Spectrum Overselling: An Optimal Auction Design Perspective

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***Abstract*—This paper presents the design of a novel optimal spectrum auction mechanism that enables the Primary User (PU) of a channel to intelligently 'oversell' it to a carefully chosen set of unlicensed Secondary Users (SUs) while collecting suitable payments from them. The paper shows that, under inherent uncertainties associated with communication processes, our proposed overselling methodology leads to the enhancement of important performance criteria, such as the PU's utility and spectrum utilization, beyond what is permitted by traditional approaches to Dynamic Spectrum Sharing (DSS). Analytical characterization of important properties sustained by our mechanism, such as conditions under which spectrum overselling becomes optimal, have been presented and computationally efficient techniques to implement the proposed methodology have been described. Numerous simulation results have been presented to provide important insights.**

***Index Terms*—Dynamic Spectrum Sharing, Spectrum Overselling, Auction Design**

I. Introduction

Spectrum sharing is a vital technology for solving the spectrum scarcity problem and supporting the demands of next generation applications in 5G networks and beyond [1]. In support of such a technology, Primary Users (PUs), who hold long-term licenses of a spectrum range, can allow spectrum that is unused by them, referred to as *spectrum holes*, to be accessed by unlicensed Secondary Users (SUs) via Dynamic spectrum sharing (DSS) techniques [2]. To enable such a paradigm, for example, the works in [3]–[8] investigate the design of market structures that can support DSS, [9] employs game theory [10] to study and characterize spectrum sharing strategies, and [11] investigates signal processing aspects associated with DSS.

While allowing SUs to use unused spectrum belonging to PUs (i.e., spectrum holes) can improve spectrum utilization, the degree of enhancements achieved depend on communication characteristics of SUs. *What if an SU that has been permitted to use a PU's channel, does not eventually need to transmit as planned?* Such events, which are inherently hard to predict, can lead to under-exploitation of spectrum holes and greatly sacrifice spectrum utilization– a problem which has remained grossly underexplored. E.g., past work on designing spectrum auctions for allocating PU's spectrum to SUs in a market environment (e.g., [3]–[8]) has overlooked performance degradations that can occur when SUs that have been allocated spectrum may not subsequently have their planned communication needs. While [12], [13] have touched upon the above question, it should be noted that the issue, however, largely remains unaddressed to date with a lack of rigorous analytical models for the problem. We seek to fill this void in this paper.

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In particular, to alleviate the above problem, which stems from uncertainties associated with the communication characteristics of SUs, we propose to intelligently *oversell* a PU's channel to multiple SUs in a region such that the chances of having an SU which eventually needs to transmit data using the channel is enhanced. As we show, among other advantages, this enables the improvement of spectrum utilization beyond what is permitted by traditional approaches to DSS. To enable such a paradigm in a market environment, in this paper, we build on the field of mechanism design [10], [14] to present the design of an auction mechanism that would allow a PU to 'optimally' oversell its spectrum to SUs for maximizing its revenue. To the best of our knowledge, design of such a mechanism has remained unexplored by past work. Specifically, the main contributions of the paper are as follows:

- We present the design of a novel spectrum auction mechanism that would enable a PU to optimize its revenue by intelligently overselling its channel to a carefully chosen set of SUs in a region (based on their communication characteristics and bidding behaviors) and subsequently collecting suitable payments from them. Our mechanism, which has been analytically characterized, enforces desirable properties such as truthfulness of bidding behaviors.
- We analytically characterize important properties of the outcome sustained by our auction mechanism, including characterization of conditions under which the PU's channel should be oversold.
- We present computationally efficient techniques to implement our designed auction mechanism.
- Numerous simulation results have been presented that provide important insights, including demonstration of the benefits that our mechanism brings for the PU's utility and spectrum utilization.

II. Formulation of the Auction Design Problem

Consider a PU which conducts an auction to sell an unused channel that it owns to a set of SUs (denoted as $\{1, \cdots, N\}$) by soliciting bids from them. Based on the bidding behavior of the SUs and the uncertainties associated with their communication characteristics, consider the PU to select a set of SUs to (over) sell the channel to and determine the payments that they should make in return. Specifically, to formulate the problem, suppose that the true valuation that SU i, $i \in \{1, \cdots, N\}$, has for the channel is $v_i \in [a_i, b_i]$. To model the PU's uncertainty regarding SU i's true valuation, consider it to be a random variable with $f_i : [a_i, b_i] \to \mathbb{R}_+$ being its probability density function (pdf) and $F_i : [a_i, b_i] \to [0, 1]$ being its cumulative distribution function (cdf). Further, to model uncertainties regarding SUs' communication characteristics, suppose that SU i that has been allocated the channel, eventually needs to transmit on the channel with a probability q_i. Such uncertainties can arise from inherent randomness in SUs' traffic patterns and even from uncertainties regarding their presence at the intended location and time for attempting transmission.

To enhance spectrum utilization and optimize the PU's utility under such communication uncertainties of the SUs, in this paper, we design an optimal spectrum auction mechanism that will allow the PU to judiciously oversell the channel to more than one SU (knowing that, as long as one of the SUs which have been assigned the channel transmits, the channel would be used properly). Formally, denoting the vector of bids that the PU receives as $\mathbf{v} = (v_1, \cdots, v_N)$, our auction mechanism can be described by the following two functions:

- *SU selection function*, $\mathbf{p}(\mathbf{v}) = \big(p_1(\mathbf{v}), \cdots, p_N(\mathbf{v})\big)$, where $p_i(\mathbf{v})$ is the probability that the PU assigns the channel to SU i, and
- *Payment function*, $\mathbf{x}(\mathbf{v}) = \big(x_1(\mathbf{v}), \cdots, x_N(\mathbf{v})\big)$, where $x_i(\mathbf{v})$ is the payment that SU i makes to the PU.

The goal of our auction design problem is to design the functions $\mathbf{p}(\mathbf{v})$ *and* $\mathbf{x}(\mathbf{v})$ *such that they allow the PU to optimally oversell its channel to maximize the expected utility that it gets from the auction while satisfying certain constraints.*

Next, we describe the expected utilities of the PU (seller) and the SUs (buyers/bidders) from our auction mechanism and then formulate the auction design problem.

A. Expected utilities of the PU and SUs

The expected utility of the PU ($U_0(\mathbf{p}, \mathbf{x})$) from the auction mechanism can be defined as

$$U_0(\mathbf{p}, \mathbf{x}) = \int_V \sum_{i=1}^{N} x_i(\mathbf{v}) f(\mathbf{v}) d\mathbf{v} \tag{1}$$

where $V = [a_1, b_1] \times \cdots \times [a_N, b_N]$ denotes the set of all possible combinations of SUs' valuations, $f(\mathbf{v}) = \prod_{i=1}^{N} f_i(v_i)$ is the joint density function on V for the vector of valuations $\mathbf{v} = (v_1, \cdots, v_N)$, and $d\mathbf{v} = dv_1 \cdots dv_N$. Further, the expected utility that SU i, $i \in \{1, \cdots, N\}$, gets from the auction from bidding its true valuation $v_i \in [a_i, b_i]$ is

$$U_i(\mathbf{p}, \mathbf{x}, v_i) = \int_{V_{-i}} \big(v_i\, p_i(\mathbf{v}) q_i g_{-i}(\mathbf{v}) - x_i(\mathbf{v})\big) f_{-i}(\mathbf{v}_{-i}) d\mathbf{v}_{-i} \tag{2}$$

where $V_{-i} = [a_1, b_1] \times \cdots \times [a_{i-1}, b_{i-1}] \times [a_{i+1}, b_{i+1}] \times \cdots \times [a_N, b_N]$ denotes the set of all possible combinations of SUs' valuations other than SU i, $f_{-i}(\mathbf{v}_{-i}) = \prod_{j \in \{1, \cdots, N\}, j \neq i} f_j(v_j)$ is the joint density function on V_{-i} for the vector of valuations $\mathbf{v}_{-i} = (v_1, \cdots, v_{i-1}, v_{i+1}, \cdots, v_N)$, $d\mathbf{v}_{-i} = dv_1 \cdots dv_{i-1} dv_{i+1} \cdots dv_N$, and q_i, as defined earlier, is the probability of SU i that has been allocated the channel eventually needing to transmit using it. In (2),

$$g_{-i}(\mathbf{v}) = \prod_{j=1, j \neq i}^{N} \big(1 - p_j(\mathbf{v})\, q_j\big) \tag{3}$$

is the probability of all SUs, without consideration of SU i, *not* transmitting on the channel for a given $\mathbf{p}$. Moreover, given that the *true* valuation of SU i is $v_i \in [a_i, b_i]$, the expected utility that the SU gets from bidding a *falsified* valuation $w_i \in [a_i, b_i]$ while hoping to make an undue profit is

$$\begin{aligned} \tilde{U}_i(\mathbf{p}, \mathbf{x}, w_i) = \int_{V_{-i}} & \big(v_i\, p_i(\mathbf{v}_{-i}, w_i)\, q_i\, g_{-i}(\mathbf{v}_{-i}, w_i) \\ & - x_i(\mathbf{v}_{-i}, w_i)\big) f_{-i}(\mathbf{v}_{-i})\, d\mathbf{v}_{-i} \end{aligned} \tag{4}$$

where $(\mathbf{v}_{-i}, w_i) = (v_1, \cdots, v_{i-1}, w_i, v_{i+1}, \cdots, v_N)$.

B. Auction Design as an Optimization Problem

The task of designing our spectrum auction mechanism can be formulated as the following optimization problem:

$$\max_{\mathbf{p}, \mathbf{x}} \; U_0(\mathbf{p}, \mathbf{x})$$

subject to:

$$U_i(\mathbf{p}, \mathbf{x}, v_i) \geq 0, \; \forall i \in \{1, \cdots, N\} \tag{5a}$$

$$U_i(\mathbf{p}, \mathbf{x}, v_i) \geq \tilde{U}_i(\mathbf{p}, \mathbf{x}, w_i), \; \forall i, \forall v_i, w_i \in [a_i, b_i] \tag{5b}$$

$$0 \leq p_i(\mathbf{v}) \leq 1, \; \forall i \in \{1, \cdots, N\} \tag{5c}$$

The above three constraints are explained below:

- *Individual-Rationality (IR) constraint* (5a) justifies the participation of SUs in the auction by ensuring that their expected utilities are non-negative.
- *Incentive-Compatibility (IC) constraint* (5b) disincentivizes SUs from lying about their valuations of the channel during bidding by ensuring that honest reporting of valuations form a Nash Equilibrium (NE).
- *Selection Parameter constraint* (5c) ensures that the probability with which an SU is selected for being assigned the channel follows proper probabilistic definitions. *Note that there is no constraint that restricts the channel to be allocated to at most one SU to allow overselling.*

Next, we analyze the above auction design problem.

III. Analysis of the Auction Design Problem

For a given bid v_i, let us define the probability of SU i, $i \in \{1, \cdots, N\}$, *successfully* transmitting on the channel, which would require SU i to be assigned the channel and to eventually transmit using it while other SUs do not, as

$$Q_i(\mathbf{p}, v_i) = \int_{V_{-i}} [p_i(\mathbf{v}_{-i}, v_i) q_i g_{-i}(\mathbf{v}_{-i}, v_i)] f_{-i}(\mathbf{v}_{-i}) d\mathbf{v}_{-i} \tag{6}$$

Using (6), we first present a simplified characterization of (5b).

Lemma 1: *The IC constraint in* (5b) *holds if and only if the following two conditions hold* $\forall i \in \{1, \cdots, N\}$:

$$\textit{if } v_i \geq w_i, \; \textit{then } Q_i(\mathbf{p}, v_i) \geq Q_i(\mathbf{p}, w_i) \tag{7a}$$

$$\int_{a_i}^{v_i} Q_i(\mathbf{p}, w_i) dw_i = U_i(\mathbf{p}, \mathbf{x}, v_i) - U_i(\mathbf{p}, \mathbf{x}, a_i) \tag{7b}$$

Proof: Consider $v_i, w_i \in [a_i, b_i]$ with $v_i \geq w_i$, $i \in \{1, \cdots, N\}$. Now, if v_i is SU i's true valuation for the channel while it bids the falsified valuation w_i, the expected utility that the SU gets can be expressed using (2), (4) and (6) as

$$\tilde{U}_i(\mathbf{p}, \mathbf{x}, w_i) = U_i(\mathbf{p}, \mathbf{x}, w_i) + (v_i - w_i)\, Q_i(\mathbf{p}, w_i) \tag{8}$$

To ensure that SU i does not have an incentive to bid such a falsified valuation w_i, imposing the IC constraint (5b) we get

$$U_i(\mathbf{p}, \mathbf{x}, v_i) \geq U_i(\mathbf{p}, \mathbf{x}, w_i) + (v_i - w_i)\, Q_i(\mathbf{p}, w_i) \tag{9}$$

Similarly, considering w_i to be the true valuation of SU i for the channel and v_i to be a falsified valuation that the SU bids, the IC constraint implies that

$$U_i(\mathbf{p}, \mathbf{x}, w_i) \geq U_i(\mathbf{p}, \mathbf{x}, v_i) + (w_i - v_i)\, Q_i(\mathbf{p}, v_i) \tag{10}$$

From (9) and (10), we get

$$\begin{aligned}(v_i - w_i)\, Q_i(\mathbf{p}, w_i) \leq U_i(\mathbf{p}, \mathbf{x}, v_i) - U_i(\mathbf{p}, \mathbf{x}, w_i) \leq \\ (v_i - w_i)\, Q_i(\mathbf{p}, v_i)\end{aligned} \tag{11}$$

Note that (7a) is clearly implied by (11). Further, letting $v_i = w_i + \delta$, the above inequality can be rewritten as:

$$\delta\, Q_i(\mathbf{p}, w_i) \leq U_i(\mathbf{p}, \mathbf{x}, w_i+\delta) - U_i(\mathbf{p}, \mathbf{x}, w_i) \leq \delta\, Q_i(\mathbf{p}, w_i+\delta) \tag{12}$$

Clearly, (12) implies that $Q_i(\mathbf{p}, w_i)$ is Riemann-integrable, from which (7b) follows.

Now, we show that (7a) and (7b) imply the IC constraint. Using (7b), and noting that $v_i \geq w_i$, we get

$$U_i(\mathbf{p}, \mathbf{x}, v_i) = U_i(\mathbf{p}, \mathbf{x}, w_i) + \int_{w_i}^{v_i} Q_i(\mathbf{p}, r_i) dr_i$$

As $r_i \geq w_i$, using (7a), the above expression implies that

$$U_i(\mathbf{p}, \mathbf{x}, v_i) \geq U_i(\mathbf{p}, \mathbf{x}, w_i) + \int_{w_i}^{v_i} Q_i(\mathbf{p}, w_i) dr_i$$

which reduces to the IC constraint in (9). Similarly, considering $v_i < w_i$ in the above analysis implies the IC constraint in (10), proving the 'only if' part and concluding the proof. ■

Using Lemma 1, we can simplify the optimization problem in (5) to the form given in the following theorem.

THEOREM 1: *For* $(\mathbf{p}, \mathbf{x})$ *to represent an optimal auction mechanism,* $\mathbf{p}$ *should be such that it maximizes*

$$\int_V \sum_{i=1}^{N} \left(\left(v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} \right) p_i(\mathbf{v})\, q_i\, g_{-i}(\mathbf{v}) \right) f(\mathbf{v})\, d\mathbf{v} \tag{13}$$

subject to constraint (5c)*, and the payment made by SU i,* $i \in \{1, \cdots, N\}$*, should follow*

$$x_i(\mathbf{v}) = p_i(\mathbf{v}) q_i v_i g_{-i}(\mathbf{v}) - \int_{a_i}^{v_i} [p_i(\mathbf{v}_{-i}, w_i) q_i\, g_{-i}(\mathbf{v}_{-i}, w_i)] dw_i \tag{14}$$

Proof: The PU's expected utility (1) can be re-written as:

$$\begin{aligned} U_0(\mathbf{p}, \mathbf{x}) &= \int_V \sum_{i=1}^{N} [x_i(\mathbf{v}) - v_i\, p_i(\mathbf{v})\, q_i\, g_{-i}(\mathbf{v})] f(\mathbf{v})\, d\mathbf{v} \\ &\quad + \int_V \sum_{i=1}^{N} [v_i\, p_i(\mathbf{v})\, q_i\, g_{-i}(\mathbf{v})] f(\mathbf{v})\, d\mathbf{v} \\ &= -\sum_{i=1}^{N} \int_{a_i}^{b_i} U_i(\mathbf{p}, \mathbf{x}, v_i) f_i(v_i)\, dv_i \\ &\quad + \sum_{i=1}^{N} \int_V [v_i\, p_i(\mathbf{v})\, q_i\, g_{-i}(\mathbf{v})] f(\mathbf{v})\, d\mathbf{v} \ \big(\text{using (2)}\big) \end{aligned} \tag{15}$$

Now, using (7b), we have

$$\begin{aligned} \int_{a_i}^{b_i} U_i(\mathbf{p}, \mathbf{x}, v_i) f_i(v_i)\, dv_i &= \int_{a_i}^{b_i} U_i(\mathbf{p}, \mathbf{x}, a_i) f_i(v_i)\, dv_i \\ &\quad + \int_{a_i}^{b_i} \int_{a_i}^{v_i} Q_i(\mathbf{p}, w_i)\, f_i(v_i)\, dw_i\, dv_i \\ &= U_i(\mathbf{p}, \mathbf{x}, a_i) + \int_{a_i}^{b_i} \int_{w_i}^{b_i} f_i(v_i)\, Q_i(\mathbf{p}, w_i)\, dv_i\, dw_i \\ &= U_i(\mathbf{p}, \mathbf{x}, a_i) + \int_V \big(1 - F_i(v_i)\big) p_i(\mathbf{v})\, q_i\, g_{-i}(\mathbf{v}) f_{-i}(\mathbf{v}_{-i})\, d\mathbf{v} \end{aligned} \tag{16}$$

Substituting (16) into (15), we get:

$$\begin{aligned} U_0(\mathbf{p}, \mathbf{x}) = \int_V \sum_{i=1}^{N} \left[\left(v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} \right) p_i(\mathbf{v})\, q_i\, g_{-i}(\mathbf{v}) \right] \\ .\, f(\mathbf{v})\, d\mathbf{v} - \sum_{i=1}^{N} U_i(\mathbf{p}, \mathbf{x}, a_i) \end{aligned} \tag{17}$$

In (17), $\mathbf{x}$ only appears in the last term of the PU's utility. Now, from (7b), it can be noted that for SU i, $i \in \{1, \cdots, N\}$, if $U_i(\mathbf{p}, \mathbf{x}, a_i) \geq 0$, we have $U_i(\mathbf{p}, \mathbf{x}, v_i) \geq 0$, $\forall v_i \in [a_i, b_i]$, which leads to the satisfaction of the IR constraint (5a). Thus, the best possible value of the last term of (17) can be obtained, which is zero since the PU seeks to maximize its utility, as well as the IR constraint can be satisfied, by having $U_i(\mathbf{p}, \mathbf{x}, a_i) = 0$, $\forall i \in \{1, \cdots, N\}$, which implies, using (7b) that

$$U_i(\mathbf{p}, \mathbf{x}, v_i) - \int_{a_i}^{v_i} Q_i(\mathbf{p}, w_i) dw_i = 0 \tag{18}$$

Substituting (2) and (6) into (18), we get (14) with the PU's utility thereby becoming (13). This proves the theorem. ■

A. Characteristics of the Auction Outcome

First, for $i \in \{1, \cdots, N\}$, let us define

$$\theta_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} \tag{19}$$

Further, for a vector of bids $\mathbf{v}$ received from N SUs, define

$$\xi^{(N)} = \sum_{i=1}^{N} \theta_i(v_i)\, p_i(\mathbf{v})\, q_i\, g_{-i}(\mathbf{v}) \tag{20}$$

Now, it should be noted that the PU's expected utility in (13) is maximized if $\mathbf{p}(\mathbf{v})$ is such that it maximizes (20) for all $\mathbf{v} \in V$ (subject to constraint (5c)). Next, in Lemma 2, we present an important characteristic of $\mathbf{p}(\mathbf{v})$ that maximizes (20).

LEMMA 2: *There always exists an SU selection strategy* $\mathbf{p}(\mathbf{v}) = \big(p_1(\mathbf{v}), \cdots, p_N(\mathbf{v})\big)$ *whose* p_i *is Boolean* $\forall i \in \{1, \cdots, N\}$ *that dominates any probabilistic solution of* (20).

Proof: The partial derivative of (20) w.r.t. p_i yields

$$\frac{\partial \xi^{(N)}}{\partial p_i} = \theta_i(v_i)\, q_i\, g_{-i}(\mathbf{v}) - q_i \sum_{j=1, j \neq i}^{N} \theta_j(v_j)\, p_j(\mathbf{v})\, q_j\, g_{-i,-j}(\mathbf{v}) \tag{21}$$

where $g_{-i,-j}(\mathbf{v}) = \prod_{k=1, k \neq i,j}^{N} (1 - p_k(\mathbf{v})\, q_k)$ is the probability of all SUs, other than i and j, *not* transmitting on the channel. Clearly, (21) is independent of p_i. This implies that, for any given $p_{-i} = (p_1, \cdots, p_{i-1}, p_{i+1}, \cdots, p_N)$, (20) varies *linearly* w.r.t. p_i. Thus, for $\mathbf{p}$ to optimize (20), for any $i \in \{1, \cdots, N\}$, we must have $p_i = 0$ (if $\frac{\partial \xi^{(N)}}{\partial p_i} < 0$) or $p_i = 1$ (if $\frac{\partial \xi^{(N)}}{\partial p_i} > 0$) or that any $p_i \in [0, 1]$ is optimal (if $\frac{\partial \xi^{(N)}}{\partial p_i} = 0$). This proves the lemma. ■

Case	Optimal (p_i, p_j)	Optimality Condition
(i)	$(1, 1)$	$q_i \leq \eta_i(\mathbf{v}); \forall i \in \{1, 2\}$
(ii)	$(0, 1)$	$0 \leq q_i \leq 1, q_j \geq \eta_j(\mathbf{v}); i, j \in \{1, 2\}, i \neq j$
(iii)	$(1, *)$	$q_i = \eta_i(\mathbf{v}), q_j \leq \eta_j(\mathbf{v}); i, j \in \{1, 2\}, i \neq j$

TABLE I
OPTIMAL SU SELECTION

Next, considering two participating SUs, we analyze how their communication uncertainties (q_is) impact the channel allocation strategy of our auction mechanism.

LEMMA 3: *In the presence of two participating SUs, our auction mechanism deterministically oversells the channel, i.e., the optimal solution to* (20) *is* $\big(p_1(\mathbf{v}), p_2(\mathbf{v})\big) = (1, 1)$, *when we have* $q_i \leq \eta_i(\mathbf{v})$, $\forall i \in \{1, 2\}$, *such that*

$$\eta_i(\mathbf{v}) = \frac{\theta_j(v_j)}{\theta_i(v_i) + \theta_j(v_j)}, \tag{22}$$

where $j \in \{1, 2\}$, $j \neq i$, *and* $\theta_i(v_i)$ *follows* (19).

Proof: For $N = 2$, (20) becomes:

$$\xi^{(2)} = \theta_1(v_1)\, p_1\, q_1\, \big(1 - p_2\, q_2\big) + \theta_2(v_2)\, p_2\, q_2\, \big(1 - p_1\, q_1\big) \tag{23}$$

In (23), clearly, if $\theta_i(v_i)$ is negative, $i \in \{1, 2\}$, it is optimal to assign the corresponding p_i as zero. Therefore, to explore the criteria for having $p_i > 0$ as the optimal solution, we focus on $\theta_i(v_i) \geq 0$. Now, for $i, j \in \{1, 2\}$, $i \neq j$, for $p_i = 1$ to optimize (23), clearly, we must have $\frac{\partial \xi^{(2)}}{\partial p_i} \geq 0$, which yields $\theta_i(v_i)\, q_i\, (1 - p_j\, q_j) - \theta_j(v_j)\, p_j\, q_j\, q_i \geq 0$, which simplifies to $q_j \leq \eta_j / p_j$, where η_j is defined in (22). Thus, we can conclude that the optimal solution of (23) corresponds to $\big(p_1, p_2\big) = (1, 1)$ when $q_i \leq \eta_i$ $\forall i \in \{1, 2\}$. This proves the lemma. ∎

While Lemma 3 characterizes the condition for $(p_1, p_2) = (1, 1)$ to be the optimal solution of (20), Table I, considering $N = 2$, exhaustively lists $\mathbf{p}$ that optimizes (20) under q_i satisfying varying conditions, $i \in \{1, 2\}$, with $\theta_i(v_i) \geq 0$ (since, otherwise, the solution is clearly easy to obtain). Since Case (i) in the table was addressed in Lemma 3, we prove the optimality conditions corresponding to the other cases below:

- Case (ii): For $(p_i, p_j) = (0, 1)$ to be optimal, $i, j \in \{1, 2\}$, $i \neq j$, using (23), we must have $\frac{\partial \xi^{(2)}}{\partial p_i} \leq 0$ and $\frac{\partial \xi^{(2)}}{\partial p_j} \geq 0$, which implies $q_j \geq \eta_j / p_j$ and $q_i \leq \eta_i / p_i$, respectively. Now, substituting $p_i = 0$ into the latter condition, we get $q_i \leq \infty$ (i.e., $0 \leq q_i \leq 1$), and substituting $p_j = 1$ into the former condition, we get $q_j \geq \eta_j$.
- Case (iii): For $(p_i, p_j) = (1, *)$ to be optimal, $i, j \in \{1, 2\}$, $i \neq j$, where $*$ denotes any value of $p_j \in [0, 1]$, we must have $\frac{\partial \xi^{(2)}}{\partial p_i} \geq 0$ and $\frac{\partial \xi^{(2)}}{\partial p_j} = 0$, which implies $q_j \leq \eta_j / p_j$ and $q_i = \eta_i / p_i$, respectively. Now, substituting $p_i = 1$ into the latter condition, we get $q_i = \eta_i$, and since the former condition, whose R.H.S. varies from η_j to ∞ (as p_j varies from 1 to 0), must hold for any value of $p_j \in [0, 1]$, we must clearly have $q_j \leq \eta_j$.

Fig. 1 presents numerical results to corroborate Lemma 2 and Lemma 3. For the figure, we considered two SUs, numbered 1 and 2, which bid $v_1 = 15$ and $v_2 = 30$ for the channel with $q_1 = 0.7$ and $q_2 = 0.2$. The valuations of the two SUs were considered to be uniformly distributed over $[a_1, b_1] = [0, 20]$ and $[a_2, b_2] = [0, 30]$. As can be seen from the figure, the PU's utility (20) increases linearly with both p_1 and p_2. This

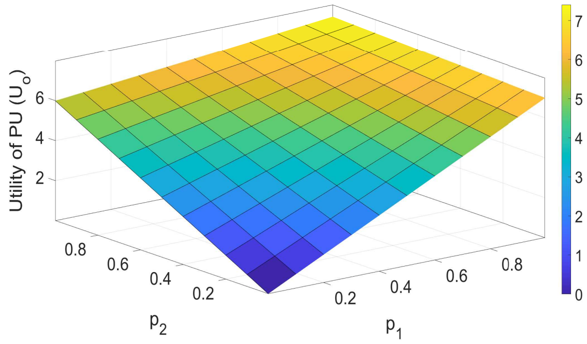


Fig. 1. PU's utility (U_0) vs. SU selection probabilities (p_1 and p_2).

corroborates the ideology of the proof of Lemma 2. Further, under the above parameters, using (19) and (22), it can be shown that $\eta_1 = 0.75$ and $\eta_2 = 0.25$. Accordingly, since $q_1 \leq \eta_1$ and $q_2 \leq \eta_2$, as can be seen from the figure, the PU's utility is maximized when $(p_1, p_2) = (1, 1)$ (i.e., when the channel is sold to both the SUs). This corroborates Lemma 3.

B. Determination of the Auction Outcome

Given the presence of N SUs in general, and their advertised bids $\mathbf{v}$, we now describe how to optimize (20) for finding the SU(s) that the channel should be (over) sold to and how to compute their payments using (14).

1) *Optimal selection of SUs*: It should be noted that (20) is non-convex in nature since its leading principal minors, viz. $D_1 = \frac{\partial^2 \xi^{(N)}}{\partial p_i^2} = 0$ and $D_2 = -\left(\frac{\partial^2 \xi^{(N)}}{\partial p_i \partial p_j}\right)^2 < 0$, which implies that the function is neither concave nor convex [15]. In such a scenario, noting that both our maximization objective (20) and constraint (5c) are polynomials, we can employ a semidefinite solver like SeDuMi [16] (to reduce our non-convex problem to solving a sequence of finite convex linear matrix inequality (LMI) problems) along with the MATLAB software Gloptipoly 3 [17]. To corroborate the solution obtained via such an approach, we employed Gloptipoly 3 to optimize (20) subject to (5c) under the parameters that were used for Fig. 1. The solution yielded by the optimization process corresponded to $(p_1, p_2) = (1, 1)$, which tallies with the optimal solution.

2) *Determination of SUs' payments*: The payment that an SU should make can be found using (14). First, as can be noted, an SU that is not assigned the channel, does not make any payment. This is due to the fact that for advertised bid $v_i \in [a_i, b_i]$ of SU i, if $p_i(v_i, \mathbf{v}_{-i}) = 0$, we have that: *a)* the first term of (14) is 0; and *b)* the second term of (14) is also 0 (over bids in $[a_i, v_i]$) since its integrand, which is clearly non-negative, is a monotonically increasing (non-decreasing) function of the bid of SU i as follows from (7a). Next, to find the payment of SU i that is assigned the channel, we first present a characteristic of the integrand in (14) in the following lemma (which can be exploited to compute the integral).

LEMMA 4: *Considering optimal solutions of* (20), *the integrand in* (14), *viz.* $p_i(\mathbf{v}_{-i}, w_i)\, q_i\, g_{-i}(\mathbf{v}_{-i}, w_i)$, *is a monotonically increasing step function of SU i's bid* w_i.

Proof: As follows from (7a), the integrand in (14) is a monotonically increasing function of bid w_i of SU i, $i \in \{1, \cdots, N\}$. Further, as follows from Lemma 2, a Boolean

p_i, $\forall i \in \{1, \cdots, N\}$, always exists that would maximize (20) while satisfying (5c), making the integrand in (14) behave as a step function w.r.t. w_i. Putting the above two characteristics together, the lemma follows. ■

Next, in Algorithm 1, we exploit Lemma 4 to compute the payment of SU i, $i \in \{1, \cdots, N\}$, that is assigned the channel. Specifically, Algorithm 1 takes as inputs the advertised bids $\mathbf{v} = (v_1, \cdots, v_N)$ and the index i of an SU whose payment needs to be computed. In Step 1, the algorithm initializes the payment $x_i(\mathbf{v})$ to the first term of (14). In steps 3-16, leveraging the characteristic presented in Lemma 4, the algorithm employs a variation of the bisection method [15] to iteratively identify points of discontinuities of (14)'s integrand over $[a_i, v_i]$ (steps 4-13), and then progressively computes the area under the integrand function between consecutive points of discontinuities (to compute the integration over consecutive discontinuity points) while gradually, following the nature of (14), subtracting the found areas from the previous value of $x_i(\mathbf{v})$ (steps 14 and 17). Since the integration in (14) ranges over $[a_i, v_i]$, the `while` loop in steps 3-16 iterates until the integrand in (14) assumes its value evaluated at SU's bid a_i.

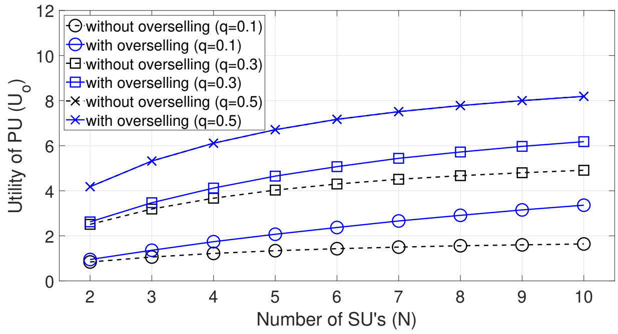
Algorithm 1 Computation of SU i's payment

Require: $\mathbf{v} = (v_1, \cdots, v_N)$; index i of the SU whose payment needs to be computed.

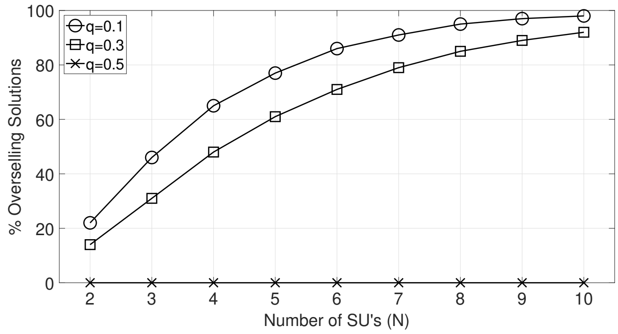
1: $x_i(\mathbf{v}) \leftarrow p_i(\mathbf{v})\, q_i\, v_i\, g_{-i}(\mathbf{v})$
2: $w \leftarrow v_i$
3: **while** $p_i(\mathbf{v}_{-i}, w)\, g_{-i}(\mathbf{v}_{-i}, w) \neq p_i(\mathbf{v}_{-i}, a_i)\, g_{-i}(\mathbf{v}_{-i}, a_i)$ **do**
4: $\quad l \leftarrow a_i$
5: $\quad r \leftarrow w$
6: $\quad$ **while** $r - l > \epsilon$ **do**
7: $\quad\quad m \leftarrow \frac{l+r}{2}$
8: $\quad\quad$ **if** $p_i(\mathbf{v}_{-i}, m)\, g_{-i}(\mathbf{v}_{-i}, m) < p_i(\mathbf{v}_{-i}, w)\, g_{-i}(\mathbf{v}_{-i}, w)$ **then**
9: $\quad\quad\quad l \leftarrow m$
10: $\quad\quad$ **else**
11: $\quad\quad\quad r \leftarrow m$
12: $\quad\quad$ **end if**
13: $\quad$ **end while**
14: $\quad x_i(\mathbf{v}) \leftarrow x_i(\mathbf{v}) - (w - l)\, p_i(\mathbf{v}_{-i}, w)\, q_i\, g_{-i}(\mathbf{v}_{-i}, w)$
15: $\quad w \leftarrow l - \epsilon$
16: **end while**
17: $x_i(\mathbf{v}) \leftarrow x_i(\mathbf{v}) - (w - a_i)\, p_i(\mathbf{v}_{-i}, w)\, q_i\, g_{-i}(\mathbf{v}_{-i}, w)$

IV. Simulation Results

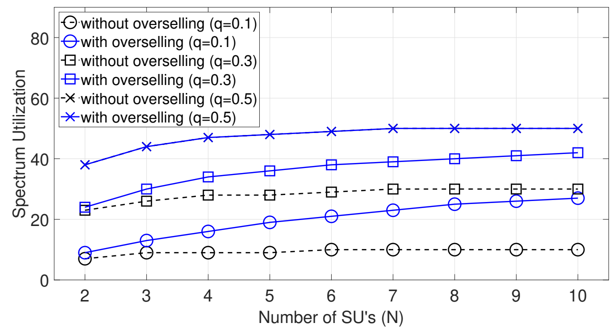
In this section, we provide simulation results to gain insights into our developed auction mechanism and show its performance advantages. In Fig. 2, we study the performance and characteristics of our developed auction mechanism with varying number of SUs (N). Specifically, in Fig. 2(a), Fig. 2(b), and Fig. 2(c), we show how the utility that the PU acquires, how the percentage of solutions where the channel is oversold (to more than one SU), and how spectrum utilization scales, respectively, with N. For the figures, we considered the bids of all SUs to be uniformly distributed over $[0, 20]$ with



(a) Utility of the PU with and without overselling (U_o) vs. N



(b) Percentage of solutions which oversold the channel vs. N



(c) Spectrum Utilization with and without overselling vs. N

Fig. 2. Performance analysis with varying no. of SUs (N)

$q = q_1 = q_2 = \cdots = q_N$. We have performed 100k iterations over randomly generated bids to get the presented results.

Note that in Fig. 2(a) and 2(c), 'with overselling' refers to the operating point obtained from our auction mechanism in Theorem 1 while 'without overselling' corresponds to the solution of (5) with the constraint $\sum_{i=1}^{N} p_i(\mathbf{v}) \leq 1$ added (to ensure that the channel is sold to at most one SU). It can be shown that in the latter case, for a given $\mathbf{v}$, since $g_{-i}(\mathbf{v}) = 1$ when SU i is assigned the channel, the optimal solution of (20) corresponds to, if $\max_{i \in \{1, \cdots, N\}} \theta_i(v_i) \geq 0$, deterministically assigning the channel to SU j such that $j = \operatorname{argmax}_{i \in \{1, \cdots, N\}} \theta_i(v_i) q_i$, and to not allocate the channel to any SU otherwise, where $\theta_i(\cdot)$ follows (19). In such a scenario, the payment of winning SU j can be found from (14) using the approach described in Section III-B2 with $g_{-j}(\cdot) = 1$. Also, under the considered simulation parameters, at $q = 0.5$, note that our mechanism dictates that the channel should not be oversold, and accordingly, the 'with overselling' and 'without

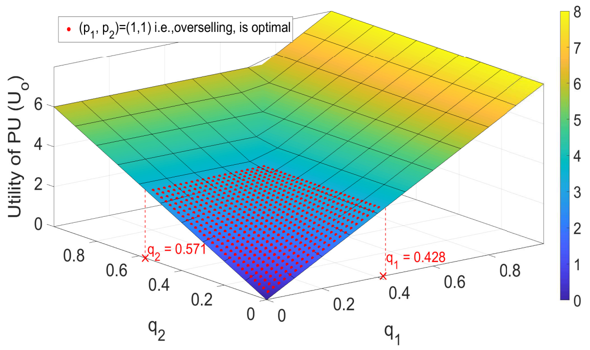


Fig. 3. Utility of the PU (U_0) vs. communication uncertainties of the SUs

overselling' solutions coincide in Figs. 2(a) and 2(c).

As can be seen from Fig. 2(a), for any given N, under values of q that allow the channel to be oversold (which are $q = 0.1$ and $q = 0.3$ in the figure), the PU's utility 'with overselling' is greater than that obtained 'without overselling'. Again, as can be seen from Fig. 2(c), under the aforementioned values of q, spectrum utilization, measured by the percentage of times the channel was used to communicate successfully (out of the 100k runs), in the 'with overselling' case is greater than that obtained 'without overselling'. *These observations clearly show the performance advantages of our proposed approach.*

Further, as can be seen from Fig. 2(a), under the aforementioned values of q that allow the channel to be oversold, the PU's utility 'with overselling' increases at a faster rate with N than the rate at which it increases 'without overselling'. This is because, with increasing N, while increase in the PU's utility 'without overselling' can only be attributed to the enhanced ability to find an SU which is willing to make a higher payment for the channel, in the 'with overselling' case, not only does the PU's utility benefit from the above advantage as N increases, but also from the ability to oversell the channel to more SUs. This latter trend is corroborated by Fig. 2(b), which shows that the percentage of solutions (out of the 100k runs) in which the channel is oversold (to more than one SU) increases with N (under values of q that allow the channel to be oversold). Implications of the observation made from Fig. 2(b) can also be noted in Fig. 2(c) which shows that spectrum utilization, under values of q that allow the channel to be oversold, increases with N (while spectrum utilization 'without overselling', as expected, barring a small initial increase, relatively remains constant as N varies).

In Fig. 3, we study the impact that communication uncertainties of SUs (q_is) have on the PU's utility and the overselling dynamics of our auction mechanism. For the figure, we consider 2 SUs, which bid $(v_1, v_2) = (14, 18)$, which were chosen uniformly from the ranges $[a_1, b_1] = [10, 20]$ and $[a_2, b_2] = [10, 30]$, respectively. As can be seen, the PU's utility monotonically increases as q_1 and q_2 increase since, as follows from (14), the payment made by winning SU i increases with q_i. Further, as noted in the figure, our auction mechanism oversells the channel $\big($i.e., chooses $(p_1, p_2) = (1, 1)\big)$ when $0 \leq q_1 \leq 0.428$ and $0 \leq q_2 \leq 0.571$. It can be noted that this tallies with Lemma 3, which also prescribes that the channel be sold to both SUs when $q_1 \leq \frac{\theta_2(v_2=18)}{\theta_2(v_1=14)+\theta_2(v_2=18)} = \frac{6}{8+6} = 0.428$ and $q_2 \leq \frac{\theta_1(v_1=14)}{\theta_2(v_1=14)+\theta_2(v_2=18)} = \frac{8}{8+6} = 0.571$, which corroborate the lemma.

V. Conclusion

This paper presented the design of a novel spectrum auction mechanism that allows a PU to intelligently oversell its channel to a set of SUs in a region based on their advertised bids and their communication characteristics. As has been shown, under inherent uncertainties associated with communication processes, our designed overselling methodology can enhance the PU's revenue and the associated spectrum utilization beyond what is permitted by schemes where overselling in not permitted. Conditions under which spectrum overselling is optimal have been analytically characterized and computationally efficient techniques to implement our mechanism have been described. Numerous simulation results, which provide important insights, have been presented. In the future, we plan to build on our current results to explore double auction formats for overselling spectrum.

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