

Dynamical Coherence Measures

Anna Vershynina

*Department of Mathematics, Philip Guthrie Hoffman Hall, University of Houston, 3551 Cullen Blvd., Houston, TX
77204-3008, USA*

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Abstract

We present several measures of the dynamic coherence of channels and investigate their properties.

1 Introduction

Quantum coherence describes the existence of quantum interference, and it is often used in thermodynamics [1, 8, 16], transport theory [21, 33], and quantum optics [10, 23], among few applications. Quantum coherence resource theory starts with free, i.e. incoherent, states, which are diagonal states in a pre-fixed basis. Free operations are some quantum channels that do not create coherence where it was absent, in other words, map the set of incoherent states to itself. Problems involving coherence included quantification of coherence [2, 19, 20, 24, 29, 30, 34], distribution [18], entanglement [5, 26], operational resource theory [4, 5, 9, 32], correlations [13, 17, 27]. See [25] for a more detailed review.

Relatively recently, static resource theories (i.e. the one mostly concerned with states and their manipulation) have been extended to regard quantum channels as the elementary generalized resource, leading to a wide open area of research of the dynamical resource theory. Static resource theory has three main components: free states, free operations and resource measures. In analogue, dynamic resource theory must have free channels, free superchannels and resource measures. Note that the dynamical theory is a generalization of the statical one since any state can be regarded as a quantum channel mapping a trivial state to a given one. Much progress has been focused on the development of the theory of entropic quantifiers of channels and operational resource theory [6, 12, 15, 22, 28, 35], to name a few references.

In Chapter 2, we start with common notions and notations: relative entropy/divergence of channels, entropy of channels, static and the dynamic coherence theories. We present a table comparing the building blocks of a static and a dynamic coherence resource theories.

In Chapter 3, we present a few measures of the dynamic coherence of channels and investigate their properties.

2 Preliminaries

2.1 Relative entropy of the channels

We consider a class of the following divergences.

2.1 Definition. A function D on a set of pairs of states is a (generalized convex) *divergence* if

- $D(\rho\|\sigma) \geq 0$, and $D(\rho\|\sigma) = 0$ if and only if $\rho = \sigma$
- (Data processing/monotonicity). For any quantum channel $\mathcal{N}$, we have $D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) \leq D(\rho\|\sigma)$
- (Stability) $D(\rho \otimes \tau\|\sigma \otimes \tau) = D(\rho\|\sigma)$.
- (Joint convexity) For $0 \leq \lambda \leq 1$, and quantum states ρ_i, σ_i , we have

$$D(\lambda\rho_1 + (1-\lambda)\rho_2\|\lambda\sigma_1 + (1-\lambda)\sigma_2) \leq \lambda D(\rho_1\|\sigma_1) + (1-\lambda)D(\rho_2\|\sigma_2) .$$

2.2 Example. (Umegaki) Quantum relative entropy, $D(\rho\|\sigma) = \text{Tr}(\rho \log \rho - \rho \log \sigma)$, and trace-distance, $\|\rho - \sigma\|_1 = \text{Tr}|\rho - \sigma|$, are examples of a generalized convex divergence.

2.3 Definition. The *quantum divergence of channels* $\mathcal{N}_{A \rightarrow B}$ and $\mathcal{M}_{A \rightarrow B}$ is defined as

$$D(\mathcal{N}\|\mathcal{M}) = \max_{\rho_{AR}} D(\mathcal{N} \otimes I(\rho)\|\mathcal{M} \otimes I(\rho)) . \quad (2.1)$$

Here the maximization is taken over all sized of a system R and all states ρ_{AR} . However, it is sufficient to consider only pure states ρ_{AR} with system R being isomorphic to system A , because of the state purification, the data-processing inequality, and the Schmidt decomposition theorem.

The relative entropy of channels was first proposed in [7], and generalized in [14]. When divergence is a trace-distance, then the divergence of channels is called a *diamond-distance* of channels.

The quantum divergence of channels satisfies the following properties [35]:

- (Non-negativity) $D(\mathcal{N}\|\mathcal{M}) \geq 0$ and $D(\mathcal{N}\|\mathcal{M}) = 0$ if and only if $\mathcal{N} = \mathcal{M}$.
- (Weak monotonicity) For any quantum channels $\mathcal{V}_i$, we have

$$D(\mathcal{V}_1 \circ \mathcal{N} \circ \mathcal{V}_2\|\mathcal{V}_1 \circ \mathcal{M} \circ \mathcal{V}_2) \leq D(\mathcal{N}\|\mathcal{M}) .$$

- (Strong monotonicity) For any super-channel Λ ,

$$D(\Lambda(\mathcal{N})\|\Lambda(\mathcal{M})) \leq D(\mathcal{N}\|\mathcal{M}) .$$

- (Joint convexity) For $0 \leq \lambda \leq 1$, and quantum channels $\mathcal{N}_i, \mathcal{M}_i$, we have

$$D(\lambda\mathcal{N}_1 + (1-\lambda)\mathcal{N}_2\|\lambda\mathcal{M}_1 + (1-\lambda)\mathcal{M}_2) \leq \lambda D(\mathcal{N}_1\|\mathcal{M}_1) + (1-\lambda)D(\mathcal{N}_2\|\mathcal{M}_2) .$$

- (Stability) $D(\mathcal{N} \otimes I\|\mathcal{M} \otimes I) = D(\mathcal{N}\|\mathcal{M})$.

The properties of the quantum divergence of channels is discussed in Chapter 3.

2.4 Definition. *Trace-norm of a linear map* is defined as

$$\|\mathcal{A}\|_1 = \max_{\rho} \|\mathcal{A}(\rho)\|_1 . \quad (2.2)$$

And a *trace-distance of channels* (CPTP linear maps) is defined as

$$\|\mathcal{N} - \mathcal{M}\|_1 = \max_{\rho} \|\mathcal{N}(\rho) - \mathcal{M}(\rho)\|_1 . \quad (2.3)$$

2.2 The diamond norm

The diamond norm (or completely bounded trace-norm) of a linear map Ω is defined as

$$\|\Omega_{A \rightarrow B}\|_{\diamond} = \max_R \|\Omega_{A \rightarrow B} \otimes I_R\|_1 = \max_{\rho_{AR}} \|\Omega_{A \rightarrow B} \otimes I_R(\rho_{AR})\|_1 .$$

Here again is sufficient to consider only systems isomorphic to the system A .

The diamond-norm satisfies the following properties [31]:

- The diamond norm of a quantum channel (CPTP) is one: if $\mathcal{N}$ is CPTP map, then $\|\mathcal{N}\|_{\diamond} = 1$.
- Since trace-norm is monotone under quantum channels, in particular partial traces, we get that for any linear map,

$$\|\Omega\|_1 \leq \|\Omega\|_{\diamond} .$$

- (Sub-multiplicativity) for any linear maps Ω and Σ

$$\|\Omega \circ \Sigma\|_{\diamond} \leq \|\Omega\|_{\diamond} \|\Sigma\|_{\diamond} ,$$

- (Multiplicativity under tensor products) for any linear maps Ω and Σ

$$\|\Omega \otimes \Sigma\|_{\diamond} = \|\Omega\|_{\diamond} \|\Sigma\|_{\diamond} .$$

- (Monotonicity under superchannels) For any superchannel Λ there exist two channels $\mathcal{M}$ and $\mathcal{K}$ [6], such that the output channel can be written as

$$\Lambda(\mathcal{N}_{A \rightarrow B})_{C \rightarrow D} = \mathcal{M}_{BE \rightarrow D} \circ (\mathcal{N}_{A \rightarrow B} \otimes I_E) \circ \mathcal{K}_{C \rightarrow AE} .$$

Then

$$\|\Lambda(\mathcal{N})\|_{\diamond} \leq \|\mathcal{M}\|_{\diamond} \|\mathcal{N} \otimes I\|_{\diamond} \|\mathcal{K}\|_{\diamond} = \|\mathcal{N}\|_{\diamond} ,$$

since $\mathcal{M}$ and $\mathcal{K}$ are channels, their diamond norm is one, and since $\|\mathcal{N} \otimes I\|_{\diamond} = \|\mathcal{N}\|_{\diamond}$.

Taking the diamond-distance as a divergence in (2.4) defines a *diamond-distance of channels*,

$$\|\mathcal{N} - \mathcal{M}\|_{\diamond} = \max_{\rho_{AR}} \|\mathcal{N} \otimes I(\rho) - \mathcal{M} \otimes I(\rho)\|_1 . \quad (2.4)$$

Recall that for the Umegaki relative entropy the Pinsker's inequality for states holds: for any states ρ, σ , we have

$$D(\rho \parallel \sigma) \geq \frac{1}{2} \|\rho - \sigma\|_1^2 .$$

Straight from definition of the quantum divergences of channels, (2.4), and the Pinsker's inequality, we obtain Pinsker's inequality for channels.

2.5 Proposition. *For any quantum channels $\mathcal{N}, \mathcal{M}$, the Pinsker's inequality for channels holds*

$$D(\mathcal{N} \parallel \mathcal{M}) \geq \frac{1}{2} \|\mathcal{N} - \mathcal{M}\|_{\diamond}^2 \geq \frac{1}{2} \|\mathcal{N} - \mathcal{M}\|_1^2 .$$

Here D is the relative entropy of channels based on the Umegaki relative entropy.

2.3 Entropy of the channel

Completely depolarizing/randomizing channel is

$$\mathcal{R}_{A \rightarrow B}(\rho_A) = \text{Tr}(\rho_A)\pi_B ,$$

where $\pi_B = I_B/|B|$ is the maximally mixed state.

The entropy of a quantum channel is defined as [35]

$$S(\mathcal{N}) = \log_2 |B| - D(\mathcal{N} \| \mathcal{R}) , \quad (2.5)$$

here D is the relative entropy of the channels.

The entropy of a quantum channel has the following properties [11, 12]:

- (Additivity) For any two quantum channels, $S(\mathcal{N} \otimes \mathcal{M}) = S(\mathcal{N}) + S(\mathcal{M})$.
- (Monotonicity) For any uniformity preserving superchannel Λ (i.e. sending a completely randomizing channel to a completely randomizing one, $\Lambda(\mathcal{R}_{A \rightarrow B}) = \mathcal{R}_{C \rightarrow D}$), we have $S(\Lambda(\mathcal{N})) \geq S(\mathcal{N})$.
- (Boundedness) The entropy of a channel could be negative, but it is bounded, $|S(\mathcal{N})| \leq \log |B|$. The lowest value is achieved for an isometry, and the highest value is achieved for a completely randomizing channel.

Note that one may take different relative entropies instead of the relative entropy, such as sandwiched Rényi entropy, the max-relative entropy, generalized divergences of several types [12]. Rényi entropy of channels satisfy all of the above properties. And if a generalized divergence monotone, then the corresponding entropy of the channel is monotone under uniformity preserving superchannel.

2.4 Static resource theories of coherence

Consider all Hilbert spaces of the same dimension d . Fix basis $\mathcal{E} = \{|j\rangle\}_{j=1}^d$ in a Hilbert space $\mathcal{H}$. The set of **incoherent states** for a fixed basis $\mathcal{E}$ is $\mathcal{I}_{\mathcal{E}} = \{\rho = \sum_j p_j |j\rangle \langle j|\}$. We drop the subscript $\mathcal{E}$ from now on.

The completely dephasing operator is defined as

$$\Delta(\rho) = \sum_j \langle j | \rho | j \rangle |j\rangle \langle j| = \sum_j X_{jj} \rho X_{jj} . \quad (2.6)$$

There are various choices to consider as free operations. We present only some operations here. See [3, 4, ?] for a comparison and analysis of various incoherent operations.

2.6 Definition. Consider a quantum channel (CPTP map) Φ with the following Kraus operators $\Phi(\rho) = \sum_n K_n \rho K_n^*$. Then Φ is

- the *maximal incoherent operation* (MIO) if $\Phi(\mathcal{I}) \subset \mathcal{I}$.
- the *incoherent operation* (IO) if $K_n \mathcal{I} K_n^* \subset \mathcal{I}$, for all n .
- the *dephasing-covariant incoherent operation* (DIO) if $\Phi \Delta = \Delta \Phi$.
- the *strictly incoherent operation* (SIO) if $K_n \Delta(\rho) K_n^* = \Delta(K_n \rho K_n^*)$, for all n .

2.5 Dynamic resource theory of coherence

Here we start with the set of all bounded linear maps $\mathcal{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$. Quantum channels now play the role of states in the static resource theory. While there is only one commonly agreed set of free incoherent states (diagonal in the pre-fixed basis), there are multiple way one can consider free channels in the dynamic resource theory, such as classical channels, or detection/creation/detection-creation(DIO) incoherent [22, 28].

2.7 Definition. A quantum channel $\mathcal{N}$ is

- *detection incoherent (DI)* iff $\Delta\mathcal{N} = \Delta\mathcal{N}\Delta$,
- *creation incoherent (CI)* iff $\mathcal{N}\Delta = \Delta\mathcal{N}\Delta$,
- *detection-creation incoherent (DCI) or DIO* iff $\Delta\mathcal{N} = \mathcal{N}\Delta$.

Here Δ is a completely dephasing operator.

A set of free superoperations also can be defined in multiple ways. Some of these definitions rely on a definition of a completely dephasing superoperator, which is commonly defined as

$$\Delta(\mathcal{N}) = \Delta \circ \mathcal{N} \circ \Delta. \quad (2.7)$$

2.8 Definition. Consider a quantum super-channel Λ with the following Kraus decomoposition $\Lambda = \sum_n \Omega_n$, where the Choi matrix of each Ω_n is of rank one. Then Φ is

- *the maximal incoherent superchannel (MISC)* if $\Delta \circ \Lambda \circ \Delta = \Lambda \circ \Delta$. Note that for a set of classical channels $\mathcal{C}$, Λ is MISC if and only if $\Lambda(\mathcal{C}) \subset \mathcal{C}$.
- *the incoherent superchannel (ISC)* if $\Delta \circ \Omega_n \circ \Delta = \Omega_n \circ \Delta$, for all n .
- *the dephasing incoherent superchannel (DISC)* if $\Lambda \circ \Delta = \Delta \circ \Lambda$.
- *the strictly incoherent superchannel (SISC)* if $\Delta \circ \Omega_n = \Omega_n \circ \Delta$, for all n .

See [22] for a more detailed discussion on these super-channels.

Another way to define a free superchannel is through the decomposition $\Lambda(\mathcal{N}) = \mathcal{M}_{BE \rightarrow D} \circ (\mathcal{N}_{A \rightarrow B} \otimes I_E) \circ \mathcal{K}_{C \rightarrow AE}$, taking channels $\mathcal{M}$ and $\mathcal{K}$ as any of the free ones (MIO/IO/DIO/...) [28, 35].

2.6 Static and dynamic resource theories of coherence

The following table presents a visual correspondence between static and dynamic theory of coherence.

Static	Dynamic
Hilbert space $\mathcal{H}$	bounded linear operators $\mathcal{B}(\mathcal{H})$
state $\rho \in \mathcal{B}(\mathcal{H})$ positive, trace-one	channel $\mathcal{N} \in \mathcal{L}(\mathcal{H}) : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ CPTP
ON basis of $\mathcal{H}$ is $\mathcal{E} = \{ j\rangle\}_{j=1}^d$	
ON basis of $\mathcal{B}(\mathcal{H})$ is $\{X_{i,j} = i\rangle\langle j \}_{i,j}$	ON basis of $\mathcal{L}$ is $\{\Theta_{i,j,k,l}(\rho) = \langle i \rho j\rangle k\rangle\langle l \}$
free incoherent states $\mathcal{I}$: $\sum_j p_j X_{jj}$	free channels $\mathcal{C}$
subset of classical states (trace-one)	classical/MIO/IO/ $\dots$ /DI/CI
completely dephasing operator $\Delta(\rho) \in \mathcal{I}$	completely dephasing superoperator $\Delta(\mathcal{N})$
free operations $\Phi(\rho) = \sum_n K_n \rho K_n^*$:	free superoperations
	$\Lambda(\mathcal{N}) = \sum_n \Omega_n$:
MIO: $\Phi(\mathcal{I}) \subseteq \mathcal{I}$	MISC: $\Delta \circ \Lambda \circ \Delta = \Lambda \circ \Delta$
IO: $K_n \mathcal{I} K_n^* \subseteq \mathcal{I}$	ISC: $\Delta \circ \Omega_n \circ \Delta = \Omega_n \circ \Delta$
DIO: $\Phi(\Delta(\rho)) = \Delta(\Phi(\rho))$	DISC: $\Delta \circ \Lambda = \Lambda \circ \Delta$
SIO: $K_n \Delta(\rho) K_n^* = \Delta(K_n \rho K_n^*)$	SISC: $\Delta \circ \Omega_n = \Omega_n \circ \Delta$
SIO $\subset$ (DIO or IO) $\subset$ MIO	SISC $\subset$ (DISC or ISC) $\subset$ MISC
	$\Lambda(\mathcal{N}) = \mathcal{M}_{BE \rightarrow D} \circ (\mathcal{N}_{A \rightarrow B} \otimes I_E) \circ \mathcal{K}_{C \rightarrow AE}$
	$\mathcal{M}$ and $\mathcal{K}$ are free channels (MIO/IO/ $\dots$ /DI/CI)

3 Coherence measures

3.1 Definition. Define a function $f(\mathcal{N}, \mathcal{M})$ between two channels $\mathcal{N}, \mathcal{M}$, viewed as a distance between them, such that

- $f(\mathcal{N}, \mathcal{M}) \geq 0$, and $f(\mathcal{N}, \mathcal{M}) = 0$ if and only if $\mathcal{N} = \mathcal{M}$.
- (Weak monotonicity) For quantum channels, $f(\mathcal{V} \circ \mathcal{N} \circ \mathcal{U}, \mathcal{V} \circ \mathcal{M} \circ \mathcal{U}) \leq f(\mathcal{N}, \mathcal{M})$. It is possible to restrict this condition for when $\mathcal{V}, \mathcal{U}$ are free (DI, CI, or DCI) channels only.
- (Joint convexity). For $0 \leq \lambda \leq 1$, we have $f(\lambda \mathcal{N}_1 + (1 - \lambda) \mathcal{N}_2, \lambda \mathcal{M}_1 + (1 - \lambda) \mathcal{M}_2) \leq \lambda f(\mathcal{N}_1, \mathcal{M}_1) + (1 - \lambda) f(\mathcal{N}_2, \mathcal{M}_2)$.
- (Monotone under tensor product with the identity operator). For quantum channels $\mathcal{N}, \mathcal{M}$, we have $f(\mathcal{N} \otimes I, \mathcal{M} \otimes I) \leq f(\mathcal{N}, \mathcal{M})$.
- (Monotone under tensor product with the dephasing operator). For quantum channels $\mathcal{N}, \mathcal{M}$, and the completely dephasing operator Δ , we have $f(\mathcal{N} \otimes \Delta, \mathcal{M} \otimes \Delta) \leq f(\mathcal{N}, \mathcal{M})$.

The following lemma can be obtained from the proof of Theorem 9 in [28].

3.2 Lemma. Trace-one distance of channels, $f(\mathcal{N}, \mathcal{M}) = \|\mathcal{N} - \mathcal{M}\|_1$, satisfy the above conditions.

The proof of the following lemma is similar to the proof of the lemma above, but we provide it for the completeness sake below.

3.3 Lemma. *Quantum divergence of channels (see Definition 2.3), $f(\mathcal{N}, \mathcal{M}) = D(\mathcal{N} \parallel \mathcal{M})$, satisfy the above conditions, including strong monotonicity, for any divergence from Definition 2.1. In particular, the relative entropy of channels and the diamond-distance of channels are such functions.*

Proof. We only need to check the monotonicity under tensor product with the dephasing operator, since other properties are satisfied by [35].

Let $\mathcal{N}, \mathcal{M}$ be two quantum channels. Then

$$D(\mathcal{N}_{A \rightarrow B} \otimes \Delta_C \parallel \mathcal{M}_{A \rightarrow B} \otimes \Delta_C) = \max_{\rho_{ACR}} D(\mathcal{N}_{A \rightarrow B} \otimes \Delta_C \otimes I_R(\rho_{ACR}) \parallel \mathcal{M}_{A \rightarrow B} \otimes \Delta_C \otimes I_R(\rho_{ACR})) \quad (3.1)$$

$$= \max_{\rho} D([\mathcal{N} \otimes I \otimes I][I \otimes \Delta \otimes I](\rho) \parallel [\mathcal{M} \otimes I \otimes I][I \otimes \Delta \otimes I](\rho)) \quad (3.2)$$

$$= \max_{\sigma_{ACR} = I_A \otimes \Delta_C \otimes I_R(\rho)} D(\mathcal{N} \otimes I \otimes I(\sigma) \parallel \mathcal{M} \otimes I \otimes I(\sigma)) . \quad (3.3)$$

Any state ρ_{ACR} can be written as $\rho_{ACR} = \sum_{i,j,k,l} \alpha_{i,j,a,b} |a\rangle \langle b|_{AR} \otimes |i\rangle \langle j|_C$, where $\{|i\rangle\}$ (or $\{|j\rangle\}$) is the fixed basis of the operator Δ . Then

$$I_A \otimes \Delta_C \otimes I_R(\rho_{ACR}) = \sum_{i,j,k,l} \alpha_{i,j,a,b} |a\rangle \langle b|_{AR} \otimes |i\rangle \langle j|_C \quad (3.4)$$

$$=: \sum_i p_i \sigma_i \otimes |i\rangle \langle i| , \quad (3.5)$$

here we denoted states σ_i such that $\text{Tr} \sigma_i = 1$, then $\sum_i p_i = 1$.

Therefore,

$$D(\mathcal{N} \otimes I \parallel \mathcal{M} \otimes I) = \max_{\sigma_{iAR}, p_i} D \left(\sum_i p_i [\mathcal{N} \otimes I](\sigma_i)_{AR} \otimes |i\rangle \langle i|_C \parallel \sum_i p_i [\mathcal{M} \otimes I](\sigma_i)_{AR} \otimes |i\rangle \langle i|_C \right) \quad (3.6)$$

$$\leq \max_{\sigma_{iAR}, p_i} \sum_i p_i D([\mathcal{N} \otimes I](\sigma_i)_{AR} \otimes |i\rangle \langle i|_C \parallel [\mathcal{M} \otimes I](\sigma_i)_{AR} \otimes |i\rangle \langle i|_C) \quad (3.7)$$

$$= \max_{\sigma_i, p_i} \sum_i p_i D(\mathcal{N} \otimes I(\sigma_i) \parallel \mathcal{M} \otimes I(\sigma_i)) \quad (3.8)$$

$$\leq \max_{\sigma_{AR}} D(\mathcal{N}_{A \rightarrow B} \otimes I_R(\sigma_{AR}) \parallel \mathcal{M}_{A \rightarrow B} \otimes I_R(\sigma_{AR})) \quad (3.9)$$

$$= D(\mathcal{N} \parallel \mathcal{M}) . \quad (3.10)$$

In the first inequality we used joint convexity of quantum relative entropy. In the second equality we used stability of the divergence. $\square$

3.4 Theorem. *Let f be a function satisfying properties in Definition 3.1. Consider sets DI , CI , and DCI , as the sets of free operations $\mathcal{C}$ (recall Definition 2.7). Define the following coherence measures:*

$$C_f^{DI}(\mathcal{N}) = \min_{\Omega \in DI} f(\Delta \mathcal{N}, \Delta \Omega), \quad (3.11)$$

$$C_f^{CI}(\mathcal{N}) = \min_{\Omega \in CI} f(\mathcal{N} \Delta, \Omega \Delta), \quad (3.12)$$

$$C_f^{DCI}(\mathcal{N}) = \min_{\Omega \in DCI} f(\mathcal{N}, \Omega). \quad (3.13)$$

These coherence norm satisfy the following properties:

I $C(\mathcal{N}) = 0$ if and only if $\mathcal{N} \in \mathcal{C}$.

II Let $\Lambda^{AA'}(\mathcal{N}) = \Phi_2^{AA'} \circ (\mathcal{N}^A \otimes I^{A'}) \circ \Phi_1^{AA'}$ be a superchannel such that $\Phi_i^{AA'} \in \mathcal{C}$. Also define $\Lambda^A(\mathcal{N}) = \text{Tr}_{A'} \circ \Lambda^{AA'}(\mathcal{N})$. Then the coherence measure is monotone under either Λ^A or $\Lambda^{AA'}$,

$$C(\Lambda(\mathcal{N})) \leq C(\mathcal{N}) .$$

III For any quantum channels $\mathcal{N}, \mathcal{K}$ and $0 \leq \lambda \leq 1$, the coherence measure is convex

$$C(\lambda\mathcal{N} + (1-\lambda)\mathcal{K}) \leq \lambda C(\mathcal{N}) + (1-\lambda)C(\mathcal{K}) .$$

Some particular cases were discussed previously. In [28] it was shown that (3.11) is a coherence measure when f is a norm that is sub-multiplicative on quantum channels, sub-multiplicative with respect to the tensor products, and such that $\|\Omega\| \leq 1$ for any $\Omega \in DI$, in particular, for the diamond norm. Additionally, it was shown that (3.11) is a coherence measure when f is a trace-one norm. In [35] it was shown that (3.13) is a coherence measure for relative entropy of channels.

Note that one may think that all three coherence measures (3.11)-(3.13) can be written as the minimal distance to the free set $\mathcal{C}$, i.e one can define $C(\mathcal{N}) = \min_{\Omega \in \mathcal{C}} F(\mathcal{N}, \Omega)$, with the distance function defined as $F(\mathcal{N}, \mathcal{M}) = f(\Delta\mathcal{N}, \Delta\mathcal{M})$ for $\Omega \in DI$, and similarly for the other two measures. However, the first two measures are not distance measures: the simplest property, $F(\mathcal{N}, \mathcal{M}) = 0$ if and only if $\mathcal{N} = \mathcal{M}$, is not true.

Proof. 1. If $\mathcal{N} \in \mathcal{C}$, then $C(\mathcal{N}) = 0$. On the other hand, if $C(\mathcal{N}) = 0$, then there exists $\Omega \in \mathcal{C}$ such that

- for $\mathcal{C} = DI$: $\Delta\mathcal{N} = \Delta\Omega = \Delta\Omega\Delta = \Delta\mathcal{N}\Delta$, therefore $\mathcal{N} \in DI$.
- for $\mathcal{C} = CI$: $\mathcal{N}\Delta = \Omega\Delta = \Delta\Omega\Delta = \Delta\mathcal{N}\Delta$, therefore $\mathcal{N} \in CI$.
- for $\mathcal{C} = DCI$: $\mathcal{N} = \Omega \in DCI$.

2. Note that a composition of two maps in $\mathcal{C}$ is in $\mathcal{C}$. I.e. if $\Omega_1, \Omega_2 \in DI$, then $\Omega_1\Omega_2 \in DI$ since

$$\Delta\Omega_1\Omega_2 = \Delta\Omega_1\Delta\Omega_2 \tag{3.14}$$

$$= \Delta\Omega_1\Delta\Omega_2\Delta \tag{3.15}$$

$$= \Delta\Omega_1\Omega_2\Delta . \tag{3.16}$$

The first equality holds since $\Omega_1 \in DI$, i.e. $\Delta\Omega_1 = \Delta\Omega_1\Delta$. The second equality holds since $\Omega_2 \in DI$, i.e. $\Delta\Omega_2 = \Delta\Omega_2\Delta$. The last equality holds since again $\Omega_1 \in DI$ and therefore $\Delta\Omega_1\Delta = \Delta\Omega_1$.

Similarly, if $\Omega_1, \Omega_2 \in CI$, then $\Omega_1\Omega_2 \in CI$ since

$$\Omega_1\Omega_2\Delta = \Omega_1\Delta\Omega_2\Delta \tag{3.17}$$

$$= \Delta\Omega_1\Delta\Omega_2\Delta \tag{3.18}$$

$$= \Delta\Omega_1\Omega_2\Delta . \tag{3.19}$$

The first equality holds since $\Omega_2 \in CI$, i.e. $\Omega_2\Delta = \Delta\Omega_2\Delta$. The second equality holds since $\Omega_1 \in CI$, i.e. $\Omega_1\Delta = \Delta\Omega_1\Delta$. The last equality holds since again $\Omega_2 \in CI$ and therefore $\Delta\Omega_2\Delta = \Omega_2\Delta$.

Similarly, if $\Omega_1, \Omega_2 \in DI$, then $\Omega_1\Omega_2 \in DCI$ since

$$\Delta\Omega_1\Omega_2 = \Omega_1\Delta\Omega_2 \quad (3.20)$$

$$= \Omega_1\Omega_2\Delta. \quad (3.21)$$

The first equality holds since $\Omega_1 \in DCI$, i.e. $\Delta\Omega_1 = \Omega_1\Delta$. The second equality holds since $\Omega_2 \in DI$, i.e. $\Delta\Omega_2 = \Omega_2\Delta$.

Let $\Phi \in CI$ be any channel in DI. Then

$$C_f^{DI}(\mathcal{N} \circ \Phi) = \min_{\Omega \in DI} f(\Delta\mathcal{N}\Phi, \Delta\Omega) \quad (3.22)$$

$$\leq \min_{\Omega \in DI} f(\Delta\mathcal{N}\mathcal{K}, \Delta\Omega\Phi) \quad (3.23)$$

$$\leq \min_{\Omega \in DI} f(\Delta\mathcal{N}, \Delta\Omega) \quad (3.24)$$

$$= C_f^{DI}(\mathcal{N}). \quad (3.25)$$

In the first inequality we used that the composition of DI maps is DI. In the second inequality we used that the function f is weakly monotone for quantum channels.

Let $\Phi \in CI$ be any channel in CI. Then

$$C_f^{CI}(\mathcal{N} \circ \Phi) = \min_{\Omega \in CI} f(\mathcal{N}\Phi\Delta, \Omega\Delta) \quad (3.26)$$

$$\leq \min_{\Omega \in CI} f(\mathcal{N}\Phi\Delta, \Omega\Phi\Delta) \quad (3.27)$$

$$= \min_{\Omega \in CI} f(\mathcal{N}\Delta\Phi\Delta, \Omega\Delta\Phi\Delta) \quad (3.28)$$

$$\leq \min_{\Omega \in CI} f(\mathcal{N}\Delta, \Omega\Delta) \quad (3.29)$$

$$= C_f^{CI}(\mathcal{N}). \quad (3.30)$$

In the first inequality we used that the composition of CI maps is CI. The last inequality is true, since the function f is weakly monotone on quantum channels.

Let $\Phi \in DCI$ be any channel in DCI. Then

$$C_f^{DCI}(\mathcal{N} \circ \Phi) = \min_{\Omega \in DCI} f(\mathcal{N}\Phi, \Omega) \quad (3.31)$$

$$\leq \min_{\Omega \in DCI} f(\mathcal{N}\Phi, \Omega\Phi) \quad (3.32)$$

$$\leq \min_{\Omega \in DCI} f(\mathcal{N}, \Omega) \quad (3.33)$$

$$\leq C_f^{DCI}(\mathcal{N}). \quad (3.34)$$

In the first inequality we used that the composition of DCI maps is DCI. The last inequality is true, since the function f is weakly monotone on quantum channels.

3. Let $\Phi \in DI$. Then

$$C_f^{DI}(\Phi \circ \mathcal{N}) = \min_{\Omega \in DI} f(\Delta \Phi \mathcal{N}, \Delta \Omega) \quad (3.35)$$

$$\leq \min_{\Omega \in DI} f(\Delta \Phi \mathcal{N}, \Delta \Phi \Omega) \quad (3.36)$$

$$= \min_{\Omega \in DI} f(\Delta \Phi \Delta \mathcal{N}, \Delta \Phi \Delta \Omega) \quad (3.37)$$

$$\leq \min_{\Omega \in DI} f(\Delta \mathcal{N}, \Delta \Omega) \quad (3.38)$$

$$= C_f^{DI}(\mathcal{N}) . \quad (3.39)$$

The last inequality is true, since the relative entropy of channels is weakly monotone.

Let $\Phi \in CI$. Then

$$C_f^{CI}(\Phi \circ \mathcal{N}) = \min_{\Omega \in CI} f(\Phi \mathcal{N} \Delta, \Omega \Delta) \quad (3.40)$$

$$\leq \min_{\Omega \in CI} f(\Phi \mathcal{N} \Delta, \Phi \Omega \Delta) \quad (3.41)$$

$$\leq \min_{\Omega \in CI} f(\mathcal{N} \Delta, \Omega \Delta) \quad (3.42)$$

$$= C_f^{CI}(\mathcal{N}) . \quad (3.43)$$

The last inequality is true, since the relative entropy of channels is weakly monotone.

Let $\Phi \in DCI$. Then

$$C_f^{DCI}(\Phi \circ \mathcal{N}) = \min_{\Omega \in DCI} f(\Phi \mathcal{N}, \Omega) \quad (3.44)$$

$$\leq \min_{\Omega \in DCI} f(\Phi \mathcal{N}, \Phi \Omega) \quad (3.45)$$

$$\leq \min_{\Omega \in DCI} f(\mathcal{N}, \Omega) \quad (3.46)$$

$$= C_f^{DCI}(\mathcal{N}) . \quad (3.47)$$

The last inequality is true, since the relative entropy of channels is weakly monotone.

4. Consider

$$C_f^{DI}(\mathcal{N} \otimes I) = \min_{\Omega \in DI} f(\Delta(\mathcal{N} \otimes I), \Delta \Omega) \quad (3.48)$$

$$\leq \min_{\Omega = \Omega' \otimes I \in DI} f(\Delta(\mathcal{N} \otimes I), \Delta(\Omega' \otimes I)) \quad (3.49)$$

$$= \min_{\Omega' \in DI} f(\Delta \mathcal{N} \otimes \Delta, \Delta \Omega' \otimes \Delta) \quad (3.50)$$

$$\leq \min_{\Omega' \in DI} f(\Delta \mathcal{N}, \Delta \Omega') \quad (3.51)$$

$$= C_f^{DI}(\mathcal{N}) . \quad (3.52)$$

The last inequality holds since f is monotone under tensor product with the dephasing operator.

Similarly,

$$C_f^{CI}(\mathcal{N} \otimes I) = \min_{\Omega \in CI} f((\mathcal{N} \otimes I)\Delta, \Omega\Delta) \quad (3.53)$$

$$\leq \min_{\Omega=\Omega' \otimes I \in CI} f((\mathcal{N} \otimes I)\Delta, (\Omega' \otimes I)\Delta) \quad (3.54)$$

$$= \min_{\Omega' \in CI} f(\mathcal{N}\Delta \otimes \Delta, \Omega'\Delta \otimes \Delta) \quad (3.55)$$

$$\leq \min_{\Omega' \in CI} f(\mathcal{N}\Delta, \Omega'\Delta) \quad (3.56)$$

$$= C_f^{CI}(\mathcal{N}) . \quad (3.57)$$

Also,

$$C_f^{DCI}(\mathcal{N} \otimes I) = \min_{\Omega \in DCI} f(\mathcal{N} \otimes I, \Omega) \quad (3.58)$$

$$\leq \min_{\Omega=\Omega' \otimes I \in DCI} f(\mathcal{N} \otimes I, \Omega' \otimes I) \quad (3.59)$$

$$\leq \min_{\Omega' \in DCI} f(\mathcal{N}, \Omega') \quad (3.60)$$

$$= C_f^{DCI}(\mathcal{N}) . \quad (3.61)$$

In the last inequality we used that f is monotone under tensor product with the identity operator.

From 2-4, for any quantum superchannel $\Lambda^{AA'}(\mathcal{N}) = \Phi_2^{AA'} \circ (\mathcal{N}^A \otimes I^{A'}) \circ \Phi_1^{AA'}$ with free $\Phi_i \in \mathcal{C}$, all coherence measures are monotone. Also, since partial trace channel is DI, CI, and DCI, by 3, for any quantum superchannel $\Lambda^A = \text{Tr}_{A'} \circ \Lambda^{AA'}$, all coherence measures are monotone,

$$C_f^{\mathcal{C}}(\Lambda(\mathcal{N})) \leq C_f^{\mathcal{C}}(\mathcal{N}) .$$

5. Let $0 \leq \lambda \leq 1$. Denote $\Omega_1 \in CI$ and $\Omega_2 \in CI$ as channels such that

$$C_f^{DI}(\mathcal{N}) = f(\Delta\mathcal{N}, \Delta\Omega_1) , \quad C_f^{DI}(\mathcal{K}) = f(\Delta\mathcal{K}, \Delta\Omega_2) .$$

Then

$$C_f^{DI}(\lambda\mathcal{N} + (1-\lambda)\mathcal{K}) = \min_{\Phi \in DI} f(\Delta(\lambda\mathcal{N} + (1-\lambda)\mathcal{K}), \Delta\Phi) \quad (3.62)$$

$$\leq f(\Delta(\lambda\mathcal{N} + (1-\lambda)\mathcal{K}), \Delta(\lambda\Omega_1 + (1-\lambda)\Omega_2)) \quad (3.63)$$

$$= f(\lambda\Delta\mathcal{N} + (1-\lambda)\Delta\mathcal{K}, \lambda\Delta\Omega_1 + (1-\lambda)\Delta\Omega_2) \quad (3.64)$$

$$\leq \lambda f(\Delta\mathcal{N}, \Delta\Omega_1) + (1-\lambda)f(\Delta\mathcal{K}, \Delta\Omega_2) \quad (3.65)$$

$$= \lambda C_f^{DI}(\mathcal{N}) + (1-\lambda)C_f^{DI}(\mathcal{K}) . \quad (3.66)$$

Here, in the second inequality we used the joint convexity of f .

Similar results are straightforward for CI and DCI. □

3.1 Measures defined with the completely dephasing superoperator

While there is one definition of a completely dephasing superoperator (2.7), there could be others. We present several dynamical coherence measures based on general definitions of a completely dephasing operator. A few examples are considered later.

3.1.1 Coherence based on a quantum divergence of channels

Let Δ be a super-channel, such that $\Delta(\mathcal{N})$ is free for any channel $\mathcal{N}$ and it preserves free channels. For any quantum divergence of channels (2.4), define coherence measure as

$$C_{re}(\mathcal{N}) = D(\mathcal{N} \| \Delta(\mathcal{N})) .$$

Then from the properties of the divergence of channels, we have

1. (Non-negativity) $C_{re}(\mathcal{N}) \geq 0$.
2. If $C_{re}(\mathcal{N}) = 0$, then $\mathcal{N} = \Delta(\mathcal{N})$ is free. If $\mathcal{N}$ is free, then $\Delta(\mathcal{N}) = \mathcal{N}$, and $C_{re}(\mathcal{N}) = 0$.
3. (Monotonicity) Let free super-channels be the ones that commute with Δ . Then since the quantum divergence of channels is monotone under any superchannels, $C_{re}(\mathcal{N})$ is monotone.
4. (Convexity) C_{re} is convex, since quantum divergence of channels is jointly convex.

3.1.2 Coherence based on entropy

Let $\mathcal{C}$ be a set of free channels containing randomizing channel and Δ be a superchannel preserving free channels and mapping all channels to the free ones. For any divergence of channels (2.4), define coherence measure as

$$C_e(\mathcal{N}) = S(\Delta(\mathcal{N})) - S(\mathcal{N}) = D(\mathcal{N} \| \mathcal{R}) - D(\Delta(\mathcal{N}) \| \mathcal{R}) .$$

Recall that the entropy of channel is monotone under uniformity preserving superchannel. Then we have

1. (Non-negativity) Since Δ is a uniformity preserving superchannel, then $S(\Delta(\mathcal{N})) \geq S(\mathcal{N})$ and $C_e(\mathcal{N}) \geq 0$.
2. If the channel $\mathcal{N} \in \mathcal{C}$ is free, then $\Delta(\mathcal{N}) = \mathcal{N}$, and $C_e(\mathcal{N}) = 0$.
3. (Monotonicity) Let free superchannels Λ be the ones that commutes with Δ and preserve free channels. Then we have

$$C_e(\mathcal{N}) - C_e(\Lambda(\mathcal{N})) = S(\Delta(\mathcal{N})) - S(\mathcal{N}) - S(\Delta(\Lambda(\mathcal{N}))) + S(\Lambda(\mathcal{N})) \quad (3.67)$$

$$= S(\Lambda(\mathcal{N})) - S(\mathcal{N}) + S(\Delta(\mathcal{N})) - S(\Delta(\Lambda(\mathcal{N}))) \quad (3.68)$$

$$= S(\Lambda(\mathcal{N})) - S(\mathcal{N}) + S(\Delta(\mathcal{N})) - S(\Lambda(\Delta(\mathcal{N}))) \quad (3.69)$$

$$= S(\Lambda(\mathcal{N})) - S(\mathcal{N}) \geq 0 . \quad (3.70)$$

The last inequality is true since Λ is uniformity preserving.

3.1.3 Example 1

Define the set of free channels $\mathcal{C}$ as $\mathcal{N}(\rho) = \sum_{i,k} \alpha_{i,k} \Theta_{i,i,k,k}(\rho)$, for $\sum_k \alpha_{i,k} = 1$ and $\alpha_{i,k} = \alpha_{k,i}$. Then, a free channel $\mathcal{N} \in \mathcal{C}$ has the form:

$$\mathcal{N}(\rho) = \sum_{i,k} \alpha_{i,k} \langle i | \rho | i \rangle |k\rangle \langle k| = \sum_k \text{Tr}(M_k \rho) |k\rangle \langle k| , \quad (3.71)$$

for $\sum_k \alpha_{i,k} = 1$ and $\sum_i \alpha_{i,k} = 1$, and $M_k = \sum_i \alpha_{i,k} |i\rangle \langle i|$. Note that these channels are unital, $\mathcal{N}(I) = I$, since $\sum_i \alpha_{i,k} = 1$. Also, a free channel $\mathcal{N}$ maps any state to an incoherent state, $\mathcal{N}(\rho) \in \mathcal{I}$.

For example,

1. $\alpha_{i,k} = \frac{1}{|B|}$, then $\mathcal{N}(\rho) = \frac{1}{|B|}I = \pi_B = \mathcal{R}(\rho)$. So the depolarizing channel is free.
2. For $\alpha_{i,k} = \delta_{i,k}$, we have $\mathcal{N}(\rho) = \Delta(\rho)$. Therefore, the dephasing operator is free.

These free channels are a subset of what is called detection-creation-incoherent measurement in [35], $\mathcal{N}(\rho) = \sum_k \text{Tr}(M_k \rho) |k\rangle \langle k|$, where $M_k = \sum_i \alpha_{i,k} |i\rangle \langle i|$ are such that $M_k \geq 0$ and $\sum_k M_k = I$.

Recall, the completely dephasing operator is defined as

$$\Delta(\rho) = \sum_j \langle j| \rho |j\rangle |j\rangle \langle j| = \sum_j X_{jj} \rho X_{jj} = \sum_j \text{Tr}(X_{jj}^* \rho) X_{jj} . \quad (3.72)$$

Define a completely dephasing superoperator Δ as

$$\Delta(\mathcal{N}) = \sum_{i,k} \Theta_{i,i,k,k} \mathcal{N} \Theta_{i,i,k,k} . \quad (3.73)$$

Then

$$\Delta(\mathcal{N})(\rho) = \sum_{i,k} \Theta_{i,i,k,k} \mathcal{N} \Theta_{i,i,k,k}(\rho) \quad (3.74)$$

$$= \sum_{i,k} \langle i| \rho |i\rangle \Theta_{i,i,k,k} \mathcal{N}(|k\rangle \langle k|) \quad (3.75)$$

$$= \sum_{i,k} \langle i| \rho |i\rangle \langle i| \mathcal{N}(|k\rangle \langle k|) |i\rangle |k\rangle \langle k| . \quad (3.76)$$

For unital quantum channels ($\mathcal{N}(I) = I$), we have

$$\text{Tr} \Delta(\mathcal{N})(\rho) = \sum_{i,k} \langle i| \rho |i\rangle \langle i| \mathcal{N}(|k\rangle \langle k|) |i\rangle \quad (3.77)$$

$$= \sum_i \langle i| \rho |i\rangle \langle i| \mathcal{N}(I) |i\rangle \quad (3.78)$$

$$= \text{Tr} \rho . \quad (3.79)$$

Also, for a unital quantum channel $\mathcal{N}$, which is equal to its adjoint, $\mathcal{N} = \mathcal{N}^*$, we have $\langle i| \mathcal{N}(|k\rangle \langle k|) |i\rangle = \langle k| \mathcal{N}(|i\rangle \langle i|) |k\rangle$, and $\sum_k \langle i| \mathcal{N}(|k\rangle \langle k|) |i\rangle = \langle i| \mathcal{N}(I) |i\rangle = 1$. Therefore, for any unital quantum channel that is equal to its adjoint, the output channel $\Delta(\mathcal{N}) \in \mathcal{C}$ is free.

Additionally note that, if $\mathcal{N} \in \mathcal{C}$ is free, i.e. $\mathcal{N}(\rho) = \sum_{i,l} \alpha_{i,l} \langle i| \rho |i\rangle |l\rangle \langle l|$, then we have $\mathcal{N}(|k\rangle \langle k|) = \sum_l \alpha_{k,l} |l\rangle \langle l|$, and therefore

$$\Delta(\mathcal{N})(\rho) = \sum_{i,k} \langle i| \rho |i\rangle \langle i| \mathcal{N}(|k\rangle \langle k|) |i\rangle |k\rangle \langle k| \quad (3.80)$$

$$= \sum_{i,k} \alpha_{k,i} \langle i| \rho |i\rangle |k\rangle \langle k| \quad (3.81)$$

$$= \sum_{i,k} \alpha_{i,k} \langle i| \rho |i\rangle |k\rangle \langle k| \quad (3.82)$$

$$= \mathcal{N}(\rho) \quad (3.83)$$

Then $\Delta(\mathcal{N}) = \mathcal{N}$ for $\mathcal{N} \in \mathcal{C}$. And $\mathcal{R} \in \mathcal{C}$, therefore Δ is uniformity preserving.

In conclusion, let us consider the set $\mathcal{D}$ of all unital quantum channels that are equal to their adjoint. Then, Δ maps of channels in $\mathcal{D}$ into the set of free channels $\mathcal{C}$, and it preserves all free channels.

Thus, C_{re} and C_e of channels in $\mathcal{D}$ are good measures of channel coherence.

3.1.4 Example 2

Define the set of free channels $\mathcal{C}_2$ as $\mathcal{N}(\rho) = \sum_{i,k} \alpha_{i,k} \Theta_{i,i,k,k}(\rho)$, for $\sum_k \alpha_{i,k} = 1$ and $\sum_i \alpha_{i,k} = 1$, which is a larger set of free channels than in the Example 1. Therefore, the completely randomizing channel $\mathcal{R}$ is free.

Define the completely dephasing superchannel

$$\Delta(\mathcal{N})(\rho) = \sum_{i,k} \langle i | \rho | i \rangle \langle k | \mathcal{N}(|i\rangle \langle i|) | k \rangle | k \rangle \langle k | \quad (3.84)$$

$$= \Delta(\mathcal{N}(\Delta(\rho))) . \quad (3.85)$$

Then for a free channel $\mathcal{N}(\rho) = \sum_{i,k} \alpha_{i,k} \langle i | \rho | i \rangle | k \rangle \langle k |$,

$$\Delta(\mathcal{N})(\rho) = \sum_{i,k} \langle i | \rho | i \rangle \langle k | \mathcal{N}(|i\rangle \langle i|) | k \rangle | k \rangle \langle k | \quad (3.86)$$

$$= \sum_{i,k} \langle i | \rho | i \rangle \alpha_{i,k} | k \rangle \langle k | \quad (3.87)$$

$$= \mathcal{N}(\rho) . \quad (3.88)$$

Therefore, Δ preserves free channels.

Moreover, for any unital channel $\mathcal{N}$, $\Delta(\mathcal{N}) \in \mathcal{C}$ is free, since treating $\langle k | \mathcal{N}(|i\rangle \langle i|) | k \rangle = \alpha_{i,k}$, we have $\sum_k \langle k | \mathcal{N}(|i\rangle \langle i|) | k \rangle = \text{Tr}(\mathcal{N}(|i\rangle \langle i|)) = 1$, and $\sum_i \langle k | \mathcal{N}(|i\rangle \langle i|) | k \rangle = \langle k | \mathcal{N}(I) | k \rangle = 1$.

Thus, considering the set $\mathcal{D}$ of all unital channels, we have that Δ maps all channels in $\mathcal{D}$ into the set of free channels $\mathcal{C}$, and it preserves all free channels.

Thus, C_{re} and C_e of channels in $\mathcal{D}$ are good measures of channel coherence.

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