_2 \geq \dots \geq r_\ell$,
- (b) $a_i \geq r_i$ for all $1 \leq i \leq \ell$,
- (c) $r_i > r_{i+1}$ if $a_i \geq a_{i+1}$.

There is a simple bijection between $\text{PD}(n)$ and the set of all bounded compatible sequences where $a_i < n$ for each i , see [3]. Given a bounded compatible sequence $(\mathbf{a}, \mathbf{r})$, we may construct a pipe dream by putting a cross-tile at position $(r_i, a_i + 1 - r_i)$ for each $1 \leq i \leq \ell$ and fill the remaining positions with elbow-tiles. Conversely, given a pipe dream, we may construct a bounded compatible sequence as follows: scan the pipe dream from bottom to top and within each row left to right, and whenever we encounter a cross-tile at position (r, c) we append $(r + c - 1, r)$ to the compatible sequence. For example, the corresponding bounded compatible sequences for the pipe dreams in $\text{PD}^{\text{red}}(1432)$ on p.6 are as follows; the vector $\mathbf{a}$ is recorded in the top row and $\mathbf{r}$ in the bottom row.

$$\begin{pmatrix} 3 & 2 & 3 \\ 3 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 3 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 2 & 3 \\ 2 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 3 & 2 \\ 2 & 2 & 1 \end{pmatrix}.$$

2.6 Reduced BPD-PD bijection

Both reduced pipe dreams and reduced bumpless pipe dreams give combinatorial formulas for Schubert polynomials $\mathfrak{S}_\pi$, $\pi \in S_\infty$ which are important polynomials in the study of Schubert calculus [3, 10]. Explicitly,

$$\mathfrak{S}_\pi = \sum_{D \in \text{PD}^{\text{red}}(\pi)} \prod_{(r,c) \in \text{cross}(D)} x_r = \sum_{D \in \text{BPD}^{\text{red}}(\pi)} \prod_{(r,c) \in \text{blank}(D)} x_r.$$

For this reason, there exists a weight-preserving bijection between $\text{PD}^{\text{red}}(\pi)$ and $\text{BPD}^{\text{red}}(\pi)$, where the weight of a PD or BPD is its monomial contribution to the Schubert polynomial indexed by its permutation.

In [8], such an explicit direct bijection $\varphi : \text{BPD}^{\text{red}}(\pi) \rightarrow \text{PD}^{\text{red}}(\pi)$ is given using an iterative algorithm. To find the image of a BPD under φ , the algorithm computes in each iteration the position of one crossing in the corresponding PD. For a detailed description of this process, see [8, Definition 3.1]. For explicit examples, see [8, Example 3.4]. This bijection is weight-preserving; in particular, for $D \in \text{BPD}^{\text{red}}(\pi)$, the number of blank tiles in row k equals the number of cross-tiles in row k of $\varphi(D)$.

Because the bijection is weight-preserving and there is a unique lowest weight monomial that corresponds to the Lehmer code of the permutation in each Schubert polynomial, the permutation BPD is mapped to the **bottom pipe dream**, the unique pipe dream with all crosses left-justified.

3 Characterizing TSSCPP as pseudo-Yamanouchi pipe dreams

This section focuses on our first theorem: a characterization of TSSCPP as a subset of all (reduced and non-reduced) pipe dreams.

3.1 Mapping TSSCPP into pipe dreams

Recall the bijection of [Proposition 2.7](#) from TSSCPP to the TSSCPP boolean triangles of [Definition 2.6](#). As TSSCPP boolean triangles are triangular arrays with entries in $\{0, 1\}$, we can transform them to pipe dreams (reduced and non-reduced), since these are triangular arrays of tiles with two choices for each spot. There are several possibilities for how to do this; we choose to correspond each 1 to a cross-tile $\boxplus$ and each 0 to an elbow-tile $\boxminus$. There are also several choices for orientation of the triangle. We set the following convention.

Given a TSSCPP boolean triangle b of order n , create a triangular array $y_{i,j}$, $1 \leq i \leq n-1$, $1 \leq j \leq n-i$ of zeros and ones where $y_{i,j} = b_{n-i,i+j-1}$. That is, rotate b counterclockwise and justify to the corner.

$$\begin{array}{cccccccccc}
 b_{n-1,1} & b_{n-1,2} & b_{n-1,3} & \cdots & b_{n-1,n-1} & y_{1,1} & y_{1,2} & y_{1,3} & \cdots & y_{1,n-1} \\
 b_{n-2,2} & b_{n-2,3} & \cdots & b_{n-2,n-1} & & y_{2,1} & y_{2,2} & \cdots & y_{2,n-2} & \\
 b_{n-3,3} & \cdots & b_{n-3,n-1} & & & y_{3,1} & \cdots & y_{3,n-3} & & \\
 & \ddots & & & & & \ddots & & & \\
 b_{1,n-1} & & & & & y_{n-1,1} & & & &
 \end{array}$$

The inequality of [Definition 2.6](#) translates to the following:

$$1 + \sum_{k=1}^i y_{j-k,k} \geq \sum_{k=1}^{i+1} y_{j-k+1,k} \quad \text{for all } 1 \leq i < j \leq n-1.$$

Now we turn each 1 into a cross-tile $\boxplus$ and each 0 into an elbow-tile $\boxminus$. We call the pipe dreams that lie in this image the TSSCPP pipe dreams. See [Figure 2](#) for an example. Note that permutation TSSCPP boolean triangles have weakly decreasing rows; this corresponds to left-justified crosses in the associated pipe dream.

3.2 A Yamanouchi-like condition on bounded compatible sequences

In this subsection, we prove [Theorem 3.3](#) characterizing TSSCPP pipe dreams. We also prove [Lemma 3.2](#), which will be used in [Section 4](#).

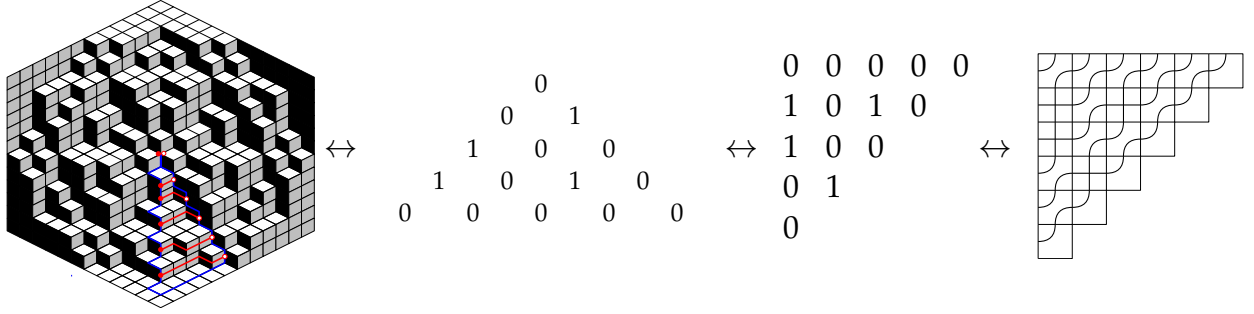


Figure 2: An example of transforming a TSSCPP to a pipe dream.

Definition 3.1. Given a bounded compatible sequence $(\mathbf{a}, \mathbf{r}) = ((a_1, \dots, a_\ell), (r_1, \dots, r_\ell))$, define $\text{count}(k, j)(\mathbf{a})$ (or $\text{count}(k, j)$ when $\mathbf{a}$ is understood) to be the number of j that appear in $a_1, \dots, a_k$. We say that $(\mathbf{a}, \mathbf{r})$ is **pseudo-Yamanouchi** if for all $1 \leq k \leq \ell$, $1 \leq j \leq n - 2$, $1 + \text{count}(k, j) \geq \text{count}(k, j + 1)$. We also say that a pipe dream is pseudo-Yamanouchi if its corresponding bounded compatible sequence is so.

Lemma 3.2. For any π , the bottom pipe dream is pseudo-Yamanouchi.

Proof. The bottom pipe dream is the unique pipe dream of π with all left-justified cross-tiles. Thus the bounded compatible sequence $(\mathbf{a}, \mathbf{r})$ is either empty or $\mathbf{a}$ is made up of increasing runs such that it can be written for some $m \geq 1$ as

$$\mathbf{a} = (j_1, j_1 + 1, \dots, j_1^* - 1, j_1^*, j_2, j_2 + 1, \dots, j_2^* - 1, j_2^*, \dots, j_m, j_m + 1, \dots, j_m^* - 1, j_m^*)$$

where $j_1 > j_2 > \dots > j_m$ and $j_i^* \geq j_i$ for all $1 \leq i \leq m$. Because $j_1 > j_2 > \dots > j_m$, each increasing run needs to start with a smaller number than the previous.

Suppose $(\mathbf{a}, \mathbf{r})$ is not pseudo-Yamanouchi. Choose the smallest k such that there exists a j for which $1 + \text{count}(k, j) < \text{count}(k, j + 1)$. Since $\text{count}(k, j)$ is a non-negative increasing function of k , it must be that $1 + \text{count}(k - 1, j) = \text{count}(k - 1, j + 1)$ and $a_k = j + 1$, since we chose k to be the smallest value with the property. If we are in row $j + 1$ ($r_k = j + 1$), then this is the first time that $j + 1$ has appeared in $\mathbf{a}$, so $\text{count}(k, j + 1) = 1$ and thus cannot be greater than $1 + \text{count}(k, j)$. If $r_k > j + 1$, then $a_{k-1} = j$, since the cross-tiles are left-justified, and thus $\text{count}(k - 1, j) + 1 = \text{count}(k, j)$. But $1 + \text{count}(k - 1, j) \geq \text{count}(k - 1, j + 1) = \text{count}(k, j + 1) - 1$. So finally, $\text{count}(k, j) \geq \text{count}(k, j + 1) - 1$, which is a contradiction. $\square$

Theorem 3.3. $\text{TSSCPP}(n)$ is in bijection with the set of pseudo-Yamanouchi pipe dreams in $\text{PD}(n)$.

Proof. We identify a TSSCPP with the 0-1 triangular array $(y_{i,j})_{1 \leq i \leq n-1, 1 \leq j \leq n-i}$ satisfying the inequalities $1 + \sum_{k=1}^i y_{j-k,k} \geq \sum_{k=1}^{i+1} y_{j-k+1,k}$ for all $1 \leq i < j \leq n - 1$, as described in

Section 3.1. These inequalities mean the following: for any position (i, j) in the corresponding pipe dream, the number of crosses in the same diagonal as (i, j) at or below row i can be at most one more than the number of crosses in the previous diagonal at or below row i . Therefore it suffices to check this property when (i, j) is a cross to decide whether the pipe dream is TSSCPP.

Now suppose (a, r) is an entry of a pseudo-Yamanouchi compatible sequence. Then by the definition of the reading order, all crosses that appear at or below row r in the $(a - 1)$ st diagonal of the corresponding pipe dream appear before (a, r) in the compatible sequence. Therefore the inequality for the $(r, a - r + 1)$ position is implied by the pseudo-Yamanouchi property. The converse is true by a similar argument. $\square$

4 Main result

In this section, we prove our main result, [Theorem 1.1](#). The proof uses the following theorem and lemmas, the first of which is due to Yibo Gao. We will need the following terminology. A permutation π **avoids** a permutation π' if there is no subsequence of π having the same relative order as π' .

Theorem 4.1 ([7, Theorem 4.1]). *If $\pi \in S_n$ avoids 1432, then any two reduced pipe dreams of π are connected by simple slides.*

Lemma 4.2. *Suppose $D \in \text{PD}(n)$ is pseudo-Yamanouchi and $D' \in \text{PD}(n)$ is related to D by a simple slide. Then D' is pseudo-Yamanouchi.*

Proof. Suppose $D \in \text{PD}(n)$ is pseudo-Yamanouchi. Let $(\mathbf{a}, \mathbf{r}) = ((a_1, \dots, a_\ell), (r_1, \dots, r_\ell))$ be its associated bounded compatible sequence. Suppose for some $1 < i < n$, D has a $\boxplus$ tile at position $(r_i, a_i + 1 - r_i)$ and no $\boxplus$ tiles at positions $(r_i, a_i + 2 - r_i)$, $(r_{i-1}, a_i - r_i)$, or $(r_{i-1}, a_i + 1 - r_i)$. Then a simple slide may be applied to D , resulting in another pipe dream D' with $\boxplus$ tile at position $(r_{i-1}, a_i + 1 - r_i)$ and no $\boxplus$ tiles at positions $(r_i, a_i + 1 - r_i)$, $(r_i, a_i + 2 - r_i)$, or $(r_{i-1}, a_i - r_i)$. That is, the simple slide moves the $\boxplus$ tile up one unit and to the right one unit and there were no other $\boxplus$ tiles in these intermediate squares. This preserves the diagonal but decrements the row index, creating a new bounded compatible sequence $(\mathbf{a}', \mathbf{r}') = ((a'_1, \dots, a'_\ell), (r'_1, \dots, r'_\ell))$ such that

$$a'_k = \begin{cases} a_k & k < i \\ a_{k+1} & i \leq k < m \\ a_i & k = m \\ a_k & m < k \leq \ell \end{cases} \quad (4.1)$$

where $m > i$ is uniquely chosen so that $(\mathbf{a}', \mathbf{r}')$ satisfies the conditions of a bounded compatible sequence. Let $\tilde{j} := a_i$. So $\mathbf{a}$ and $\mathbf{a}'$ differ only in that $\tilde{j}$ has slid to the right from index i to m .

Recall $\text{count}(k, j)(\mathbf{a})$ denotes the number of j that appear in $a_1, \dots, a_k$. By [Theorem 3.3](#), $(\mathbf{a}, \mathbf{r})$ is pseudo-Yamanouchi, so for all $1 \leq k \leq \ell$, $1 \leq j \leq n-2$, $1 + \text{count}(k, j)(\mathbf{a}) \geq \text{count}(k, j+1)(\mathbf{a})$. We need only check that $(\mathbf{a}', \mathbf{r}')$ is also pseudo-Yamanouchi. The only values of j we need to consider are $\tilde{j}-1$, $\tilde{j}$, and $\tilde{j}+1$.

By (4.1), $\text{count}(k, j)(\mathbf{a}') = \text{count}(k, j)(\mathbf{a})$ for all values of j when $k < i$ or $k \geq m$. Thus we need only check the pseudo-Yamanouchi inequality for k in the range $i \leq k < m$.

Suppose $i \leq k < m$. Since $(\mathbf{a}, \mathbf{r})$ and $(\mathbf{a}', \mathbf{r}')$ are related by a simple slide, we know there is no cross in D at position $(r_{i-1}, a_i - r_i)$. That is, $a_{m-1} \neq \tilde{j}-1$. Furthermore, $a_k \neq \tilde{j}-1$ for all $i \leq k < m$. Thus $\text{count}(k, \tilde{j}-1)(\mathbf{a}') = \text{count}(k, \tilde{j}-1)(\mathbf{a})$ for all $i \leq k < m$ while $\text{count}(k, \tilde{j})(\mathbf{a}') = \text{count}(k, \tilde{j})(\mathbf{a}) - 1$ in this same range. So

$$\text{count}(k, \tilde{j}-1)(\mathbf{a}') = \text{count}(k, \tilde{j}-1)(\mathbf{a}) \geq \text{count}(k, \tilde{j})(\mathbf{a}) - 1 = \text{count}(k, \tilde{j})(\mathbf{a}').$$

So the pseudo-Yamanouchi condition is more than satisfied when comparing diagonals $\tilde{j}-1$ and $\tilde{j}$.

Since $(\mathbf{a}, \mathbf{r})$ and $(\mathbf{a}', \mathbf{r}')$ are related by a simple slide, we also know there is no cross in D at position $(r_i, a_i + 2 - r_i)$. That is, $a_{i+1} \neq \tilde{j}+1$. Furthermore, $a_k \neq \tilde{j}+1$ for all $i \leq k < m$. Thus $\text{count}(k, \tilde{j}+1)(\mathbf{a}') = \text{count}(k, \tilde{j}+1)(\mathbf{a})$ for all $i \leq k < m$ while $\text{count}(k, \tilde{j})(\mathbf{a}') = \text{count}(k, \tilde{j})(\mathbf{a}) - 1$ in this same range. So

$$1 + \text{count}(k, \tilde{j})(\mathbf{a}') = 1 + \text{count}(k, \tilde{j})(\mathbf{a}) \geq \text{count}(k, \tilde{j}+1)(\mathbf{a}) = \text{count}(k, \tilde{j}+1)(\mathbf{a}').$$

Thus, the pseudo-Yamanouchi condition is satisfied on diagonals $\tilde{j}$ and $\tilde{j}+1$.

Therefore, $(\mathbf{a}', \mathbf{r}')$ is pseudo-Yamanouchi, implying D' is pseudo-Yamanouchi. $\square$

Lemma 4.3. *If $\pi \in S_n$ avoids 1432, then all reduced pipe dreams of π are pseudo-Yamanouchi.*

Proof. Choose $\pi \in S_n$ that avoids 1432. Using the previous two lemmas, we know that simple slides preserve the pseudo-Yamanouchi property and that all reduced pipe dreams in $\text{PD}^{\text{red}}(\pi)$ are connected by simple slides. So we need only show one reduced pipe dream is pseudo-Yamanouchi, and then all of them are. By [Lemma 3.2](#), the bottom (permutation) pipe dream is pseudo-Yamanouchi. Thus the lemma is proved. $\square$

Proof of Theorem 1.1. Let $\pi \in S_n$. The explicit bijection $\varphi : \text{PD}^{\text{red}}(\pi) \rightarrow \text{BPD}^{\text{red}}(\pi)$ of [8] discussed in [Section 2.6](#) is weight-preserving; in particular, for $D \in \text{PD}^{\text{red}}(\pi)$, the number of cross-tiles in row k equals the number of blank tiles in row k of $\varphi(D)$. By [Theorem 3.3](#), TSSCPP are characterized as the set of pseudo-Yamanouchi pipe dreams in $\text{PD}(n)$. Thus whenever such pipe dreams are reduced, φ produces a BPD with the same weight. Thus we have an injection $\varphi : \text{TSSCPP}^{\text{red}}(\pi) \hookrightarrow \text{ASM}^{\text{red}}(\pi)$ given by transforming the TSSCPP to its corresponding reduced pipe dream, mapping it to a reduced BPD using φ , and then transforming to an ASM using the bijection described in [Section 2.4](#).

Suppose π avoids 1432. Then by [Lemma 4.3](#), all pipe dreams in $\text{PD}^{\text{red}}(\pi)$ are pseudo-Yamanouchi, so $\text{TSSCPP}^{\text{red}}(\pi)$ is in bijection with $\text{PD}^{\text{red}}(\pi)$. So the above injection is a bijection between $\text{TSSCPP}^{\text{red}}(\pi)$ and $\text{ASM}^{\text{red}}(\pi)$. $\square$

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An embedding of the skein action on set partitions into the skein action on matchings

Jesse Kim^{*1}

¹*Department of Mathematics, University of California San Diego, CA*

Abstract. Rhoades defined a skein action of the symmetric group on noncrossing set partitions which generalized an action of the symmetric group on matchings. The $\mathfrak{S}_n$ -action on matchings is made possible via the Ptolemy relation, while the action on set partitions is defined in terms of a set of skein relations that generalize the Ptolemy relation. The skein action on noncrossing set partitions has seen applications to coinvariant theory and coordinate rings of partial flag varieties. We will show how Rhoades' $\mathfrak{S}_n$ -module can be embedded into the $\mathfrak{S}_n$ -module generated by matchings, thereby explaining how Rhoades' generalized skein relations all arise from the Ptolemy relation.

1 Introduction

This extended abstract concerns an action of $\mathfrak{S}_n$ on the vector space spanned by the set of noncrossing set partitions of $n : \{1, 2, \dots, n\}$ first defined by Rhoades [6]. This action is defined in terms of three *skein relations*, the simplest of which is the *Ptolemy relation* shown below.

$$\begin{array}{c} \bullet & & \bullet \\ & \diagdown & / \\ & \bullet & \\ & / & \diagdown \\ \bullet & & \bullet \end{array} \mapsto \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \begin{array}{c} \bullet & \bullet \\ \hline \bullet & \bullet \end{array}$$

The original motivation for defining this action was to give an algebraic proof of certain cyclic sieving results on noncrossing set partitions, first proven by Reiner, Stanton, and White [5] and Pechenik [4] via direct enumeration. Rhoades' action has since been found within coinvariant rings and coordinate rings of certain partial flag varieties [1, 3], strengthening the claim that it is an action worth studying in its own right. Rhoades' action generalizes a similar action on the vector space with basis given by noncrossing matchings. In this extended abstract, we will show how Rhoades' action can be found within the action on noncrossing matchings and thereby explain how the three skein relations all arise from the Ptolemy relation.

^{*}jykim@ucsd.edu.

More precisely, A *matching* on n is a collection of disjoint size-two subsets of n . Let M_n denote the set of all matchings on n . The symmetric group $\mathfrak{S}_n$ acts naturally on M_n as follows. If $\sigma \in \mathfrak{S}_n$ and $m = \{\{a_1, b_1\}, \dots, \{a_k, b_k\}\}$ is a matching, then

$$\sigma \star m = \{\{\sigma a_1, \sigma b_1\}, \dots, \{\sigma a_k, \sigma b_k\}\}. \quad (1.1)$$

We can extend this action to an action on $\mathbb{C} M_n$, the $\mathbb{C}$ -vector space with basis given by matchings of n . For our purposes, it will be more useful to consider a sign-twisted version of this action, where

$$\sigma \circ m = \text{sign } \sigma \{\{\sigma a_1, \sigma b_1\}, \dots, \{\sigma a_k, \sigma b_k\}\}. \quad (1.2)$$

A matching is *noncrossing* if it does not contain two subsets $\{a, c\}$ and $\{b, d\}$ with $a < b < c < d$. The action on matchings does not descend to an action on NCM_n , the set of all noncrossing matchings of n , since permuting elements in a noncrossing matching could introduce crossings. However, we can linearize and define an action on $\mathbb{C} NCM_n$, the $\mathbb{C}$ -vector space with basis given by noncrossing matchings of n . For any noncrossing matching m and adjacent transposition $s_i = (i, i+1)$, define

$$s_i \cdot m = \begin{cases} s_i \circ m & \text{if } s_i \circ m \text{ is noncrossing} \\ m & \text{otherwise} \end{cases}. \quad (1.3)$$

Here $\circ$ denotes the action on all matchings and m' is the matching where the subsets of m containing i and $i+1$, call them $\{i, a\}$ and $\{i+1, b\}$ have been replaced with $\{i, i+1\}$ and $\{a, b\}$ and all other subsets remain the same. In other words, $s_i \circ m$, m , and m' form a trio of matchings that differ only in a Ptolemy relation. There exists an $\mathfrak{S}_n$ -equivariant linear projection $p_M : \mathbb{C} M_n \rightarrow \mathbb{C} NCM_n$ given for any matching m by

$$m \mapsto w^{-1} \cdot w \circ m, \quad (1.4)$$

where w is any permutation for which $w \circ m$ is noncrossing. The kernel of this projection is spanned by sums of matchings which differ only by a Ptolemy relation, i.e.

$$\begin{aligned} & \{\{a_1, a_2\}, \{a_3, a_4\}, \{a_5, a_6\}, \dots, \{a_{2k-1}, a_{2k}\}\} \\ & \quad \{\{a_1, a_3\}, \{a_2, a_4\}, \{a_5, a_6\}, \dots, \{a_{2k-1}, a_{2k}\}\} \\ & \quad \{\{a_1, a_4\}, \{a_2, a_3\}, \{a_5, a_6\}, \dots, \{a_{2k-1}, a_{2k}\}\} \end{aligned} \quad (1.5)$$

for any distinct $a_1, \dots, a_{2k} \in n$. This projection can be thought of as a way to “resolve” crossings in a matching and obtain a sum of noncrossing matchings.

Analogously, let Π_n denote the family of all set partitions of n , and let $\mathbb{C} \Pi_n$ be the $\mathbb{C}$ -vector space with basis given by Π_n . We can define an action of $\mathfrak{S}_n$ on $\mathbb{C} \Pi_n$ analogous to the action on $\mathbb{C} M_n$. A set partition is *noncrossing* if there do not exist

distinct blocks A and B and elements $a, c \in A, b, d \in B$ with $a < b < c < d$. Let $\text{NCP } n$ denote the set of all noncrossing set partitions, and let $\mathbb{C} \text{NCP } n$ be the corresponding vector space. Rhoades [6] defined an action of $\mathfrak{S}_n$ on $\mathbb{C} \text{NCP } n$ as follows. For any noncrossing set partition π and adjacent transposition s_i ,

$$s_i \cdot \pi = \begin{cases} -\pi & i \text{ and } i+1 \text{ are in the same block of } \pi \\ -\pi' & \text{at least one of } i \text{ and } i+1 \text{ is in a singleton block of } \pi \\ \sigma \pi' & i \text{ and } i+1 \text{ are in different size 2 or larger blocks of } \pi \end{cases}$$

where π' is the set partition obtained by swapping which blocks i and $i+1$ are in, and σ is defined for any almost-noncrossing (i.e. the crossing can be removed by a single adjacent transposition) partition π by $\sigma \pi = \pi - \pi_2 - \pi_3 - \pi_4$ where, if the crossing blocks in σ are $\{i, a_1, \dots, a_k\}$ and $\{i+1, b_1, \dots, b_l\}$, then π_2, π_3 and π_4 are obtained from π by replacing these blocks with

- $\{i, i+1\}$ and $\{a_1, \dots, a_k, b_1, \dots, b_l\}$ for π_2
- $\{i, i+1, a_1, \dots, a_k\}$ and $\{b_1, \dots, b_l\}$ for π_3
- $\{i, i+1, b_1, \dots, b_l\}$ and $\{a_1, \dots, a_k\}$ for π_4

when $k, l \geq 2$. If $k = 1$ then $\pi_4 = 0$ instead and if $l = 1$ then $\pi_3 = 0$ instead. The sum of partitions given by $\sigma \pi$ is best described with a picture, see Figure 1. The three possibilities (depending on whether $k, l \geq 2$) are the three skein relations mentioned earlier. A more detailed description of this action can be found in [6]. Rhoades showed that we again have an $\mathfrak{S}_n$ -equivariant linear projection $p_\Pi : \mathbb{C} \Pi n \rightarrow \mathbb{C} \text{NCP } n$ given for any set partition π by

$$\pi \mapsto w^{-1} \cdot w \circ \pi, \quad (1.6)$$

where w is any permutation for which $w \circ \pi$ (here $\circ$ denotes the action of $\mathfrak{S}_n$ on all set partitions) is noncrossing. The kernel of this projection is generated by the set of all elements of the form $w \circ s_i \circ \pi - \sigma \pi$ (the skein relations) for any permutation w and almost noncrossing set partition π .

Rhoades was able to determine the $\mathfrak{S}_n$ -irreducible structure of this module. In particular, the span of all singleton-free noncrossing set partitions with exactly k blocks is an $\mathfrak{S}_n$ -irreducible of shape $(k, k, 1^{n-2k})$, and the span of all noncrossing set partitions with exactly s singletons and exactly k non-singleton blocks is isomorphic to an induction product of $S_{k, k, 1^{n-2k-s}}$ with the sign representation of $\mathfrak{S}_s$. Similarly, if we restrict the action on $\mathbb{C} \text{NCP } n$ to the span of noncrossing perfect matchings, i.e. noncrossing matchings of $2n$ with exactly n pairs, then this action gives a model for the $\mathfrak{S}_n$ -irreducible of shape (n, n) called the SL_2 -web basis. If we instead restrict to the span of noncrossing matchings with exactly k pairs, we get a submodule isomorphic to the

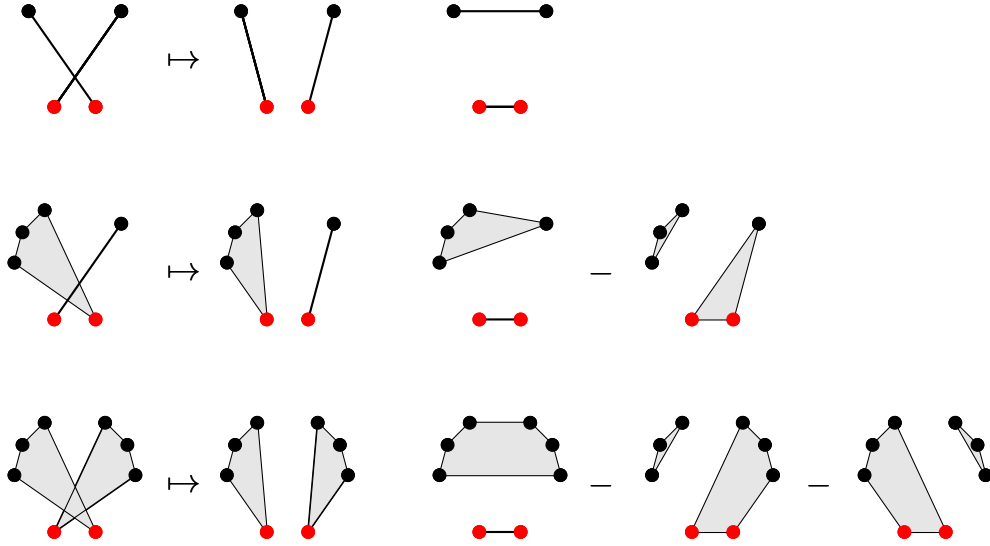


Figure 1: The three skein relations defining the action of $\mathfrak{S}_n$ on $\mathbb{C} \text{NCP } n$. The red vertices are adjacent indices $i, i+1$ and the shaded blocks have at least three elements. The symmetric 3-term relation obtained by reflecting the middle relation is not shown.

induction product of $S^{k,k}$ and a sign representation of $\mathfrak{S}_{n-2k}$. By the Pieri rule, this induction product is a direct sum of three irreducible submodules, exactly one of which is isomorphic to $S^{k,k,1^{n-2k}}$. Therefore there exists an embedding of $\mathbb{C} \text{NCP } n_0$, the span of all singleton-free noncrossing set partitions in $\mathbb{C} \text{NCP } n$, into $\mathbb{C} \text{NCM } n$. If we additionally require that our embedding send set partitions with k blocks to matchings with k pairs, then this embedding is unique. Our main result is to explicitly describes the embedding as follows:

Theorem 1. The linear map $f_n : \mathbb{C} \text{NCP } n_0 \rightarrow \mathbb{C} \text{NCM } n$ defined by

$$f_n(\pi) = \sum_{m \in M_\pi(n)} m$$

is a $\mathfrak{S}_n$ -equivariant embedding of vector spaces. Here $M_\pi(n)$ is defined to be the set of all matchings m in $M(n)$ for which each block of π contains exactly one pair in m .

For an example of this map, if $\pi = \{\{1,2,3\}, \{4,5\}\}$ then

$$f_n(\pi) = \{\{1,2\}, \{4,5\}\} + \{\{1,3\}, \{4,5\}\} + \{\{2,3\}, \{4,5\}\}$$

is a sum of 3 matchings in $\mathbb{C} \text{NCM } n$.

2 The embedding

In order to prove that our map is an embedding, it will be helpful to introduce a multiplicative structure to work with. To do so we will introduce three graded commutative $\mathbb{C}$ -algebras R_n , A_n , and M_n , all with $\mathfrak{S}_n$ -actions. If we forget the multiplicative structure, the underlying $\mathfrak{S}_n$ -modules of R_n , A_n , and M_n will contain a copy of $\mathbb{C} \Pi_n$, $\mathbb{C} M_n$, and $\mathbb{C} NCM_n$ respectively. In the case of M_n , this copy will be all of M_n . The structure of this proof is best explained via a commutative diagram, see Figure 2. We will define a map $h_n \circ \iota_\Pi : \mathbb{C} \Pi_n \rightarrow M_n$, and show that its kernel is equal to the kernel of p_Π . The desired embedding f_n then follows from the first isomorphism theorem.

$$\begin{array}{ccccc}
 & & h_n & & \\
 & \nearrow & & \searrow & \\
 R_n & \xrightarrow{g_n} & A_n & \xrightarrow{q} & M_n \\
 & \uparrow \iota_\Pi & & & \uparrow \sim \\
 & \mathbb{C} M_n & \xrightarrow{p_M} & \mathbb{C} NCM_n & \\
 \uparrow \iota_\Pi & & & & \uparrow f_n \\
 \mathbb{C} \Pi_n & \xrightarrow{p_\Pi} & \mathbb{C} NCP_n & &
 \end{array}$$

Figure 2: A commutative diagram of the maps used in the following proofs. All maps shown are $\mathfrak{S}_n$ -equivariant linear maps. Maps between R_n , A_n , and M_n are also morphisms of $\mathbb{C}$ -algebras. The desired embedding is shown as a dashed arrow.

We begin with the definition of R_n .

Definition 1. Let $n \in \mathbb{N}$. Define R_n to be the unital graded commutative $\mathbb{C}$ -algebra freely generated by nonempty subsets of n where each generator has degree 1. Define a degree-preserving action of $\mathfrak{S}_n$ on R_n by

$$\pi \cdot \{a_1, \dots, a_k\} = \text{sign } \pi \{ \pi a_1, \dots, \pi a_k \}$$

for any permutation $\pi \in \mathfrak{S}_n$ and generator $\{a_1, \dots, a_k\} \in R_n$.

The vector space structure of R_n can be thought of as formal sums of multiset collections of subsets of n by interpreting each product of generators as a multiset collection of the corresponding subsets. It is in this sense that it contains a copy of $\mathbb{C} \Pi_n$, as set partitions of n are particular collections of subsets of n . To be precise, there exists an $\mathfrak{S}_n$ -module embedding $\iota_\Pi : \mathbb{C} \Pi_n \hookrightarrow R_n$, given by sending any singleton-free set partition π to the product of its blocks.

The ring A_n is a subring of R_n designed to model matchings in much the same way which R_n models set partitions. It is defined as follows.

Definition 2. Let $n \in \mathbb{N}$ and define A_n to be the $\mathfrak{S}_n$ -invariant subalgebra of R_n freely generated by the size two subsets of n . The subring A_n is invariant under the $\mathfrak{S}_n$ -action of R_n , and thus inherits a graded $\mathfrak{S}_n$ -action from R_n .

Like R_n , the ring A_n can be thought of as the ring of multiset collections of size-two subsets of n . As matchings are particular collections of size-two subsets of n , we again have an $\mathfrak{S}_n$ -module embedding $\iota_M : \mathbb{C} M_n \hookrightarrow A_n$, given by

$$\{\{a_1, b_1\}, \dots, \{a_k, b_k\}\} \mapsto \{a_1, b_1\} \cdots \{a_k, b_k\}$$

for any matching $\{\{a_1, b_1\}, \dots, \{a_k, b_k\}\}$.

Our final ring, M_n , is defined as a quotient of A_n in the following way.

Definition 3. Define I_n to be the ideal of A_n generated by elements of the following forms

- $\{a, b\} \cdot \{a, b\}$
- $\{a, b\} \cdot \{a, c\}$
- $\{a, b\} \cdot \{c, d\} \quad \{a, c\} \cdot \{b, d\} \quad \{a, d\} \cdot \{b, c\}$

for any distinct $a, b, c, d \in n$. Then I_n is a homogeneous $\mathfrak{S}_n$ -invariant ideal of A_n , so define M_n to be the graded $\mathfrak{S}_n$ -module $M_n : A_n / I_n$. Let $q : A_n \rightarrow M_n$ be the quotient map.

The first two types of elements listed in the definition of I_n serve the purpose of removing collections of size-two subsets which are not actually matchings. The third is the Ptolemy relation used to define the action of $\mathfrak{S}_n$ on $\mathbb{C} NCM_n$, so quotienting by this ideal gives an $\mathfrak{S}_n$ -module isomorphic to $\mathbb{C} NCM_n$, as per the following argument.

Proposition 1. There is an $\mathfrak{S}_n$ -module isomorphism from $\mathbb{C} NCM_n$ to M_n , given by

$$\{\{a_1, b_1\}, \dots, \{a_k, b_k\}\} \mapsto \{a_1, b_1\} \cdots \{a_k, b_k\}$$

for any noncrossing matching $\{\{a_1, b_1\}, \dots, \{a_k, b_k\}\}$.

Proof. Let $q : A_n \rightarrow M_n$ be the quotient map. Consider the map $q \circ \iota_M : \mathbb{C} M_n \rightarrow M_n$. The kernel of $q \circ \iota_M$ is spanned by the elements of $\mathbb{C} M_n$ which are sent under ι_M to a multiple of $\{a, b\} \cdot \{c, d\} \quad \{a, c\} \cdot \{b, d\} \quad \{a, d\} \cdot \{b, c\}$ for some distinct $a, b, c, d \in n$. Therefore the kernel of $q \circ \iota_M$ is equal to the kernel of p_M by 1.5 and the surrounding discussion. The image of $q \circ \iota_M$ is all of M_n . To see this, note that products of generators of A_n form a vector space basis for A_n , and every such basis element is either in the image of ι_M or in I_n . We therefore have

$$\mathbb{C} NCM_n \sim \mathbb{C} M_n / \ker p_M \quad \mathbb{C} M_n / \ker q \circ \iota_M \sim \text{im } q \circ \iota_M = M_n \quad (2.1)$$

where the isomorphism on the left is induced by the map p_M and the isomorphism on the right is induced by the map $q \circ \iota_M$. Composing these isomorphisms gives the stated map. $\square$

The following definition is the key idea behind our main theorem.

Definition 4. Let $n \in \mathbb{N}$. Define the $\mathbb{C}$ -algebra map $g_n : R_n \rightarrow A_n$ by

$$g_n(A) = \sum_{\{a,b\} \subseteq A} \{a,b\}$$

for generators $A \in R_n$. Singleton sets are sent to 0 by g_n . Define $h_n : A_n \rightarrow M_n$ where q is the quotient map $A_n \rightarrow M_n$.

We give the definition in terms of R_n , A_n , and M_n for simplicity and ease of proofs later, but the map we really care about is $h_n \circ \iota_\Pi : \mathbb{C} \Pi_n \rightarrow M_n$. Under this map, a set partition π is sent to the product of its blocks, then each block is sent to the sum of all size-two subsets it contains. After distributing, we get a sum of all ways to pick a size two subset from each block. Composing with the isomorphism between M_n and $\mathbb{C} \text{NCM}_n$ we get the sum of all matchings such that each block of π contains exactly one pair of the matching, as in [Theorem 2](#).

The $\mathfrak{S}_n$ -equivariance of h_n is simple to check. We have the following calculation.

Lemma 1. Let $i, j \geq 2$ and let $p_1, p_2, \dots, p_i$ and $q_1, q_2, \dots, q_j$ be in Π_n . Then the element of R_n

$$\begin{aligned} \kappa_n : & \{p_1, \dots, p_i\} \cdot \{q_1, \dots, q_j\} - \{p_1, \dots, p_{i-1}\} \cdot \{q_1, \dots, q_j, p_i\} \\ & - \{p_1, \dots, p_i, q_j\} \cdot \{q_1, \dots, q_{j-1}\} - \{p_1, \dots, p_{i-1}, q_j\} \cdot \{q_1, \dots, q_{j-1}, p_i\} \\ & - \{p_1, \dots, p_{i-1}, q_1, \dots, q_{j-1}\} \cdot \{p_i, q_j\} \end{aligned} \quad (2.2)$$

lies in the kernel of h_n .

When $i, j > 2$, the element κ_n corresponds to the five-term skein relation depicted in Figure 1. If either i equals 2, then $\{p_1, \dots, p_{i-1}\} = \{p_1\}$ is a one element set and therefore sent to 0 by h_n , removing the term containing $\{p_1\}$ corresponds to the four-term skein relation depicted in Figure 1. Similarly, if j equals 2 or i and j both equal two, removing the terms in κ_n which are individually sent to 0 corresponds to the four or three-term skein relation depicted in Figure 1.

Proof. The proof of this lemma is a mostly straightforward calculation from the definition of h_n . $h_n(\kappa_n)$ can be shown to be a sum of relations in I_n . $\square$

The preceding lemma allows for the identification of the kernel of $h \circ \iota_\Pi$ and p_Π mentioned in the beginning of this section.

Proposition 2. The kernel of $h \circ \iota_\Pi$ is generated by the set of all elements of the form

$$w \circ s_i \circ \pi - \sigma \circ \pi$$

i.e., the skein relations, for any permutation w and singleton-free almost noncrossing set partition π .

Proof. By Lemma 1, all such elements lie in the kernel. Since restricting $\mathbb{C} \Pi_{n-0}$ to noncrossing set partitions with exactly k blocks gives an irreducible submodule [6, Proposition 5.2], and each such irreducible is sent to degree k in M_n , it suffices to demonstrate that each irreducible is not sent identically to 0. But for each noncrossing set partition $\pi \in \Pi_{n-0}$, since π does not have singleton blocks, $h_n \circ \iota_\Pi \pi$ is a nonvanishing sum of noncrossing matchings of n , which form a basis for M_n by Proposition 1, and is therefore nonzero. $\square$

We can now prove our main result.

Theorem 2. *The linear map $f_n : \mathbb{C} \text{NCP}_{n-0} \rightarrow \mathbb{C} \text{NCM}_n$ defined by*

$$f_n \pi = \sum_{m \in M_\pi} m$$

is a $\mathfrak{S}_n$ -equivariant embedding of vector spaces. Here M_π is defined to be the set of all matchings m in M_n for which each block of π contains exactly one pair in m .

Proof. By Proposition 2, the kernel of $h \circ \iota_\Pi$ is equal to the kernel of p_Π . So we have

$$\begin{aligned} \mathbb{C} \text{NCP}_{n-0} &\sim \mathbb{C} \Pi_{n-0} / \ker p_\Pi \\ &\sim \mathbb{C} \Pi_{n-0} / \ker h \circ \iota_\Pi \sim \text{im } h \circ \iota_\Pi \subset M_n \sim \mathbb{C} \text{NCM}_n \end{aligned} \quad (2.3)$$

where the isomorphism on the left is induced by p_Π and the middle isomorphism is induced by $h \circ \iota_\Pi$, and the isomorphism on the right is as defined in Proposition 1. Chasing these isomorphisms and inclusions results in the map f_n . $\square$

3 The image

We have an embedding $f_n : \mathbb{C} \text{NCP}_{n-0} \hookrightarrow \mathbb{C} \text{NCM}_n$, so it is a natural question to ask for a description of the image of f_n within $\mathbb{C} \text{NCM}_n$. Via the commutative diagram in Figure 2, we have an isomorphism of images

$$\text{im } h_n \sim \text{im } f_n. \quad (3.1)$$

So it is equivalent to describe the image of h_n , and the multiplicative structure of M_n will make describing the image of h_n easier. It turns out to have a simple description, the proof of which requires a few lemmas.

Lemma 2. *Let $A \subseteq n$. Then $h_n A^2 = 0$.*

Proof. Applying the definition of h_n gives

$$h_n(A)^2 = \sum_{\substack{a,b \in n \\ a/b}} \sum_{\substack{c,d \in n \\ c/d}} \{a,b\} \cdot \{c,d\} \quad (3.2)$$

Using the defining relation of M_n that

$$\{a,b\} \cdot \{a,c\} = 0$$

we have

$$\sum_{\substack{a,b \in n \\ a/b}} \sum_{\substack{c,d \in n \\ c/d}} \{a,b\} \cdot \{c,d\} = \frac{1}{3} \sum_{\substack{a,b,c,d \in n \\ a,b,c,d \text{ distinct}}} \{a,b\} \cdot \{c,d\} - \{a,c\} \cdot \{b,d\} - \{a,d\} \cdot \{b,c\}. \quad (3.3)$$

The right hand side of the above equation equals 0 because

$$\{a,b\} \cdot \{c,d\} - \{a,c\} \cdot \{b,d\} - \{a,d\} \cdot \{b,c\} = 0$$

for any distinct $a,b,c,d \in n$. □

The proof of the following two lemmas are similar.

Lemma 3. Let A, B be disjoint subsets of n . Then

$$h_n(A) \cdot \left(\sum_{\substack{a \in A \\ b \in B}} \{a,b\} \right) = 0.$$

Lemma 4. Let $B_1, \dots, B_k$ be the blocks of a singleton free set partition of n . Then

$$h_n \left(\prod_{i=1}^k B_i \right) = h_n \left(n \cdot \prod_{i=1}^{k-1} B_i \right)$$

We can now describe the image of h_n .

Theorem 3. Let H_n be the ideal of M_n generated by $h_n(n)$. Then

$$\text{im } h_n = H_n.$$

Proof. It is immediate from [Lemma 4](#) that the image of h_n is contained in H_n , so it suffices to show that the vector space dimension

$$\dim H_n \leq \dim \text{im } h_n = \dim \text{im } f_n = \#NCP(n, 0) \quad (3.4)$$

To bound the dimension of H_n , computation shows that for any $a \in n$,

$$h_n \cdot \{1, a\} = h_n - \left(\sum_{\substack{b \in n \\ b \neq a, 1}} \{a, b\} \right).$$

Therefore H_n is spanned by elements of the form

$$h_n \cdot \{a_1, a_2\} \cdot \{a_3, a_4\} \cdots \{a_{2k-1}, a_{2k}\}$$

where the sets $\{a_1, a_2\}, \dots, \{a_{2k-1}, a_{2k}\}$ form a noncrossing matching of $\{2, \dots, n\}$. These elements are not linearly independent, however. Consider the element $\tilde{f}_n \tilde{\pi}$ of M_n given by

$$\tilde{f}_n \tilde{\pi} := \prod_{B \in \tilde{\pi}} h_n \cdot B$$

for any singleton free noncrossing set partition $\tilde{\pi}$ of $\{2, \dots, n\}$ (i.e. the “image of f_n ” if such a set partition were in the domain of f_n). Computation shows that

$$h_n \cdot \tilde{f}_n \tilde{\pi} = 0$$

The collection of $\tilde{f}_n \tilde{\pi}$ for singleton-free noncrossing set partitions π of $\{2, \dots, n\}$ is linearly independent. To see this, note that any linear relation among the $\tilde{f}_n \tilde{\pi}$ would also be a linear relation among $f_{n-1} \pi$ where π is the set partition of $n-1$ obtained by decrementing the indices in $\tilde{\pi}$. But f_{n-1} is an embedding and singleton-free noncrossing set partitions are linearly independent in $\mathbb{C} \text{NCP } n-1_0$. The dimension of H_n is therefore bounded by

$$\begin{aligned} \dim H_n &\leq \#\{\text{noncrossing matchings of } \{2, \dots, n\}\} \\ &\quad - \#\{\text{singleton-free noncrossing set partitions of } \{2, \dots, n\}\}. \end{aligned} \quad (3.5)$$

Noncrossing matchings of $\{2, \dots, n\}$ are in bijection with noncrossing set partitions of n in which only the block containing 1 may be a singleton (though it may be larger). Given a noncrossing set partition, simply take the matching that matches the largest and smallest element of each block not containing 1. Singleton-free noncrossing set partitions of $\{2, \dots, n\}$ are in bijection with set partitions of n in which $\{1\}$ is the unique singleton block. We therefore have

$$\dim H_n \leq \# \text{NCP } n_0$$

as desired. □

4 Future directions

One of the goals motivating this paper is to find new combinatorially nice bases for $\mathfrak{S}_n$ -irreducibles which arise from existing bases in an analogous way to the skein action. More specifically, suppose we have a basis for S^λ which is indexed by certain structures on the set k , where $k = |\lambda|$ (e.g. noncrossing perfect matchings, in the case of this paper). We can create a basis for the induction product of S^λ with a sign representation of $\mathfrak{S}_{n-k}$ indexed by all ways to put a certain structure on a k -element subset of n . The Pieri rule tells us which $\mathfrak{S}_n$ irreducibles this decomposes into. In particular, there will be one copy of $\lambda, 1^{n-k}$. How do we isolate that irreducible?

It is optimistic to think that there will be a method that works in any sort of generality, but perhaps analogs could be found in certain specific cases. For example, an analog might exist for the web basis for $S^{k,k,k}$ introduced by Kuperberg [2]. The web basis consists of planar bipartite graphs embedded in a disk with n boundary vertices all of degree 1, interior vertices are degree 3, all boundary vertices are part of the same bipartition, and no cycles of length less than 6 exist. One potential candidate for a basis for $S^{k,k,k,1^{n-3k}}$ is as follows.

Conjecture 1. *Let A be the set of all planar bipartite graphs embedded in a disk for which the following conditions hold*

- *There are n vertices on the boundary of the disk, and there exists a bipartition in which all of these vertices are in the same part.*
- *Every interior vertex is connected to a boundary vertex.*
- *Every interior vertex in the same bipartition as the boundary vertices is degree 3. These are called negative interior vertices.*
- *Every interior vertex not in the same bipartition as the boundary vertices is degree at least 3. These are called positive interior vertices.*
- *The number of positive interior vertices minus the number of negative interior vertices is exactly k .*
- *No cycles of length less than 6 exist.*

Then $|A|$ is equal to the dimension of $S^{k,k,k,1^{n-3k}}$.

The set A can be thought of as consisting of webs for which the condition of interior vertices being degree 3 has been partially relaxed. The conjecture can be shown to hold for $k = 2$ and any n , as well as $n = 10, k = 3$ via direct enumeration. If the above conjecture is true, it suggests the following question.

Question 1. Does there exist a combinatorially nice action of $\mathfrak{S}_n$ on $\mathbb{C} A$ which creates a $\mathfrak{S}_n$ module isomorphic to $S^{k,k,k,1^{n-3k}}$? If so, what does the unique embedding into $S^{k,k,k}$ induced with a sign representation of $\mathfrak{S}_{n-3k}$ look like?

A positive answer to this question might help elucidate how to apply similar methods more generally.

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Recursions and Proofs in Cataland

Theo Douvropoulos^{*1} and Matthieu Josuat-Vergès²

¹*Department of Mathematics and Statistics, University of Massachusetts, Amherst, USA*

²*IRIF, CNRS, Université Paris-Cité, France*

Abstract. We give the first type-independent proof of the Kreweras-style formulas for the enumeration of noncrossing partitions in a real reflection group W , with respect to parabolic type. This answers a central open question in Coxeter-Catalan combinatorics, originally asked by Athanasiadis-Reiner in 2003, special cases of which have been open even longer. Our proof also covers the m -Fuss version of the problem, as well as similar Loday-style formulas for the refined-by-type enumeration of faces of the m -cluster complex of W . It proceeds by developing a family of combinatorial recursions that completely determine the enumeration and proving their algebraic counterparts.

Keywords: reflection groups, noncrossing partitions, cluster complex, parabolic types

1 Introduction

Some of the most fascinating results about the symmetric group S_n are special cases of theorems that hold for all (finite) Coxeter groups W , or more generally complex reflection groups; this is the world of Coxeter combinatorics. *Coxeter-Catalan* combinatorics in particular study the poset of *W -noncrossing partitions* $NC\ W : 1, c \leq_R$ of elements $w \in W$ that lie below a Coxeter element c under the absolute order $\leq_R$ of W . It is a lattice, it has many applications outside of combinatorics –in particular it encodes $K\ \pi, 1$ spaces for the braid group $B\ W$ – and when $W = S_n$, it is isomorphic to the lattice of noncrossing partitions due to Kreweras which are enumerated by the Catalan numbers.

One of the main open problems in Coxeter-Catalan combinatorics (for more than twenty years [3, §7]) has been to explain, i.e. give case-free proofs (ones that do not rely on the classification of Coxeter groups) for the remarkable product formula (1.1) below for the enumeration of chains in $NC\ W$. In this extended abstract, we present the first such proof, for which we developed a framework of techniques in [10, 12, 9], and which is finally presented at the full version of this paper [11]. In fact, we simultaneously prove these chain-counting formulas *and* analogous product formulas (1.2) that constitute refinements of the f -vectors of cluster complexes; we had proved the latter ones in a case-by-case way in [12]. In both cases the formulas are indexed by a *parabolic type*, which is a W -orbit of flats $X \in \mathcal{L}_W/W$ in the intersection lattice of the reflection arrangement of W , see Section 2.

^{*}douvropoulos@math.umass.edu.

Theorem 1.1. For an irreducible real reflection group W and a type $X \in \mathcal{L}_W/W$, the number $\text{Krew}_{W,X}^{\text{NC}} m$ of length- m chains in $\text{NC } W$ whose first element has type X is given by

$$\text{Krew}_{W,X}^{\text{NC}} m = \frac{\prod_{i=1}^{\dim X} mh \cdot (1 - b_i^X)}{N_X : W_X}, \quad (1.1)$$

and the number $\text{Lod}_{W,X}^{\text{NC}} m$ of faces of type X in the cluster complex $\mathcal{Y}(W, m)$ is given by

$$\text{Lod}_{W,X}^{\text{NC}} m = \frac{\prod_{i=1}^{\dim X} mh \cdot (1 - b_i^X)}{N_X : W_X}, \quad (1.2)$$

where N_X and W_X are respectively the setwise and pointwise stabilizers of X , b_i^X its Orlik-Solomon exponents, and h the Coxeter number of W .

There is a version of the theorem for reducible groups W as well, which is an immediate consequence of it, but we will avoid it here to keep the notation simple. The terminology we chose for these numbers stands for Kreweras and Loday who respectively in [15] and [16] calculated them for the symmetric group $W = S_n$ and for $m = 1$.

There is a natural way to define parking functions associated to m -chains of $\text{NC } W$; they carry a natural W -action and the resulting module (see [2, 19]) is called the *m -Fuss noncrossing parking space* and is given as

$$\text{Park}_W^{\text{NC}} m : \bigoplus_{X \in \mathcal{L}_W/W} \text{Krew}_{W,X}^{\text{NC}} m \cdot \uparrow_{W_X}^W \mathbf{1}.$$

A sibling object to it is the so called *algebraic parking space* $\text{Park}_W^{\text{alg}} m : \mathbb{C}[V] / \Theta$ introduced in [2] as the quotient of the ambient polynomial ring over a system of parameters $\Theta : \theta_1, \dots, \theta_n$ of homogeneous degrees $\deg \theta_i = mh - 1$ that carry the reflection representation of W (the existence of such h.s.o.p. relies on Rouquier's shift functors for rational Cherednik algebras). The numbers given by the product formula (1.1) of Theorem 1.1 are naturally structure coefficients for $\mathbb{C}[V] / \Theta$ so that the theorem can be equivalently phrased as the W -isomorphism between the two parking spaces; that is, we prove that $\text{Park}_W^{\text{NC}} m \sim_W \text{Park}_W^{\text{alg}} m$. This is a big part of a series of open problems, listed as *Parking Space Conjectures* in [2]; our approach however cannot currently deal with the full version of them which includes an extra cyclic structure on the modules.

While this project was in preparation, Galashin et al [14] resolved another long open problem in Coxeter-Catalan combinatorics and defined *rational* noncrossing partitions for real reflection groups and proved enumerative formulas for them. Among their results is a case-free proof of the special case $X = V$ of Theorem 1.1 for Weyl groups. It is unclear at the moment whether their techniques can be extended to our setting; they involve certain character evaluations in the Hecke algebra that are uniformly understood

when it comes to lifts of the Coxeter element (the $X = V$ case) but not (yet) in general. On the other hand, the algebraic recursions we describe in [Proposition 3.4](#) are given essentially in the rational setting; if they are combined with a refined model of rational noncrossing partitions that keeps track of the parabolic type, they may be able to extend the *full* [Theorem 1.1](#) to the rational case.

A comparison of recursions

We prove [Theorem 1.1](#) by expanding the ideas of [7]. There is a natural recursion on the numbers $\text{Krew}_{W,X}^{\text{NC}}(m)$ if one counts length- m chains with respect to the parabolic type of their k -th element. The main ingredient of our proof is to show that the same recursion is satisfied by the right hand side of (1.1). To facilitate this, we will formally define the *algebraic Kreweras numbers* $\text{Krew}_{W,X}^{\text{alg}}(m)$ and the *algebraic Loday numbers* $\text{Lod}_{W,X}^{\text{alg}}(m)$ as

$$\text{Krew}_{W,X}^{\text{alg}}(m) := \frac{\prod_{i=1}^{\dim X} (mh - 1 - b_i^X)}{N(X) : W_X} \quad \text{and} \quad \text{Lod}_{W,X}^{\text{alg}}(m) := \frac{\prod_{i=1}^{\dim X} (mh - 1 - b_i^X)}{N(X) : W_X}, \quad (1.3)$$

where again $N(X)$ and W_X are respectively the setwise and pointwise stabilizers of X , b_i^X its Orlik-Solomon exponents, and h the Coxeter number of W , see [Section 2](#) for details. The name stems from the fact that the numbers $\text{Krew}_{W,X}^{\text{alg}}(m)$ are naturally structure coefficients of the algebraic parking space [2] but we will not use that here.

In the following sections we prove a series of combinatorial recursions that relate the numbers $\text{Krew}_{W,W}^{\text{NC}}(m)$ and $\text{Lod}_{W,X}^{\text{NC}}(m)$ with corresponding Kreweras and Loday numbers for smaller rank groups W' or parabolic types Z of smaller dimension ($\dim Z < \dim X$). We prove a sufficient part of those recursions for the algebraic Kreweras and Loday numbers as well and we show inductively in [Section 6](#) that

$$\text{Krew}_{W,X}^{\text{NC}}(m) = \text{Krew}_{W,X}^{\text{alg}}(m) \quad \text{and} \quad \text{Lod}_{W,X}^{\text{NC}}(m) = \text{Lod}_{W,X}^{\text{alg}}(m), \quad (1.4)$$

for all W, X , and m , which is [Theorem 1.1](#).

The linear term

The recursion on chains that we described above (and that we present in [Section 3](#) in detail) has a significant drawback: assuming knowledge of the chain counts $\text{Krew}_{W',X'}^{\text{NC}}(m)$ for all previously considered groups and parabolic types, it can determine all coefficients of the polynomial (in m) $\text{Krew}_{W,X}^{\text{NC}}(m)$ *apart from its linear term*. To remedy this we build a separate recursion in which we show that the linear term is as prescribed by formula (1.1). This relies on our work in [12] which relates the (combinatorial) Kreweras numbers with the Loday numbers, which themselves have a natural combinatorial recursion

(Proposition 4.1). A technically difficult part of this project is to prove the algebraic version (Lemma 5.2) of the recursion for the Loday numbers; we only need to do it for the linear coefficient but even for only this, the proof requires many non-trivial steps.

In almost all the statements we give sketches of the proof arguments but we try to encompass all the components that go into them. The full version of this project, with complete proofs, will appear in [11].

2 Real reflection groups and their invariants

A real reflection group is a finite subgroup $W \leq \text{GL } V$ of invertible linear transformations of some space $V \sim \mathbb{R}^n$, that is generated by Euclidean reflections. The arrangement of the fixed hyperplanes of the reflections of W is called the *reflection arrangement* $\mathcal{A}_W$ of W and the rank of W is defined as the rank of $\mathcal{A}_W$. We say that W is *irreducible* if it leaves no non-trivial subspace of V invariant. Arbitrary intersections $X : \cap_{i \in I} H_i$ of the reflection hyperplanes $H_i \in \mathcal{A}_W$ are called *flats* and they form the intersection lattice $\mathcal{L}_W$ of W ; their pointwise stabilisers W_X are called *parabolic subgroups*; they act as reflection groups on their *moved spaces* $M : X^\perp$. The group W acts on the lattice $\mathcal{L}_W$ and its orbits $X \in \mathcal{L}_W / W$ form the *parabolic types* of W ; they generalize the notion of *cycle type* in S_n .

The real reflection groups W are precisely the finite Coxeter groups; they come with a system of simple generators $S : \{s_1, \dots, s_n\}$ (where n is the rank of W) that correspond to the reflecting hyperplanes bounding a selected chamber of $\mathcal{A}_W$. The product of the simple generators $c : s_1 \cdots s_n$ and any element conjugate to it will be called a *Coxeter element* of W and its order $h : |c|$ the *Coxeter number* of W .

Any parabolic subgroup W_X is conjugate to a *standard parabolic subgroup* $\langle J \rangle$ generated by a subset $J \subseteq S$. The parabolic type of a subset $J \subseteq S$ is the parabolic type of $\langle J \rangle$. We will make use of the numbers $N_{X,Y}$ which count the number of subsets $I \subseteq S$ of parabolic type X that are contained in a *fixed* subset $J \subseteq S$ of parabolic type Y .

We define the (absolute) *reflection length* $_R w$ of an element $w \in W$ as the smallest number k for which we can write a factorization $w = t_1 \cdots t_k$ in reflections t_i . Then we have a natural order $\leq_R$ on W where $u \leq_R v$ if and only if $_R u \leq_R u^{-1}v \leq_R v$; we call it the *absolute reflection order* of W . We define the *noncrossing partition lattice* $NC \ W$ as the interval $[1, c]_{\leq_R}$ in W below a fixed Coxeter element c under $\leq_R$. Every element $w \in W$ has a parabolic type defined as the W -orbit of its fixed space $V^w \in \mathcal{L}_W / W$; this allows the definition of the combinatorial Kreweras numbers in Theorem 1.1.

There is an associated object to $NC \ W$ called the *m-cluster complex* $\Upsilon \ W, m$; it is a simplicial complex whose vertices are indexed by m -colored almost positive roots [13] and whose faces correspond to sets of pairwise compatible vertices (under a certain relation). In [12] we define a parabolic type to each face F of $\Upsilon \ W, m$; essentially this involves taking the product of the reflections that make up F , in an appropriate order,

and then taking the Kreweras complement of that product. This allows us to define the Loday numbers in [Theorem 1.1](#).

A lot of our proofs –for instance [Lemma 3.3](#), but not the complete [Theorem 1.1](#)– apply also to the wider class of [complex reflection groups](#). These are finite subgroups $W \leq \text{GL } \mathbb{C}^n$ generated by their elements $t \in W$ that satisfy $\text{codim } V^t = 1$, which we then also call *reflections*. We will consider another intermediate family, the [well generated](#) complex reflection groups W which may be generated by exactly $\text{rank } W$ -many of their reflections; it is precisely these groups that have natural analogs of Coxeter elements.

Coxeter numbers and Orlik-Solomon exponents associated to flats

To each flat $X \in \mathcal{L}_W$ we may associate various numerical invariants that play an important role in this paper. The restricted reflection arrangements $\mathcal{A}^X$ are always free [\[18\]](#) and therefore their characteristic polynomials factor with positive integer roots

$$\chi \mathcal{A}^X, t = \prod_{i=1}^{\dim X} (t - b_i^X),$$

which are known as the [Orlik-Solomon exponents](#) b_i^X of X . When $X = V$, the exponents are given as $b_i^V = d_i - 1$, in terms of the [invariant degrees](#) d_i of W .

To each pair of flats $X \supset Z$ we may assign a sequence of $(\dim X - \dim Z)$ -many Coxeter numbers as follows. Assume that $W_Z = W_1 \times \cdots \times W_r$ is the decomposition into irreducibles and that h_i are the Coxeter numbers of the W_i 's. Write also X_i for the intersections of X with the moved spaces M_i of the W_i (i.e. $W_{X_i} = W_i \cap W_X$). Then we write $\{h_i(X, Z) \mid i = 1, \dots, (\dim X - \dim Z)\}$ for the [multiset of Coxeter numbers](#):

$$\{h_i(X, Z)\} : \{ \underbrace{h_1, \dots, h_1}_{\dim X_1 \text{ -times}}, \dots, \underbrace{h_r, \dots, h_r}_{\dim X_r \text{ -times}} \}.$$

Similarly we will need to consider the Orlik-Solomon exponents $b_i^{X,Z}$ of the arrangements $\mathcal{A}_Z^X$; we make sure to order the Coxeter numbers and the OS exponents consistently. A special case will be particularly important. For any hyperplane $Z \in \mathcal{A}^X$, the [relative Coxeter number](#) $h(X, Z)$ is defined as the Coxeter number of the unique irreducible component of W_Z that does not belong to W_X ; it is equal to $h_1(X, Z)$ with the previous notation. Generalizing a result of Orlik-Solomon-Terao [\[17\]](#) it can also be given as $h(X, Z) = |\mathcal{A}_Z^X| - |\mathcal{A}_Z^H| - 1$ for any hyperplane H such that $H \cap X = Z$.

3 Recursions for Kreweras numbers

We prove in this section identical recursions for the combinatorial ([Proposition 3.1](#)) and algebraic ([Proposition 3.4](#)) Kreweras numbers.

Proposition 3.1. *For a well generated complex reflection group W , parabolic type $X \in \mathcal{L}_{\mathcal{A}_W}/W$, and for any positive integers m, k, r such that $m \geq k + r$, we have that*

$$\text{Krew}_{W, X}^{\text{NC}}(m) = \sum_{\substack{Y \in \mathcal{L}_W/W \\ \dim Y \leq X}} \text{Krew}_{W_Y, X}^{\text{NC}}(k) \cdot \text{Krew}_{W, Y}^{\text{NC}}(r).$$

Sketch. This is just the result of counting length- m chains $1 \leq_R w_1 \leq \dots \leq_R w_m \leq_R c$ in $\text{NC } W$ such that $V^{w_1} \sim X$ with respect to the parabolic type $Y : V^{w_k}$ of their k -th element w_k . $\square$

Remark 3.2. In the right hand side of Proposition 3.1, the number $\text{Krew}_{W_Y, X}^{\text{NC}}$ is equal to 0 whenever $\dim Y \geq X$ apart from the case that $Y = X$. This is important in the proof of the main result in Section 6.

The proof of the following algebraic recursion (Proposition 3.4) is a core ingredient of this paper and relies on previous work on arrangement Laplacians and their spectrum. It was shown in [8] that the characteristic polynomial of the $\mathcal{A}$ -Laplacian $L_{\mathcal{A}}(\omega)$, for arbitrary weights ω , is given in terms of the Laplacians of the localizations $\mathcal{A}_Y$. In the setting of Theorem 1.1, the restricted reflection arrangements $\mathcal{A}^X$ and a special selection of weights were considered in [9]. The Laplacian recursion for the restrictions $\mathcal{A}^X$ with weights $\omega_Z : h \in X, Z$ —see Section 2—gives essentially the following Lemma 3.3.

Lemma 3.3 ([9]). *For any complex reflection arrangement $\mathcal{A}$ and flat $X \in \mathcal{L}_{\mathcal{A}}$, we have that*

$$\prod_{i=1}^{\dim X} t - kh_i b_i^X = \sum_{Y \subseteq X} \prod_{i=1}^{\dim X - \dim Y} (kh_i(X, Y) - b_i^{X, Y}) \cdot t^{\dim Y},$$

where $h, h_i(X, Y), b_i^X, b_i^{X, Y}$ are Coxeter numbers and OS-exponents associated to $\mathcal{A}^X$ and $\mathcal{A}_Y^X$.

We are now ready to prove the analogous recursion of Proposition 3.1 for the algebraic Kreweras numbers. Notice that here we do not require that W be real. Moreover, the following statement holds for all complex reflection groups; we state it only for the well generated ones because even though the definition (1.3) of the algebraic Kreweras numbers makes sense for all complex types, it is more meaningful in the well generated case where it encodes structure coefficients of the algebraic parking space.

Proposition 3.4. *For a well generated complex reflection group W , parabolic type $X \in \mathcal{L}_{\mathcal{A}_W}/W$, and for any positive integers m, k, r , such that $m \geq k + r$, we have that*

$$\text{Krew}_{W, X}^{\text{alg}}(m) = \sum_{\substack{Y \in \mathcal{L}_W/W \\ \dim Y \leq X}} \text{Krew}_{W_Y, X}^{\text{alg}}(k) \cdot \text{Krew}_{W, Y}^{\text{alg}}(r).$$

Sketch. Rewriting the summation in terms of flats as opposed to orbits of flats, it is enough to show –compare with (1.3)– that

$$\prod_{i=1}^{\dim X} (mh - 1 - b_i^X) = \sum_{Y \subseteq X} \prod_{i=1}^{\dim X - \dim Y} (kh_i^{X,Y} - 1 - b_i^{X,Y}) \cdot \prod_{i=1}^{\dim Y} (rh - 1 - b_i^Y).$$

If we now set $t = -rh - 1$ and replace k with $-k$, this becomes

$$\prod_{i=1}^{\dim X} (t - kh - b_i^X) = \sum_{Y \subseteq X} \prod_{i=1}^{\dim X - \dim Y} (kh_i^{X,Y} - 1 - b_i^{X,Y}) \cdot \prod_{i=1}^{\dim Y} (t - b_i^Y), \quad (3.1)$$

which is equivalent with [Lemma 3.3](#) after a Möbius inversion described in [\[9, §3\]](#). $\square$

4 Interdependence between Kreweras and Loday numbers

In this section we present relations between the Kreweras and Loday numbers that allow the complete determination of one family from the other. These relations hold identically in the combinatorial ([Proposition 4.1](#)) and algebraic ([Proposition 4.3](#)) setting.

Proposition 4.1. *For a real reflection group W and parabolic type $X \in \mathcal{L}_{\mathcal{A}_W}/W$, we have that*

$$\text{Lod}_{W,X}^{\text{NC}}(m) = \sum_{\substack{Y \in \mathcal{L}_W/W \\ \dim Y \leq X}} N_{X,Y} \cdot \text{Krew}_{W,Y}^{\text{NC}}(m).$$

Sketch. This is proven in detail in [\[12, Prop. 8.1\]](#) which builds upon the work of [\[6\]](#). We relate faces of type X in $Y/W, m$ with chains in $\text{NC } W$ that start at a *standard* parabolic element of type X but must then continue first with another standard parabolic (this explains the term $N_{X,Y}$). $\square$

In [\[12, §10\]](#) we discuss how this statement should be seen as a refinement of an f -to- h transformation, perhaps due to a shelling of the cluster complex $Y/W, m$ via the chains in $\text{NC } W$ generalizing the work of Athanasiadis-Tzanaki in [\[4\]](#).

The following [Proposition 4.3](#) was originally a conjecture of Drew Armstrong (personal communication). It can be proven in many ways; for Weyl groups it has an interpretation via Ehrhart theory and the inside-out polytopes of Beck and Zaslavsky [\[5\]](#), in [\[12\]](#) we reduce it to a representation-theoretic description of tensoring with the sign character (see [\[12, Lem. 3.4, Lem. 5.3, Lem. 7.3\]](#)) but here we present a proof that is solely based in hyperplane arrangement theory. The following [Lemma 4.2](#) is known as Kung’s identity (see [\[1\]](#)) and is immediate after expanding the characteristic polynomials and standard manipulations of the Möbius functions.

Lemma 4.2. *For a hyperplane arrangement $\mathcal{A}$, and parameters s, t , we have*

$$\chi_{\mathcal{A}, st} = \sum_{Y \in \mathcal{L}_{\mathcal{A}}} \chi_{\mathcal{A}_Y, s} \cdot \chi_{\mathcal{A}^Y, t}.$$

Proposition 4.3. *For a real reflection group W and parabolic type $X \in \mathcal{L}_{\mathcal{A}_W}/W$, we have that*

$$\text{Lod}_{W, X}^{\text{alg}} m = \sum_{\substack{Y \in \mathcal{L}_W/W \\ \dim Y \leq X}} N_{X, Y} \cdot \text{Krew}_{W, Y}^{\text{alg}} m.$$

Sketch. Rewriting the summation in terms of flats of the reflection arrangement $\mathcal{A} : \mathcal{A}_W$ instead of orbits of flats, it is enough to show –compare with (1.3)– that

$$\prod_{i=1}^{\dim X} mh - 1 - b_i^X = \sum_{Y \subseteq X} c_{\mathcal{A}_Y^X} \cdot \prod_{i=1}^{\dim Y} mh - 1 - b_i^Y,$$

where $c_{\mathcal{A}_Y^X}$ denotes the number of chambers in $\mathcal{A}_Y^X$. This is precisely Kung’s identity (Lemma 4.2) for the arrangement $\mathcal{A}^X$ after setting $t = mh - 1$ and $s = -1$. $\square$

5 Recursions for Loday numbers

In this section, we return to Loday numbers and present identical recursions for them; combinatorial ones (Proposition 3.1) and algebraic ones (Proposition 3.4). We start with the combinatorial case which generalizes the Fomin-Reading recursion [13, Prop. 8.3].

Proposition 5.1 ([12, Prop. 11.3]). *For a real reflection group W , with arrangement $\mathcal{A}$ and simple generators S , and for any parabolic type $X \in \mathcal{L}_{\mathcal{A}}/W$ with $\dim X = k$, we have that*

$$\text{Lod}_{W, X}^{\text{NC}} m = \sum_{s \in S} \frac{mh - 2}{2k} \cdot \text{Lod}_{W_s, X \cap W_s}^{\text{NC}} m.$$

Sketch. This is proven in detail in [12, Prop. 11.3]; it relies on a double counting of pairs F, v of faces F of $Y \in W, m$ and vertices $v \in F$. $\square$

The previous Proposition can be extended to a double counting of pairs F, K where the face F has type X and the face $K \subseteq F$ has fixed dimension; in that case the factor $mh - 2 / 2k$ will also be replaced by a Loday number $\text{Lod}_{W, K}^{\text{NC}} m$.

The next lemma is the most technically complicated part of this project. Proposition 5.1 is the only combinatorial recursion for which we cannot uniformly prove the full algebraic analog. However, we will only need the linear term case (see Section 6). To simplify the presentation we prove only the case $X = V$ but the same argument works in general.

Lemma 5.2. *For a real reflection group W , with arrangement $\mathcal{A}$ and simple generators S , and for any parabolic type $X \in \mathcal{L}_{\mathcal{A}}/W$ with $\dim X = k$, we have that*

$$[m^1] \left(\text{Lod}_{W, X}^{\text{alg}} \quad m \right) = [m^1] \left(\frac{mh}{2k} \cdot \sum_{s \in S} \text{Lod}_{W_s, X \cap W_s}^{\text{alg}} \quad m \right).$$

Sketch for $X = V$. Again, we can rewrite the summation as one over flats and then we would have to show –compare with (1.3)– that

$$[m^1] \left(\prod_{i=1}^n mh \quad d_i \right) = [m^1] \left(\frac{mh}{n} \cdot \sum_{L \in \mathcal{L}_{\mathcal{A}}^1} \prod_{i=1}^{n-1} (mh_i \quad W_L \quad d_i \quad W_L) \right),$$

where the sum is over 1-dimensional flats L . We can inductively assume the result to be true for the parabolic subgroups W_L and the above equation becomes then equivalent to

$$[m^1] \left(n! \cdot \prod_{i=1}^n mh \quad d_i \right) = [m^1] \left(\sum_{\mathcal{F}} \left(\prod_{i=1}^n (mh_i \quad \mathcal{F} \quad 2) \right) \right),$$

where the summation is over all complete flags $\mathcal{F} = \{\mathbf{0}\} = X_0 \subseteq X_1 \subseteq \dots \subseteq X_n = V$, with $X_i \in \mathcal{L}_{\mathcal{A}}$ and $\dim X_i = i$, and where $h_i \quad \mathcal{F}$ is the relative Coxeter number $h(X_i, X_{i-1})$ for the pair of flats X_i and X_{i-1} in $\mathcal{F}$. It is easy to extract the linear terms from both sides of the equation; we must now show that

$$n! \cdot h \cdot |W| \sum_{i=1}^n \frac{1}{d_i} = 2^{n-1} \cdot \sum_{\mathcal{F}} \sum_{i=1}^n h_i \quad \mathcal{F}, \quad (5.1)$$

where we have used that $|W| = d_1 \cdots d_n$.

Any pair of flats $Z \subseteq X$, with $\dim X = \dim Z + 1$ can be extended to a complete flag $\mathcal{F}$. Moreover, by [8, Prop. 4.3] the number of complete flags in a simplicial arrangement $\mathcal{K}$ of rank n equals $c(\mathcal{K}) \cdot n! / 2^n$ where $c(\mathcal{K})$ denotes the number of chambers of $\mathcal{K}$. Since in our case both arrangements $\mathcal{A}_X$ and $\mathcal{A}^Z$ are simplicial, there will be $c(\mathcal{A}_X) \cdot \text{codim } X / 2^{\text{codim } X}$ (respectively $c(\mathcal{A}^Z) \cdot \dim Z / 2^{\dim Z}$) complete flags in the interval V, X (resp. $Z, \mathbf{0}$) in $\mathcal{L}_{\mathcal{A}}$. We can thus rewrite the RHS of (5.1) as

$$\begin{aligned} \text{RHS}((5.1)) &= \sum_{\substack{X \in \mathcal{L}_{\mathcal{A}} \\ X \neq \mathbf{0}}} \text{codim } X! \cdot c(\mathcal{A}_X) \cdot \sum_{Z \in \mathcal{A}^X} \dim Z! \cdot c(\mathcal{A}^Z) \cdot h(X, Z) \\ &= \sum_{\substack{Z \in \mathcal{L}_{\mathcal{A}} \\ Z \neq V}} (n-1-\dim Z)! \cdot \dim Z! \cdot c(\mathcal{A}_Z) \cdot c(\mathcal{A}^Z) \cdot |\mathcal{A}_Z|, \end{aligned} \quad (5.2)$$

where the second equality is a reordering of the summation and an application of the definition of $h(X, Z)$ in Section 2 and where $|\mathcal{A}_Z|$ denotes the number of hyperplanes in $\mathcal{A}_Z$.

Now, the quantity $c_{\mathcal{A}_Z} \cdot c_{\mathcal{A}^Z} \cdot |\mathcal{A}_Z|$ counts, by a standard double counting of faces and chambers in $\mathcal{A}$, the triples (C, H, F) of chambers $C \in \mathcal{C}_{\mathcal{A}}$, hyperplanes $H \in \mathcal{A}$, and faces $F \subseteq C$ such that $\text{span } F = Z$ and $F \subseteq H$. Rewriting (5.2) as a sum over such triples, we get that (below $C \cap H, \mathbf{0}$ denotes an interval in the *face lattice* of $\mathcal{A}$)

$$\begin{aligned} \text{RHS ((5.1))} &= \sum_{C \in \mathcal{C}_{\mathcal{A}}} \sum_{H \in \mathcal{A}} \sum_{F \in C \cap H, \mathbf{0}} (n-1-\dim F)! \cdot \dim F! \\ &= |W| \cdot \sum_{H \in \mathcal{A}} \sum_{F \in C_0 \cap H, \mathbf{0}} (n-1-\dim F)! \cdot \dim F! \\ &= |W| \cdot \sum_{H \in \mathcal{A}} \sum_{J \subseteq \text{cosup } H} (n-1-|J|)! \cdot |J|!. \end{aligned} \quad (5.3)$$

Here the second equality holds because in the reflection arrangement $\mathcal{A}$ the intersection patterns of hyperplanes and chambers are all the same; so we only need to address the fundamental chamber C_0 . The third equality above is a consequence of [10]; any face of the fundamental chamber corresponds to a subset $J \subseteq S$ and, in particular, the face $C_0 \cap H$ corresponds to the complement of the support of H , which we denote $\text{cosup } H$.

It is an easy consequence of Pascal's rule for binomial coefficients –in particular, the interpretation of $\text{cosup } H$ is irrelevant– that

$$\sum_{J \subseteq \text{cosup } H} (n-1-|J|)! \cdot |J|! = \frac{n!}{n-|\text{cosup } H|}.$$

Combining this with (5.1) and (5.3) it is sufficient to show that

$$h \cdot \sum_{i=1}^n \frac{1}{d_i} = \sum_{H \in \mathcal{A}} \frac{1}{|\text{supp } H|}, \quad (5.4)$$

which should be seen as a generalization of the formula $hn = 2|\mathcal{A}|$ and is itself a consequence of the main theorem of [10] and (3.1). We use [10] for its case-free proof of Chapoton's formula that counts the number of reflections of *full* support and we use (3.1) to compare the coefficient of k^1 in its two sides when $X = V$ and $t = 1$. $\square$

6 The main result

We are now ready to give the proof of our main theorem by combining the combinatorial and algebraic recursions of the previous sections. The inductive proof relies on the following simple lemma that addresses the structure of Propositions 3.1 and 3.4.

Lemma 6.1. *Assume that for two polynomials $P(x), Q(x) \in \mathbb{C}[X]$ of the same degree n , and for all triples m, k, r of positive integers such that $m = k + r$, we have*

$$P(m) - P(k) - P(r) = Q(m) - Q(k) - Q(r).$$

Then, $P(x)$ and $Q(x)$ have all their coefficients equal apart from possibly the linear term.

Proof. The function $L(x) = P(x) - Q(x)$ is a polynomial in x and by the assumption it satisfies $L(k) = r = L(k) - L(r)$ for any positive integers k, r . This means that in fact it is a linear function in x (with constant term 0). $\square$

We are now ready to prove [Theorem 1.1](#) as formulated in (1.4).

Proof of Theorem 1.1. Our proof proceeds via two nested inductions. First we assume that

$$\text{Krew}_{W,Z}^{\text{NC}} = \text{Krew}_{W,Z}^{\text{alg}} \quad \text{and} \quad \text{Lod}_{W,Z}^{\text{NC}} = \text{Lod}_{W,Z}^{\text{alg}}, \quad (6.1)$$

for all real reflection groups W of rank less than or equal to n and for all parabolic types $Z \in \mathcal{L}_W/W$; the base case $n = 1$ is trivially true.

Now, we pick an irreducible W such that $\text{rank } W = n = 1$ and we run a separate induction on its flats. We assume that (6.1) is true for all types $Z \in \mathcal{L}_W/W$ with $\dim Z \leq k$ (the case $k = 0$, i.e. $Z = 0$, is trivially true) and we pick some parabolic type X with $\dim X = k + 1$. Combining our inductive assumptions with [Proposition 5.1](#) and [Lemma 5.2](#) we must have that

$$[m^1] \left(\text{Lod}_{W,X}^{\text{NC}}(m) \right) = [m^1] \left(\text{Lod}_{W,X}^{\text{alg}}(m) \right). \quad (6.2)$$

Now [Propositions 4.1](#) and [4.3](#) again in conjunction with our inductive assumptions imply that also

$$[m^1] \left(\text{Krew}_{W,X}^{\text{NC}}(m) \right) = [m^1] \left(\text{Krew}_{W,X}^{\text{alg}}(m) \right). \quad (6.3)$$

Looking at the Kreweras recursions now, we rewrite the left hand side of [Proposition 3.1](#) as

$$\text{Krew}_{W,X}^{\text{NC}}(m) = \text{Krew}_{W,X}^{\text{NC}}(k) + \text{Krew}_{W,X}^{\text{NC}}(r),$$

and this leaves in the right hand side only expressions involving smaller rank groups or lower dimension flats; similarly for [Proposition 3.4](#). This means that combining our inductive assumptions and these two Propositions we will have for the polynomials $P(m) = \text{Krew}_{W,X}^{\text{NC}}(m)$ and $Q(m) = \text{Krew}_{W,X}^{\text{alg}}(m)$ that

$$P(m) = P(k) + P(r) = Q(m) = Q(k) + Q(r),$$

for all positive integers m, k, r such that $m = k + r$. Now, we just showed in (6.3) that the coefficients of the *linear* terms of $P(m)$ and $Q(m)$ are equal and combining the above expression with [Lemma 6.1](#) we must have that *all* coefficients are equal. In other words, we have shown that

$$\text{Krew}_{W,X}^{\text{NC}}(m) = \text{Krew}_{W,X}^{\text{alg}}(m),$$

and by applying [Propositions 4.1](#) and [4.3](#) again with our inductive assumptions we must also have that

$$\text{Lod}_{W,X}^{\text{NC}}(m) = \text{Lod}_{W,X}^{\text{alg}}(m).$$

This completes the inductive argument and so it completes the proof as well. $\square$

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Polyhedral models for K -theory of toric and flag varieties

Leonid Monin^{*1} and Evgeny Smirnov^{†2,3}

¹EPFL, Lausanne, Switzerland

²HSE University, Moscow, Russia

³Independent University of Moscow, Russia

Abstract. In 1992, Pukhlikov and Khovanskii provided a description of the cohomology ring of toric variety as a quotient of the ring of differential operators on spaces of virtual polytopes. Later Kaveh generalized this construction to the case of cohomology rings for full flag varieties.

In this paper we extend Pukhlikov–Khovanskii type presentation to the case of K -theory of toric and flag varieties. First we study the Gorenstein duality quotients of the group algebra of free abelian group (possibly of infinite rank). Then we specialize to the K -ring of integer (virtual) polytopes with a fixed normal fan. Finally we show that the K -theory of toric and flag varieties can be realized as polytope K -rings and describe the classes of toric orbits or Schubert varieties in them.

Keywords: K -theory, toric varieties, flag varieties

1 Introduction

This paper is devoted to the study of K -theory of toric and flag varieties. In [12], A. Khovanskii and A. Pukhlikov described the cohomology ring of a toric variety X as the quotient of the ring of differential operators with constant coefficients modulo the annihilator of the volume polynomial of the moment polytope of X . This construction was generalized by K. Kaveh [5] who observed that the cohomology ring of a full flag variety can be obtained by applying the same construction to a Gelfand–Zetlin polytope. This description was later used by V. Kiritchenko, E. Smirnov, and V. Timorin [8] to provide a polyhedral model for Schubert calculus on full flag varieties.

In this paper we generalize the above results to the K -theory. First, in Section 2 we generalize the construction of algebras with Gorenstein duality pairing via Macaulay inverse systems (see [7]) to the case of shift operators with constant coefficients. Then we use this to define a polytope K -ring by considering the algebra of shift operators

^{*}leonid.monin@epfl.ch.

[†]esmirnov@hse.ru. The study has been funded within the framework of the HSE University Basic Research Program.

with constant coefficients modulo the annihilator of the Ehrhart polynomial and study its properties in the case when polytope is integrally simple.

In [Section 3](#) we prove that the K -theory of a toric variety coincides with the K -ring of its moment polytope. Finally, in [Section 4](#) we extend this to full flag varieties showing that K -theory of G/B for $G = \mathrm{GL}(n)$ is given by the K -ring of Gelfand–Zetlin polytope. Moreover, we identify the structure sheaves of Schubert varieties of G/B in the K -ring of Gelfand–Zetlin polytope. This provides a polyhedral model for K -theoretic Schubert calculus generalizing results of [\[8\]](#).

This is a report on the forthcoming paper [\[10\]](#). In particular, we skip most of the proofs in this text and refer the reader to [\[10\]](#) for more details and complete proofs.

2 Polytope algebra

In this section we associate an algebra with duality to any function $f: \Lambda \rightarrow \mathbf{k}$ on a lattice. We then study a particular example of this construction given by the Ehrhart function on the lattice of integral (virtual) polytopes with a given normal fan.

2.1 Algebra associated to a function on a lattice

Let $\mathbf{k}$ be any field and let A be a commutative algebra with identity over $\mathbf{k}$. In further subsections of the paper we will stick with $\mathbf{k} = \mathbb{Q}$, however we decided to work with more general setting here. A linear function $\ell: A \rightarrow \mathbf{k}$ defines a symmetric, bilinear pairing on A via:

$$\langle a, b \rangle_\ell := \ell(a \cdot b) \text{ for any } a, b \in A.$$

Definition 2.1. A pairing $\langle \cdot, \cdot \rangle_\ell$ on algebra A is called *Gorenstein duality pairing* if it is non-degenerate.

The main objective of this subsection is to give a construction of algebras with Gorenstein duality pairing from a $\mathbf{k}$ -valued function on a lattice. Everywhere in this text by a lattice we mean a (possibly infinitely generated) free abelian group.

Let Λ be a lattice and let $\mathbf{k}$ be any field. We will denote by $\mathbf{k}^\Lambda$ the set of maps $f: \Lambda \rightarrow \mathbf{k}$. Let us further denote by $\mathbf{k}[\Lambda]$ the group algebra of Λ . The group algebra $\mathbf{k}[\Lambda]$ is acting on the set of functions $\mathbf{k}^\Lambda$ via shift operators. That is, for $t = \sum_{i=1}^r a_i \lambda_i \in \mathbf{k}[\Lambda]$ and $f \in \mathbf{k}^\Lambda$ we have

$$t \cdot f(x) = \sum_{i=1}^r a_i f(x - \lambda_i), \text{ for any } x \in \Lambda.$$

In what follows we identify the group algebra $\mathbf{k}[\Lambda]$ with the algebra of shift operators with constant coefficients $\mathrm{Sh}(\Lambda)$ on Λ . We will further denote by t_x the shift operator by $-x$ and by $D_x = 1 - t_x$ the corresponding difference operator for any $x \in \Lambda$.

Theorem 2.2. Let $f \in \mathbf{k}^\Lambda$ be any function, then $\text{Ann } f = \{s \in \text{Sh } \Lambda \mid s \cdot f \equiv 0\}$ is an ideal in $\text{Sh } \Lambda$. Moreover, the quotient algebra $A_f : \text{Sh } \Lambda / \text{Ann } f$ has a Gorenstein duality pairing defined by a function

$$\ell_f : \text{Sh } \Lambda \rightarrow \mathbf{k}, \quad \ell_f : t \mapsto t \cdot f \neq 0.$$

In particular, linear function ℓ_f descends to a well-defined linear function on A_f .

The construction from [Theorem 2.2](#) is quite general as the following Theorem shows.

Theorem 2.3. Let A be a commutative algebra generated by invertible elements $a_1, \dots, a_s \in A^\times$ with Gorenstein duality pairing given by a function $\ell : A \rightarrow \mathbf{k}$. Then $A \simeq A_f$ with

$$f : \mathbb{Z}^s \rightarrow \mathbf{k}, \quad f : (n_1, \dots, n_s) \mapsto \ell(a_1^{n_1} \dots a_s^{n_s}).$$

Remark 2.4. A version of [Theorem 2.2](#) is also true for any commutative semigroup with cancellation Λ . In this more general form [Theorem 2.2](#) is closely related to the explicit version of Macaulay duality recently studied in [7] (see also [4, 6] for related results).

Note that for a general function $f : \Lambda \rightarrow \mathbf{k}$ the ideal $\text{Ann } f$ might be trivial (see [Example 2.5](#)). In fact, as illustrated by [Lemma 2.6](#), for most functions f , $\text{Ann } f = 0$. However, for some classes of functions f , the ideal A_f is non-trivial. For instance, if f is a polynomial on the lattice Λ , the quotient algebra A_f is Artinian, i.e. is a finite dimensional vector space over $\mathbf{k}$.

Example 2.5. Let $g(x) = e^{ax}$ be a double exponential function on one-dimensional lattice $\mathbb{Z}$ and let $T = \sum_{i=1}^s \lambda_i t_{k_i}$ with $k_1 < \dots < k_s$. Then $|t \cdot g(x)| > 0$ for $x \gg 0$, in particular $t \cdot g \neq 0$ so $\text{Ann } g$ is trivial.

Lemma 2.6. Let $\Lambda = \mathbb{Z}$ be a one-dimensional lattice. Then for each nontrivial shift operator $T = \sum_{i=1}^k \lambda_i t_{r_i}$ the functions $f : \mathbb{Z} \rightarrow \mathbf{k}$ annihilated by T form a finite dimensional vector space.

In particular, if $\mathbf{k}$ is countable, the set of functions f with nontrivial annihilator $\text{Ann } f \neq 0$ is countable, while the set of all functions $\mathbf{k}^\mathbb{Z}$ is uncountable.

Proposition 2.7. Let $\Lambda \simeq \mathbb{Z}^r$ be of finite rank, and let f be a polynomial function on Λ . Then A_f is an Artinian algebra.

Proof. Let $D_i = 1 - t_i$ be the standard difference operators for $i = 1, \dots, r$. Then Laurent monomials $D_1^{k_1} \dots D_r^{k_r}$ form a basis of $\mathbf{k}^\Lambda$. The statement follows from the fact that for a polynomial f of degree d , one has $D_1^{k_1} \dots D_r^{k_r} \cdot f = 0$ for $|k_1| + \dots + |k_r| > d$. $\square$

We finish this subsection with a description of the relation between two algebras A_f, A_g defined by pairs (Λ_g, g) and (Λ_f, f) such that there is a lattice homomorphism $\sigma : \Lambda_g \rightarrow \Lambda_f$ with $g = \sigma^* f$. Our description is parallel to [8, Proposition 2.4].

Proposition 2.8. There exists an abelian group $M_{f,g}$ with an epimorphism $\pi : A_f \rightarrow M_{f,g}$ and a monomorphism $\iota : A_g \rightarrow M_{f,g}$ such that $\pi \tilde{\alpha} \tilde{\beta} = \iota \alpha \beta$ whenever $\pi \tilde{\alpha} = \iota \alpha$ and $\pi \tilde{\beta} = \iota \beta$.

2.2 Ehrhart polynomial and the polytope algebra

In this subsection we briefly recall the definition of the Ehrhart polynomial on the space of virtual polytopes. For more detailed background on virtual polytopes we refer to [11].

We will call a polytope $\Delta \subset \mathbb{R}^d$ integral if all vertices of Δ belong to the integer lattice $\mathbb{Z}^d \subset \mathbb{R}^d$. Denote by $\mathcal{P}$ the set of all integral polytopes in $\mathbb{R}^d$. The set $\mathcal{P}$ has a structure of an abelian semigroup with respect to Minkowski addition:

$$\Delta_1 + \Delta_2 = \{x + y \mid x \in \Delta_1, y \in \Delta_2\}.$$

It is easy to check that $\mathcal{P}$ has the cancellation property, i.e.

$$\Delta_1 + \Delta = \Delta_2 + \Delta \text{ if and only if } \Delta_1 = \Delta_2.$$

Thus $\mathcal{P}$ is embedded into its Grothendieck group, which we denote by $\mathcal{P}$ i.e. the group of formal differences of elements in $\mathcal{P}$. The space of integral virtual polytopes $\mathcal{P}$ has a structure of a free abelian group.

A virtual polytope Δ is uniquely described by its support function $H_\Delta : \mathbb{R}^{d \vee} \rightarrow \mathbb{R}$ given by

$$H_\Delta(\psi) := \min_{x \in \Delta} \psi(x),$$

for a convex polytope Δ and extended by linearity to virtual polytopes. For an integral virtual polytope Δ its support function H_Δ is a piecewise linear function which attains integer values on $\mathbb{Z}^{d \vee}$. The cones of linearity of H_Δ form a fan in $\mathbb{R}^{d \vee}$. For a given fan $\Sigma \subset \mathbb{R}^{d \vee}$, let us denote by $\mathcal{P}_\Sigma$ the set of integer convex polytopes Δ with H_Δ linear on the cones of Σ . Similarly, by $\mathcal{P}_\Sigma$ we denote the set of integer virtual polytopes with support function linear on the cones of Σ . Clearly, $\mathcal{P}_\Sigma$ is a free abelian group and $\mathcal{P}_\Sigma$ is its subsemigroup.

Let $e_1, \dots, e_r$ be the primitive ray generators of Σ . Then a virtual polytope is uniquely determined by the evaluation of H_Δ on $e_1, \dots, e_r$. This defines an embedding $\mathcal{P}_\Sigma \hookrightarrow \mathbb{Z}^r$ as a sublattice. Notice however that $\mathcal{P}_\Sigma$ is in general a proper sublattice of $\mathbb{Z}^r$. In fact, it can both have smaller rank and be non-saturated.

An Ehrhart polynomial $\text{Ehr} : \mathcal{P} \rightarrow \mathbb{Z}$ is the unique polynomial on the space of virtual polytopes such that

$$\text{Ehr}(\Delta) = |\Delta \cap \mathbb{Z}^d|, \text{ for any } \Delta \in \mathcal{P}.$$

It is shown in [12] that the polynomial Ehr is well defined, i.e. that the number of lattice points is the restriction of a polynomial function on $\mathcal{P}$.

Definition 2.9. Let Σ be a complete rational fan and $\mathcal{P}_\Sigma$ be the corresponding space of integer virtual polytopes. We define the polytope K -ring to be the algebra K_Σ defined by the pair $(\mathcal{P}_\Sigma, \text{Ehr})$

$$K_\Sigma = \text{Sh } \mathcal{P}_\Sigma / \text{Ann } \text{Ehr}.$$

Note that since $\mathcal{P}_\Sigma$ is a finite rank lattice and Ehr is a polynomial, K_Σ is an Artinian algebra by [Proposition 2.7](#).

2.3 Structure of polytope K-ring

In this subsection we will study the algebra K_Σ in more detail. First let us assume that Σ is a smooth fan with $|\Sigma(1)| = r$. In this case, the evaluation of H_Δ on integer ray generators of Σ canonically identifies the lattice $\mathcal{P}_\Sigma$ with $\mathbb{Z}^r$. We will denote by $\Delta_1, \dots, \Delta_r$ the corresponding basis of $\mathcal{P}_\Sigma$. In other words Δ_i is a virtual polytope in $\mathcal{P}_\Sigma$ such that

$$H_{\Delta_i}(e_j) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise,} \end{cases}$$

where $e_1, \dots, e_r$ are the primitive ray generators of Σ . We further denote by $t_1, \dots, t_r$ the shift operators with respect to $\Delta_1, \dots, \Delta_r$.

Theorem 2.10. *Let $\Sigma \subset \mathbb{R}^n$ be a smooth complete fan with $|\Sigma(1)| = r$ and primitive ray generators $e_1, \dots, e_r$. Then the polytope ring A_Σ is given by*

$$A_f \simeq \mathbb{Q}[t_1^{\pm 1}, \dots, t_r^{\pm 1}] / I + J,$$

where I is generated by products $1 - t_{i_1}^{-1} \cdots 1 - t_{i_s}^{-1}$ such that $\rho_{i_1}, \dots, \rho_{i_s} \in \Sigma(1)$ are distinct and do not form a cone in Σ and $J = \langle \prod_{i=1}^r t_i^{\langle u, e_i \rangle} - 1 \mid u \in \mathbb{Z}^n \rangle$.

Remark 2.11. The statement of Theorem 2.10 is also true over $\mathbb{Z}$. We will treat this case in the full version [10].

The difference operators appearing in the definition of ideal I will play an important role in what follows. We will denote them by $D_i = 1 - t_i^{-1}$. More generally, for any cone $\sigma \in \Sigma$ we denote by D_σ the corresponding product of difference operators

$$D_\sigma = \prod_{\rho_i \in \sigma} D_i.$$

The proof of Theorem 2.10 is done in two steps. The first part is to show that the relations I and J are satisfied in the algebra K_Σ . Indeed, the relations J are satisfied since

$$\prod_{i=1}^r t_i^{\langle u, e_i \rangle} \cdot \text{Ehr } \Delta = \text{Ehr} \left(\Delta + \sum_{i=1}^r \langle u, e_i \rangle \Delta_i \right) = \text{Ehr } \Delta + u = \text{Ehr } \Delta \text{ for any } u \in \mathbb{Z}^n.$$

The relations from I are subject of the following lemma.

Lemma 2.12. *Let Σ be a smooth fan and $\Delta \in \mathcal{P}_\Sigma$ an integer (possibly virtual) polytope. Then*

$$D_{i_1} \cdots D_{i_s} \cdot \text{Ehr } \Delta = \text{Ehr } F_\sigma,$$

where σ is a cone spanned by $\rho_{i_1}, \dots, \rho_{i_s}$ and F_σ is a corresponding (virtual) face of Δ . In particular, if $\rho_{i_1}, \dots, \rho_{i_s}$ do not form a cone, $D_{i_1} \cdots D_{i_s} \cdot \text{Ehr } \Delta = 0$.

Since both sets of relations I and J are satisfied in K_Σ , there exists a surjection

$$\psi: \mathbb{Z} t_1^{\pm 1}, \dots, t_r^{\pm 1} / I \cup J \rightarrow A_\Sigma, \quad \psi: t_i \mapsto \Delta_i.$$

To show that ψ is an isomorphism, we will then use a Morse theoretic argument which is parallel to results in [13].

In what follows we will also have to work with non-smooth fans Σ and their K -rings K_Σ . It is convenient in this case to reduce to the smooth case by considering any smooth subdivision Σ' of Σ and using Proposition 2.8. Indeed, $\mathcal{P}_\Sigma \subset \mathcal{P}_{\Sigma'}$ and the Ehrhart polynomial on $\mathcal{P}_\Sigma$ is the restriction of Ehrhart polynomial on $\mathcal{P}_{\Sigma'}$. Thus there is an abelian group $M_{\Sigma', \Sigma}$ with an epimorphism $\pi: K_{\Sigma'} \rightarrow M_{\Sigma', \Sigma}$ and a monomorphism $\iota: K_\Sigma \rightarrow M_{\Sigma', \Sigma}$ such that $\pi \tilde{\alpha} \tilde{\beta} = \iota \alpha \beta$ whenever $\pi \tilde{\alpha} = \iota \alpha$ and $\pi \tilde{\beta} = \iota \beta$. This allows to perform the computations in $K_{\Sigma'}$ instead of K_Σ which we understand better.

3 Toric varieties

In this section we will apply the results of Section 2 to computation of the K -theory of toric varieties. In what follows we assume the basic knowledge of toric geometry and refer to [2] for further details and references.

For a smooth algebraic variety X we denote by $K_0 X$ the free abelian group generated by isomorphism classes of coherent sheaves on X up to the relation $\mathcal{V} = \mathcal{U} = \mathcal{W}$ whenever there is a short exact sequence $0 \rightarrow \mathcal{V} \rightarrow \mathcal{W} \rightarrow \mathcal{U} \rightarrow 0$. The subgroup generated by classes of vector bundles is denoted by $K^0 X$. For a smooth variety X the inclusion $K^0 X \hookrightarrow K_0 X$ is an isomorphism. In this case, we define the ring structure on $K_0 X$ via $\mathcal{V} \mathcal{U} = \mathcal{V} \otimes \mathcal{U}$. In what follows we will work with rational K -theory $K_0 X \otimes_{\mathbb{Z}} \mathbb{Q}$. To simplify the notation we will denote the rational K -theory of X by $K_0 X$. Finally, K -theory admits a proper push-forward. In particular, for a trivial map $f: X \rightarrow \text{pt}$, the pushforward $f_*: K_0 X \rightarrow K_0 \text{pt} \simeq \mathbb{Q}$ is a linear function on $K_0 X$ which is equal to the holomorphic Euler characteristic on the classes of sheaves:

$$f_* \mathcal{F} = \chi(X, \mathcal{F}).$$

For a more detailed introduction to K -theory we refer to [9].

Let $T \simeq \mathbb{C}^{*n}$ be an algebraic torus, $M \simeq \mathbb{Z}^n$ its character lattice and $N = \text{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ its dual lattice. We denote by $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$, $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ vector spaces spanned by M and N respectively. Let further Σ be a smooth complete fan and Y_Σ the corresponding toric variety. We denote by $\Sigma(1) = \{\rho_1, \dots, \rho_r\}$ the set of rays of Σ and by $D_1, \dots, D_r$ the corresponding T -invariant divisors. Finally, for a cone $\sigma \in \Sigma$ of dimension ≥ 1 we will denote by X_σ the closure of the T -orbit corresponding to σ .

The main input from toric geometry for us comes from the computation of holomorphic Euler characteristic of line bundles on Y_Σ in terms of combinatorics of polytopes.

More concretely, every line bundle $\mathcal{L}$ on Y_Σ can be linearized, and this is equivalent (as a sheaf) to $\mathcal{O}(\sum_{i=1}^r h_i D_i)$ for some $h_1, \dots, h_r \in \mathbb{Z}$. Therefore, there is a surjection

$$\mathcal{P}_\Sigma \rightarrow \text{Pic } Y_\Sigma, \quad \Delta_h \mapsto \mathcal{O}\left(\sum_{i=1}^r h_i D_i\right).$$

We will denote the line bundle corresponding to a polytope $\Delta \in \mathcal{P}_\Sigma$ by $\mathcal{L}_\Delta$.

Proposition 3.1. *Let $\Delta \in \mathcal{P}_\Sigma$ be an integer virtual polytope and $\mathcal{L}_\Delta$ the corresponding line bundle on Y_Σ . Then $\chi(Y_\Sigma, \mathcal{L}_\Delta) = \text{Ehr } \Delta$.*

Theorem 3.2. *Let Σ be a smooth, complete fan and let Y_Σ be the corresponding toric variety. Then we have an isomorphism*

$$K_0(Y_\Sigma) \simeq K_\Sigma = \text{Sh } \mathcal{P}_\Sigma / \text{Ann Ehr}.$$

Proof. First, since Y_Σ has an algebraic cell decomposition, the Euler characteristic provides a Gorenstein duality pairing on $K_0(Y_\Sigma)$, i.e. the Euler pairing

$$\langle \mathcal{F}, \mathcal{G} \rangle_{Eu} : \chi(Y_\Sigma, \mathcal{F} \otimes \mathcal{G}), \quad \mathcal{F}, \mathcal{G} \in K_0(Y_\Sigma)$$

is non-degenerate. Moreover, $K_0(Y_\Sigma)$ is generated by the Picard lattice $\text{Pic } Y_\Sigma$. Therefore, by [Theorem 2.3](#) we get $K_0(Y_\Sigma) \simeq \text{Sh } \text{Pic } Y_\Sigma / \text{Ann } \chi$. Finally the theorem follows from the correspondence of $\mathcal{P}_\Sigma$ with the Picard lattice $\text{Pic } Y_\Sigma$ and [Proposition 3.1](#). $\square$

As a corollary of [Theorems 2.10](#) and [3.2](#) we obtain the following statement.

Corollary 3.3. *Let Σ be a smooth fan and Y_Σ the corresponding toric variety. Then*

$$K_0(Y_\Sigma) \simeq \mathbb{Q}[t_1^{\pm 1}, \dots, t_r^{\pm 1}] / I + J,$$

where I is generated by monomials $D_{i_1} \cdots D_{i_s}$ such that $\rho_{i_1}, \dots, \rho_{i_s} \in \Sigma(1)$ are distinct and do not form a cone in Σ and $J = \left\langle \prod_{i=1}^r t_i^{\langle u, e_i \rangle} - 1 \mid u \in \mathbb{Z}^n \right\rangle$.

The above presentation for $K_0(Y_\Sigma)$ can be also obtained by applying the Chern character isomorphism to the usual presentation of the Chow ring of Y_Σ .

We will finish this section with a description of the classes of structure sheaves $\mathcal{O}_{X_\sigma}$ for orbit closures in the polytope K-ring K_Σ .

Proposition 3.4. *Let Σ be a smooth fan and Y_Σ the corresponding toric variety. Let further $\sigma \in \Sigma$ be a cone and $X_\sigma \subset Y_\Sigma$ the corresponding orbit closure. Then the class of $\mathcal{O}_{X_\sigma}$ is represented in K_Σ by the operator $D_\sigma = \prod_{\rho_i \in \sigma} D_i$.*

Proof. Indeed, since $K_0(Y_\Sigma)$ is generated by $\text{Pic } Y_\Sigma$, it is enough to check that

$$\chi(Y_\Sigma, \mathcal{O}_{X_\sigma} \otimes \mathcal{L}_\Delta) = D_\sigma \cdot \text{Ehr } \Delta$$

for any $\Delta \in \mathcal{P}_\Sigma$. Hence the proposition follows from [Lemma 2.12](#) and the fact that $\chi(Y_\Sigma, \mathcal{O}_{X_\sigma} \otimes \mathcal{L}_\Delta) = \chi(X_\sigma, \mathcal{L}_\Delta|_{X_\sigma}) = \text{Ehr } F_\sigma$, where F_σ is a face of Δ corresponding to σ . $\square$

4 Full flag varieties

Let G be a reductive algebraic group of rank n . Fix a Borel subgroup $B \subset G$; let T be the maximal torus corresponding to B . Denote by $\mathfrak{X} = \mathfrak{X}(T)$ the weight lattice of G (i.e., the character lattice of T). For a dominant highest weight $\lambda = \lambda_1, \dots, \lambda_n \in \mathfrak{X}$, the dimension of the corresponding irreducible G -module V_λ is computed using the *Weyl dimension formula*:

$$\dim V_\lambda = \prod_{\alpha \in \Delta^+} \frac{\langle \lambda + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle},$$

where the product is taken over the system of positive roots Δ^+ , $\langle \cdot, \cdot \rangle$ is the Cartan pairing on $\mathfrak{X}$, and $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$. Thus, $\dim V_\lambda$ is a polynomial in $\lambda_1, \dots, \lambda_n$. It is called the *Weyl polynomial* and denoted by $F_G(\lambda)$. It is an inhomogeneous polynomial of degree $|\Delta^+|$.

Example 4.1. If $G = \mathrm{GL}(n)$, the Weyl polynomial equals $F_{\mathrm{GL}(n)}(\lambda) = \prod_{i < j} \frac{\lambda_i - \lambda_j - i - j}{j - i}$.

The previous discussion immediately implies the following theorem about the K -group of the full flag variety G/B .

Theorem 4.2. *We have an isomorphism $K_0(G/B) \simeq \mathrm{Sh}(\mathfrak{X} / \mathrm{Ann} F_G)$.*

Proof. The proof is analogous to the proof of [Theorem 3.2](#). Indeed, G/B has an algebraic cell decomposition and $K_0(G/B)$ is generated by $\mathrm{Pic}(G/B)$ which can be identified with $\mathfrak{X}$. Therefore, the statement follows from [Theorem 2.3](#) and Borel–Weil–Bott theorem which states that $\chi(G/B, \mathcal{L}_\lambda) = \dim V_\lambda = F_G(\lambda)$. $\square$

4.1 Gelfand–Zetlin polytopes

In this subsection we study $K_0(G/B)$ for $G = \mathrm{GL}(n)$ in more detail. For this we shall need the definition of Gelfand–Zetlin polytopes.

Take a strictly decreasing sequence of integers $\lambda = \lambda_1 > \lambda_2 > \dots > \lambda_n$. Consider a triangular tableau of the following form (it is called a *Gelfand–Zetlin tableau*):

$$\begin{array}{ccccccc} \lambda_n & & \lambda_{n-1} & & \lambda_{n-2} & & \dots & & \lambda_1 \\ & x_{1,n-1} & & x_{1,n-2} & & \dots & & & x_{11} \\ & & x_{2,n-2} & & \dots & & & x_{21} & \\ & & & \ddots & \vdots & & & & \\ & & & & x_{n-1,1} & & & & \end{array} \quad (4.1)$$

We will interpret x_{ij} , where $i + j \leq n$, as coordinates in $\mathbb{R}^N$, where $N = \frac{n(n-1)}{2}$. This tableau can be viewed as a set of inequalities on the coordinates in the following way:

for each triangle $\begin{smallmatrix} a & & b \\ & c & \end{smallmatrix}$ in this tableau, impose the inequalities $a \leq c \leq b$. This system of inequalities defines a bounded nondegenerate polytope in $\mathbb{R}^N$. This polytope is called a *Gelfand–Zetlin polytope*; we will denote it by $\text{GZ } \lambda$.

Gelfand–Zetlin polytopes were introduced by I. M. Gelfand and M. L. Zetlin¹ in 1950 (cf. [3]). The integer points in $\text{GZ } \lambda$ index a special basis, called the Gelfand–Zetlin basis, in the irreducible representation V_λ with the highest weight λ of the group $\text{GL } n$. In particular, the number of integer points in $\text{GZ } \lambda$ equals $\dim V_\lambda$. This can be viewed as follows: the map taking each row of the Gelfand–Zetlin tableau into its sum

$$\text{pr}: \mathbb{R}^N \rightarrow \mathbb{R}^{n-1} \sim \mathfrak{X}, \quad x_{11}, x_{12}, \dots, x_{n-1,1} \mapsto \left(\sum_{i=1}^{n-1} x_{1i}, \sum_{i=1}^{n-2} x_{2i}, \dots, x_{n-1,1} \right)$$

projects $\text{GZ } \lambda$ onto the weight polytope of V_λ .

The following proposition is immediate.

Proposition 4.3. *For a given n , all Gelfand–Zetlin polytopes have the same normal fan. The Ehrhart polynomial of $\text{GZ } \lambda$ is equal to the Weyl polytope of type A_{n-1} :*

$$\text{Ehr } \text{GZ } \lambda = F_{\text{GL } n} \prod_{i < j} \frac{\lambda_i - \lambda_j - i - j}{j - i}.$$

We denote the lattice of (possibly virtual) integer Gelfand–Zetlin polytopes by $\mathcal{P}_{\text{GZ}}$. [Theorem 4.2](#) together with [Proposition 4.3](#) implies the following corollary.

Corollary 4.4. *Let $\text{Fl } n = \text{GL } n / B$ be the variety of complete flags in $\mathbb{C}^n$. Its K-group is isomorphic to $K_0 \text{Fl } n \sim K_{\text{GZ}} \text{Sh } \mathcal{P}_{\text{GZ}} / \text{Ann Ehr } \text{GZ } \lambda$.*

4.2 Faces of Gelfand–Zetlin polytopes

Let us describe the set of faces of the Gelfand–Zetlin polytope. The polytope is defined by a set of inequalities, represented by (4.1). Each face is obtained by turning some of these inequalities into equalities. In particular, each facet is defined by a unique equation: $x_{ij} = x_{i-1,j-1}$ or $x_{ij} = x_{i-1,j}$ for some pair (i, j) , where $i + j \leq n$. (We suppose that $x_{0,k} = \lambda_k$). Denote the facets of the first type by F_{ij} .

Definition 4.5. A face F of $\text{GZ } \lambda$ is called a *Kogan face* if it is obtained as the intersection of facets F_{ij} for some i, j . Equivalently, F is a Kogan face if it contains the vertex defined by the equations

$$\lambda_n = x_{1,n-1} = \dots = x_{n-1,1}, \quad \lambda_{n-1} = x_{1,n-2} = \dots = x_{n-2,1}, \quad \dots, \quad \lambda_2 = x_{1,1}.$$

¹Sometimes also spelled Cetlin or Tsetlin.

We will represent Kogan faces of the Gelfand–Zetlin polytope symbolically by diagrams obtained from Gelfand–Zetlin tableaux by replacing all λ_i 's and x_{ij} 's by dots, where each equality of type $x_{ij} = x_{i-1,j-1}$ is represented by an edge joining these dots. We shall assign to each Kogan face F a word $\underline{w} \in F$ in the alphabet $s_1, \dots, s_{n-1}$ of Coxeter generators of the symmetric group S_n , as follows. We mark the edge going from $x_{i-1,j-1}$ to x_{ij} by a simple transposition $s_{i-j-1} \in S_n$ (recall that $1 \leq i, j$ and $i - j \leq n$), as shown on Figure 1, and take the word in $s_1, \dots, s_{n-1}$ obtained by reading the letters on the edges from bottom to top from left to right.



Figure 1: Diagrams of Kogan faces

Definition 4.6. Let $\underline{w} = s_{i_1}, \dots, s_{i_k}$ be a word. The *Demazure product* $\delta \underline{w}$ of $\underline{w}$ is the permutation defined inductively as follows: $\delta s_i = s_i$, and $\delta \underline{w}, s_i$ equals $\delta \underline{w} s_i$ if $\ell(\delta \underline{w} s_i) > \ell(\delta \underline{w})$, and $\delta \underline{w}$ otherwise. Note if $\underline{w}$ is a reduced word, then $\delta \underline{w} = s_{i_1} \dots s_{i_k}$.

Definition 4.7. Let F be a Kogan face of codimension k , and let $\underline{w} \in F = s_{i_1}, \dots, s_{i_k}$ be the corresponding word. We shall say that F *corresponds* to the permutation $\delta \underline{w} \in S_n$. A Kogan face is said to be *reduced* if the word $\underline{w} \in F$ is reduced, and *non-reduced* otherwise.

Example 4.8. Diagrams on Figure 1 produce the words s_3, s_2, s_1, s_3 and s_3, s_2, s_3, s_1, s_3 respectively. Both of these faces correspond to the permutation $s_3 s_2 s_1 s_3 = 4231$. The left of them is reduced, while the right one is not.

Finally, we can assign a collection of Kogan faces to each permutation $w \in W$. Denote by $\mathcal{F}_w$ the set of all Kogan faces F corresponding to w , and by Γ_w the (set-theoretic) union of all these faces.

4.3 Characters of Demazure modules

In this subsection we give a definition of Demazure modules and recall a theorem by Kiritchenko, Smirnov, and Timorin relating their characters to faces of Gelfand–Zetlin polytopes.

Let $G \subset GL(n, \mathbb{C})$. We denote the Weyl group of G by $W \simeq S_n$. For $w \in W$, let $X_w = \overline{BwB/B} \subset G/B$ be the corresponding Schubert variety; in particular, for the longest element w_0 we have $X_{w_0} = G/B$. Let $\mathcal{L}_\lambda$ be the G -equivariant line bundle on G/B defined as $G \times^B \mathbb{C}_{-\lambda} \rightarrow G/B$, and let $\mathcal{L}_w^\lambda$ stand for the restriction of $\mathcal{L}_\lambda$ to X_w .

Definition 4.9. Let $V_\lambda = H^0(G/B, \mathcal{L}_\lambda^*)$ be the irreducible representation with the highest weight λ . For $w \in W$, define a *Demazure module* $V_w \lambda$ as follows. Take a vector of weight $w\lambda$ in V_λ (such a vector is unique up to a scalar) and consider the B -module of V_λ generated by this vector. We denote this B -module by $V_w \lambda$.

The following theorem is well-known (cf., for instance, [1, Sec. 3.3]).

Theorem 4.10. For $\lambda \in \mathfrak{X}$, $w \in W$, we have a B -module isomorphism $H^0(X_w, \mathcal{L}_w \lambda) \sim V_w \lambda$.

The character of a Demazure module $\text{char } V_w \lambda$ is an element of the group algebra $A(T) : \mathbb{Z} \mathfrak{X}$ of the character group. The following theorem provides a relation between characters of Demazure modules and faces of Gelfand–Zetlin polytopes.

Theorem 4.11 ([8, Theorem 5.1]). The character of a Demazure module $V_w \lambda$ is obtained as the sum over all integer points of the set of Kogan faces of $GZ \lambda$ corresponding to w :

$$\text{char } V_w \lambda = \sum_{x \in \Gamma_w \cap \mathbb{Z}^N} e^{\text{pr } x}.$$

Specializing the character at 0, we obtain the dimension formula for $V_w \lambda$:

Corollary 4.12. The Euler characteristic of $\mathcal{L}_w \lambda$ is equal to

$$\chi(\mathcal{L}_w \lambda) = \dim V_w \lambda = \# \Gamma_w \cap \mathbb{Z}^N.$$

We finish this section by identifying classes of structure sheaves of Schubert varieties in K_{GZ} . Since the Gelfand–Zetlin fan is not smooth it is more convenient for us to work with the module $M_{\widetilde{GZ}, GZ}$ which is associated to its smooth subdivision (See Proposition 2.8 and the end of Section 2.3). Recall that there is an epimorphism $\pi: K_{\widetilde{GZ}} \rightarrow M_{\widetilde{GZ}, GZ}$. Finally, define $\mathcal{D}_w \in K_{\widetilde{GZ}}$ via

$$\mathcal{D}_w = \sum_{\Gamma \in \mathcal{F}_w} (-1)^{\ell(w) - \dim \Gamma} D_\Gamma.$$

Theorem 4.13. Let $w \in W$. Then the class $\mathcal{O}_w$ of the structure sheaf of Schubert variety X_w is represented in K_{GZ} by $\pi(\mathcal{D}_w)$.

Proof. Since $K_0(G/B)$ is generated by $\text{Pic } G/B$, it is enough to check that $\chi(\mathcal{L}_w \lambda) = \mathcal{D}_w \cdot \text{Ehr } GZ \lambda$. Therefore the theorem follows from Corollary 4.12 and the fact that $\mathcal{D}_w \cdot \text{Ehr } GZ \lambda = \# \Gamma_w \cap \mathbb{Z}^N$ by the inclusion-exclusion formula. $\square$

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The Kromatic Symmetric Function: A K -theoretic analogue of X_G

Logan Crew¹, Oliver Pechenik¹, and Sophie Spirkl^{*1}

¹Department of Combinatorics and Optimization, University of Waterloo, Ontario, Canada

Abstract. Schur functions are a basis of the symmetric function ring that represent Schubert cohomology classes for Grassmannians. Replacing the cohomology ring with K -theory yields a rich combinatorial theory of inhomogeneous deformations, where Schur functions are replaced their K -analogues, the basis of *symmetric Grothendieck functions*. We introduce and initiate a theory of the *Kromatic symmetric function* $\overline{X}_G$, a K -theoretic analogue of the chromatic symmetric function X_G of a graph G . The Kromatic symmetric function is a generating series for graph colorings in which vertices may receive any nonempty set of distinct colors such that neighboring color sets are disjoint.

Our main result lifts a theorem of Gasharov (1996) to this setting, showing that when G is a claw-free incomparability graph, $\overline{X}_G$ is a positive sum of symmetric Grothendieck functions. This result suggests a topological interpretation of Gasharov's theorem. We then show that the Kromatic symmetric functions of path graphs are not positive in any of several K -analogues of the e -basis of symmetric functions, demonstrating that the Stanley–Stembridge conjecture (1993) does not have such a lift to K -theory and so is unlikely to be amenable to a topological perspective. We also define a vertex-weighted extension of $\overline{X}_G$ and show that it admits an edge deletion-contraction relation. Finally, we give a K -analogue for $\overline{X}_G$ of the classic monomial-basis expansion of X_G .

Keywords: chromatic symmetric function, K -theory, Stanley–Stembridge conjecture, deletion-contraction, symmetric Grothendieck function

1 Introduction

The *chromatic symmetric function* X_G of a graph G was introduced by R. Stanley [28] as a generalization of G.D. Birkhoff's *chromatic polynomial* [4]. While the chromatic polynomial enumerates proper graph colorings by the number of colors used, X_G also records

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how many times each color is used. A recent boom of research regarding X_G has focused on the *Stanley–Stembridge conjecture* [31], which proposes (in a reformulation by M. Guay-Paquet [17]) that unit interval graphs have chromatic symmetric functions that expand positively in the e -basis of the ring Sym of symmetric functions. In the last few years, various special cases of this conjecture have been established through direct combinatorial analysis, including the cases of *lollipop graphs* [12] and many *claw-free* graphs [18]. Another approach has been to consider various generalizations of the chromatic symmetric function and corresponding lifts of the Stanley–Stembridge conjecture. Examples of this latter approach include the *chromatic quasisymmetric function* and *Shareshian–Wachs conjecture* of [27] (further studied in, e.g., [1, 2, 6, 7]), and D. Gebhard–B. Sagan’s [16] chromatic symmetric function in noncommuting variables combined with notions of e -positivity and *appendable e -positivity* (further studied in, e.g., [3, 10, 13]). Our work provides a novel generalization of X_G in the same vein.

An important appearance of the ring of symmetric functions Sym is as the cohomology of complex *Grassmannians* (parameter spaces for linear subspaces of a vector space) or more precisely for the classifying space BU . Here, the *Schubert classes* derived from a natural cell decomposition of BU are represented by the *Schur function* basis s_λ of Sym . A richer perspective into the topology of BU is obtained by replacing cohomology with a generalized cohomology theory. In particular, there has been much focus on studying the associated combinatorics of the K -theory ring (see, e.g., [5, 23, 25, 32]). In this context, many of the classical objects of symmetric function theory are seen to have interesting K -analogues, often resembling “superpositions” of classical objects. For example, classical *semistandard Young tableaux* are replaced by *set-valued tableaux* (allowing multiple labels per cell), while Schur functions are replaced by *Grothendieck polynomials* $\bar{s}_\lambda$ (inhomogeneous deformations of s_λ).

Our work introduces a K -analogue of the chromatic symmetric function X_G , enumerating colorings of the graph G that assign a nonempty set of distinct colors to each vertex such that adjacent vertices receive disjoint sets. While our *Kromatic symmetric function* $\bar{X}_G$ is new, similar functions have been previously considered. The first such function was originally discussed by Stanley [29] in the context of graph analogues of symmetric functions, with connections to the real-rootedness of polynomials. Recently, as part of his effort to refine Schur-positivity results and the Stanley–Stembridge conjecture, B.-H. Hwang [19] studied a similar quasisymmetric function for graphs endowed with a fixed map $\alpha : V \rightarrow \mathbb{N}$ that dictates the size of the set of colors each vertex receives. To connect chromatic quasisymmetric functions of vertex-weighted graphs to *horizontal-strip LLT polynomials*, F. Tom [33] has considered a variant for fixed α with repeated colors allowed. Our work appears to be the first to connect these ideas to the combinatorics of K -theoretic Schubert calculus. (However, [24] is similar in spirit to our work, developing a K -theoretic analogue of the *Postnikov–Shapiro algebra* [26], an apparently unrelated invariant of graphs).

In this extended abstract, having introduced the *Kromatic symmetric function*, we begin to develop its combinatorial theory. Further details and proofs are given in the full manuscript [8]. We show that the Kromatic symmetric function $\overline{X}_G$ for any graph G expands positively in a K -theoretic analogue (that we also introduce) of the *monomial basis* of Sym . In this expansion, the coefficients enumerate coverings of the graph by (possibly overlapping) *stable sets*. We further extend the definition of $\overline{X}_G$ to a vertex-weighted setting, where we give a deletion-contraction relation analogous to that developed by the first and last authors [9] for the vertex-weighted version of X_G .

Our main result is that the Kromatic symmetric function of a *claw-free incomparability graph* expands positively in the symmetric Grothendieck basis $\overline{s}_\lambda$ of Sym , lifting to K -theory a celebrated result of V. Gasharov [15] that such graphs have Schur-positive chromatic symmetric functions. While all known proofs of Gasharov’s theorem are representation-theoretic or purely combinatorial, the existence of our K -theoretic analogue suggests that both results likely also have an interpretation in terms of the topology of Grassmannians. Precisely, for each claw-free incomparability graph G , there should be a subvariety of the Grassmannian whose cohomology class is represented by X_G and whose K -theoretic structure sheaf class is represented by $\overline{X}_G$. It would be very interesting to have an explicit construction of such subvarieties.

On the other hand, we show that the Kromatic symmetric functions $\overline{X}_{P_n}$ of path graphs P_n generally do not expand positively in either of two K -theoretic deformations we propose for the e -basis of Sym . This fact suggests that the Stanley–Stembridge conjecture, if true, is not naturally interpreted in terms of the cohomology of Grassmannians and is unlikely to be amenable to such topological tools from Schubert calculus. We hope these observations can play a similar role to [11] in limiting the range of potential avenues of attack on the Stanley–Stembridge conjecture.

This extended abstract is organized as follows. In Section 2, we provide an overview of the background and notation used from symmetric function theory (Section 2.1), K -theoretic Schubert calculus (Section 2.2), and graph theory (Section 2.3). In Section 3, we formally introduce the Kromatic symmetric function $\overline{X}_G$ and give its basic properties, including a formula for the expansion in a new K -analogue of the monomial-basis of Sym and a deletion-contraction relation for a vertex-weighted generalization. We also give our main theorem that the Kromatic symmetric functions of claw-free incomparability graphs expand positively in symmetric Grothendieck functions, lifting the main result of [15]. In Section 4, we introduce two different K -theoretic analogues of the e -basis of Sym and show that the Kromatic symmetric function $\overline{X}_{P_3}$ of a 3-vertex path graph P_3 is not positive in either analogue, casting doubt on hopes for a Schubert calculus-based approach to the Stanley–Stembridge conjecture.

2 Background

Throughout this work, $\mathbb{N}$ denotes the set of (strictly) positive integers. We write $[n]$ for the set of positive integers $\{1, 2, \dots, n\}$. If S is any set, 2^S denotes the power set of all subsets of S .

2.1 Partitions and Symmetric Functions

In this section, we give a brief overview of necessary background material necessary. Further details can be found, e.g., in the textbooks of Stanley [30], Manivel [22], and Macdonald [21].

An **integer partition** $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ is a finite nonincreasing sequence of positive integers. We define $\ell(\lambda)$ to be the length of the sequence λ (so above, $\ell(\lambda) = k$). Define $r_i(\lambda)$ to be the number of occurrences of i as a part of λ (so, for example, $r_1(2, 1, 1) = 2$). If

$$\sum_{i=1}^{\ell(\lambda)} \lambda_i = n,$$

we say that λ is a partition of n , and we write $\lambda \vdash n$. The **Young diagram of shape** λ is a set of squares called **cells**, left- and top-justified (i.e., in “English notation”), so that the i th row from the top contains λ_i cells. For example, the Young diagram of shape $(2, 2, 1)$ is . Let $C(\lambda)$ denote the set of cells of the Young diagram of shape λ . If $c \in C(\lambda)$

is a cell of the Young diagram of shape λ , we write $c^\uparrow$ for the cell immediately above c (assuming it exists), $c^\rightarrow$ for the cell immediately right of c , etc. We write $\lambda^\top$ for the **transpose** of λ , the integer partition whose Young diagram is obtained from that of λ by exchanging rows and columns.

Let $S_{\mathbb{N}}$ denote the set of all permutations of the set $\mathbb{N}$ fixing all but finitely-many elements. A **symmetric function** $f \in \mathbb{C}[[x_1, x_2, \dots]]$ is a power series of bounded degree such that for each permutation $\sigma \in S_{\mathbb{N}}$, we have $f(x_1, x_2, \dots) = f(x_{\sigma(1)}, x_{\sigma(2)}, \dots)$. The set $\text{Sym} \subset \mathbb{C}[[x_1, x_2, \dots]]$ of symmetric functions forms a $\mathbb{C}$ -vector space. Furthermore, if Λ^d denotes the set of symmetric functions that are homogeneous of degree d , then each Sym^d is a vector space, and

$$\text{Sym} = \bigoplus_{d=0}^{\infty} \text{Sym}^d$$

as graded vector spaces.

The dimension of Sym^d as a $\mathbb{C}$ -vector space is equal to the number of integer partitions of d , and many bases of symmetric functions are conveniently indexed by integer partitions. Below we provide some commonly used bases that will be used in this paper.

Definition 2.1. The following are bases of Sym :

- the *monomial symmetric functions* $\{m_\lambda\}$, defined as

$$m_\lambda = \sum x_{i_1}^{\lambda_1} \cdots x_{i_{\ell(\lambda)}}^{\lambda_{\ell(\lambda)}},$$

where the sum ranges over all distinct monomials formed by choosing distinct positive integers $i_1, \dots, i_{\ell(\lambda)}$;

- the *augmented monomial symmetric functions* $\{\tilde{m}_\lambda\}$, defined as

$$\tilde{m}_\lambda = \left(\prod_{i=1}^{\infty} r_i^{\lambda_i} i! \right) m_\lambda;$$

- the *elementary symmetric functions* $\{e_\lambda\}$, defined by

$$e_n = \prod_{i_1 < \cdots < i_n} x_{i_1} \cdots x_{i_n}; \quad e_\lambda = e_{\lambda_1} \cdots e_{\lambda_{\ell(\lambda)}};$$

- and the *complete homogeneous symmetric functions* $\{h_\lambda\}$, defined by

$$h_n = \prod_{i_1 \leq \cdots \leq i_n} x_{i_1} \cdots x_{i_n}; \quad h_\lambda = h_{\lambda_1} \cdots h_{\lambda_{\ell(\lambda)}}.$$

The space of symmetric functions is equipped with a natural inner product $\langle \cdot, \cdot \rangle$; it may be defined by $\langle h_\lambda, m_\mu \rangle = \delta_{\lambda, \mu}$, where $\delta_{\bullet, \bullet}$ denotes the *Kronecker delta function*.

We will also need the basis of Schur functions. A *Young tableau* of shape λ is a function $T : C(\lambda) \rightarrow \mathbb{N}$, typically visualized by writing the value $T(c)$ in the cell c . A Young tableau T of shape λ is *semistandard* if for each cell $c \in C(\lambda)$, we have $T(c) \leq T(c \rightarrow)$ and $T(c) < T(c \downarrow)$ whenever the cells in question exist. We write $\text{SSYT}(\lambda)$ for the set of all semistandard Young tableaux of shape λ . The *Schur function* s_λ is defined by

$$s_\lambda = \sum_{T \in \text{SSYT}(\lambda)} x^T, \quad \text{where } x^T = \prod_{c \in C(\lambda)} x_{T(c)}.$$

As λ ranges over integer partitions, the Schur functions are another basis of Sym . The inner product on Sym also satisfies $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda, \mu}$.

When $f \in \text{Sym}$ is a symmetric function and $\{b_\lambda\}$ is a basis of symmetric functions indexed by integer partitions λ , the notation $b_\mu f$ denotes the coefficient of b_μ when f is expanded in the b -basis. A symmetric function $f \in \text{Sym}$ is said to be *b-positive* if $b_\mu f$ is nonnegative for every integer partition μ .

2.2 K -theoretic Schubert calculus

The **Grassmannian** $\Gamma_k = \text{Gr}_k \mathbb{C}^\infty$ is the parameter space of k -dimensional vector subspaces of the space of all eventually-zero sequences of complex numbers. The space Γ_k can be given the structure of a projective Ind-variety and has a cell decomposition into cells Γ_λ indexed by partitions with at most k parts. Each Γ_λ induces a cohomology class $\sigma_\lambda \in H^*(\Gamma_k)$ and classically we have $H^*(\Gamma_k) \cong \text{Sym}_k = \text{Sym} \cap \mathbb{C}[x_1, \dots, x_k]$ with the isomorphism taking the class of the cell σ_λ to the Schur polynomial $s_\lambda(x_1, \dots, x_k)$.

Each cell-closure in Γ_k also has a structure sheaf, inducing a class in the representable **K -theory** ring $K^0(\Gamma_k)$. These K -theoretic classes are represented by inhomogeneous symmetric polynomials called *Grothendieck polynomials* $\bar{s}_\lambda(x_1, \dots, x_k)$.

A **set-valued tableau** of shape λ is a filling T of each cell of $C(\lambda)$ with a nonempty set of positive integers. The set-valued tableau T is **semistandard** if for each cell $c \in C(\lambda)$, we have $\max T(c) \leq \min T(c \rightarrow)$ and $\max T(c) < \min T(c \downarrow)$ whenever the cells in question exist. In other words, T is semistandard if every Young tableau formed by choosing one number from the set of each cell is semistandard. Let $\text{SV}(\lambda)$ denote the set of all semistandard set-valued tableaux of shape λ . The **symmetric Grothendieck function** $\bar{s}_\lambda$ is

$$\bar{s}_\lambda = \sum_{T \in \text{SV}(\lambda)} (-1)^{|T| - \ell(\lambda)} x^T,$$

where $|T| = \sum_{c \in C(\lambda)} |T(c)|$ and $x^T = \prod_{c \in C(\lambda)} \prod_{i \in T(c)} x_i$. Note that $\bar{s}_\lambda$ contains terms of degree greater than or equal to $|\lambda|$, and that the sum of all of its lowest-degree terms is equal to s_λ .

2.3 Graphs and Coloring

Here, we recall basic notions, terminology, and notations from graph theory. For further details, see, e.g., the textbooks [14, 34].

A **graph** G consists of a set V of **vertices**, and a set E of unordered pairs of distinct vertices called **edges**. All graphs in this paper are *simple*, so there are no loops, and no multi-edges. When $\{v_1, v_2\} \in E(G)$, we will typically denote this edge by $v_1 v_2$ and say v_1 and v_2 are **adjacent**. Two graphs G, G' are **isomorphic** if there is a bijection $\phi : V(G) \rightarrow V(G')$ such that $uv \in E(G)$ if and only if $\phi(u)\phi(v) \in E(G')$. In this paper, we consider graphs up to isomorphism.

The **complete graph** K_d with d vertices is the graph such that $V(K_d) = [d]$, and

$$E(K_d) = \{vw : v, w \in [d], v \neq w\}.$$

The n -vertex **path** P_n has vertex set $V(P_n) = [n]$ and edge set $E(P_n) = \{uv : u, v \in [n], v - u = 1\}$. The **claw** $K_{1,3}$ has vertex set $V(K_{1,3}) = [4]$ and edge set $E(K_{1,3}) = \{12, 13, 14\}$.

A **stable set** (or **independent set**) of a graph G is a set $S \subseteq V(G)$ of vertices such that for each $v, w \in S$, $vw \notin E(G)$. A **clique** of a graph G is a set $S \subseteq V(G)$ of vertices such that for each $v, w \in S$, $vw \in E(G)$.

For $\alpha : V(G) \rightarrow \mathbb{N}$ a vertex weighting of the graph G , the **α -clan graph of G** is the graph $C_\alpha(G)$ obtained by blowing up each vertex v into a clique of $\alpha(v)$ vertices. More formally, $C_\alpha(G)$ has vertex set $V(C_\alpha(G)) = \{v, i : v \in V(G), i \in \alpha(v)\}$. In $C_\alpha(G)$, the vertices v, i and w, j are adjacent either if $vw \in E(G)$ or if both $v = w$ and $i \neq j$.

Given a vertex $v \in V(G)$, its **open neighborhood** $N(v)$ is defined by $N(v) = \{w : vw \in E(G)\}$. The **contraction** of a graph G by an edge $vw \in E(G)$, notated G/e , is the graph with vertex set $V(G/e) = V(G) \setminus \{v, w\} \cup \{v^*\}$, and edge set $E(G/e) = E(G) \setminus vN(v) \cup wN(w) \cup v^*N(v) \cup v^*N(w)$.

A **coloring** of a graph G is a function $\kappa : V(G) \rightarrow \mathbb{N}$. A coloring κ of G is **proper** if $\kappa(a) \neq \kappa(b)$ whenever $ab \in E(G)$.

The **chromatic symmetric function**[28] of a graph G is the power series

$$X_G = \sum_{\kappa} \prod_{v \in V(G)} x_{\kappa(v)}$$

where the first sum ranges over all proper colorings κ of G . Note that, for every graph G , $X_G \in \text{Sym}$.

3 The Kromatic Symmetric Function

A **vertex-weighted graph** (G, α) consists of a graph G together with a function $\alpha : V(G) \rightarrow \mathbb{N}$; we call α the **weight function** on the vertices of G . A **proper α -coloring** of G is a function $\kappa : V(G) \rightarrow 2^{\mathbb{N}} \setminus \{\emptyset\}$ assigning to each $v \in V(G)$ a set of $\alpha(v)$ distinct colors in $\mathbb{N}$, subject to the constraint that when $uv \in E(G)$, we have $\kappa(u) \cap \kappa(v) = \emptyset$. Note that these conditions are equivalent to saying that any choice of a single element from each $\kappa(v)$ yields a proper coloring of G . A **proper set-coloring** of G is a proper α -coloring for some weight function on the vertices of G .

The **chromatic symmetric function** of the vertex-weighted graph (G, α) is

$$X_G^\alpha = \sum_{\kappa} \prod_{v \in V(G)} \prod_{i \in \kappa(v)} x_i,$$

where the first sum runs over all proper α -colorings of G . Note that up to a scalar factor depending only on α , the chromatic symmetric function X_G^α equals the chromatic symmetric function $X_{C_\alpha(G)}$ of the α -clan graph of G .

Definition 3.1. The **Kromatic symmetric function** of a graph G is the symmetric power series

$$\bar{X}_G = \sum_{\alpha} X_G^\alpha,$$

where α ranges over all weightings of the vertex set $V \setminus G$.

In other words, $\overline{X}_G$ enumerates all colorings of G by nonempty sets of colors, such that adjacent vertices receive disjoint sets of colors. Note that $\overline{X}_G$ is not a homogeneous symmetric function, but rather consists of X_G plus terms of degree higher than $|V \setminus G|$.

Remark 3.2. Stanley [29] considered a function Y_G related to $\overline{X}_G$, although with two differences. Firstly, Y_G uses the rescaled power series $X_{C_\alpha \setminus G}$ in place of X_G^α . Secondly, Y_G allows $\alpha_v = 0$, whereas the Kromatic symmetric function $\overline{X}_G$ only considers strictly positive vertex weightings. We are unaware of any further study of the functions Y_G since their introduction in [29].

Although vertex weightings are used in the definition of $\overline{X}_G$, the function $\overline{X}_G$ is independent of any particular vertex weighting. We will find it useful to also consider a vertex-weighted analogue of $\overline{X}_G$. Let α and w be independent vertex weightings on G . Define

$$X_{G,w}^\alpha = \sum_{\kappa} \prod_{v \in V \setminus G} \left(\prod_{i \in \kappa \setminus v} x_i \right)^{w_v},$$

where again the first sum runs over all proper α -colorings of G . Finally, we define the *vertex-weighted Kromatic symmetric function* of the vertex-weighted graph (G, w) to be

$$\overline{X}_{G,w} = \sum_{\alpha} X_{G,w}^\alpha,$$

where the sum is over all weight functions α .

For λ an integer partition, let K_λ denote the vertex-weighted complete graph $K_{\ell(\lambda), w}$, where $w_i = \lambda_i$ for each i . It is straightforward to see that $X_{K_\lambda} = \widetilde{m}_\lambda$, the augmented monomial symmetric function. Thus, by analogy, we define

$$\widetilde{m}_\lambda : \overline{X}_{K_\lambda} = \sum_{\alpha} \widetilde{m}_{\lambda_1^{\alpha_1}, \dots, \lambda_{\ell(\lambda)}^{\alpha_{\ell(\lambda)}}},$$

where the sum is over all vertex weightings α of $K_{\ell(\lambda)}$. We call $\widetilde{m}_\lambda$ the *K-theoretic augmented monomial symmetric function*. To justify this definition, we show that the Kromatic symmetric function of every graph (even every vertex-weighted graph) is a positive sum of K -theoretic augmented monomial symmetric functions.

First, we need some additional definitions. We define a *stable set cover* C of a graph G to be a collection of stable sets of G such that every vertex of $V \setminus G$ is in at least one element of C . In symbols, this means that

$$\bigcup_{S \in C} S = V \setminus G;$$

note that this union is not required to be disjoint. We write $\text{SSC}(G)$ for the family of all stable set covers of G . For $C \in \text{SSC}(G)$, if G is endowed with a vertex-weighting w , let

$\lambda \vdash C$ be the partition of length $|C|$ whose parts are $\sum_{v \in S} w_v$ for $S \in C$. Finally, let the **color class** of the color i in a proper set-coloring κ be $\{v \in V(G) : i \in \kappa(v)\}$, the set of vertices of G that receive color i under κ .

Proposition 3.3. *For any vertex-weighted graph (G, w) , we have*

$$\overline{X}_{G,w} = \sum_{C \in \text{SSC}(G)} \overline{m}_\lambda \cdot C.$$

The Kromatic symmetric function for vertex-weighted graphs also admits a deletion-contraction relation, analogous to that of [9] for the chromatic symmetric function.

Given $S \subseteq V(G)$ and $v \in V(G)$ with $v \notin S$, we let $vS \subseteq E(G)$ denote $\{vs : s \in S\}$.

Proposition 3.4. *Let (G, w) be a vertex-weighted graph, and let v_1 and v_2 be vertices such that $e = v_1v_2 \notin E(G)$. Then*

$$\overline{X}_{G,w} = \overline{X}_{G/e,w/e} + \overline{X}_{G \cup e,w} - \overline{X}_{G \cup e \cup v_1N_{v_2},w^1} - \overline{X}_{G \cup e \cup v_2N_{v_1},w^2} + \overline{X}_{G^*,w^*} \quad (3.1)$$

where

- $w^1_{v_1} = w_{v_1}$, $w^1_{v_2} = w_{v_2}$, and for all other $v \in V(G)$, $w^1_v = w_v$.
- $w^2_{v_2} = w_{v_1}$, $w^2_{v_1} = w_{v_2}$, and for all other $v \in V(G)$, $w^2_v = w_v$.
- $G^* = V(G) \cup v_3, E(G) \cup \{v_1v_2, v_1v_3, v_2v_3\} \cup v_3N_{v_1} \cup v_3N_{v_2}$.
- $w^*_{v_3} = w_{v_1} + w_{v_2}$, and for all other $v \in V(G)$, $w^*_v = w_v$.

The deletion-contraction relation of Proposition 3.4 can be used to yield algorithmically the $\overline{m}_\lambda$ -expansion of a Kromatic symmetric function $\overline{X}_{G,w}$.

Corollary 3.5. *Arbitrarily applying Proposition 3.4 recursively to a graph (G, w) (repeatedly applying it to any nonedge of each graph formed if possible) terminates in a sum of Kromatic symmetric functions of vertex-weighted complete graphs, yielding the $\overline{m}_\lambda$ expansion of $\overline{X}_{G,w}$.*

Our main result is the following theorem.

Theorem 3.6. *If G is a claw-free incomparability graph, then $\overline{X}_G$ is Grothendieck-positive.*

Proof. (Sketch.) We use the Jacobi-Trudi formula of Iwao [20, Proposition 4.4] to write a dual Grothendieck function as a sum of products of complete homogeneous symmetric functions. Then, for any graph G , the inner product of this expression with $\overline{X}_G$ yields a formula for the coefficient of $\overline{s}_\lambda$ in the Grothendieck expansion of $\overline{X}_G$ in terms of its monomial expansion. In the case that G is a claw-free incomparability graph, we then extend Gasharov's [15] theory of P -arrays to collect terms in this expansion and show that the coefficient of any $\overline{s}_\lambda$ is nonnegative. $\square$

We would be very interested in a solution to the following.

Problem 3.7. *For each claw-free incomparability graph G , find a corresponding subvariety V_G of the Grassmannian such that the cohomology class of V_G is represented in Sym by X_G and the structure sheaf class of V_G is represented by $\overline{X}_G$.*

4 Analogues of the Stanley–Stembridge conjecture

The previous section shows that Schur-positivity of X_G when G is a claw-free incomparability graph lifts to an analogue for $\overline{X}_G$. It is natural to ask if it is similarly possible to lift the *Stanley–Stembridge conjecture* — claiming that such X_G are e -positive — to the Kromatic setting. However, it appears that the answer is “no.”

We propose two definitions for a lift of the e -basis to the K -theoretic setting. On one hand, e -basis elements in usual symmetric function theory may be defined in terms of fillings of single-column Young diagrams, so we may lift this formula.

Definition 4.1. The *tableau K -elementary symmetric function* $\bar{e}_\lambda$ is given by

$$\bar{e}_n = \bar{s}_{1^n} \quad \text{and} \quad \bar{e}_\lambda = \bar{e}_{\lambda_1} \cdots \bar{e}_{\lambda_\ell \lambda}.$$

On the other hand, we may also define $e_n = \frac{1}{n!} X_{K_n}$, and lift this characterization.

Definition 4.2. The *graph K -elementary symmetric function* is given by

$$\bar{e}'_n = \frac{1}{n!} \overline{X}_{K_n} \quad \text{and} \quad \bar{e}'_\lambda = \bar{e}'_{\lambda_1} \cdots \bar{e}'_{\lambda_\ell \lambda}.$$

It is reasonable to hope (for extending the Stanley–Stembridge conjecture) that $\overline{X}_G$ is positive in one of these K -theoretic e -bases, whenever G is a claw-free incomparability graph, or even just when G is a unit interval graph. However, one can compute that $\overline{X}_{P_3}$ is not positive in either K -theoretic e -basis, dashing such hopes.

The terms of $\overline{X}_{P_3}$ that are homogeneous of degree 3 must come from tableau or graph K -elementary symmetric functions of degree 3, and have coefficients corresponding to e -expansion of X_{P_3} . Since $X_{P_3} = 3e_3 + e_{21}$, one sees that the terms of $\overline{X}_{P_3}$ for $|\lambda| = 3$ in the $\bar{e}$ -basis are $3\bar{e}_3 + \bar{e}_{21}$, and in the $\bar{e}'$ -basis are $3\bar{e}'_3 + \bar{e}'_{21}$. However, we now run into problems with the $|\lambda| = 4$ terms. In particular, both $\bar{e}_{21}$ and $\bar{e}'_{21}$ are supported on the monomial $x_1^2 x_2^2$, with two distinct variables each of degree 2. However, it is easy to check that there is no proper set-coloring of P_3 using exactly 1 twice and 2 twice; thus, these monomials must be cancelled by $\bar{e}$ or $\bar{e}'$ terms with strictly negative coefficients.

That this breakdown is so fundamental suggests that it may not be possible to reasonably generalize e -positivity to the Kromatic symmetric function, in stark contrast with the generalization of Schur-positivity given in [Theorem 3.6](#). This suggests that the Stanley–Stembridge is not amenable to a topological interpretation along the lines of [Problem 3.7](#).

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Foata-like bijections and science fiction

Sam Armon^{*1}

¹Department of Mathematics, University of Southern California

Abstract. A central open problem in algebraic combinatorics is to find a combinatorial formula for the Kostka-Macdonald polynomials $\tilde{K}_{\lambda\mu}(q, t)$, which describe the expansion of the Macdonald polynomial $\tilde{H}_{\mu}(Z; q, t)$ in the Schur basis. Haiman proved that $\tilde{K}_{\lambda\mu}(q, t)$ has nonnegative integer coefficients by proving that the dimension of the Garsia-Haiman module $\mathcal{H}_{\mu}$ equals $n!$, demonstrating an intricate relationship between the Kostka-Macdonald polynomials and this module. This relationship was further expounded upon by Bergeron and Garsia, whose "science fiction" heuristics conjecture certain intersection properties of Garsia-Haiman modules which mirror observed symmetries in the Kostka-Macdonald coefficients. The most potent of these heuristics is the $\frac{n!}{k}$ conjecture, which asserts that the dimension of the intersection of k Garsia-Haiman modules should have dimension $\frac{n!}{k}$. We solve the special case of the $\frac{n!}{k}$ conjecture where the indexing partitions have hook shape by constructing an explicit basis for the intersection, using two maps in the spirit of Foata's.

Keywords: Macdonald polynomials, Garsia-Haiman modules, bijective combinatorics

1 Introduction

The (transformed) Macdonald polynomials $\{\tilde{H}_{\mu}(Z; q, t) : \mu \text{ an integer partition}\}$ form a notable basis for the ring of symmetric functions, as they simultaneously generalize the Schur functions, Hall-Littlewood polynomials, and Jack symmetric functions, among others. As such, they have become a central object of study since their introduction by Macdonald in [11]. Of particular interest is their expansion in the basis of Schur functions $\{s_{\lambda}(Z)\}$:

$$\tilde{H}_{\mu}(Z; q, t) = \sum_{\lambda} \tilde{K}_{\lambda\mu}(q, t) s_{\lambda}(Z).$$

A priori we have $\tilde{K}_{\lambda\mu}(q, t) \in \mathbb{Q}(q, t)$, but Macdonald's *positivity theorem* asserts that $\tilde{K}_{\lambda\mu}(q, t)$ is in fact a polynomial in q and t with nonnegative integer coefficients. It remains an important open problem to provide a combinatorial proof of Macdonald positivity — e.g. writing $\tilde{K}_{\lambda\mu}(q, t)$ as a q, t -weighted sum over some set of combinatorial objects — but Haiman [10] resolved the positivity theorem algebraically, using a representation-theoretic analog $\mathcal{H}_{\mu}$ of $\tilde{H}_{\mu}(Z; q, t)$.

^{*}armon@usc.edu.

For μ an integer partition of $n \in \mathbb{N}$, the *Garsia-Haiman module* $\mathcal{H}_\mu$ is a bigraded symmetric group module defined as follows (see [7] for full details). Let $X = x_1, x_2, \dots, x_n$, $Y = y_1, y_2, \dots, y_n$, and define

$$\mathfrak{I}_\mu = \left\{ f \in \mathbb{Q}[X, Y] : f \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}; \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n} \Delta_\mu = 0 \right\}$$

to be the ideal of polynomials whose corresponding differential operator annihilates a certain $\mathfrak{S}_n$ -alternating, doubly homogeneous polynomial $\Delta_\mu \in \mathbb{Q}[X, Y]$, so that $\mathfrak{I}_\mu$ is $\mathfrak{S}_n$ -invariant. The Garsia-Haiman module is defined as $\mathcal{H}_\mu = \mathbb{Q}[X, Y] / \mathfrak{I}_\mu$; it inherits the diagonal action of $\mathfrak{S}_n$ on $\mathbb{Q}[X, Y]$, and is afforded a bigrading since Δ_μ is doubly homogeneous, so we may write

$$\mathcal{H}_\mu = \bigoplus_{i,j} \mathcal{H}_{\mu(i,j)}.$$

Haiman made the following connection between $\mathcal{H}_\mu$ and the Macdonald polynomials:

Theorem 1 ([9]). *If the dimension of $\mathcal{H}_\mu$ is $n!$, then the bigraded Frobenius series of $\mathcal{H}_\mu$ given by*

$$\text{Frob}_{\mathcal{H}_\mu}(Z; q, t) = \sum_{i,j} q^i t^j \text{ch } \mathcal{H}_{\mu(i,j)},$$

where ch is the Frobenius map which sends the irreducible $\mathfrak{S}_n$ -module S^λ to the Schur function s_λ , equals the transformed Macdonald polynomial $\tilde{H}_\mu(Z; q, t)$.

Due to known identities involving Macdonald polynomials it is necessary that $\mathcal{H}_\mu$ afford a bigraded version of the regular representation, hence the dimension condition, but in fact this is also sufficient.

Haiman proved that $\dim \mathcal{H}_\mu = n!$ in [10], thus resolving the Macdonald positivity theorem by realizing $\tilde{K}_{\lambda_\mu}(q, t)$ as encoding the doubly graded multiplicity of the irreducible $\mathfrak{S}_n$ -module S^λ in $\mathcal{H}_\mu$. Haiman's proof, however, is algebro-geometric in nature, and does not afford an explicit basis for $\mathcal{H}_\mu$. The construction of such a basis in general remains an open problem, although some special cases have been solved, in particular when μ has hook shape (see [1],[2],[3],[4],[8],[12]).

With an eye towards a combinatorial proof of Macdonald positivity, it is desirable to study $\mathcal{H}_\mu$ more combinatorially. Along these lines, Bergeron and Garsia undertook a speculative study of some of the remarkable intersection properties possessed by the modules $\mathcal{H}_\mu$ in [5]. Their most potent assertion is the following $\frac{n!}{k}$ conjecture:

Conjecture 1 ([5]). *Let λ be an integer partition of $n - 1$, and let $\mu^1, \dots, \mu^k$ be partitions of n each obtained from λ by removing a removable cell from the Young diagram of λ . Then,*

$$\dim \left(\bigcap_{i=1}^k \mathcal{H}_{\mu^i} \right) = \frac{n!}{k}.$$

In the case that λ has hook shape — i.e. $\lambda = a, 1, 1, \dots, 1 = a, 1^\ell$ for some $a \geq 2, \ell \geq 1$ — we resolve the $\frac{n!}{k}$ conjecture by constructing an explicit basis for $\mathcal{H}_{a, 1^{\ell-1}} \cap \mathcal{H}_{a-1, 1^\ell}$.

2 The $\frac{n!}{k}$ conjecture for hook shapes

Adin, Remmel and Roichman define a plethora of different bases for $\mathcal{H}_\mu$ when μ has hook shape in [1]. To tackle the $\frac{n!}{k}$ conjecture in this special case, we adopt a different viewpoint on their *Artin basis*. This basis is indexed by permutations $w \in \mathfrak{S}_n$ and each basis element encodes information about certain inversions in w which depend on the partition μ . We restate this basis in terms of *standard fillings* of μ .

Throughout, we identify a partition $\mu = \mu_1 \geq \mu_2 \geq \dots$ with its Young diagram, opting for French (coordinate) notation, so that μ is depicted diagrammatically as an array of bottom- and left-justified cells with μ_i cells in row i . We call a cell $c \in \mu$ *removable* if the diagram $\mu - \{c\}$ still has partition shape.

Definition 1. For any partition μ of $n \in \mathbb{N}$, a *standard filling* of μ is a bijective map $S : \mu \rightarrow \{1, 2, \dots, n\}$, i.e. an assignment of the numbers $1, 2, \dots, n$ to the cells of (the Young diagram of) μ . The set of all standard fillings of μ is denoted $SF \mu$.

By assigning any arbitrary order to the cells of μ , we obtain a bijection between $SF \mu$ and $\mathfrak{S}_n$, i.e. $|SF \mu| = n!$. For instance, the $3! = 6$ standard fillings of $\mu = 2, 1$ are:

$$\begin{array}{|c|c|} \hline 3 & \\ \hline 1 & 2 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 2 & \\ \hline 1 & 3 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 3 & \\ \hline 2 & 1 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & \\ \hline 2 & 3 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 2 & \\ \hline 3 & 1 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & \\ \hline 3 & 2 \\ \hline \end{array}.$$

For any standard filling S , we let $S_{i,j}$ denote the entry in row i , column j of S . We now hone in on the case where $\mu = a, 1^\ell$ has hook shape.

Definition 2. Given $S \in SF a, 1^\ell$, define a *row inversion* to be a pair $S_{1,i} > S_{1,j}$ where $i < j$. Similarly, define a *column inversion* to be a pair $S_{j,1} > S_{i,1}$ for $i < j$ (note the swapped indices). Denote the collection of row (resp. column) inversions in S by

$$\begin{aligned} \text{rowInv } S &= \{ t, r : t > r, t \text{ left of } r \text{ in row 1 of } S \}, \\ \text{collnv } S &= \{ d, c : d > c, d \text{ above } c \text{ in column 1 of } S \}. \end{aligned}$$

Definition 3. Given $S \in SF a, 1^\ell$, define the polynomial $\varphi_S(X, Y) \in \mathbb{Q}[X, Y]$ by

$$\varphi_S(X, Y) = \prod_{d, c \in \text{collnv } S} x_d \prod_{t, r \in \text{rowInv } S} y_r \quad (2.1)$$

(cf. [1], Definition 1.6).

This definition is seen to be equivalent to [1] by associating a standard filling S of μ with its reading word $w_S \in \mathfrak{S}_n$. Note also that $\varphi_S(X, Y)$ encodes only the information of the *larger* elements in column inversions, and of the *smaller* elements in row inversions:

Example 1. The following standard filling of $\mu = 5, 1^4$:

$$S \quad \begin{array}{|c|} \hline 4 \\ \hline 1 \\ \hline 9 \\ \hline 7 \\ \hline 5 & 6 & 3 & 2 & 8 \\ \hline \end{array}$$

has

$$\begin{aligned} \text{colInv } S &= \{ \textcolor{blue}{4}, 1, \textcolor{blue}{9}, 7, \textcolor{blue}{9}, 5, \textcolor{blue}{7}, 5 \}, \\ \text{rowInv } S &= \{ 5, \textcolor{red}{3}, 5, \textcolor{red}{2}, 6, \textcolor{red}{3}, 6, \textcolor{red}{2}, 3, \textcolor{red}{2} \}, \end{aligned}$$

so $\varphi_S(X, Y) = \textcolor{blue}{x}_4 \textcolor{blue}{x}_7 \textcolor{blue}{x}_9^2 \textcolor{red}{y}_2^3 \textcolor{red}{y}_3^2$.

Theorem 2 ([1], Corollary 1.7). The set

$$\{\varphi_S(X, Y) : S \in SF(a, 1^\ell)\}$$

forms a basis for the Garsia-Haiman module $\mathcal{H}_{a, 1^\ell}$.

Equipped with this basis, we prove the following special case of the $\frac{n!}{k}$ conjecture:

Theorem 3 ($\frac{n!}{k}$ for hook shapes). The $\frac{n!}{k}$ conjecture holds for the hook shape $\lambda = a, 1^\ell$, i.e.

$$\dim \mathcal{H}_{a, 1^{\ell-1}} \cap \mathcal{H}_{a-1, 1^\ell} = \frac{n!}{2}.$$

As a hook shape has at most two removable boxes (one at the end of the first row, and one at the end of the first column), this result is as strong as possible for hook shapes since the case $k \geq 3$ is trivial.

The Artin basis of [1] is as "nice" as can be in the sense that, not only do we prove that $\mathcal{H}_{a, 1^{\ell-1}} \cap \mathcal{H}_{a-1, 1^\ell}$ has the correct dimension, we actually obtain an explicit basis for the intersection by finding exactly $\frac{n!}{2}$ standard fillings of $a, 1^{\ell-1}$ and of $a-1, 1^\ell$ which yield the same basis element.

To ease the notational burden, we fix once and for all the partitions $\mu = a, 1^{\ell-1}$ and $\rho = a-1, 1^\ell$, for some $a \geq 2$ and $\ell \geq 1$. Define the following subsets of $SF \mu$ and $SF \rho$:

$$SF_{<} \mu = \{S \in SF \mu : S_{1,1} < S_{1,a}\}, \quad SF_{<} \rho = \{T \in SF \rho : T_{\ell-1,1} < T_{1,1}\}. \quad (2.2)$$

Then clearly $|SF_{<} \mu| = |SF_{<} \rho| = \frac{n!}{2}$. We show in turn that

$$\{\varphi_S X, Y : S \in SF_{<} \mu\} = \{\varphi_T X, Y : T \in SF_{<} \rho\} \quad (2.3)$$

by defining a bijective map $\theta : SF_{<} \mu \rightarrow SF_{<} \rho$ which satisfies $\varphi_{\theta S} = \varphi_S$.

Heuristically, the standard fillings in (2.2) are the "right" candidates for a basis of $\mathcal{H}_\mu \cap \mathcal{H}_\rho$ because no standard filling of ρ has a row inversion whose elements are a units apart, and conversely no standard filling of μ has a column inversion whose elements are $\ell - 1$ units apart.

As a first naïve step, we may define a simple auxiliary map which translates between standard fillings of μ and of ρ :

Definition 4. For any $S \in SF_{<} \mu$, define a standard filling $\text{bump } S \in SF_{<} \rho$ by moving each entry in the first column of S up one row, and then pushing each remaining entry in the first row to the left one column.

Then $\text{bump} : SF_{<} \mu \rightarrow SF_{<} \rho$ is a bijection whose inverse is given by first pushing the first row to the right, and then pushing the first column down. The problem, however, is that bump does not restrict to a bijection $SF_{<} \mu \rightarrow SF_{<} \rho$:

Example 2. For $\mu = 4, 1^3$ and $\rho = 3, 1^4$, we have

$$\text{bump} \left(\begin{array}{|c|c|c|c|} \hline 5 & & & \\ \hline 1 & & & \\ \hline 2 & & & \\ \hline 6 & 4 & 3 & 7 \\ \hline \end{array} \right) = \begin{array}{|c|c|c|} \hline 5 \\ \hline 1 \\ \hline 2 \\ \hline 6 \\ \hline 4 & 3 & 7 \\ \hline \end{array} .$$

Then $S \in SF_{<} \mu$ since $6 < 7$, yet $\text{bump } S \notin SF_{<} \rho$ since $5 > 4$.

In addition to correcting this deficiency, we must also ensure that the row and column inversions in the resulting standard filling align with the original. Thus after performing bump , we must take the following into consideration:

1. There may be row inversions in S , created by $u = S_{1,1}$, which are no longer present in $\text{bump } S$. We thus must rearrange the first row so as to reintroduce these lost row inversions.
2. After rearranging the first row of $\text{bump } S$, there is a new entry v in row 1, column 1, so we may have introduced some column inversions which were not present in S . Thus we must rearrange the first column both in order to negate these newly introduced column inversions, and in order to ensure that the $(\ell - 1, 1)$ -entry is less than the $(1, 1)$ -entry, so that the resulting filling lies in $SF_{<} \rho$.

To this end, we define a map arm_u which reintroduces the row inversions removed in 1., and a map leg_v which negates the column inversions introduced in 2.; by appropriately composing these maps with bump, we obtain a map which proves (2.3).

Definition 5. For a fixed $u \in \mathbb{N}$, consider the set of words

$$\mathcal{A}_u = \{w_1 w_2 \cdots w_m : m \in \mathbb{N}, w_m > u\}.$$

For any $w = w_1 w_2 \cdots w_m \in \mathcal{A}_u$, define a new word $\text{arm}_u w$ as follows: let $b_1, b_2, \dots, b_j$ denote the indices for which $w_{b_i} < u$. If there are no such indices, then define $\text{arm}_u w = w$. Otherwise,

1. Draw a vertical bar immediately to the left of w_{b_i} if either $w_{b_i-1} > u$, or if $i = 1$ and $b_1 = 1$.
2. Within each newly created block which contains at least one of the w_{b_i} 's, move the leftmost entry which is $> u$ to the front of the block, immediately right of the (leftmost) vertical bar.

Define the resulting word to be $\text{arm}_u w$.

In context, for $S \in SF_{< \mu}$ and $u = S_{1,1}$, we wish to perform arm_u on the word $S_{1,2} S_{1,3} \cdots S_{1,a}$ consisting of the entries in the first row of S which lie strictly right of the first column (note that by construction this word lies in $\mathcal{A}_u$). We then determine which entries in row 1 of S are $< u$, as these correspond precisely to the row inversions in S that are lost in $\text{bump } S$ when u gets bumped into the second row. A larger number is then shuffled to the left of each of these entries in order to reintroduce the lost row inversions.

Example 3. Let $w = 49263187 \in \mathcal{A}_5$. To compute $\text{arm}_5 w$ we draw vertical bars to the left of 4, 2, and 3, and within each block shuffle the leftmost number > 5 to the front:

$$\begin{array}{cccccccc} | & 4 & 9 & | & 2 & 6 & | & 3 & 1 & 8 & 7 \\ \curvearrowright & & & \curvearrowright & & & \curvearrowright & & & & \end{array} \mapsto \begin{array}{cccccccc} | & 9 & 4 & | & 6 & 2 & | & 8 & 3 & 1 & 7 \end{array}$$

So, $\text{arm}_5 w = 94628317$. Note also that in the word $5w = 549263187$, we have inversions $5 > 4, 5 > 2, 5 > 3, 5 > 1$ which are not present in w . However, when we perform arm_5 on w , we obtain new inversions $9 > 4, 6 > 2, 8 > 3, 8 > 1$ and no others, so the smaller entries in the inversions of $5w$ and of $\text{arm}_5 w$ are the same.

After shuffling around the first row of $\text{bump } S$ to correct for the lost row inversions, we must appropriately reorder the first column, for which we need the following map.

Definition 6. For a fixed $v \in \mathbb{N}$, consider the set of words

$$\mathcal{L}_v = \{w_1 w_2 \cdots w_m : m \in \mathbb{N}, w_1 < v\}.$$

For any $w = w_1 w_2 \cdots w_m \in \mathcal{L}_v$, define a new word $\text{leg}_v w$ as follows: let $c_1, c_2, \dots, c_k$ denote the indices for which $w_{c_i} > v$. If there are no such indices, then define $\text{leg}_v w = w$. Otherwise,

1. Draw a vertical bar immediately to the right of w_{c_i} if either $w_{c_i-1} < v$, or if $i = k$ and $c_k = m$.
2. Within each newly created block which contains at least one of the w_{c_i} 's, move the rightmost entry which is $< v$ to the end of the block, immediately left of the (rightmost) vertical bar.

Define the resulting word to be $\text{leg}_v w$.

Note that leg_v is in a sense "dual" to arm_u , in that arm_u shuffles larger entries to the left, and leg_v shuffles smaller entries to the right. Again, for a bit of context, we wish to perform leg_v on the first column of a filling after performing bump, and then $\text{arm}_{S_{1,1}}$, on some $S \in SF_{< \mu}$, when a new entry is introduced into row 1, column 1. Denoting the resulting filling by T , we will have $v = T_{1,1} < T_{2,1}$ by construction, so that $T_{2,1}T_{3,1} \cdots T_{\ell-1,1} \in \mathcal{L}_v$, and leg_v then shuffles smaller numbers further up in the first column to negate the column inversions created by v .

Example 4. Let $w = 48731926 \in \mathcal{L}_5$. To compute $\text{leg}_5 w$ we draw vertical bars to the right of 7, 9, and 6, and within each block shuffle the rightmost number < 5 to the end:

$$\begin{array}{cccc|cccc|cccc} 4 & 8 & 7 & & 3 & 1 & 9 & & 2 & 6 & & \\ & \text{green} & \text{green} & & & \text{green} & \text{green} & & & & & \end{array} \mapsto \begin{array}{cccc|cccc|cccc} 8 & 7 & 4 & & 3 & 9 & 1 & & 6 & 2 & & \end{array}$$

So, $\text{leg}_5 48731926 = 87439162$. Since in practice we read the entries in the first column from bottom to top, the coinversions in the resulting word will correspond to column inversions. With this in mind, note that word $5w$ has coinversions $5 < 8, 5 < 7, 5 < 9, 5 < 6$ which are not present in w . However, when we perform leg_5 on w , we remove the coinversions $4 < 8, 4 < 7, 1 < 9, 2 < 6$ and no others, so the larger entries in the coinversions of $5w$ and of $\text{leg}_5 w$ agree.

Remark 1. Note the similarity of arm_u and leg_v to the maps γ_x introduced by Foata [6], which he used to bijectively prove that the major index and inversion number are equidistributed over $\mathfrak{S}_n$. We point out that these maps are not, in fact equivalent; one immediate difference is the manner in which the respective maps factor a word w into subwords. In Example 3, for instance, Foata's map γ_5 would factor w as $|49|26|3|187$ and cyclicly rotate the entries within each block. When performing arm_5 on w , however, the factorization is different and the manner in which we reorder each block is as well.

By composing these maps, we obtain our desired bijection $\theta : SF_{< \mu} \rightarrow SF_{< \rho}$:

Definition 7. Given $S \in SF_{< \mu}$, let $u = S_{1,1}$, and let v be the leftmost entry in the first row of S which is $> u$. Then define

$$\theta(S) = \text{leg}_v \circ \text{arm}_u \circ \text{bump}(S),$$

where arm_u acts only on the first row of bump S — that is, we replace the first row of bump S with $\text{arm}_u S_{1,2}S_{1,3} \cdots S_{1,a}$ — and leg_v acts on the first column of arm_u bump S , strictly above the first row, so that these entries are replaced by $\text{leg}_v S_{1,1}S_{2,1} \cdots S_{\ell,1}$, entered from bottom to top.

Since $S_{1,a} > u$ and $u < v$ for any $S \in SF_{<} \mu$, each step in the composition is defined; thus θ is well-defined. Furthermore we have $\theta S_{\ell,1,1} < v = \theta S_{1,1}$ by construction, so that $\theta S \in SF_{<} \rho$.

Example 5. Let $\mu = 5, 1^4$ and $\rho = 4, 1^5$, and let

$$S = \begin{array}{|c|c|c|c|c|} \hline 9 \\ \hline 1 \\ \hline 7 \\ \hline 4 \\ \hline 5 & 6 & 3 & 2 & 8 \\ \hline \end{array} \in SF_{<} \mu.$$

Then in the language of the above definition, we have $u = 5, v = 6$, so that

$$\theta S = \text{leg}_6 \circ \text{arm}_5 \circ \text{bump } S = \text{leg}_6 \circ \text{arm}_5 \left(\begin{array}{|c|c|c|c|c|} \hline 9 \\ \hline 1 \\ \hline 7 \\ \hline 4 \\ \hline 5 \\ \hline 6 & 3 & 2 & 8 \\ \hline \end{array} \right)$$

$$\text{leg}_6 \left(\begin{array}{|c|c|c|c|c|} \hline 9 \\ \hline 1 \\ \hline 7 \\ \hline 4 \\ \hline 5 \\ \hline 6 & 8 & 3 & 2 \\ \hline \end{array} \right) = \begin{array}{|c|c|c|c|} \hline 1 \\ \hline 9 \\ \hline 4 \\ \hline 7 \\ \hline 5 \\ \hline 6 & 8 & 3 & 2 \\ \hline \end{array} \in SF_{<} \rho.$$

Note that $\varphi_S = \varphi_{\theta S} = x_7^2 x_9^4 y_2^3 y_3^2$.

Finally, we have:

Theorem 4. For partitions $\mu = a, 1^{\ell-1}$ and $\rho = a-1, 1^\ell$, the map $\theta : SF_{<} \mu \rightarrow SF_{<} \rho$ defined above satisfies $\varphi_S = \varphi_{\theta S}$ for every $S \in SF_{<} \mu$. Furthermore θ is a bijection, and $\dim \mathcal{H}_\mu \cap \mathcal{H}_\rho \geq \frac{n!}{2}$.

Proof. (Sketch) By construction, we have

$$\{r : t, r \in \text{rowInv } S \text{ for some } t\} = \{r' : t', r' \in \text{rowInv } \theta S \text{ for some } t'\}$$

and

$$\{d : d, c \in \text{collnv } S \text{ for some } c\} = \{d' : d', c' \in \text{collnv } \theta S \text{ for some } c'\},$$

and these sets uniquely determine φ_S (resp. $\varphi_{\theta S}$), so $\varphi_S = \varphi_{\theta S}$.

We furthermore have

$$\theta S = \theta T \Rightarrow \varphi_{\theta S} = \varphi_{\theta T} \Rightarrow \varphi_S = \varphi_T \Rightarrow S = T,$$

so θ is injective and therefore also bijective, thus

$$\{\varphi_S : S \in SF_{<} \mu\} \subset \mathcal{H}_\mu \cap \mathcal{H}_\rho.$$

□

The reverse assertion, that $\dim \mathcal{H}_\mu \cap \mathcal{H}_\rho \leq \frac{n!}{2}$, follows from a comparatively clunkier argument.

Theorem 5. For $\mu = a, 1^{\ell-1}$, $\rho = a-1, 1^\ell$, we have $\dim \mathcal{H}_\mu \cap \mathcal{H}_\rho \leq \frac{n!}{2}$.

Proof. (Sketch) We show that $\varphi_S \notin \mathcal{H}_\rho$ for any $S \in SF_\mu - SF_{<} \mu$ by demonstrating that from any such polynomial φ_S , one can infer that there must be at least a distinct entries in the first row of S , i.e. that φ_S can not be achieved from any filling in SF_ρ (resp. $\varphi_T \notin \mathcal{H}_\mu$ for any $T \in SF_\rho - SF_{<} \rho$ by an analogous argument). □

Combining [Theorems 4](#) and [5](#) proves this special case of the $\frac{n!}{k}$ conjecture.

Remark 2. Garsia and Haiman noted in [\[7\]](#) that one may define the Garsia-Haiman module as

$$D_\mu = \text{span}\left\{f \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}; \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n} \Delta_\mu : f \in \mathbb{Q}[X, Y]\right\},$$

the span of all partial derivatives of all orders of Δ_μ , and an isomorphism $\mathcal{H}_\mu \xrightarrow{\sim} sD_\mu$ is given by $f[X, Y] \mapsto f \frac{\partial}{\partial X}, \frac{\partial}{\partial Y} \Delta_\mu$.

It is worth remarking that the "science fiction" heuristics in [\[5\]](#), which establish remarkable connections between Garsia-Haiman modules and the Kostka-Macdonald coefficients $\tilde{K}_{\lambda\mu}(q, t)$, are formulated in terms of D_μ , rather than $\mathcal{H}_\mu$. The restriction of the Artin basis to $\mathcal{H}_\mu \cap \mathcal{H}_\rho$ does not immediately yield a basis for $D_\mu \cap D_\rho$ in the sense that

$$\{\varphi_S \frac{\partial}{\partial X}, \frac{\partial}{\partial Y} \Delta_\mu : S \in SF_{<} \mu\} \not\subset \{\varphi_T \frac{\partial}{\partial X}, \frac{\partial}{\partial Y} \Delta_\rho : T \in SF_{<} \rho\}.$$

In fact, experimental evidence suggests that these sets are disjoint.

As the above isomorphism is bidegree-complementing, the optimistic (some would say naïve) conjecture would be that the complement $SF_\mu - SF_{<} \mu$ (resp. ρ) indexes a basis for $D_\mu \cap D_\rho$. However, it is not true in general that $\varphi_S \frac{\partial}{\partial X}, \frac{\partial}{\partial Y} \Delta_\mu \in D_\rho$ for arbitrary $S \in SF_\mu - SF_{<} \mu$, so this basis is not particularly well-equipped to make this paradigm shift.

Acknowledgements

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Double Dimers and Super Ptolemy Relations

Gregg Musiker^{*1}, Nick Ovenhouse⁺², and Sylvester W. Zhang^{‡1}

¹*School of Mathematics, University of Minnesota, Minneapolis, MN, USA*

²*Department of Mathematics, Yale University, New Haven, CT, USA*

Abstract. Ptolemy’s theorem relates the lengths of the diagonals and sides of a quadrilateral inscribed in a circle, and this is the inspiration for the mutation relation in a cluster algebra associated to a triangulated surface. A super-symmetric version of the Ptolemy relation was introduced recently by Penner and Zeitlin, involving anti-commuting variables. Previous work of the first author and Schiffler gave a formula for cluster variables in terms of perfect matchings of some planar graph. Motivated by this, we investigate certain algebraic expressions, obtained via iterating the super Ptolemy relation, that may be given as a sum over double dimer covers of this graph.

Keywords: cluster algebras, super algebras, snake graphs, dimer models

1 Introduction

Ptolemy’s theorem is a well-known identity between the lengths of the diagonals and sides of a quadrilateral inscribed in a circle, as exemplified in [Figure 1](#).

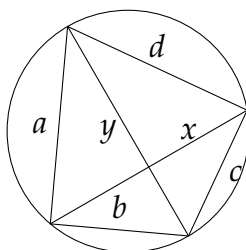


Figure 1: A quadrilateral inscribed in a circle. Ptolemy’s relation says $xy = ac + bd$.

In a way, the Ptolemy relation is like a “skein relation”, in the sense that it realizes a product of crossing edges by a sum of products of non-crossing ones. More generally, if a polygon is inscribed in a circle, then using the Ptolemy relation iteratively, one may express a product of any number of crossing diagonals as an expression involving only products of non-crossing ones. For example, consider the inscribed pentagon in [Figure 2](#), and the product of the lengths of the three diagonals shown.

^{*}musiker@umn.edu. Gregg Musiker was partially supported by Grants DMS-1745638, DMS-1854162.

[†]nicholas.ovenhouse@yale.edu. Nick Ovenhouse was partially supported by Grant DMS-1745638.

[‡]swzhang@umn.edu. Sylvester Zhang was partially supported by Grant DMS-1745638.

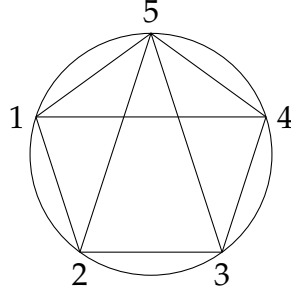


Figure 2: A pentagon inscribed in a circle, and three distinguished diagonals.

Let x_{ij} be the length of the diagonal between vertices labeled i and j (including the boundary sides $x_{i,i+1}$). Then using the Ptolemy relation twice, we can re-write the product of the three diagonals shown in Figure 2 as follows.

$$\begin{aligned}
 x_{14}x_{25}x_{35} &= x_{12}x_{45}x_{35} + x_{15}x_{24}x_{35} \\
 &= x_{12}x_{45}x_{35} + x_{15}x_{24}x_{35} \\
 &= x_{12}x_{45}x_{35} + x_{15}x_{23}x_{45} + x_{34}x_{25} \\
 &= x_{12}x_{45}x_{35} + x_{15}x_{23}x_{45} + x_{15}x_{34}x_{25}
 \end{aligned}$$

Note that the diagonals $(2,5)$ and $(3,5)$ give a triangulation of the pentagon. By dividing both sides of the equation above by $x_{25}x_{35}$, we obtain

$$x_{14} = \frac{1}{x_{25}x_{35}} (x_{12}x_{45}x_{35} + x_{15}x_{23}x_{45} + x_{15}x_{34}x_{25})$$

In particular, this expresses x_{14} as a subtraction-free expression in terms of the lengths of the edges in the triangulation. This is an example of the following fact.

Theorem 1. [10] *Let T be a fixed triangulation of a polygon inscribed in a circle. The length of any diagonal not in T is expressible as a Laurent polynomial with positive integer coefficients in terms of the lengths of the diagonals in T .*

2 Perfect Matching Formula for Diagonal Lengths

In [6], the first author and Schiffler showed that when one uses the Ptolemy relation to re-write the length of a diagonal in terms of lengths from a triangulation, the resulting Laurent polynomial expression is a sum over perfect matchings of some planar graph. We will recall this result below, after some definitions.

Definition 1. A *snake graph* is a planar graph consisting of a sequence of square faces, where each square is attached either above or to the right of the previous one.

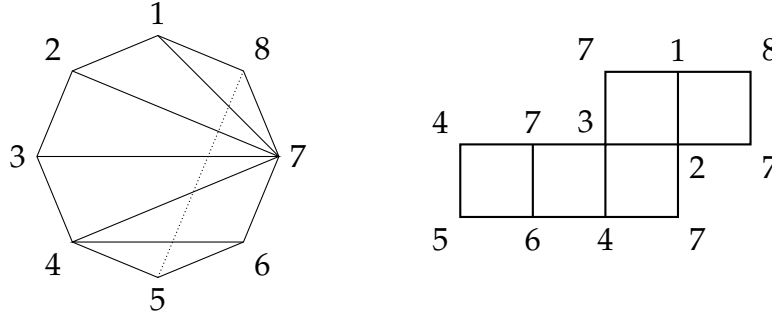


Figure 3: The snake graph G_γ for $\gamma = 5, 8$.

Remark 1. In terms of the language of Young diagrams, a snake graph is the same thing as a *border strip* skew Young diagram. That is, a skew Young diagram λ/μ which contains no 2×2 block of boxes.

To each diagonal in a triangulated polygon, we will associate a particular snake graph, with some edge labeling.

Definition 2. Let T be a triangulation of an inscribed polygon and $\gamma = a, b$ an arc not in T . Let $t_1, t_2, \dots, t_k$ be the diagonals in T which cross γ . For each t_i , let S_{t_i} be the quadrilateral in T containing t_i as its diagonal. For each pair of adjacent diagonals, their corresponding tiles $S_{t_i}, S_{t_{i+1}}$ share a common edge. Connect all $S_{t_1}, \dots, S_{t_k}$ by identifying the common edges, and call the resulting graph the *snake graph* of γ , denoted G_γ .

Remark 2. Note that in order to attach the tiles as described in Definition 2, the even-numbered tiles (the second, fourth, etc.) must have opposite orientation from the corresponding quadrilateral in T .

Example 1. Figure 3 shows an example of the snake graph associated to a diagonal in an octagon. The vertices of the snake graph are labeled by the corresponding vertices in the polygon. Although edge labels are not shown in the figure, assume each edge is labeled by x_{ij} , the length of the corresponding diagonal.

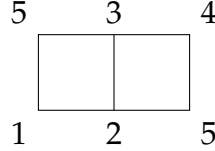
Definition 3. Let G be a graph. A *perfect matching* of G (also called a *dimer cover*) is a subset M of the edges of G , such that each vertex is incident to exactly one edge of M . We will write $D(G)$ for the set of dimer covers of G . If G has an edge-weighting, then $\text{wt } M$ is defined as the product of the edge weights in M .

Theorem 2. [6, Theorem 3.1] Let T be a triangulation of an inscribed polygon, and let γ be a diagonal not in T . Then the length x_γ is given by

$$x_\gamma = \frac{1}{x_{i_1} x_{i_2} \cdots x_{i_k}} \sum_{M \in D(G_\gamma)} \text{wt } M$$

where $x_{i_1}, x_{i_2}, \dots, x_{i_k}$ are the lengths of the diagonals which γ crosses.

Example 2. Consider the triangulated pentagon shown in Figure 2, with triangulation consisting of the diagonals 2,5 and 3,5. The snake graph of $\gamma = 1,4$ is shown below.



This graph has three dimer covers, corresponding to the three terms in the expression given earlier:

$$x_{14} \frac{1}{x_{25}x_{35}} x_{12}x_{45}x_{35} \quad x_{15}x_{23}x_{45} \quad x_{15}x_{34}x_{25}$$

3 The Super Ptolemy Relation

In this section, we will review a simplified version of the setup from [9]. Informally, in addition to variables x_{ij} representing diagonal lengths, we will also consider some extra non-commuting variables.

Definition 4. A *super algebra* is an associative algebra $A = A_0 \oplus A_1$ with a $\mathbb{Z}_2$ -grading. That is, $A_i A_j \subseteq A_{i+j}$. A super algebra is called *super-commutative* (or often just *commutative*) if $xy = (-1)^{ij}yx$ whenever $x \in A_i$ and $y \in A_j$. In other words, elements of A_0 , called *even*, are central (they commute with everything), and elements of A_1 , called *odd*, anti-commute with each other.

Given a triangulation T of an inscribed polygon, and a choice of orientation of each of its edges, we define a commutative superalgebra A_T as follows. Recall that x_{ij} denotes the length of diagonal (i,j) . If the triangulation T has edges $e_1, \dots, e_{2n-3}$, let $C^\infty(x_{e_1}, \dots, x_{e_{2n-3}})$ be the commutative algebra of all smooth functions in the x_{e_i} variables corresponding to these lengths. Also let E be the exterior algebra on the $(n-2)$ -dimensional space with basis vectors θ_{ijk} , corresponding to triangles in T with vertices i, j, k . Then we define A_T to be

$$A_T : C^\infty(x_{e_1}, \dots, x_{e_{2n-3}}) \otimes E$$

By definition, the generators of A_T correspond to edges and triangles from a triangulation T . Penner and Zeitlin defined a super Ptolemy transformation [9], which can

be iterated to produce new elements x_{ij} and θ_{ijk} of A_T corresponding to diagonals and triangles which are not in T . We will describe this transformation now.

When two triangulations are related by a flip, as in Figure 4, one can define new elements of A_T by the following “super Ptolemy relations” [9, Theorem B]:

$$ef \quad ac \quad bd \quad \sqrt{abcd} \sigma \theta \quad \sigma' \quad \frac{\sqrt{bd} \sigma - \sqrt{ac} \theta}{\sqrt{ac} \quad bd} \quad \theta' \quad \frac{\sqrt{bd} \theta \quad \sqrt{ac} \sigma}{\sqrt{ac} \quad bd} \quad (3.1)$$

A flip alters the orientation of one of the edges, as depicted in Figure 4. In particular, the order of multiplying the odd variables σ and θ are dictated by the orientation of the edge being flipped, as in Figure 4.

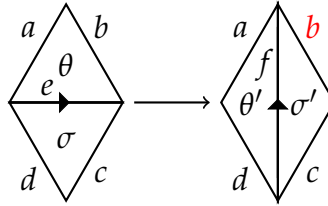


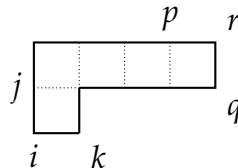
Figure 4: Super Ptolemy relation. The orientation of the edge b is changed by the flip.

4 Double Dimer Covers

Similar to the ordinary Ptolemy relation, one may iteratively apply the super Ptolemy relation to write x_{ij} in terms of the original generators, corresponding to a fixed triangulation T . In [4], the authors showed that this can be calculated using the same snake graph, but summing over double dimer covers, which we define as follows.

Definition 5. Let G be a graph. A *double dimer cover* is a multi-set M of edges in G , such that every vertex of G is contained in exactly two elements of M . In other words, a double dimer cover is the superposition of two dimer covers. Denote $DD(G)$ the set of all double dimer covers of G .

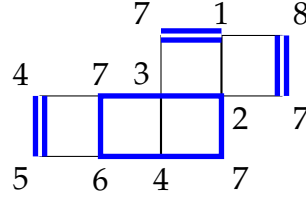
We will now define the weight of a double dimer cover of a snake graph G_γ . Observe that every double dimer cover consists of doubled edges and cycles. For each cycle we associate a product of two odd variables to it. Let c be a cycle, with i, j, k being the three vertices of on the bottom-left corner of c and p, q, r being the three vertices on the top-right corner of c , as illustrated below.



Associate a product of two odd variables to c by $\nu(c) = \theta_{ijk}\theta_{pqr}$, which are the odd variables corresponds to the triangles i, j, k and p, q, r . We then define the weight of a double dimer cover $M \in DD(G_\gamma)$ to be

$$\text{wt } M = \prod_{e \in M} \sqrt{x_e} \prod_{c \text{ a cycle in } M} \nu(c) \quad (4.1)$$

Example 3. Consider the following double dimer cover of the snake graph in Figure 3.



Its weight is $x_{45}x_{17}x_{78}\sqrt{x_{67}x_{46}x_{47}x_{27}x_{23}x_{37}}\theta_{467}\theta_{237}$.

In [4], we proved the following result, generalizing Theorem 2.

Theorem 3. [4, Theorem 6.1] *Let T be a triangulation T , and let $\gamma = i, j$ be a diagonal not in T . Then x_{ij} is given by*

$$x_\gamma = x_{ij} = \frac{1}{x_{i_1}x_{i_2} \cdots x_{i_k}} \sum_{M \in DD(G_\gamma)} \pm \text{wt } M$$

where $x_{i_1}, x_{i_2}, \dots, x_{i_k}$ are the lengths of the diagonals which γ crosses. In particular, all x_γ 's are Laurent polynomials in $\sqrt{x_{ij}}$'s and polynomials in θ_{ijk} 's.¹

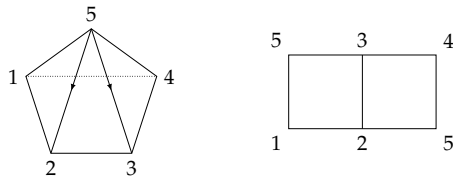
There is a stronger version of Theorem 3, which says there is a special orientation on T , called the *default orientation*, and a total ordering on the odd variables, called the *positive order* (See Section 2.2 of [4] for details), with the property that all terms in the sum in Theorem 3 have positive sign when the odd variables in $\text{wt } M$ are multiplied in the positive order.

In [3, Proposition 1.4], the authors showed that every choice of orientation is equivalent to a default orientation, after negating some of the odd θ_{ijk} variables. This allows one to explicitly determine the signs in Theorem 3 for an arbitrary choice of orientation.

Remark 3. For a snake graph G_γ , there is a special family of double dimer covers which only uses doubled edges and no cycles. The weights of these double dimer covers are exactly the same as that of the (single) dimer cover using the same set of edges (but only once). Therefore the set of double dimer covers of G_γ naturally contains the set of (single) dimer covers as a weight-preserving subset.

¹In [4, Theorem 6.1 part (b)], formulas for some odd variables are given, however, they are not Laurent polynomials.

Example 4. Consider the same pentagon as in Figure 2 and its snake graph as in Example 2.


$$x_{14} \frac{1}{x_{25}x_{35}} \left(\begin{array}{cccc} x_{12}x_{45}x_{35} & x_{15}x_{23}x_{45} & x_{15}x_{34}x_{25} & x_{15}\sqrt{x_{23}x_{25}x_{45}x_{34}}\theta_{235}\theta_{345} \\ x_{45}\sqrt{x_{15}x_{12}x_{23}x_{35}}\theta_{125}\theta_{235} & \sqrt{x_{15}x_{12}x_{25}x_{45}x_{34}x_{35}}\theta_{125}\theta_{345} \end{array} \right)$$

		
$x_{12}x_{35}x_{45}$	$x_{15}x_{23}x_{45}$	$x_{15}x_{25}x_{34}$

		
$x_{15}\sqrt{x_{23}x_{25}x_{45}x_{34}}\theta_{235}\theta_{345}$	$x_{45}\sqrt{x_{15}x_{12}x_{23}x_{35}}\theta_{125}\theta_{235}$	$\sqrt{x_{15}x_{12}x_{25}x_{45}x_{34}x_{35}}\theta_{125}\theta_{345}$

The partial order on the set of dimer covers is defined as follows. Number the squares of the snake graph $1, 2, \dots, N$ from bottom-left to top-right. For a dimer cover M , consider the double dimer cover $M \cup M_0$. Define a subset $s(M) \subseteq N$ as the set of squares which are surrounded by cycles of $M \cup M_0$. The partial order is then defined as $M \leq M'$ if $s(M) \subseteq s(M')$. An example is shown in Figure 6. Denote $P_D(G)$ the poset of dimer covers on G .

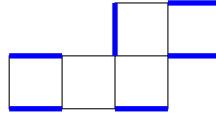


Figure 5: The minimal dimer cover of a snake graph.

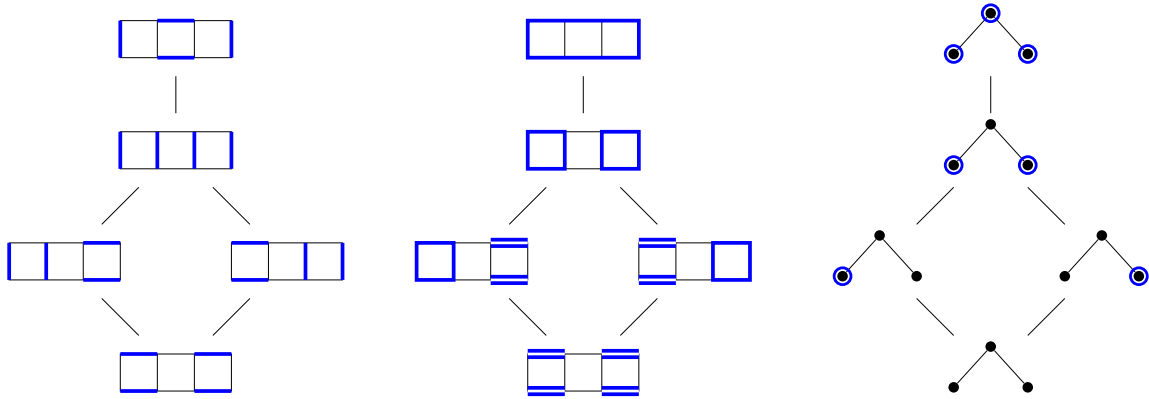
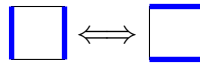


Figure 6: Left: The poset of dimer covers of a snake graph. Middle: the corresponding double dimer covers, after superimposing the minimal dimer cover. Right: Lattice of order ideals of the fence poset, where a blue circle indicate elements in the order ideal.

Alternatively, we can construct the $P_D G_\gamma$ inductively by “flipping” the dimer covers on tiles of the snake graph. If a tile of snake graph is covered by two dimer edges, then one can perform a flip on that tile as follows to obtain a new dimer cover.



Let M be a dimer cover of G_γ and M' is obtained from M via a flip, then either M' is covered by M or M is covered by M' . If we require the poset to have a unique minimum element, then the above rule and the choice of minimal element (the minimal dimer cover M_0) are sufficient to determine the entire poset.

Moreover, for a snake graph G , the poset $P_D G$ is a distributive lattice and can be realized as the poset of order ideals of the *fence poset* associated to G , denoted F_G . See the right image in Figure 6. We refer the reader to Section 9 of [4] for more details about fence posets.

In the case of double dimer covers, there exists a similar natural partial order structure. We first define the *minimal double dimer cover* $\tilde{M}_0$ of a snake graph G_γ to be the superposition of two identical copies of the minimal dimer covers of G_γ . In other words, the minimal double dimer cover looks exactly the same as the minimal single dimer cover but with doubled edges instead.

For a snake graph G , denote $P_{DD} G$ the poset of double dimer covers of G . We can

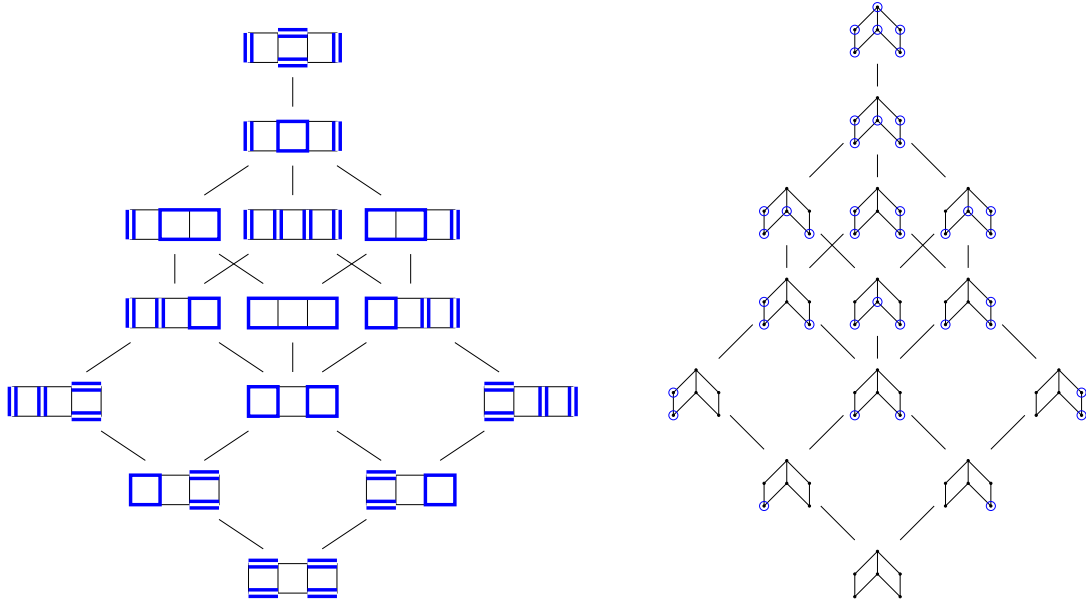


Figure 7: Left: the partial order on double dimer covers of a snake graph. Right: the poset of order ideals of the corresponding doubled fence poset.

then define $P_{DD} G$ in the same way by performing flips (of a pair of single edges) on the tiles of G . Note we are allowed to flip a pair of singled edges inside some doubled edges. The possible local flips are summarized as follows.

$$\square \iff \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} \quad \square \iff \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} \quad \square \iff \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} \quad \square \iff \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}$$

Similar to the single dimer case, these flips corresponds to covering relations in the poset. Therefore starting with the minimal double dimer cover $\tilde{M}_0$ and applying flips inductively generates the entire partial order.² See Figure 7 for example.

Similar to the single dimer case, the poset $P_{DD} G$ is also a distributive lattice. In this case, it can be realized as the poset of order ideals on the *doubled fence poset* that is the product of the fence poset associated to G and a two element chain: $\{0, 1\} \times F_G$. See Figure 7 for example.

6 Special Cases: Zig-zag and Straight Snake Graphs

We conclude with a discussion on two special cases of snake graphs: the zig-zag and the straight snake graphs.

²Note that we can define $P_{DD} G$ in a way similar to the first definition we gave for $P_D G$, by superimposing every double dimer cover with the minimal double dimer cover. However it is more complicated to illustrate in the case of double dimer covers, therefore we omit this definition.

Zig-zag Snakes Graphs. Zig-zag snake graphs corresponds to the longest arc in a fan triangulation, i.e. the triangulation where all triangles share a common vertex. Let Z_n be the zig-zag snake graph with n tiles. Then the number of double dimer covers of Z_n is the binomial coefficient $\binom{n+2}{2}$. Moreover, there is a q -analogue of this statement, given in the following theorem, a special case of a more general result from work in progress of the authors and R. Schiffler [2].

Theorem 4. For a double dimer cover $M \in P_{DD} Z_n$, let $\text{rk } M$ denote its rank in the poset. Then we have

$$\sum_{M \in P_{DD} Z_n} q^{\text{rk } M} = \binom{n+2}{2}_q$$

In fact, the poset $P_{DD} Z_n$ is isomorphic to the poset of lattice paths on the $2 \times n$ board, whose rank generating function is well known to be the q -binomial coefficient. See Figure 8 for an illustration.

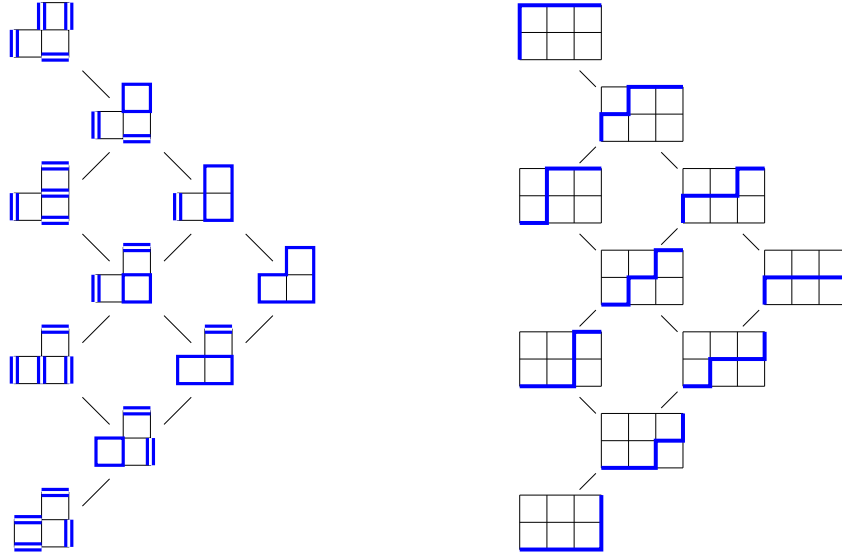


Figure 8: Left: The poset of double dimer covers on a zig-zag snake graph. Right: The poset of lattice path on $2 \times n$ grid.

Straight Snakes Graphs and Super Fibonacci Numbers. Another interesting family of snake graphs are the straight snake graphs. Let L_n denote the straight snake graph that is a horizontal row of n tiles. The number of dimer covers of L_n gives a nice combinatorial interpretation of Fibonacci numbers. More precisely, $|D L_n| = f_{n+1}$ where $f_0 = f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$. In the case of double dimer covers, this gives rise to a generalization which we call *super Fibonacci numbers*.

Definition 6. Consider the family of double dimer covers $\mathbb{D} L_n$ on L_n such that every $M \in \mathbb{D} L_n$ has at most one cycle, and that cycle has odd length. Further, define $\mathbb{D}^0 L_n$ to be the set of double dimers without cycles and $\mathbb{D}^1 L_n := \mathbb{D} L_n \setminus \mathbb{D}^0 L_n$. Denote $y_n = |\mathbb{D}^1 L_{n-1}|$ and recall that $|\mathbb{D}^0 L_{n-1}| = |D L_{n-1}| = f_n$. Finally, define the n -th super Fibonacci number to be

$$F_n := |\mathbb{D}^0 L_{n-1}| + |\mathbb{D}^1 L_{n-1}| \varepsilon = f_n + y_n \varepsilon$$

where $\varepsilon^2 = 0$.

Super Fibonacci numbers live in the super algebra $\mathbb{Z} \otimes \wedge \varepsilon$ with one odd generator, i.e. the algebra of dual numbers.

Remark 4. The reason why we forbid multiple cycles and cycles of even length is related to the hyperbolic geometry of the annulus, which we do not explain here. See Section 11 of [4] for more details. In fact, in [5] we showed that F_n 's can be interpreted as certain “super-lengths” on a marked annulus.

The F_n 's satisfy several identities, generalizing those of the classical Fibonacci numbers. To better state these identities, we make a simple adjustment to the definition of super Fibonacci numbers as follows. Define

$$\tilde{y}_n := \begin{cases} y_n & \text{if } n \text{ is odd.} \\ y_n - 1 & \text{if } n \text{ is even.} \end{cases}$$

and $\mathcal{F}_n := f_n + \tilde{y}_n \varepsilon$. In terms of double dimer covers, $\mathcal{F}_n$ corresponds to disallowing the double dimer cover that is the longest cycle.

Theorem 5. The ‘simpler’ super Fibonacci numbers $\mathcal{F}_n$ satisfy the recurrence

$$\mathcal{F}_0 = 1 - \varepsilon, \quad \mathcal{F}_1 = 1, \quad \mathcal{F}_{n+1} = 1 + \varepsilon \mathcal{F}_n + \mathcal{F}_{n-1},$$

and its generating function is

$$\sum_n \mathcal{F}_n x^n = \frac{1 - \varepsilon}{1 - (1 + \varepsilon)x - x^2}$$

The ratio of Fibonacci numbers has a nice continued fraction expansion:

$$\frac{f_n}{f_{n-1}} = \cfrac{1, \dots, 1}{n \text{ times}} + 1 + \cfrac{1}{1 + \cfrac{1}{\dots}}$$

There is a similar identity for ratios of the $\mathcal{F}_n$'s.

Theorem 6.

$$\frac{\mathcal{F}_n}{\mathcal{F}_{n-1}} = \cfrac{1}{1} \cfrac{\underbrace{\varepsilon, \dots, 1}_{n \text{ times}}}{\varepsilon} \cfrac{1}{1} \cfrac{\varepsilon}{1} \cfrac{1}{\varepsilon} \cfrac{1}{1} \cfrac{1}{\varepsilon} \cfrac{1}{1} \dots$$

In addition, by analysing the generating function, we are able to write the super Fibonacci numbers in terms of the golden ratio:

Theorem 7. Let $\Phi_{\pm} = \frac{1 \pm \sqrt{5}}{2}$. $\left(\frac{1 \pm \frac{1}{\sqrt{5}}}{2} \right) \varepsilon = \frac{1 \pm \sqrt{5}}{2} \left(1 \pm \frac{1}{\sqrt{5}} \varepsilon \right)$. Then we have

$$\mathcal{F}_n = \frac{1}{\sqrt{5}} \left(1 - \frac{6}{5} \varepsilon \right) \left(\Phi^{n-1} - \Phi_-^{n-1} \right)$$

Remark 5. Note that y_n is the OEIS sequence A054454 as well as the third column of the triangular array A054453. The super Fibonacci numbers also appeared in [8] under the name of “shadow sequences”. An analogue of Theorem 6 has also been obtained in [1] for shadow Fibonacci numbers as an example of super continued fractions.

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Generalized Pitman–Stanley flow polytopes

William T. Dugan^{*1}, Maura Hegarty^{2†},
Alejandro H. Morales^{1‡}, and Annie Raymond^{1§}

¹Department of Mathematics and Statistics, UMass Amherst, Amherst, MA, USA

²Operations Research Center, MIT, Cambridge, MA, USA

Abstract. The Pitman–Stanley polytope is well-studied and related to plane partitions with entries 0 and 1. We study a generalization of this polytope related to plane partitions with entries $0, 1, \dots, m$. We show that this generalization is realized as a flow polytope of a grid graph. We give multiple characterizations of its vertices and give formulas for the number of vertices and faces. We also study formulas for its number of lattice points and volume.

Résumé. Le polytope de Pitman–Stanley a été très étudié et a des connexions aux partitions de plan avec entrées 0 et 1. Nous étudions une généralisation de ce polytope avec des connexions aux partitions de plan avec entrées $0, 1, \dots, m$. Nous démontrons que cette généralisation peut être réalisée comme un polytope de flot d’un graphe en forme de grille. Nous donnons plusieurs caractérisations de ses sommets ainsi que des formules pour le nombre de sommets et de faces. Nous étudions également des formules pour le nombre de points à coordonnées entières et le volume.

Keywords: Pitman–Stanley polytope, flow polytope, plane partitions

1 Introduction

The eponymous Pitman–Stanley polytope introduced in [11] is a well-studied polytope in geometric, algebraic, and enumerative combinatorics. This polytope is defined as follows. For a positive integer n and vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{N}^n$, let

$$\text{PS}_n \mathbf{a}, \mathbf{b} : \{ \mathbf{x} \in \mathbb{R}_{\geq 0}^n \mid b_1 \cdots b_i \leq x_1 \cdots x_i \leq a_1 \cdots a_i \text{ for } i = 1, \dots, n \},$$

i.e., $\text{PS}_n \mathbf{a}, \mathbf{b}$ consists of the nonnegative vectors $\mathbf{x}$ in $\mathbb{R}^n$ that are between vectors $\mathbf{a}$ and $\mathbf{b}$ in *dominance order*. (Recall that, for $\mathbf{v}, \mathbf{w} \in \mathbb{N}^n$, we say that $\mathbf{v}$ *dominates* $\mathbf{w}$ if $\sum_{j=1}^i v_j \geq \sum_{j=1}^i w_j$ for every $i = 1, \dots, n$.)

^{*}wdugan@umass.edu.

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Postnikov showed that $\text{PS}_n \mathbf{a} : \text{PS}_n \mathbf{a}, \mathbf{0}$ is a *generalized permutohedron* [10], and Baldoni and Vergne [1] realized $\text{PS}_n \mathbf{a}$ as a flow polytope of a graph (see Figure 1a).

Important results of [11] include that the polytope $\text{PS}_n \mathbf{a}$ is combinatorially equivalent to a product of simplices. We denote by Δ_d the d -dimensional simplex. Given $\mathbf{a} = a_1, \dots, a_n$ with $a_1 > 0$, its *signature* $\text{sgn } \mathbf{a} : c_1, \dots, c_k$ is a sequence of nonnegative integers such that $\chi \mathbf{a} : 1, 0^{c_1-1}, 1, 0^{c_2-1}, \dots, 1, 0^{c_k-1}$ where 1 and 0 represent a positive a_i and zero a_i respectively. Then the polytope $\text{PS}_n \mathbf{a}$ is combinatorially equivalent to the product of simplices $\Delta_{c_1} \times \dots \times \Delta_{c_k}$. In particular, if each $a_i > 0$, then $\text{PS}_n \mathbf{a}$ is combinatorially equivalent to an n -cube.

The volume of the polytope $\text{PS}_n \mathbf{a}$ is of interest in probability [11, §2] and it is known that $\text{vol PS}_n \mathbf{1} = \frac{n!}{n!} = 1$, the number of *parking functions* of size n [11, Thm. 11]. Recall that a *plane partition* of skew shape λ/μ is an array π of nonnegative integers of shape λ/μ that is weakly decreasing in rows and columns. Pitman and Stanley also showed that the lattice points of $\text{PS}_n \mathbf{a}, \mathbf{b}$ are in correspondence with plane partitions of skew shape $\theta \mathbf{a}, \mathbf{b} : a_1 \dots a_n, \dots, a_1 a_2, a_1 / b_1 \dots b_n, \dots, b_1 b_2, b_1$ with entries 0 and 1 [11, Thm. 12].

When $\mathbf{b} = \mathbf{0}$, note that $\theta \mathbf{a}, \mathbf{b}$ is a straight shape and the number of lattice points of $\text{PS}_n \mathbf{1}$ is given by the *Catalan number* $C_n = \frac{1}{n+1} \binom{2n}{n}$. Pitman and Stanley also mentioned in [11, Sec. 5] how their polytope can be generalized so that lattice points correspond to plane partitions of the same shape with entries $0, 1, \dots, m$ for a fixed positive integer m . The m th *generalized Pitman–Stanley polytope* is defined as follows.

$$\text{PS}_n^m \mathbf{a}, \mathbf{b} : \{ x_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq m \in \mathbb{R}_{\geq 0}^{nm} \mid b_1 \dots b_i \leq x_{1m} \dots x_{im} \leq \dots \leq x_{11} \dots x_{i1} \leq a_1 \dots a_i \text{ for } i = 1, \dots, n \},$$

i.e., $\text{PS}_n^m \mathbf{a}, \mathbf{b}$ consists of nonnegative matrices x_{ij} in $\mathbb{R}^{nm}$ whose columns form a *multi-chain* of length m between $\mathbf{a}$ and $\mathbf{b}$ in dominance order. Note that $\text{PS}_n \mathbf{a}, \mathbf{b} = \text{PS}_n^1 \mathbf{a}, \mathbf{b}$.

This generalization is the object of this extended abstract whose full version will appear in [2, 3]. We show that the polytope $\text{PS}_n^m \mathbf{a}, \mathbf{b}$ is also a flow polytope of a *grid graph* $G_{n,m}$ denoted as $\mathcal{F}_{G_{n,m}} \mathbf{a}, \mathbf{b}$. See both Figures 1a and 1b for $m = 1$ and Figure 1c for general m . This allows us to use the theory of flow polytopes (e.g., [1, 5, 9]) to study the faces, lattice points, and volume of $\text{PS}_n^m \mathbf{a}, \mathbf{b}$. In Lemma 2.5, we give a bijection between the lattice points of $\mathcal{F}_{G_{n,m}} \mathbf{a}, \mathbf{b}$ and plane partitions of skew shape $\theta \mathbf{a}, \mathbf{b}$ with entries $0, 1, \dots, m$. Note that Liu–Mészáros–St. Dizier [7, 8] have realized *Gelfand–Tsetlin polytopes*, whose lattice points correspond to (skew) semistandard Young tableaux with bounded entries, as flow polytopes. These tableaux and plane partitions of skew shape with bounded entries are related but not equivalent (e.g., see [12, §7.10, 7.22]) and this extends to the associated polytopes.

Vertices and faces: Theorem 3.2 gives a characterization of the vertices of $\text{PS}_n^m \mathbf{a}, \mathbf{b}$ as flows of $\mathcal{F}_{G_{n,m}} \mathbf{a}, \mathbf{b}$ whose support is a subgraph of $G_{n,m}$ that is a forest. This

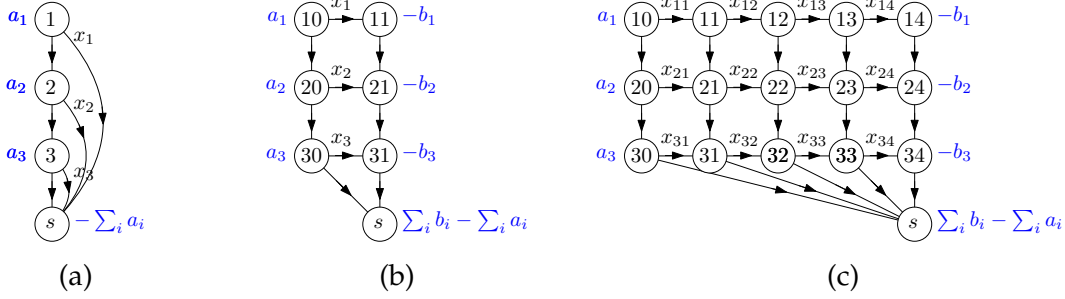


Figure 1: The first graph realizes $\text{PS}_n^1 \mathbf{a}, \mathbf{0}$ for $n = 3$ as shown by Baldoni and Vergne [1]. The second and third graphs are $G(n, 1)$ and $G(n, m)$ which realize as flow polytopes the original and generalized Pitman–Stanley polytopes $\text{PS}_n^1 \mathbf{a}, \mathbf{b}$ and $\text{PS}_n^m \mathbf{a}, \mathbf{b}$, respectively, for $n = 3, m = 1$ and $n = 3, m = 4$, respectively.

result follows from a general result of flow polytopes by Gallo and Sodini [4]. This characterization gives an explicit characterization of the flows on the grid corresponding to vertices in terms of *unsplittable flows* (see Corollary 3.5).

Since the lattice points of $\text{PS}_n^m \mathbf{a}, \mathbf{b}$ are in bijection with plane partitions, in Theorem 3.7 we characterize the plane partitions corresponding to vertices, which we call *vertex plane partitions* (see Definition 3.6).

After giving a flow and plane partition interpretation of the vertices of $\text{PS}_n^m \mathbf{a}, \mathbf{b}$, we turn to enumerating them. Let $v^{n,m} \mathbf{a}$ be the number of vertices of the polytope $\text{PS}_n^m \mathbf{a} : \text{PS}_n^m \mathbf{a}, \mathbf{0}$. We show that the number $v^{n,m} \mathbf{a}$ is a polynomial in m and show that this number has the following nonnegative expansion in the basis $\binom{m}{k}^1$ in terms of plane partitions of staircase shape $\delta_n = n, n-1, \dots, 1$. Let $p_{n,k}$ be the number of vertex plane partitions of staircase shape δ_n with the set of entries equal to $\{0, 1, \dots, k-1\}$.

Theorem 1.1. For $\mathbf{a} \in \mathbb{N}^n$ such that $\chi \mathbf{a} = \mathbf{1}$, we have that $v^{n,m} \mathbf{a} = \sum_{k=1}^{n-1} \binom{n-1}{k}^1 p_{n,k} \binom{m}{k}^1$. In particular, $p_{n, n-1}$ equals the number of shifted Standard Young Tableaux (SYT) of staircase shape $n, n-1, \dots, 1$ [14, A003121].

Moreover, in Theorem 4.4, we give a recurrence for the number of unsplittable flows of $\text{PS}_n^m \mathbf{a}, \mathbf{b}$ by fixing the flows on the first column. As a corollary, we obtain a recurrence for the number $v^{n,m} \mathbf{a}$ of vertices in terms of a generalization of dominance order called 1-dominance (see Definition 4.2).

Corollary 1.2. Let $\mathbf{a} \in \mathbb{N}^n$, then $v^{n,m} \mathbf{a} = \sum_{\mathbf{j}} v^{n,m-1} \mathbf{j}$, where the sum is over $\mathbf{j} \in \{0, 1\}^n$ such that $\chi \mathbf{a} = \mathbf{1}$ -dominates $\mathbf{j}$.

This recurrence allows us to compute generating functions for $\sum_{m \geq 0} v^{n,m} \mathbf{a} x^m$, for fixed n . Lastly, we give in Theorem 5.5 a recurrence for the number of d -faces of $\text{PS}_n^m \mathbf{a}$ similar to Corollary 1.2.

Lattice points, volume: Since lattice points of $\text{PS}_n^m \mathbf{a}, \mathbf{b}$ correspond to plane partitions with bounded entries, their number is given by a determinant formula of Kreweras [6].

Corollary 1.3. *The number of lattice points of $\text{PS}_n^m \mathbf{a}, \mathbf{b}$ has the following formula*

$$\# \text{PS}_n^m \mathbf{a}, \mathbf{b} \cap \mathbb{Z}^{mn} = \det \left[\begin{pmatrix} a_1 & \cdots & a_{n-i} & 1-b_1 & \cdots & 1-b_{n-j} & 1 & m \\ & & i-j & m & & & & \end{pmatrix} \right]_{i,j=1}^n.$$

By replacing a_i and b_j by $t \cdot a_i$ and $t \cdot b_j$ this gives a formula for the Ehrhart polynomial of $\text{PS}_n^m \mathbf{a}, \mathbf{b}$ whose leading term in t gives the volume of the polytope. The case $\mathbf{a} = \mathbf{1}$ and $\mathbf{b} = \mathbf{0}$ is particularly nice for both lattice points and volume [11, Thm. 14].

Corollary 1.4 ([11]). *The polytope $\text{PS}_n^m \mathbf{1}$ has the following number of lattice points and volume*

$$\# \text{PS}_n^m \mathbf{1} \cap \mathbb{Z}^{mn} = \prod_{1 \leq i < j \leq n} \frac{2m-i-j+1}{i}, \quad \text{vol } \text{PS}_n^m \mathbf{1} = \text{SYT } n^m \cdot \prod_{i=1}^m i! \cdot n^{n-m-1} \cdot i,$$

where $\text{SYT } n^m$ is the number of SYT of shape $n \times m$.

Pitman and Stanley asked in [11] for a combinatorial proof of the volume formula above. We give such a proof for the case $m = 2$ and give a bijection between SYT of shape $n \times m$ and the pieces of a subdivision of $\text{PS}_n^m \mathbf{1}$ coming from the *Lidskii formula* for flow polytopes [1, 9]. See [3] for details.

2 $\text{PS}_n^m \mathbf{a}, \mathbf{b}$ as a flow polytope of a grid graph

Given a connected acyclic directed graph $G = (V, E)$ and a netflow vector $\mathbf{a} \in \mathbb{Z}^{|V|}$, let $\mathcal{F}_G \mathbf{a}$ denote the *flow polytope* of graph G with *net flow* $\mathbf{a}$, i.e., the set of flows on G with net flow $\mathbf{a}$ (see, e.g., [9] for a detailed definition).

In [1, Example 16], Baldoni–Vergne showed that $\text{PS}_n \mathbf{a}$ is *integrally equivalent* to a flow polytope. We extend this result to $\text{PS}_n^m \mathbf{a}, \mathbf{b}$. The underlying directed graph is the following grid graph with a sink (see Figure 1c).

Definition 2.1. *For positive integers n and m , let $G_{n,m}$ be a directed graph on the vertex set $V = \{i, j \mid 1 \leq i \leq n, 0 \leq j \leq m\} \cup \{s\}$ and the following directed edges:*

- $i, j \rightarrow i, j-1$ and $i, j \rightarrow i-1, j$ for $i \in \{1, 2, \dots, n-1\}$ and $j \in \{0, 1, \dots, m-1\}$.
- $n, j \rightarrow n, j-1$ and $n, j \rightarrow s$ where $j \in \{1, 2, \dots, m-1\}$, and $n, m \rightarrow s$.
- $i, m \rightarrow i-1, m$ where $i \in \{1, 2, \dots, n-1\}$.

Definition 2.2. *For $\mathbf{a} = (a_1, a_2, \dots, a_n)$ and $\mathbf{b} = (b_1, b_2, \dots, b_n)$ in $\mathbb{R}_{\geq 0}^n$, and the graph $G_{n,m}$ as defined above, denote by $\mathcal{F}_{G_{n,m}} \mathbf{a}, \mathbf{b}$ the flow polytope on $G_{n,m}$ where:*

- vertices $1,0, 2,0, 3,0, \dots, n,0$ have respective net flows $a_1, a_2, \dots, a_n$,
- vertices $1,m, 2,m, 3,m, \dots, n,m$ have respective net flows $-b_1, -b_2, \dots, -b_n$,
- the sink vertex s has net flow $-\sum_{i=1}^n a_i - \sum_{i=1}^n b_i$, and all other vertices have net flow 0.

Example 2.3. Flow polytope $\mathcal{F}_{G_{3,5}} \mathbf{a}, \mathbf{b}$ is illustrated in Figure 1c.

Theorem 2.4. The polytope $\text{PS}_n^m \mathbf{a}, \mathbf{b}$ is integrally equivalent to the flow polytope $\mathcal{F}_{G_{n,m}} \mathbf{a}, \mathbf{b}$.

Proof sketch. Let $\Phi : \mathcal{F}_{G_{n,m}} \mathbf{a}, \mathbf{b} \rightarrow \text{PS}_n^m \mathbf{a}, \mathbf{b}$ be defined as follows:

$$x_{i,j} = f_{i,j-1, i,j}, \quad \text{for } i \in \{1, 2, \dots, n\} \text{ and } j \in \{1, 2, \dots, m\},$$

that is, $x_{i,j}$ is the flow of the horizontal edge $i, j-1, i, j$. See Figure 1c. One can check that Φ gives the desired integral equivalence. $\square$

We denote integral equivalence of polytopes by $\equiv$.

Lemma 2.5. There is a bijection between integral flows for $\text{PS}_n^m \mathbf{a}, \mathbf{b} \equiv \mathcal{F}_{G_{n,m}} \mathbf{a}, \mathbf{b}$ and plane partitions of shape $\theta \mathbf{a}, \mathbf{b}$ with entries at most m .

Proof. Let $\pi = \pi_{ij}$ be a plane partition of shape $\theta \mathbf{a}, \mathbf{b}$ with entries at most m . Consider a column j with (non-empty) entries $\pi_{f_j, j}, \pi_{f_j-1, j}, \dots, \pi_{i_j, j}$ where f_j depends on $\mathbf{b}$. (In the case when $\mathbf{b} = \mathbf{0}$, then $f_j = 1$ for every column.) This corresponds to the *trajectory* of one unit of flow in the graph $G_{n,m}$ that starts at vertex $n-1-i_j, 0$, goes to $n-1-i_j, \pi_{i_j, j}$, then to $n-1-i_j-1, \pi_{i_j, j}$, then to $n-1-i_j-1, \pi_{i_j-1, j}$, then to $n-1-i_j-2, \pi_{i_j-1, j}$, and so on. At the end, two cases happen: if $f_j = 1$, it keeps going until it reaches $n, \pi_{1, j}$ from where it goes to the sink s , or if $f_j \geq 2$, until it reaches $n-2-f_j, \pi_{f_j, j}$ from where it goes to the vertex $n-2-f_j, m$ with negative netflow $-b_{n-2-f_j}$ if $f \geq 2$. In other words, the entries of a column of π give the columns of $G_{n,m}$ where a unit of flow starting at vertex $n-1-i_j, 0$ goes down.

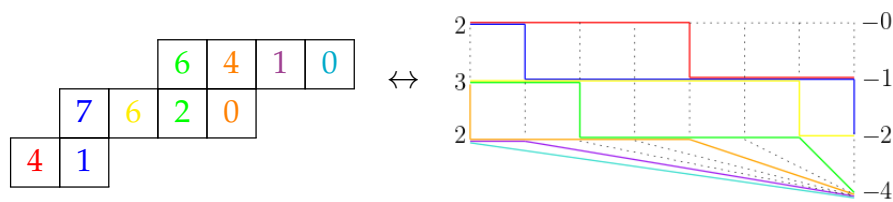


Figure 2: Constructing an integral flow on $G_{3,7}$ from a skew plane partition.

For the inverse map see the long version of the paper [2]. See Figure 2 for an example. $\square$

Figure 2 shows the *trajectory decomposition* of an integer $\mathbf{a}, \mathbf{b}$ -flow.

As a corollary one can count the lattice points of $\text{PS}_n^m \mathbf{a}, \mathbf{b}$ with a determinant.

Proof of Corollary 1.3. By Lemma 2.5, the number of lattice points of $\text{PS}_n^m \mathbf{a}, \mathbf{b}$ equals the number of plane partitions of shape $\theta \mathbf{a}, \mathbf{b}$ with entries at most m . The result then follows by the following result of Kreweras [6, §2.3.7]: the number of plane partitions of shape λ/μ with entries at most m equals $\det \begin{bmatrix} \lambda_i - \mu_j & m \\ i-j & m \end{bmatrix}_{i,j=1}^\ell$. $\square$

3 Flow and plane partition characterization of vertices

The dimension of the flow polytope $\mathcal{F}_G \mathbf{a}$ is $\beta_1 G : |E| - |V| + 1$ [1, §1.1]. Hille studied flow polytopes and their faces in [5] in the context of *quivers*. The following notion of his is important to understand the faces of $\mathcal{F}_G \mathbf{a}$.

Definition 3.1 (*$\mathbf{a}$ -valid¹*). Given a flow polytope $\mathcal{F}_G \mathbf{a}$ with underlying graph G , a subgraph H of G is *$\mathbf{a}$ -valid* if the edges of H are the support of some $\mathbf{a}$ -flow on G .

The following characterization of vertices of $\mathcal{F}_G \mathbf{a}$ appeared in earlier work of Gallo and Sodini [4, Thm. 3.1].

Theorem 3.2 ([4]). Vertices of $\mathcal{F}_G \mathbf{a}$ are in correspondence with $\mathbf{a}$ -valid subforests of G .

We can use Theorem 3.2 to give a local criterion (at each vertex of G) for whether or not a flow is a 0-cell of the corresponding flow polytope, as follows.

Definition 3.3. For an integer $\mathbf{a}, \mathbf{b}$ -flow on $G_{n,m}$, the trajectories of two units of flow *split* (resp. *merge*) if

- at a vertex where the two trajectories meet, they leave (resp. enter) through different edges,
- or if at a vertex with negative (resp. positive) net flow where the two trajectories meet, exactly one leaves (resp. enters) through an edge.

An integer $\mathbf{a}, \mathbf{b}$ -flow $\mathbf{x}$ in $\mathcal{F}_{G_{n,m}} \mathbf{a}, \mathbf{b}$ is an *unsplittable flow* if no two of its trajectories split.

Example 3.4. Figure 3a (left) is an example of an unsplittable flow.

From the proof of Lemma 2.5, recall that any integral flow for $\mathcal{F}_{G_{n,m}} \mathbf{a}, \mathbf{b}$ can be seen as non-crossing trajectories of units of flow. Through this interpretation, we see that the i th trajectory from the top ends at the same vertex in any integral flow for $\mathcal{F}_{G_{n,m}} \mathbf{a}, \mathbf{b}$. Note that this does not depend on m . See Figure 3c. The next result gives a characterization of the vertices of $\mathcal{F}_{G_{n,m}} \mathbf{a}, \mathbf{b}$ in terms of unsplittable flows. See Figure 3d for an example.

Corollary 3.5. Consider an integral flow for $\mathcal{F}_{G_{n,m}} \mathbf{a}, \mathbf{b}$ with $\mathbf{a}, \mathbf{b} \in \mathbb{N}^n$. That flow corresponds to a vertex of $\mathcal{F}_{G_{n,m}} \mathbf{a}, \mathbf{b}$ if and only if the following conditions are satisfied.

¹In Hille [5], these are called *regular flows*.

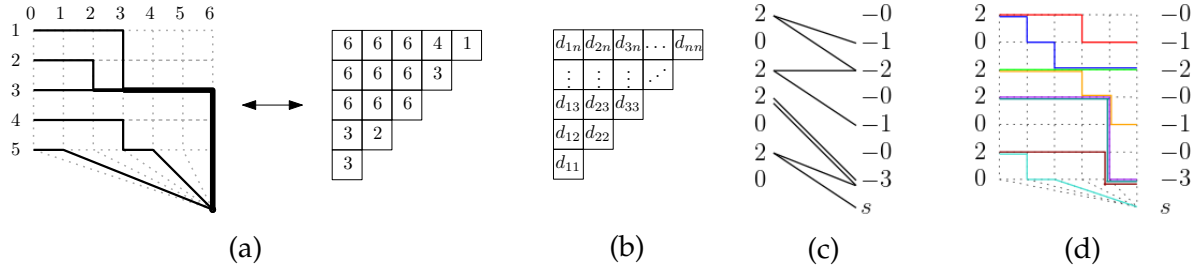


Figure 3: (a) an unsplittable flow and the corresponding vertex plane partition, (b) plane partition encoding where flows corresponding to vertices descend, (c) Starting and ending vertices for all trajectories for any flow in $\mathcal{F}_{G_{5,m}}$ $2, 0, 2, 2, 0, 2, 0, 0, 1, 2, 0, 1, 0, 3$ for any m , and (d) example of a vertex flow from Corollary 3.5.

- (i) The trajectories of any two units of flow starting at the same vertex with positive net flow do not merge.
- (ii) The trajectories of any two units of flow ending at the same vertex with negative net flow (including sink s) do not split.
- (iii) The trajectories of any two units of flow starting and ending at different vertices can split at most once and merge at most once.

In particular, if $\mathbf{b} = \mathbf{0}$, flows corresponding to vertices of $\mathcal{F}_{G_{n,m}}$ $\mathbf{a}$ are unsplittable flows.

Proof. First note that if any of (i), (ii) or (iii) does not hold for some integral flow for $\mathcal{F}_{G_{n,m}}$ $\mathbf{a}, \mathbf{b}$, then the flow contains a cycle. By Theorem 3.2, this implies that this flow does not correspond to a vertex of $\mathcal{F}_{G_{n,m}}$ $\mathbf{a}, \mathbf{b}$. Contrapositively, this means that if a flow is a vertex, then (i), (ii), (iii) must all hold.

For the converse, suppose that (i), (ii), (iii) all hold for some integral flow and that this flow contains a cycle between the trajectories of two units of flow. Note that any cycle contains both a merge and a split. Therefore, if the trajectories of those two units started or ended at the same vertex of $G_{n,m}$, we would have a contradiction to (i) and (ii) respectively. Finally, note that if the two units started and ended at different vertices, there must be at least one merge before the cycle and a split after the cycle, which means that in total, there are at least two merges and two splits, a contradiction to (iii). Therefore, if (i), (ii), (iii) hold for some integral flow, then the flow contains no cycles, and by Theorem 3.2, it corresponds to a vertex. $\square$

Next, we give a plane partition characterization for the vertices.

Definition 3.6. Let n and m be nonnegative integers. A plane partition $\pi = \pi_{ij}$ of staircase shape $\delta_n = (n, n-1, \dots, 1)$ with entries at most m is a *vertex plane partition* if (i) either $\pi_{i,j} < \pi_{i-1,j-1}$ or $\pi_{i,j} \geq \pi_{i-1,j}$, and (ii) if $\pi_{i,j} = \pi_{i,j-1}$ then $\pi_{i-1,j} = \pi_{i-1,j-1}$.

Theorem 3.7. *Let n and m be nonnegative integers, then the vertices of $PS_n^m \mathbf{a} \equiv \mathcal{F}_{G_{n,m}} \mathbf{a}$ with $\chi \mathbf{a} = \mathbf{1}$ are in correspondence with the vertex plane partitions.*

Proof. As was explained in Lemma 2.5, any lattice point $\mathbf{x}$ in $\mathcal{F}_{G_{n,m}} \mathbf{a}$ corresponds to some integer flow in $G_{n,m}$. Moreover, we know that if an integral flow is a vertex of $\mathcal{F}_{G_{n,m}} \mathbf{a}$, then no two trajectories split. Let d_{tr} be the index of the column in $G_{n,m}$ where the flow starting at $(t, 0)$ (for $0 \leq t \leq n$) descends from row r to row $r-1$. Since $\chi \mathbf{a} = \mathbf{1}$, we get a plane partition of shape $(n, n-1, \dots, 1)$ recording the numbers d_{tr} (see Figure 3b). We already know that $d_{tr} \geq d_{t-1,r}$, and $d_{t,r} \geq d_{t,r-1}$ for any flow. For an unsplittable flow, we know that if the unit flows starting respectively at $(t, 0)$ and $(t', 0)$ where $t < t'$ join, then they never split again. Moreover, if in row r of $G_{n,m}$, those two flows haven't joined yet, then the flow for t' must descend to row $r-1$ before the flow for t descends into row r , i.e., $d_{t',r} < d_{t,r-1}$. If on the other hand those two flows have already joined in or by row r , then they will descend together in row r , and they will also descend together in all subsequent rows, i.e., $d_{t,r'} = d_{t',r'}$ for all $r \leq r' \leq n$. Thus we obtain the conditions for a vertex plane partition $\pi = \pi_{ij}$, $\pi_{ij} = d_{j,n-1-i}$ (see Figure 3b). This argument can be generalized for the skew case, $\mathbf{b} \neq \mathbf{0}$ [2]. $\square$

Example 3.8. *Figure 3a illustrates the bijection between an unsplittable flow of the polytope $PS_5^6 \mathbf{1} \equiv \mathcal{F}_{G_{5,6}} \mathbf{1}$ and a vertex plane partition.*

4 Enumeration of number of vertices

Let $v^{n,m} \mathbf{a}, \mathbf{b}$ be the number of unsplittable flows of $PS_n^m \mathbf{a}, \mathbf{b}$, and let $v^{n,m} \mathbf{a} := v^{n,m} \mathbf{a}, \mathbf{0}$, which by Corollary 3.5 gives the number of vertices of $PS_n^m \mathbf{a}$. In this section, we give expansions and recursions for $v^{n,m} \mathbf{a}, \mathbf{b}$.

4.1 Nonnegative combinations of binomial coefficients

We show that $v^{n,m} \mathbf{a}$ is a polynomial in m by giving a nonnegative expansion in terms of the polynomial basis $\binom{m-1}{k}$.

Proof of Theorem 1.1. We know from Theorem 3.7 that $v^{n,m} \mathbf{a}$ counts the number of vertex plane partitions. We partition these according to how many different values they use. Consider all the vertex plane partitions using k values. You can further partition these by the k values among $\{0, 1, \dots, m\}$ that they use. There are $\binom{m-1}{k-1}$ such classes, and each class contains the same number of vertex plane partitions. Since δ_n has $\binom{n-1}{2}$ cells, there can be at most that many different labels within a plane partition, and so the result holds. For the bijection between plane partitions counted by $p_{n, \binom{n-1}{2}}$ and shifted SYT of staircase shape δ_n , see Figure 4 and the long version of [2]. $\square$

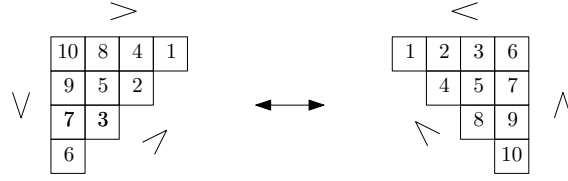


Figure 4: Example of correspondence between vertex plane partitions with distinct entries and shifted SYT of staircase shape.

4.2 Recurrences by fixing the flow of a column of $G_{n,m}$

For some flow $\mathbf{x}$ on $G_{n,m}$ and a column c , for $0 \leq c \leq m-1$, of horizontal arcs $i, c, i, c+1$ for $1 \leq i \leq n$ in $G_{n,m}$, let $\mathbf{x}_{\bullet c} : x_{1c}, x_{2c}, \dots, x_{nc}$ be the vector recording the flow on column c . Further, we let $v^{n,m}_{\mathbf{a}, \mathbf{b}, c, \mathbf{u}}$ for $0 \leq c \leq m-1$ be the number of vertices $\mathbf{x}$ in $PS_n^m(\mathbf{a}, \mathbf{b})$ such that $\mathbf{x}_{\bullet c} = \mathbf{u}$. Note that, for $\mathbf{a} \in \mathbb{N}^n$, by partitioning all different possible flows $\mathbf{x}$ on $G_{n,m}$ by their flow $\mathbf{x}_{\bullet c}$ for some column $0 \leq c \leq m-1$, we have that $v^{n,m}_{\mathbf{a}, \mathbf{b}} = \sum_{\mathbf{u} \in \mathbb{N}^n: \mathbf{a} \text{ dominates } \mathbf{u}} v^{n,m}_{\mathbf{a}, \mathbf{b}, c, \mathbf{u}}$. Indeed, as explained earlier, if $\mathbf{a} \in \mathbb{N}^n$, then unsplittable flows in $PS_n^m(\mathbf{a}, \mathbf{b})$ are integral. Furthermore, since flow can only go down and right, we can immediately note that $v^{n,m}_{\mathbf{a}, \mathbf{b}, c, \mathbf{u}} = 0$ if $\mathbf{a}$ does not dominate $\mathbf{u}$. So instead of summing over all vectors in $\mathbb{N}^n$, we already know that

$$v^{n,m}_{\mathbf{a}, \mathbf{b}} = \sum_{\substack{\mathbf{u} \in \mathbb{N}^n: \\ \mathbf{a} \text{ dominates } \mathbf{u}}} v^{n,m}_{\mathbf{a}, \mathbf{b}, c, \mathbf{u}}.$$

Knowing where $\mathbf{a}$ is 0 and where it is positive, which is captured by $\chi(\mathbf{a})$, allows to further make this sum smaller.

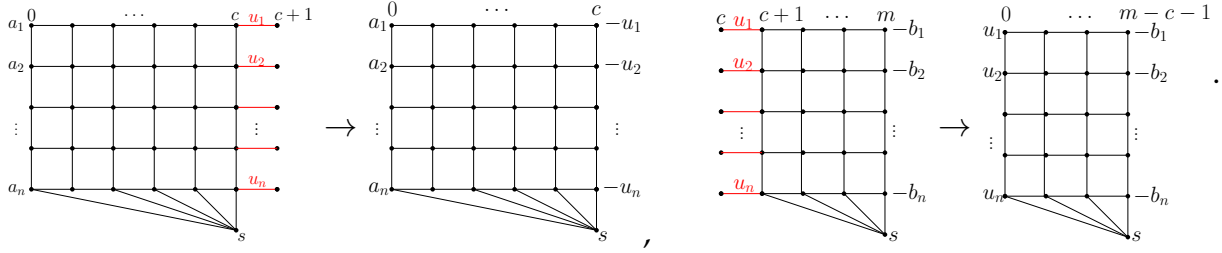
Note that, because of unsplittability, for $\mathbf{a}, \mathbf{a}' \in \mathbb{N}^n$, if $\chi(\mathbf{a}) = \chi(\mathbf{a}')$, then we have that $v^{n,m}_{\mathbf{a}} = v^{n,m}_{\mathbf{a}'}$.

In this section, we refine our understanding of when $v^{n,m}_{\mathbf{a}, \mathbf{b}, c, \mathbf{u}} = 0$ to give a recursion to calculate $v^{n,m}_{\mathbf{a}, \mathbf{b}}$ based on the following: we can calculate $v^{n,m}_{\mathbf{a}, \mathbf{b}, c, \mathbf{u}}$ by calculating the product of the number of flows on two graphs obtained after deleting column c of horizontal arcs in $G_{n,m}$.

Lemma 4.1. Fix positive integers n, m and let $1 \leq c \leq m-2$. Then

$$v^{n,m}_{\mathbf{a}, \mathbf{b}} = \sum_{\substack{\mathbf{u} \in \mathbb{N}^n: \\ \mathbf{a} \text{ dominates } \mathbf{u} \text{ and} \\ \chi(\mathbf{a}) \text{ dominates } \chi(\mathbf{u})}} v^{n,c}_{\mathbf{a}, \mathbf{u}} \cdot v^{n, m-c-1}_{\mathbf{u}, \mathbf{b}}.$$

Proof. We observe that the number of unsplittable flows counted by $v^{n,m}_{\mathbf{a}, \mathbf{b}, c, \mathbf{u}}$ is the product of the number of unsplittable flows on the following two graphs:



□

For the recursions, we need to understand unsplittable flows. This will also allow us to characterize what $\mathbf{x}_{\bullet c-1}$ can be if one is given $\mathbf{x}_{\bullet c}$. The patterns of zeros will end up playing an important role through the following definition.

Definition 4.2. Let $\mathbf{a} = a_1, a_2, \dots, a_n$, $\mathbf{b} = b_1, b_2, \dots, b_n \in \mathbb{N}^n$ be such that $\mathbf{a}$ dominates $\mathbf{b}$, and $\chi \mathbf{a} : \mathbf{v}$ dominates $\chi \mathbf{b} : \mathbf{w}$. If $v_{z-2}, v_{z-3}, \dots, v_n$ dominates $w_{z-2}, w_{z-3}, \dots, w_n$ for every $z \in [n]$ such that $b_z = 0$, then we say that $\mathbf{a}$ 1-dominates $\mathbf{b}$. Note that 1-dominance does not yield a poset.

Example 4.3. The vector $(3, 2, 0, 0, 2, 3, 4)$ 1-dominates $(3, 0, 1, 0, 1, 0, 6)$.

Theorem 4.4. Let $\mathbf{a}, \mathbf{b} \in \mathbb{N}^n$. Then

$$v^{n,m}(\mathbf{a}, \mathbf{b}) = \sum_{\substack{\mathbf{j} \in \{0,1\}^n: \\ \chi \mathbf{a} \text{ 1-dominates } \mathbf{j}}} v^{n,m-1}(\mathbf{w}_{\mathbf{j}}, \mathbf{b}),$$

where $\mathbf{w}_{\mathbf{j}}$ is the vector defined as follows. Let i_0 be the smallest index such that $j_{i_0} = 1$. Then $w_{\mathbf{j}, i_0} = \sum_{i=1}^{i_0} a_i$. Further, for any $i_1 < i_2$ such that $j_{i_1} = j_{i_2} = 1$, and $j_i = 0$ for all $i_1 < i < i_2$, $w_{\mathbf{j}, i_2} = \sum_{i=i_1}^{i_2} a_i$. All other components of $\mathbf{w}_{\mathbf{j}}$ are zero. Note that $\chi \mathbf{w}_{\mathbf{j}} = \mathbf{j}$.

In particular when $\mathbf{b} = \mathbf{0}$, we get the recurrence in [Corollary 1.2](#) for the number of vertices of $\text{PS}_n^m(\mathbf{a})$.

Remark 4.5. This result allows to compute $v^{n,m}(\mathbf{a}, \mathbf{b})$ via a transfer-matrix [\[13, Sec. 4.7\]](#) approach: adding entries of powers of an adjacency matrix of a digraph with vertices $\mathbf{j} \in \{0,1\}^n$ and arcs $\mathbf{j}, \mathbf{k}$ if $\mathbf{j}$ 1-dominates $\mathbf{k}$. This allows us to find generating functions like

$$\sum_{m \geq 0} v^{5,m}(\mathbf{1}) x^m = \frac{1 - 16x + 4x^2 - 48x^3 + 62x^4 - 20x^5 + 88x^6 - 14x^7 + 37x^8 - 8x^9 + 4x^{10}}{1 - x^{16}}.$$

5 Enumeration of d -faces of $\text{PS}_n^m(\mathbf{a})$

In this section, we use a similar recurrence to the one in [Theorem 4.4](#), i.e., fixing the flow in the first column of $G_{n,m}$, to give a recurrence for the number of higher-dimensional faces of $\text{PS}_n^m(\mathbf{a}) = \mathcal{F}_{G_{n,m}}(\mathbf{a})$. We assume that $\mathbf{b} = \mathbf{0}$.

First, we need the following description of the faces of flow polytopes $\mathcal{F}_G \mathbf{a}$ due to Hille [5, part of Thm. 3.2] for the case $a_i \geq 0$. A graph G is $\mathbf{a}$ -valid if it admits an $\mathbf{a}$ -flow.

Theorem 5.1 ([5]). *Let $\mathcal{F}_G \mathbf{a}$ be a flow polytope and $d \geq 0$ be a nonnegative integer. Then the d -dimensional faces of $\mathcal{F}_G \mathbf{a}$ are of the form $\mathcal{F}_H \mathbf{a}$ where H is an $\mathbf{a}$ -valid connected subgraph of G such that $\beta_1(H) = |E(H)| - |V(H)| + 1 = d$, after forgetting the directions of edges of H .*

Let $f_d^{n,m} \mathbf{a}$ be the number of d -dimensional faces of $PS_n^m \mathbf{a}$. For $\mathbf{u} \in \{0,1\}^n$, let $f_d^{n,m} \mathbf{a}, \mathbf{u}$ be the number of d -dimensional faces of $PS_n^m \mathbf{a}$ corresponding to $\mathbf{a}$ -valid subgraphs of $G_{n,m}$ with support $\mathbf{u}$ in the m th column.

Lemma 5.2. *Fix positive integers n, m , and let $1 \leq c \leq m - 2$. Then:*

$$f_d^{n,m} \mathbf{a} = \sum_{\mathbf{u} \in \{0,1\}^n} \sum_{i=0}^d f_i^{n,c} \chi(\mathbf{a}, \mathbf{u}) \cdot f_{d-i}^{n,m-c-1} \mathbf{u} \quad (5.1)$$

Proof sketch. The proof follows the same approach as Lemma 4.1 but using Theorem 5.1. $\square$

In the next theorem, we obtain a more explicit recurrence by fixing $c = 1$ in the previous lemma. We need the following notation and omit the proof.

Definition 5.3. *Let $\mathbf{j}$ and $\mathbf{a}$ be vectors of the same length such that $\mathbf{a}$ has signature $\text{sgn } \mathbf{a} = (b_1 - 1, b_2 - 1, \dots, b_\ell - 1)$. Then the **block structure of $\mathbf{j}$ according to $\mathbf{a}$** , denoted $\text{sgn}_{\mathbf{a}} \mathbf{j}$, is the tuple $(B_1, \dots, B_\ell)$ where B_1 consists of the first b_1 elements of $\mathbf{j}$, B_2 consists of the next b_2 elements of $\mathbf{j}$ and so on.*

Example 5.4. *Pick $\mathbf{a}$ with $\chi(\mathbf{a}) = (1, 1, 0, 1, 0, 0, 1, 0, 1)^T$ and let $\mathbf{j} = (1, 0, 0, 0, 1, 1, 1, 0, 0)^T$. Then the signature of $\mathbf{a}$ is $\text{sgn } \mathbf{a} = (0, 1, 2, 1, 0)$. Hence the block structure of $\mathbf{j}$ according to $\mathbf{a}$ is the tuple $(1, 0, 0, 0, 1, 1, 1, 0, 0)$.*

Theorem 5.5. *Let n and m be positive integers, $\mathbf{a} \in \mathbb{N}^n$, and $d \in \{0, 1, \dots, nm\}$ then for $m > 1$:*

$$f_d^{n,m} \mathbf{a} = \sum_{\mathbf{j} \in \{0,1\}^n} \sum_{k=0}^d \binom{\beta_{\mathbf{j},\mathbf{a}}}{k} f_{d-k-\gamma_{\mathbf{j},\mathbf{a}}}^{n,m-1} \mathbf{j},$$

where $\beta_{\mathbf{j},\mathbf{a}}$ is the number of nonzero blocks of $\text{sgn}_{\mathbf{a}} \mathbf{j}$, $\gamma_{\mathbf{j},\mathbf{a}} = b_{\mathbf{j}} - \beta_{\mathbf{j},\mathbf{a}}$, and $b_{\mathbf{j}}$ is the total number of 1's appearing in $\mathbf{j}$. For $m = 1$, we have $f_d^{n,1} \mathbf{j} = x^{d-k} \prod_{i=1}^k (x^{c_i} - 1)$ where $c_1, \dots, c_k = \text{sgn } \mathbf{j}$.

Corollary 5.6. *For $\mathbf{a} \in \mathbb{N}^n$ with $\chi(\mathbf{a}) = \mathbf{1}$, then $f_d^{n,m} \mathbf{a} = \sum_{\mathbf{j} \in \{0,1\}^n} \sum_{k=0}^d \binom{b_{\mathbf{j}}}{k} f_{d-k}^{n,m-1} \mathbf{j}$, where $b_{\mathbf{j}}$ is the number of ones appearing in $\mathbf{j}$.*

Example 5.7. *For $m = n = 2$, $f_d^{2,2} \mathbf{1}, \mathbf{1} = \left(f_d^{2,1} \mathbf{1}, \mathbf{1} - 2 \cdot f_{d-1}^{2,1} \mathbf{1}, \mathbf{1} - f_{d-2}^{2,1} \mathbf{1}, \mathbf{1} \right) \left(f_d^{2,1} \mathbf{1}, 0 - f_{d-1}^{2,1} \mathbf{1}, 0 \right) \left(f_d^{2,1} \mathbf{0}, 1 - f_{d-1}^{2,1} \mathbf{0}, 1 \right) - f_d^{2,1} \mathbf{0}, 0$. One can use this to obtain the f -vector $(1, 10, 21, 18, 7, 1)$ of $PS_2^2 \mathbf{1}, \mathbf{1}$.*

Acknowledgements

We were inspired by [7, 8] to extend a link between skew plane partitions with bounded parts and flow polytopes.

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On linear intervals in the alt $\mathcal{T}$ -Tamari lattices

Cesar Ceballos^{*1}, and Clément Chenevière^{+2,3}

¹*Institute of Geometry, TU Graz, Graz, Austria*

²*Institut de Recherche Mathématique Avancée, Université de Strasbourg, Strasbourg, France*

³*Ruhr-Universität Bochum, Bochum, Germany*

Abstract. Given a lattice path p , the $\mathcal{T}$ -Tamari lattice and the $\mathcal{D}$ -Dyck lattice are two natural examples of partial order structures on the set of lattice paths that lie weakly above p . In this paper, we introduce a more general family of lattices, called alt $\mathcal{T}$ -Tamari lattices, which contains these two examples as particular cases. Unexpectedly, we show that all these lattices have the same number of linear intervals.

Résumé. Étant donné un chemin discret p , le treillis $\mathcal{T}$ -Tamari et le treillis $\mathcal{D}$ -Dyck sont deux ordres partiels très naturels sur l'ensemble des chemins qui restent toujours au-dessus de p . Dans cet article, nous définissons une famille plus générale de treillis, appelés les treillis alt $\mathcal{T}$ -Tamari, qui contient ces deux ordres partiels comme cas particuliers. Nous montrons que, de façon surprenante, tous ces treillis possèdent le même nombre d'intervalles linéaires.

Keywords: Tamari lattice, Dyck lattice, linear intervals.

1 Introduction

The classical Tamari lattice is a partial order on Catalan objects which has inspired a vast amount of research in various mathematical fields [11]. One direction of research which has received a lot of attention in recent years regards its number of intervals [7], which is conjectured to be equal to the dimension of the alternating component in the study of trivariate diagonal harmonics [10]. Motivated by this intriguing connection, Bergeron introduced a generalization of the Tamari lattice called the m -Tamari lattice, and conjectured that its number of intervals again coincides with the dimension of the alternating component in higher trivariate diagonal harmonics [1]. A formula for their enumeration and connections to representation theory can be found in [2, 3]. A further generalization of the Tamari lattice, which includes the m -Tamari lattice, is the $\mathcal{T}$ -Tamari lattice introduced by Préville-Ratelle and Viennot [12]. These lattices are indexed by a lattice path p , and their number of intervals is connected to the enumeration of non-separable planar maps as shown in [9].

^{*}cesar.ceballos@tugraz.at. Supported by the Austrian Science Fund FWF, grants P 33278 and I 5788.

⁺ccheneviere@unistra.fr. Supported by the French National Research Agency, grants ANR-19-CE48-0011 and ANR-21-CE48-0020.

Inspired by the enumeration of intervals in the classical Tamari lattice and its generalizations, and guided by computer experimentation, Chapoton proposed to study the enumeration of the simpler class of linear intervals (intervals which are chains). This led to the work of the second author in [8], where he provides an explicit simple formula for the number of linear intervals in the classical Tamari lattice, and shows that their enumeration coincides with the enumeration of linear intervals in the Dyck lattice. The Dyck lattice, sometimes called the Stanley lattice, is perhaps the most natural poset on Dyck paths, defined by $P \leq Q$ if Q is weakly above P . In [8], the author also defines a new family of posets called alt Tamari posets, which contain the Tamari lattice and the Dyck lattice as particular cases. He shows that all alt Tamari posets have the same number of linear intervals of any given length.

In this paper, we generalize the results in [8] by introducing a new family of posets called alt α -Tamari posets. We show that they are lattices, and that they all have the same number of linear intervals of any given length. Figure 1 and Figure 2 illustrate the three different alt α -Tamari lattices for $\alpha = ENEENN$. In each case, the number of linear intervals of length k is given by $\binom{n}{k}$ where $n = 0, 1, 2, 3 = 16, 24, 16, 3$. For instance, 16 represents the trivial intervals of length 0, which are just the elements of each poset; there are 24 linear intervals of length 1, which correspond to the cover relations (edges in the figures); there are 16 linear intervals of length 2, and 3 linear intervals of length 3. The fact that these numbers coincide is somewhat surprising, since the posets look quite different. As a warm up exercise, the reader is invited to find the 3 linear intervals of length 3 in each of the figures.

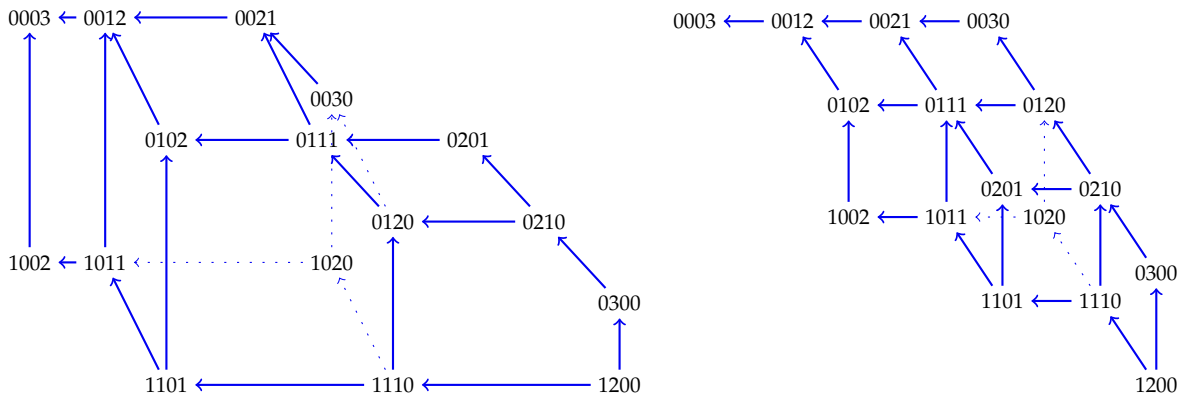


Figure 1: The α -Tamari lattice and α -Dyck lattice for $\alpha = ENEENN$. They are the alt α -Tamari lattices Tam_α for $\alpha = (2,0,0)$ and $\alpha = (0,0,0)$, respectively.

As the figures suggest, the alt α -Tamari lattices possess a rich underlying geometric structure, which seems to be realizable as a polytopal complex in some Euclidean space.

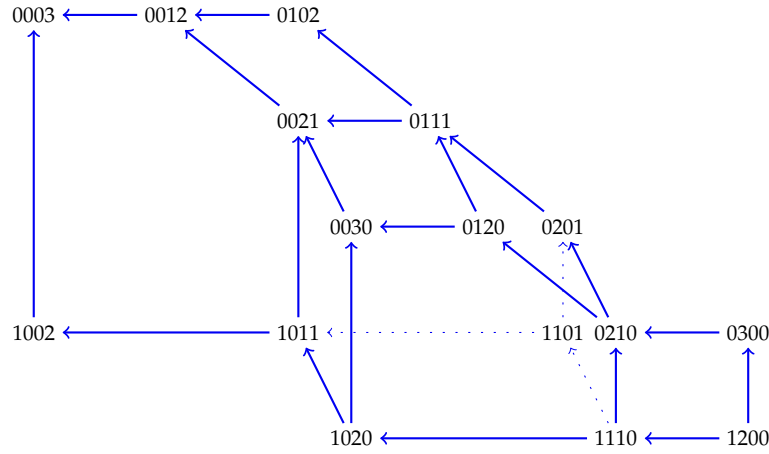


Figure 2: The alt γ -Tamari lattice Tam_{γ} for $ENEENN$ and $(1,0,0)$.

This was shown to be true for γ -Tamari lattices in [5], where polytopal complex realizations induced by some arrangements of tropical hyperplanes are provided. Similar geometric realizations in this general context will be presented in forthcoming work.

2 The γ -Dyck lattice

Let γ be a lattice path on the plane consisting of a finite number of north and east unit steps. We may represent a path γ as a word in the letters E and N for east and north steps respectively. We may as well represent γ as a sequence of non negative integers $i_0, i_1, \dots, i_k$, where k is the number of north steps of γ , i_0 is the number of initial east steps, and $i_i \geq 0$ is the number of consecutive east steps immediately following the i -th north step of γ . For instance, the path $ENEENNENEEE$ would correspond to the sequence $(1,2,0,1,3)$, while $ENEENN$ corresponds to $(1,2,0,0)$.

A γ -path is a lattice path using north and east steps, with the same endpoints as γ , that is weakly above γ . Alternatively, $i_0, \dots, i_k$ is a γ -path if and only if $i_j \geq i_{j-1} - i_{j-2}$ for all $0 \leq j \leq k$. The elements of the posets in Figure 1 and Figure 2 are labelled by γ -paths using this representation, where we omit the commas and parentheses for simplicity. For instance, the label 1200 is the minimal path $(1,2,0,0)$, which corresponds to $ENEENN$.

Definition 1. The γ -Dyck lattice Dyck_{γ} is the poset on γ -paths where $P \leq Q$ if Q is weakly above P .

2.1 Left and right intervals in the $\mathcal{D}_n$ -Dyck lattice

We focus on the special class of linear intervals in a poset. An interval $[P, Q]$ is *linear* if it is totally ordered, or equivalently if it is a chain of the form $P = P_0 < P_1 < \dots < P_k = Q$. The *length* of such a linear interval is defined to be k . A linear interval of length zero (containing only one element) is said to be *trivial*. The non trivial linear intervals of the $\mathcal{D}_n$ -Dyck lattice can be easily characterized into two different classes.

Definition 2. An interval $[P, Q]$ in $\mathcal{D}_n$ is a *left interval* if Q is obtained from P by transforming a subpath $E \rightarrow N$ into $N \rightarrow E$ for some $n \geq 1$. It is a *right interval* if Q is obtained from P by transforming a subpath $E \rightarrow N$ into $N \rightarrow E$ for some $n \geq 1$.

Proposition 3. The left and right intervals in the previous definition are linear intervals of length n . Moreover, all non trivial linear intervals in $\mathcal{D}_n$ are either left or right intervals. Only covering relations are both left and right (when $n = 1$).

Corollary 4. Left intervals of length n in $\mathcal{D}_n$ are in bijection with n -paths marked at a north step preceded by n east steps. Right intervals of length n in $\mathcal{D}_n$ are in bijection with n -paths marked at an east step followed by n north steps.

3 The $\mathcal{T}_n$ -Tamari lattice

The $\mathcal{T}_n$ -Tamari lattices are a generalization of the Tamari lattice. They were defined in terms of n -paths by Préville-Ratelle and Viennot in [12]. An alternative description in terms of n -trees was presented in [6].

3.1 On n -paths

For a lattice point p on a n -path γ , define its n -altitude $\text{alt}_n(p)$ to be the maximum number of horizontal steps that can be added to the right of p without crossing γ . Given a valley EN of γ , let p be the lattice point between the east and north steps. Let q be the next lattice point of γ such that $\text{alt}_n(q) = \text{alt}_n(p) + 1$, and $\gamma_{p,q}$ be the subpath of γ that starts at p and ends at q . Let γ' be the path obtained from γ by switching $\gamma_{p,q}$ with the east step E that precedes it. The n -rotation of γ at the valley p is defined to be γ' . An example is illustrated in Figure 3.

Definition 5. The $\mathcal{T}_n$ -Tamari poset Tam_n is the reflexive transitive closure of n -rotations on n -paths.

Theorem 6 ([12]). The $\mathcal{T}_n$ -Tamari poset is a lattice. Its covering relations are exactly n -rotations.

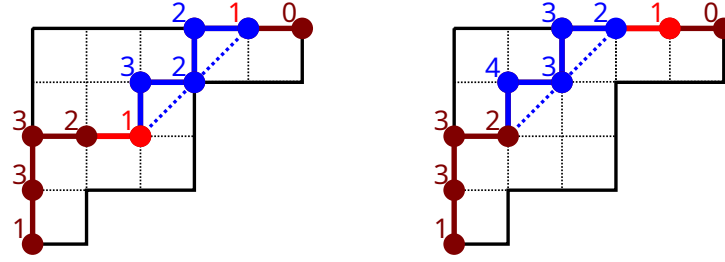


Figure 3: The rotation operation of a $\prec$ -path. Each node is labelled with its $\prec$ -altitude.

Another approach to define the $\prec$ -Tamari lattice is to introduce the $\prec$ -elevation of a subpath as the difference of $\prec$ -altitude between its ending point and its starting point. We thus write $\text{elev } E = 1$ for an east step E and $\text{elev } N_i = i$ if N_i is the i -th north step of a $\prec$ -path. For any subpath A of $\prec$, we then have $\text{elev } A = \sum_{a \in A} \text{elev } a$ as the sum of the $\prec$ -elevation of the steps of A .

The $\prec$ -excursion of a north step N of a $\prec$ -path is defined as the shortest subpath A of $\prec$ that starts with this N and such that $\text{elev } A = 0$. It follows from the definition of the $\prec$ -excursion that exchanging the east step E of a valley with the $\prec$ -excursion that follows it is exactly a covering relation in Tam .

This alternative description is useful to generalize the $\prec$ -Tamari lattice in [Section 4](#).

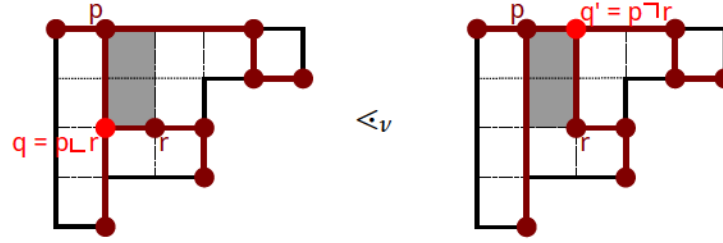
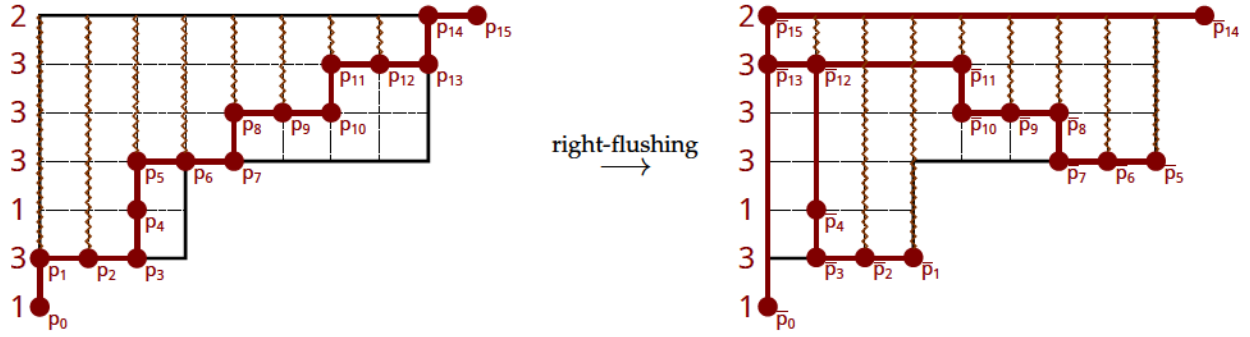
3.2 On $\prec$ -trees

One can also define a poset on $\prec$ -trees which is isomorphic to the $\prec$ -Tamari lattice.

We denote by F the Ferrers diagram that lies weakly above $\prec$ in the smallest rectangle containing $\prec$. Let L denote the set of lattice points inside F . We say that two points $p, q \in L$ are $\prec$ -incompatible if p is strictly southwest or strictly northeast of q , and the smallest rectangle containing p and q lies entirely in F . Otherwise, p and q are said to be $\prec$ -compatible. A $\prec$ -tree is a maximal collection of pairwise $\prec$ -compatible elements in L . In particular, the vertex at the top-left corner of F is $\prec$ -compatible with everyone else, and belongs to every $\prec$ -tree. Connecting two consecutive elements in the same row or column allows us to visualize $\prec$ -trees as classical rooted binary trees [6]. The vertex at top-left corner of F is always the *root*. An example of a $\prec$ -tree and the rotation operation which we now describe is shown in [Figure 4](#).

Let T be a $\prec$ -tree and $p, r \in T$ be two elements which do not lie in the same row or same column. We denote by $p \sqcap r$ the smallest rectangle containing p and r , and write $p \sqcup r$ (resp. $p \sqcap r$) for the lower left corner (resp. upper right corner) of $p \sqcap r$.

Let $p, q, r \in T$ be such that $q \in p \sqcup r$ and no other elements besides p, q, r lie in $p \sqcap r$. The $\prec$ -rotation of T at q is defined as the set $T' = T \setminus \{q\} \cup \{q'\}$, where $q' \in p \sqcap r$. As proven in [6, Lemma 2.10], the rotation of a $\prec$ -tree is also a $\prec$ -tree.

Figure 4: The rotation operation of a v -tree.Figure 5: Right flushing bijection from v -paths to v -trees.

Definition 7. The rotation poset of v -trees is the reflexive transitive closure of v -rotations.

Theorem 8 ([6]). The v -Tamari lattice is isomorphic to the rotation poset of v -trees.

A bijection between these two posets is given by the *right flushing bijection* introduced in [6]. This bijection maps a v -path $\mu = (\mu_0, \dots, \mu_k)$ to the unique v -tree with $\mu_i + 1$ nodes at height i . This tree can be recursively obtained by adding $\mu_i + 1$ nodes at height i from bottom to top, from right to left, avoiding forbidden positions. The forbidden positions are those above a node that is not the left most node in a row (these come from the initial points of the east steps in the path μ). In Figure 5, the forbidden positions are the ones that belong to the wiggly lines. Note that the order of the nodes per row is reversed.

The inverse of the right flushing bijection is called the *left flushing bijection*, and can be described similarly, adding points from left to right, from bottom to top, avoiding the forbidden position given by the wiggly lines.

3.3 Left and right intervals in the v -Tamari lattice

The description of the v -Tamari lattice on v -trees gives an easy description of its linear intervals.

Definition 9. An interval $[T, T']$ in the v -Tamari lattice on v -trees is a left interval if T' is obtained from T by applying ℓ rotations at nodes $q_0, \dots, q_{\ell-1}$ which are (maybe not directly)

consecutive in the same row, from left to right, for example $\bar{p}_{13}, \bar{p}_{12}$ in Figure 5. It is a right interval if T is obtained from T by applying α -rotations at nodes $q_0, \dots, q_{i-1}$ which are (maybe not directly) consecutive in the same column, from bottom to top, for example $\bar{p}_3, \bar{p}_4$ in Figure 5.

Proposition 10. *The left and right intervals in the previous definition are linear intervals of length α . Moreover, all non trivial linear intervals in Tam_α are either left or right intervals.*

Remark 11. *The left flushing of a left interval on the rotation lattice of α -trees produces a left interval P, Q of α -paths in Tam_α , where P is of the form AE^kBC with B some α -excursion and Q is of the form ABE^kC . In other words, P is a α -path with a valley preceded by k east steps.*

The left flushing of a right interval on the rotation lattice of α -trees produces a right interval P, Q of α -paths in Tam_α , where P is of the form $AEB_1 \dots B_kC$ with $B_1, \dots, B_k$ being k consecutive α -excursions, and Q is of the form $AB_1 \dots B_kEC$.

4 The alt α -Tamari lattice

Given a fixed path γ , the α -Dyck lattice and the α -Tamari lattice are two posets defined on α -paths with quite similar covering relations. In both cases, a covering relation consists of swapping the east step of a valley with a subpath that follows it. We can in fact define a whole family of posets that are described in a similar way, and we call them the alt α -Tamari posets. We also prove that they are lattices.

4.1 On α -paths

Let $\gamma = \gamma_0, \dots, \gamma_k$ be a fixed path. We say that $\mathbf{a} = a_1, \dots, a_k$ is an increment vector with respect to γ if $a_i \geq \gamma_i$ for all $1 \leq i \leq k$.

Similarly as the α -elevation, we introduce a notion of α -elevation of a subpath and declare that the α -elevation $\text{elev}_\gamma N_i$ or γN_i of the i -th north step N_i of any α -path is equal to γ_i and that $\text{elev}_\gamma E = 1$ for an east step E . Thus, for any subpath A of γ , its α -elevation is $\text{elev}_\gamma A = a_A$.

The α -excursion of a north step N of a α -path γ is defined as the smallest subpath A of γ that starts with this N and such that $\text{elev}_\gamma A = 0$. Let EN be a valley of γ and A be the α -excursion of N . Let γ' be the path obtained from γ by switching the step E and the subpath A . We say that $\gamma \rightarrow \gamma'$ is a α -rotation.

Definition 12. *Let $\mathbf{a}$ be an increment vector with respect to γ . The alt α -Tamari poset $\text{Tam}_\alpha(\mathbf{a})$ is the reflexive transitive closure of α -rotations on the set of α -paths.*

Remark 13. *For a fixed path γ , there are two extreme choices of increment vector $\mathbf{a}$. If $a_i = \gamma_i$ for all $1 \leq i \leq k$, the alt α -Tamari lattice coincides with the α -Tamari lattice. If $a_i = 0$ for all $1 \leq i \leq k$, the alt α -Tamari lattice coincides with the α -Dyck lattice.*

For a general increment vector $\check{\nu}$ with respect to $\check{\nu}$, it is not a priori clear that Tam is a lattice. This is a consequence of the following results.

Theorem 14. *Let $\check{\nu}_0 = \sum_{i=0}^k \check{\nu}_i$ and $\check{\nu}_i = \sum_{j=1}^i \check{\nu}_j$. Then $\check{\nu} = \check{\nu}_0, \check{\nu}_1, \dots, \check{\nu}_k$ is a path below $\check{\nu}$ whose endpoints are the same as $\check{\nu}$. Let 1 be the $\check{\nu}$ -path with all north steps at the beginning. The following hold:*

1. *The alt $\check{\nu}$ -Tamari poset $\text{Tam}_{\check{\nu}}$ is the restriction of $\text{Tam}_{\check{\nu}}$ to the subset of paths above 1 .*
2. *The restriction of $\text{Tam}_{\check{\nu}}$ to the subset of paths above 1 is the interval $[1, \check{\nu}]$ in $\text{Tam}_{\check{\nu}}$.*
3. *The alt $\check{\nu}$ -Tamari poset $\text{Tam}_{\check{\nu}}$ is a lattice.*

Proof. Firstly, we have $\sum_{i=0}^j \check{\nu}_i = \sum_{i=0}^j \sum_{k=1}^i \check{\nu}_k = \sum_{k=1}^j \sum_{i=k}^j \check{\nu}_k = \sum_{k=1}^j (j-k+1) \check{\nu}_k = \sum_{k=1}^j \check{\nu}_k = \check{\nu}_j$, where equality holds for $j \leq k$. Then, $\check{\nu}$ -rotations of a $\check{\nu}$ -path coincide with $\check{\nu}$ -rotations of $\check{\nu}$.

Secondly, the condition on $\check{\nu}$ ensures that any $\check{\nu}$ -rotation of a $\check{\nu}$ -path can be achieved as a sequence of $\check{\nu}$ -rotations. Thus, all $\check{\nu}$ -paths are indeed above 1 in $\text{Tam}_{\check{\nu}}$, thus $\text{Tam}_{\check{\nu}}$ is indeed an interval in a lattice, and it is therefore a lattice. $\square$

4.2 On $\check{\nu}$ -trees

The alt $\check{\nu}$ -Tamari lattice $\text{Tam}_{\check{\nu}}$ is the interval $[1, \check{\nu}]$ in $\text{Tam}_{\check{\nu}}$. So, it can be described as the rotation lattice of $\check{\nu}$ -trees that are above the $\check{\nu}$ -tree T corresponding to $\check{\nu}$ in $\text{Tam}_{\check{\nu}}$. These trees can be described as maximal collections of pairwise compatible elements in a shape $F_{\check{\nu}}$, which we will now describe. This point of view will be essential to show that all alt $\check{\nu}$ -Tamari lattices have the same number of linear intervals of any length.

Let $\check{\nu}$, $\check{\nu}$, and $\check{\nu}$ as in [Theorem 14](#). Let $F_{\check{\nu}}$ be the Ferrers diagram that lies weakly above $\check{\nu}$. We consider the lattice path $\hat{\nu}$ that starts at the lowest right corner of $F_{\check{\nu}}$ (the point with coordinates $(\check{\nu}_0, 0)$) which consists of the sequence of west and north steps

$$W^0 N W^1 N W^2 \dots N W^k, \quad \text{for } i = i_0, i_1, \dots, i_k.$$

We define $F_{\check{\nu}}$ to be the subset of $F_{\check{\nu}}$ consisting of the boxes that are not below $\hat{\nu}$, and denote by $L_{\check{\nu}}$ its set of lattice points. A $\check{\nu}$ -tree is a maximal collection of pairwise $\check{\nu}$ -compatible elements in $L_{\check{\nu}}$. An example is illustrated on the right of [Figure 6](#).

Proposition 15. *The $\check{\nu}$ -trees are exactly the $\check{\nu}$ -trees that are above T in $\text{Tam}_{\check{\nu}}$, where T is the $\check{\nu}$ -tree corresponding to $\check{\nu}$ under the right flushing bijection with respect to $\check{\nu}$.*

Definition 16. *The rotation poset of $\check{\nu}$ -trees is the reflexive transitive closure of $\check{\nu}$ -rotations on $\check{\nu}$ -trees.*

Theorem 17. *The alt $\check{\nu}$ -Tamari lattice $\text{Tam}_{\check{\nu}}$ is isomorphic to the rotation poset of $\check{\nu}$ -trees.*

Proposition 18. *For each $\mathbb{A}$ -path $\alpha = (a_0, a_1, \dots, a_k)$, there is a unique $\mathbb{A}$ -tree T that has exactly $a_i + 1$ nodes at height i , for $0 \leq i \leq k$.*

We call the bijection in this proposition the $\mathbb{A}$ -right flushing bijection from $\mathbb{A}$ -paths to $\mathbb{A}$ -trees. It can be described in exactly the same way as the right flushing bijection from Section 3.2: we recursively add $a_i + 1$ nodes to the tree inside the shape F_i , from right to left, from bottom to top, while avoiding the forbidden positions above a node which is not the left most node in a row. Figure 6 shows the image of the path $\alpha = (1, 0, 1, 1, 3, 2, 1, 2)$ for $\max = 1, 0, 2, 2, 0, 3, 0$ (left) and for $\max = 0, 0, 1, 2, 0, 1$ (right), where the base path is $\beta = (3, 1, 0, 2, 2, 0, 3, 0)$.

4.3 Left and right intervals in the alt $\mathbb{A}$ -Tamari lattice

Since $\text{Tam}^{\mathbb{A}}$ is an interval in $\text{Tam}^{\vee}$, its linear intervals are linear intervals in $\text{Tam}^{\vee}$.

Proposition 19. *The non trivial linear intervals in $\text{Tam}^{\mathbb{A}}$ are either left or right intervals $[T, T']$ where T and T' are both $\mathbb{A}$ -trees.*

We identify a left interval $[T, T']$ of length n in $\text{Tam}^{\mathbb{A}}$ with the bottom tree T marked at the nodes $q_0, q_1, \dots, q_n$, where T' is obtained by consecutively rotating $q_0, \dots, q_{n-1}$ in T . Here, $q_0, \dots, q_n$ are consecutive initial nodes (left most nodes) of T in some row different than the top. A $\mathbb{A}$ -marked $\mathbb{A}$ -tree is a tuple $(T, q_0, \dots, q_n)$ obtained this way.

For simplicity in the figures, we represent a left interval by marking an L (horizontal L) in T , which connects the nodes $q_0, \dots, q_n$ to each other and the node q_0 to the node directly above it. Figure 6 shows two trees with six different marked L 's. They correspond to six left intervals of lengths 1, 1, 1, 1, 2, and 3.

Similarly, we identify a right interval $[T, T']$ of length n in $\text{Tam}^{\mathbb{A}}$ with the top tree T' marked at the nodes $q_0, q_1, \dots, q_n$, where T is obtained by consecutively rotating down $q_0, \dots, q_{n-1}$ in T' . Here, $q_0, \dots, q_n$ are consecutive final nodes (top most nodes) of T' in some column different than the left most. In addition, the points q_i have the condition that they are not the bottom lattice point of a north step in the path $\hat{\alpha}$; otherwise, a down rotation could not be performed at the node q_{i-1} . A $\mathbb{A}$ -marked $\mathbb{A}$ -tree is a tuple $(T, q_0, \dots, q_n)$ obtained this way.

For simplicity in the figures, we represent a right interval by marking an $\mathbb{A}$ (vertical $\mathbb{A}$) in T' , which connects the nodes $q_0, \dots, q_n$ to each other and the node q_0 to the node directly to its left. Figure 7 shows two trees with four different marked $\mathbb{A}$'s. They correspond to four right intervals of lengths 1, 1, 1, and 3.

Proposition 20. *The following hold:*

1. *Left intervals in $\text{Tam}^{\mathbb{A}}$ are in bijection with $\mathbb{A}$ -marked $\mathbb{A}$ -trees $(T, q_0, \dots, q_n)$.*
2. *Right intervals in $\text{Tam}^{\mathbb{A}}$ are in bijection with $\mathbb{A}$ -marked $\mathbb{A}$ -trees $(T, q_0, \dots, q_n)$.*

In both cases, the length of the interval is equal to n .

5 Bijections between linear intervals

As we have seen in [Proposition 18](#), a $\mathbb{P}$ -tree T is completely characterized by its *row vector* $r = r_0, r_1, \dots, r_k$, where r_i is the number of nodes of T at height i . The *horizontal flushing bijection* $\mathcal{H}$ is the bijection from $\mathbb{P}$ -trees to $\mathbb{P}$ -trees that preserves the number of nodes at each height (preserving the row vector). This naturally extends to a map $\mathcal{H}$ from $\mathbb{P}$ -marked $\mathbb{P}$ -trees to $\mathbb{P}$ -marked $\mathbb{P}$ -trees, defined by

$$\mathcal{H}(T, q_0, \dots, q) = (\mathcal{H}(T), q_0, \dots, q),$$

where q_i is the node of $\mathcal{H}(T)$ corresponding to q_i in T under the bijection $\mathcal{H}$. An example is illustrated in [Figure 6](#).

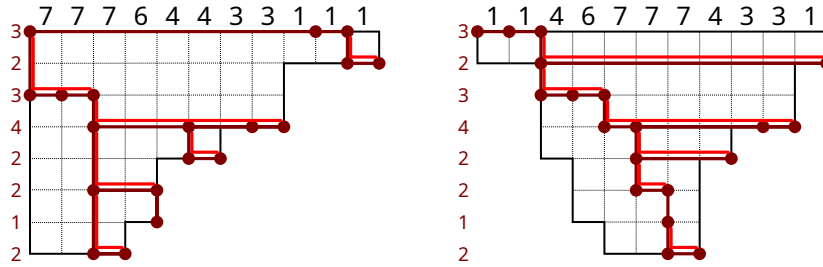


Figure 6: Bijection between left intervals for $\mathbb{P}^{max}$ and $(0, 0, 1, 2, 0, 1, 0)$.

Proposition 21. *The map $\mathcal{H}$ is a bijection from $\mathbb{P}$ -marked $\mathbb{P}$ -trees to $\mathbb{P}$ -marked $\mathbb{P}$ -trees.*

Theorem 22. *For a fixed $\mathbb{P}$, all alt $\mathbb{P}$ -Tamari lattices have the same number of left intervals of length n .*

In order to define a bijection between right intervals of two alt $\mathbb{P}$ -Tamari lattices we need a more subtle idea. For this, we use a bijection that "preserves" a modified column vector of $\mathbb{P}$ -trees.

Let $\mathbb{P}$ and $\mathbb{P}'$ as in [Theorem 14](#), and $m = 0, \dots, k$. The paths $\hat{\cdot}$ and $\check{\cdot}$ start at the origin $(0, 0)$ and end at the coordinate (m, k) . Given a $\mathbb{P}$ -tree T , the *column vector* of T is the vector $c(T) = (\bar{c}_0, \bar{c}_1, \dots, \bar{c}_m)$ such that $\bar{c}_i$ counts the number of nodes in T with x -coordinate equal to i .

The $\mathbb{P}$ -tree T is completely determined by $c(T)$. It can be reconstructed via the *down flushing bijection*, which recursively adds points to T in $F_{\mathbb{P}}$ from right to left, from bottom to top, avoiding forbidden positions appearing on the left of previously added nodes that are not the top node in their column (the wiggles forbidden lines are horizontal in this case).

We call a lattice point in $F_{\mathbb{P}}$ *irrelevant* if it is the bottom point of a north step in the path $\hat{\cdot}$ (recall this is the left boundary path of the shape $F_{\mathbb{P}}$). All other lattice

points are called *relevant*. Given a $\mathcal{A}$ -tree T , the $\mathcal{A}$ -column vector of T is the vector $c \in T = c_1, \dots, c_m$ such that c_i counts the number of relevant nodes in T with x -coordinate equal to i . Since the only relevant point with x -coordinate equal to zero is the top left corner (corresponding to the root of every tree), we note that $c_0 = 1$ for every tree, and is omitted in the vector $c \in T$. In [Figure 7](#), the irrelevant points are drawn green, and the relevant points brown; the $\mathcal{A}$ -column vectors of the two examples in this figure are $(1, 4, 1, 2, 2, 1, 1, 1, 1, 2, 1)$ (left) and $(1, 2, 2, 2, 1, 4, 1, 1, 1, 1, 1)$ (right). Note that one is a permutation of the other.

Proposition 23. A $\mathcal{A}$ -tree T is completely determined by its $\mathcal{A}$ -column vector $c \in T$.

Let $b = b_1, \dots, b_m$ be such that b_i counts the number of boxes in column i in the diagram $F_{\mathcal{A}}$. Note that the vector b is just a permutation of the vector c . We define the permutation σ_b as the permutation determined by $b = \sigma_b(c)$, which sends the j -th appearance of a number b in b to the j -th appearance of b in c .

Proposition 24. Let T be a $\mathcal{A}$ -tree and α another increment vector with respect to $\mathcal{A}$. There exists a unique T be a $\mathcal{A}$ -tree T such that $c \in T = \sigma_{\alpha}(c)$, and the irrelevant nodes in T and T have the same heights.

We define the *vertical flushing bijection* σ_{α} as the bijection between $\mathcal{A}$ -trees and $\mathcal{A}$ -trees that preserves the heights of the irrelevant nodes and that transforms the $\mathcal{A}$ -column vector according to the permutation σ_{α} . That is,

$$\sigma_{\alpha}(T) = T, \text{ where } c \in T = \sigma_{\alpha}(c) \in T.$$

An example is illustrated in [Figure 7](#).

This map naturally extends to a map σ_{α} from $\mathcal{T}$ -marked $\mathcal{A}$ -trees to $\mathcal{T}$ -marked $\mathcal{A}$ -trees, defined by

$$\sigma_{\alpha}(T, q_0, \dots, q) = (\sigma_{\alpha}(T), q_0, \dots, q),$$

where q_i is the node of $\sigma_{\alpha}(T)$ corresponding to q_i in T under the bijection σ_{α} . An example is illustrated in [Figure 7](#).

Proposition 25. The map σ_{α} is a bijection from $\mathcal{T}$ -marked $\mathcal{A}$ -trees to $\mathcal{T}$ -marked $\mathcal{A}$ -trees.

Theorem 26. For a fixed $\mathcal{A}$, all alt $\mathcal{A}$ -Tamari lattices have the same number of right intervals of length n .

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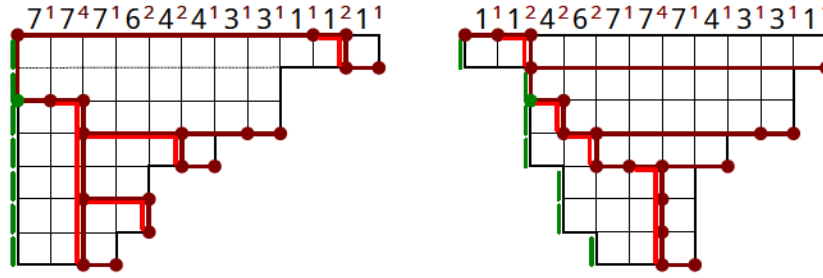


Figure 7: Bijection between right intervals for δ^{max} and $\delta = (0, 0, 1, 2, 0, 1, 0)$.

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RSK for 3-free posets

Kyle Celano^{1*}

¹Department of Mathematics, University of Miami, Miami, FL

Abstract. A long-standing open problem is to find an RSK-like correspondence between permutations and pairs of tableaux coming from Gasharov's decomposition of Stanley's chromatic symmetric functions into Schur functions. In this, we present such a correspondence RSK_P for incomparability graphs of 3-free posets P that moreover preserve the descent and inversion statistics. We then extend RSK_P to bijections from proper colorings and multicolorings providing new combinatorial proofs for the Schur expansions of Gasharov for the chromatic symmetric function, of Shareshian–Wachs for the chromatic quasisymmetric function, and of Hwang for the multichromatic quasisymmetric function, and its refinement to equivalence classes of acyclic orientations in the case that P is 3-free.

Keywords: symmetric functions, RSK algorithms, chromatic symmetric functions

1 Introduction

The famous Robinson-Schensted-Knuth RSK correspondence is a bijection

$$\mathfrak{S}_n \rightarrow \bigsqcup_{\lambda \vdash n} SYT_\lambda \times SYT_\lambda \quad (1.1)$$

where $\mathfrak{S}_n$ is the set of permutations on $n : \{1, \dots, n\}$ and SYT_λ is the set of standard Young tableaux of shape λ . This is a *row-insertion* algorithm taking each letter w_i of a permutation $w = w_1 \cdots w_n \in \mathfrak{S}_n$ and inserting into a pair (T_{i-1}, R_{i-1}) of size $i-1$ by a *bumping* procedure in T_{i-1} and a *recording* procedure in R_{i-1} to make a pair (T_i, R_i) of size i . The bijection takes w to (T_n, R_n) (see [12, Chapter 7]).

A general form of RSK is a bijection

$$\mathcal{P}^n \rightarrow \bigsqcup_{\lambda \vdash n} SSYT_\lambda \times SSYT_\lambda \quad (1.2)$$

where $SSYT_\lambda$ is the set of semistandard Young tableaux of shape λ and where $\mathcal{P}^n$ is the set of size n generalized permutations, which are multisets $\left\{ \begin{smallmatrix} u_i \\ v_i \end{smallmatrix} \right\}_{i=1}^n$ of biletters $\begin{smallmatrix} u_i \\ v_i \end{smallmatrix}$

*kac265@miami.edu.

where u_i and v_i are positive integers. These can be written as two-line arrays so that if

$$\begin{pmatrix} u \\ v \end{pmatrix} \quad \begin{pmatrix} u_1 & u_2 & \dots & u_n \\ v_1 & v_2 & \dots & v_n \end{pmatrix}$$

then u is weakly increasing and if $u_i = u_j$ and $i < j$, then $v_i < v_j$. This general *RSK* inserts the *bottom* word v into the first tableau and records the procedure with *top* word u .

If we restrict to when the top word u is $12 \cdots n$, we get a bijection from the set $\mathbb{P}^n$ of length n words on the alphabet $\mathbb{P}$ of positive integers

$$\mathbb{P}^n \rightarrow \bigsqcup_{\lambda \vdash n} \text{SSYT}_\lambda \times \text{SYT}_\lambda. \quad (1.3)$$

If we restrict to when the bottom word v is a permutation in $\mathfrak{S}_n$, we get another bijection

$$\{ u, v \in \mathcal{P}^n \mid v \in \mathfrak{S}_n \} \rightarrow \bigsqcup_{\lambda \vdash n} \text{SYT}_\lambda \times \text{SSYT}_\lambda. \quad (1.4)$$

The bijection (1.3) provides a combinatorial proof of the decomposition of the complete homogeneous symmetric function h_{1^n} in terms of Schur functions

$$h_{1^n} = \sum_{\lambda \vdash n} f^\lambda s_\lambda$$

and the bijection (1.1) provides a bijective proof of

$$n! = \sum_{\lambda \vdash n} f^\lambda$$

where f^λ is the number of standard Young tableaux of shape $\lambda \vdash n$.

Gasharov [5] obtains the following decomposition of Stanley's chromatic symmetric function [11] for the incomparability graph $\text{Inc } P$ of a $(3, 1)$ -free poset P ,

$$X_{\text{Inc } P}(\mathbf{x}, t) = \sum_{\lambda \vdash n} f_\lambda^P s_\lambda \quad (1.5)$$

where f_λ^P counts a certain class of Young tableaux $\mathcal{T}_{P, \lambda}$ called *P-tableaux* of shape λ .¹ This specializes to

$$n! = \sum_{\lambda \vdash n} f_\lambda^P f^\lambda. \quad (1.6)$$

Gasharov obtains (1.5) by applying Jacobi–Trudi and then a sign-reversing involution. A long-standing open problem then is to find an *RSK-like* bijection proving (1.5) and (1.6) directly.

¹Definitions to be given in the next section

Progress has been made on this problem for certain classes of 3-1-free posets by Sundquist–Wagner–West and Chow [13, 4] for a large class of subposets and Kim–Pylyavskyy [8] for a slightly smaller one.

In this extended abstract², we first note that proper colorings can be represented as generalized permutations whose bottom word is a permutation and hence (1.5) is equivalent to existence of a bijection analogous to that in (1.4). We then construct a relatively simple RSK-like bijection when P is 3-free (or, length at most 1): this row-insertion algorithm inserts bottom letters of a generalized permutation representing the coloring and records the procedure with the top word, just as in RSK. Although the previous algorithms apply to much broader classes of posets, ours is much simpler.

Also, for 3-free natural unit interval orders, our RSK gives a bijective proof of the Shareshian–Wachs [9, 10] refinement of (1.5) for chromatic quasisymmetric functions. Hwang [7] further generalizes (1.5) to multichromatic quasisymmetric functions, which involve multicolorings of a graph. By viewing these multicolorings as generalized permutations, we extend our RSK to a bijective map from multicolorings to pairs of semistandard P -tableaux and semistandard Young tableaux, thereby attaining an analogue to a dual form of (1.2). This provides a bijective proof of Hwang’s generalization in the case of 3-free natural unit interval orders. Moreover, we can restrict our RSK to colorings of Hwang’s equivalence classes of acyclic orientations, which provides a bijective proof of the Schur expansion for his pieces of the chromatic quasisymmetric function in the case of 3-free natural unit interval orders.

This extended abstract is organized as follows: In Section 2, we provide the relevant definitions and background for chromatic (quasi)symmetric functions and P -tableaux. In Section 3, we describe the RSK for 3-free posets on permutations (Definition 3.1, Theorem 3.4) and then show how to extend RSK to obtain bijective proofs of the Schur expansions of Gasharov, Shareshian–Wachs (Theorem 3.10) and Hwang (Theorems 3.13 and 3.14).

2 Chromatic Symmetric Functions and P -tableaux

Stanley’s chromatic symmetric function [11] is defined for a graph $G = (n, E)$ to be

$$X_G(\mathbf{x}) = \sum_{\kappa \in \mathcal{K}(G)} x_{\kappa(1)} \cdots x_{\kappa(n)}$$

where $\mathcal{K}(G)$ is the set of proper colorings of G . The incomparability graph of a poset P , denoted $\text{Inc}(P)$, is the graph whose vertices are the elements of P and whose edges correspond to pairs of incomparable elements of P .

²See [2] for the full paper



Figure 1: A poset P and a P -tableaux

A P -tableau³ is a Young tableau of size $|P| = n$ filled with the elements of P such that

- each element of P appears exactly once,
- if x is immediately to the left of y , then $x \not\prec_P y$, and
- if x is immediately above of y , then $x <_P y$.

See Figure 1 for an example.

Let $\mathcal{T}_{P,\lambda}$ be the set of P -tableaux of shape λ and $f_P^\lambda = |\mathcal{T}_{P,\lambda}|$. Recall that a poset is (a, b) -free if it contains no induced subposet that is the disjoint union of an a -chain and a b -chain. For a partition λ , let λ^* denote its conjugate. Then, Gasharov showed the following by applying the Jacobi–Trudi identity and then a sign-reversing involution.

Theorem 2.1 ([5, Theorem 4]). *For a $(3, 1)$ -free poset P , $X_{\text{Inc } P} = \sum_{\lambda \vdash n} f_P^\lambda s_{\lambda^*}$.*

A row-strict Young tableau is one where the rows strictly increase and the columns weakly increase. Let RST_λ be the set of row-strict tableaux of shape λ . If $R \in RST_\lambda$, then its conjugate $R^* \in SSYT_{\lambda^*}$. Hence, $s_{\lambda^*} = \sum_{R \in RST_\lambda} x^{m_1 R} x^{m_2 R} \dots$ where $m_i R$ is the number of times i appears in R . We identify the set of permutations $\mathfrak{S}_n$ with the subset $\{\kappa_w\}_{w \in \mathfrak{S}_n} \subset \mathcal{K}_G$ given by $\kappa_w(i) = w^{-1}(i)$ for $w \in \mathfrak{S}_n$ (why we do this is will be apparent later, see Remark 2.4). Thus, our problem of finding an appropriate RSK that would provide direct bijective proofs of Theorem 2.1 and (1.6) is as follows.

Problem 2.2. *For a $(3, 1)$ -free poset P with $G = \text{Inc } P$, find an RSK-like bijection*

$$RSK_P : \mathcal{K}_G \rightarrow \bigsqcup_{\lambda \vdash n} \mathcal{T}_{P,\lambda} \times RST_\lambda$$

which restricts to a bijection

$$RSK_P : \mathfrak{S}_n \rightarrow \bigsqcup_{\lambda \vdash n} \mathcal{T}_{P,\lambda} \times SYT_\lambda.$$

³This is the conjugate version of Gasharov’s original definition [5]. We make this change to more naturally describe the algorithm as a row-insertion algorithm.

A *natural unit interval order* (or, NUIO) is a $\mathbf{3}$ - $\mathbf{1}$ -free and $\mathbf{2}$ - $\mathbf{2}$ -free poset with a certain natural labeling. Let $G = \text{Inc } P$ for a natural unit interval order P . The Shareshian–Wachs [9, 10] *chromatic quasisymmetric function* for G is

$$X_G(\mathbf{x}, t) = \sum_{\kappa \in \mathcal{K}_G} t^{\text{des } \kappa} x_{\kappa_1} \cdots x_{\kappa_n}$$

where $\text{des } \kappa$ is the number of edges $ij \in E$ such that $i < j$ and $\kappa_i > \kappa_j$. This quasisymmetric function specializes to Stanley’s chromatic symmetric function by setting $t = 1$. Shareshian and Wachs prove that $X_G(\mathbf{x}, t)$ is in fact a symmetric function. For a P -tableaux $T \in \mathcal{T}_P$ define

$$\text{inv}_G T = \#\{xy \in E \mid x < y \text{ and } x \text{ is to the right of } y\}.$$

Let $f_{P,i}^\lambda$ be the number of P -tableaux T with $\lambda \vdash T = \lambda$ and $\text{inv}_G T = i$.

Theorem 2.3 ([10, Theorem 6.3]). *For a natural unit interval order P ,*

$$X_{\text{Inc } P}(\mathbf{x}, t) = \sum_{i \geq 0} t^i \sum_{\lambda \vdash n} f_{P,i}^\lambda s_{\lambda^*}. \quad (2.1)$$

Hence, for RSK_P to prove (2.1) bijectively, it must additionally satisfy:

$$\text{If } P \text{ is a NUIO and } \text{RSK}_P \kappa = T, R \text{ then } \text{des } \kappa = \text{inv}_G T. \quad (2.2)$$

Remark 2.4. For $w \in \mathfrak{S}_n$, the Shareshian–Wachs G -inversion statistic [9, 10] is

$$\text{inv}_G w = \#\{\{w_i, w_j\} \in E \mid i < j, w_i > w_j\}.$$

The G -inversion statistic $\text{inv}_G w$ is a natural extension of the classical inversion $\text{inv } w$ statistic to the graphs we are considering. If $\kappa_w \in \mathcal{K}_G$ is defined by $\kappa_w i = w^{-1} i$, then $\text{des } \kappa_w = \text{inv}_G w$. Hence, (2.2) for the restriction to $\mathfrak{S}_n$ is that if $\text{RSK}_P w = T, R$, then $\text{inv}_G w = \text{inv}_G T$.

In the next section, we will present an RSK_P solving Problem 2.2 which additionally satisfies (2.2) for 3-free posets.

Remark 2.5. The class of 3-free posets have been studied extensively in the literature of chromatic quasisymmetric functions under a few different, but equivalent, definitions.

- They are *length* 1 or 0 posets and their incomparability graphs are unit interval orders with bipartite complement as in Abreu–Nigro [1].
- They correspond to *abelian* Hessenberg varieties as in Harada–Precup [6].
- They correspond to Dyck paths with *bounce number* 1 or 0 as in Cho–Huh [3].

The commonality between each of the definitions and interpretations of 3-free posets is that the resulting theory for chromatic quasisymmetric functions and Hessenberg varieties is quite nice and allows for one to prove e -positivity in a variety of ways. Therefore, one should expect an RSK for 3-free posets to be just as nice.

3 RSK for 3-free posets

In this section, we assume that P is a 3-free poset with incomparability graph G . Hence, any P -tableau T has at most 2 rows.

3.1 RSK_P for permutations

We will define a row-insertion algorithm that inserts the letters of permutation w $w_1 \cdots w_n$ one by one to make a series of pairs $T_1, R_1, \dots, T_n, R_n$ of P -tableau T_i and standard Young tableau R_i with same shape.

To do so, we must describe a series of *local moves* on P -arrays (which are defined to be P -tableaux in columns only). Given a $1, 2, 1$ subshape of a P -array T , we change it into a $(2, 1, 1)$ shape with the same entries

$$\begin{array}{|c|c|c|} \hline z & x & \omega \\ \hline & y & \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|} \hline \tilde{z} & \tilde{x} & \tilde{\omega} \\ \hline \tilde{y} & & \\ \hline \end{array}$$

Define the change according to the 4 cases as shown in Figure 2.

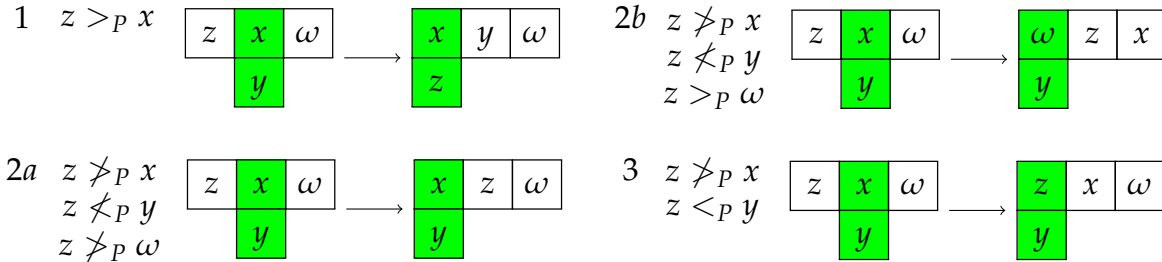


Figure 2: The 4 local moves

Definition 3.1 (RSK for 3-free P). Let P be a 3-free poset and let $w \in \mathfrak{S}_n$.

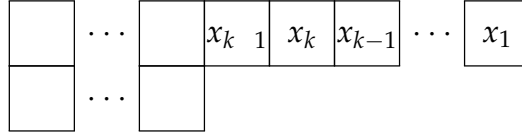
1. For w_1 , define

$$T_1 = \begin{array}{|c|} \hline w_1 \\ \hline \end{array} \quad R_1 = \begin{array}{|c|} \hline 1 \\ \hline \end{array}$$

2. Suppose $w_1, \dots, w_{i-1}$ have been inserted forming a pair T_{i-1}, R_{i-1} of P -tableau and standard Young tableaux, respectively, of the same shape λ with $|\lambda| = i - 1$.

- (a) If w_i is not smaller in P than the last number of the first row of T_{i-1} , then place w_i at the end of the row creating T_i .

(b) Otherwise, suppose T_{i-1} looks like



and let x_k be the leftmost element of the first row such that $x_j >_P w_i$ for all $j \leq k$ (so, $x_{k-1} \not>_P w_i$). Then replace x_k by w_i and place x_k in the cell directly below its old cell (this is empty because P is 3-free). Then, apply the *local moves* moving this new domino to the left until we have a partition shape. Call this new tableau T_i .

In either case, if c is the new cell of T_i , set $R_i \leftarrow c \cup i$.

We write $RSK_P w \mapsto (T, R)$ for the result after inserting the final letter of w .

Example 3.2. Let P be as in Figure 1 and suppose we have

$$T_5, R_5 = \left(\begin{array}{|c|c|c|c|c|c|} \hline 3 & 4 & 2 & 5 & 6 & \\ \hline \end{array}, \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 5 & \\ \hline \end{array} \right)$$

and we want to insert $w_6 = 1$. Then we need 3 local moves, which we indicate with an arrow.

$$\begin{array}{|c|c|c|c|c|c|} \hline 3 & 4 & 2 & 1 & 6 & \\ \hline & & 5 & & & \end{array} \xrightarrow{3} \begin{array}{|c|c|c|c|c|c|} \hline 3 & 4 & 2 & 1 & 6 & \\ \hline & & 5 & & & \end{array} \xrightarrow{2b} \begin{array}{|c|c|c|c|c|c|} \hline 3 & 1 & 4 & 2 & 6 & \\ \hline & 5 & & & & \end{array} \xrightarrow{2a} \begin{array}{|c|c|c|c|c|c|} \hline 1 & 3 & 4 & 2 & 6 & \\ \hline & 5 & & & & \end{array}$$

Example 3.3. Let P be as in Figure 1 and let $w = 361452$. We show the full RSK_P for w in Figure 3. If we apply a local move during the insertion step to compute T_i , we show them to the left of the corresponding step.

Theorem 3.4. For 3-free P , the above algorithm $w \mapsto (T, R)$ is a bijection

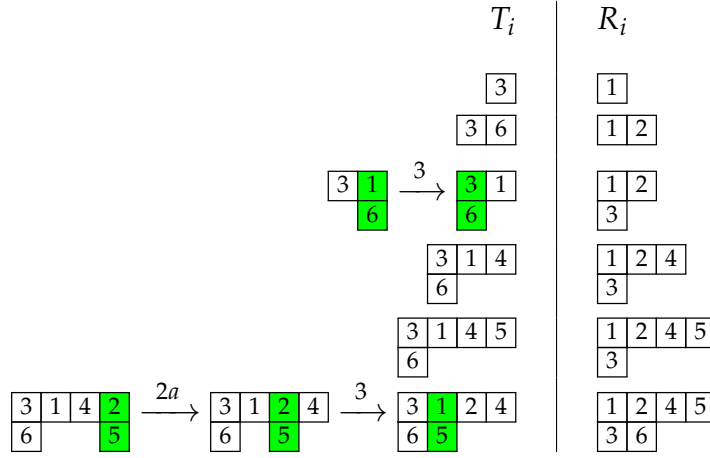
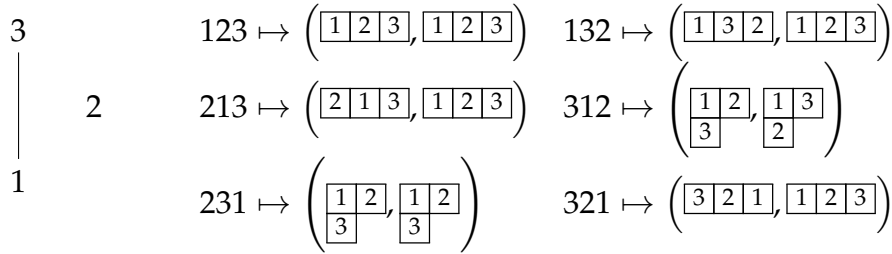
$$RSK_P : \mathfrak{S}_n \rightarrow \bigsqcup_{\lambda \vdash n} \mathcal{T}_{P,\lambda} \times \text{SYT}_\lambda$$

such that if P is additionally a natural unit interval order and $RSK_P w \mapsto (T, R)$, then $\text{inv}_G w = \text{inv}_G T$.

Proof (Idea). One needs to check that

- (1) local moves create P -tableaux,
- (2) each local move has a unique *inverse* local move, and
- (3) each local move preserves G -inversions when P is a natural unit interval order.

Then (1) plus 3-free gives a well-definition; (2) gives the bijective property; and (3) plus the definition of the insertion gives (2.2). $\square$

Figure 3: RSK_P 361452Figure 4: RSK_P for a poset P on 3

Example 3.5. In Example 3.3, we have $\text{inv}_G w = 5$ and $\text{inv}_G T = 5$. See Figure 4 for a complete example for $n = 3$.

Hence, Theorem 3.4 provides a new combinatorial proof of (1.6) when P is $\mathbf{3}$ -free. For $w \in \mathfrak{S}_n$, let $\text{DES } w$ be the usual descent set of a permutation. For a standard Young tableau R define $\text{DES } R = \{i \in [n-1] \mid i+1 \text{ is below } i\}$. Then classical RSK has the property that if $RSK(w) = (T, R)$, then $\text{DES } w = \text{DES } R$. For a poset P , define $\text{DES}_P w = \{i \in [n-1] \mid w_i >_P w_{i+1}\}$. Then RSK_P also “preserves descents”, meaning the following.

Proposition 3.6. For $\mathbf{3}$ -free posets P , if $RSK_P(w) = (T, R)$, then $\text{DES}_P w = \text{DES } R$.

Example 3.7. In Example 3.3, we have $\text{DES}_P w = \{2, 5\} = \text{DES } R$.

Remark 3.8. Chow [4] points out that the Sundquist–Wagner–West algorithm preserves descents only on a subclass of the posets considered in [13]. This smaller subclass does not contain $\mathbf{3}$ -free posets. The Kim–Pylyavskyy algorithm satisfies the similar property that $\text{DES } R = \{n-d \mid d \in \text{DES}_P w\}$.

3.2 Proper colorings

While section 3.1 was just for permutations, it serves as the foundation for an RSK_P of colorings. For each proper coloring $\kappa \in \mathcal{K}_G$, form the generalized permutation $\{\kappa_i^i\}_{i=1}^n$ of biletters. Then write this as a two-line array

$$\begin{pmatrix} \bar{\kappa} \\ w \end{pmatrix} \begin{pmatrix} \kappa w_1 & \kappa w_2 & \dots & \kappa w_n \\ w_1 & w_2 & \dots & w_n \end{pmatrix}$$

so that $\bar{\kappa} = \kappa w_1 \leq \kappa w_2 \leq \dots \leq \kappa w_n$ and if $\kappa w_i = \kappa w_j$ and $i < j$, then $w_i > w_j$.

Example 3.9. Let P be as in Figure 1 and let κ be the coloring of $\text{Inc } P$ below

$$\begin{array}{ccccccc} \kappa & 3 & 2 & 1 & 3 & 2 & 1 \\ G & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \quad \begin{pmatrix} \bar{\kappa} \\ w \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 & 2 & 3 & 3 \\ 6 & 3 & 5 & 2 & 4 & 1 \end{pmatrix}$$

Then, the general RSK_P inserts the bottom word w (which is a permutation) to get a P -tableau T as in Definition 3.1 and records the procedure in R with the top word $\bar{\kappa}$. Our choice of 2-line array ensures that if $\kappa w_i = \kappa w_{i-1}$, then $w_i >_P w_{i-1}$, so κw_i and κw_{i-1} will be in separate rows of R , and hence R will be *row-strict*.

Theorem 3.10. For 3-free P , the above algorithm $\kappa \mapsto (T, R)$ is bijection

$$RSK_P : \mathcal{K}_G \rightarrow \bigsqcup_{\lambda \vdash n} \mathcal{T}_{P,\lambda} \times RST_\lambda$$

such that if $RSK_P w = (T, R)$ and P is additionally a natural unit interval order, then $\text{des } \kappa = \text{inv}_G T$. Moreover, restricting this map to $\mathfrak{S}_n$ gives the bijection in Theorem 3.4.

Theorem 3.10 hence provides new combinatorial proofs of the Schur bases expansions given in Theorems 2.1 and 2.3 when P is 3-free.

Remark 3.11. By applying P, ω -partition reciprocity, the monomials of $\omega X_G \mathbf{x}, t$ are seen to be indexed by pairs $(\bar{\mathbf{o}}, \kappa)$ of acyclic orientations $\bar{\mathbf{o}}$ and weak colorings κ of $\bar{\mathbf{o}}$, meaning $x \rightarrow y$ in $\bar{\mathbf{o}}$ implies that $\kappa x \leq \kappa y$. In the full paper [2], we extend RSK_P to this collection.

3.3 Proper multicolorings

We now come to the fullest generality of RSK_P , mirroring the dual of the RSK bijection (1.2). A *proper multicoloring* of a graph G is a map κ that assigns each vertex i a finite set $\kappa_i \subset \mathbb{P}$ such that if $ij \in E$ then $\kappa_i \cap \kappa_j = \emptyset$. Let $\mathcal{MK}_n G$ be the set of proper colorings so that $\sum_{v \in G} |\kappa_v| = n$. We can turn a multicoloring $\kappa \in \mathcal{MK}_n G$ into a generalized permutation $\begin{pmatrix} \bar{\kappa} \\ w \end{pmatrix}$ of length n analogously to how we did colorings. See Example 3.12.

Example 3.12. Let P be as in Figure 1 and let κ be the multicoloring of $\text{Inc } P$ below

$$\begin{array}{cccccc} & & 7 & & & \\ & 4 & 5 & 6 & 4 & \\ \kappa & 3 & 2 & 1 & 3 & 2 & 1 \\ G & 1 & \text{---} 2 & \text{---} 3 & \text{---} 4 & \text{---} 5 & \text{---} 6 \end{array} \quad \left(\begin{array}{c} \bar{\kappa} \\ w \end{array} \right) \quad \left(\begin{array}{cccccccccc} 1 & 1 & 2 & 2 & 3 & 3 & 4 & 5 & 5 & 6 & 7 \\ 6 & 3 & 5 & 2 & 4 & 1 & 1 & 5 & 2 & 3 & 2 \end{array} \right)$$

Hence, the analogous version of (1.2)⁴ is a bijection

$$RSK_P : \mathcal{MK}_n(G) \rightarrow \bigsqcup_{\lambda \vdash n} \mathcal{SST}_{P,\lambda} \times RST_\lambda$$

where $\mathcal{SST}_{P,\lambda}$ is the set of *semistandard* P -tableaux (meaning you can repeat elements of P) of shape λ . For a multicoloring κ of G , let the *type* μ of a multicoloring κ be the composition $\mu = |\kappa^{-1}(1)|, |\kappa^{-1}(2)|, \dots$ and let $\text{des } \kappa$ be the number of pairs $\{a, i, b, j\}$ such that if $a \in \kappa^{-1}(i)$, $b \in \kappa^{-1}(j)$, $ij \in E(G)$ and $i < j$, then $a > b$. The definition of $\text{inv}_G T$ is easily extended to $T \in \mathcal{SST}_{P,\lambda}$.

Theorem 3.13. *If P is 3-free, then for each n , RSK_P is a bijection*

$$RSK_P : \mathcal{MK}_n(G) \rightarrow \bigsqcup_{\lambda \vdash n} \mathcal{SST}_{P,\lambda} \times RST_\lambda$$

such that if $RSK_P \kappa = (T, R)$, then the type of κ is the content of T . If P is additionally a natural unit interval order, then $\text{des } \kappa = \text{inv}_G T$.

When P is 3-free, Theorem 3.13 provides new bijective proofs of Gasharov's Schur expansion [5, Theorem 4] of the multichromatic symmetric function $X_G \mathbf{x}; \mu$ and when P is a natural unit interval order, Hwang's expansion [7, Theorem 4.19] of the multichromatic quasisymmetric function $X_G \mathbf{x}, t; \mu$ of type μ . These are both defined analogously to their chromatic versions,

$$X_G \mathbf{x}, t; \mu = \sum_{\lambda \vdash n} |\mathcal{SST}_{P,\lambda,\mu}| s_\lambda \quad X_G \mathbf{x}, t; \mu = \sum_{i \geq 0} t^i \sum_{\lambda \vdash n} |\mathcal{SST}_{P,\lambda,\mu,i}| s_\lambda$$

where $\mathcal{SST}_{P,\lambda,\mu,i}$ is the set of $T \in \mathcal{SST}_{P,\lambda}$ with content μ and $\text{inv}_G T = i$ and $\mathcal{SST}_{P,\lambda,\mu} = \bigsqcup_{i \geq 0} \mathcal{SST}_{P,\lambda,\mu,i}$. In particular, we get a direct combinatorial proof that for each composition $\mu = \mu_1, \mu_2, \dots$ of n , we have

$$\binom{n}{\mu} = \sum_{\lambda \vdash n} |\mathcal{SST}_{P,\lambda,\mu}| f^\lambda.$$

⁴More precisely, a *dual* form of RSK .

3.3.1 Equivalence Classes of Acyclic Orientations

Hwang [7] gives a further refinement of $X_G(\mathbf{x}, t; \mu)$, but we will just discuss the case of $\mu = 1^n$ for clarity here. A *local flip* of an acyclic orientation $\bar{o}$ changes $a \rightarrow b \leftarrow c$ to $a \leftarrow b \rightarrow c$ provided a and c are covered by b in the poset induced by $\bar{o}$. Say $\bar{o}'$ is *equivalent* to $\bar{o}$ if it is related by a sequence of local flips, and let $\bar{o}$ be its equivalence class. Note that if $\bar{o}' \in \bar{o}$ then $\text{des } \bar{o}' = \text{des } \bar{o}$. Let $\mathcal{K}(G, \bar{o})$ be the set of colorings that are *compatible* (in a sense) with an $\bar{o}' \in \bar{o}$ and define a quasisymmetric function

$$X_{G, \bar{o}}(\mathbf{x}) = \sum_{\kappa \in \mathcal{K}(G, \bar{o})} x_\kappa$$

Note that $X_G(\mathbf{x}, t) = \sum_{\bar{o}} t^{\text{des } \bar{o}} X_{G, \bar{o}}(\mathbf{x}, t)$. When $G = \text{Inc } P$ for a natural unit interval order P , Hwang uses noncommutative symmetric function theory to prove that this is a symmetric function with Schur expansion

$$X_{G, \bar{o}}(\mathbf{x}) = \sum_{T \in \mathcal{T}_{P, \bar{o}}} s_\lambda(T).$$

where $\mathcal{T}_{P, \bar{o}}$ is some set of P -tableaux that are *compatible* (in a sense) with an $\bar{o}' \in \bar{o}$. All $T \in \mathcal{T}_{P, \bar{o}}$ have $\text{inv}_G(T) = \text{des } \bar{o}$. We prove this expansion bijectively using RSK_P . Let $\mathcal{T}_{P, \bar{o}, \lambda}$ be the set of $T \in \mathcal{T}_{P, \bar{o}}$ with shape λ .

Theorem 3.14. *Let P be a 3-free natural unit interval order. For each acyclic orientation $\bar{o}$ of $G = \text{Inc } P$, RSK_P restricts to a bijection*

$$\text{RSK}_P : \mathcal{K}(G, \bar{o}) \rightarrow \bigsqcup_{\lambda \vdash n} \mathcal{T}_{P, \bar{o}, \lambda} \times \text{RST}_\lambda$$

Remark 3.15. We show that RSK_P can be extended to the *beast* poset (Figure 5a) or more generally the *beastly* poset⁵ (Figure 5b). The beast poset is explicitly avoided in the previous algorithms [13, 8].

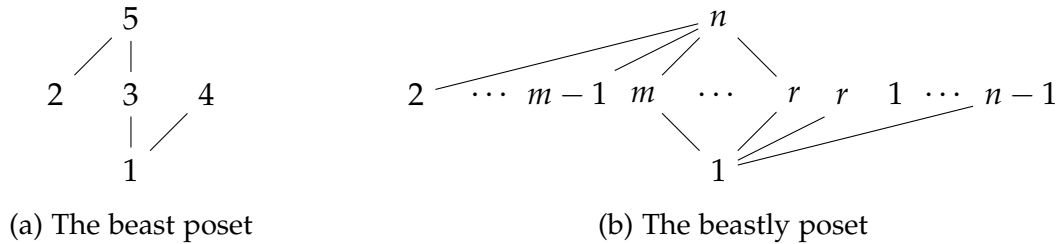


Figure 5

⁵This is the second e -positive class of posets considered by Cho and Huh [3] with 3-free being the first.

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Towards Butler's conjecture

Donghyun Kim^{*1}, Seung Jin Lee^{†2}, and Jaeseong Oh^{‡3}

¹Department of Mathematics, Sungkyunkwan University, Suwon, Korea

²Department of Mathematical Sciences, Research institute of Mathematics, Seoul National University, Seoul, Korea

³School of Computational Sciences, Korea Institute for Advanced Study, Seoul, Korea

Abstract. For a partition ν , let $\lambda, \mu \subseteq \nu$ be two distinct partitions such that $|\nu/\lambda| = |\nu/\mu| - 1$. Butler conjectured that the divided difference $I_{\lambda, \mu} X; q, t = T_\lambda \tilde{H}_\mu X; q, t - T_\mu \tilde{H}_\lambda X; q, t / T_\lambda - T_\mu$ of modified Macdonald polynomials of two partitions λ and μ is Schur positive. By introducing a new LLT equivalence called column exchange rule, we give a combinatorial formula for $I_{\lambda, \mu} X; q, t$, which is a positive monomial expansion. We also prove Butler's conjecture for some special cases.

Keywords: Butler's conjecture, Science Fiction conjecture, Modified Macdonald polynomials, Macdonald intersection polynomials, column exchange rule, Butler permutations, q, t -Kostka polynomial

1 Introduction

In his seminal paper [19], Macdonald introduced the *Macdonald P -polynomials* $P_\mu X; q, t$ which are q, t -extension of Schur functions indexed by partitions μ . The *modified Macdonald polynomials* $\tilde{H}_\mu X; q, t$ are introduced as a combinatorial version of the Macdonald P -polynomials.

For a partition $\mu \vdash n$, the Garsia–Haiman module V_μ is defined by the subspace in the polynomial ring $\mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n]$ spanned by partial derivatives of the polynomial Δ_μ analogous to the Vandermonde determinant:

$$\Delta_\mu : \det \left[x_i^{p_j} y_i^{q_j} \right]_{i,j=1,\dots,n},$$

where p_j, q_j runs over cells in μ and the symmetric group $\mathfrak{S}_n$ acts diagonally permuting x and y variables [8]. Later, Haiman [13] proved that the Garsia–Haiman module plays

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[†]lsjin@snu.ac.kr. S. J. Lee was supported by the National Research Foundation of Korea (NRF) grant funded by the Korean government (MSIT) (No. 2019R1C1C1003473).

[‡]jsoh@kias.re.kr. J. Oh was supported by KIAS Individual Grant (CG083401) at Korea Institute for Advanced Study.

a role as a representation-theoretic model for the modified Macdonald polynomial, i.e., the (bigraded) Frobenius characteristic of V_μ coincides with $\tilde{H}_\mu X; q, t$.

Bergeron and Garsia [4] studied relationships among Garsia–Haiman modules called *Science Fiction conjecture*. Among the variety of its implications, they studied the intersection of Garsia–Haiman modules. To elaborate, Science Fiction conjecture implies the following.

Conjecture 1.1. Let ν be a partition and $\mu^1, \dots, \mu^k \subseteq \nu$ be k distinct partitions such that $|\nu/\mu^1| + \dots + |\nu/\mu^k| = 1$. Then the bigraded $\mathfrak{S}_n$ -module $\bigcap_{i=1}^k V_{\mu^i}$ is of dimension $\frac{n!}{k}$, and its Frobenius characteristic is

$$\text{Frob} \left(\bigcap_{i=1}^k V_{\mu^i}, q, t \right) = \sum_{i=1}^k \prod_{j \in \mu^i} \frac{T_{\mu^j}}{T_{\mu^j} - T_{\mu^i}} \tilde{H}_{\mu^i} X; q, t, \quad (1.1)$$

where $T_\mu := \prod_{i,j \in \mu} t^{i-1} q^{j-1}$.

The first assertion is called $\frac{n!}{k}$ -conjecture, and Armon recently proved the $\frac{n!}{2}$ -conjecture for hook shapes [2]. The second implication of Science Fiction conjecture gives a formula for the Frobenius characteristic of the intersection of the Garsia–Haiman modules as a linear combination of the modified Macdonald polynomials. We will call the symmetric function in the right-hand side of (1.1) an *Macdonald Intersection polynomial* of $\mu^1, \dots, \mu^k$ and denote it by $I_{\mu^1, \dots, \mu^k} X; q, t$. In the companion paper of the authors [15], we study a remarkable connection between $I_{\mu^1, \dots, \mu^k} X; q, t$, ∇e_{k-1} which is the Frobenius characteristic of the diagonal harmonics [14], and the Shuffle formula $D_{k-1} X; q, t$ in [10, 7]. In this paper, we focus more on the case when $k = 2$, which is related to Butler’s conjecture. In 1994, Butler observed a surprising behavior of the modified Macdonald polynomials.

Conjecture 1.2. (Butler’s conjecture [6]) Let ν be a partition and $\lambda, \mu \subseteq \nu$ be two distinct partitions such that $|\nu/\lambda| + |\nu/\mu| = 1$. Then the Macdonald intersection polynomial $I_{\lambda, \mu} X; q, t = \frac{T_\lambda \tilde{H}_\mu X; q, t - T_\mu \tilde{H}_\lambda X; q, t}{T_\lambda - T_\mu}$ is Schur positive.

Note that Science Fiction conjecture implies Butler’s conjecture, as mentioned in [4]. To be more precise, the Macdonald intersection polynomial $I_{\lambda, \mu} X; q, t$ is given as a Frobenius characteristic of an $\mathfrak{S}_n$ -module $V_\lambda \cap V_\mu$, thus Schur positive.

We divide the main results in this paper into three parts. The first part gives a combinatorial formula (F -expansion, monomial expansion) for $I_{\lambda, \mu} X; q, t$ (Theorem 4.1, Corollary 4.2). The second part is to prove partial cases for Butler’s conjecture (Theorem 5.1, Corollary 5.2). The third part provides combinatorial formulas for q, t -Kostka polynomials which are consistent with Butler’s conjecture (Theorems 6.1 and 6.2).

2 Generalization of modified Macdonald polynomials

A (general) *diagram* D is a collection of points (cells) in $\mathbb{Z} \times \mathbb{Z}$, and a *bottom cell* of D is a cell located in the lowest position on each column. A *filled diagram* D, f is a diagram together with a filling

$$f : D \setminus \{\text{bottom cells of } D\} \rightarrow \mathbb{F}$$

that assigns a scalar in $\mathbb{F}$ for each cell of D that is not a bottom cell. Throughout this paper, we let $\mathbb{F} = \mathbb{C}[q, t]$. We visualize D, f by writing the corresponding value of f for each cell. For example, see [Figure 1](#).

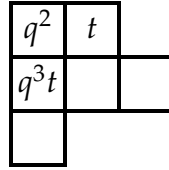


Figure 1: A filled diagram

We denote a diagram by its row numbers for each column: $D = D^1, D^2, \dots$, where D^j is the set of i 's such that $(i, j) \in D$. Throughout this paper, we assume that for every diagram D , each column D^i is an interval $[a, b] = \{a, a+1, \dots, b\}$ for some $a \leq b$. For example, the diagram in [Figure 1](#) can be represented as $D = [1, 3], [2, 3], [2, 2]$, and the Young diagram for a partition $\mu = \mu_1, \dots, \mu_\ell$ can be written as $\mu = \mu'_1, \dots, \mu'_{\mu_1}$, where μ' is the conjugate of μ .

Definition 2.1. We give a total order on cells in D row by row, top to bottom, and left to right within each row. We denote such total order with $N_D : D \rightarrow [|D|]$. For a filled diagram D, f , we define functions $\text{inv}_D : \mathfrak{S}_{|D|} \rightarrow \mathbb{F}$ and $\text{maj}_{D,f} : \mathfrak{S}_{|D|} \rightarrow \mathbb{F}$ as follows: For a permutation $w \in \mathfrak{S}_{|D|}$, we say that a pair u, v of cells in D is an *inversion* with respect to w if $w_{N_D(u)} > w_{N_D(v)}$ and either

- $u = (i, j), v = (i, j')$ and $j < j'$ or
- $u = (i, j), v = (i-1, j')$ and $j > j'$.

Then we define

$$\text{inv}_D(w) := \prod_{u,v} q,$$

where the product is over all pairs u, v of cells in D that are inversions with respect to w . For a permutation $w \in \mathfrak{S}_{|D|}$, we say that a cell $u = (i, j)$ in D is a *descent* with respect to w if $w_{N_D(u)} > w_{N_D(v)}$ where $v = (i-1, j)$ is the cell just below u . Then we define

$$\text{maj}_{D,f}(w) := \prod_u f(u)$$

where the product is over all cells that are descents with respect to w . Finally we define a function $\text{stat}_{D,f} : \mathfrak{S}_{|D|} \rightarrow \mathbb{F}$ as

$$\text{stat}_{D,f} w : \text{inv}_D w \text{ maj}_{D,f} w .$$

We defined $\text{stat}_{D,f} w$ for a permutation $w \in \mathfrak{S}_{|D|}$. However, we note that $\text{stat}_{D,f} w$ can be defined in the same way for any length $|D|$ word of positive integers w . We define a generalization of the modified Macdonald polynomial $\tilde{H}_{D,f}$ for a filled diagram D, f as

$$\tilde{H}_{D,f}(X; q, t) = \sum_{w \in \mathfrak{S}_n} \text{stat}_{D,f} w F_{\text{iDes } w} ,$$

where $F_S X$ is the fundamental quasisymmetric functions.

Given a partition μ , we define the *standard filling* of μ ,

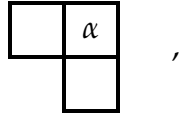
$$f_\mu^{\text{st}} : \mu \setminus \{\text{bottom cells of } \mu\} \rightarrow \mathbb{F}$$

by $f_\mu^{\text{st}} u = q^{-\text{arm}_\mu u} t^{\text{leg}_\mu u - 1}$. Then the modified Macdonald polynomial for the filled diagram (μ, f_μ^{st}) is the usual modified Macdonald polynomial

$$\tilde{H}_{\mu, f_\mu^{\text{st}}}(X; q, t) = \tilde{H}_\mu(X; q, t) ,$$

by Haglund–Haiman–Loehr formula [12].

Example 2.2. Let D, f be a filled diagram as depicted below,



where $\alpha \in \mathbb{F}$. Then the statistics are given in the following table.

$w \in \mathfrak{S}_3$	123	132	213	231	312	321
$\text{iDes } w$	$\emptyset$	$\{2\}$	$\{1\}$	$\{1\}$	$\{2\}$	$\{1, 2\}$
$\text{inv}_D w$	1	1	q	1	q	q
$\text{maj}_{D,f} w$	1	α	1	α	1	α

Thus, the modified Macdonald polynomial for D, f is

$$\tilde{H}_{D,f}(X; q, t) = F_\emptyset + q + \alpha F_{\{1\}} + q + \alpha F_{\{2\}} + q\alpha F_{\{1,2\}} .$$

Two distinct filled diagrams D, f and D', f' can give the same modified Macdonald polynomials. For example, given a filled diagram D, f where

$$D = D^1, D^2, \dots, D^\ell .$$

Consider a diagram

$$D' = D^2, \dots, D^\ell, D^1 \uparrow 1,$$

where $I = 1 \dots \{a \uparrow 1 : a \in I\}$ for an interval I . In other words, D' is a diagram obtained by moving the leftmost column of D to the end of the right and placing it one cell up. Cells in D and D' are naturally in bijection, and we give a filling f' on D' inherited from the filling f on D . We define cycling $D, f \rightarrow D', f'$. Then the following is well-known.

Lemma 2.3. (Cycling rule) *Let D, f be a filled diagram and D', f' be cycling D, f . Then we have*

$$\tilde{H}_{D,f}(X; q, t) = \tilde{H}_{D',f'}(X; q, t).$$

In [Section 3](#) and in the omitted proofs in [Section 5](#), we exhibit a variety of relations between the modified Macdonald polynomials (or LLT polynomials).

3 Column exchange rule and Butler permutations

3.1 Column exchange rule

We introduce the *column exchange rule*, which concerns the equidistribution of two statistics over permutations. We use this rule to exchange the columns of filled diagrams.

Proposition 3.1. (column exchange rule) *Let D, f and D', f' be filled diagrams that are the same except for the i -th and $i \uparrow 1$ -th columns. In addition, let i -th, and $i \uparrow 1$ -th column of D, f is of the form $\mu \uparrow n, m$ and of D', f' is of the form $\lambda \uparrow m, n$ for some $n > m$, and $f_\mu : \mu \rightarrow \mathbb{F}$ and $f_\lambda : \lambda \rightarrow \mathbb{F}$ be fillings such that*

- $f_\mu(i, 1) = f_\lambda(i, 2)$ for $i > m \uparrow 1$,
- $\alpha : q^{-1}f_\mu(m \uparrow 1, 1) = f_\lambda(m \uparrow 1, 2)$, and
- $f_\mu(i, 1) = \alpha f_\mu(i, 2) = \alpha f_\lambda(i, 1) = f_\lambda(i, 2)$ for $1 < i \leq m$,

Then there is a stat, iDes, and content-preserving bijection $\phi_i^{D,f}$ from $\mathfrak{S}_{|D|}$ to $\mathfrak{S}_{|D'|}$. In particular, we have

$$\tilde{H}_{D,f}(X; q, t) = \tilde{H}_{D',f'}(X; q, t).$$

For example, filled diagrams μ, f_μ and λ, f_λ depicted in [Figure 2](#) satisfy the conditions in [Proposition 3.1](#). In what follows, we abuse our notation to write $\phi_i^{D,f} = \phi_i$ for convenience.

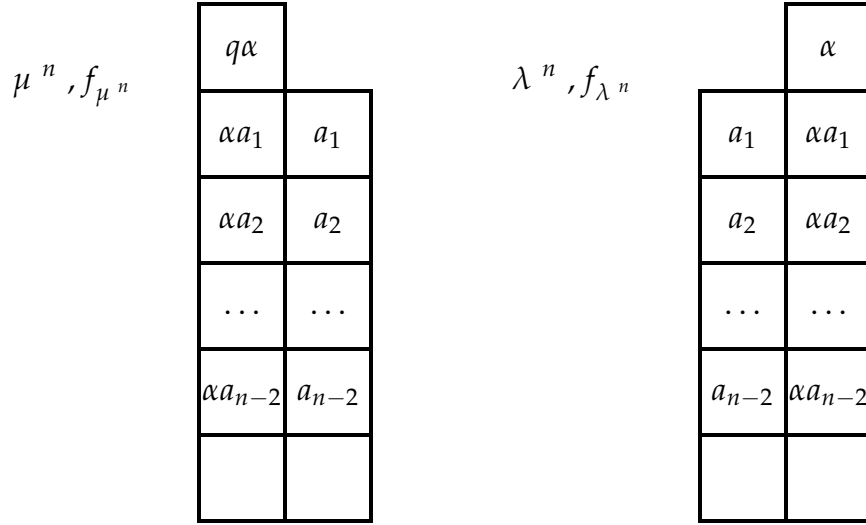


Figure 2: Filled diagrams μ, f_μ and λ, f_λ

Remark 3.2. When $D, f = \mu, f_\mu^{\text{st}}$ for a partition μ and standard filling f_μ^{st} , the implication

$$\tilde{H}_{D,f}(X; q, t) = \tilde{H}_{D',f'}(X; q, t)$$

of the above Proposition follows from [11, Theorem 5.1.1]. Our proof gives a bijective proof of their theorem, answering a question of Haglund [1]. In addition, we would like to highlight a few differences and advantages of the column exchange rule (and its proof).

First of all, the bijection ϕ preserves not only stat, iDes, but also content. This fact will be a crucial point in Section 5, and in the upcoming paper [15]. Moreover, the bijections ϕ and ψ appear in the construction of the map ζ , which will be the main ingredient of Proposition 3.4. Finally, the conditions in Proposition 3.1 is more flexible than the conditions in [11, Theorem 5.1.1].

3.2 Butler permutations

For a directed perfect matching $M = \{a_1, a_2, \dots, a_{2n-1}, a_{2n}\}$, we say an arc a, b is in the *forward direction* if $a < b$ and in the *reverse direction* otherwise. An arc a, b is *nested* by the other arc c, d if $\min\{c, d\} < a, b$ and $\max\{c, d\} > a, b$. We say that the pair of arcs a, b and c, d is *crossing* if $\min\{a, b\} < \min\{c, d\} < \max\{a, b\} < \max\{c, d\}$ or $\min\{c, d\} < \min\{a, b\} < \max\{c, d\} < \max\{a, b\}$. If not, we say the pair is *noncrossing*.

Definition 3.3. For a permutation $w = w_1 \dots w_{2n} \in \mathfrak{S}_{2n}$. Consider a directed perfect matching M on $\{0, 1, \dots, 2n-1\}$ with directed arcs $\alpha_1 w, \alpha_2 w, \dots$ as follows:

- $\alpha_1 w$ is a directed arc from w_n to $n-1$,
- $\alpha_i w$ is a directed arc from w_{n-2i-1} to $w_{n-2i-1-1}$ for $2 \leq i \leq \frac{n}{2}$, and
- $\alpha_{\frac{n}{2}-1} w$ is a directed arc from 0 to w_1 .

Let $k w$ the smallest integer k such that the pair of arcs $\alpha_k w$ and $\alpha_{k-1} w$ is noncrossing if exists. We say that w is a *Butler permutation* if either

- $\alpha_{k-1} w$ is nested by $\alpha_k w$ and $\alpha_k w$ is in the reverse direction, or
- $\alpha_{k-1} w$ is not nested by $\alpha_k w$ and $\alpha_k w$ is in the forward direction.

If there is no such k , we let $k w$ to be $\frac{n}{2}$. We say that w is a *Butler permutation* if $\alpha_k w$ is in the reverse direction. We denote the set of Butler permutations of length $2n$ by $\mathfrak{B}_{2n}$.

Let w be a word of weight $\alpha = \alpha_1, \alpha_2, \dots$, i.e. number of i 's in w is α_i . The *standardization* $\text{std } w$ of w is given by replacing 1's with the numbers $1, \alpha_1$ in increasing order from left to right, replacing 2's with the numbers in $\alpha_1-1, \alpha_1-\alpha_2$ in increasing order from left to right, and so on. For example, we have $\text{std } 3155 = 2134$.

3.3 Main Lemma

Proposition 3.4. Suppose $n > m$. Let $\bar{\mu} = n, m \setminus \{1\}$, $\bar{\nu} = m-1, m \setminus \{1\} \subseteq \mu$, and $\bar{\lambda} = m, n \setminus \{1\}$. Let $f_{\bar{\mu}} : \bar{\mu} \rightarrow \mathbb{F}$ and $f_{\bar{\lambda}} : \bar{\lambda} \rightarrow \mathbb{F}$ be fillings such that

- $f_{\bar{\mu}}(i, 1) = f_{\bar{\lambda}}(i, 2)$ for $i > m-1$,
- $\alpha : q^{-1} f_{\bar{\mu}}(m-1, 1) = f_{\bar{\lambda}}(m-1, 2)$,
- $f_{\bar{\mu}}(i, 1) = \alpha f_{\bar{\mu}}(i, 2) = \alpha f_{\bar{\lambda}}(i, 1) = f_{\bar{\lambda}}(i, 2)$ for $2 < i \leq m$, and
- $f_{\bar{\mu}}(2, 1) = \alpha f_{\bar{\lambda}}(2, 1)$.

Then, for any filled diagram D, f_D ,

$$\frac{\tilde{H}_{D, \bar{\mu}, f_D, f_{\bar{\mu}}}(X; q, t) - \alpha \tilde{H}_{D, \bar{\lambda}, f_D, f_{\bar{\lambda}}}(X; q, t)}{1 - \alpha} = \sum_w \text{stat}_{\bar{\mu}, f_{\bar{\mu}}} F_{\text{Des } w},$$

where the sum is over all permutations $w \in \mathfrak{S}_{n-m-1+|D|}$ such that $\text{std } w \downarrow_{\bar{\mu}}^{D, \bar{\mu}} \in \mathfrak{B}_{2m}$.

4 Combinatorial formulas for $I_{\lambda, \mu} X; q, t$.

4.1 Proof of a combinatorial formula for the F -expansion for $I_{\lambda, \mu} X; q, t$.

Theorem 4.1. For a partition ν , let $\mu, \lambda \subseteq \nu$ be two distinct partitions such that $|\nu/\mu| = |\nu/\lambda| - 1$. Moreover, suppose the cells ν/μ , and ν/λ are in i -th and j -th column respectively for $i < j$. Then the fundamental quasisymmetric expansion of $I_{\lambda, \mu} X; q, t$ is given by

$$I_{\lambda, \mu} X; q, t = \sum_{w \in \mathcal{B}_{\mu, \lambda}} \text{stat}_{\mu, f_{\mu}^{\text{st}} w} F_{\text{Des } w} X. \quad (4.1)$$

Here, $\mathcal{B}_{\mu, \lambda}$ is the set of permutations w of $\mathfrak{S}_{|D|}$ such that restriction of

$$\text{std} \circ \phi_{\nu_1-1} \cdots \circ \phi_j \circ \phi_1 \circ \cdots \circ \phi_{i-1} w$$

to the first $m-1$ rows of the last two columns is a Butler permutation.

We sketch a proof of [Theorem 4.1](#) by giving an example. Let $\lambda = 4, 3, 3, 2, 1$ and $\mu = 4, 4, 3, 1, 1$. First of all, the filled diagram $\lambda, f_{\lambda}^{\text{st}}$ is

t				
$q^{-2}t^2$	$q^{-1}t$	t		
$q^{-3}t^3$	$q^{-2}t^2$	$q^{-1}t^2$	t	

After applying the column exchange rule to move the second column to the left and the fourth column to the right, then cycling rule to move the first column to the last, we obtain the filled diagram $D_{\lambda}, f_{D_{\lambda}}$

$q^{-1}t$				$q^{-1}t$
$q^{-2}t^2$	t			$q^{-2}t^2$
$q^{-3}t^3$	$q^{-1}t^2$		$q^{-1}t$	

On the other hand, the filled diagram μ, f_μ^{st} is

$q^{-1}t$	t			
$q^{-2}t^2$	$q^{-1}t^2$	t		
$q^{-2}t^3$	$q^{-1}t^3$	t^2		

After applying the column exchange rule to move the fourth column to the left and the second column to the right, then cycling rule to move the first column to the last, we obtain the filled diagram D_μ, f_{D_μ}

$q^{-1}t$			$q^{-1}t$	
$q^{-2}t^2$	t		$q^{-1}t^2$	
$q^{-3}t^3$	$q^{-1}t^2$		$q^{-3}t^3$	

One can check that the filled diagrams D_λ, f_{D_λ} and D_μ, f_{D_μ} are the same except for the last two columns. In addition, the last two columns satisfy the condition of [Proposition 3.4](#) where $\alpha = q^2t^2$.

We end this section with two implications of [Theorem 4.1](#). The first one is a monomial expansion of $I_{\lambda,\mu} X; q, t$, and the second one is about the specialization $I_{\lambda,\mu} X; 1, 1$.

Corollary 4.2. *For a partition ν , let $\mu, \lambda \subseteq \nu$ be two distinct partitions such that $|\nu/\lambda| = |\nu/\mu| - 1$. Then the monomial expansion of $I_{\lambda,\mu} X; q, t$ is given by*

$$I_{\lambda,\mu} X; q, t = \sum_{\nu \vdash n} \sum_{w \in \mathcal{B}_{\mu,\lambda,\nu}} \text{stat}_{\mu, f_\mu^{st}} w \, m_\nu X. \quad (4.2)$$

Here, $\mathcal{B}_{\mu,\lambda,\nu}$ is the set of words w of weight ν whose standardization $\text{std } w$ is a Butler permutation of μ, λ .

Corollary 4.3. *For a partition ν , let $\mu, \lambda \subseteq \nu$ be two distinct partitions such that $|\nu/\lambda| = |\nu/\mu| - 1$. Then we have*

$$I_{\lambda,\mu} X; 1, 1 = h_{2, 1^{n-2}} X,$$

which is independent of μ and λ . In particular, this is consistent with $\frac{n!}{2}$ -conjecture.

5 Schur positivity of $I_{\lambda,\mu} X; q, t$

The first Schur positivity result is the Schur positivity of $I_{\lambda,\mu} X; q, t$ when we move a cell in the first or the second row of λ to obtain μ . The proof of the following theorem relies on the theory of LLT polynomials [16]. More precisely, we used Schur positivity of LLT polynomials [9] and LLT equivalences given in [17, 20, 18].

Theorem 5.1. *For a partition ν , let $\mu, \lambda \subseteq \nu$ be two distinct partitions such that $|\nu/\lambda| = |\nu/\mu| = 1$. Moreover, suppose ν/λ is a cell in the first or the second row. Then $I_{\lambda,\mu} X; q, t$ positively expands in LLT polynomials. In particular, $I_{\lambda,\mu} X; q, t$ is Schur positive.*

As a corollary, we prove Butler's conjecture for $q = 1$ or $t = 1$ (for general μ and λ).

Corollary 5.2. *Let λ be a partition and μ be a partition obtained from λ by moving a cell of a partition λ to the upper row. Then $I_{\lambda,\mu} X; q, 1$ (or $I_{\lambda,\mu} X; 1, q$) is Schur positive.*

6 Combinatorial formulas for q, t -Kostka polynomials

The modified q, t -Kostka polynomial $\tilde{K}_{\mu,\nu} q, t$ is the Schur coefficient of the modified Macdonald polynomial:

$$\tilde{H}_{\mu} X; q, t = \sum_{\lambda} \tilde{K}_{\mu,\lambda} q, t s_{\lambda} X.$$

This section provides combinatorial formulas for q, t -Kostka polynomials when μ is a two-column partition or a hook.

6.1 Two-column case

For $n \geq 1$, we define a directed weighted graph G_n on 2-bounded partitions p of size between n and $2n$, where there is a directed edge from p to q if and only if $p \Rightarrow q$ is a strong marked cover. The weight of each edge is the spin statistic for each corresponding strong marked cover. Then one may think of vertical strong marked tableaux of 2^n as paths of the (hexagonal) graph, defined on 2-bounded partitions with edges corresponding to strong marked cover, from top to bottom. For a path P , we define the *end partition* $\text{End } P$ of P as the 2-bounded partition corresponding to the end point of P and *weight* $\text{wt } P$ of the path P as the sum of weights of the edges in P . We now give a combinatorial formula for 2-Schur expansion for the modified Macdonald polynomials of two-column partitions. The formula is motivated by the *vertical dual Pieri rule*, and the *shift invariance* of k -Schur functions studied by Blasiak, Morse, Pun, and Summers [5]. While working on this paper, we found that our formula is equivalent to Zabrocki's formula [21].

Theorem 6.1. *Let n and m be positive integers with $n \geq 2m$. Then we have*

$$\omega \left(\tilde{H}_{2^m, 1^{n-2m}}(X; q, t) \right) = \sum_P \text{wt } P \prod_{i \in L(P)} \frac{q}{t^i} s_{\text{End } P}^2, \quad (6.1)$$

where ω is the conjugate map on the symmetric function ring given by $h_n \rightarrow e_n$ for $n \geq 1$ and $s_\mu^2(X)$ is the 2-Schur function. In particular, this formula is consistent with Butler's conjecture.

6.2 Hook case

In [3], Assaf defined the bijections $\mathcal{A}_k$ for $k \geq 1$. Using these bijections, she gave a combinatorial formula for the Schur expansion of the modified Macdonald polynomials indexed by hook partitions. This formula is indeed consistent with Butler's conjecture. By showing this, we provide the following formula, which only involves the 'ordinary' major statistic.

Theorem 6.2. *For $\mu = n - k, 1^k$ a hook partition, we have*

$$\tilde{H}_\mu(X; q, t) = \sum_{\lambda \vdash n} \sum_{w \in \text{SS } \lambda} \left(\prod_{\substack{1 \leq i \leq k-1, \\ i \in \text{Des } w}} t^{\text{maj } w|_k} \prod_{\substack{k \leq i \leq n-1, \\ i \in \text{Des } w}} \frac{q^{n-i}}{t^i} \right) s_\lambda(X),$$

where $\text{SS } \lambda$ is the set of superstandard tableaux of shape λ . In particular, this formula is consistent with Butler's conjecture.

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Semistandard Parking Functions and a Finite Shuffle Theorem

Nicolle González^{*1}, José Simental^{†2}, and Monica Vazirani^{‡3}

¹*Department of Mathematics, University of California, Berkeley, USA*

²*Instituto de Matemáticas, Universidad Nacional Autónoma de México, CDMX, México*

³*Department of Mathematics, University of California, Davis, USA*

Abstract. We introduce the higher rank rational q, t -Catalan polynomials and prove these are equal to finite truncations of the Hikita polynomial. We also generalize results of Gorsky-Mazin-Vazirani and construct an explicit bijection between semistandard parking functions and affine compositions. Using these results we prove a finite analogue of the Rational Shuffle Theorem in the context of spherical double affine Hecke algebras.

Keywords: Catalan polynomial, Shuffle Theorem, elliptic Hall algebra, Hikita polynomial, DAHA, spherical DAHA.

1 Introduction

The *Catalan numbers* C_n are some of the most ubiquitous quantities throughout mathematics, naturally counting objects across a vast array of fields. Their many generalizations, ranging from the so-called *rational Catalan numbers* $C_{m,n}$ to their bivariate q, t counterpart $C_{m,n}(q, t)$, have played a central role in the deep connections between q, t -combinatorics and important problems arising in the K-theory of Hilbert schemes, the homology of torus knots, the geometry of Gieseker varieties, and other areas of algebraic geometry, topology, and representation theory.

A particularly important family of objects counted by Catalan numbers is the set of *Dyck paths*, lattice paths in a square that do not cross the diagonal. Particular labelings of these paths give rise to *parking functions*. Originally introduced by Konheim and Weiss [12] in their study of hashing problems, parking functions were afterwards generalized by Armstrong, Loehr and Warrington [2] to the rational setting. A new perspective on parking functions, inspired by Anderson [1] and studied extensively by Gorsky-Mazin-Vazirani [7], gives an explicit bijection between the set of parking functions $\text{PF}_{m,n}$ and a certain set of affine permutations. More recently, motivated by the

^{*}nicolles@berkeley.edu.

[†]simental@im.unam.mx.

[‡]vazirani@math.ucdavis.edu. M.Vazirani was partially supported by Simons Foundation Grant 707426.

representation theory of quantizations of the Gieseker moduli space [5] Simental, jointly with Etingof-Krylov-Losev, generalized these constructions further by introducing the *higher rank rational Catalan numbers* $C_{m,n}^r$ and *higher rank semistandard parking functions* $\text{SSPF}^r_{m,n}$. As in the classical and rational cases, $|\text{SSPF}^r_{m,n}| = C_{m,n}^r$.

We generalize this further by extending the bijection given in [7] to a bijection between $\text{SSPF}^r_{m,n}$ and affine compositions, which through an explicit standardization procedure, allows us to define a “dinv” statistic on these new sets of objects. We use this statistic to introduce the *higher rank rational q, t -Catalan polynomial* $C_{m,n}^r(X; q, t)$. In [6] we further show how the dinv statistic corresponds precisely to the dimension of an associated affine space in an affine paving of a parabolic Springer fiber.

Tying much of this theory together are the celebrated *Shuffle Theorems*. In the classical setting, the Shuffle Theorem gives a combinatorial formula for the bigraded Frobenius character of the space of diagonal harmonics. While the Frobenius character of this representation was computed by Haiman geometrically in the early 2000’s [10] and its combinatorial expression, given as a sum over parking functions, was conjectured by Haglund-Haiman-Loehr-Remmel-Ulyanov [9] shortly thereafter, the conjecture was open for 15 years until Carlsson and Mellit [4] proved it. Unexpectedly arising in their studies of knot invariants via Cherednik algebras, Gorsky and Neguț proposed a rational generalization of the Shuffle Conjecture. In particular, they conjectured that a certain *elliptic Hall algebra* element acting on 1 inside the ring of symmetric functions gave rise to the Frobenius character of a certain bigraded S_n representation. The combinatorial formula for this character, eponymously named the *Hikita polynomial*, had previously been computed by Hikita [11] as a certain sum over $\text{PF}_{m,n}$. Using similar methods as in the classical case, Mellit [13] successfully proved this Rational Shuffle Theorem.

We connect our results to the Shuffle Theorems by first proving that our higher rank rational q, t -Catalan polynomial corresponds precisely to the truncation of the Hikita polynomial to a finite number of variables. Then, using the fact that the elliptic Hall algebra arises as the inverse limit of the spherical *double affine Hecke algebra* (DAHA) SH^r , we show there is an action of SH^r on the ring of symmetric polynomials whose action on 1 results in these higher rank rational q, t -Catalan polynomials. Since in the $r \rightarrow \infty$ limit this agrees with the rational generalization of Gorsky-Neguț, our construction gives a finite analogue of the Rational Shuffle Theorem.

2 Background

Let $m, n \in \mathbb{Z}_{>0}$ satisfy $\gcd(m, n) = 1$. An (m, n) -*Dyck path* is a lattice path consisting of only north and west steps from $(n, 0)$ to $(0, m)$ bounded by the lines $x = 0$, $y = 0$, and $mx - ny = mn$. Denote by $\mathcal{D}_{m,n}$ the set of all (m, n) -Dyck paths and let $\mathcal{N} \mathcal{D}_{m,n}$ be the

set of *north* steps for any $D \in \mathcal{D}_{m,n}$.

Let $v_1, v_2, \dots, v_m$ and $u_1, u_2, \dots, u_n$ be the north and west steps of $D \in \mathcal{D}_{m,n}$ read from $(n, 0)$ to $(0, m)$. We denote by $\text{dinv } D$ the number of **diagonal inversions** of D , which consist of pairs (u_r, v_s) over all $1 \leq r \leq n$ and $1 \leq s \leq m$, such that both u_r and v_s are intersected by some line ℓ of slope $-\frac{m}{n}$ and v_s lies southeast of u_r in D . The **area** of a Dyck path $D \in \mathcal{D}_{m,n}$ is the number of lattice cells fully contained between the path D and the diagonal line $mx - ny = mn$. We set $\text{co-dinv } D = \frac{m-1}{2} \frac{n-1}{2} - \text{area } D$.

Example 1. The Dyck path D in [Figure 2](#) has $\text{area } D = 2$, $\text{dinv } D = 4$ and thus $\text{co-area } D = \frac{m-1}{2} \frac{n-1}{2} - 2 = 4$, $\text{co-dinv } D = \frac{m-1}{2} \frac{n-1}{2} - 4 = 2$.

It is a well known fact that the number of (m, n) -Dyck paths for coprime m, n is given by the *rational Catalan numbers*, $C_{m,n} = \frac{1}{n} \binom{n}{m}^{m-1}$. The (q, t) -Catalan number is the bivariate deformation given by the polynomial

$$C_{m,n}(q, t) = \sum_{D \in \mathcal{D}_{m,n}} q^{\text{area } D} t^{\text{dinv } D}. \quad (2.1)$$

2.1 Parking Functions and the Hikita Polynomial

An (m, n) -**parking function** is a pair (D, φ) , where D is an (m, n) -Dyck path and $\varphi : \mathcal{N}(D) \rightarrow \{1, \dots, m\}$ is a bijection that is strictly decreasing when reading upward on consecutive north steps. We will denote by $\text{PF}_{m,n}$ the set of (m, n) -parking functions.

For example, in [Figure 1](#), the three rightmost diagrams correspond to all possible $(3, 2)$ -parking functions for the given Dyck path.

Parking functions are in bijection with a certain set of affine permutations. To explain this bijection we recall the *affine symmetric group*,

$$\tilde{S}_m = \left\{ \sigma : \mathbb{Z} \rightarrow \mathbb{Z} \mid \sigma \text{ is a bijection, } \sigma(x+m) = \sigma(x) + m, \text{ and } \sum_{i=1}^m \sigma(i) = \binom{m+1}{2} \right\}.$$

By the periodicity condition $\sigma \in \tilde{S}_m$ is completely determined by $\sigma(1), \dots, \sigma(m)$. We will use window notation and write $\sigma = \sigma(1), \dots, \sigma(m)$. We say $\sigma \in \tilde{S}_m$ is **n -stable** if $\sigma(x+n) > \sigma(x)$ for all $x \in \mathbb{Z}$ and denote by $\tilde{S}_m^n$ the set of all n -stable affine permutations.

In [7], the third author jointly with Gorsky and Mazin study a bijection (see [Section 3.1](#))

$$\mathcal{A} : \text{PF}_{m,n} \rightarrow \tilde{S}_m^n, \quad (2.2)$$

Using this, for $\sigma = \mathcal{A}(D, \varphi)$, we define

$$\text{co-dinv } (D, \varphi) = |\{ (i, h) \in \{1, \dots, m\} \times \{1, \dots, n\} \mid \sigma(i) - h < \sigma(i) \}|. \quad (2.3)$$

and set $\text{dinv } (D, \varphi) = \frac{m-1}{2} \frac{n-1}{2} - \text{co-dinv } (D, \varphi)$.

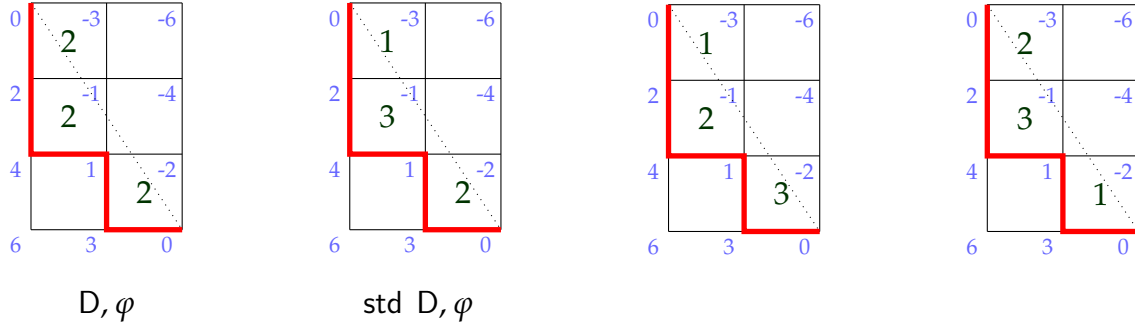


Figure 1: A $3, 2$ -semistandard parking function D, φ and all $3, 2$ -parking functions for D , of which the second from the left is the unique standardization $\text{std } D, \varphi$ determined by γ . A box with upper right corner x, y is marked by $\gamma(x, y)$.

Gessel defined the *fundamental quasisymmetric functions* for any subset $S \in \{1, \dots, m\}$ as the series

$$Q_S X : \sum_{\substack{i_1 \leq \dots \leq i_m \\ j \in S \Rightarrow i_j < i_{j-1}}} x_{i_1} \dots x_{i_m}. \quad (2.4)$$

These functions interpolate between elementary and complete symmetric functions and form a basis for the space of quasisymmetric functions in an infinite number of variables.

Let $\text{area } D, \varphi : \text{area } D$ for any $D, \varphi \in \text{PF } m, n$. Then, the *Hikita polynomial* is the q, t -symmetric function given by

$$\mathcal{H}_{m,n} X; q, t : \sum_{D, \varphi \in \text{PF } m, n} q^{\text{area } D, \varphi} t^{\text{dinv } D, \varphi} Q_{\text{Des } \sigma^{-1}} X, \quad (2.5)$$

where $\text{Des } \sigma^{-1} : \{1 \leq j \leq m \mid \sigma^{-1}(j-1) < \sigma^{-1}(j)\}$ denotes the *descent set* of σ^{-1} for $\sigma \in \mathcal{A} D, \varphi \in \tilde{S}_m^n$. When paired with the complete symmetric function $h_m X$ under the Hall inner product, the Hikita polynomial returns the rational q, t -Catalan number, that is $\langle \mathcal{H}_{m,n}, h_m \rangle = C_{m,n}(q, t)$.

2.2 Higher Rank Catalans and Semistandard Parking Functions

Motivated by the representation theory of quantizations of the Gieseker moduli space, in [5] Etingof, Krylov, Losev and the second author introduced the following generalization of parking functions.

Definition 2. Let $r, m, n \in \mathbb{Z}_{>0}$. A *rank r semistandard m, n -parking function* is a pair D, φ with $D \in \mathcal{D}_{m,n}$ and a function $\varphi : \mathcal{N}_D \rightarrow \{1, \dots, r\}$ that is weakly decreasing

when reading upward on consecutive north steps. Let $\text{SSPF}^r_{m,n}$ be the set of all rank r semistandard m, n -parking functions.

In the special case when $r = m$, and φ is a bijection, then the filling will automatically become standard, recovering the original set of m, n -parking functions $\text{PF}_{m,n}$ considered in [2].

Given any $D, \varphi \in \text{SSPF}^r_{m,n}$, we let $\text{area } D, \varphi := \text{area } D$, define the **weight** of D, φ as the composition of m given by $\text{wt } D, \varphi := (|\varphi^{-1}(1)|, |\varphi^{-1}(2)|, \dots, |\varphi^{-1}(r)|)$. Let $\text{SSPF}^r_{\mathbf{w}, m, n}$ denote the set of all rank r semistandard m, n -parking functions with weight $\mathbf{w}$. Note that for any $D, \varphi \in \text{PF}_{m,n}$, we have $\text{wt } D, \varphi = 1^m$.

Example 3. Consider D, φ , the semistandard $5, 4$ -parking function in Figure 2 (left). Then D, φ has rank $r = 3$, $\text{area } D, \varphi = 2$, and $\text{wt } D, \varphi = 1, 3, 1$. By comparison, the $5, 4$ -parking function in Figure 2 (right) has the same area but rank 5 and weight 1^5 .

In [5, Theorem 2.28], Etingof, Krylov, Losev and the second author defined the **rank r m, n -Catalan number** $C^r_{m,n}$ to be the cardinality of $\text{SSPF}^r_{m,n}$ and proved that when m and n are coprime,

$$C^r_{m,n} := |\text{SSPF}^r_{m,n}| = \frac{1}{n} \binom{nr}{m} \binom{m-1}{m}. \quad (2.6)$$

3 Higher Rank Rational q, t -Catalan Polynomial

As mentioned above, there exists a map $\mathcal{A} : \text{PF}_{m,n} \rightarrow \tilde{S}^n_m$. We will recall this bijection and construct a generalization for $\text{SSPF}_{m,n}$. Denote by $[r]$ the set $\{1, \dots, r\} \subset \mathbb{Z}$.

Definition 4. Let $m, r \in \mathbb{Z}_{>0}$. An m, r -**affine composition** is a function $f : \mathbb{Z} \rightarrow \mathbb{Z}$ satisfying the following properties:

- (1) $f(x) - m = f(x - r)$ for all $x \in \mathbb{Z}$.
- (2) The set $f^{-1}(r)$ has exactly one element from each residue class mod m .
- (3) $\sum_{x \in f^{-1}(r)} x = \frac{m-1}{2}$.

Denote by $\mathcal{AC}_{m,r}$ the set of all m, r -affine compositions. See Examples 5 and 7.

Just as for affine permutations, by the periodicity of condition (1) above, we use window notation and write $f = f(1), \dots, f(m-r)$. The **weight** of f is given by

$$\text{wt } f = (|f^{-1}(1)|, |f^{-1}(2)|, \dots, |f^{-1}(r)|).$$

Note that an affine composition of weight 1^m is simply an affine permutation. As before, $f \in \mathcal{AC}_{m,r}$ is **n -stable** if $f(x) - n \geq f(x)$ for all $x \in \mathbb{Z}$. Then, let $\mathcal{AC}_{\mathbf{w}, m, r}$ be the set of all m, r -affine compositions with weight $\mathbf{w}$ and denote by $\mathcal{AC}^n_{\mathbf{w}, m, r}$ the subset of $\mathcal{AC}_{\mathbf{w}, m, r}$ consisting of those affine compositions that are n -stable.

Example 5. Let $m, r \in \mathbb{Z}_{>0}$ and consider the (m, r) -affine composition $f \in \mathcal{AC}(m, r)$ that in window notation is $f = (1, 1, \dots, 1)_r$. Note that $f^{-1} = (r, 0, \dots, 0)_m$ so that f is indeed an affine composition of weight $(m, 0, \dots, 0) \in \mathbb{Z}_{\geq 0}^r$. We leave it to the reader to verify that f is the function $f(x) = r \lfloor \frac{x-1}{m} \rfloor + 1$.

3.1 Semistandard Parking Functions are Affine Compositions

We now describe $\mathcal{A} : \text{SSPF}^r(m, n) \rightarrow \mathcal{AC}^n(m, r)$ which is a generalization of the map $\mathcal{A} : \text{PF}(m, n) \rightarrow \tilde{S}_m^n$.

Given any $(D, \varphi) \in \text{SSPF}^r(m, n)$, index the north steps $v_1, \dots, v_m \in \mathbb{N}^D$ by their topmost coordinate, so that $v_j = (a_j, j)$ for each $1 \leq j \leq m$ and in particular $a_m = 0$. Define the function $f_\varphi : \mathbb{Z} \rightarrow \mathbb{Z}$ as follows:

Step 1: Define $\gamma : \mathbb{Z}^2 \rightarrow \mathbb{Z}$ by $\gamma(x, y) = mn - mx - ny$.

Step 2: Set $\tilde{f} = \gamma \circ v_j = \varphi \circ v_j$ for $j \in [m]$, so that $\tilde{f}^{-1}(i) = \gamma \circ \varphi^{-1}(i)$ for $i \in [r]$.

Step 3: For any $x \notin \{\gamma \circ v_j\}_{1 \leq j \leq m}$, write $x = \gamma \circ v_j - pm$ for some $p \in \mathbb{Z}$ and $j \in [m]$ and set $\tilde{f}(x) = \tilde{f} \circ v_j - pr$.

Step 4: Let $k = \frac{n(m-1) - m - 1}{2} - \sum_{j=1}^m a_j$, and set $f_\varphi(x) = \tilde{f}(x) + k$ for all $x \in \mathbb{Z}$.

Then we set $\mathcal{A}(D, \varphi) = f_\varphi$. Note that since m, n are coprime then the set $\{\gamma \circ v_j\}_{1 \leq j \leq m}$ contains exactly one element per residue class mod m . Thus $\tilde{f}$ is well-defined.

Theorem 6. Let $r, m, n \in \mathbb{Z}_{>0}$ with $\gcd(m, n) = 1$. The map $\mathcal{A} : \text{SSPF}^r(m, n) \rightarrow \mathcal{AC}^n(m, r)$ sending $(D, \varphi) \mapsto f_\varphi$ is a weight-preserving bijection. Hence, given any r -part weak composition $\mathbf{w}$ of m , the construction above gives an isomorphism of sets,

$$\mathcal{A}_{\mathbf{w}} : \text{SSPF}_{\mathbf{w}}^r(m, n) \rightarrow \mathcal{AC}_{\mathbf{w}}^n(m, r).$$

In the special case when $r = m$ and $\mathbf{w} = 1^m$, so that $\mathcal{AC}_{1^m}^n(m, m) = \tilde{S}_m^n$, Theorem 6 recovers the map $\mathcal{A} : \text{PF}(m, n) \rightarrow \tilde{S}_m^n$ constructed in [7]. For example, in Figure 4 we list the affine permutations for all the $(3, 2)$ -parking functions.

Example 7. Set $m = 5, n = 4$ and $r = 3$. Let (D, φ) be the rank 3 semistandard $(5, 4)$ -parking function in Figure 2 (left). Note that $\text{wt}(D, \varphi) = (1, 3, 1)$. From the figure we can read that

$$\gamma \circ \varphi^{-1}(1) = \{0\}, \quad \gamma \circ \varphi^{-1}(2) = \{3, 4, 7\}, \quad \gamma \circ \varphi^{-1}(3) = \{6\}.$$

Thus, setting $\tilde{f} = \gamma \circ v_j = 5p : \tilde{f} \circ v_j = 3p$ we find that $\tilde{f}$ is the following map:

$$\begin{array}{cccccccc|cccc|cccc} \dots & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \dots \\ \dots & -3 & -4 & -1 & -1 & 1 & 0 & -1 & 2 & 2 & 4 & 3 & 2 & 5 & 5 & 7 & \dots \end{array}$$

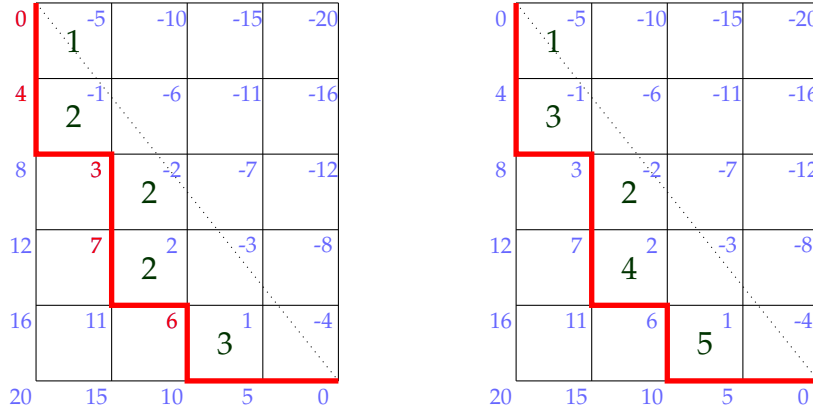


Figure 2: A semistandard $5,4$ -parking function with rank $r = 3$ (left), and its standardization (right). A box with upper right corner (x, y) is marked by $\gamma(x, y)$.

Now since the x -coordinates of the north steps $\{v_j = a_j, j\}_{1 \leq j \leq 7}$ are given by $a_1 = 2$, $a_2 = a_3 = 1$, and $a_4 = a_5 = 0$, it follows that $k = 1$. Thus, setting $f_\varphi(x) = \tilde{f}(x) - 1$ we find that $f_\varphi = -1, 2, 2, 4, 3, 3$, thus $\text{wt } f_\varphi = \text{wt } \tilde{f} = 1, 3, 1$, as expected. Notice that by construction $f_\varphi^{-1}(3) = \{\gamma(v_j) - 1\} = \{-1, 2, 3, 5, 6\}$ which, modulo 5, is equivalent to the set $\{1, \dots, 5\}$. Summing the entries in this set we get $\frac{6}{2}$, hence f_φ is indeed a $5, 3$ -affine composition, which by inspection can be seen to be 4-stable.

3.2 Standardization and Diagonal Inversions

The bijection in Theorem 6 allows us to compute co-dinv via Equation (2.3), provided that $D, \varphi \in \text{PF}(m, n)$. Thus, in order to define the diagonal inversions for any semistandard parking function we utilize a standardization map from $\text{SSPF}(m, n) \rightarrow \text{PF}(m, n)$.

For $\mathbf{w} = w_1, \dots, w_r$, an r -part weak composition of m , let $S_{\mathbf{w}} = S_{w_1} \times \dots \times S_{w_r}$ be the parabolic subgroup of $\tilde{S}_m$ indexed by $\mathbf{w}$. Denote by $S_{\mathbf{w}} \backslash \tilde{S}_m^{\min}$ the set of minimal length right coset representatives and let $S_{\mathbf{w}} \backslash \tilde{S}_m^n = S_{\mathbf{w}} \backslash \tilde{S}_m^{\min} \cap \tilde{S}_m^n$.

Proposition 8. Let $m, r \in \mathbb{Z}_{>0}$. Given $\mathbf{w} = w_1, \dots, w_r$, an r -part weak composition of m , let $_{\mathbf{w}}f$ be the affine composition of weight $\mathbf{w}$ given by $_{\mathbf{w}}f : \underbrace{1, \dots, 1}_{w_1}, \underbrace{2, \dots, 2}_{w_2}, \dots, \underbrace{r, \dots, r}_{w_r}$. Then

the map

$$S_{\mathbf{w}} : S_{\mathbf{w}} \backslash \tilde{S}_m^{\min} \rightarrow \mathcal{AC}_{\mathbf{w}}(m, r)$$

sending $\sigma \mapsto _{\mathbf{w}}f \circ \sigma$ is a bijection. Moreover, $S_{\mathbf{w}}$ preserves n -stability and thus restricts to a bijection $S_{\mathbf{w}}^n : S_{\mathbf{w}} \backslash \tilde{S}_m^n \rightarrow \mathcal{AC}_{\mathbf{w}}^n(m, r)$ for any $n \in \mathbb{Z}_{>0}$ coprime to m .

Definition 9. For any $D, \varphi \in \text{SSPF}_{\mathbf{w}}^r(m, n)$ the *standardization* of D, φ is the parking

function $\text{std } D, \varphi \in \text{PF } m, n$ given by,

$$\text{std } D, \varphi : \mathcal{A}^{-1} \mathcal{S}_{\mathbf{w}}^{-1} \mathcal{A}_{\mathbf{w}} D, \varphi .$$

Theorem 10. For any $D, \varphi \in \text{SSPF}_{\mathbf{w}}^r m, n$, its standardization $\text{std } D, \varphi$ is the m, n -parking function D, α with $\alpha : \mathcal{N} D \rightarrow \{1, \dots, m\}$ the unique bijection satisfying:

- (1) $\alpha \varphi^{-1} a < \alpha \varphi^{-1} b$ for all $1 \leq a < b \leq r$, and
- (2) $\alpha v_i < \alpha v_j$ whenever $\varphi v_i < \varphi v_j$ and $\gamma v_i < \gamma v_j$.

As seen in Figure 1 conditions 1 and 2 ensure a unique standard representative of a semistandard parking function D, φ .

Hence, for any rank r semistandard m, n -parking function D, φ we set

$$\text{dinv } D, \varphi : \text{dinv } \text{std } D, \varphi . \quad (3.1)$$

Example 11. Consider the rank $r = 3$ semistandard $5, 4$ -parking function D, φ in Figure 2 (left) with preimages $\varphi^{-1} 1 = \{v_5\}$, $\varphi^{-1} 2 = \{v_2, v_3, v_4\}$, and $\varphi^{-1} 3 = \{v_1\}$. Since $\gamma v_2 = 7$, $\gamma v_3 = 3$, and $\gamma v_4 = 4$, then $\alpha v_5 < \alpha v_3 < \alpha v_4 < \alpha v_2 < \alpha v_1$. Hence α is the unique map from $\mathcal{N} D \rightarrow \{1, \dots, 5\}$ given in Figure 2 (right).

Example 12. Consider the rank 2 semistandard $3, 2$ -parking function in Figure 1 (left). We have that $\mathcal{A} \text{std } D, \varphi = 1, 2, 3$, the identity permutation which does not have inversions and thus has $\text{codinv } 0$. Then $\text{dinv } D, \varphi = \frac{2-1}{2} \frac{3-1}{2} = 0 = 1$. By comparison, the affine permutations corresponding to the two rightmost parking functions in Figure 1 are $1, 3, 2$ and $2, 1, 3$ both of which have inversions.

While we do not expand upon this here, it is important to note that the affine spaces in an affine paving of a parabolic affine Springer fiber are indexed by the set $\text{SSPF}^r m, n$ and the co-dinv statistic is the dimension of the corresponding affine space. Each parabolic affine Springer fiber admits a surjection from the affine Springer fiber in the full affine flag variety, and the latter variety has an affine paving indexed by parking functions, see [7]. The standardization procedure is related to properties of this surjection, see [6] for details.

Definition 13. Let $m, n, r \in \mathbb{Z}_{>0}$ with $\gcd m, n = 1$. The *rank r rational q, t -Catalan polynomial* is the polynomial in $\mathbb{Q}[q, t, x_1, \dots, x_r]$ given by

$$C_{m,n}^r(x_1, \dots, x_r; q, t) : \sum_{D, \varphi \in \text{SSPF}^r m, n} q^{\text{area } D} t^{\text{dinv } D, \varphi} x^{\text{wt } \varphi} .$$

Example 14. It can be shown that $C_{m,n}^1(x_1; q, t) = x_1^m C_{m,n}(q, t)$, where $C_{m,n}(q, t)$ is the usual q, t -Catalan polynomial. In this sense, Definition 13 generalizes that of q, t -Catalan polynomials. As seen in Figure 3, for $r = 2, m = 3, n = 2$ we have

$$C_{3,2}^2 = x_1^3 x_2^3 q t + x_1^2 x_2^2 q + 1 t .$$

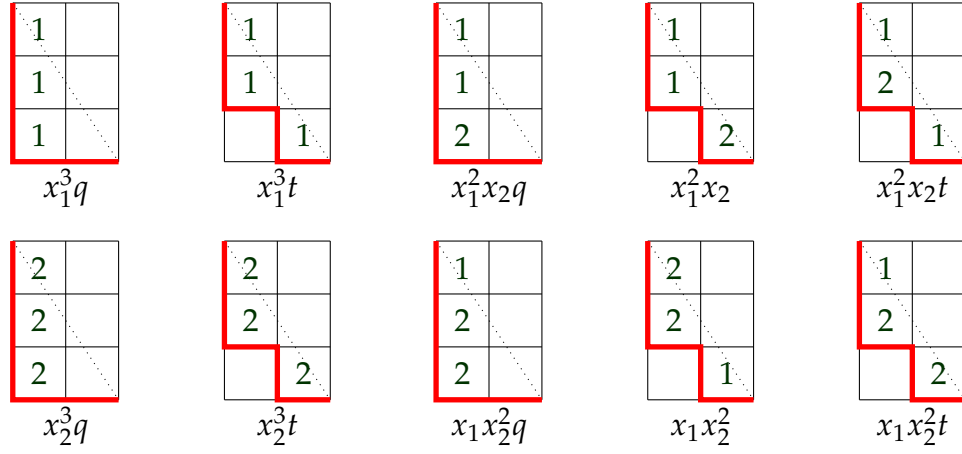


Figure 3: All the semistandard $3,2$ -parking functions of rank 2 and their contribution to $C_{3,2}^2(x_1, x_2; q, t)$.

Using the standardization procedure above it can be shown that the restriction to r variables of Gessel's quasisymmetric function $Q_{\text{Des } \sigma^{-1}} X|_{x_1, \dots, x_r}$ is naturally indexed by the rank r semistandard parking functions D, φ that standardize to D, α , where $\mathcal{A}(D, \alpha) = \sigma$ and $\text{Des } \sigma^{-1}$ is the descent set of σ^{-1} . Namely,

$$Q_{\text{Des } \sigma^{-1}} X|_{x_1, \dots, x_r} = \sum_{D, \varphi \in \text{std}^{-1}(D, \alpha)} x^{\text{wt } D, \varphi}. \quad (3.2)$$

Combining (2.5) and (3.2) we obtain the following theorem.

Theorem 15. *Given any $m, n, r \in \mathbb{Z}_{>0}$ with $\gcd(m, n) = 1$, the truncation of the Hikita polynomial to r variables is given by the rank- r rational q, t -Catalan polynomial,*

$$\mathcal{H}_{m,n}(X; q, t)|_{x_1, \dots, x_r} = C_{m,n}^r(x_1, \dots, x_r; q, t).$$

Thus, $C_{m,n}^r(x_1, \dots, x_r; q, t)$ is a q, t -symmetric, Schur positive polynomial, and hence also symmetric in $x_1, \dots, x_r$.

Example 16. Let us compute the Hikita polynomial $\mathcal{H}_{3,2}(X; q, t)$. There are four $3,2$ -parking functions D, φ , which are listed together with their associated affine permutations $\mathcal{A}(D, \varphi)$ in Figure 4. From here, we compute the Schur expansion:

$$\mathcal{H}_{3,2}(X; q, t) = qQ_{\{3\}} + Q_{\{2\}} + tQ_{\emptyset} + Q_{\{1\}} = q + t s_3 + s_{2,1}.$$

It is straightforward to verify that the truncation to 2-variables of this polynomial coincides with the computation of $C_{3,2}^2(x_1, x_2; q, t)$ in Example 14.

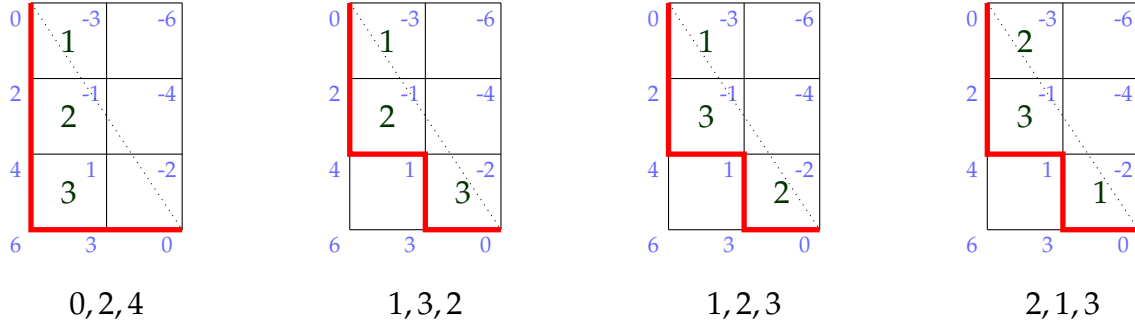


Figure 4: The $3,2$ -parking functions and their associated affine permutations under the bijection $\mathcal{A}$, which can easily be read from $\gamma_{x,y}$ for each lattice point x,y on D .

4 Spherical DAHA and a Finite Shuffle Theorem

The *elliptic Hall algebra* $\mathcal{E}$ is the $\mathbb{C}[q, t]$ -algebra consisting of an infinite family of generators $\{P_{m,n} \mid m, n \in \mathbb{Z}_{\geq 0}^2 \setminus \{0,0\}\}$ modulo some relations, see e.g. [14]. Although $\mathcal{E}$ originally arose as the Hall algebra of the category of coherent sheaves on an elliptic curve, it has since had various topological and combinatorial incarnations and played a central role in connecting q, t -combinatorics and the Shuffle Theorems to many recent results in the theory of homological and geometric invariants for knots and links.

There is a well-known geometric action of $\mathcal{E}$ on the ring of symmetric functions $\Lambda_{q,t}$ whose fixed points correspond to the *modified Macdonald polynomials* $\tilde{H}_\lambda(X; q, t)$ [14]. Denote this representation by V^{geom} and by $P_{m,n}^\infty f$ the image of $f \in \Lambda_{q,t}$ under this action. The *Rational Shuffle Theorem*, conjectured by Gorsky and Neguț [8] and proved by Mellit [13], states the following.

Theorem 17 ([13]). *For coprime $m, n \in \mathbb{Z}_{>0}$, $P_{m,n}^\infty 1 = \mathcal{H}_{m,n}(X; q, t)$.*

The *spherical DAHA* SH^r is the spherical subalgebra of the double affine Hecke algebra of type $\mathfrak{gl}^r$. We consider the subalgebra SH^r of SH^r generated by elements $P_{m,n}^r$ with $m, n \in \mathbb{Z}_{\geq 0}^2 \setminus \{0,0\}$. Schiffmann-Vasserot show that under the natural map $\text{SH}^r \rightarrow \text{SH}^{r-1}$ sending $P_{m,n}^r \mapsto P_{m,n}^{r-1}$, the algebra $\mathcal{E}$ arises as an inverse limit of the spherical DAHAs. Namely,

$$\mathcal{E} \sim \varprojlim_r \text{SH}^r \quad (4.1)$$

under which

$$P_{m,n} = \varprojlim_r P_{m,n}^r.$$

The algebra $\text{SH } r$ comes equipped with a faithful polynomial representation on $\mathbb{Q}[q, t, x_1, \dots, x_r]^{S_r}$, denoted $\text{Pol } r$, that is compatible with (4.1) above and the inverse limit $\Lambda_{q,t} \sim \varprojlim_r \mathbb{Q}[q, t, x_1, \dots, x_r]^{S_r}$. Thus, there is an induced polynomial representation V^{alg} of $\mathcal{E}$ on $\Lambda_{q,t}$. This representation is nontrivially isomorphic to the geometric one of Schiffmann-Vasserot.

Proposition 18 ([14]). *There is an isomorphism of $\mathcal{E}$ representations $\Phi : V^{\text{alg}} \rightarrow V^{\text{geom}}$ given by the plethystic substitution that sends $F[X; q, t] \mapsto F\left[\frac{X}{1-t^{-1}}; q, t^{-1}\right]$.*

By combining this proposition with Theorems 15 and 17 we obtain the following finite analogue of the Rational Shuffle Theorem.

Theorem 19. *Given any $m, n, r \in \mathbb{Z}_{>0}$ with $\gcd(m, n) = 1$, there exists an action of $\text{SH } r$ on $\mathbb{Q}[q, t, x_1, \dots, x_r]^{S_r}$ satisfying*

$$P_{m,n}^r = 1 - C_{m,n}^r(x_1, \dots, x_r; q, t).$$

In particular, this action is nontrivially isomorphic to the polynomial representation $\text{Pol } r$.

This action is compatible with the geometric action of $\mathcal{E}$ on $\Lambda_{q,t}$ under all the inverse limits above, with $P_{m,n}^\infty = \varprojlim_r P_{m,n}^r$ as operators, so that as $r \rightarrow \infty$ it recovers the Rational Shuffle Theorem.

Given the complicated nature of plethystic substitution, finding the explicit action of $\text{SH } r$ on $\mathbb{Q}[q, t, x_1, \dots, x_r]^{S_r}$ in Theorem 19 is quite difficult. In particular, even describing the action for $r = 1$ explicitly is unclear since although $\text{SH } 1$ is straightforward, the rational Catalan numbers $C_{m,n}^r(q, t)$ are not. Nonetheless, it would be very interesting to find an explicit description of this action.

5 The Non-Coprime Case and a Bizley-Type Formula

Naturally, one asks how much of these constructions can be extended to the non-coprime case. Unfortunately it is neither easy nor obvious how to proceed. A starting point, however, is the computation of $C_{m,n}^r = |\text{SSPF}^r(m, n)|$ in the case m, n are not coprime. Below we give a Bizley-type formula for these numbers.

Theorem 20. *Let $m, n, r \in \mathbb{Z}_{>0}$ with $\gcd(m, n) = 1$. There is an equality of formal power series:*

$$1 - \sum_{k=1}^{\infty} C_{mk, nk}^r z^k = \exp\left(\sum_{k=1}^{\infty} \frac{1}{nk} \binom{nkr}{mk} z^k\right).$$

One can apply similar techniques to obtain a Bizley-type formula for the number of parking functions. This is, however, much more elegantly expressed by Aval and Bergeron in [3], who adapt Bizley’s proof incorporating symmetric function arguments to find the Frobenius characteristic of the S_{mk} -representation $\text{CPF } mk, nk$. One can adapt the arguments in this paper to show that

$$C_{mk,nk}^r = \text{Frob } \text{CPF } mk, nk \mid x_1 \cdots x_{mk-1}, x_{mk-1} \cdots 0.$$

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A Catalanimal formula for Macdonald polynomials

J. Blasiak^{*1}, M. Haiman^{†2}, J. Morse^{‡3}, A. Pun^{§3}, and G. H. Seelinger^{¶4}

¹Department of Mathematics, Drexel University, Philadelphia, PA

²Department of Mathematics, University of California, Berkeley, CA

³Department of Mathematics, University of Virginia, Charlottesville, VA

⁴Department of Mathematics, University of Michigan, Ann Arbor, MI

Abstract. Catalanimals are rational functions encoding virtual GL_l character series. When truncated to the polynomial characters, they have been shown to recover many important symmetric function quantities with coefficients in $\mathbb{Q}[q, t]$ that arise from the study of diagonal harmonics, including $\nabla^m e_k$ and, more generally, $\nabla^m s_\lambda$. Providing Catalanimal representatives of these quantities gave the necessary tools to show $\nabla^m s_\lambda$ is essentially a positive q, t -weighted sum of distinguished LLT polynomials, thereby resolving the Loehr-Warrington conjecture. Missing from this story was a Catalanimal description of the modified Macdonald polynomials $\tilde{H}_\mu$, which are intimately linked to the ∇ operator. In this abstract, we give a Catalanimal style expression for the modified Macdonald polynomials and provide a positivity conjecture on the entire GL_l character series.

Keywords: Symmetric functions, Macdonald polynomials, Catalanimals

1 Introduction

Macdonald polynomials are a basis of symmetric functions with coefficients in $\mathbb{Q}[q, t]$ exhibiting deep connections to representation theory and algebraic geometry. In particular, specific specializations of the q, t parameters recover various widely-studied bases of symmetric functions, such as Hall-Littlewood polynomials, Jack polynomials, q -Whittaker functions, and Schur functions, among others.

Since their introduction, the Macdonald Positivity Conjecture was central to the study of Macdonald polynomials. Precisely, the Macdonald Positivity Conjecture was that the Schur expansion coefficients of the modified Macdonald polynomials $\tilde{H}_\mu(X; q, t)$ lie in

^{*}jblasiak@gmail.com. Supported by NSF Grant DMS-1855784

[†]mhaiman@math.berkeley.edu.

[‡]morsej@virginia.edu. Supported by NSF Grant DMS-1855804 and Simons Foundation-821999.

[§]ayp6e@virginia.edu. Supported by NSF Grant DMS-1855804.

[¶]ghseeli@umich.edu. Supported by NSF Grant DMS-1855804.

$\mathbb{Z}_{\geq 0}[q, t]$. In [8], it was shown that the Macdonald polynomials are doubly graded characters of certain symmetric group modules, known as Garsia-Haiman modules. In [16], the Macdonald Positivity Conjecture was proved by resolving the $n!$ -Conjecture for the Garsia-Haiman modules.

Out of this study arose the space of diagonal harmonics, a bigraded S_k -module containing all the Garsia-Haiman modules associated to partitions of k as submodules. It was conjectured in [9] and proven in [17] that the doubly graded character of the diagonal harmonics can be succinctly described as ∇e_k for e_k the k th elementary symmetric function and ∇ an operator on symmetric functions with eigenfunctions given by the Macdonald polynomials. Precisely,

$$\nabla \tilde{H}_\mu = t^{n(\mu)} q^{n(\mu^*)} \tilde{H}_\mu \text{ where } n(\mu) = \sum_i i - 1 - \mu_i. \quad (1.1)$$

Haglund et. al [13] conjectured a combinatorial identity expressing ∇e_k as a q, t -weighted sum of LLT polynomials, which are q -analogues of skew Schur functions related to Kazhdan-Lusztig polynomials and Fock space representations of $\mathcal{U}_q \hat{\mathfrak{sl}}_n$. This identity, now known as the Shuffle Theorem, was proven in [6]. After being conjectured in 2005, this formula was generalized in many directions, including the Rational Shuffle Theorem (conjectured in [1], proven in [21]), the Compositional Shuffle Theorem (conjectured in [14], proven in [6]), the Delta Theorem (conjectured in [15], proven in [7]), and the Loehr-Warrington Conjecture ([19]) for ∇s_μ , where s_μ is the Schur function indexed by any partition μ .

In several previous works, we provide a general class of functions called *Catalanimals*, special cases of which give quantities like $\nabla^m s_\mu$ and expressions in the Rational Shuffle Theorem. Catalanimals are defined with 3 “root ideals” and a weight, denoted $H, R_q, R_t, R_{qt}, \gamma$. Catalanimals are properly thought of as infinite series of virtual GL_1 characters, i.e. linear combinations of terms χ_λ for $\lambda \in \mathbb{Z}^l$ satisfying $\lambda_1 \geq \dots \geq \lambda_l$ for χ_λ an irreducible GL_1 -character. To recover symmetric functions, we define pol_X to be the truncation of this series to terms χ_λ satisfying $\lambda_l \geq 0$, called polynomial terms. This produces an element of Λ via the identification $\chi_\lambda \leftrightarrow s_\lambda$.

Using Catalanimals, we gave an alternative proof of the Rational Shuffle Theorem and provided a generalization whose combinatorics are controlled by Dyck paths below a line of irrational slope in [5]. In [4], we provide a proof of the Extended Delta Conjecture by expressing it as a simple sum of Catalanimals. In [2], we used the Catalanimals introduced in [3] to resolve and generalize the Loehr-Warrington Conjecture for $\nabla^m s_\lambda$. Missing from these works was any treatment of Macdonald polynomials, the foundation upon which these shuffle theorems lie. In this extended abstract, we now fill in this missing piece by providing a Catalanimal style formula for a modified Macdonald

polynomial. Our Catalanimal-style formula reads

$$\mathbf{H}_\mu(\mathbf{z}; q, t) = \sigma \left(\frac{\prod_{\alpha_{ij} \in R_\mu \setminus \widehat{R}_\mu} (1 - q^{\text{arm } \mu(i)} t^{-\text{leg } \mu(i)} z_i / z_j) \prod_{\alpha_{ij} \in \widehat{R}_\mu} (1 - q t z_i / z_j)}{\prod_{\alpha_{ij} \in R} (1 - q z_i / z_j) \prod_{\alpha_{ij} \in R_\mu} (1 - t z_i / z_j)} \right), \quad (1.2)$$

where σ is the Weyl symmetrization operator defined in Section 2.2 and $R_\mu, \widehat{R}_\mu$ are subsets of the set of positive roots for GL_l explained in Section 3. Under polynomial truncation, our formula recovers $\omega \tilde{H}_\mu(X; q, t)$.

The proof begins with the Haglund-Haiman-Loehr formula [11] for $\tilde{H}_\mu(X; q, t)$ as a weighted sum of LLT polynomials. We then apply the operator ∇ and combine this with a formula for ∇ on an LLT polynomial established in our recent work [3].

This work serves as an extended abstract for a future paper in which we will elaborate on and further generalize this result.

2 Background

2.1 Partitions and symmetric functions

The (French style) diagram of a partition $\mu = \mu_1 \geq \dots \geq \mu_k > 0$ is the set $\{ (i, j) \in \mathbb{Z}^2 : 1 \leq j \leq k, 1 \leq i \leq \mu_j \}$. We identify (i, j) with the lattice square or *box* whose northeast corner has coordinates $(x, y) = (i, j)$, and refer to this box as being in *column* i and *row* j . We set $|\mu| = \mu_1 + \dots + \mu_k$ and let $\ell(\mu) = k$ be the number of non-zero parts. We write μ^* for the transpose of a partition μ . The *arm* and *leg* of a box $b \in \mu$ are the number of boxes in μ strictly east of b and strictly north of b , respectively.

Example 2.1.1. Below we give the diagram of the partition $\mu = 2, 2, 1$ and we fill box b with $(\text{arm } b, \text{leg } b)$:

0,0	
1,1	0,0
1,2	0,1

For this μ , we have $|\mu| = 5$ and $\ell(\mu) = 3$.

Let $\Lambda = \Lambda(X)$ be the algebra of symmetric functions in infinitely many variables $X = x_1, x_2, \dots$, with coefficients in the field $\mathbf{k} = \mathbb{Q}(q, t)$. We follow Macdonald's notation [20] for the graded bases of Λ , and the automorphism $\omega: \Lambda \rightarrow \Lambda$ given on Schur functions by $\omega s_\lambda = s_{\lambda^*}$. We also work with series and symmetric functions in finitely many variables $\mathbf{z} = z_1, \dots, z_l$. If $f(X) \in \Lambda$ is a formal symmetric function, then $f(\mathbf{z})$ or $f(z_1, \dots, z_l)$ denotes its specialization with $X = z_1, \dots, z_l, 0, 0, \dots$.

Given a symmetric function $f \in \Lambda$ and any expression A involving indeterminates, the plethystic evaluation $f(A)$ is defined by writing f as a polynomial in the power-sums

p_k and evaluating with $p_k \mapsto p_k A$, where $p_k A$ is the result of substituting a^k for every indeterminate a occurring in A . The variables q, t from our ground field $\mathbf{k}$ count as indeterminates.

By convention, the name of an alphabet $X = x_1, x_2, \dots$ stands for $x_1 x_2 \dots$ inside a plethystic evaluation. Then $f X = f x_1 x_2 \dots = f x_1, x_2, \dots = f X$. For example, the evaluation $f X / 1 - t^{-1}$ is the image of $f X$ under the $\mathbf{k}$ -algebra automorphism of Λ that sends p_k to $p_k / 1 - t^{-k}$.

The *modified Macdonald polynomials* $\tilde{H}_\mu = \tilde{H}_\mu X; q, t$ of [10] are defined in terms of the Macdonald polynomials $Q_\mu X; q, t$ [20, VI (4.12)] or their integral forms $J_\mu X; q, t$ [20, VI (8.3)] by

$$\tilde{H}_\mu X; q, t = t^{n_\mu} J_\mu \frac{X}{1 - t^{-1}}; q, t^{-1} = t^{n_\mu} \left(\prod_{b \in \mu} (1 - q^{\text{arm } b} t^{-\text{leg } b}) \right) Q_\mu \frac{X}{1 - t^{-1}}; q, t^{-1}, \quad (2.1)$$

where $n_\mu = \sum_i i - 1 \mu_i$. The $\tilde{H}_\mu X; q, t$ also have a direct combinatorial description [11], which we will recall in [Theorem 4.1.1](#).

When $q = 0$, the modified Macdonald polynomials reduce to the *modified Hall-Littlewood polynomials*

$$\tilde{H}_\mu X; 0, t = t^{n_\mu} Q_\mu \frac{X}{1 - t^{-1}}; t^{-1}, \quad (2.2)$$

where the Hall-Littlewood polynomials $Q_\mu X; t$ are as defined in [20, III (2.11)]. At $t = 1$ and $t = \infty$, the $\tilde{H}_\mu X; 0, t$ specialize to the complete homogeneous symmetric functions $\tilde{H}_\mu X; 0, 1 = h_\mu X$ and Schur functions $\tilde{H}_\mu X; 0, \infty = s_\mu X$.

Example 2.1.2. The Macdonald polynomial $\tilde{H}_{22} X; q, t$ has the following expansion into the basis of Schur functions.

$$\tilde{H}_{22} X; q, t = s_4 + q t s_{31} + q^2 t^2 s_{22} + q t q^2 t s_{211} + q^2 t^2 s_{1111} \quad (2.3)$$

2.2 Weyl symmetrization and related operators

The Weyl symmetrization operator σ for GL_l is defined by

$$\sigma f z_1, \dots, z_l = \sum_{w \in S_l} w \left(\frac{f z_1, \dots, z_l}{\prod_{i < j} (1 - z_j / z_i)} \right) = \sum_{w \in S_l} w \left(\frac{f z_1, \dots, z_l}{\prod_{\alpha_{ij} \in R} (1 - z_j / z_i)} \right), \quad (2.4)$$

where $f \in \mathbf{k} z_1^{\pm 1}, \dots, z_l^{\pm 1}$ is a Laurent polynomial, S_l acts by permuting the variables $z_1, \dots, z_l$, and $R = R \text{ GL}_l = \{\alpha_{ij} : 1 \leq i < j \leq l\}$ denotes the set of positive roots for GL_l , with $\alpha_{ij} = \epsilon_i - \epsilon_j \in \mathbb{Z}^l$.

When $\mathbf{z}^\nu = z_1^{\nu_1} \dots z_l^{\nu_l}$ for a dominant weight ν (a weight $\nu \in \mathbb{Z}^l$ is dominant if $\nu_1 \geq \dots \geq \nu_l$), $\sigma \mathbf{z}^\nu = \chi_\nu$ is an irreducible GL_l character. For an arbitrary weight

$\gamma \in \mathbb{Z}^l$, either $\sigma \mathbf{z}^\gamma = \pm \chi_\nu$ for a suitable dominant weight ν , or $\sigma \mathbf{z}^\gamma = 0$. We extend σ to an operator on formal $\mathbf{k}$ -linear combinations $\sum_{\gamma \in \mathbb{Z}^l} c_\gamma \mathbf{z}^\gamma$ by applying it term by term, giving an infinite formal linear combination of irreducible GL_l characters $\sum_\nu a_\nu \chi_\nu = \sum_{\gamma \in \mathbb{Z}^l} c_\gamma \sigma \mathbf{z}^\gamma$. This makes sense because for each dominant weight ν , the set of monomials $\mathbf{z}^\gamma$ such that $\sigma \mathbf{z}^\gamma = \pm \chi_\nu$ is finite.

Recall that the *polynomial characters* of GL_l are the irreducible characters χ_ν for which ν is a partition, that is, $\nu_l \geq 0$. Given any formal $\mathbf{k}$ -linear combination $\sum_\nu a_\nu \chi_\nu$ of irreducible GL_l characters, we define its *polynomial truncation* by

$$\text{pol}_X \left(\sum_\nu a_\nu \chi_\nu \right) = \sum_{\nu_l \geq 0} a_\nu s_\nu X. \quad (2.5)$$

In principle, the right hand side is an infinite formal sum of symmetric functions, but, for instance, if $\sum_\nu a_\nu \chi_\nu$ is homogeneous of degree d , then the right hand side is an ordinary symmetric function, homogeneous of degree d .

2.3 LLT polynomials

We recall the definition and basic properties of LLT polynomials [18], using the ‘attacking inversions’ formulation from [13].

A *skew diagram* is a difference $\nu = \lambda / \mu$ of partition diagrams $\mu \subseteq \lambda$. The *content* of a box $b = (i, j)$ in row j , column i of a (skew) diagram is $c(b) = i - j$.

Let $\nu = (\nu_1, \dots, \nu_k)$ be a tuple of skew diagrams. We consider the set of boxes in ν to be the disjoint union of the sets of boxes in the ν_i , and define the *adjusted content* of a box $a \in \nu_i$ to be $\tilde{c}(a) = c(a) - i\epsilon$, where ϵ is a fixed positive number such that $k\epsilon < 1$.

A *diagonal* in ν is the set of boxes of a fixed adjusted content, that is, a diagonal of fixed content in one of the ν_i .

The *reading order* on ν is the total ordering $<$ on the boxes of ν such that $a < b \Rightarrow \tilde{c}(a) \leq \tilde{c}(b)$ and boxes on each diagonal increase from southwest to northeast. See [Example 2.3.2](#). An *attacking pair* is an ordered pair of boxes a, b in ν such that $a < b$ in reading order and $0 < \tilde{c}(b) - \tilde{c}(a) < 1$.

A *semistandard tableau* on the tuple ν is a map $T: \nu \rightarrow \mathbb{Z}$ which restricts to a semistandard Young tableau on each component ν_i . The set of these is denoted $\text{SSYT } \nu$. An *attacking inversion* in T is an attacking pair a, b such that $T(a) > T(b)$. The number of attacking inversions in T is denoted $\text{inv } T$.

Definition 2.3.1. The *LLT polynomial* indexed by a tuple of skew diagrams ν is the generating function

$$\mathcal{G}_\nu(X; q) = \sum_{T \in \text{SSYT } \nu} q^{\text{inv } T} \mathbf{x}^T, \quad (2.6)$$

where $\mathbf{x}^T = \prod_{a \in \nu} x_{T(a)}$. By [13, 18], $\mathcal{G}_\nu(X; q)$ is known to be symmetric.

Example 2.3.2. Let $\nu = 32 / 10, 33 / 11$ and let $\nu_1, \dots, \nu_8$ denote the boxes of ν in increasing reading order as illustrated in (2.7). We can visualize this by aligning the shapes of ν along content diagonals and reading diagonals top to bottom and bottom left to top right along each diagonal.

(2.7)

The tuple ν has attacking pairs

$$\{ \nu_2, \nu_3, \nu_3, \nu_4, \nu_4, \nu_5, \nu_4, \nu_6, \nu_5, \nu_7, \nu_6, \nu_7, \nu_7, \nu_8 \}. \quad (2.8)$$

Let $T \in \text{SSYT } \nu$ be as follows:

(2.9)

T has attacking inversions $\{ \nu_3, \nu_4, \nu_7, \nu_8 \}$, so $\text{inv } T = 2$. Also, $\mathbf{x}^T = x_1^3 x_2^2 x_3 x_4^2$.

3 Catalananimals

In [3], we introduced *Catalananimals*—infinite series of irreducible GL_l characters in variables $\mathbf{z} = z_1, \dots, z_l$, defined by

$$H_{R_q, R_t, R_{qt}, \lambda} = \sigma \left(\frac{z_1^{\lambda_1} \cdots z_l^{\lambda_l} \prod_{\alpha_{ij} \in R_{qt}} (1 - q t z_i / z_j)}{\prod_{\alpha_{ij} \in R_q} (1 - q z_i / z_j) \prod_{\alpha_{ij} \in R_t} (1 - t z_i / z_j)} \right), \quad (3.1)$$

depending on a weight $\lambda \in \mathbb{Z}^l$ and subsets R_q, R_t, R_{qt} of the set of positive roots $R = \{\epsilon_i - \epsilon_j \mid 1 \leq i < j \leq l\}$ for GL_l . Our convention is always to expand the denominator factors $1 - c z_i / z_j$ for $c \in \mathbf{k}$ and $i < j$ as geometric series $1 - c z_i / z_j = 1 - c z_i / z_j + \cdots$ before applying σ .

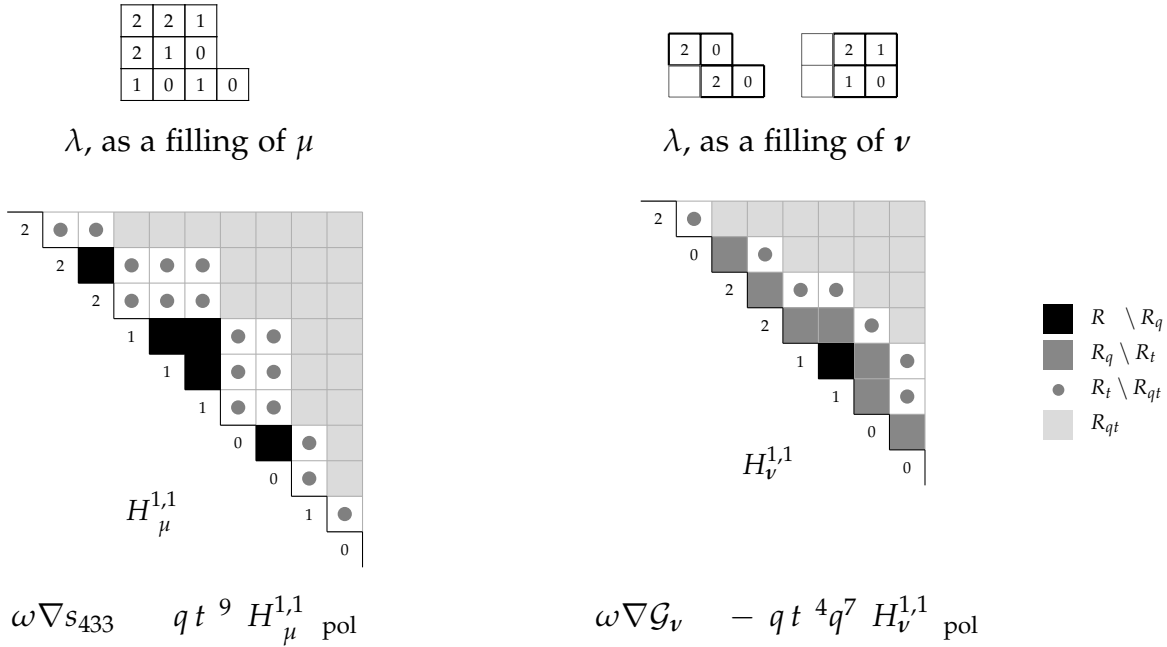


Figure 1: (i) The Catalanimal $H_\mu^{1,1}$ for $\mu = 433$. (ii) The Catalanimal $H_\nu^{1,1}$ for $\nu = 32/10, 33/11$. These are illustrated by drawing the root sets in an $\ell \times \ell$ grid labeled by matrix-style coordinates, with the sets $R \setminus R_q, R_q \setminus R_t, R_t \setminus R_{qt}, R_{qt}$ specified according to the legend on the right; the weight λ is written on the diagonal with λ_1 in the upper left.

Distinguished among Catalanimals are the *LLT Catalanimals*, denoted $H_\nu^{1,1}$ in [3], whose polynomial part is equal to $\omega \nabla$ applied to the LLT polynomial $\mathcal{G}_\nu$, up to a q, t -monomial scalar factor. The Catalanimal $H_\nu^{1,1}$ has root sets $R \supseteq R_q \supseteq R_t \supseteq R_{qt}$, determined as follows using the same attacking inversion combinatorics as in the definition of the LLT polynomial $\mathcal{G}_\nu$:

- $R \setminus R_q \leftrightarrow$ pairs of boxes in the same diagonal in ν ,
- $R_q \setminus R_t \leftrightarrow$ the attacking pairs in ν ,
- $R_t \setminus R_{qt} \leftrightarrow$ pairs going between adjacent diagonals ν ,

where the boxes of ν are numbered $1, \dots, l$ in reading order (see [Example 2.3.2](#)). The weight λ is obtained by filling each diagonal D of ν with the value

$1 - \chi(D \text{ contains a row start}) - \chi(D \text{ contains a row end})$, where $\chi(P) = 1$ if P is true or 0 if P is false, and then reading this filling in the reading order—see [Figure 1](#).

Example 3.1.1. Consider $\nu = \nu^1$ for $\nu^1 = 111$. To construct $H_\nu^{1,1}$, we note that every box is on a different diagonal in ν and no boxes attack each other, so $R_q = R_t = R$. Furthermore, in reading order, boxes 1 and 2 are adjacent and boxes 2 and 3 are adjacent, so $R_{qt} = \{\alpha_{13}\}$. Finally, every box is both a row start and a row end, so $\lambda = 111$. Therefore, $H_\nu^{1,1}$ is given by

$$H_\nu^{1,1} = R, R, \{\alpha_{13}\}, 111 = \sigma \left(\frac{z^{111} (1 - qt z_1/z_3)}{\prod_{1 \leq i < j \leq 3} (1 - q z_i/z_j) (1 - t z_i/z_j)} \right) \quad (3.2)$$

Taking the polynomial part, this becomes

$$\begin{aligned} \text{pol}_X H_\nu^{1,1} = R, R, \{\alpha_{13}\}, 111 \\ = s_{111} q t q^2 q t t^2 s_{21} q t q^3 q^2 t q t^2 t^3 s_3 \\ \omega \nabla e_3 \end{aligned} \quad (3.3)$$

Note that the LLT polynomial with $\nu = \nu^1$ satisfying $\nu^1 = 111$ is equal to e_3 .

For the purposes of this work, we will need only LLT Catalananimals for ν a tuple of ribbon shapes. A *ribbon* is a connected skew shape containing no 2×2 block of boxes. If we only consider ribbons with m boxes such that the bottom-right box has content -1 , then the contents of all the boxes are consecutive integers $-1, -2, \dots, -m$. Define the *descent set* of a ribbon ν to be the set of contents c_u of the boxes $u = i, j \in \nu$ such that the box $v = i, j-1$ directly below u also belongs to ν .

Then, to a partition μ and a subset $D \subset \{i, j \in \mu \mid j > 1\}$, we associate a tuple of ribbons $\nu = \nu^1, \dots, \nu^k$ where $k = \mu_1$ is the number of columns of μ , and ν^i has size μ_i^* , box contents $\{-1, -2, \dots, -\mu_i^*\}$, and descent set $\text{Des } \nu^i = \{-j \mid i, j \in D\}$. See [Example 4.1.2](#) for examples of $\nu = \nu^1, \dots, \nu^k$ for fixed μ as D varies. Let $\mu^1, \dots, \mu^l$ denote the boxes of $\nu = \nu^1, \dots, \nu^k$ in increasing reading order. Also, let A_μ denote the number of attacking pairs in $\nu = \nu^1, \dots, \nu^k$. Note that both the reading order of the boxes of $\nu = \nu^1, \dots, \nu^k$ and the number of attacking pairs depend only on μ and not on D . Then, for any box $b = i, j \in \mu$, let $\text{south } b = i, j-1$ and let $V_\mu = \{\mu^i, \mu^j \mid \text{south } \mu^i = \mu^j\}$ be the set of vertical dominoes in μ .

In this case, the root sets for the LLT Catalananimal indexed by $\nu = \nu^1, \dots, \nu^k$ become R_q , R_t , and R_{qt} .

$$R_t = R_\mu \stackrel{\text{def}}{=} \{\alpha_{ij} \in R \mid \tilde{c}(\mu^i) = 1 \leq \tilde{c}(\mu^j)\} \quad (3.4)$$

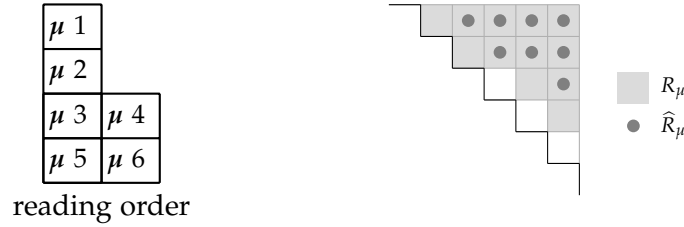
$$R_{qt} = \hat{R}_\mu \stackrel{\text{def}}{=} \{\alpha_{ij} \in R \mid \tilde{c}(\mu^i) = 1 < \tilde{c}(\mu^j)\} \quad (3.5)$$

Then, we have the following special case of [\[3, Corollary 8.4.1\]](#).

Theorem 3.1.2. For a partition μ and a subset $D \subset \{i, j \in \mu \mid i > 1\}$, let $\nu = \nu_{\mu, D}$ and $S = \{\mu_i, \mu_j \in V_\mu \mid \mu_i \in D\}$. Then, we have the following formula for the operator ∇ applied to the LLT polynomial $\mathcal{G}_\nu X; q$:

$$\nabla \mathcal{G}_\nu X; q = \omega \text{pol}_X \sigma \left(-qt^{|\mu| - \mu_1 - |D|} q^{A_\mu} \times \frac{z_1 \cdots z_l \prod_{\mu_i, \mu_j \in V_\mu \setminus D} z_i / z_j \prod_{\alpha_{ij} \in \widehat{R}_\mu} 1 - qt z_i / z_j}{\prod_{\alpha_{ij} \in R} 1 - q z_i / z_j \prod_{\alpha_{ij} \in R_\mu} 1 - t z_i / z_j} \right) \quad (3.6)$$

Example 3.1.3. For $\mu = 2, 2, 1, 1$, $l = 6$, the root sets R_μ and $\widehat{R}_\mu$ are visualized below drawn in an $l \times l$ grid, labeled by matrix-style coordinates and specified by the legend.



4 Macdonald Catalanimals

4.1 The Haglund-Haiman-Loehr formula

Haglund-Haiman-Loehr [11] gave a formula for the modified Macdonald polynomials $H_\mu X; q, t$ as a positive sum of LLT polynomials indexed by a tuple of ribbons. We now recall this formula.

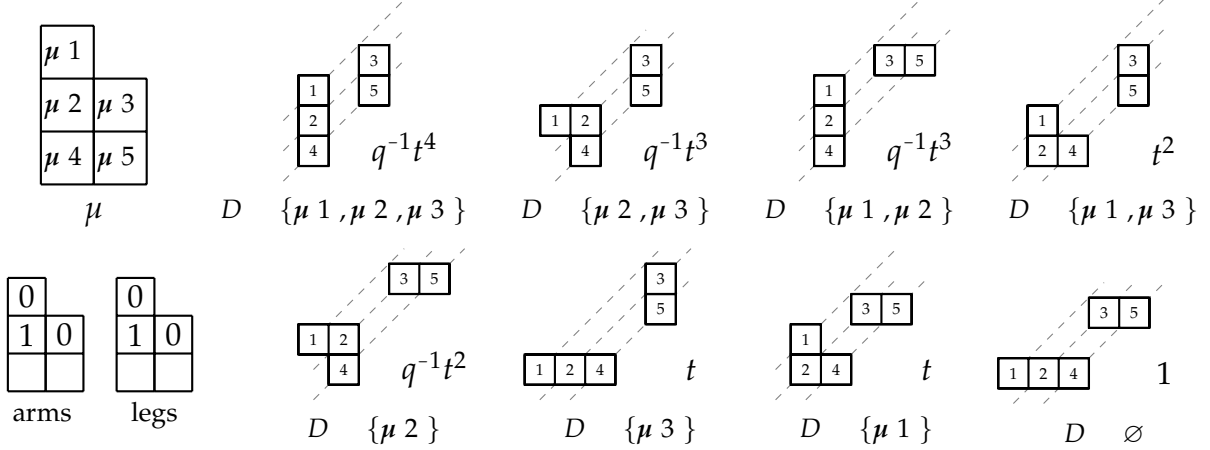
Theorem 4.1.1. For partition μ ,

$$\tilde{H}_\mu X; q, t = \sum_D \left(\prod_{u \in D} q^{-\text{arm } u} t^{\text{leg } u - 1} \right) \mathcal{G}_{\nu_{\mu, D}} X; q, \quad (4.1)$$

where the sum runs over all subsets $D \subset \{i, j \in \mu \mid j > 1\}$.

Example 4.1.2. We illustrate the theorem for $\mu = 2, 2, 1$. Below we draw $\nu_{\mu, D}$ for the 8 possible values of D with boxes labeled in reading order and with the corresponding

coefficient $\prod_{u \in D} q^{-\text{arm } u} t^{\text{leg } u} = 1$.



4.2 Proving the Macdonald Catalanimal formula

Applying ∇ to (4.1) and substituting our Catalanimal formula (3.6) for $\nabla \mathcal{G}_\nu$ for each summand, the resulting sum simplifies and yields the following formula for the modified Macdonald polynomials $\tilde{H}_\mu(X; q, t)$ (previewed in (1.2)):

Theorem 4.2.1. *For every partition μ ,*

$$\tilde{H}_\mu(X; q, t) = \omega \text{pol}_X \mathbf{H}_\mu(\mathbf{z}; q, t) \quad (4.2)$$

for $\mathbf{H}_\mu$ given by

$$\mathbf{H}_\mu(\mathbf{z}; q, t) = \sigma \left(\frac{\prod_{\alpha_{ij} \in R_\mu \setminus \hat{R}_\mu} (1 - q^{\text{arm } \mu_i} t^{-\text{leg } \mu_i} z_i/z_j) \prod_{\alpha_{ij} \in \hat{R}_\mu} (1 - qt z_i/z_j)}{\prod_{\alpha_{ij} \in R} (1 - q z_i/z_j) \prod_{\alpha_{ij} \in R_\mu} (1 - t z_i/z_j)} \right). \quad (4.3)$$

The work of [11, 12] actually gives many formulae for $\tilde{H}_\mu$, one for each rearrangement of the columns of μ . In turn, following the same proof technique, we obtain a Catalanimal style formula for $\tilde{H}_\mu$ for each rearrangement of the columns of μ . More details will be given in the full version of this paper.

4.3 Positivity of the Macdonald Catalanimal

Finally, we provide a conjecture on the series $\mathbf{H}_\mu$ in (4.3). It is a theorem of [16] that the expansion $\tilde{H}_\mu(X; q, t) = \sum_{\lambda} \tilde{K}_{\lambda, \mu}(q, t) s_\lambda$ has positive coefficients $\tilde{K}_{\lambda, \mu}(q, t) \in \mathbb{Z}_{\geq 0}[q, t]$. However, computer experimentation suggests that $\mathbf{H}_\mu(\mathbf{z}; q, t)$ exhibits a stronger positivity as a series of GL_L -characters.

Conjecture 4.3.1. For every partition μ of l , the series $\mathbf{H}_\mu \mathbf{z}; q, t$ is a positive sum of irreducible GL_l characters; that is, the coefficients in

$$\mathbf{H}_\mu \mathbf{z}; q, t = \sum_{\nu} \mathbf{K}_{\nu, \mu} q, t \chi_{\nu} \quad (4.4)$$

are polynomials $\mathbf{K}_{\nu, \mu} q, t \in \mathbb{Z}_{\geq 0} q, t$ with non-negative integer coefficients.

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Combinatorial invariance for lower intervals using hypercube decompositions

Grant T. Barkley^{*1} and Christian Gaetz^{†2}

¹Harvard University, Cambridge MA

²Cornell University, Ithaca NY

Abstract. We give a new proof of the combinatorial invariance conjecture for lower intervals of the symmetric group. This conjecture posits that Kazhdan–Lusztig polynomials associated to intervals in the Bruhat order depend only on the poset structure of the interval. For lower intervals of the symmetric group, this was originally shown by Brenti using special matchings. Our proof uses a different combinatorial structure, called a hypercube decomposition, which was recently introduced by Blundell, Buesing, Davies, Veličković, and Williamson as an approach to proving combinatorial invariance for arbitrary intervals. Instead of studying the Kazhdan–Lusztig polynomials directly, we apply hypercube decompositions to the related family of $\tilde{R}$ -polynomials. We prove a new, explicit combinatorial recurrence for $\tilde{R}$ -polynomials using certain hypercube decompositions.

Keywords: Bruhat order, Kazhdan–Lusztig polynomials, R -polynomials

1 Introduction

Kazhdan–Lusztig polynomials $P_{u,v} q$ are polynomials associated to a pair u, v of elements of a Coxeter group W . They were first introduced in [17] as transition functions between the standard basis of the Hecke algebra of W and the Kazhdan–Lusztig basis. The polynomials can also be defined geometrically: if X_u, X_v are the Schubert cells associated to u and v in a flag variety, and $x \in X_u$ is any point, then $P_{u,v} q$ is the Poincaré polynomial of the stalk at x of the intersection cohomology sheaf on the Schubert variety $\overline{X}_v$. Recall that the Bruhat order is a partial order on Coxeter group elements such that $u \leq v$ if and only if X_u is contained in the Schubert variety $\overline{X}_v$. In particular, $P_{u,v} q = 0$ unless $u \leq v$. The *combinatorial invariance conjecture* is the assertion, proposed by Dyer [13] and Lusztig, that if the Bruhat intervals u, v and u', v' are isomorphic as posets, then $P_{u,v} q = P_{u',v'} q$.

^{*}gbarkley@math.harvard.edu. GTB was supported by NSF grant DMS-1854512.

[†]crgaetz@gmail.com. CG was partially supported by a Klarman Postdoctoral Fellowship at Cornell University and by an NSF Postdoctoral Research Fellowship (DMS-2103121).

If $u \leq u' \leq e$, so that both u, v and u', v' are lower intervals in Bruhat order, then the combinatorial invariance conjecture is a theorem. This was first proven when $W = S_n$ by Brenti [6] using *special matchings* of the Hasse diagram of e, v . Du Cloux [10] then extended the result to a large class of Coxeter groups using an analysis of isomorphisms between lower intervals. Finally combinatorial invariance of lower intervals was shown for all Coxeter groups by Delanoy [11] and independently by Brenti–Caselli–Marietti [7] via a classification of all possible special matchings of these posets. Little is known about special matchings in intervals that are not lower, and there exist non-lower intervals which have no special matchings. As a result, the technique used by these authors does not extend to all intervals, and there are few other cases where combinatorial invariance is known. These include intervals with no subinterval isomorphic to Bruhat order on S_3 [4], intervals of S_n with length up to 8 [13, 15, 16], intervals in the infinite group $\tilde{A}_2$ [8], and intervals where $P_{u,v}(q) = 1$ [9, 12].

Here we will give a new proof of combinatorial invariance for lower intervals in the symmetric group, which is quite distinct from the earlier proofs using special matchings. Our method gives hope for proving combinatorial invariance for an arbitrary interval of S_n , because the combinatorial structure it relies upon exists in every such interval (in contrast to special matchings). Following the conventions in this area, we will say that a notion is *combinatorial* if it depends only on the isomorphism class of the poset u, v .

Our proof is inspired by recent work of Blundell, Buesing, Davies, Veličković, and Williamson [3], in which the authors created a machine learning model to predict $P_{u,v}(q)$ from the Bruhat graph associated to u, v . They found that the model placed more importance on edges that formed a subgraph shaped like the edges of a hypercube. A more refined analysis led them to the notion of a *hypercube decomposition*. A hypercube decomposition of u, v is a subinterval u, z whose complement contains many hypercubes (see Definition 2.5). The authors conjecture an intriguing combinatorial recursion for $P_{u,v}(q)$ that should hold for any hypercube decomposition. They also show that the formula is true for a certain (non-combinatorial) *standard hypercube decomposition* which exists in any interval of the symmetric group.

In this paper we do not prove their conjecture, or even study the polynomials $P_{u,v}(q)$ directly at all. Instead we prove a more explicit recursion for a related family of polynomials, the $\tilde{R}$ -polynomials. For elements $u \leq v$ of a Coxeter group W , Theorem 5.1.4 of [2] implies that combinatorial invariance of $\tilde{R}_{u,z}(q)$ for all $z \in u, v$ is equivalent to combinatorial invariance of $P_{u,z}(q)$ for all $z \in u, v$. We also specialize to certain *strong* hypercube decompositions (see Definition 2.6), which include the standard hypercube decomposition. In the next section we define a combinatorial polynomial $\tilde{H}_{u,z,v}(q)$ associated to a hypercube decomposition and depending on the polynomials $\tilde{R}_{u,x}$ for $x \leq z$.

Main Theorem. Let $W = S_n$. Then for any $v \in W$, and any $z \leq v$ such that e, z gives a strong hypercube decomposition of e, v , we have

$$\tilde{R}_{e,v}(q) = \tilde{H}_{e,z,v}(q).$$

Furthermore, for any $u \leq v$ in S_n , there exists $u \leq z \leq v$ such that u, z gives a strong hypercube decomposition of u, v , and we have

$$\tilde{R}_{u,v}(q) = \tilde{H}_{u,z,v}(q). \quad (*)$$

As a result of these two facts, the polynomial $\tilde{R}_{e,v}(q)$ depends only on the isomorphism class of e, v as a poset.

The first fact alone is not enough to imply combinatorial invariance for lower intervals: without the second fact (for $u = e$), our combinatorial recursion for $\tilde{R}_{e,v}$ may not reach a base case. But also the second fact of the theorem alone does not imply any cases of combinatorial invariance, because the z produced there is not defined combinatorially.

In light of the theorem statement, one might ask whether $(*)$ always holds for any strong hypercube decomposition of u, v . The answer is no; a (minimal) counterexample is given by

$$u = 132546, z = 612345, v = 651234.$$

However, we make the following conjecture, which when combined with the main theorem implies combinatorial invariance. For polynomials P and Q , we write $P \leq Q$ to mean that the coefficient of q^k in P is less than or equal to that in Q , for all k .

Conjecture. For any interval u, v in S_n , and any strong hypercube decomposition u, z , we have

$$\tilde{R}_{u,v} \leq \tilde{H}_{u,z,v}.$$

The main theorem is a summary of [Theorems 1, 2, and 3](#). To state these theorems requires some details on hypercube decompositions, which are given in the next section.

The authors have generalized the results announced in this extended abstract in order to prove new interesting cases of combinatorial invariance. For the details, see [\[1\]](#). That same reference contains the details of many proofs which are omitted here.

2 Hypercube decompositions

Let W be a Coxeter group with generating set S and reflection set $T = \{wsw^{-1} \mid w \in W, s \in S\}$. Elements of these sets are called simple generators and reflections, respectively. For the [Main Theorem](#) it will suffice to take $W = S_n$, $S = \{12, 23, \dots, n -$

$1, n\}$, and $T = \{ab \mid a \neq b\}$, but some of the results hold for any Coxeter group. Here we write ab or a, b to denote the transposition swapping a and b . We will also use one-line notation for permutations: 3124 and $3, 1, 2, 4$ both denote the permutation sending $1 \mapsto 3, 2 \mapsto 1, 3 \mapsto 2, 4 \mapsto 4$. The length of a Coxeter group element w is the minimal number of simple generators needed to express w and is denoted $\ell(w)$.

Definition 2.1. The *Bruhat graph* Ω_W is the directed graph with vertices given by the elements of W and an edge from u to v if and only if $u < v$ and vu^{-1} is a reflection. We make this an edge-labeled graph by labeling the edge $u \rightarrow v$ by the unique reflection $t \in T$ such that $tu = v$. We denote this by $u \xrightarrow{t} v$.

We say $u \leq v$ in *Bruhat order* if there is path from u to v in the Bruhat graph. The Bruhat graph $\Omega_{u,v}$ is the induced subgraph of Ω_W on the elements of $[u, v]$.

From this description there are two facts that aren't obvious about the Bruhat graph:

Proposition 2.2 ([14, Proposition 3.3]). *The Bruhat order is a graded poset with rank function ℓ . In particular, the Hasse diagram of Bruhat order is a subgraph of Ω_W . Furthermore, the unlabeled graph underlying $\Omega_{u,v}$ is determined by the poset isomorphism class of $[u, v]$.*

As a result of this proposition, we are allowed to use the (unlabeled) edges of $\Omega_{u,v}$ in constructing combinatorial invariants. The following definition introduces our main object of study, the $\tilde{R}$ -polynomials.

Definition 2.3. A *reflection order* is a total ordering $\prec$ on the reflections T satisfying certain properties (see [2, Section 5.2]). If $W = S_n$, reflection orders are exactly those such that whenever $1 \leq a < b < c \leq n$, we have either

$$ab \prec ac \prec bc \text{ or } bc \prec ac \prec ab$$

Fix a reflection order $\prec$. An *increasing path* of length k in the Bruhat graph $\Omega_{u,v}$ is a path $u \xrightarrow{t_1} u_1 \xrightarrow{t_2} \cdots \xrightarrow{t_k} u_k = v$ such that $t_1 \prec t_2 \prec \cdots \prec t_k$ in the reflection order. The $\tilde{R}$ -polynomial is defined by

$$\tilde{R}_{u,v}(q) = \sum_{k=0}^{\ell(v)-\ell(u)} \# \text{ of length-}k \text{ increasing paths in } \Omega_{u,v} \cdot q^k.$$

Proposition 2.4 ([2, Theorems 5.1.4 and 5.3.4]). *The $\tilde{R}$ -polynomials are independent of the choice of reflection order. Furthermore, the Kazhdan–Lusztig polynomial $P_{u,v}(q)$ can be computed from the family $\tilde{R}_{u,z}(q)$ as z varies in $[u, v]$. In particular, combinatorial invariance for $\tilde{R}$ -polynomials implies combinatorial invariance for Kazhdan–Lusztig polynomials.*

The proposition implies that the Kazhdan–Lusztig polynomials can be computed from $\Omega_{u,v}$ as an abstract digraph with its edge-labeling by elements of T . We will be especially interested in certain configurations of edges introduced in [3]. First we define a useful family of directed graphs. The k -hypercube is the Hasse diagram of the Boolean lattice with k atoms, equivalently the 1-skeleton of an k -dimensional hypercube. The 2-hypercube is also called a *diamond*.

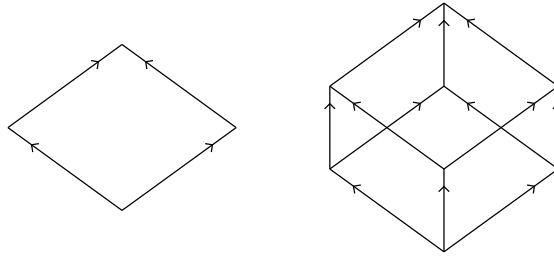


Figure 1: The 2-hypercube and the 3-hypercube.

Fix $x \in u, v$. Let Y be a collection of k elements of $\Omega_{u,v}$ which each have incoming edges from x . We say that Y spans a hypercube at x if Y is the set of atoms of a subgraph of $\Omega_{u,v}$ which is isomorphic to the k -hypercube. If Y spans a unique hypercube at x , then we define $\theta_x Y$ to be the top vertex of the hypercube spanned by Y .

Definition 2.5. Let u, v be a Bruhat interval. Let $z \in u, v$. We say that $I = u, z$ is a **hypercube decomposition** (following [3]) if these two properties hold:

(D) If $y \begin{matrix} \nearrow \\ \nwarrow \end{matrix} w \begin{matrix} \nwarrow \\ \nearrow \end{matrix} y'$ is a subgraph of $\Omega_{u,v}$ and x, y, y' are all in I , then w is in I .

(H) If $x \in I$ and $Y = \{y_1, y_2, \dots, y_k\} \subseteq u, v$ is such that $x \rightarrow y_i$ and $y_i \notin I$ for all i , and the $\{y_i\}$ are pairwise incomparable, then Y spans a unique hypercube.

For x in a hypercube decomposition I , define the set $Y_x : \{y \in u, v \setminus I \mid x \rightarrow y\}$. For any antichain $Y \subseteq Y_x$, property (H) implies that $\theta_x Y$ is well-defined. The function θ_x is called the *hypercube map*; in [3] its domain is extended to all subsets of Y_x . Note that $\theta_x \emptyset = x$. To simplify exposition, in the remainder of this abstract we will assume that if $Y, Y' \subseteq Y_x$ are distinct antichains, then $\theta_x Y \neq \theta_x Y'$. We also define for each $x \in I$ a subset $Y_x^\circ \subseteq Y_x$ which is the unique antichain satisfying $\theta_x Y_x^\circ = v$ if it exists. If such a subset does not exist, we set $Y_x^\circ : \emptyset$.

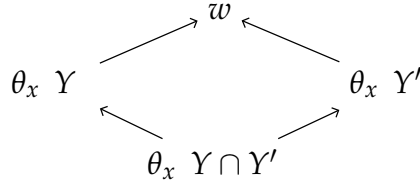
We can now define the polynomial

$$\tilde{H}_{u,z,v} q : \sum_{\substack{x \in I \\ \theta_x Y_x^\circ \subseteq v}} q^{|Y_x^\circ|} \tilde{R}_{u,x} q ,$$

where the sum is taken over $x \in u, z$ such that there is an antichain contained in Y_x which spans a hypercube with top v . We are almost ready to state our theorem computing $\tilde{R}_{e,v}$. We first need a special condition on a hypercube decomposition.

Definition 2.6. We say that the hypercube decomposition I of u, v is *strong* if it satisfies the following condition:

(U) For all $x \in I$ and antichains $Y, Y' \subseteq Y_x$ with $|Y| + |Y'| = |Y \cap Y'| + 1$, if



is a subgraph of $\Omega_{u,v}$, then $Y \cup Y'$ is an antichain and $w = \theta_x Y \cup Y'$.

Theorem 1. Let $W = S_n$ and $z \leq v$ any two elements such that $I = e, z$ gives a strong hypercube decomposition of e, v . Then

$$\tilde{R}_{e,v} = \tilde{H}_{e,z,v}.$$

Proof sketch. The left-hand side is the generating function which counts increasing paths γ from e to v , weighted by $q^{|\gamma|}$, where $|\gamma|$ is the number of edges in γ . The right hand side is the generating function which counts pairs (x, γ') , where x is in I and γ' is an increasing path from e to x . Such a pair is weighted by $q^{|\gamma'| + |Y_x^\circ|}$. To prove the inequality, it will suffice to give a bijective, weight-preserving map from the objects counted by the left-hand side to the objects counted by the right-hand side.

Given an increasing path γ from e to v , we define x to be the last vertex of γ which is in I . The path γ' is defined as the initial section of γ which ends at x . We would like to show that γ is the unique completion of γ' to an increasing path from e to v , and furthermore that $|\gamma| = |\gamma'| + |Y_x^\circ|$. This reduces to two facts: that every hypercube subgraph of Ω_W contains a unique increasing path, and that once γ leaves I , every remaining vertex used by γ is in the image of θ_x . The proof of the latter fact uses (U), but it also requires that we define increasing paths using a reflection order with special properties. That such a reflection order exists is a consequence of property (D) and uses the fact that e, v is a lower Bruhat interval.

We would also like to know that the map $\gamma \mapsto x, \gamma'$ is surjective. This amounts to checking, for each $x \in J$ with $\theta_x Y_x^\circ = v$, that the unique increasing path in the hypercube spanned by Y_x° may be concatenated to the end of any increasing path from e to x to produce a path that is still increasing. Again this relies on a specially constructed reflection order. See [1] for more details. $\square$

If the only strong hypercube decomposition is the trivial one where $z = v$, then this formula gives us no information. Fortunately, in the symmetric group, there are always nontrivial strong hypercube decompositions. If $v = v_1, v_2, \dots, v_n$ is the one-line notation for v , and i is the minimal index such that $v_i < i$, then the *standard hypercube decomposition* of e, v is $I = \{x \in e, v \mid x_i = i\}$. One can similarly define the standard hypercube decomposition in any nontrivial interval of S_n .

The following is a strengthening of [3, Theorem 5.1], and its proof can be extracted from the explicit description of the hypercube map in Section 5 of that paper.

Theorem 2. *For every $u < v$ in S_n , the standard hypercube decomposition of u, v is strong.*

Finally we have the following result applying to non-lower intervals. Remarkably, Brenti proved essentially the same result in [5, Corollary 3.9], long before the invention of hypercube decompositions. Our proof is quite different from Brenti's, and proceeds in the same way as that of Theorem 1.

Theorem 3. *For every $u < v$ in S_n , the standard hypercube decomposition $I = u, z$ of u, v satisfies*

$$\tilde{R}_{u,v} q = \tilde{H}_{u,z,v} q.$$

Theorem 3, when combined with our Conjecture, implies that

$$\tilde{R}_{u,v} = \min_{\substack{u \leq z < v \\ \text{strong hyp. dec.}}} \tilde{H}_{u,z,v}.$$

Here the comparison of polynomials is taken coefficient-wise. If the Conjecture is true, then this gives a recursion for $\tilde{R}_{u,v}$ depending only on the unlabeled graph structure for $\Omega_{u,v}$, hence proving the combinatorial invariance conjecture for S_n .

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Equivariant log-concavity of independence sequences of claw-free graphs

Shiyue Li^{*1}

¹*Department of Mathematics, Brown University, Providence, RI 02912, USA*

Abstract. We show that the graded vector space spanned by independent vertex sets of any claw-free graph is strongly equivariantly log-concave, viewed as a graded permutation representation of the graph automorphism group. Our proof reduces the problem to the equivariant hard Lefschetz theorem on the cohomology of a product of projective lines, inspired by a combinatorial map of Krattenthaler. Both the result and the proof generalize our previous result on graph matchings. This also gives a strengthening and a new proof of results of Hamidoune, and Chudnovsky–Seymour.

Keywords: equivariant log-concavity, claw-free graphs, independence sequence

1 Introduction

A graph G is claw-free if no induced subgraph is the bipartite graph $K_{1,3}$. An independent set of a graph G is a set of nonadjacent vertices. The independence sequence of a claw-free graph is *log-concave*: for all $1 \leq k \leq \ell$, the numbers I_j of independent sets of size j satisfies that

$$I_{k-1}I_{\ell-1} \leq I_kI_\ell.$$

This was first proven by Hamidoune [5]. Then, Chudnovsky and Seymour [3] proved it by showing that the generating polynomial has only real roots, which is well-known to imply log-concavity with no internal zeros.

It is often interesting to ask if a certain behavior of a mathematical object respects the underlying symmetry. The notion of equivariant log-concavity was introduced by Gedeon, Proudfoot and Young [4] as a natural categorification of logarithmic concavity. Recently, it is used to study various log-concavity behaviors with respect to a natural group action in the contexts of topology, geometry and combinatorics.

Let Γ be a finite group, a Γ -representation $V_\bullet$ is **strongly equivariantly log-concave** if for all $1 \leq k \leq \ell$,

$$V_{k-1} \otimes V_{\ell-1} \subseteq V_k \otimes V_\ell$$

as a Γ -subrepresentation.

We highlight some known equivariantly log-concave graded representations that are of combinatorial, geometric, and topological interests in the literature:

^{*}shiyue_li@brown.edu.

- Theorem 1.1.** (A) The $V_\bullet^n$ given by the q -binomial coefficients for a fixed n as a $GL_n \mathbb{F}_q$ -representation is strongly equivariantly log-concave [11, Proposition 6.7].
- (B) The rational cohomology $H^* \text{Conf } n, \mathbb{C}, \mathbb{Q}$ of the configuration space of n points in $\mathbb{C}$ as an S_n -representation is strongly equivariantly log-concave for degrees ≤ 14 [10].
- (C) The rational cohomology $H^* \text{Conf } n, \mathbb{R}^3, \mathbb{Q}$ of the configuration space of n points in $\mathbb{C}$ as an S_n -representation is strongly equivariantly log-concave for degrees ≤ 14 [10].
- (D) The $V_\bullet^n$ of even degrees of the intersection homology of the complex affine hypertoric variety of the root system of $\mathfrak{sl}_n$, viewed as an S_n -representation is strongly equivariantly log-concave for degrees ≤ 14 [10].
- (E) The $V_\bullet^G$ given by matchings in a graph G as an $\text{Aut } G$ -representation is strongly equivariantly log-concave [9].
- (F) The $V_\bullet^n$ given by k -subsets in n as an S_n -representation is strongly equivariantly log-concave (as a special case of [9]).

The aim of this paper is to study the equivariant log-concavity of the following graded representation. Let G be a claw-free graph. Let $\mathbb{I}_k$ denote the set of independent vertex sets of size k . The automorphism group $\text{Aut } G$ naturally acts on all independent vertex sets, and each $\mathbb{I}_k$ is invariant under this action. Define the graded representation of $\text{Aut } G$

$$V_\bullet^G = \bigoplus_{k \geq 0, I \in \mathbb{I}_k} \mathbb{C} I,$$

and it admits a grading given by cardinalities. We have the following theorem.

Theorem 1.2. For any claw-free graph G , the graded $\text{Aut } G$ -representation $V_\bullet^G$ is strongly equivariantly log-concave.

Remark 1.3. Our proof uses combinatorics inspired by the work of Krattenthaler [8] to reduce the problem to the equivariant hard Lefschetz theorem on a Boolean algebra, or the cohomology of a product of projective lines, a generalization of the method in the author's previous work on graph matchings [9]. The result specializes to our previous result on graph matchings by taking the line graph $L G$ of a graph G : The line graph $L G$ of a graph G consists of vertices each for every edge in G and edges each for every common vertex shared by two edges in G . For example, every cycle graph C_n with n edges has its line graph isomorphic to itself, and the line graph of K_4 is the 1-skeleton of the hypersimplex $\Delta_{2,4}$. A matching on G of size k yields an independent vertex set in $L G$ of size k . Line graphs are claw-free, by construction, but not all claw-free graphs are line graphs.

Remark 1.4. Taking dimensions immediately covers the previous results of Hamidoune, and Chudnovsky–Seymour, thus giving new proofs to these results.

Remark 1.5. Communicated by Eric Ramos and Nick Proudfoot, the group consisting of Melody Chan, Chris Eur, Dane Miyata, Nick Proudfoot, Eric Ramos, Lorenzo Vecchi, Claudia Yun, was studying if the graded $\text{Aut } T$ -representation of the independence sequence of a tree T is strongly equivariantly log-concave. They provided a counterexample, the star graph with 6 leaves, to disprove the statement. Note that this counterexample is “claw-ful”, quite the opposite of “claw-free”. Morally speaking, the enigmatic “claw” structure seems to be an obstruction to the equivariant log-concavity of independence sequence of a tree, but the lack thereof turns out to be crucial in our proof of [Theorem 1.2](#).

Remark 1.6. During FPSAC 2023, an audience member and later the FPSAC committee chairs pointed out a third proof, prior to this paper, of log-concavity of the independence sequences of claw-free graphs by [\[15\]](#), of which we were not aware. The construction is similar to our construction in the reduction to the injectivity of the raising operator of an $\mathfrak{sl}_2 \mathbb{C}$ -representation. One of the standard proofs of the hard Lefschetz theorem for a compact Kähler manifold is by exhibiting an $\mathfrak{sl}_2 \mathbb{C}$ -action on the relevant vector space, where the hard Lefschetz operator coincides with the raising operator of the $\mathfrak{sl}_2 \mathbb{C}$ -action. See, for example, Chern’s proof [\[2\]](#). Our setup features the presence of a group action on the graded vector space, which naturally arises from the underlying objects, and is respected by the hard Lefschetz operator. A priori, the raising operator of a $\mathfrak{sl}_2 \mathbb{C}$ -representation or the hard Lefschetz operator needs not be equivariant with respect to the natural group action. For example, at the end of [\[15\]](#), Wagner recovers the well-known log-concavity of Stirling numbers of the second kind by constructing a claw-free graph for each n . However, we have shown, in an unpublished joint work with Siddharth Kannan, that the graded permutation representation of S_n spanned by partitions of the set n is not S_n -equivariantly log-concave. The S_n -equivariant embeddings fail when $n = 7, 8$.¹ It would be fascinating to see if there exist other natural categorifications of the Stirling numbers of the second kind, perhaps inspired by Wagner’s construction, that are S_n -equivariantly log-concave.

2 Proof of the main theorem

In this section, we prove the main theorem. The main idea is to construct a family of $\text{Aut } G$ -equivariant injections

$$V_{k-1} \otimes V_{\ell-1} \hookrightarrow V_k \otimes V_\ell$$

¹Explicit computations are available at our GitHub repository: <https://github.com/shiyue-li/eqStir/blob/main/eqStir.sage>

for all $1 \leq k \leq \ell$ by reducing to the equivariant hard Lefschetz operator on a Boolean algebra, or the cohomology of a product of projective lines, via the combinatorics of the independent vertex sets. This method is inspired by Krattenthaler's combinatorial proof of the log-concavity of graph matching sequence [8].

Fix a graph G and $1 \leq k \leq \ell$, for each pair I, J in $\mathbb{I}_{k-1} \times \mathbb{I}_{\ell-1}$, consider the induced subgraph on the symmetric difference of I and J , i.e., $I \setminus J \cup J \setminus I$, denoted by $G_{I,J}$. The components in $G_{I,J}$ can only be either a path or a cycle, because G is claw-free and I, J are independent vertex sets. Consider all the components in $G_{I,J}$ that are paths of even lengths, i.e., paths that contain odd number of vertices in $I \cup J$, denoted as $C_{I,J}$. ("C" for "chains".) Color vertices from I with blue, and J with pink. Note that each path in $C_{I,J}$ has both endpoints colored blue or pink. Now we do some counting: Let $P_{I,J}$ resp. $B_{I,J}$ be the number of paths with pink resp. blue endpoints in $C_{I,J}$. We have that

$$P_{I,J} - B_{I,J} = |C_{I,J}|, \text{ and } P_{I,J} - B_{I,J} = \ell - 1 - k - 1 \geq 2.$$

From these, we know

$$2B_{I,J} \leq P_{I,J} - 2 = |C_{I,J}| - 2, \text{ and therefore, } B_{I,J} \leq \frac{|C_{I,J}|}{2} - 1. \quad (2.1)$$

Our next step is to decompose each of $V_{k-1} \otimes V_{\ell-1}$ and $V_k \otimes V_\ell$ into a direct sum of Boolean algebras on certain partitions in $\mathbb{I}_{k-1} \times \mathbb{I}_{\ell-1}$ and $\mathbb{I}_k \times \mathbb{I}_\ell$.

Definition 2.1. Two pairs I, J, I', J' of independent vertex sets are equivalent if $I \cup J = I' \cup J'$ and I resp. J agrees with I' resp. J' outside of $C_{I,J}$ or $C_{I',J'}$.

One verifies using arguments in [9] that this indeed gives partitions $\Pi_{k-1,\ell-1}$ and $\Pi_{k,\ell}$ on $\mathbb{I}_{k-1} \times \mathbb{I}_{\ell-1}$ and $\mathbb{I}_k \times \mathbb{I}_\ell$ respectively.

For any part $P \in \Pi_{k-1,\ell-1}$ and each pair I, J in P , we associate a set of pairs of independent vertex sets in $\mathbb{I}_{k-1} \times \mathbb{I}_{\ell-1}$ as follows. For each path in $C_{I,J}$ with endpoints colored blue, we swap the colors on all the vertices in this path from pink to blue and from blue to pink. This swapping produces a path in $C_{I,J}$ with endpoints colored pink. Now collect all the blue resp. pink vertices in $G_{I,J}$ and record that as I' resp. J' . Since I' resp. J' now has k resp. ℓ vertices, the pair I', J' is in $\mathbb{I}_k \times \mathbb{I}_\ell$. Repeat for every path in $C_{I,J}$, we obtain a subset $N_{I,J}$ in $\mathbb{I}_k \times \mathbb{I}_\ell$. See Figure 1 for an example on the 6-cycle graph for $k = \ell = 2$. Using a similar argument as in [9, Section 2.2], we verify that $N_{I,J}$ is a part of $\Pi_{k,\ell}$, denoted as P' . Now define a map

$$\Phi_{k,\ell}: V_{k-1} \otimes V_{\ell-1} \rightarrow V_k \otimes V_\ell, \quad I \otimes J \mapsto \frac{1}{|N_{I,J}|} \sum_{I',J' \in N_{I,J}} I' \otimes J'.$$

Using similar argument as in [9, Section 2.2], one verifies that $\Phi_{k,\ell}$ is $\text{Aut } G$ -equivariant.

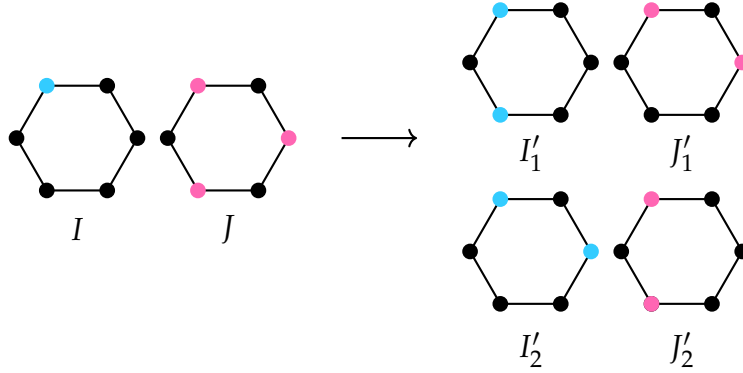


Figure 1

To show injectivity, we consider the following vector space for any part P in $\Pi_{k-1, \ell-1}$

$$V_{k-1, \ell-1} P : \text{Span}_{\mathbb{F}} \{I \otimes J \mid I, J \in P\}.$$

We now realize $V_{k-1, \ell-1} P$ as a categorification of the B_P th level of the Boolean lattice on C_P . Consider the map

$$\beta_P: P \rightarrow \binom{C_P}{B_P}, \quad I, J \mapsto \text{the set of paths with blue endpoints in } C_{I,J}.$$

It is well-defined by the construction of paths of blue endpoints in C_P and bijective using a similar argument in [9]. Next, we consider the vector space

$$V_{C_P, B_P} : \text{Span}_{\mathbb{F}} \left\{ B \mid B \in \binom{C_P}{B_P} \right\},$$

and define

$$\underline{\beta}_P: V_{k-1, \ell-1} P \rightarrow V_{C_P, B_P}, \quad I \otimes J \mapsto \text{the set of paths with blue endpoints in } C_{I,J}.$$

It is an isomorphism of vector spaces, because β_P is a bijection on the bases.

Then, we do the same procedure for $\Pi_k \times \Pi_\ell$. We define vector spaces $V_{k, \ell} P'$, $V_{C_{P'}, B_{P'}}$ and the maps $\beta_{P'}$ and $\underline{\beta}_{P'}$ similar to those for P . Note that, by construction,

$$B_{P'} = B_P + 1 \quad \text{and} \quad C_{P'} = C_P.$$

Finally, for each P in $\Pi_{k-1, \ell-1}$, define the linear map

$$L_P: V_{C_P, B_P} \rightarrow V_{C_{P'}, B_{P'}-1}, \quad B \mapsto \frac{1}{|C_P| - B_P} \sum_{B \subseteq B' \in \binom{C_{P'}}{B_P-1}} B'.$$

Crucially, L_P is a hard Lefschetz operator on the Boolean algebra spanned by all subsets of C_P , where the grading is given by cardinality. It is injective for degrees

$$B_P \leq |C_P|/2 - 1.$$

This operator and its injectivity on the lower half graded pieces have been studied in various contexts. We invite the reader to see proofs of various flavors: [12], [13, The hard Lefschetz theorem], [6, Proposition 7], [7], [14, Theorem 4.7] and [1, Theorem 1.1(3)].

By construction, the following diagram commutes:

$$\begin{array}{ccc} V_{k-1, \ell-1} P & \xrightarrow[\sim]{\beta_P} & V_{C_P, B_P} \\ \downarrow \Phi_{k, \ell} & & \downarrow L_P \\ V_{k, \ell} P' & \xrightarrow[\sim]{\beta_{P'}} & V_{C_{P'}, B_{P'}-1}. \end{array}$$

Therefore, $\Phi_{k, \ell}$ is injective from $V_{k-1, \ell-1} P$ to $V_{k, \ell} P'$.

Note that by construction,

$$V_{k-1} \otimes V_{\ell-1} = \bigoplus_{P \in \Pi_{k-1, \ell-1}} V_{k-1, \ell-1} P \sim \bigoplus_{P \in \Pi_{k-1, \ell-1}} V_{C_P, B_P}.$$

Then the last sentence of the previous paragraph implies that $\Phi_{k, \ell}$ is injective on the tensor product $V_{k-1} \otimes V_{\ell-1}$.

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Stembridge codes and Chow rings

Hsin-Chieh Liao^{*1}

¹Department of Mathematics, University of Miami, USA

Abstract. It is well known that the Eulerian polynomial is the Hilbert series of the cohomology of the permutohedral variety. We answer a question of Stembridge on finding a geometric explanation of the *permutation representation* this cohomology carries. Our explanation involves an $\mathfrak{S}_n$ -equivariant bijection between a basis for the Chow ring of the Boolean matroid and codes introduced by Stembridge. There are analogous results for the stellohedral variety. We provide a geometric explanation of the permutation representation that its cohomology carries. This involves the augmented Chow ring of a matroid introduced by Braden, Huh, Matherne, Proudfoot and Wang. Along the way, we also obtain some new results on augmented Chow rings.

Keywords: Eulerian polynomial, representation of the symmetric group, permutohedron, stellohedron, augmented Chow ring of a matroid, building set

1 Introduction

Consider the $n - 1$ -dimensional permutohedron

$$\Pi_n : \text{conv}\{\sigma_1, \dots, \sigma_n \in \mathbb{R}^n : \sigma = \sigma_1\sigma_2 \dots \sigma_n \in \mathfrak{S}_n\}.$$

Its normal fan $\Sigma_n = \Sigma \Pi_n$ can be obtained from the *braid arrangement* $H_{i,j} : \{x \in \mathbb{R}^n : x_i = x_j\}$ for $1 \leq i < j \leq n$. The toric variety X_{Σ_n} associated to Σ_n is called the *permutohedral variety*. An intriguing fact about X_{Σ_n} is that the Hilbert series of its cohomology $H^* X_{\Sigma_n}$ is the Eulerian polynomial.

The cohomology $H^* X_{\Sigma_n}$ carries a representation of $\mathfrak{S}_n$ induced by the $\mathfrak{S}_n$ -action on Σ_n . Using work of Procesi [12], Stanley [14] computed its Frobenius series, which shows this representation is a permutation representation. Stembridge [15] introduced a combinatorial object called a *code* and showed that the representation of $\mathfrak{S}_n$ on the space generated by codes has the same Frobenius series as $H^* X_{\Sigma_n}$. He then asked if there is a geometric explanation of the permutation representation on $H^* X_{\Sigma_n}$ (see [15, p.317], [16, p.296, Problem 11.2]).

In this extended abstract¹, we answer Stembridge's question by identifying $H^* X_{\Sigma_n}$ with the Chow ring of the Boolean matroid and then finding a permutation basis for the

^{*}h.liao@math.miami.edu.

¹See [7] for the full paper.

induced action of $\mathfrak{S}_n$ on the Chow ring. We show that a basis of Feichtner–Yuzvinsky for general matroids serves the purpose when we apply it to Boolean matroids. We do this by constructing an $\mathfrak{S}_n$ -equivariant bijection between this basis and Stembridge codes.

There is a parallel story involving the *stellohedron* $\tilde{\Pi}_n$. The story begins with the *binomial Eulerian polynomial*, which Postnikov, Reiner, Williams [11] show is equal to the Hilbert series of the cohomology of the toric variety associated to $\tilde{\Pi}_n$. Shareshian and Wachs [13] show that the representation of $\mathfrak{S}_n$ on this cohomology is a permutation representation. We provide a geometric explanation for this, which involves the *augmented Chow ring of matroids* introduced by Braden, Huh, Matherne, Proudfoot, Wang [3], [2]. We introduce *extended codes* in order to obtain results analogous to those for the permutohedron mentioned above. Along the way, we also obtain some general results for the augmented Chow rings and the augmented Bergman fans of matroids.

2 Eulerian story: Permutohedra

2.1 The $\mathfrak{S}_n$ -module structure on $H^* X_{\Sigma_n}$

Given a d -dimensional simple polytope P with normal fan ΣP and dual polytope P^* , the h -polynomial $h_{P^*}(t)$ of P^* agrees with the Hilbert series of the cohomology $H^* X_{\Sigma P}$ of the toric variety $X_{\Sigma P}$ (see [14] eq. (26)). It is well-known that the h -polynomial $h_{\Pi_n^*}(t)$ for the dual permutohedron Π_n^* is the Eulerian polynomial $A_n(t)$, hence one has

$$A_n(t) = h_{\Pi_n^*}(t) = \sum_{j=0}^{n-1} \dim H^{2j} X_{\Sigma_n}(t)^j.$$

The cohomology $H^* X_{\Sigma_n}$ carries an $\mathfrak{S}_n$ -representation induced by the $\mathfrak{S}_n$ -action on Σ_n . Using the work of Procesi [12], Stanley [14] computed the Frobenius series of this representation:

$$\sum_{n \geq 0} \sum_{j=0}^{n-1} \text{ch} \left(H^{2j} X_{\Sigma_n} \right) t^j z^n = \frac{1 - t H(z)}{H(tz) - tH(z)}, \quad (2.1)$$

where ch is the *Frobenius characteristic*, $H(z) = \sum_{n \geq 0} h_n \mathbf{x} z^n$ and h_n is the complete homogeneous symmetric function of degree n . From (2.1), one can see that $H^* X_{\Sigma_n}$ carries a permutation representation of $\mathfrak{S}_n$.

A *Stembridge code* is a sequence α over $\{0, 1, 2, \dots\}$ with marks such that if $m(\alpha)$ is the maximum number appearing in α then for each $k \in m(\alpha)$

- k occurs at least twice in α ;
- a mark is assigned to the i th occurrence of k for a unique $i \geq 2$.

For all $k \in m \setminus \alpha$, let f_k be the number of occurrences of k in α to the left of the marked k . (So $f_k = i - 1$.) We let $\text{ind } \alpha, f$ denote the Stembridge code. The *index* of α, f is

$$\text{ind } \alpha, f : \sum_{k \in m \setminus \alpha} f_k.$$

Let $\mathcal{C}_n = \bigcup_{j=0}^{n-1} \mathcal{C}_{n,j}$ where $\mathcal{C}_{n,j}$ is the set of Stembridge codes of length n with index j .

Example 2.1. A Stembridge code $\alpha, f = 11320\hat{2}\hat{3}\hat{1}2$ consists of $\alpha = 11320231$ with $f_1 = 2, f_2 = 1, f_3 = 1$ and $\text{ind } \alpha, f = 4$. There are 6 codes in $\mathcal{C}_3$:

α, f	000	01 $\hat{1}$	10 $\hat{1}$	1 $\hat{1}$ 0	1 $\hat{1}$ 1	11 $\hat{1}$
$\text{ind } \alpha, f$	0	1	1	1	1	2

For $\sigma \in \mathfrak{S}_n$, define $\sigma \cdot \alpha_1 \alpha_2 \dots \alpha_n, f = \alpha_{\sigma^{-1}(1)} \alpha_{\sigma^{-1}(2)} \dots \alpha_{\sigma^{-1}(n)}, f$. The action induces a graded representation $V_n = \bigoplus_{j=0}^{n-1} V_{n,j}$ of $\mathfrak{S}_n$, where $V_{n,j} = \mathbb{C} \mathcal{C}_{n,j}$. Its graded Frobenius series was computed by Stembridge and it agrees with (2.1). That is

$$Q_n(x, t) : \sum_{j=0}^{n-1} \text{ch } V_{n,j} t^j = \sum_{j=0}^{n-1} \text{ch } (H^{2j} X_{\Sigma_n}) t^j. \quad (2.2)$$

Therefore $V_n \sim_{\mathfrak{S}_n} H^* X_{\Sigma_n}$. Stembridge [15, p.317] then asked for a geometric explanation of $H^* X_{\Sigma_n}$ being a permutation representation, or more explicitly in [16, p.296, Problem 11.2], whether there is a basis of $H^* X_{\Sigma_n}$ permuted by $\mathfrak{S}_n$ that induces the representation we are looking at. Although there is no obvious direct connection between Stembridge codes and $H^* X_{\Sigma_n}$, it is natural that we expect such basis to have similar combinatorial structure as Stembridge codes. We will present such a basis in Section 2.3.

2.2 Building sets and Chow rings of atomic lattices

Here we recall the background about building sets, nested set complexes, and Chow rings of the atomic lattice from [6]. For any poset P and $X \in P$, we write $P_{\leq X} = \{Y \in P : Y \leq X\}$.

Let $\mathcal{L}$ be an atomic lattice. A subset $\mathcal{G} \subseteq \mathcal{L} - \{\hat{0}\}$ is a *building set* of $\mathcal{L}$ if for any $X \in \mathcal{L} - \{\hat{0}\}$ with subset $\max \mathcal{G}_{\leq X} = \{G_1, \dots, G_k\}$, there is a poset isomorphism

$$\varphi_X : \prod_{i=1}^k \hat{0}, G_i \longrightarrow \hat{0}, X$$

with $\varphi_X(\hat{0}, \dots, \hat{0}, G_i, \hat{0}, \dots, \hat{0}) = G_i$ for $i = 1, \dots, k$.

Given a building set $\mathcal{G}$ of $\mathcal{L}$, we say a subset $N \subseteq \mathcal{G}$ is *nested* (or $\mathcal{G}$ -*nested*) if for any pairwise incomparable elements $G_1, \dots, G_t \in N$ ($t \geq 2$), their join $G_1 \vee \dots \vee G_t \notin \mathcal{G}$. Notice that the collection of all $\mathcal{G}$ -nested sets forms an abstract simplicial complex $\mathcal{N}(\mathcal{L}, \mathcal{G})$.

which is called the *nested set complex*. If $\mathcal{G}$ contains the maximal element $\hat{1}$ then $\mathcal{N}(\mathcal{L}, \mathcal{G})$ is a cone with apex $\{\hat{1}\}$, in this case the base of the cone is called the *reduced nested set complex* $\widetilde{\mathcal{N}}(\mathcal{L}, \mathcal{G})$.

The nested set complex $\mathcal{N}(\mathcal{L}, \mathcal{G})$ can be viewed as a generalization of the order complex $\Delta(\mathcal{L} - \{\hat{0}\})$. Indeed, $\mathcal{L} - \{\hat{0}\}$ is the maximal building set, $\mathcal{N}(\mathcal{L}, \mathcal{L} - \{\hat{0}\})$ is $\Delta(\mathcal{L} - \{\hat{0}\})$ and $\widetilde{\mathcal{N}}(\mathcal{L}, \mathcal{L} - \{\hat{0}\})$ is $\Delta(\mathcal{L} - \{\hat{0}, \hat{1}\})$.

Definition 2.2 ([6]). Let $\mathcal{L}$ be an atomic lattice, $\mathfrak{A}(\mathcal{L})$ be the set of atoms and $\mathcal{G}$ be a building set of $\mathcal{L}$. Then the *Chow ring of $\mathcal{L}$ with respect to $\mathcal{G}$* is the $\mathbb{Q}$ -algebra

$$D(\mathcal{L}, \mathcal{G}) := \mathbb{Q}[x_G : G \in \mathcal{G}] / I + J$$

where $I = \langle \prod_{i=1}^t G_i : \{G_1, \dots, G_t\} \notin \mathcal{N}(\mathcal{L}, \mathcal{G}) \rangle$ and $J = \langle \sum_{G \geq H} x_G : H \in \mathfrak{A}(\mathcal{L}) \rangle$.

In particular, the *Chow ring of a matroid M* as defined in [1], [3], [2] is the special case that $\mathcal{L}$ is the lattice of flats of M and $\mathcal{G}$ is the maximal building set, i.e. $D(\mathcal{L}, \mathcal{G}) = D(\mathcal{L}, M) = \mathbb{Q}[x_G : G \in \mathcal{G}] / I$ is the Stanley-Reisner ring of $\mathcal{N}(\mathcal{L}, \mathcal{G})$. For basic notions of matroid theory and Stanley-Reisner rings we refer the readers to [10] and [9] respectively.

Feichtner and Yuzvinsky [6] also found a Gröbner basis for the ideal $I + J$ which gives the following basis for $D(\mathcal{L}, \mathcal{G})$.

Proposition 2.3 ([6]). *The following monomials form a basis for $D(\mathcal{L}, \mathcal{G})$*

$$\left\{ \prod_{G \in N} x_G^{a_G} : N \text{ is nested, } a_G < d_{G', G} \right\}$$

where G' is the join of $N \cap \mathcal{L}_{<G}$ and $d_{G', G}$ is the minimal number d such that $H_1, \dots, H_d \in \mathfrak{A}(\mathcal{L})$ satisfies $G' \vee H_1 \vee \dots \vee H_d = G$. In particular, if $\mathcal{L}$ is a geometric lattice, then $d_{G', G} = \text{rk } G - \text{rk } G'$.

2.3 Stembridge codes and FY-basis for Chow ring of B_n

In this section, we shall answer Stembridge's question. We identify $H^*(X_{\Sigma_n})$ with the Chow ring of the Boolean matroid, and use the facts in Section 2.2 to find a basis permuted by the $\mathfrak{S}_n$ -action.

The following lemma from [16, p.251(1.5)] allows us to identify $H^*(X_{\Sigma_n})$ with $\mathbb{Q}$ quotient of Stanley-Reisner ring of the boundary $\partial\Pi_n^*$ of the dual permutohedron.

Lemma 2.4 ([4], [16]). *Let P be a simple n -lattice polytope in an Euclidean n -space V with normal fan $\Sigma(P)$ or equivalently the face fan of P^* . Let $K[\partial P^*]$ be the Stanley-Reisner ring of ∂P^* over a field K of characteristic 0. Let $\theta_i = \sum_{v \in V(P^*)} \langle v, e_i \rangle x_v$ for $i = 1, \dots, n$, where $\{e_i\}$ is the standard basis in $\mathbb{Z}^n \subset V$, and $V(P^*)$ is the set of vertices of P^* . If a finite group G acts on $\Sigma(P)$ simplicially and freely, then*

$$H^*(X_{\Sigma(P)}, K) \sim K[\partial P^*] / \langle \theta_1, \dots, \theta_n \rangle \text{ as } KG\text{-modules.}$$

Let M be a loopless matroid on ground set n with a lattice of flats $\mathcal{L} M$. For each $S \subseteq n$, write $e_S = \sum_{i \in S} e_i$. The *Bergman fan* Σ_M of M is a fan in $\mathbb{R}^n / \langle e_n \rangle$ consists of cones $\sigma_{\mathcal{F}}$ indexed by all flags $\mathcal{F} = \{F_1 \subsetneq \dots \subsetneq F_k\}$ in $\mathcal{L} M - \{\emptyset, n\}$, where

$$\sigma_{\mathcal{F}} = \mathbb{R}_{\geq 0} \{e_{F_1}, \dots, e_{F_k}\}.$$

The *Bergman complex* is the simplicial complex obtained by intersecting Σ_M with the unit sphere centered at 0. Obviously, the Bergman complex is a geometric realization of the order complex $\Delta \mathcal{L} M - \{\emptyset, n\}$.

The *Chow ring* of M has the following two presentations

$$A M : \frac{\mathbb{Q} x_F \mid F \in \mathcal{L} M - \{\emptyset\} / \langle x_F x_G : F, G \text{ are incomparable in } \mathcal{L} M \rangle}{\langle \sum_{F: i \in F} x_F : 1 \leq i \leq n \rangle} \quad (2.3)$$

$$\frac{\mathbb{Q} x_F \mid F \in \mathcal{L} M - \{\emptyset, n\} / \langle x_F x_G : F, G \text{ are incomparable in } \mathcal{L} M \rangle}{\langle \sum_{F: i \in F} x_F - \sum_{F: j \in F} x_F : i \neq j \rangle}. \quad (2.4)$$

The presentation (2.3) is a special case of [Definition 2.2](#), and (2.4) is obtained from (2.3) by eliminating x_E . The presentation (2.4) is used in [\[1\]](#), [\[3\]](#), [\[2\]](#). Note that the numerator in (2.4) is the Stanley-Reisner ring of Σ_M , or equivalently the Bergman complex of M .

Now let M be the Boolean matroid B_n ; the flats of B_n are all subsets of n and $\mathcal{L} B_n$ is the Boolean lattice. It is well-known that the order complex $\Delta \mathcal{L} B_n - \{\emptyset, n\}$ is $\partial \Pi_n^*$. After some easy calculations, applying [Lemma 2.4](#) to (2.4) shows that

$$A B_n = \frac{\mathbb{Q} \partial \Pi_n^*}{\langle \theta_1, \dots, \theta_{n-1} \rangle} \sim_{\mathfrak{S}_n} H^* X_{\Sigma_n}, \mathbb{Q}.$$

By applying [Proposition 2.3](#) to (2.3), the Feichtner–Yuzvinsky basis of $A B_n$ is given by

$$FY B_n : \left\{ x_{F_1}^{a_1} x_{F_2}^{a_2} \dots x_{F_k}^{a_k} : \begin{array}{l} \emptyset \subsetneq F_1 \subsetneq F_2 \subsetneq \dots \subsetneq F_k \subseteq n, \\ 1 \leq a_i \leq |F_i| - |F_{i-1}| - 1 \end{array} \right\}.$$

Note that $|F_i| - |F_{i-1}| \geq 2$ for all i . We see that $\mathfrak{S}_n$ permutes $FY B_n$ and makes $A B_n$ an $\mathfrak{S}_n$ -module. It turns out $FY B_n$ has similar structure as the codes $\mathcal{C}_n$.

Theorem 2.5. *There is a bijection $\phi : FY B_n \rightarrow \mathcal{C}_n$ that respects the $\mathfrak{S}_n$ -actions and takes the degree of the monomials to the index of the corresponding codes.*

The bijection is defined as follows. Given $u = x_{F_1}^{a_1} \dots x_{F_k}^{a_k} \in FY B_n$, let $\phi u = \alpha, f$, where $\alpha_i = \begin{cases} j & \text{if } i \in F_j - F_{j-1} \text{ for } i \in n \text{ and } f_j = a_j \text{ for } j \in k. \\ 0 & \text{if } i \in n - F_k \end{cases}$

Example 2.6. The basis for $A^2 B_4$ and the corresponding codes in $\mathcal{C}_{4,2}$:

$$\begin{array}{cccccccccccc}
 x_{12}x_{1234} & x_{13}x_{1234} & x_{14}x_{1234} & x_{23}x_{1234} & x_{24}x_{1234} & x_{34}x_{1234} & x_{123}^2 & x_{124}^2 & x_{134}^2 & x_{234}^2 & x_{1234}^2 \\
 \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\
 1\hat{1}2\hat{2} & 12\hat{1}\hat{2} & 12\hat{2}\hat{1} & 21\hat{1}\hat{2} & 21\hat{2}\hat{1} & 2\hat{2}1\hat{1} & 11\hat{1}0 & 110\hat{1} & 101\hat{1} & 011\hat{1} & 11\hat{1}1
 \end{array}$$

It is easy to see that the bijection respects the $\mathfrak{S}_4$ -action on both sets.

Theorem 2.5 shows that $A^j B_n \otimes \mathbb{C}$ and $V_{n,j}$ are isomorphic permutation representation for all $0 \leq j \leq n-1$. To our knowledge, this is the first found basis that bears an explicit resemblance to codes.

3 Binomial Eulerian story: stellohedra

We will now switch track to a parallel story. The *binomial Eulerian polynomial* $\tilde{A}_n(t) = \sum_{k=1}^n \tilde{A}_k t^k$ shares many similar combinatorial and geometric properties with the Eulerian polynomial (see Shareshian and Wachs [13]). Geometrically, it is shown by Postnikov, Reiner, Williams [11] that $\tilde{A}_n(t)$ is the h -polynomial of the dual of the *stellohedron* $\tilde{\Pi}_n$. Let $\Delta_n = \text{conv}\{0, e_i : i \in [n]\}$ where e_i is the standard basis vector in $\mathbb{R}^n$. The *stellohedron* $\tilde{\Pi}_n$ is the simple polytope obtained from Δ_n by truncating all faces not containing 0 in the inclusion order. Let $\tilde{\Sigma}_n$ be its normal fan. The Hilbert series of $H^* X_{\tilde{\Sigma}_n}$ satisfies

$$\tilde{A}_n(t) = h_{\tilde{\Pi}_n^*}(t) = \sum_{j \geq 0} \dim H^{2j} X_{\tilde{\Sigma}_n} t^j.$$

Shareshian and Wachs [13] introduced the symmetric function analogue of $\tilde{A}_n(t)$,

$$\tilde{Q}_n(\mathbf{x}, t) = h_n(\mathbf{x}) + t \sum_{k=1}^n h_{n-k}(\mathbf{x}) Q_k(\mathbf{x}, t) \quad (3.1)$$

and showed that if we consider the simplicial action of $\mathfrak{S}_n$ on $\tilde{\Pi}_n^*$, the induced representation of $\mathfrak{S}_n$ on $H^* X_{\tilde{\Sigma}_n}$ has the following graded Frobenius series.

Theorem 3.1 ([13]). *For all $n \geq 1$, we have $\sum_{j=0}^n \text{ch}(H^{2j} X_{\tilde{\Sigma}_n}) t^j = \tilde{Q}_n(\mathbf{x}, t)$.*

From (3.1), since $Q_k(\mathbf{x}, t)$ is h -positive, so is $\tilde{Q}_n(\mathbf{x}, t)$. Thus $H^* X_{\tilde{\Sigma}_n}$ also carries a permutation representation of $\mathfrak{S}_n$.

3.1 Extended codes

We introduce an analog of Stembridge codes called the *extended codes*.

An *extended code* α, f is a marked sequence like a Stembridge code. The sequence α is over $\{0, 1, \dots\} \cup \{\infty\}$ with ∞ 's working as 0's in codes; $m \leq \alpha$ and f are defined as in Stembridge codes. Define the *index* of the extended code α, f as

$$\text{ind } \alpha, f = \begin{cases} -1 & \text{if } \alpha = \infty \dots \infty; \\ \sum_{k \in m \leq \alpha} f(k) & \text{otherwise.} \end{cases}$$

Let $\tilde{\mathcal{C}}_{n,j}$ be the set of extended codes of length n with index j and $\tilde{\mathcal{C}}_n = \bigcup_{j=-1}^{n-1} \tilde{\mathcal{C}}_{n,j}$.

Example 3.2. The following are all the extended codes of length 3:

$$\begin{aligned} \tilde{\mathcal{C}}_{3,-1} &= \{\infty\infty\infty\}, \quad \tilde{\mathcal{C}}_{3,0} = \{0\infty\infty, \infty 0\infty, \infty\infty 0, \infty 00, 0\infty 0, 00\infty, 000\}, \\ \tilde{\mathcal{C}}_{3,1} &= \{1\hat{1}\infty, 1\infty\hat{1}, \infty 1\hat{1}, 01\hat{1}, 10\hat{1}, 1\hat{1}0, 1\hat{1}1\}, \quad \tilde{\mathcal{C}}_{3,2} = \{11\hat{1}\}. \end{aligned}$$

For $\sigma \in \mathfrak{S}_n$, define $\sigma \cdot \alpha_1 \alpha_2 \dots \alpha_n, f = \alpha_{\sigma(1)} \alpha_{\sigma(2)} \dots \alpha_{\sigma(n)}, f$. This induces an graded $\mathfrak{S}_n$ -representation on $\tilde{V}_n = \bigoplus_{j=0}^n \tilde{V}_{n,j-1}$ where $\tilde{V}_{n,j-1} = \mathbb{C} \tilde{\mathcal{C}}_{n,j-1}$. We compute its Frobenius series and obtain a result parallel to Stembridge codes.

Theorem 3.3. For $n \geq 1$, we have $\sum_{j=0}^n \text{ch } \tilde{V}_{n,j} t^j = \tilde{Q}_n(x, t)$.

Combining with [Theorem 3.1](#), we see $\tilde{V}_n \sim_{\mathfrak{S}_n} H^* X_{\tilde{\Sigma}_n}$ as permutation modules. One can also ask if there is a permutation basis for $H^* X_{\tilde{\Sigma}_n}$ which has similar combinatorial structure as extended codes. We will show such a basis exists in what follows.

3.2 Augmented Chow ring of B_n and $H^* X_{\tilde{\Sigma}_n}$

Braden, Huh, Matherne, Proudfoot, and Wang [\[3\]](#) recently introduced the *augmented Bergman fan* and the *augmented Chow ring* of a matroid. They showed that the augmented Bergman fan of B_n is the normal fan $\tilde{\Sigma}_n$ of the stellohedron $\tilde{\Pi}_n$. Therefore the corresponding spherical complex is the boundary complex $\partial \tilde{\Pi}_n^*$. Below we use [Lemma 2.4](#) to identify $H^* X_{\tilde{\Sigma}_n}$ with the augmented Chow ring of B_n .

Let M be a loopless matroid on n with lattice of flats $\mathcal{L}(M)$ and the collection of independent subsets $\mathcal{I}(M)$. The *augmented Chow ring* of M is defined as

$$\tilde{A}(M) := \frac{\mathbb{Q} \left[\{x_F\}_{F \in \mathcal{L}(M) \setminus n} \cup \{y_1, y_2, \dots, y_n\} \right] / I_1 \quad I_2}{\langle y_i - \sum_{F: i \notin F} x_F \rangle_{i=1,2,\dots,n}} \quad (3.2)$$

where $I_1 = \langle x_F x_G : F, G \text{ are incomparable in } \mathcal{L}(M) \rangle$, $I_2 = \langle y_i x_F : i \notin F \rangle$. There is a simplicial fan associated with $\tilde{A}(M)$ called the *augmented Bergman fan* of M .

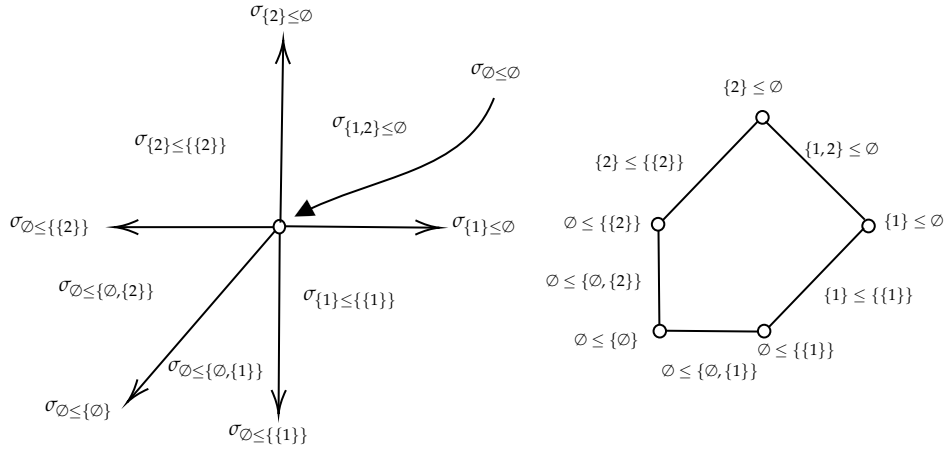
Let $I \in \mathcal{I}(M)$ and $\mathcal{F} = F_1 \subsetneq \dots \subsetneq F_k$ be a flag in $\mathcal{L}(M)$. We say I is *compatible with*

$\mathcal{F}$, denoted by $I \leq \mathcal{F}$, if $I \subseteq F_1$. Note that for the empty flag $\emptyset$, we have $I \leq \emptyset$ for any $I \in \mathcal{I} M$. The *augmented Bergman fan* $\tilde{\Sigma}_M$ of M is a simplicial fan in $\mathbb{R}^n$ consisting of cones $\sigma_{I \leq \mathcal{F}}$ indexed by compatible pairs $I \leq \mathcal{F}$, where $\mathcal{F}$ is a flag in $\mathcal{L} M - \{n\}$ and

$$\sigma_{I \leq \mathcal{F}} = \mathbb{R}_{\geq 0} \left(\{e_i\}_{i \in I} \cup \{-e_n \setminus F\}_{F \in \mathcal{F}} \right).$$

The corresponding simplicial complex is called the *augmented Bergman complex*.

Example 3.4. The augmented Bergman fan $\tilde{\Sigma}_{B_2}$ of the Boolean matroid B_2 in $\mathbb{R}^2$ and the corresponding augmented Bergman complex:



One can show $\tilde{\Sigma}_{B_2}$ is the normal fan of $\tilde{\Pi}_2$. Thus the augmented Bergman complex is $\partial \tilde{\Pi}_2^*$.

Note that the numerator in (3.2) is the Stanley-Reisner ring of $\tilde{\Sigma}_M$, since for each $i \in [n]$, $F \in \mathcal{L} M - \{n\}$ the rays $\sigma_{\{i\} \leq \emptyset}$ corresponds to y_i , and $\sigma_{\emptyset \leq \{F\}}$ corresponds to x_F .

Now consider the case of B_n , the augmented Bergman fan $\tilde{\Sigma}_{B_n}$ is $\tilde{\Sigma}_n$. Some easy calculations show that all conditions in Lemma 2.4 hold. Therefore we have

$$\tilde{A}(B_n) = \frac{\mathbb{Q}[\partial \tilde{\Pi}_n^*]}{\langle \theta_1, \dots, \theta_n \rangle} \sim_{\mathfrak{S}_n} H^*(X_{\tilde{\Sigma}_n}, \mathbb{Q}).$$

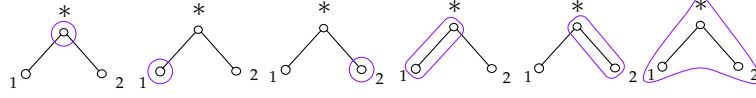
However, there was no analogue of Feichtner–Yuzvinsky’s result known for the augmented Chow ring. In order to overcome this, we turn our focus on another way of constructing the stellohedron $\tilde{\Pi}_n$ — as the graph associahedron of n -star graph $K_{1,n}$.

3.3 Stellohedron as the graph associahedron of $K_{1,n}$

Let $G = (V, E)$ be a simple graph. The *graphical building set* $\mathcal{B}(G)$ is the set of nonempty subsets I of V such that the induced subgraph of G on I is connected. In fact, $\mathcal{B}(G)$ is a building set of the Boolean lattice over V .

The n -star graph $K_{1,n}$ has vertex set $V = [n] \cup \{*\}$ and edge set $E = \{\{i, *\} : i \in [n]\}$.

Example 3.5. $\mathcal{B} K_{1,2}$ consists of the following elements:

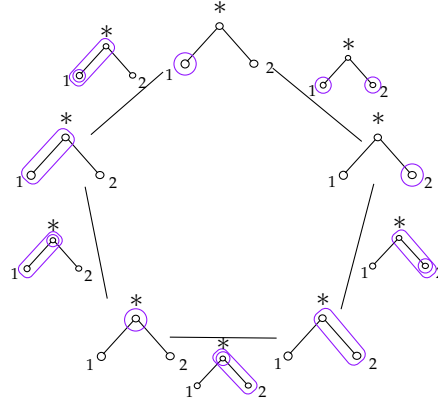


For graphical building sets $\mathcal{B} G$, the nested sets $N \subset \mathcal{B} G$ can be characterized by:

- (1) for all $I, J \in N$, either $I \subset J$, $I \supset J$ or $I \cap J = \emptyset$.
- (2) for all $I, J \in N$ if $I \cap J = \emptyset$, then $I \cup J \notin \mathcal{B} G$ (not “connected”).

From [11, Theorem 6.5], $\widetilde{\mathcal{N}} \mathcal{L} B_{n-1}, \mathcal{B} K_{1,n}$ is combinatorially equivalent to $\partial \widetilde{\Pi}_n^*$.

Example 3.6. The reduced nested set complex with respect to $\mathcal{B} K_{1,2}$ is $\partial \widetilde{\Pi}_2^*$:

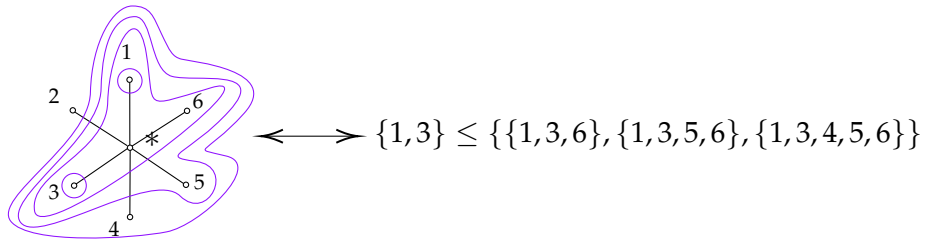


Comparing Example 3.6 and Example 3.4 leads us to a combinatorial proof of the following result².

Proposition 3.7. *The augmented Bergman fan (complex) of B_n is combinatorially equivalent to the dual stellohedron $\widetilde{\Pi}_n^*$. Consequently, there is a poset isomorphism between their face lattices.*

The isomorphism is as follows. Each nested set represents a face of $\partial \widetilde{\Pi}_n^*$, and it corresponds to a compatible pair $I \leq \mathcal{F}$ which represents a cone in $\widetilde{\Sigma}_{B_n}$.

Example 3.8. For $n = 6$, the following is a nested set with respect to $\mathcal{B} K_{1,6}$ and the corresponding compatible pair.



²This is a weaker result than [3, Proposition 2.6] which states that the augmented Bergman fan of B_n is the normal fan of the stellohedron.

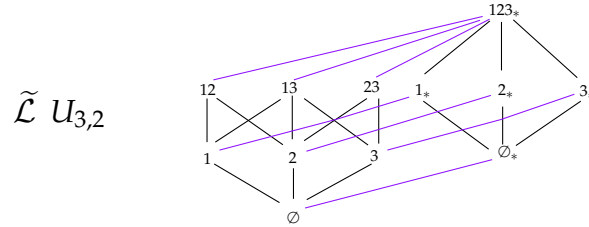
We show in Section 3.4 that the isomorphism in Proposition 3.7 holds even when the lattice is not a Boolean lattice.

3.4 Face Structure of augmented Bergman fan of M

Let $M, \mathcal{L}_M, \mathcal{I}_M$ be defined as in Section 3.2. The collection $\mathcal{I}_M$ forms an abstract simplicial complex called the *independence complex*; here we identify $\mathcal{I}_M$ with the face lattice of the independence complex. We construct a new poset $\tilde{\mathcal{L}}_M$ from $\mathcal{L}_M$ and $\mathcal{I}_M$ in the following ways:

- As a set, $\tilde{\mathcal{L}}_M = \mathcal{L}_M \uplus \mathcal{I}_M$. Denote by F_* the flat F in $\tilde{\mathcal{L}}_M$.
- For $I \in \mathcal{I}_M$, define $I < \text{cl}_M I_*$ where $\text{cl}_M I$ is the closure of I in M . The relations inside $\mathcal{L}_M$ and $\mathcal{I}_M$ stay the same.

Example 3.9. Consider the uniform matroid $U_{3,2}$, then the new poset is



Lemma 3.10. Take $\mathcal{G} = \{\{1\}, \dots, \{n\}\} \cup \{F_*\}_{F \in \mathcal{L}_M}$ as the building set in $\tilde{\mathcal{L}}_M$. Then the nested sets are of the form $\{\{i\}\}_{i \in I} \cup \{F_*\}_{F \in \mathcal{F}}$ for some compatible pair $I \leq \mathcal{F}$ where $I \in \mathcal{I}_M$ and $\mathcal{F}$ is a flag of arbitrary flats.

With this lemma, we could recover $\tilde{\Sigma}_M$ as the reduced nested set complex with respect to the building set $\mathcal{G}$.

Theorem 3.11. $\tilde{\mathcal{N}}_{\tilde{\mathcal{L}}_M, \mathcal{G}}$ is combinatorially equivalent to the augmented Bergman fan (complex) of M . Consequently, there is a poset isomorphism between their face lattices:

$$\{\{i\}\}_{i \in I} \cup \{F_*\}_{F \in \mathcal{F}} \longleftrightarrow \sigma_{I \leq \mathcal{F}}$$

for compatible pair $I \leq \mathcal{F}$ where $I \in \mathcal{I}_M$ and flag $\mathcal{F} \subset \mathcal{L}_M - \{n\}$.

Furthermore, if we consider the Chow ring $D_{\tilde{\mathcal{L}}_M, \mathcal{G}}$, then

$$D_{\tilde{\mathcal{L}}_M, \mathcal{G}} = \frac{\mathbb{Q} \left[\{x_F\}_{F \in \mathcal{F}_M} \cup \{y_i\}_{i \in n} \right] / I_1 \quad I_2}{\langle y_i - \sum_{F: i \in F} x_F \rangle_{i \in n} \quad \langle \sum_{F: \emptyset \subset F} x_F \rangle} \\ \frac{\mathbb{Q} \left[\{x_F\}_{F \in \mathcal{F}_M \setminus n} \cup \{y_i\}_{i \in n} \right] / I_1 \quad I_2}{\langle y_i - \sum_{F: i \notin F} x_F \rangle_{i \in n}} \\ \tilde{A}_M$$

Therefore, the augmented Chow ring of M can be viewed as the Chow ring of $\tilde{\mathcal{L}} M$ with respect to $\mathcal{G}$. We can apply [Proposition 2.3](#) to obtain a basis for $\tilde{A} M$.

Corollary 3.12. *The augmented Chow ring $\tilde{A} M$ of M has the following basis*

$$\tilde{FY} M : \left\{ x_{F_1}^{a_1} x_{F_2}^{a_2} \cdots x_{F_k}^{a_k} : \begin{array}{l} \emptyset \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k \\ 1 \leq a_1 \leq \text{rk } F_1, a_i \leq \text{rk } F_i - \text{rk } F_{i-1} - 1 \text{ for } i \geq 2 \end{array} \right\}$$

Remark 3.13. After this work is done, we know from [8, Section 5.1] the fact that $\tilde{A} M$ can be expressed as $D \tilde{\mathcal{L}} M, \mathcal{G}$ has also been discovered independently by Chris Eur, who further noticed that $\tilde{\mathcal{L}} M$ is the lattice of flats of the *free coextension* $M^* \leftarrow e^*$ of M . This fact and the results in this section were later included in Eur, Huh, and Larson's paper [5, Lemma 5.14, Section 7.2].

3.5 Back to Boolean matroids

From [Corollary 3.12](#), the basis $\tilde{FY} B_n$ consists of monomials $x_{F_1}^{a_1} \cdots x_{F_k}^{a_k}$ indexed by a chain in the Boolean lattice with exponent a_i satisfying

- $|F_i| - |F_{i-1}| \geq 2$ for all $i \geq 2$.
- $1 \leq a_1 \leq |F_1|$ and $a_i \leq |F_i| - |F_{i-1}| - 1$ for all $i \geq 2$.

Example 3.14. $\tilde{FY} B_3 = \{1 \mid x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123} \mid x_1 x_{123}, x_2 x_{123}, x_3 x_{123}, x_{12}^2, x_{13}^2, x_{23}^2, x_{123}^2 \mid x_{123}^3\}$.

Similar to $FY B_n$, $\mathfrak{S}_n$ also acts on $\tilde{FY} B_n$ naturally and makes $\tilde{A} B_n$ a permutation module. The following result is analogous to [Theorem 2.5](#).

Theorem 3.15. *There is a bijection $\tilde{\phi} : \tilde{FY} B_n \rightarrow \tilde{\mathcal{C}}_n$ that respects the $\mathfrak{S}_n$ -actions and takes the degree of the monomials to the index -1 of the corresponding extended codes.*

The bijection is defined as follows. Given $u = x_{F_1}^{a_1} \cdots x_{F_k}^{a_k} \in \tilde{FY} B_n$, let $\tilde{\phi} u = \alpha, f$, where if $a_1 = 1$, then $\alpha_i = \begin{cases} j-1 & \text{if } i \in F_j - F_{j-1} \text{ for } i \in n, \text{ and } f(j) = a_j - 1 \text{ for } j \in k; \\ \infty & \text{if } i \in n - F_k \end{cases}$ else if $a_1 \geq 2$, then $\alpha_i = \begin{cases} j & \text{if } i \in F_j - F_{j-1} \text{ for } i \in n, \text{ and } f(1) = a_1 - 1, f(j) = a_j \text{ for } 2 \leq j \leq k. \\ \infty & \text{if } i \in n - F_k \end{cases}$

Example 3.16. Let $u_1 = x_{14} x_{1247} x_{1245679}^2$, $u_2 = x_{14}^2 x_{1247} x_{1245679}^2 \in \tilde{FY} B_9$, then $\tilde{\phi} u_1 = 01\infty 022\hat{1}\infty\hat{2}$ and $\tilde{\phi} u_2 = 12\infty\hat{1}33\hat{2}\infty\hat{3}$.

[Theorem 3.15](#) shows that $\tilde{A}^j B_n \otimes \mathbb{C}$ and $\tilde{V}_{n,j-1}$ are isomorphic permutation representations of $\mathfrak{S}_n$. This answers our question posed in the end of [Section 3.1](#). This also gives a new proof of Shareshian and Wachs' result ([Theorem 3.1](#)).

Acknowledgements

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Interpolating between classic and bumpless pipe dreams

Allen Knutson^{*1} and Gabe Udell^{†1}

¹*Department of Mathematics, Cornell University*

Abstract. We introduce *hybrid pipe dreams*, in which some rows are specified to contain classic pipe dream tiles (plus a couple new ones) and some to contain bumpless pipe dream tiles (albeit upside down). For each of the 2^n possibilities of row specification, there is a sum over hybrid pipe dreams to compute each double Schubert polynomial. We give a bijection to swap the specifications on two adjacent rows, demonstrating the equivalence of the many formulæ.

Keywords: Schubert polynomials, pipe dreams

1 Hybrid pipe dreams compute Schubert polynomials

1.1 Hybrid pipe dreams and the formula

Define the **classic pipe dream tiles** to be



the **(upside down) bumpless pipe dream tiles** to be



and in each case call the shaded tile(s) **weighty**.

Given $\pi \in S_n$ a permutation, and $\tau \in \{C, B\}^n$ a word in **Classic** and **Bumpless**, define a **τ -hybrid pipe dream for π** to be an $n \times n$ square made of tiles, such that

- if two tiles share an edge, they must agree as to whether it is blank or not. Hence the pipes end only on the boundary of the square
- the i th row is called “a classic row” or “a bumpless row” depending on τ_i , and uses only tiles from one set or the other
- each classic row has a number on the left edge, and is blank on the right edge, with the opposite being true for bumpless rows; this number is the **row label**

^{*}allenk@math.cornell.edu. AK was partially supported by NSF Grant DMS-1953948.

[†]gru5@cornell.edu.

- the top edges are labeled with $\pi^{-1} 1, \pi^{-1} 2, \dots, \pi^{-1} n$ from left to right
- the bottom edges are blank
- the labeled boundary edges on the left and right are labeled $1 \dots n$ in order, counterclockwise starting from the Northwest corner
- the numbers at the two ends of a pipe agree, and no two pipes cross twice.

Some examples are below. Our first main theorem is the following:

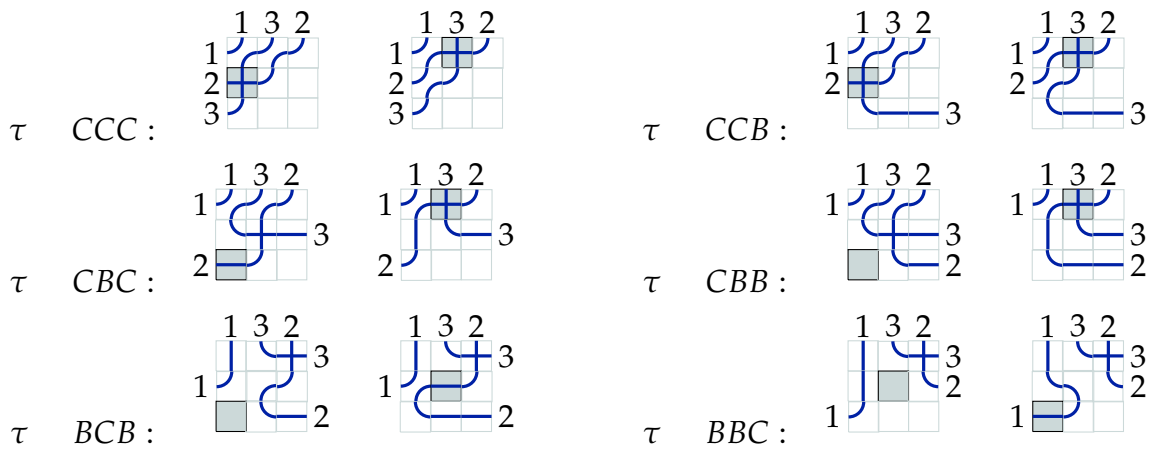
Theorem 1. For $\pi \in S_n$ a permutation, and any word $\tau \in \{C, B\}^n$, the double Schubert polynomial $S_\pi x_1, \dots, x_n, y_1, \dots, y_n$ can be computed as a sum over the τ -hybrid pipe dreams δ for π :

$$S_\pi = \sum_{\delta} \prod \{x_i - y_j : i, j \in [n] \text{ and the tile in physical column } j \text{ and row labeled } i \text{ is weighty}\}$$

In the case $\tau = C^n$, this reproduces the pipe dream formula in [3, Prop. 6.2] (and [1] in the $y_i \equiv 0$ setting). In the case $\tau = B^n$, this reproduces the bumpless pipe dream formula in [7] (albeit upside down) and is equivalent to the alternating sign matrix formula from [9, 8]. The pipes are labeled differently here (by π^{-1}) than they are in [1, 3, 6, 7], but the placements of tiles are exactly the same.

The number of terms in this sum is $S_\pi|_{x_i \equiv 1, y_j \equiv 0}$, independent of τ . Our other main theorem (in Section 2), which implies this one, will be a bijective correspondence.

Here are the hybrid pipe dreams for $\pi = 132$ and various choices of τ .



1.2 The all-classic case $\tau = C^n$

The boundary conditions mean that every edge on North and West is crossed by a pipe, whereas every edge on South and East is blank.

Proposition 1. A τ -hybrid pipe dream for $\tau \in C^n$ has

- only j-tiles  on the antidiagonal, and only blank tiles  below the antidiagonal;
- only crosses  and elbows  above the antidiagonal.

In particular the  and r-tile , despite being allowed, are never used.

Proof. Consider the rightmost column, from top to bottom. The Northeast tile has a number on its North edge and blank on its East edge, and cannot be a -tile (as those are forbidden in classic rows), so must be a -tile. The next tile below it is blank on North and East, so must be , and the same argument holds all the way down the rightmost column.

Now consider the first row, from right to left, starting just West of the Northeast tile. The next tile has pipes crossing its North and East side, so must be a  or an . The same argument holds all the way to the left in the first row.

Remove the last column and first row and do induction on the remaining $(n-1) \times (n-1)$ square. $\square$

This picture is slightly different *mathematically* from the classic pipe dreams in, say, [1] which fill the fourth quadrant and have only crossing and elbow tiles. That formulation is based on the forward-stability of Schubert polynomials, $S_\pi = S_{\pi \oplus I_m}$ (where the $\oplus$ is on the permutation matrices). However, even in [1] they omit “the ‘sea’ of wavy strands” and obtain as pictures *exactly* the C^n -hybrid pipe dreams here.

1.3 The all-bumpless case $\tau \in B^n$

There is less to say here – our definition of B^n -hybrid pipe dreams exactly matches that of [7], except for being upside down and for the shift in labeling (they have $\pi(1) \dots \pi(n)$ along the East side, where we have $n \dots 1$).

1.4 Repeating letters

We inverted the labeling principally to simplify the description of the labels on the boundary (as $1 \dots n$ counterclockwise). There is another benefit (though not one we will pursue here); in the case that π has descents in at most the positions $k_1, k_1 - k_2, \dots, k_1 - k_2 - k_3 \dots - k_d$ it is natural to write π^{-1} as a **string** with content $1^{k_1} 2^{k_2} \dots d^{k_d} d^{1 - n - \sum k_i}$. Put another way, π is the shortest representative inside a coset $S_n / \prod_i S_{k_i}$ (where $k_{d+1} : n - \sum_{i=1}^d k_i$), which is an equivalence class within which the *positions* are ambiguous, and its inverse has ambiguous *values*. If one allows in this way multiple pipes to have the same label, one must forbid those pipes from crossing even *once*.

Placing the ambiguity on values is more natural in these scattering-diagram-inspired contexts, in that physics is comfortable with indistinguishable particles but not with ambiguous positions.

2 Equivalence of the 2^n formulæ

In this section we give combinatorial proofs of the equivalence of the 2^n formulæ. For this purpose, we introduce **decorated hybrid pipe dreams**, as hybrid pipe dreams with decorations of x or $-y$ on the weighty tiles. Following the conventions in [5] (which introduced this notion in the bumpless case), we write the set of decorated τ -hybrid pipe dreams for π as

$$\widetilde{\text{HPD}} \tau, \pi : \{ D, f \mid D \in \text{HPD} \tau, \pi, f : \text{weighty } D \rightarrow \{x, -y\} \}$$

where $\text{HPD} \tau, \pi$ is the set of τ -hybrid pipe dreams for π , and $\text{weighty } D$ is the set of weighty tiles in D . Define the **monomial of a decorated hybrid pipe dream** to be

$$\text{mon } D, f : \prod_{\substack{i,j \in \text{weighty } D \\ f(i,j) = x}} x_i \prod_{\substack{i,j \in \text{weighty } D \\ f(i,j) = -y}} -y_j, \quad \text{so } S_\pi \sum_{D, f \in \widetilde{\text{HPD}} \tau, \pi} \text{mon } D, f$$

as follows from [Theorem 1](#).

We prove the equivalence of these 2^n formulæ by constructing a bijection

$$\phi : \widetilde{\text{HPD}} \tau_1, \pi \rightarrow \widetilde{\text{HPD}} \tau_2, \pi$$

for any $\tau_1, \tau_2 \in \{C, B\}^n$ while making sure that $\text{mon } \phi(D, f) = \phi(\text{mon } D, f)$ for all $D, f \in \widetilde{\text{HPD}} \tau_1, \pi$. We will do this in two steps (each then iterated many times). First, we handle the case where τ, τ' differ only in the last letter. That is, we describe how to change the last row from classic to bumpless, and in fact, we do this in such a way that $\phi(D, f)$ differs from D, f only in its final row. Second, we handle the case where τ' is the same as τ but with some (consecutive) substring CB replaced by BC . That is, we describe how to switch the order of two adjacent rows, one bumpless and one classic. We do this in such a way that $\phi(D, f)$ differs from D, f only in the rows that were swapped. The composite bijections ϕ are automatically coherent in the sense that any triangle $\widetilde{\text{HPD}} \tau_1, \pi \rightarrow \widetilde{\text{HPD}} \tau_2, \pi \rightarrow \widetilde{\text{HPD}} \tau_3, \pi$ commutes.

2.1 Monk's rule and the Gao–Huang bijection

Hybrid pipe dreams were created to help understand the connection between pipe dreams and bumpless pipe dreams, but they are not the first tool for doing so. Another approach to this connection involves Monk's rule.

Theorem 2 (Monk's rule for double Schubert polynomials). *For π a permutation, α a positive integer, $t_{a,b}$ the transposition of a and b , and $\succ$ the Bruhat covering relation,*

$$x_\alpha - y_{\pi \alpha} S_\pi = \sum_{\substack{s < \alpha \\ \pi t_{s,\alpha} \succ \pi}} S_{\pi t_{s,\alpha}} - \sum_{\substack{l > \alpha \\ \pi t_{\alpha,l} \succ \pi}} S_{\pi t_{\alpha,l}}.$$

Monk's rule for single Schubert polynomials can be proven bijectively using either pipe dreams [1, 2, 5] or bumpless pipe dreams [5] by using, for either $\tau \in C^n$ or $\tau \in B^n$, maps $x_\alpha \rightsquigarrow: \text{HPD } \tau, \pi \rightarrow \bigcup_{\substack{\alpha < l \\ \pi t_{\alpha,l} \succ \pi}} \text{HPD } \tau, \pi t_{\alpha,l}$ and $m_{s,\beta} : \text{HPD } \tau, \pi t_{s,\beta} \rightarrow \bigcup_{\substack{l > \beta \\ \pi t_{\beta,l} \succ \pi}} \text{HPD } \tau, \pi t_{\beta,l}$ when $s < \beta, \pi t_{s,\beta} \succ \pi$. In [4], Gao and Huang constructed a bijection ϕ_{GH} between bumpless and classic pipe dreams preserving mon D, f when all decorations are x . Furthermore, they proved their bijection to be canonical in the sense that it commutes with $x_\alpha \rightsquigarrow$ and $m_{s,\beta}$. When $\tau_1 \in B^n, \tau_2 \in C^n$ and either all decorations being x or all decorations being $-y$, we get specializations ϕ_x, ϕ_y of the hybrid pipe dream bijection. We conjecture that $\phi_{GH} D = \phi_x D = \phi_y D^T$ for any bumpless pipe dream D , where D^T denotes the transpose of a pipe dream or bumpless pipe dream.

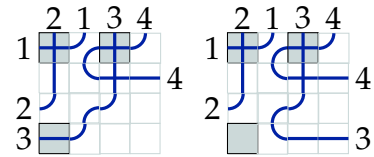
2.2 Changing the bottom row

A single picture shows the full gamut of possibilities for the bottom rows, in the C and B cases:



Note that this correspondence preserves the positions of the weighty tiles, and hence the associated monomials (assuming we keep the same decoration).

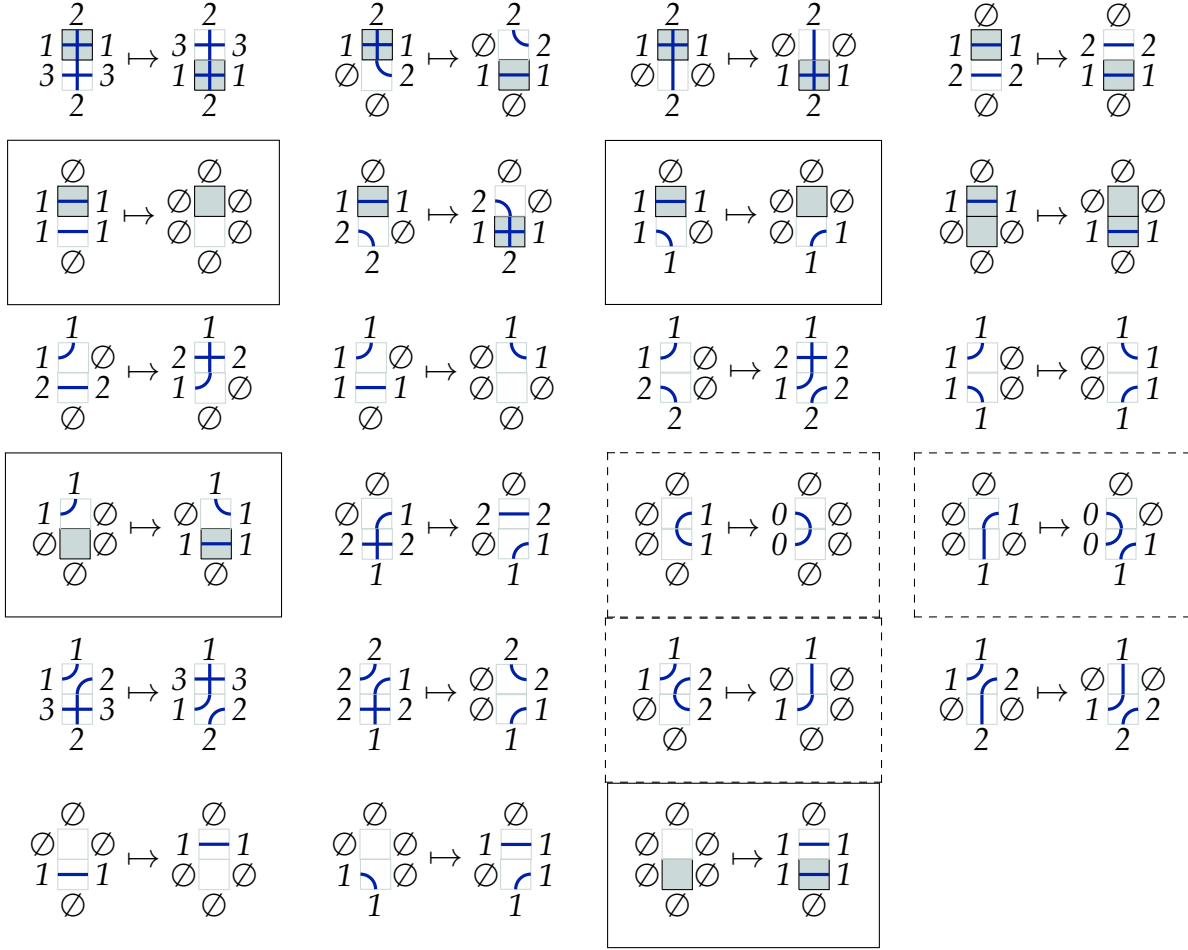
We give an in-context example of the correspondence:



2.3 Swapping a classic row with an adjacent bumpless row

Define a $\overset{C}{B}$ **section** to be a tiled $2 \times k$ rectangle that could appear in rows i and $i+1$ (and k contiguous columns) of some τ -hybrid pipe dream, where $\tau_i \in C$ and $\tau_{i+1} \in B$. Define $\overset{B}{C}$ **sections** similarly. We say the pipe connecting i on the top of a hybrid pipe dream to i on a side is **labeled i** or is **the i pipe**. Where a pipe enters/exits an $\overset{C}{B}$ section ($\overset{B}{C}$ section) we will place a number indicating the label of the pipe. These numbers are just to distinguish which pipes connect to each other outside the $\overset{C}{B}$ section ($\overset{B}{C}$ section). That is, if we take an $\overset{C}{B}$ section ($\overset{B}{C}$ section) and apply a permutation to each number on its boundary, we will not distinguish this from the original.

Theorem 3. *Assuming all the decorations are $-ys$, there exists uniquely a monomial-preserving bijection $\phi_y : \widehat{HPD} \tau_1 CB \tau_2, \pi \rightarrow \widehat{HPD} \tau_1 BC \tau_2, \pi$ changing only those two rows, using the replacements below:*



The proof is a not-too-laborious case check, determining all the ways to glue a classic tile atop a bumpless tile (or vice versa), then to specify the possible connectivities; finally to match the 23 of each, preserving the net connectivity (up to potential pipes entering and leaving on the same side!) and the number of weighty tiles. It takes some additional analysis to confirm that the replacement tiles glue together. Uniqueness follows (slightly non-trivially) from the rest of the lemmas and theorems of this section and the observation that each of the 23 pairs of tiles has a distinct connectivity (i.e. no two have the same labels around the outside). The four boxed replacements are the ones that would not preserve the monomial if the weighty tiles were labeled x rather than $-y$. We spend the rest of the abstract dealing with them (and bringing the dash-boxed replacements along for the ride).

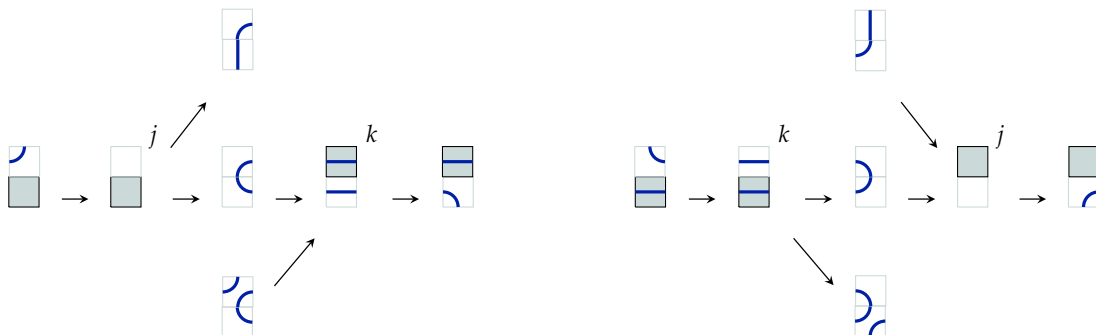
When a $\begin{smallmatrix} C \\ B \end{smallmatrix}$ section or $\begin{smallmatrix} B \\ C \end{smallmatrix}$ section is of (full) width n , call it **full**. The major property that full $\begin{smallmatrix} C \\ B \end{smallmatrix}$ sections ($\begin{smallmatrix} B \\ C \end{smallmatrix}$ sections) have, that non-full ones need not, is that a full $\begin{smallmatrix} C \\ B \end{smallmatrix}$ section ($\begin{smallmatrix} B \\ C \end{smallmatrix}$ section) has blank labels on the right edge of its top row and the left edge of its bottom row (left edge of its top row and right edge of its bottom row) and blank labels on its other vertical edges, whereas an arbitrary $\begin{smallmatrix} C \\ B \end{smallmatrix}$ section ($\begin{smallmatrix} B \\ C \end{smallmatrix}$ section) may have any combination of blank and non-blank labels on its vertical edges.

Two $\begin{smallmatrix} C \\ B \end{smallmatrix}$ sections ($\begin{smallmatrix} B \\ C \end{smallmatrix}$ sections) are said to be the same if they have the same tiles and the same connectivity. As in [Theorem 3](#), we put labels on $\begin{smallmatrix} C \\ B \end{smallmatrix}$ sections ($\begin{smallmatrix} B \\ C \end{smallmatrix}$ sections) in order to show their connectivity and not to indicate the labels the pipes may have had within some larger hybrid pipe dream.

We attach decorations f to $\begin{smallmatrix} C \\ B \end{smallmatrix}$ sections ($\begin{smallmatrix} B \\ C \end{smallmatrix}$ sections) in the same way that we did with hybrid pipe dreams. To assign weights to $\begin{smallmatrix} C \\ B \end{smallmatrix}$ sections ($\begin{smallmatrix} B \\ C \end{smallmatrix}$ sections) in the context of a complete hybrid pipe dream we need to know the labels on the two rows. However, for the purpose of verifying that our bijection preserves weights, we will say the classic row is labeled c and the bumpless row is labeled b so that we attach the weight x_c to weighty tiles decorated with x in the classic row and x_b to weighty tiles decorated with x in the bump less row.

Although the bijection in [Theorem 3](#) is not weight-preserving for arbitrary decorated $\begin{smallmatrix} C \\ B \end{smallmatrix}$ section, most of the replacement rules in the bijection described in [Theorem 3](#) will still be a part of the more general bijection we present in [Theorem 5](#). The leftover columns will be the ones appearing in the seven boxed and dash-boxed rules in [Theorem 3](#). To define the row swapping bijection for arbitrary decorated $\begin{smallmatrix} C \\ B \end{smallmatrix}$ sections we will (1) organize these leftover tiles into chunks which are described in [Definition 1](#), (2) define the bijection on these chunks in [Theorem 4](#), and finally (3) show that everything outside those chunks can be dealt with using the local replacement rules from [Theorem 3](#).

Definition 1. We say that a $\begin{smallmatrix} C \\ B \end{smallmatrix}$ section is **obstructive** if it can be made with the following non-deterministic finite state machine (on the left) by starting at one of the leftmost 5 nodes and ending at one of the rightmost 5 nodes. In the finite state machine j and k mean to repeat that node j or k times respectively where j and k may be any non-negative integer including 0.



The state machine diagram defining **obstructive** $\begin{smallmatrix} B \\ C \end{smallmatrix}$ sections is the one on the right.

Theorem 4. *There is a unique weight preserving bijection between decorated obstructive $\begin{smallmatrix} C \\ B \end{smallmatrix}$ sections and decorated obstructive $\begin{smallmatrix} B \\ C \end{smallmatrix}$ sections such that the labels along the top and bottom edges are preserved.*

Proof. Every obstructive $\begin{smallmatrix} C \\ B \end{smallmatrix}$ section ($\begin{smallmatrix} B \\ C \end{smallmatrix}$ section) begins with a possibly empty sequence of columns with weighty tiles in the bumpless (classic) row and non-weighty tiles in the classic (bumpless) row. After this, there will be a single column with no weighty tiles. Finally, the $\begin{smallmatrix} C \\ B \end{smallmatrix}$ section ($\begin{smallmatrix} B \\ C \end{smallmatrix}$ section) ends with a possibly empty sequence of columns with weighty tiles in the classic (bumpless) row and non-weighty tiles in the bumpless (classic) row. An obstructive $\begin{smallmatrix} C \\ B \end{smallmatrix}$ section ($\begin{smallmatrix} B \\ C \end{smallmatrix}$ section) is completely determined by the labels on its top and bottom edges together with the position of its single non-weighty column. Further, for each possible boundary of an obstructive $\begin{smallmatrix} C \\ B \end{smallmatrix}$ section ($\begin{smallmatrix} B \\ C \end{smallmatrix}$ section) and each i between 1 and the width of the $\begin{smallmatrix} C \\ B \end{smallmatrix}$ section ($\begin{smallmatrix} B \\ C \end{smallmatrix}$ section), there exists an obstructive $\begin{smallmatrix} C \\ B \end{smallmatrix}$ section ($\begin{smallmatrix} B \\ C \end{smallmatrix}$ section) with those boundary labels that has a non-weighty i th column.

Let D be an obstructive $\begin{smallmatrix} C \\ B \end{smallmatrix}$ section ($\begin{smallmatrix} B \\ C \end{smallmatrix}$ section) with width k and let f be some decoration on it. Then, $\text{mon } D, f = x_c^a x_b^{k-1-a-m} \prod_{j=1}^m 1 - y_{i_j}$ for some $0 \leq m \leq k-1$, some $0 \leq a \leq k-1-m$, and some sequence $i_1, i_2, \dots, i_m \in \{1, 2, \dots, k\}$. We claim that mon restricts to a bijection from $\begin{smallmatrix} C \\ B \end{smallmatrix}$ sections ($\begin{smallmatrix} B \\ C \end{smallmatrix}$ sections) with fixed labels along their top and bottom edges to monomials of the form in the previous sentence. Once proven, this claim implies that there is a unique weight preserving bijection between decorated obstructive $\begin{smallmatrix} C \\ B \end{smallmatrix}$ sections and decorated obstructive $\begin{smallmatrix} B \\ C \end{smallmatrix}$ sections. Suppose we have some $i_1, \dots, i_m \in k$ with $0 \leq m \leq k-1$, and some $0 \leq a \leq k-1-m$. Given labels along the top and bottom edges we want to show there is a unique obstructive decorated $\begin{smallmatrix} C \\ B \end{smallmatrix}$ section

D, f and a unique obstructive decorated $\begin{smallmatrix} B \\ C \end{smallmatrix}$ section D', f' with those labels and with $\text{mon } D, f = \text{mon } D', f' = x_c^a x_b^{k-1-a-m} \prod_{j=1}^m 1 - y_{i_j}$. To achieve that monomial, there needs to be a weighty tile in both D and D' decorated with $-y$ in the i_j th column for each $j = 1, \dots, m$. The a factors of x_c in the monomial must come from the rightmost (leftmost) a columns of D (D') which are not already reserved for giving $-y_{i_j}$ weights. The $k-1-a-m$ factors of x_b must come from the leftmost (rightmost) $k-1-a-m$ columns of D (D') which are not already reserved for giving $-y_{i_j}$ weights. We have thus determined the weights we'll need to have coming from $m + a + k-1-a-m = k-1$ of the k columns of D (D'). Hence, the unique remaining column of D (D') will be the unique non-weighty column of D (D'). Setting a column as the non-weighty column uniquely determines what D and D' should each be based on the labels along the top and bottom edges. In the process of determining where the non-weighty column of D and D' had to be to achieve the monomial $x_c^a x_b^{k-1-a-m} \prod_{j=1}^m 1 - y_{i_j}$, we specified which columns would contribute which weights. This tells us exactly how to decorate D and D' because each column of an obstructive $\begin{smallmatrix} C \\ B \end{smallmatrix}$ section ($\begin{smallmatrix} B \\ C \end{smallmatrix}$ section) has at most one weighty

tile. We have thus shown that mon gives a weight preserving bijection between obstructive decorated $\frac{C}{B}$ sections ($\frac{B}{C}$ sections) with set labels on the top and bottom edges, and monomial weights of the form $x_c^a x_b^{k-1-a-m} \prod_{j=1}^m -y_{i_j}$ for appropriate a, m, i_j . We conclude that for any decorated $\frac{C}{B}$ section D, f , there is a unique decorated $\frac{B}{C}$ section D', f' such that the top and bottom edge labels of D match those of D' and $\text{mon } D, f = \text{mon } D', f'$. $\square$

Columns of a full $\frac{C}{B}$ section ($\frac{B}{C}$ section) which aren't contained in some obstructive $\frac{C}{B}$ section ($\frac{B}{C}$ section) turn out to be completely determined by the connectivity of the full $\frac{C}{B}$ section ($\frac{B}{C}$ section). To aid in the discussion of the connectivity of hybrid pipe dreams we introduce the following definitions.

Definition 2. Define the i th edge content $C_i D$ of either a $\frac{C}{B}$ section or $\frac{B}{C}$ section D to be x, y where x is the label of the pipe passing from the tile in the $(i-1)$ st column of the classic row of D to the i th column of the classic row of D (or $\emptyset$ if there is no such pipe) and y is the label of the pipe passing from the i th column of the bumpless row of D to the $(i-1)$ st column of the bumpless row of D (and $\emptyset$ if there is no such pipe).

Define the boundary ∂D of D to be the list of labels along the top, bottom, left, and right boundary edges of D where D may be either a $\frac{C}{B}$ section or a $\frac{B}{C}$ section.

If D is a $\frac{C}{B}$ section ($\frac{B}{C}$ section) such that for all $\frac{C}{B}$ sections ($\frac{B}{C}$ sections) D' such that $\partial D' = \partial D$, $C_i D' = x, y$ we call i **separating** for D .

Lemma 1. If D is a $\frac{C}{B}$ section ($\frac{B}{C}$ section) and $C_i D = x, y$ with $x \neq y$ then i is separating for D and in fact $C_i D' = x, y$ for any $\frac{C}{B}$ section ($\frac{B}{C}$ section) D' with $\partial D = \partial D'$. In particular, if we introduce the following conditions:

- (i) the label x appears on the top edge of D in a column right of the i th column (or on the right edge of the classic row)
- (ii) the label x appears on the bottom edge of D in a column left of the i th column (or on the right edge of the classic row)
- (iii) the label y appears on the top edge of D in the i th column or to the left of it (or on the left edge of the bumpless row)
- (iv) the label y appears on the bottom edge of D in a column right of the i th column (or on the right edge of the bumpless row),

then

1. $C_i D = x, y$ with $x \neq y$ and $x \neq \emptyset \neq y$ iff conditions (i), (ii), (iii), and (iv) all hold.
2. $C_i D = x, \emptyset$ with $x \neq \emptyset$ iff conditions (i) and (ii) hold and for all y , either condition (iii) does not hold or condition (iv) does not hold.

3. $C_i D \not\subset \emptyset, y$ with $y \not\subset \emptyset$ iff conditions (iii) and (iv) hold, and for all x , either condition (i) does not hold or condition (ii) does not hold.

Lemma 2. *If the label $x \not\subset \emptyset$ appears on the top edge of the i th column of D and the bottom edge of the j th column of D for some $j \geq i$ or the right edge of the bumpless row of D when D has width j , then $i - 1, i, \dots, j$ are all separating for D . Additionally, if the label $x \not\subset \emptyset$ appears on the bottom edge of the i th column and the left edge of the bumpless row of D then $1, 2, \dots, i$ are all separating for D .*

Definition 3. *Let D be a $\frac{C}{B}$ section ($\frac{B}{C}$ section) of width n . The interval a, b is irreducible in D if no $i \in a, b - 1$ is separating for D . An irreducible interval a, b in D is called maximal if $a - 1$ and b are separating for D .*

Lemma 3. *Let F be a full $\frac{C}{B}$ section ($\frac{B}{C}$ section) and let D be the $\frac{C}{B}$ section ($\frac{B}{C}$ section) sitting in columns a through b of F where $b > a$. If a, b is a maximal irreducible interval for F then D is an obstructive $\frac{C}{B}$ section ($\frac{B}{C}$ section).*

Proof. Suppose that a, b is a maximal irreducible interval for F .

Say a pipe p enters D from the top edge of the i th column (with $i \in a, b$ of course). If p exits D from the bottom edge of a column right of i (or exits from the bottom of i) then $i - 1$ and i are both separating for D by Lemma 2. In that case, irreducibility implies that $i - 1 \not\subset a - 1$ and $i \not\subset b$ which would then contradict the assumption that $b > a$. We conclude that p must exit D either to the left of the i th column or through a vertical edge of D . If p exits through the right edge of the bumpless row of D then $b - 1$ is separating for D by Lemma 2 so either D is not irreducible or $b \not\subset a - 1$. Either case is a contradiction so we conclude that p exits through the left edge of the classic row of D or somewhere left of the i th column. If p exits D in the j th column for some $a \leq j < i$ then $i - 1$ will be separating for D by Lemma 1 parts 1 and 2. Irreducibility then implies $i - 1 \not\subset a - 1$ which contradicts our assumption that $a \leq j < i$. Since p can't exit D in the bottom of column i and it can't exit D in a column $j < i$ or a column $j > i$, the only remaining possibility is that p exits D through the left edge of the classic row of D and $i \not\subset a$. In summary, if there is a pipe entering D from a top edge, it must enter from the top edge of column a and exit through the left edge of the classic row of D .

Suppose a pipe p exits D from the bottom edge of the i th column of D . By the preceding paragraph, p cannot have entered D from a top edge, so p must have entered D from the right edge of the classic row of D or the left edge of the bumpless row of D . Suppose towards a contradiction that p entered through the left edge of the bumpless row. Then, a will be separating for D by Lemma 2. Since $b - 1 > a$, that contradicts irreducibility. Therefore, we conclude that p can only have entered D from the right edge of the classic row. If i is less than b , then Lemma 1 shows $b - 1$ to be separating for D . Thus, we conclude $i \not\subset b$ by irreducibility. Our takeaway is that if a pipe p exits

D from a bottom edge, it must be entering from the right edge of the classic row and exiting from the bottom edge of column b .

In addition to what we know already, Lemma 1 tells us that for all $i = a, \dots, b-1$, $C_i D = \emptyset, \emptyset$ or $C_i D = x, x$ for some x . With all the restrictions on the contents of D we can now go through the list of 2×1 $\begin{smallmatrix} C \\ B \end{smallmatrix}$ section columns and find that if D is

a $\begin{smallmatrix} C \\ B \end{smallmatrix}$ section, the possible columns are: , , , , , , , and . If D is a $\begin{smallmatrix} B \\ C \end{smallmatrix}$

section, the possible tiles will be: , , , , , , , and . This does not yet tell us that D is an obstructive $\begin{smallmatrix} C \\ B \end{smallmatrix}$ section ($\begin{smallmatrix} B \\ C \end{smallmatrix}$ section). The problem is that D could start with a column which is not a start state of the finite state machine or end with a column that is not an end state of the finite state machine. Call a column of a $\begin{smallmatrix} C \\ B \end{smallmatrix}$ section ($\begin{smallmatrix} B \\ C \end{smallmatrix}$ section) problematic if it is included in the finite state machine for obstructive $\begin{smallmatrix} C \\ B \end{smallmatrix}$ sections ($\begin{smallmatrix} B \\ C \end{smallmatrix}$ sections). In the finite state machine for $\begin{smallmatrix} C \\ B \end{smallmatrix}$ sections ($\begin{smallmatrix} B \\ C \end{smallmatrix}$ sections), the valid start states are exactly the columns which can be preceded by some non-problematic tile and the valid end states are exactly the columns that can be followed by some non-problematic tile. The columns which are not valid start states cannot appear at the beginning of a full $\begin{smallmatrix} C \\ B \end{smallmatrix}$ section ($\begin{smallmatrix} B \\ C \end{smallmatrix}$ section). The columns which are not valid end states cannot appear at the end of a full $\begin{smallmatrix} C \\ B \end{smallmatrix}$ section ($\begin{smallmatrix} B \\ C \end{smallmatrix}$ section). Hence, if D starts with an invalid start state/ends with an invalid end state, it will be preceded/followed by a sequence of problematic tiles beginning/ending with a valid start/end column. This larger $\begin{smallmatrix} C \\ B \end{smallmatrix}$ section ($\begin{smallmatrix} B \\ C \end{smallmatrix}$ section) will be obstructive. If this containment is proper, then either $a-1$ or b won't be separating. That would contradict maximality, so we conclude that D is an obstructive $\begin{smallmatrix} C \\ B \end{smallmatrix}$ section ($\begin{smallmatrix} B \\ C \end{smallmatrix}$ section). $\square$

Theorem 5. *There is a weight preserving bijection between decorated full $\begin{smallmatrix} C \\ B \end{smallmatrix}$ sections and decorated full $\begin{smallmatrix} B \\ C \end{smallmatrix}$ sections.*

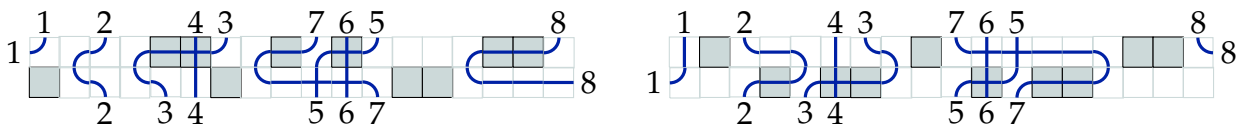
Proof. To map a $\begin{smallmatrix} C \\ B \end{smallmatrix}$ section D to a $\begin{smallmatrix} B \\ C \end{smallmatrix}$ section we first split it up into smaller pieces, splitting it up between columns i and $i+1$ whenever i is separating for D . This leaves us with maximal irreducible intervals which by Lemma 3 will either be single columns or obstructive $\begin{smallmatrix} C \\ B \end{smallmatrix}$ sections. Obstructive $\begin{smallmatrix} C \\ B \end{smallmatrix}$ sections (including ones which are single columns) are mapped to obstructive $\begin{smallmatrix} B \\ C \end{smallmatrix}$ sections in the manner described in Theorem 4 and all other single columns are mapped to single columns in the manner described in Theorem 3. The 23 replacement rules in Theorem 3 are characterized by the property that

$$C_i \phi_y D = \begin{cases} \emptyset, \emptyset & C_i D = x, x \text{ for some } x \\ x, x \text{ for some } x & C_i D = \emptyset, \emptyset \\ C_i D & \text{else.} \end{cases} \quad \text{This property also holds on}$$

the leftmost and rightmost boundary edges for the bijection between decorated obstructive

tive $\frac{C}{B}$ sections ($\frac{B}{C}$ sections). Thus, when we apply our construction on each maximal irreducible interval, the results will fit together. This construction sends maximal irreducible intervals to maximal irreducible intervals so it is indeed a bijection. $\square$

As an example of Theorem 5, the following $\frac{C}{B}$ section and $\frac{B}{C}$ section are mapped to one another when all decorations are x .



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Rigorous Analytic Combinatorics in Several Variables in SageMath

Benjamin Hackl^{*1}, Andrew Luo⁺², Stephen Melczer^{‡2},
Jesse Selover^{§3}, and Elaine Wong^{¶4}

¹*Institute of Mathematics and Scientific Computing, University of Graz, Austria
Department of Mathematics, University of Klagenfurt, Austria*

²*Department of Combinatorics and Optimization, University of Waterloo, Canada*

³*Department of Mathematics, University of Massachusetts, Amherst, USA*

⁴*Oak Ridge National Laboratory, Tennessee, USA*

Abstract. We introduce the new `sage_acsv` package for the SageMath computer algebra system, allowing users to rigorously compute asymptotics for a large variety of multivariate sequences with rational generating functions. Using Sage’s support for exact computations over the algebraic number field, this package provides the first rigorous implementation of algorithms from the theory of analytic combinatorics in several variables.

Résumé. Nous introduisons le nouveau package `sage_acsv` pour le système de calcul formel SageMath, permettant aux utilisateurs de calculer rigoureusement les asymptotiques pour une grande variété de séquences multivariées avec des fonctions génératrices rationnelles. Utilisant le support de Sage pour les calculs exacts sur le champ de nombres algébriques, ce package fournit la première implémentation rigoureuse d’algorithmes de la théorie de la combinatoire analytique à plusieurs variables.

Keywords: ACSV, asymptotics, analytic combinatorics, generating functions, computer algebra

^{*}benjamin.hackl@uni-graz.at.

⁺j92luo@uwaterloo.ca.

[‡]smelczer@uwaterloo.ca.

[§]jselover@umass.edu.

[¶]wongey@ornl.gov – This manuscript has been authored in part by UT-Battelle, LLC, under contract DE-AC05-00OR22725 with the US Department of Energy (DOE). The publisher, by accepting the article for publication, acknowledges that the U.S. Government retains a non-exclusive, paid up, irrevocable, worldwide license to publish or reproduce the published form of the manuscript, or allow others to do so, for U.S. Government purposes. The DOE will provide public access to these results in accordance with the [DOE Public Access Plan](#).

1 Introduction

The field of *Analytic Combinatorics in Several Variables* (ACSV) [7, 11] adapts methods from complex analysis in several variables, algebraic and differential geometry, topology, and computer algebra to create effective methods to study the asymptotic properties of a multivariate sequence f_i $i \in \mathbb{N}^d$ using analytic properties of its *generating function*,

$$F(z) = \sum_{i \in \mathbb{N}^d} f_i z^i = \sum_{i_1, \dots, i_d \in \mathbb{N}^d} f_{i_1, \dots, i_d} z_1^{i_1} \cdots z_d^{i_d},$$

when this series represents a complex analytic function near the origin. The `sage_acsv` package takes a multivariate rational function $F(z)$ specified as an explicit symbolic fraction and a *direction vector* $r \in \mathbb{Z}_{>0}^d$ and returns an asymptotic expansion for the *r*-diagonal sequence¹ of the power series coefficients f_{nr} $n \geq 0$ of F , whose indices are natural number multiples of r . Our package works under conditions that are verifiable and hold generically, except that F must be *combinatorial*, meaning that all but a finite number of its power series coefficients are non-negative.

Remark 1. It is unknown whether combinatoriality is a computationally decidable property, even in the univariate case. However, this property holds often in combinatorial contexts (for instance, the multivariate generating function of any combinatorial class is combinatorial because its series coefficients count something).

The `sage_acsv` package can be installed in any recent SageMath installation (preferably version 9.4 or later) by running the command `sage -pip install sage-acsv` from any Sage instance with access to the internet, or by downloading its source code at https://github.com/ACSVMath/sage_acsv and placing the module in the appropriate Python search path.

Example 2. The $(1, 1)$ -diagonal of the combinatorial rational function

$$F(x, y) = \frac{1}{1 - x - y} = \sum_{a, b \geq 0} \binom{a+b}{a} x^a y^b$$

forms the sequence $f_{n,n} = \binom{2n}{n}$ of central binomial coefficients. After installing the package, running the code

```
sage: from sage_acsv import diagonal_asymptotic
sage: var('x, y')
(x, y)
sage: diagonal_asymptotic(1/(1 - x - y), as_symbolic=True)
4^n/(sqrt(pi)*sqrt(n))
```

¹The theory of ACSV shows how asymptotics typically vary uniformly with small perturbations of r , so it is also possible to derive asymptotics varying near fixed directions, and even limit theorems, using similar techniques.

verifies that the required assumptions of ACSV hold and proves that

$$\binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi n}} \left(1 + O\left(\frac{1}{n}\right)\right). \quad \triangleleft$$

The asymptotic behavior of a multivariate rational diagonal sequence under our assumptions (to be detailed below) is specified by a sum of terms of the form $C\rho^n \pi n^\alpha$ where C and ρ are algebraic numbers and α is a rational number. By default, the `diagonal_asy` command returns expansions of this form as a list of tuples containing elements of Sage's Algebraic Field, which stores exact representations of the quantities. The optional flag `as_symbolic` invoked in [Example 2](#) tells `diagonal_asy` to return a symbolic sum involving n which makes it easier to view (and approximate) asymptotics but makes it difficult to access the exact algebraic numbers involved.

Example 3. The asymptotic behavior of the sequence

$$f_n = \sum_{k=0}^n \binom{n}{k}^2 \binom{n-k}{k}^2,$$

which is the main diagonal of

$$F(w, x, y, z) = \frac{1}{1 - w - x - y - z - xyz - yz - y - z - 1},$$

was used by Apéry [1] in his celebrated proof of the irrationality of $\zeta(3)$. Running

```
sage: from sage_acsv import diagonal_asy
sage: var('w, x, y, z, t')
(w, x, y, z, t)
sage: G = 1
sage: H = 1 - w*(1 + x)*(1 + y)*(1 + z)*(1 + y + z + y*z + x*y*z)
sage: diagonal_asy(G/H, as_symbolic=True)
1.225275868941647? * 33.97056274847714? ^n / (pi ^ 1.5 * n ^ 1.5)
```

gives the dominant asymptotic behavior of f_n in terms of algebraic numbers represented by decimal approximations. Running

```
sage: asm_vals = diagonal_asy(G/H)
sage: add([a.radical_expression() ^ n * b * c * d.radical_expression()
          for (a, b, c, d) in asm_vals])
(12*sqrt(2) + 17) ^ n * sqrt(17/2*sqrt(2) + 12) / (4*pi ^ 1.5 * n ^ 1.5)
```

represents these algebraic quantities in radicals using Sage's capability for computing symbolic radical expressions, proving

$$f_n \sim \frac{(12\sqrt{2} + 17)^n}{\sqrt{\pi n^3}} \left(\frac{\sqrt{12 + 17/\sqrt{2}}}{4} + O\left(\frac{1}{n}\right) \right).$$

We note that such radical expressions are not certified and, of course, it is not always possible (or, even if possible, useful) to write the algebraic numbers appearing in asymptotics for general multivariate diagonals in radicals. $\triangleleft$

The, by now well-established, theory of univariate analytic combinatorics [4] shows that to determine asymptotics of the power series coefficients of a generating function $f(z)$, one should find the *dominant singularities* of f (those closest to the origin), determine the type of singularity (pole of finite order, logarithmic or algebraic branch point, essential singularity, etc.), and then use known *transfer theorems* to determine the contribution of each singularity to the asymptotics for the corresponding sequence. Dominant singularities are generalized from one to several variables by the following definition.

Definition 4 (minimal point). Let $F(z) = \sum_{i \in \mathbb{N}^d} f_i z^i$ be a convergent power series with domain of convergence $\mathcal{D}$. A singularity $w \in \mathbb{R}^d$ of $F(z)$ is said to be *minimal* if w lies in the boundary $\partial\mathcal{D}$. Equivalently, w is minimal if there does not exist another singularity $w' \in \mathbb{R}^d$ of $F(z)$ with $|w'_i| < |w_i|$ for all $1 \leq i \leq d$.

Although the theory of ACSV applies to meromorphic functions, we focus here on the case when $F(z) = G(z)/H(z)$ is a rational function so that we can use tools from polynomial system solving (rational functions can also capture the behavior of algebraic functions through embeddings in higher dimensions [6]). We always assume that G and H are coprime polynomials, which implies that the set of singularities of F is defined by the *singular variety* $\mathcal{V} = \{z \in \mathbb{C}^d : H(z) = 0\}$.

Unlike the univariate case, in at least two variables there are always an infinite number of minimal points when F admits singularities. This reflects, in part, the fact that minimal points are important to the asymptotics of f_i and there are an infinite number of ways for the index i to approach infinity by varying the direction vector r . Thankfully, it is possible in generic circumstances to determine a finite set of singularities, depending on r , that dictate asymptotics of the r -diagonal of F .

Definition 5 (critical point). A point $w \in \mathbb{C}_*^d : \mathbb{C} \setminus \{0\}^d$ with nonzero coordinates is a (*simple smooth*) *critical point* of F for the direction $r = r_1, \dots, r_d$ if the gradient $\nabla H(w)$ is nonzero and

$$\begin{aligned} H(w) &= 0, \\ r_k z_1 H_{z_1}(w) - r_1 z_k H_{z_k}(w) &= 0 \text{ for } 2 \leq k \leq d, \end{aligned} \tag{1.1}$$

where H_{z_k} denotes the partial derivative of H with respect to the variable z_k .

The theory of ACSV shows how minimal critical points, when they exist, typically determine asymptotics (see [Theorem 9](#) below). The command

```
diagonal_asy(F, r, linear_form, return_points, as_symbolic)
```

computes asymptotics of the r -diagonal f_{nr} of $F(z) = G(z)/H(z)$ as $n \rightarrow \infty$, under the following assumptions:

1. $H(0) \neq 0$, so that $G(z)/H(z)$ has a convergent power series expansion near the origin,
2. H and all of its partial derivatives do not simultaneously vanish, so that all poles of F are simple and $\mathcal{V}$ forms a manifold,
3. the smooth critical point system (1.1) has a finite number of solutions, at least one of which is minimal, and
4. the explicit matrix $\mathcal{H} = \mathcal{H}(w)$ defined by (2.1) below is nonsingular at all minimal critical points w .

The algorithm verifies these assumptions computationally, so the user need not worry about them. As mentioned above, we further require that

5. the power series expansion of F is *combinatorial*, meaning that it admits at most a finite number of negative coefficients,

however it is unknown whether combinatoriality is decidable, even for univariate rational functions, so **the user must know this** through other means in order for the output to be proven correct (for instance, if the input is a multivariate generating function for a counting sequence then this is satisfied). Table 1 summarizes the possible input parameters of the function `diagonal_asy`.

Multivariate generating functions enumerating combinatorial classes are, by their definition, combinatorial. The form of F can also be used to prove combinatoriality, for instance if $F(z) = G(z)/(1 - I(z))$ where G and I are polynomials with non-negative coefficients and I vanishes at the origin.

Remark 6. If H and its partial derivatives simultaneously vanish because H is not square-free then minimal critical points can still be determined by replacing H with its square-free part (the product of its irreducible factors); asymptotics can also be determined using a minor generalization of the formula given below to higher order poles. If $\mathcal{V}$ admits points where it is not locally a manifold, but (1.1) still has a finite set of solutions and admits a smooth minimal critical point w such that all other solutions of (1.1) with the same coordinate-wise modulus are also smooth, then this approach can still determine asymptotics. In general, if nonsmooth minimal points affect asymptotic behavior, then more advanced results are required [7, Part III].

Table 1: Parameters for the `diagonal_asy` command.

Parameter	Type	Description
$F(z)$	Combinatorial rational function in $\mathbb{Z}[z_1, \dots, z_d]$	Ratio of coprime polynomials $G(z)$ and $H(z)$.
r (optional)	List of d positive integers	Direction to compute asymptotics. If none is provided, use $r = 1$.
<code>linear_form</code> (optional)	Linear polynomial in $\mathbb{Z}[z_1, \dots, z_d]$	Integer linear form to be used in the algorithm. If none provided, generate one at random.
<code>return_points</code> (optional)	True or False	Flag to also return coordinates of minimal critical points determining asymptotics. Default is False.
<code>as_symbolic</code> (optional)	True or False	Flag to return asymptotics as a symbolic sum involving n . Default is False.

Example 7. The sequence alignment problem in molecular biology compares evolutionary relationships between species by measuring differences in their DNA sequences. Pemantle and Wilson [10] study one such problem whose behavior is encoded by the main diagonal of the combinatorial multivariate rational generating function

$$\frac{x^2y^2 - xy + 1}{1 - x - y - xy - xy^2 - x^2y - x^2y^3 - x^3y^2}.$$

We immediately prove dominant asymptotic behavior for the main diagonal by running

```
sage: from sage_acsv import diagonal_asy
sage: var('x, y')
(x, y)
sage: G = x^2*y^2 - x*y + 1
sage: H = 1 - (x + y + x*y - x*y^2 - x^2*y + x^2*y^3 + x^3*y^2)
sage: F = G/H
sage: asm_vals = diagonal_asy(F, as_symbolic=True)
0.9430514023983397? * 4.518911369262258? ^ n / (sqrt(pi) * sqrt(n))
```

where now the leading constant is an algebraic number of degree 10 and the exponential growth constant is an algebraic number of degree 5. ◀

A test notebook working through these examples, and further ones, is available with the source code of the package at https://github.com/ACSVMath/sage_acsv.

Past Work

The theoretical underpinnings of our software began with the work of Pemantle and Wilson [9] on smooth ACSV. This early paper used easy to understand explicit contour deformations of complex integrals to derive asymptotics, however it relies on stronger assumptions than we need. The specific asymptotic result we use was originally proven in Baryshnikov and Pemantle [2] using the notion of *hyperbolic cones* to enable more advanced deformations, which was then studied from an algorithmic viewpoint by Melczer and Salvy [8]. The latter two authors created a preliminary Maple implementation of the algorithm which was not rigorous, as it did not use certified numerics. A previous implementation by Raichev [12], currently included as a core module in SageMath, can also compute² the asymptotic contributions of minimal critical points (including in some nonsmooth situations). However, this package requires the user to independently find and certify the minimal critical points (the hardest step of the analysis), so it also does not rigorously determine asymptotics. In contrast, our package uses Sage’s exact computations for algebraic numbers to rigorously decide the (in)equalities necessary to prove computed asymptotics under our assumptions.

2 Algorithmic Details

In this section we quickly recap the theoretical background of ACSV, and then describe how our package works in more detail.

2.1 Recap of ACSV

We start by representing the r -diagonal sequence by a Cauchy integral,

$$f_{nr} = \frac{1}{2\pi i} \int_C F(z) \frac{dz}{z^{nr+1}},$$

where C is any product of circles sufficiently close to the origin, with $dz = dz_1 \cdots dz_d$ and $z^{nr+1} = z_1^{nr_1+1}, \dots, z_d^{nr_d+1}$. Under our assumptions, it is possible to deform C away from the origin and replace the Cauchy integral by a residue integral localized to smooth minimal critical points (except at points that yield an exponentially negligible error). The definition of critical points is crafted so that the resulting residue integral can be analyzed using the classical saddle-point method. Our asymptotic formulas depend on a certain matrix, which we now define.

²Unfortunately, as of the publication of this article, changes to the underlying SageMath codebase have broken some functionality of Raichev’s package.

Definition 8 (phase Hessian matrix). If $\mathbf{w}$ is a smooth critical point, then the *phase Hessian matrix* $\mathcal{H} = \mathcal{H}(\mathbf{w})$ at $\mathbf{w}$ is the $(d-1) \times (d-1)$ matrix defined by

$$\mathcal{H}_{i,j} = \begin{cases} V_i V_j & U_{i,j} - V_j U_{i,d} - V_i U_{j,d} & V_i V_j U_{d,d}, & i \neq j, \\ V_i & V_i^2 & U_{i,i} - 2V_i U_{i,d} & V_i^2 U_{d,d}, & i = j, \end{cases} \quad (2.1)$$

where

$$U_{i,j} = \frac{w_i w_j H_{z_i z_j}(\mathbf{w})}{w_d H_{z_d}(\mathbf{w})} \quad \text{and} \quad V_i = \frac{r_i}{r_d}.$$

The key ACSV theorem in our context is the following.

Theorem 9 (Melczer [7, Theorem 5.1]). Suppose that the system of polynomial equations (1.1) admits a finite number of solutions, exactly one of which, $\mathbf{w} \in \overline{\mathbb{Q}}_*^d$, is minimal. Suppose further that $H_{z_d}(\mathbf{w}) \neq 0$, and that the phase Hessian matrix $\mathcal{H}$ at $\mathbf{w}$ has nonzero determinant. Then, as $n \rightarrow \infty$,

$$f_{nr} = \mathbf{w}^{-nr} n^{1-d/2} \frac{2\pi r_d^{1-d/2}}{\sqrt{\det \mathcal{H}}} \left(\frac{-G(\mathbf{w})}{w_d H_{z_d}(\mathbf{w})} + O\left(\frac{1}{n}\right) \right).$$

If the zero set of H contains a finite number of points with the same coordinate-wise modulus as $\mathbf{w}$, all of which satisfy the same conditions as $\mathbf{w}$, then an asymptotic expansion of f_{nr} is obtained by summing the right hand side of this expansion at each point.

Remark 10. If $G(\mathbf{w}) = 0$, then the leading asymptotic term in Theorem 9 will vanish. In many cases dominant asymptotics can usually still be determined by computing higher-order terms using (increasingly complicated) explicit formulas, however (in nongeneric situations, or when G and H were not reduced to be coprime) it is possible for f_{nr} to grow exponentially slower than $\mathbf{w}^{-nr}$, and a local analysis near $\mathbf{w}$ can only prove $f_{nr} = O(\mathbf{w}^{-nr} n^{-k})$ for any positive integer k .

In order to determine which critical points are minimal we use the following result.

Lemma 11 (Melczer and Salvy [8]). If F is combinatorial and $\mathbf{y} \in \mathbb{C}_*^d$ is a minimal critical point then so is $(|y_1|, \dots, |y_d|)$. Furthermore, $\mathbf{w} \in \mathbb{R}_{>0}^d$ is a minimal critical point if and only if the system

$$\begin{aligned} H(\mathbf{z}) &= H(tz_1, \dots, tz_d) = 0, \\ z_1 H_{z_1}(\mathbf{z}) - r_1 \lambda &= \dots = z_d H_{z_d}(\mathbf{z}) - r_d \lambda = 0, \end{aligned} \quad (2.2)$$

has a solution $(\mathbf{z}, \lambda, t) \in \mathbb{R}^{d+2}$ with $\mathbf{z} = \mathbf{w}$ and $t = 1$ and no solution with $\mathbf{z} = \mathbf{w}$ and $0 < t < 1$.

[Theorem 9](#) and [Lemma 11](#) display our overall strategy: encode the (generically finite [7, Section 5.3.4]) set of solutions to the polynomial system (2.2) in a convenient manner, determine which solutions with $t = 1$ have $z_1, \dots, z_d$ positive and real, and then use the solutions with $t \in [0, 1]$ to eliminate those that are not minimal. Under our assumptions there will be at most one positive real minimal critical point: it remains to find the other critical points with the same coordinate-wise modulus and then add the asymptotic contributions of each given by [Theorem 9](#).

2.2 Kronecker Representations

The key to an efficient algorithm is the representation used to encode the solutions of the extended critical point system (2.2). Following Melczer and Salvy [8], we use a *Kronecker representation*, which is also known as a *rational univariate representation* (see Melczer and Salvy [8] for background on the Kronecker representation and its history in computer algebra). A *Kronecker representation* of a zero-dimensional algebraic set

$$S = \{z \in \mathbb{C}^d : f_1(z) = \dots = f_d(z) = 0\}$$

consists of

- a new variable u given by a separating integer linear form $u = \kappa \cdot z$ in the original variables z for some $\kappa \in \mathbb{Z}^d$,
- a square-free integer polynomial $P \in \mathbb{Z}[u]$, and
- integer polynomials $Q_1, \dots, Q_d \in \mathbb{Z}[u]$,

such that the points z defined by $z_i = Q_i(u) / P'(u)$ as u runs through the roots of P give the elements of S . The Kronecker representation of a zero-dimensional variety encodes its points using the univariate polynomial P , and is constructed so that the degrees and maximum coefficient sizes of P and the Q_i can be efficiently bounded [13, 14].

Although specialized, Gröbner-free, algorithms to compute a Kronecker representation [5] exist, to the best of our knowledge they have not been implemented in Sage. We thus use a lexicographical Gröbner basis computation to determine our Kronecker representations. The ability to compute Kronecker representations in Sage may be of independent interest to some users.

Example 12. Determining the asymptotics of the Apéry-3 Sequence requires solving an *extended critical point system*

$$\begin{array}{rcll} xyz - xz & x - \lambda & 0, \\ xyz - yz & y - \lambda & 0, \\ xyz - xz - yz & z - \lambda & 0, \\ xyz - xz - yz & x & y & z - 1 & 0, \\ xyz t^3 - xz t^2 - yz t^2 & xt & yt & zt - 1 & 0. \end{array}$$

Using the linear form $u = x + t$, running

```
sage: from sage_acsv import kronecker
sage: var('x, y, z, t, lambda_')
(x, y, z, t, lambda_)
sage: kronecker([x*y*z - x*z + x - lambda_, x*y*z - y*z + y - lambda_,
....:          x*y*z - x*z - y*z + z - lambda_,
....:          x*y*z - x*z - y*z + x + y + z - 1,
....:          x*y*z*t^3 - x*z*t^2 - y*z*t^2 + x*t + y*t + z*t - 1],
....:          [x, y, z, t, lambda_], x + t)
(u^8 - 18*u^7 + 146*u^6 - 692*u^5 + 2067*u^4 - 3922*u^3 + 4553*u^2
- 2925*u + 790,
[10*u^7 - 153*u^6 + 1046*u^5 - 4081*u^4 + 9589*u^3 - 13270*u^2 +
9844*u - 2985,
10*u^7 - 154*u^6 + 1061*u^5 - 4180*u^4 + 9954*u^3 - 14044*u^2 +
10714*u - 3380,
-u^7 + 11*u^6 - 56*u^5 + 157*u^4 - 182*u^3 - 140*u^2 + 527*u -
335,
8*u^7 - 139*u^6 + 1030*u^5 - 4187*u^4 + 10021*u^3 - 14048*u^2 +
10631*u - 3335,
-12*u^7 + 181*u^6 - 1231*u^5 + 4801*u^4 - 11275*u^3 + 15548*u^2 -
11452*u + 3440])
```

gives the Kronecker representation

$$\begin{array}{lcl} P(u) & u^8 - 18u^7 + 146u^6 - 692u^5 + 2067u^4 - 3922u^3 + 4553u^2 - 2925u + 790, \\ Q_x(u) & 10u^7 - 153u^6 + 1046u^5 - 4081u^4 + 9589u^3 - 13270u^2 + 9844u - 2985, \\ Q_y(u) & 10u^7 - 154u^6 + 1061u^5 - 4180u^4 + 9954u^3 - 14044u^2 + 10714u - 3380, \\ Q_z(u) & -u^7 + 11u^6 - 56u^5 + 157u^4 - 182u^3 - 140u^2 + 527u - 335, \\ Q_t(u) & 8u^7 - 139u^6 + 1030u^5 - 4187u^4 + 10021u^3 - 14048u^2 + 10631u - 3335, \\ Q_\lambda(u) & -12u^7 + 181u^6 - 1231u^5 + 4801u^4 - 11275u^3 + 15548u^2 - 11452u + 3440. \end{array} \triangleleft$$

After computing a Kronecker representation of the critical point system, we use Sage's solver over the Real Algebraic Field to determine the real roots of $P(u)$, identify which correspond to critical points with positive real coordinates, filter out those

that are not minimal by examining solutions with t coordinate in $[0, 1]$, and then identify which critical points have the same coordinate-wise modulus. All identities and inequalities are verified exactly by working over the Algebraic Field in Sage (numeric approximations to sufficient accuracy are used to decide inequalities, unless symbolic computations are absolutely necessary to prove equality). Further details can be found in the documentation for our package.

2.3 Future Work

There are many ways to generalize this work, including: computing the higher-order terms in the expansion given by [Theorem 9](#), working with nonsmooth critical points, Gröbner-free methods of computing Kronecker representations, and generalizing to larger classes of meromorphic functions where computations are still effective. From a complexity point of view, the most expensive operation to find asymptotics is typically identifying critical points with the same coordinate-wise modulus. Our package currently uses a naive method relying on Sage’s built-in support for algebraic numbers, however work is ongoing to adapt a more efficient method described in Melczer and Salvy [8] in the specialized ACSV setting. Furthermore, we use Sage’s interface to Singular for necessary Gröbner basis computations. Work adapting newer packages, such as `msolve` [3], is also ongoing.

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A bijectionist's toolkit

Alexander Grosz^{*1}, Tobias Kietreiber^{†2},
Stephan Pfannerer^{‡2}, and Martin Rubey^{§2}

¹Technical University of Munich, Germany

²Institute of Discrete Mathematics and Geometry, Technische Universität Wien, Austria

Abstract. We present a tool, integrated into SageMath 10.0, which supports the combinatorialist in the discovery of an explicit bijection between two finite sets given various constraints, or by demonstrating that no such bijection can exist. As an application we present a conjecture which may have been hard to discover otherwise.

1 Introduction

Let $\mathcal{A} = A_i \mid i \in I$ and $\mathcal{B} = B_i \mid i \in I$ be two sequences of finite sets of the same size. One commonly employed approach for finding an explicit bijection between the two sets is to refine the problem by introducing new parameters or by stipulating that the bijection has other additional properties. For example, one may want to require that it intertwines natural actions or operations on $\mathcal{A}$ and $\mathcal{B}$ respectively or, provided $\mathcal{A} = \mathcal{B}$, that it is an involution. Also, one may want to find a parameter on $\mathcal{A}$ which has the same distribution as a natural parameter on $\mathcal{B}$ but satisfies additional constraints.

Thus, the first task is to discover such properties, and then determine experimentally whether the requirements can be met at all. To the best of the authors' knowledge, this is usually done by ad-hoc methods. For example, one might list the first few objects in $\mathcal{A}$ and $\mathcal{B}$ with pencil and paper or perhaps using a computer program and then inspect them manually.

While it is straightforward to check whether there are natural subsets of A_i and B_i that are still in bijection, other constraints may be very hard to verify or falsify. Moreover, databases like the OEIS [4] and FindStat [3] now enable us, in principle, to generate a

^{*}alexander.grosz@tum.de. Alexander Grosz was supported by the Austrian Science Fund (FWF) P29275 and the Alexander von Humboldt Foundation with funds from the German Federal Ministry of Education and Research (BMBF).

[†]tobias.kietreiber@tuwien.ac.at. Tobias Kietreiber was supported by the Austrian Science Fund (FWF) P29275.

[‡]stephan.pfannerer@tuwien.ac.at. Stephan Pfannerer was supported by the Austrian Science Fund (FWF) P29275 and is a recipient of a DOC Fellowship of the Austrian Academy of Sciences.

[§]martin.rubey@tuwien.ac.at. Martin Rubey was supported by the Austrian Science Fund (FWF) P29275.

huge number of conjectural refinements, but it is infeasible to look into them by hand. The program we describe here seeks to simplify and automate this task.

The ‘bijectionist’s toolkit’ is included in SageMath [5] beginning with version 10.0.¹

2 Motivation

Before we present the features of the program in detail, let us describe the problem that originally motivated the development of the tool. Let $A \subseteq B \subseteq \mathfrak{S}_n$ and let $\text{rot} : \mathfrak{S}_n \rightarrow \mathfrak{S}_n$ the ‘rotation’-action on A , that is, conjugation with the long cycle $1, \dots, n$. Explicitly:

$$\text{rot } \pi \cdot i = \begin{cases} \pi \cdot i - 1 & \text{if } \pi \cdot i = 1, \\ n & \text{otherwise,} \end{cases}$$

where we set $\pi \cdot n = 1$. Additionally, let $\tau : B \rightarrow \mathbb{N}$ be the length of a longest increasing subsequence in a permutation, that is, the largest number k such that $\pi \cdot i_1 < \dots < \pi \cdot i_k$ for some sequence $1 \leq i_1 < \dots < i_k \leq n$.

Then, driven by a problem from invariant theory, we want to find a statistic on permutations, that is, a function $s : A \rightarrow \mathbb{N}$, which has the same distribution as τ but is additionally invariant under rot :

$$\sum_{\pi \in \mathfrak{S}_n} q^{s(\pi)} = \sum_{\pi \in \mathfrak{S}_n} q^{\tau(\pi)}, \text{ and} \\ s(\text{rot } \pi) = s(\pi) \text{ for all } \pi \in \mathfrak{S}_n.$$

Given these constraints the program is then able to produce a complete list of minimal subdistributions, that is, non-empty inclusion-wise minimal subsets $\tilde{A}$ of A together with subsets $\tilde{Z} \subset \mathbb{N}$ of the same cardinality, such that necessarily

$$\sum_{\pi \in \tilde{A}} q^{s(\pi)} = \sum_{z \in \tilde{Z}} q^z.$$

To illustrate, the program finds that the set of four permutations

$$1, 2, 3, 4, \quad 2, 3, 4, 1, \quad 3, 4, 1, 2 \quad \text{and} \quad 4, 1, 2, 3$$

must be mapped by s to the four values 1, 2, 3 and 4, in some order. A part of this is easy to explain: these are exactly the four permutations which are invariant under rotation, so all other orbits of rot have cardinality at least 2. However, there is only a single permutation with length of longest increasing subsequence equal to 1 and a single

¹The full documentation is available at <https://doc.sagemath.org/html/en/reference/combinat/sage/combinat/bijectionist.html>

permutation with the length of a longest increasing subsequence equal to 4. Thus, these two values can only be images of permutations which are invariant under rotation.

Somewhat to our disappointment, even with the help of our tool we were unable to find an explicit description of the statistic s . Still, possibly encouraged by the fact that the program demonstrated that the statistic exists for $n \leq 9$, the existence eventually could be proven [1].

Note that the existence of such a statistic is equivalent to the existence of a bijection $S : \mathfrak{S}_n \rightarrow \mathfrak{S}_n$ such that $\tau \circ S = \text{rot } \pi = \tau \circ S^{-1}$.

3 Features

Let us now describe the main features of the tool in full generality. All types of constraints can be combined.

Basic setup

Let A and B be finite sets of equal size, and let $\tau : B \rightarrow Z$ for an arbitrary set Z . Then

```
sage: bij = Bijectionist(A, B, tau)
```

initializes `bij` to represent all functions $s : A \rightarrow Z$ such that there is a bijection $S : A \rightarrow B$, such that the following diagram commutes:

$$\begin{array}{ccc} A & & \\ \downarrow s & \searrow s & \\ B & \xrightarrow{\tau} & Z \end{array}$$

As a special case we allow τ to be the identity map on B . In this case `bij` represents all bijections $S : A \rightarrow B$, and we must have $Z = B$ and $S = s$.

Constraints

Statistics

We may supply a pair (or several pairs) of statistics $\alpha : A \rightarrow W$ and $\beta : B \rightarrow W$ on A and B respectively into an arbitrary set W :

```
sage: bij.set_statistics((alpha, beta))
```


Doing so, we stipulate that the following diagram commutes:

$$\begin{array}{ccc} A & & \\ s \downarrow & \searrow \alpha & \\ B & \xrightarrow{\beta} & W \end{array}$$

In words, we require that S is an intertwining bijection between the statistics α and β .

Intertwining relations

We may supply a pair (or several pairs) of k -ary maps $\pi : A^k \rightarrow A$ and $\rho : Z^k \rightarrow Z$, thus requiring that they are intertwined by s :

```
sage: bij.set_intertwining_relations((k, pi, rho))
```

Put differently, the following diagram must then commute:

$$\begin{array}{ccc} A^k & \xrightarrow{\pi} & A \\ s^k \downarrow & & \downarrow s \\ Z^k & \xrightarrow{\rho} & Z \end{array}$$

One may want to think of π as a map which composes objects in A , and ρ as the effect this should have on the statistic we are looking for.

Quadratic relations

We may supply a pair (or several pairs) of maps $\phi : A \rightarrow Z$ and $\psi : Z \rightarrow A$,

```
sage: bij.set_quadratic_relations((phi, psi))
```

thus requiring that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{s} & Z \\ \phi \downarrow & & \downarrow \psi \\ Z & \xleftarrow{s} & A \end{array}$$

In particular, if ϕ and ψ are the identity maps (and therefore $Z = A = B$ and $S = s$), this requires that S is an involution on A .

Constant blocks

For a set partition P of A , setting

```
sage: bij.set_constant_blocks(P)
```

we stipulate that $s : A \rightarrow Z$ is constant on the blocks of P . Note that this makes sense only if τ is not injective. To illustrate, this accommodates the condition of the motivating example that the statistic should be invariant under rot , by letting P be the corresponding orbit decomposition.

Homomesy constraints

For a set partition Q of A , setting

```
sage: bij.set_homomesic(Q)
```

we assert that the average of s on each block of Q should be constant. Of course, this requires that Z is a field.

Solutions

The three most important ways to explore the solution space are as follows.

Fibers of the statistics

The instruction

```
sage: bij.statistics_fibers()
```

simply returns the preimages of $\alpha : A \rightarrow W$ and $\beta : B \rightarrow W$. Note that it does not assert that a statistic s exists.

All solutions

Issuing

```
sage: list(bij.solutions_iterator())
```

the program lists all functions $s : A \rightarrow Z$ which satisfy all the given constraints, or raises an error if no such map exists. Frequently, however, the number of such functions will be very large. Thus, and in particular if one is only interested in checking whether the constraints can be satisfied at all, it is more useful to use `next(bij.solutions_iterator())`, which only outputs a single solution, if it exists. In any case, this is computationally expensive.

Minimal subdistributions

```
sage: list(bij.minimal_subdistributions_iterator())
```

lists all minimal subdistribution implied by the given constraints, or raises an error if no solution exists. More precisely, it returns all non-empty inclusion-wise minimal subsets $\tilde{A}$ of A of together with submultisets $\tilde{Z} \subset Z$ of the same cardinality, such that we have $s \tilde{A} = \tilde{Z}$ as multisets for all possible solutions. In particular, if τ is the identity map on B , then we obtain a pair of subsets of minimal cardinality of A and B respectively which are in bijection for all maps s satisfying the given constraints. This is computationally expensive.

4 A sample application

A treasure trove for the experimental combinatorialist is the database FindStat. It should not come as a surprise that combining the toolkit with FindStat yields some potentially interesting conjectures.

Recently, Elder, Lafrenière, McNicholas, Striker and Welch [2] used the database to find homomesies on permutations, that is, statistics $s : \mathfrak{S}_n \rightarrow \mathbb{Z}$ and bijections $m : \mathfrak{S}_n \rightarrow \mathfrak{S}_n$ such that there exists a constant $c \in \mathbb{Q}$ with

$$\frac{1}{|\mathcal{O}|} \sum_{\pi \in \mathcal{O}} s(\pi) = c$$

for all orbits $\mathcal{O}$ of m . Using a brute force search, they discovered more than a hundred combinations of maps and conjecturally homomesic statistics. Remarkably, they were then able to prove that all of these conjectures are actually true.

Using our toolkit, we can take this a step further and look for statistics τ which are *equidistributed* to a homomesic statistic, given a map m :

```
sage: n = 5
sage: A = B = Permutations(n)
sage: Q = DiscreteDynamicalSystem(A, m).cycles()
sage: bij = Bijectionist(A, B, tau)
sage: bij.set_homomesic(Q)
sage: next(bij.solutions_iterator())
```

If this yields a solution, we can be quite confident that s exists. For example, if m is the rotation action `rot` considered before, iterating through the statistics in FindStat yields www.findstat.org/St001377, ‘The major index minus the number of inversions of a permutation’ as a candidate.

As a next step, we can use FindStat again to refine our conjecture, by iterating over all pairs of equidistributed statistics α and β , such that

```
sage: bij.set_statistics((alpha, beta))
sage: next(bij.solutions_iterator())
```

still gives a solution. In the case at hand, there are very many such pairs. One particularly natural refinement is to use the major index for both α and β . We now look at the minimal subdistributions in the refined setting, for $n = 4$:

```
sage: list(bij.minimal_subdistributions_iterator())
[[([4, 3, 2, 1], [0]),
 ([4, 1, 2, 3], [0]),
 ([3, 4, 1, 2], [0]),
 ([2, 3, 4, 1], [0]),
 ([2, 1, 4, 3], [0]),
 ([1, 2, 3, 4], [0]),
 ([2, 1, 3, 4], [3, 1, 2, 4], [-2, -1]),
 ...
```

We can, in fact, query FindStat again with this data:

```
sage: findstat(list(bij.minimal_subdistributions_iterator()))
0: St001377oMp00064oMp00073 (quality [100, 100])
...
```

Although this is a plausible result, it turns out that this statistic is not homomesic with respect to rotation. However, we can make the following conjecture, which we tested up to and including $n = 6$.

Conjecture 1. Let $\text{maj} : \mathfrak{S}_n \rightarrow \mathbb{N}$ be the major index and let $\text{inv} : \mathfrak{S}_n \rightarrow \mathbb{N}$ be the number of inversions of a permutation. Furthermore, let $\text{rot} : \mathfrak{S}_n \rightarrow \mathfrak{S}_n$ be the conjugation of a permutation with the long cycle $1, \dots, n$.

There is a statistic $s : \mathfrak{S}_n \rightarrow \mathbb{Z}$ which is homomesic with respect to rot , and such that the pairs of statistics (s, maj) and $(\text{maj} - \text{inv}, \text{maj})$ are equidistributed, that is,

$$\sum_{\pi \in \mathfrak{S}_n} q^{s(\pi)} t^{\text{maj}(\pi)} = \sum_{\pi \in \mathfrak{S}_n} q^{\text{maj}(\pi) - \text{inv}(\pi)} t^{\text{maj}(\pi)}.$$

5 Implementation details

In this section, we outline the basic ideas behind the implementation. We model the problem as a 0-1 integer linear program that represents the statistic s . Given a (possibly trivial) set partition P of A and given Z , we define binary variables $x_{p,z}$ for each $p \in P$ and $z \in Z$. The assignment $x_{p,z} = 1$ then indicates that $\forall a \in p : s(a) = z$, thereby encoding that s is constant on the blocks of P .

To ensure that s and τ have the same distribution, and that the bijection $S : A \rightarrow B$ realising this equidistribution respects the statistics $\alpha : A \rightarrow W$ and $\beta : B \rightarrow W$ we add the following constraints:

$$\begin{aligned} \sum_{p \in P} m_w(p) x_{p,z} &= n_w(z), & \forall w \in W \ \forall z \in Z \\ \sum_{z \in Z} x_{p,z} &= 1, & \forall p \in P \end{aligned}$$

where $m_w(p) = |\{a \in p : \alpha(a) = w\}|$ and $n_w(z) = |\{b \in B : \beta(b) = w \text{ and } \tau(b) = z\}|$.

A solution to this integer program immediately yields a solution for $s : A \rightarrow Z$. To generate all possible solutions for s , we iteratively prohibit the current solution by adding the linear constraint $\sum_{p \in P} x_{p,s(p)} < |P|$, where $s(p)$ denotes the common image of all $a \in p$.

The minimal subdistributions can be algorithmically determined by means of a dynamic optimization problem that interacts with the model above. In every iteration, we guess a smallest subset of blocks p that potentially defines a minimal subdistribution. By an auxiliary integer program we find a minimal selection of blocks with compatible distributions according to all the solutions of the base problem we have encountered so far. We then attempt to confirm the conjecture that the incumbent subdistribution holds for any solution by testing the feasibility of the base problem after prohibiting the specific guess at hand. In case of infeasibility, we terminate with the given guess; otherwise we add the information of the new solution to create a new guess for a minimal selection of blocks. Analogously, we can determine an improved partition P' of A that is implicitly forced by other restrictions of the statistic.

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