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# Extremal sections and projections of certain convex bodies: a survey

**Abstract:** We survey results concerning sharp estimates on volumes of sections and projections of certain convex bodies, mainly $\ell_p$-balls, by and onto lower-dimensional subspaces. This subject emerged from geometry of numbers several decades ago and since then has seen the development of a variety of probabilistic and analytic methods, showcased in this chapter.

**Keywords:** Volume, convex sets, sections, projections, $p$-norms, cube, simplex

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## 1 Introduction

### 1.1 Prologue

*How small can the volume of a slice of the unit cube be?* This question, asked by Good in the 1970s in the context of its applications in geometry of numbers, has turned out to be rather influential, prompting the development of several important methods, as well as spurring the community to solve further problems and enter research directions of independent interest in convex geometry, with strong ties to probability. Those most notably include the *dual* question of extremal volume projections, which in the simplest non-trivial case of hyperplane projections naturally translates into probabilistic Khinchin-type inequalities. Intriguingly, questions on extremal volume sections can be similarly translated into the same probabilistic language.

The purpose of this survey is thus twofold: In addition to striving to give a systematic account of the known results, our second goal is to illustrate intertwined Fourier analytic, geometric, and probabilistic methods underpinning the old and recent approaches.

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## 1.2 The motivating example

We begin with recalling Good's question (following [14, 132]). Suppose we are given $n$ linear forms $L_i(x) = \sum_{j=1}^{k} a_{ij}x_j$, $i = 1,\dots,n$, in $k$ variables. When does the system $|L_i(x)| \le 1$, $i \le n$, admit a non-trivial integral solution? The cornerstone result in geometry of numbers, Minkowski's (first) theorem, provides a link to volume: If $K$ is a symmetric convex body in $\mathbb{R}^d$ of volume at least $2^d$, then it contains a non-trivial lattice point (see, e. g., [98, Chapter 2]). Let $A = [a_{ij}]_{i\le n, j\le k}$ be the $n \times k$ matrix whose $i$-th row determines $L_i$. Thus, immediately, if $k \ge n$ and $\det(A) \le 1$ when $k = n$, then the answer to Good's question is affirmative because the set

$$K = \{x \in \mathbb{R}^k,\ \big|L_i(x)\big| \le 1, i \le n\} = \{x \in \mathbb{R}^k,\ Ax \in [-1,1]^n\}$$

is the preimage of the cube $[-1,1]^n$ under the linear map $A\colon \mathbb{R}^k \to \mathbb{R}^n$ (unbounded if $A$ is singular and of volume exactly $2^k \det(A)^{-1}$ otherwise when $k = n$). The case $k < n$ is more interesting. Suppose $A$ is of full rank $k$. Then the image of $K$ under $A$ is the section of the cube $[-1,1]^n$ by the $k$-dimensional linear subspace $A(\mathbb{R}^k)$. How small can its volume be? Good's conjecture confirmed later by Vaaler in [132] says that it is at least $2^k$ (the volume of the $k$-dimensional subcube $[-1,1]^k \times \{0\}^{n-k}$). Thus, if $\det(A^\top A) \le 1$, we obtain

$$\mathrm{vol}(K) \ge \sqrt{\det(A^\top A)}\,\mathrm{vol}(K) = \mathrm{vol}\big(A(K)\big) \ge 2^k,$$

also asserting in view of Minkowski's theorem that the initial system of inequalities admits a non-trivial integral solution, provided the convenient sufficient condition $\det(A^\top A) \le 1$. From a geometric point of view, it now seems natural and interesting to ask further questions about the maximal volume sections for the cube, as well as other sets.

## 1.3 Preliminaries and overview

We endow $\mathbb{R}^n$ with the standard inner product $\langle x,y\rangle = \sum_{j=1}^{n} x_j y_j$ between two vectors $x = (x_1,\dots,x_n)$ and $y = (y_1,\dots,y_n)$ in $\mathbb{R}^n$ and denote by $|x| = \sqrt{\langle x,x\rangle}$ the induced standard Euclidean norm. Its closed centered unit ball is denoted $B_2^n$, and for the unit sphere we write $S^{n-1} = \partial B_2^n$. Moreover, we write $e_1,\dots,e_n$ for the standard basis vectors, $e_1 = (1,0,\dots,0)$, $e_2 = (0,1,0,\dots,0)$, etc. As usual, for a set $A$ in $\mathbb{R}^n$, $A^\perp = \{x \in \mathbb{R}^n, \langle x,a\rangle = 0\ \forall a \in A\}$ is its orthogonal complement, with the convention that for a vector $u$ in $\mathbb{R}^n$, $u^\perp = \{u\}^\perp$ is the hyperplane perpendicular to $u$. Dilates are denoted by $\lambda A = \{\lambda a,\ a \in A\}$ for a scalar $\lambda$. In particular, if $-A = A$, the set $A$ is called (origin-)symmetric. The Minkowski or algebraic sum of two sets is $A + B = \{a + b,\ a \in A, b \in B\}$. The orthogonal projection onto an affine or linear subspace $H$ in $\mathbb{R}^n$ is denoted by $\mathrm{Proj}_H$. Volume, i. e., the $k$-dimensional Lebesgue measure in $\mathbb{R}^n$, is denoted by $\mathrm{vol}_k(\cdot)$, identified with $k$-dimensional Hausdorff measure (normalized so that cubes with side length

1 have volume 1). Recall that a body in $\mathbb{R}^n$ is a compact set with non-empty interior. For a symmetric convex body $K$ in $\mathbb{R}^n$, its Minkowski functional is $\|x\|_K = \sup\{t \geq 0,\ x \in tK\}$, $x \in \mathbb{R}^n$, the norm whose unit ball is $K$. A function $f\colon \mathbb{R}^n \to \mathbb{R}_+$ is called log-concave if it is of the form $e^{-V}$ for a convex function $V\colon \mathbb{R}^n \to (-\infty, +\infty]$. We refer for instance to the monographs [4, 31].

To put it fairly generally, given a body $B$ in $\mathbb{R}^n$ and $1 \leq k \leq n$, the two questions of our main interest will be:

| (I) What are the minimal and maximal volume *sections* $\mathrm{vol}_k(B \cap H)$ among all $k$-dimensional subspaces $H$ in $\mathbb{R}^n$? | (II) What are the minimal and maximal volume *projections* $\mathrm{vol}_k(\mathrm{Proj}_H(B))$ among all $k$-dimensional subspaces $H$ in $\mathbb{R}^n$? |
|---|---|

We note the obvious fact that in contrast to Question (I), Question (II) does not change if we translate the body $B$.

It is worth recalling two classical convexity-type results allowing to compare such volumes in the codimension 1 case, $k = n-1$ (despite not yielding direct answers to these questions).

**Theorem 1** (Busemann [33]). *Let $K$ be a symmetric convex body in $\mathbb{R}^n$. Then the function*

$$x \mapsto \frac{|x|}{\mathrm{vol}_{n-1}(K \cap x^\perp)}, \quad x \neq 0,$$

*extended by* 0 *at* $x = 0$ *defines a norm on* $\mathbb{R}^n$.

The surface area measure $\sigma_K$ of a convex body $K$ in $\mathbb{R}^n$ is a Borel measure on the unit sphere $S^{n-1}$ defined as follows: For $E \subset S^{n-1}$, $\sigma_K(E)$ equals the volume of the part of the boundary $\partial K$ where normal vectors belong to $E$ (in other words, $\sigma_K$ is the pushforward of the $(n-1)$-dimensional Hausdorff measure on $\partial K$ via the Gauss map $\nu_K\colon \partial K \to S^{n-1}$).

**Theorem 2** (Cauchy–Minkowski). *Let $K$ be a convex body in $\mathbb{R}^n$. Then for every unit vector $\theta \in S^{n-1}$, we have*

$$\mathrm{vol}_{n-1}(\mathrm{Proj}_{\theta^\perp}(K)) = \frac{1}{2} \int_{S^{n-1}} |\langle \theta, \xi \rangle| \mathrm{d}\sigma_K(\xi).$$

*In particular, the function $x \mapsto |x| \,\mathrm{vol}_{n-1}(\mathrm{Proj}_{x^\perp}(K))$, $x \neq 0$, extended by* 0 *at* $x = 0$, *defines a norm on* $\mathbb{R}^n$.

Let us explain this formula in the case of polytopes. Suppose we are given a convex polytope $P$ in $\mathbb{R}^n$ and we want to project it onto a hyperplane $\theta^\perp$, where $\theta$ is a unit vector. Let $\mathcal{F}_\mathcal{P}$ be the set of faces of $P$. If $F \in \mathcal{F}_P$, then $\mathrm{vol}_{n-1}(\mathrm{Proj}_{\theta^\perp}(F)) = \mathrm{vol}_{n-1}(F) \cdot |\langle \theta, n(F) \rangle|$, where $n(F)$ is the unit outer-normal vector to $F$. Note that in $\mathrm{Proj}_{\theta^\perp}(P)$ every point is *covered* two times, so one gets the following expression for the volume of projection:

$$\mathrm{vol}_{n-1}(\mathrm{Proj}_{\theta^\perp} P) = \frac{1}{2} \sum_{F \in \mathcal{F}_P} \mathrm{vol}_{n-1}(F) \cdot \big|\langle \theta, n(F) \rangle\big|.$$

The Cauchy–Minkowski formula is a straightforward generalization of this formula to general convex bodies. For further background and proofs, we refer for instance to [56, Theorem 8.1.10 and (A.49)].

For $p > 0$ and a vector $x = (x_1, \dots, x_n)$ in $\mathbb{R}^n$, we define the $\ell_p$-norm of $x$ (quasinorm when $0 < p < 1$) and its (closed) unit ball by

$$\|x\|_p = \left( \sum_{j=1}^n |x_j|^p \right)^{1/p}, \quad \|x\|_\infty = \max_{j \le n} |x_j|, \quad B_p^n = \{x \in \mathbb{R}^n,\ \|x\|_p \le 1\}.$$

The cube $B_\infty^n = [-1, 1]^n$ often warrants the more convenient volume 1 normalization

$$Q_n = \frac{1}{2} B_\infty^n = \left[ -\frac{1}{2}, \frac{1}{2} \right]^n.$$

The known results about extremal volume hyperplane sections and projections of $\ell_p$-balls are summarized in Tables 1 and 2 (that is, the known answers to Questions (I) and (II) when $B = B_p^n$ and $k = n - 1$). We shall discuss them and many more in detail in the next sections.

**Table 1:** Extremal volume hyperplane sections of $\ell_p$-balls: min / $\max_{a \in S^{n-1}} \mathrm{vol}_{n-1}(B_p^n \cap a^\perp)$.

| | $0 < p < 2$ | $2 < p < \infty$ | $p = \infty$ |
|---|---|---|---|
| min | $a = (\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}})$ [74] | $a = (1, 0, \dots, 0)$ [102] | $a = (1, 0, \dots, 0)$ [61, 62] |
| max | $a = (1, 0, \dots, 0)$ [102] | **?** | $a = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \dots, 0)$ [7] |

**Table 2:** Extremal volume hyperplane projections of $\ell_p$-balls: min / $\max_{a \in S^{n-1}} \mathrm{vol}_{n-1}(\mathrm{Proj}_{a^\perp}(B_p^n))$.

| | $p = 1$ | $1 < p < 2$ | $2 < p \le \infty$ |
|---|---|---|---|
| min | $a = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \dots, 0)$ [13, 130] | **?** | $a = (1, 0, \dots, 0)$ [24] |
| max | $a = (1, 0, \dots, 0)$ [folklore] | $a = (1, 0, , \dots, 0)$ [24] | $a = (\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}})$ [24] |

## 1.4 Existing literature and our aim

There is of course a vast body of literature on the subject. Ball's survey [14] presents stochastic comparison methods and applications of the celebrated Brascamp–Lieb inequalities to derive sharp bounds on sections. Koldobsky, Ryabogin, and Zvavich's survey [78] and Koldobsky's monograph [75] bring a common Fourier analytic treatment to bounds on both sections and projections. The aim of this chapter is to update on these and gather in one place what we know to date, as well as highlight what we would like to know around Questions (I) and (II), presenting 11 conjectures. We also showcase a unifying probabilistic point of view (via Khinchin-type inequalities) which goes hand in hand with the Fourier analytic methods, allowing to obtain additional insights and sharper results.

# 2 Sections

The goal here is to give a comprehensive account of known results concerning Question (I), with some indication of methods to which we come back in Section 4. We begin with some general remarks. A convex body $K$ in $\mathbb{R}^n$ is called *isotropic* if it has volume 1, its barycenter is at the origin, and its covariance matrix is proportional to the identity matrix, that is,

$$\mathrm{vol}_n(K) = 1, \quad \int_K x\mathrm{d}x = 0, \quad \int_K x_i x_j \mathrm{d}x = L_K^2 \delta_{ij}.$$

The positive proportionality constant $L_K$ is called the isotropic constant of $K$. Every convex body admits an affine image which is isotropic (diagonalizing the covariance matrix). It turns out that for symmetric isotropic convex bodies, volumes of all sections of a fixed dimension are comparable.

**Theorem 3** (Hensley [63]). *Fix $1 \le l \le n$. There are positive constants $c_l, c_l'$ which depend only on $l$ such that for every symmetric convex body $K$ in $\mathbb{R}^n$ which is isotropic and every $l$-codimensional subspace $H$ in $\mathbb{R}^n$, we have*

$$\frac{c_l}{L_K} \le \mathrm{vol}_{n-l}(K \cap H)^{1/l} \le \frac{c_l'}{L_K}.$$

To illustrate the key insight of Hensley's argument, let us consider the hyperplane case: We take $H = a^\perp$ for a unit vector $a$ and let

$$f(t) = \mathrm{vol}_{n-1}\big(K \cap (H + ta)\big), \quad t \in \mathbb{R}. \tag{2.1}$$

By the Brunn–Minkowski inequality, this defines a log-concave function. By the assumptions on $K$, it is even and integrates to 1. We claim that $f(t)$ is the probability density function of the random variable $\langle a, X\rangle$, where $X$ is uniform on $K$. Indeed,

$$\mathbb{P}\left(\sum_{i=1}^n a_i X_i \le s\right) = \mathbb{P}\big(\langle a, X\rangle \le s\big) = \mathrm{vol}_{n-1}\big(\{x \in K : \langle a, x\rangle \le s\}\big) = \int_{-\infty}^{s} f(t)\mathrm{d}t,$$

by Fubini's theorem. Crucially,

$$\mathrm{vol}_{n-1}(K \cap a^{\perp}) = f(0). \tag{2.2}$$

Since by isotropicity $\mathbb{E}|\langle a, X\rangle|^2 = L_K^2$, we have $\int_{\mathbb{R}} t^2 f(t)\mathrm{d}t = L_K^2$. It then remains to extremize the value of $f(0)$ among all such densities. Using a "moving mass to where it is beneficial" type of argument (see the proof of Theorem 4 below), a sharp lower bound is obtained by considering a uniform density (in higher dimensions, i. e., $l > 1$, isotropicity naturally dictates a uniform density on a Euclidean ball), whilst for a sharp upper bound, using convexity, the comparison is made against a symmetric exponential density (in higher dimensions, the argument is more complicated and not sharp anymore – see [63, Lemma 3]).

Bourgain in [29] used the property of hyperplane sections having comparable volume to obtain bounds on maximal functions. He asked whether the isotropic constants $L_K$ over all $K$ in all dimensions are uniformly bounded by a universal constant and, equivalently, whether every (symmetric) convex body of volume 1 admits a hyperplane section of volume at least a universal constant. This has become one of the central questions in asymptotic convex geometry, the hyperplane or slicing conjecture; see, e. g., [31] for a comprehensive monograph, [73] for a recent survey, and [67, 72] for the best results to date.

By Theorem 3, for two arbitrary subspaces $H_1, H_2$ of codimension $k$, we have

$$\left(\frac{\mathrm{vol}_k(K \cap H_1)}{\mathrm{vol}_k(K \cap H_2)}\right)^{1/k} \le C_k$$

with $C_k = \frac{c_k'}{c_k}$. Hensley's proof gives an upper bound on $C_k$ of order $k!$, which was improved to $\sqrt{k}$ by Ball in [8], who conjectured an optimal bound to be in fact of constant order, which remains open and turns out to be equivalent to the slicing conjecture. Implicit in his work and elucidated by V. Milman and Pajor in their seminal work [104] is the following reason for that equivalence: For a symmetric isotropic convex body $K$ in $\mathbb{R}^n$ and a $k$-codimensional subspace $H$ in $\mathbb{R}^n$, there is a symmetric $k$-dimensional convex body $C$ in $H^{\perp}$ such that

$$c_1 \frac{L_C}{L_K} \le \mathrm{vol}_{n-k}(K \cap H)^{1/k} \le c_2 \frac{L_C}{L_K},$$

where $c_1, c_2 > 0$ are universal constants. The body $C$ emerges from a generalization of Busemann's Theorem 1 to higher codimensions (see [8, 104]). Since $L_K \ge L_{B_2^n}$, if the slicing conjecture is true, then $L_C \le c_3$ for a universal constant $c_3 > 0$; thus, it in particular implies the existence of a universal constant $M > 0$ such that for all $k$-codimensional subspaces $H$, we have

$$\mathrm{vol}_{n-k}(K \cap H) \le M^k. \tag{2.3}$$

## 2.1 Cubes

Recall that $Q_n = [-\frac{1}{2}, \frac{1}{2}]^n$. As highlighted in the introduction, in the context of extremal volume sections, it has always been the cube sparking most interest and attention. The first sharp result concerned minimum volume hyperplane sections and was obtained independently by Hadwiger in [61] and by Hensley in [62].

**Theorem 4** (Hadwiger [61], Hensley [62]). *For every unit vector $a$ in $\mathbb{R}^n$, we have*

$$\mathrm{vol}_{n-1}(Q_n \cap a^\perp) \geq 1.$$

*Equality holds if and only if $a = \pm e_j$ for some $1 \leq j \leq n$.*

*Proof.* Let $K = Q_n$ and let us consider the function $f$ from (2.1). Since $Q_n$ is isotropic, the value of $\int t^2 f(t)\mathrm{d}t = \int_{Q_n} \langle x, a\rangle^2 \mathrm{d}x = \frac{1}{12}$ does not depend on $a$ (easily found by taking $a = e_1$). Moreover, $\|f\|_\infty = f(0)$, for $f$ is even and log-concave. It is therefore enough to show that for *every* probability density $f$, we have

$$\|f\|_\infty^2 \int t^2 f(t)\mathrm{d}t \geq \frac{1}{12}.$$

This goes back to Moriguti's work [106] (rederived by Ball in [7] in a slightly more general case of $p$-norms). For the proof, we can assume that $f$ is even, as otherwise we consider $g(t) = \frac{1}{2}(f(t) + f(-t))$ and $\|g\|_\infty \leq \|f\|_\infty$, whereas the second moments of $f$ and $g$ are the same. Then we move mass towards the origin, as this is beneficial: formally, take $f_0 = \|f\|_\infty \mathbf{1}_{[-c,c]}$, where $c = (2\|f\|_\infty)^{-1}$. Clearly, $\|f_0\|_\infty = \|f\|_\infty$. We have

$$\int t^2(f(t) - f_0(t))\mathrm{d}t = \int (t^2 - c^2)(f(t) - f_0(t))\mathrm{d}t \geq 0,$$

as the integrant is non-negative. □

In words, the canonical coordinate subspaces uniquely minimize the volume of hyperplane sections of the cube. Soon after, this was extended to sections of arbitrary dimension by Vaaler [132], confirming Good's conjecture.

**Theorem 5** (Vaaler [132]). *Fix $1 \leq k \leq n$. For every $k$-dimensional subspace $H$ in $\mathbb{R}^n$, we have*

$$\mathrm{vol}_k(Q_n \cap H) \geq 1.$$

*Equality holds if and only if $H$ is spanned by some $k$ standard basis vectors.*

Thus, every section of the cube has *large* volume. It is also "fat in all directions," in terms of quadratic forms; see Ball and Prodromou's work [16]. Vaaler used log-concavity and the notion of peakedness (introduced by Kanter in [69]) to make such comparison, generalizing Hensley's argument. Recently, Akopyan, Hubard, and Karasev [1] gave a different proof based on topological methods.

Thus, Vaaler's theorem gives a complete answer to Question (I) for minimal volume sections of the cube. Turning to the maximal ones, the first general upper bound for hyperplane sections was given by Hensley in [62], viz. $\mathrm{vol}_{n-1}(Q_n \cap a^\perp) \le 5$ for every unit vector $a$ in $\mathbb{R}^n$, who also conjectured that the sharp bound would be with 5 replaced by $\sqrt{2}$ attained at $a = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \dots, 0)$. This was later confirmed by Ball in his seminal work [7].

**Theorem 6** (Ball [7]). *For every unit vector $a$ in $\mathbb{R}^n$, we have*

$$\mathrm{vol}_{n-1}(Q_n \cap a^\perp) \le \sqrt{2}.$$

*Equality holds if and only if $a = (\pm e_i \pm e_j)/\sqrt{2}$ for some $1 \le i < j \le n$.*

*Sketch of the proof.* The starting point of Ball's approach was Fourier analytic: If we fix a unit vector $a$ and let $f$ be the probability density of $\langle a, X\rangle$, where $X$ is a random vector uniform on $Q_n$ (thus having i. i. d. components $X_j$ which are uniform on $[-\frac{1}{2}, \frac{1}{2}]$), then by (2.2) and the standard Fourier inversion formula,

$$\mathrm{vol}_{n-1}(Q_n \cap a^\perp) = f(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(t)\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty}\prod_{j=1}^{n}\frac{\sin(\frac{1}{2}a_j t)}{\frac{1}{2}a_j t}\mathrm{d}t = \frac{1}{\pi}\int_{-\infty}^{\infty}\prod_{j=1}^{n}\frac{\sin(a_j t)}{a_j t}\mathrm{d}t.$$

(This formula can perhaps be traced back to Pólya's work [119], and was also used by Hensley.) The next crucial idea is to apply Hölder's inequality with the weights $p_j = a_j^{-2}$ to get the bound

$$\int_{-\infty}^{\infty}\prod_{j=1}^{n}\frac{\sin(a_j t)}{a_j t}\mathrm{d}t \le \prod_{j=1}^{n}\left(\int_{-\infty}^{\infty}\left|\frac{\sin(a_j t)}{a_j t}\right|^{a_j^{-2}}\mathrm{d}t\right)^{a_j^2} = \prod_{j=1}^{n}\Psi(a_j^{-2})^{a_j^2} \tag{2.4}$$

with $\Psi(p) = \sqrt{p}\int_{-\infty}^{\infty}|\frac{\sin t}{t}|^p\mathrm{d}t$, $p \ge 1$ (a similar trick was also used in Haagerup's seminal work [60] on sharp constants in Khinchin inequalities). The most technically challenging and rather intricate is the problem of maximization of $\Psi$. The so-called Ball integral inequality which he established to finish his proof asserts that

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\left|\frac{\sin t}{t}\right|^p\mathrm{d}t \le \sqrt{\frac{2}{p}}, \quad p \ge 2, \tag{2.5}$$

with equality if and only if $p = 2$. This completes the proof in the case where all $|a_j| \le \frac{1}{\sqrt{2}}$. The complimentary case is dispensed with by a geometric argument justifying that $\mathrm{vol}_{n-1}(Q_n \cap a^\perp) \le \frac{1}{|a_j|}$ for each $j$. Indeed, projecting the section onto $e_j^\perp$ changes its volume by the factor $|\langle a, e_j\rangle|$ and it is contained in the projection of the entire cube,

$$\mathrm{vol}_{n-1}(Q_n \cap a^\perp) = \frac{1}{|\langle a, e_j\rangle|}\mathrm{vol}_{n-1}(\mathrm{Proj}_{e_j^\perp}(Q_n \cap a^\perp)) \le \frac{1}{|\langle a, e_j\rangle|}\mathrm{vol}_{n-1}(\mathrm{Proj}_{e_j^\perp}(Q_n)) = \frac{1}{|a_j|}.$$

□

We mention in passing that this integral inequality has been quite influential, with a very powerful method developed by Nazarov and Podkorytov in [109] to give a "simple" proof, as well as many extensions, generalizations, discrete versions, or even stability properties (see [5, 47, 70, 95, 100, 101]).

Quite remarkably and unexpectedly, the $\sqrt{2}$ bound allows to produce a very simple counterexample to the famous Busemann–Petty problem posed in [34]: *If for two symmetric convex bodies K and L in* $\mathbb{R}^n$, *we have* $\mathrm{vol}_{n-1}(K \cap a^\perp) \le \mathrm{vol}_{n-1}(L \cap a^\perp)$ *for every vector a, does it follow that* $\mathrm{vol}_n(K) \le \mathrm{vol}_n(L)$? Indeed, Ball observed in [9] that since the volume of the hyperplane sections of the unit volume Euclidean ball in high dimensions is roughly $\sqrt{e}$ and $\sqrt{2} < \sqrt{e}$, it suffices to take $K = Q_n$, and for $L$, it suffices to take a ball of a slightly smaller radius. This argument in fact works in all dimensions $n \ge 10$. Later, in [58], Giannopoulos used similar ideas involving cylinders to produce such elegant and simple counterexamples in dimensions $n \ge 7$. The answer to the Busemann–Petty problem is negative for $n \ge 5$ and positive for $n \le 4$. This is the result of significant work involving deep Fourier-analytic insights, see [57]. We refer for instance to Koldobsky's comprehensive monograph [75] for a full account.

The situation for upper bounds on volume of sections of more than one codimension is not fully understood. Ball obtained two general bounds.

**Theorem 7** (Ball [10]). *Fix* $1 \le k \le n$. *For every k-dimensional subspace H in* $\mathbb{R}^n$, *we have*

$$\mathrm{vol}_{n-1}(Q_n \cap H) \le \min\left\{ \sqrt{\frac{n}{k}}^{\,k}, \sqrt{2}^{\,n-k} \right\}.$$

The first bound $\sqrt{\frac{n}{k}}^{\,k}$ is optimal when $k$ divides $n$. Rogalski's question asks for the symmetric convex body of largest volume ratio (see [10]). It turns out that the bound $\sqrt{\frac{n}{k}}^{\,k}$ is equivalent to the fact that the cube is such a maximizer. The second bound $\sqrt{2}^{\,n-k}$ is better than the first one for $k \ge \frac{n}{2}$ and turns out to be sharp in this case. Both bounds rely heavily on Ball's ingenious geometric version of the Brascamp–Lieb inequality (from [30]), which provides a multi-dimensional analog of Hölder's inequality. The first bound uses it in a direct way (applied to indicator functions of intervals), whereas the second bound applies it to the Fourier analytic formula. As already mentioned, the exponential bound $\sqrt{2}^{\,n-k}$ in codimension $n-k$ is largely motivated by the slicing problem; see (2.3), providing an explicit constant for the cube.

These bounds, although sharp for many values of $k$ and $n$, leave many other cases open. A sort of folklore conjecture (see, e. g., [65, 115]) states that for arbitrary $k$ and $n$, the maximal volume section of the cube is attained at an affine cube. Specifically, given $1 \le k \le n$, let $n = k\ell + r$ with $r$ being the remainder from the division of $n$ by $k$. We define the following $k$ orthogonal vectors in $\mathbb{R}^n$:

$$u_{j+1} = e_{j\ell+1} + e_{j\ell+2} + \cdots + e_{(j+1)\ell}, \quad 0 \le j < k-r,$$
$$u_{k-r+j} = e_{(d-r)\ell+(\ell+1)j+1} + e_{(d-r)\ell+(\ell+1)j+2} + \cdots + e_{(d-r)\ell+(\ell+1)(j+1)}, \quad 0 \le j < r.$$

Let $H^*$ be the $k$-dimensional subspace spanned by them. Then

$$Q_n \cap H^* = \left\{ \sum_{j=1}^{k} t_k u_k, \ |t_1|, \dots, |t_k| \le \frac{1}{2} \right\},$$

which is an affine cube of volume

$$\mathrm{vol}_k(Q_n \cap H^*) = \prod_{j=1}^{k} |u_j| = \sqrt{\ell}^{\,k-r} \sqrt{\ell+1}^{\,r}.$$

Note that this becomes $\sqrt{n/k}^{\,k}$ when $k$ divides $n$ and $\sqrt{2}^{\,n-k}$ when $k \ge n/2$.

**Conjecture 1.** *Let $1 \le k \le n$. For every $k$-dimensional subspace $H$ in $\mathbb{R}^n$, we have*

$$\mathrm{vol}_k(Q_n \cap H) \le \mathrm{vol}_k(Q_n \cap H^*).$$

In addition to Ball's results of Theorem 7, this conjecture has recently been confirmed for planar sections, i. e., when $k = 2$, by Ivanov and Tsiutsiurupa in [66], who developed local conditions for extremal subspaces.

At the end of this subsection, we mention several loosely related extensions of these fundamental results.

### Other measures

Let $\gamma_n$ denote the standard Gaussian measure on $\mathbb{R}^n$, that is, the Borel probability measure on $\mathbb{R}^n$ with density $(2\pi)^{-n/2}e^{-|x|^2/2}$, whereas for a subspace $H$, let $\gamma_H$ be its counterpart on $H$, that is, the Borel probability measure supported on $H$ with density $(2\pi)^{-\dim H/2}e^{-|x|^2/2}$ (with respect to Lebesgue measure on $H$). Due to the lack of homogeneity, now of course the cube's side lengths may play a role. For the lower bounds, Barthe, Guédon, Mendelson, and Naor [22] established an analog of Vaaler's theorem.

**Theorem 8** (Barthe–Guédon–Mendelson–Naor [22]). *Fix $1 \le k \le n$. For every $k$-dimensional subspace $H$ in $\mathbb{R}^n$, the function*

$$t \mapsto \frac{\gamma_H(tQ_n \cap H)}{\gamma_k(tQ_k)}$$

*is non-increasing on $[0, +\infty)$. In particular (letting $t \to \infty$), for every $t > 0$, we have*

$$\gamma_H(tQ_n \cap H) \ge \gamma_k(tQ_k).$$

Their argument follows Vaaler's approach, crucially using the product structure of Gaussian measure. Using Ball's geometric form of the Brascamp–Lieb inequality, they also obtain an upper bound, similar to his bound for volume.

**Theorem 9** (Barthe–Guédon–Mendelson–Naor [22]). *Fix $1 \le k \le n$. For every k-dimensional subspace $H$ in $\mathbb{R}^n$ and every $t > 0$, we have*

$$\gamma_H(tQ_n \cap H) \le \gamma_k\Big(t\sqrt{\frac{n}{k}}Q_k\Big).$$

Again, this is sharp whenever $k$ divides $n$. The maximal-Gaussian volume hyperplane sections of cubes are not known for all values of $t$. Zvavitch has shown in [135] that the hyperplane $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \dots, 0)^\perp$ cannot be extremal for all dilates because the bound from Theorem 9 in the case $k = n-1$ is tight as $t \to \infty$. König and Koldobsky [85] found conditions on product measures ensuring that the hyperplane $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \dots, 0)^\perp$ gives the maximal volume among all hyperplanes $a^\perp$ with $\max_j |a_j| \le \frac{1}{\sqrt{2}}$. When specialized to the standard Gaussian measure, they additionally obtained that the hyperplane $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \dots, 0)^\perp$ yields maximal volume (among *all* hyperplanes) if and only if $t < t_0 = 1.253\dots$. Sharp upper bounds for $t > t_0$ are not known.

### Cylinders

Dirksen [46] studied the extremal central sections of the generalized cylinders $Z_r = Q_n \times (rB_2^m)$, $r > 0$, $m, n \ge 1$. He found sharp upper bounds in the 3-dimensional case of an ordinary cylinder, i. e., $m = 2$, $n = 1$, as well as upper bounds in the general case, sharp for large radii, developing Fourier analytic formulas and delicate integral inequalities involving Bessel functions.

### Perimeter

Answering a question of Pełczyński about hyperplane sections of maximal *perimeter* (i. e., sections with the boundary of the cube), König and Koldobsky [86] have shown that the extremal direction is the same as for the volume.

**Theorem 10** (König–Koldobsky [86]). *Let $n \ge 3$. For every unit vector $a$ in $\mathbb{R}^n$, we have*

$$\mathrm{vol}_{n-2}(\partial Q_n \cap a^\perp) \le 2\big((n-2)\sqrt{2} + 1\big).$$

This bound is attained if $a = (\pm e_i \pm e_j)/\sqrt{2}$ for some $1 \le i < j \le n$. This theorem also leads to counterexamples to a perimeter version of the Busemann–Petty problem in $n \ge 14$ dimensions. For the proof, they derive a Fourier analytic formula for the perimeter; its analysis involves new ingredients, most notably local conditions for constrained extrema, as well as subtle technical estimates around Ball's integral inequality.

## Diagonal sections

Here we consider the volume of the section by the hyperplane perpendicular to the main diagonal

$$a_n = \mathrm{vol}_{n-1}(Q_n \cap (\underbrace{1,\dots,1}_{n})^{\perp}), \quad n \geq 1.$$

Perhaps a more natural interpretation of the sequence $a_1, \dots, a_n$ is as the volumes of the sections of $Q_n$ by hyperplanes perpendicular to the diagonals of subcubes of growing dimension, for $1 \leq k \leq n$, where we have

$$\mathrm{vol}_{n-1}(Q_n \cap (\underbrace{1,\dots,1}_{k}, \underbrace{0,\dots,0}_{n-k})^{\perp}) = a_k.$$

Theorems 4 and 6 in particular assert that $a_1 \leq a_i \leq a_2$. Interestingly, the volumes of the diagonal sections form a (strictly) increasing sequence.

**Theorem 11** (Bartha–Fodor–González [18]). *We have* $a_1 < a_3 < a_4 < a_5 < \cdots < a_2$.

Their approach starts with Pólya's formula $a_n = \frac{\sqrt{n}}{\pi}\int_{-\infty}^{\infty}(\frac{\sin t}{t})^n \mathrm{d}t$ and is based on an intricate asymptotic analysis by means of the Laplace method. They first argue that the sequence $(a_n)$ increases for all $n \geq n_0$ for some $n_0$. Then, using numerical estimates, they bound $n_0$ and deal with $n \leq n_0$ by computer assisted calculations. Their arguments also show that the sequence $(a_n)$ is eventually concave.

It is tempting to believe that critical hyperplane sections must be diagonal, that is, if $a \mapsto \mathrm{vol}_{n-1}(Q_n \cap a^{\perp})$ has an extremum at a unit vector $a^*$, then $a^*$ is proportional to a diagonal $(1, \dots, 1, 0, \dots, 0)$. Ambrus [3] and Ivanov and Tsiutsiurupa [66] recently independently found an elegant local condition (with vastly different methods). Moreover, Ambrus confirmed this for $n \leq 3$ and disproved it for $n = 4$.

## Discrete version

Melbourne and Roberto [101] have derived a sharp discrete analog of Ball's upper bound for hyperplane sections.

**Theorem 12** (Melbourne–Roberto [101]). *Let* $n, \ell_1, \dots, \ell_n \geq 1$ *and* $t, k_1, \dots, k_n$ *be integers. Then*

$$\left|\left\{z \in \mathbb{Z}^n \cap \prod_{j=1}^{n}[k_j, k_j + \ell_j - 1],\ \sum_{j=1}^{n} z_j = t\right\}\right| < \sqrt{2}\frac{\prod_{j=1}^{n}\ell_j}{\sqrt{\sum_{j=1}^{n}(\ell_j^2 - 1)}}.$$

The constant $\sqrt{2}$ is the best possible, as can be seen by discretizing Ball's extremizer (by taking $\ell_1 = \ell_2 = m$, $\ell_3 = \cdots = \ell_n = 1$ and letting $m \to \infty$). Mimicking Ball's approach, the following integral inequality lies at the heart of the argument:

$$\int_{-1/2}^{1/2} \left| \frac{\sin(n\pi t)}{n \sin(\pi t)} \right|^p \mathrm{d}t < \sqrt{\frac{2}{p(n^2-1)}}, \quad p \geq 2, \; n = 2, 3, \ldots.$$

This is in fact stronger than Ball's inequality (2.5) and recovers it by letting $n \to \infty$. Melbourne and Roberto developed a new viewpoint on establishing such delicate bounds for oscillatory integrands, borrowing and combining ideas from majorization and optimal transport.

#### Chessboard cutting

It is folklore that a line can meet the interiors of no more than $2N-1$ squares of the usual $N \times N$ chessboard and this bound is tight (consider the diagonal pushed down a bit). We refer to Bárány and Frenkel's work [35] for a short argument as well as precise estimates for a 3-dimensional analog. To tackle the problem in higher dimensions, in [36] they introduced the following quantity involving volumes of hyperplane sections of the cube:

$$V_n = \max_{v \in \mathbb{R}^n} \frac{\|u\|_1}{|v|} \operatorname{vol}_{n-1}(Q_n \cap v^\perp).$$

They have shown that if the cube $[0, N]^n$ is divided into $N^n$ unit cubes in the usual way, then the maximal number of unit cubes that a hyperplane can intersect equals

$$(1 + o(1)) V_n N^{n-1}$$

for a fixed $n \geq 1$ as $N \to \infty$. Confirming a conjecture from [36], Aliev recently found the constant $V_n$ [2].

**Theorem 13** (Aliev [2]). *Let $n \geq 1$. We have $V_n = \sqrt{n} \operatorname{vol}_{n-1}(Q_n \cap (1, \ldots, 1)^\perp)$.*

In words, it is the diagonal section that maximizes $V_n$; thus, $\sqrt{n} \leq V_n \leq \sqrt{2}\sqrt{n}$ and $V_n \sim \sqrt{\frac{6}{\pi}} \sqrt{n}$ for large $n$. Aliev's argument is purely geometric with the main observation being that the hyperplane parallel to $(1, \ldots, 1)^\perp$ supports the intersection body of the cube.

#### Stability

With additional insights gained from a certain probabilistic point of view (see Section 4), Chasapis and the authors recently obtained [42] a dimension-free stability result for both lower and upper bounds for hyperplane sections.

**Theorem 14** (Chasapis–Nayar–Tkocz [42]). *There are universal constants $c_1, c_2 > 0$ such that for every unit vector $a$ in $\mathbb{R}^n$ with $a_1 \geq \cdots \geq a_n \geq 0$, we have*

$$1 + c_1|a - e_1|^2 \leq \mathrm{vol}_{n-1}(Q_n \cap a^{\perp}) \leq \sqrt{2} - c_2\left|a - \frac{e_1 + e_2}{\sqrt{2}}\right|.$$

The exponents 2 and 1 on the left- and right-hand sides, respectively, are the best possible, as can be explicitly verified for $n = 2$. In an independent work [100], Melbourne and Roberto obtained a similar result.

## 2.2 Balls of *p*-norms

We begin with a monotonicity result for the parameter $p$ discovered by Mayer and Pajor [102].

**Theorem 15** (Meyer–Pajor [102]). *Fix $1 \leq k \leq n$. For every $k$-dimensional subspace $H$ in $\mathbb{R}^n$, the function*

$$p \mapsto \frac{\mathrm{vol}_k(B_p^n \cap H)}{\mathrm{vol}_k(B_p^k)}$$

*is non-decreasing on $[0, +\infty)$. In particular, comparison with the Euclidean ball yields*

$$\mathrm{vol}_k(B_p^n \cap H) \leq \mathrm{vol}_k(B_p^k), \quad 0 < p < 2,$$
$$\mathrm{vol}_k(B_p^n \cap H) \geq \mathrm{vol}_k(B_p^k), \quad p > 2.$$

*In each inequality, equality holds if and only if $H$ is spanned by some $k$ standard basis vectors.*

Meyer and Pajor established this theorem for $p \geq 1$, which was extended later to $p < 1$ independently by Barthe [19] and Caetano [39]. Letting $p \to \infty$ recovers Vaaler's theorem, Theorem 5, for the cube sections. Vaaler's argument uses Kanter's peakedness to make a comparison between uniform and Gaussian distributions. The key point in [102] was that the same comparison holds across the whole family of probability measures with densities $\{e^{-c_p|x|^p}\}_{p>0}$. We will present this crucial idea in a probabilistic setting in Section 4.

More is known for hyperplane sections when $0 < p < 2$. Meyer and Pajor [102] found that the minimal volume hyperplane sections of the cross-polytope $B_1^n$ are attained by the diagonal directions and conjectured the same for the entire range $0 < p < 2$, confirmed later by Koldobsky in [74] in a strong Schur convexity-type result.

**Theorem 16** (Koldobsky [74]). *Let $0 < p < 2$. For every two unit vectors $a$ and $b$ in $\mathbb{R}^n$ such that $(b_1^2, \dots, b_n^2)$ majorizes $(a_1^2, \dots, a_n^2)$, we have*

$$\mathrm{vol}_{n-1}(B_p^n \cap a^\perp) \le \mathrm{vol}_{n-1}(B_p^n \cap b^\perp).$$

For background on majorization and Schur convexity, we refer for instance to [25]. In particular, since

$$\left(\frac{1}{n}, \dots, \frac{1}{n}\right) \prec (a_1^2, \dots, a_n^2) \prec (1, 0, \dots, 0),$$

for an arbitrary unit vector $a$ in $\mathbb{R}^n$, the minimal and maximal volume sections follow. What makes the range $0 < p < 2$ so much more tractable compared to $p > 2$ is the fact that the Fourier transform of $e^{-|x|^p}$ is a non-negative function of the form $t \mapsto \int_0^\infty e^{-ut^2} \mathrm{d}\mu(u)$, a Gaussian mixture. In fact, the same also holds for $e^{-|x|^p}$, which allowed the authors of [49] to bypass the Fourier analytic arguments entirely. We return to this in Section 4.

The maximal volume hyperplane sections of $B_p^n$-balls for $2 < p < \infty$ are unknown. Oleszkiewicz established in [113] that Ball's upper bound for the cube, Theorem 6, does not extend to all $p > 2$, as it fails for all $p < 26.265\dots$ and large enough dimensions (by comparing the cube's extremizing hyperplane $(1, 1, \dots, 0)^\perp$ to the diagonal one $(1, 1 \dots, 1)^\perp$ in the limit $n \to \infty$). We conjecture that in each dimension there is a unique phase transition point.

**Conjecture 2.** *For every $n \ge 3$, there is a unique $p_0(n)$ such that*

$$\max_{a \in S^{n-1}} \mathrm{vol}_{n-1}(B_p^n \cap a^\perp) = \begin{cases} \mathrm{vol}_{n-1}(B_p^n \cap (1, \dots, 1)^\perp), & 2 < p \le p_0(n), \\ \mathrm{vol}_{n-1}(B_p^n \cap (1, 1, 0, \dots, 0)^\perp), & p \ge p_0(n). \end{cases}$$

For lower-dimensional sections, there is a general bound of Barthe which extends a corresponding result for the cube from Theorem 7. The argument also crucially relies on the Brascamp–Lieb inequalities.

**Theorem 17** (Barthe [19]). *Let $p \ge 2$. Fix $1 \le k \le n$. For every $k$-dimensional subspace $H$ in $\mathbb{R}^n$, we have*

$$\mathrm{vol}_k(B_p^n \cap H) \le \left(\frac{n}{k}\right)^{k(1/2-1/p)} \mathrm{vol}_k(B_p^k).$$

As for the cube, this is sharp when $k$ divides $n$ with the same extremizing subspace.

Using a direct argument involving triangulation and convexity of certain functions, Nazarov has shown that planar sections of the cross-polytope of minimal area are attained at regular polygons.

**Theorem 18** (Nazarov [42]). *Let $n \geq 3$. For every 2-dimensional subspace $H$ in $\mathbb{R}^n$, we have*

$$\mathrm{vol}_2(B_1^n \cap H) \geq \frac{n^2 \sin^3(\frac{\pi}{2n})}{\cos(\frac{\pi}{2n})},$$

*which is optimal, attained when $B_1^n \cap H$ is a regular $2n$-gon.*

All known results from Table 1 on extremal volume hyperplane sections for $\ell_p$-balls admit robust versions (recall also Theorem 14).

**Theorem 19** (Chasapis–Nayar–Tkocz [42]). *For every $p > 0$, there is a positive constant $c_p$ such that for every $n \geq 1$ and every unit vector $a = (a_1, \ldots, a_n)$ in $\mathbb{R}^n$ with $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$, we have*

$$\frac{\mathrm{vol}_{n-1}(B_p^n \cap a^\perp)}{\mathrm{vol}_{n-1}(B_p^n \cap e_1^\perp)} \geq 1 + c_p |a - e_1|^2, \quad 2 < p \leq \infty,$$

$$\frac{\mathrm{vol}_{n-1}(B_p^n \cap a^\perp)}{\mathrm{vol}_{n-1}(B_p^n \cap (\frac{e_1 + \cdots + e_n}{\sqrt{n}})^\perp)} \geq 1 + c_p \sum_{j=1}^n (a_j^2 - 1/n)^2, \quad 0 < p < 2,$$

$$\frac{\mathrm{vol}_{n-1}(B_p^n \cap a^\perp)}{\mathrm{vol}_{n-1}(B_p^n \cap e_1^\perp)} \leq (a_1^p + (1 - a_1^2)^{p/2})^{-1/p}, \quad 0 < p < 2.$$

We finish this subsection with Vaaler's conjecture on general rather precise lower bounds which have been verified to a large extent for $\ell_p$-balls.

**Conjecture 3** (Vaaler [132]). *Let $K$ be a symmetric isotropic convex body in $\mathbb{R}^n$. Then for every non-zero subspace $H$ in $\mathbb{R}^n$ of dimension $1 \leq k \leq n$, we have*

$$\mathrm{vol}_k(K \cap H) \geq 1.$$

Noteworthily, if true, it implies the slicing conjecture (made independently of it); see Hensley's theorem, Theorem 3. Vaaler's theorem confirms this inequality for the cube (which is tight). Meyer and Pajor's sharp lower bound gives this inequality for $K = B_p^n$ with $2 < p < \infty$ and all subspaces (see [102]), as well as $1 < p < 2$ and all hyperplanes (see Schmuckenschläger's note [127]); however, these are not tight anymore.

## 2.3 Simplices

Here we discuss results concerning sections of regular simplices. It will be most convenient to consider a regular $n$-dimensional simplex of side length $\sqrt{2}$ embedded in $\mathbb{R}^{n+1}$,

$$\Delta_n = \left\{ x \in \mathbb{R}^{n+1},\ x_1, \ldots, x_{n+1} \geq 0,\ \sum_{j=1}^{n+1} x_j = 1 \right\}.$$

*Central* sections will refer to those by (affine) subspaces passing through the barycenter $(\frac{1}{n+1}, \dots, \frac{1}{n+1})$ of $\Delta_n$. In particular, if $a$ is a unit vector in $\mathbb{R}^{n+1}$ with $\sum_{j=1}^{n+1} a_j = 0$ (so parallel to the hyperplane containing $\Delta_n$), then $\Delta_n \cap a^\perp$ is a central hyperplane section of $\Delta_n$. Such sections of maximal volume have been determined by Webb in [134].

**Theorem 20** (Webb [134]). *For every unit vector $a$ in $\mathbb{R}^{n+1}$ with $\sum_{j=1}^{n+1} a_j = 0$, we have*

$$\mathrm{vol}_{n-1}(\Delta_n \cap a^\perp) \le \frac{1}{\sqrt{2}} \frac{\sqrt{n+1}}{(n-1)!}.$$

*This is attained if and only if $a^\perp$ passes through some $n-1$ vertices of $\Delta_n$.*

Webb gave two proofs, both based on an elegant probabilistic formula,

$$\mathrm{vol}_{n-1}(\Delta_n \cap a^\perp) = \frac{\sqrt{n+1}}{(n-1)!} f_a(0),$$

where $f_a$ is the probability density of $\sum_{j=1}^{n+1} a_j X_j$ with $X_j$ being i. i. d. standard exponential random variables, with density $e^{-x}$ supported on $(0, +\infty)$. Thus, his result becomes $f_a(0) \le \frac{1}{\sqrt{2}}$ with equality if and only if $n-1$ of the $a_j$ vanish. His first proof mimicked Ball's Fourier analytic approach with the crucial bound coming from Hölder's inequality and an integral inequality. His second proof was probabilistic, exploiting log-concavity.

Webb also found that the 1- and 2-dimensional central sections of $\Delta_n$ of maximal volume are attained at lines and planes passing through a vertex and an edge of $\Delta_n$, respectively (see his PhD thesis [133], as well as [94] for a different argument in the line case).

For general upper bounds on central sections, following the approach involving Ball's geometric form of the Brascamp–Lieb inequality, Dirksen [45] obtained the following result.

**Theorem 21** (Dirksen [45]). *For every $k$-dimensional subspace of $\mathbb{R}^{n+1}$ passing through the barycenter of the simplex $\Delta_n$, we have*

$$\mathrm{vol}_{k-1}(\Delta_n \cap H) \le \frac{k^{\frac{k}{2(n+1)}}}{(k-1)!}.$$

*Moreover, if $\mathrm{dist}(H, e_j) \le \sqrt{\frac{n+1-k}{n+2-k}}$ for each $j \le n+1$, then*

$$\mathrm{vol}_{k-1}(\Delta_n \cap H) \le \frac{1}{(k-1)!} \sqrt{\frac{n+1}{n+2-k}},$$

*which is sharp, attained when $H$ contains $k-1$ vertices of $\Delta_n$.*

As opposed to symmetric convex bodies for which maximum volume sections by all affine subspaces of a fixed dimension always occur when they pass through the barycenter (by the Brunn–Minkowski inequality), for the simplex such a question becomes nontrivial. Webb pointed out in [134] that combining two results of Ball yields that for fixed $1 \le k \le n$, we have

$$\mathrm{vol}_k(\Delta_n \cap H) \le \mathrm{vol}_k(F_k),$$

for all $(k+1)$-dimensional affine subspaces $H$ in $\mathbb{R}^{n+1}$, where $F_k$ is a $k$-dimensional face of $\Delta_n$, that is, the $k$-dimensional slices of $\Delta_n$ of maximal volume are exactly the $k$-dimensional faces. To explain this, fix $H$ and consider the maximum volume ellipsoid, say $\mathcal{E}^*$ contained in the convex body $K = \Delta_n \cap H$. Ball found [11] that the $n$-simplex has maximal volume ratio among all convex bodies in $\mathbb{R}^n$. The volume ratio of a convex body $C$ in $\mathbb{R}^n$ is $\mathrm{vr}(C) = (\mathrm{vol}_n(C)/\mathrm{vol}_n(\mathcal{E}))^{1/n}$, where $\mathcal{E}$ is the maximum volume ellipsoid in $C$. Thus,

$$\mathrm{vol}_k(\Delta_n \cap H) = \mathrm{vr}(K)^k \,\mathrm{vol}_k(\mathcal{E}^*) \le \mathrm{vr}(F_k)^k \,\mathrm{vol}_k(\mathcal{E}^*).$$

Moreover, Ball has shown in [12] that among all $k$-dimensional ellipsoids in $\Delta_n$, the Euclidean balls inscribed in $k$-faces have maximal volume; thus, they are the maximal volume ellipsoids in $F_k$. Therefore,

$$\mathrm{vr}(F_k)^k \,\mathrm{vol}_k(\mathcal{E}^*) \le \mathrm{vol}_k(F_k).$$

In [55], Fradelizi has given a different argument, deriving this fact from a more general result for cones in isotropic position.

Lower bounds are much less understood.

**Conjecture 4.** *For every unit vector $a$ in $\mathbb{R}^{n+1}$ with $\sum_{j=1}^{n+1} a_j = 0$, we have*

$$\mathrm{vol}_{n-1}(\Delta_n \cap a^\perp) \ge \left(\frac{n}{n+1}\right)^{n-1/2} \frac{\sqrt{n+1}}{(n-1)!},$$

*which is attained when $a^\perp$ is parallel to a face of $\Delta_n$.*

This has been confirmed in low dimensions ($n \le 4$) by Brzezinski [32]. He also noticed that a bound of the correct order but off by a multiplicative constant follows by applying Fradelizi's theorem from [55] to Webb's result stated above.

## 2.4 Complex analogs

If we consider $\mathbb{C}^n$ as a Hilbert space equipped with the standard (complex) inner product and volume (Lebesgue measure after the natural identification $\mathbb{C}^n \simeq \mathbb{R}^{2n}$), most of the results about extremal volume sections (of real spaces) considered thus far beg for their natural complex counterparts. Vaaler's theorem and its generalization of Meyer and Pajor admit such extensions, with almost the same proofs, as was pointed out in their papers.

**Theorem 22** (Vaaler [132], Meyer–Pajor [102]). *Let* $1 \le k \le n$ *and let* $H$ *be a (complex)* $k$*-dimensional subspace in* $\mathbb{C}^n$*. Then*

$$\mathrm{vol}_{2k}(B^n_{p,\mathbb{C}} \cap H) \ge \mathrm{vol}_{2k}(B^k_{p,\mathbb{C}}),$$

*when* $2 \le p \le \infty$*. The reverse inequality holds when* $0 < p \le 2$*.*

Here,

$$B^n_{p,\mathbb{C}} = \left\{ z \in \mathbb{C}^n, \left( \sum_{j=1}^n |z_j|^p \right)^{1/p} \le 1 \right\}$$

is the unit ball of the complex $\ell_p(\mathbb{C}^n)$-space; in particular, $B^\infty_{\infty,\mathbb{C}}$ is the polydisc (the Cartesian product of the unit discs in $\mathbb{C}$). In fact, their proofs yield a further extension from $B^n_{p,\mathbb{C}}$ to bodies which are $\ell_p$-sums of Euclidean spaces of arbitrary dimensions, which has been in turn significantly generalized by Eskanazis in [48] (see Theorem 26 below).

Ball's cube slicing result of Theorem 6 has been extended to the complex setting by Oleszkiewicz and Pełczyński in [114], who proved the following sharp polydisc slicing bound.

**Theorem 23** (Oleszkiewicz–Pełczyński [114]). *For every unit vector* $a$ *in* $\mathbb{C}^n$*, we have*

$$\mathrm{vol}_{2n-2}(B^n_{\infty,\mathbb{C}} \cap a^\perp) \le 2\pi^{2n-2}.$$

*Equality holds if and only if* $a = (\xi e_i + \eta e_j)/\sqrt{2}$ *for some* $1 \le i < j \le n$ *and* $\xi, \eta \in \mathbb{C}$ *with* $|\xi|, |\eta| = 1$.

The proof strategy follows the same path of the Fourier analytic formula and defactorization by means of Hölder's inequality; however, new technical challenges arise. The heart of the proof is the following analytical inequality:

$$\int_0^\infty \left| \frac{2J_1(t)}{t} \right|^p t\,\mathrm{d}t \le \frac{4}{p}, \quad p \ge 2 \tag{2.6}$$

(cf. (2.5)), where $J_1$ is the Bessel function of the first kind of order 1. Its proof rests on precise pointwise bounds on $J_1$ as well as an interpolation argument. A new different proof has been very recently given in [101]. Moreover, the upper bounds for higher codimensions of Theorem 7 can be transferred almost ad verbatim to the complex case as well (as was remarked by Barthe and Koldobsky; see [114]).

The exact analog of the sharp upper bound on the perimeter from Theorem 10 also holds, as shown by König and Koldobsky in [86].

Sharp upper bounds even on hyperplane (complex codimension 1) sections in the range $2 < p < \infty$ remain open. For the same reasons as in the real case, the range $0 < p < 2$ is more tractable and we have the following analog of Koldobsky's theorem, Theorem 16.

**Theorem 24** (Koldobsky–Zymonopoulou [80]). *Let $0 < p < 2$. For every two unit vectors $a$ and $b$ in $\mathbb{C}^n$ such that $(|b_1|^2, \ldots, |b_n|^2)$ majorizes $(|a_1|^2, \ldots, |a_n|^2)$, we have*

$$\mathrm{vol}_{2n-2}(B_{p,\mathbb{C}}^n \cap a^\perp) \le \mathrm{vol}_{2n-2}(B_{p,\mathbb{C}}^n \cap b^\perp).$$

Finally, a complex version of Busemann's theorem, Theorem 1, has been developed by Koldobsky, Paouris, and Zymonopoulou in [77], whereas a full solution to the complex Busemann–Petty problem is due to Koldobsky, König, and Zymonopoulou [76].

## 2.5 Miscellanea

We finish this section with a brief account of various results related to and motivated by sharp bounds on volumes of sections.

### Slabs

For a unit vector $a$ in $\mathbb{R}^n$ and $t > 0$, we set

$$H_{a,t} = \{x \in \mathbb{R}^n,\ \left|\langle x, a\rangle\right| \le t\}$$

to be the (symmetric) slab of width $2t$ orthogonal to the direction $a$ (in other words, a thickening/enlargement $a^\perp + tB_2^n$ of the hyperplane $a^\perp$). Answering a question of V. Milman, Barthe and Koldobsky in [23] have established the following extension of Hadwiger and Hensley's Theorem 4.

**Theorem 25** (Barthe–Koldobsky [23]). *For every unit vector $a$ in $\mathbb{R}^n$ and $0 \le t \le \frac{3}{8}$, we have*

$$\mathrm{vol}_n(Q_n \cap H_{a,t}) \ge \mathrm{vol}_n(Q_n \cap H_{e_1,t}).$$

They derived this from a sharp inequality for unimodal log-concave densities in one dimension, expanding on Hensley's approach.

In words, Hadwiger and Hensley's result is stable in that, independent of the dimension, *coordinate slabs* contain the least volume of the unit cube among all symmetric slabs of fixed width at most 3/4. This bound is in the spirit of the concentration of measure (see [28, 89, 90]), providing a sharp lower bound of *small* enlargements on the volume 1/2 half-spaces $\{x \in \mathbb{R}^n, \langle x, a\rangle \ge 0\}$ in $Q_n$. The threshold $\frac{3}{8}$ is suboptimal: in the 2-dimensional case, a direct calculation from [23] shows that at $t = \sqrt{2}-1$ the extremizing slab changes from the coordinate one to the diagonal one. The sharp behavior in higher dimensions is not clear. The paper [23] provides asymptotic results that the slabs orthogonal to the main diagonal are optimal for *large t* of the order $\sqrt{n}$ as $n \to \infty$ (developing en route very interesting conditions for convexity properties of Laplace transforms), with a precise non-asymptotic result for the range $\frac{1}{2}\sqrt{n-1} \le t \le \frac{1}{2}\sqrt{n}$ obtained recently by Moody, Stone, Zach, and Zvavitch [105].

A detailed analysis of the (local as well as global) extremal slabs in the 2- and 3-dimensional cases has been conducted by König and Koldobsky [83], whereas in [84], they obtained a complex analog of Theorem 25.

### Block subspaces

Eskenazis [48] gathered under one umbrella the results on slicing $\ell_p$-balls, both real and complex, when $0 < p < 2$, thus significantly generalizing and unifying Theorems 16, 22, and 24.

**Theorem 26** (Eskenazis [48]). *Let $m, n$ be positive integers and let $0 < p < 2$. Suppose $X = (\mathbb{R}^m, \|\cdot\|)$ is a quasinormed space which admits an isometric embedding into $L_p$. For every two unit vectors $a$ and $b$ in $\mathbb{R}^n$ such that $(b_1^2, \ldots, b_n^2)$ majorizes $(a_1^2, \ldots, a_n^2)$, we have*

$$\mathrm{vol}_{mn-m}(B_p^n(X) \cap H_a) \le \mathrm{vol}_{mn-m}(B_p^n \cap H_b).$$

Here,

$$B_p^n(X) = \left\{ x = (x_1, \ldots, x_n) \in \mathbb{R}^m \times \cdots \times \mathbb{R}^m,\ \left( \sum_{j=1}^n \|x_j\|^p \right)^{1/p} \le 1 \right\}$$

is the unit ball of the $\ell_p$-sum of $X$, whereas

$$H_a = \left\{ x = (x_1, \ldots, x_n) \in \mathbb{R}^m \times \cdots \times \mathbb{R}^m,\ \sum_{j=1}^n a_j x_j = 0 \right\}$$

is a *block* subspace of codimension $m$ in $(\mathbb{R}^m)^n$. In particular, $X = \ell_2^m$ with $m = 1, 2$ recovers Theorems 16 (when $p < 2$), 22, and 24. The point is that there is a plethora of non-Hilbertian examples treated by this general result, most notably $X = \ell_q^m$ with $p \le q \le 2$.

Eskenazis' argument builds on [49], with the key new ingredient being Lewis' representation guaranteeing that the norm on $X$ which embeds isometrically into $L_p$, $p > 0$, admits a form

$$\|x\| = \left( \int_{S^{m-1}} |\langle Ux, \theta \rangle|^p \, \mathrm{d}\mu(\theta) \right)^{1/p}, \quad x \in \mathbb{R}^m,$$

for some invertible linear map $U\colon \mathbb{R}^m \to \mathbb{R}^m$ and an isotropic Borel measure $\mu$ on the unit sphere $S^{m-1}$ [91, 126]. The restriction $p < 2$ is not needed here, but is included to bring about Gaussian mixtures (as highlighted after Theorem 16).

For the regime $p > 2$, only the case $p = \infty, X = \ell_2^m$ has been considered, i. e., sections of

$$B_\infty^n(\ell_2^m) = \underbrace{B_2^m \times \cdots \times B_2^m}_{n},$$

for which Brzezinski [32] obtained that for every $n, m \geq 2$ and every unit vector $a$ in $\mathbb{R}^n$, we have

$$\mathrm{vol}_{mn-m}(B_\infty^n(\ell_2^m) \cap H_a) \leq \frac{(m+2)^{m/2}}{2^{m/2-1} m\Gamma(m/2)}. \tag{2.7}$$

This is asymptotically sharp as $n \to \infty$ because the right-hand side equals exactly $\lim_{n\to\infty} \mathrm{vol}_{mn-m}(B_\infty^n(\ell_2^m) \cap H_{(\frac{1}{\sqrt{n}},\dots,\frac{1}{\sqrt{n}})})$. The case $m = 2$ is special in that this limit also equals $A_{m,n} = \mathrm{vol}_{mn-m}(B_\infty^n(\ell_2^m) \cap H_{(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0,\dots,0)})$, whilst for every $m > 2$, the limit is strictly larger than $A_{m,n}$. In other words, Ball's upper bound from Theorem 6 does *not* generalize to block subspace sections of $B_2^m \times \cdots \times B_2^m$ for any $m > 2$ (but it does when $m = 2$, as we have seen in Oleszkiewicz and Pełczyński's theorem, Theorem 23).

We finish with Eskanazis' conjecture on sharp lower bounds by block subspaces, generalizing Hadwiger and Hensley's theorem, Theorem 4.

**Conjecture 5** (Eskenazis [48]). *Let $m, n \geq 1$. Let $K$ be a symmetric convex body in $\mathbb{R}^m$. For every unit vector $a \in \mathbb{R}^n$, we have*

$$\mathrm{vol}_{mn-m}(K \times \cdots \times K \cap H_a) \geq \mathrm{vol}_m(K)^{n-1}.$$

### Non-central sections

In this context, perhaps the most natural question to ask is about extremal volume sections by affine subspaces at a *fixed* distance $t > 0$ from the origin. This has arguably proved to be more difficult than the question of central sections, even for the cube. Sharp results for line sections have been found in [105] for the cube and in [94] for the cross-polytope. For hyperplane sections, we have the following conjecture of V. Milman (see [83]).

**Conjecture 6** (V. Milman [83]). *The minimum and maximum of $\mathrm{vol}_{n-1}(Q_n \cap H)$ over the affine hyperplanes $H$ at a fixed distance $t > 0$ from the origin are attained when $H$ is orthogonal to a diagonal direction $(1, \dots, 1, 0, \dots, 0)$ with a suitable number of 1's depending on $t$.*

There are several partial results supporting it. König and Koldobsky verified that it holds in low dimensions ($n = 2, 3$) [83]. Moody, Stone, Zach, and Zvavitch have established that in the range $\frac{1}{2}\sqrt{n-1} < t < \frac{1}{2}\sqrt{n}$ the main diagonal direction gives the maximal section [105], later extended to all $t > \frac{1}{2}\sqrt{n-2}$ by Pournin in [116], where one of the key ideas was to employ a noteworthy combinatorial formula for sections of the cube,

$$\mathrm{vol}_{n-1}\big([0,1]^n \cap \{x \in \mathbb{R}^n,\ \langle x, a\rangle = b\}\big) = \sum_v \frac{(-1)^{\sum v_j}|a|(b - \langle v, a\rangle)^{n-1}}{(n-1)!\prod a_j},$$

where the sum is over the vertices $v$ of the cube $[0,1]^d$ such that $\langle v, a\rangle \le b$ (see also [17]). In a recent preprint [117], Pournin also showed that the main diagonal direction is strictly locally maximal for $t = \Omega(\frac{\sqrt{n}}{\log n})$, derived from general local conditions for all diagonal directions. König and Rudelson [87] obtained dimension-free lower bounds on non-central sections of the cube as well as the polydisc. König [82] treated non-central extremal volume as well as perimeter sections of the regular simplex, cube, and cross-polytope, when the distance $t$ is fairly large, also investigating local behavior for the entire range of $t$.

### Probabilistic extensions

There is a natural link between the volume of sections and negative moments of linear forms, which goes back at least to Kalton and Koldobsky's work [68]. To illustrate it, first note that the value at say $x = 0$ of a probability density $f$ on $\mathbb{R}$ which is continuous at 0 can be obtained by taking the limit of its negative moments,

$$f(0) = \lim_{q \to -1+} \frac{1+q}{2} \int |x|^q f(x)\mathrm{d}x. \tag{2.8}$$

In view of this and the basic probabilistic formula for sections (2.2), the sharp bounds for hyperplane sections of the cube from Theorems 4 and 6 can be phrased as

$$1 \le \lim_{q \to -1+} \frac{1+q}{2} \mathbb{E}\left|\sum_{j=1}^n a_j U_j\right|^q \le \sqrt{2}$$

for all unit vectors $a$ in $\mathbb{R}^n$, where $U_1, U_2, \dots$ are i. i. d. random variables uniform on $[-\frac{1}{2}, \frac{1}{2}]$. Do such inequalities remain true with a fixed $q$? The answer is known for the cube and polydisc, where a sharp phase transition of the extremizer occurs for the upper bound with diagonal directions entering the picture.

**Theorem 27** (Chasapis–König–Tkocz [41]). *Let $-1 < q < 0$. Let $U_1, U_2, \dots$ be i. i. d. random variables uniform on $[-\frac{1}{2}, \frac{1}{2}]$. For every $n \ge 1$ and unit vectors $a$ in $\mathbb{R}^n$, we have*

$$\mathbb{E}|U_1|^q \le \mathbb{E}\left|\sum_{j=1}^n a_j U_j\right|^q \le \begin{cases} \mathbb{E}|(U_1 + U_2)/\sqrt{2}|^q, & -1 < q \le q_0, \\ \lim_{m\to\infty} \mathbb{E}|(U_1 + \cdots + U_m)/\sqrt{m}|^q, & q_0 \le q < 0. \end{cases}$$

*The constant* $q_0 = -0.79\ldots$ *is given uniquely by equating the two expressions on the right-hand side.*

A similar behavior has been established for the polydisc slicing by Chasapis, Singh, and Tkocz in [43], with the phase transition "moving to the left" where the negative moments recover volume.

# 3 Projections

We turn our attention to Question (II) from the introduction about projections of extremal volume of basic convex bodies such as the cube, simplex, and cross-polytope, as well as the family of $\ell_p$-balls. As we will see, our understanding of hyperplane projections of $\ell_p$-balls is at the same level as for sections (see Tables 1 and 2), whilst in general much less is known, particularly for lower-dimensional projections. The methods also seem to shift from analytic to more of an algebraic or combinatorial nature.

## 3.1 Cubes

Thanks to Cauchy's formula from Theorem 2, extremal volume projections on hyperplanes are easy to determine, for the surface area measure of the cube $Q_n$ is the counting measure $\sum_{j=1}^n \delta_{\pm e_j}$ of the set of the $2n$ vectors $\{\pm e_j,\ j \le n\}$ outer normal to the facets of $Q_n$; thus, for every unit vector $a$ in $\mathbb{R}^n$, we have

$$\mathrm{vol}_{n-1}(\mathrm{Proj}_{a^\perp}(Q_n)) = \sum_{j=1}^n |a_j|.$$

Therefore,

$$1 \le \mathrm{vol}_{n-1}(\mathrm{Proj}_{a^\perp}(Q_n)) \le \sqrt{n},$$

by squaring and neglecting the off-diagonal terms for the lower bound and simply applying the Cauchy–Schwarz inequality for the upper bound. The former is attained if and only if $Q_n$ is projected onto a coordinate hyperplane and the latter is attained if and only if $Q_n$ is projected onto a hyperplane orthogonal to a main diagonal.

A zonotope is the Minkowski sum of intervals. Orthogonal projections of the unit cube $Q_n = [-\frac{1}{2}, \frac{1}{2}]$ are zonotopes and, conversely, every zonotope can be obtained as such a projection (of a possibly rescaled and translated cube in a sufficiently high dimension). Shephard's decomposition of zonotopes into parallelepipeds led him in [129] to the following classical formula for volume: If $v_1, \ldots, v_n$ are vectors in $\mathbb{R}^k$, then the volume of the zonotope $Z = \sum_{j=1}^n [0, v_j]$ is expressed as

$$\mathrm{vol}_k(Z) = \sum_{1 \le j_1 < \cdots < j_k \le n} \left|\det[v_{j_1}\ \ldots\ v_{j_k}]\right|,$$

where $[v_{j_1} \ \dots \ v_{j_k}]$ is the $k \times k$ matrix with columns $v_{j_1}, \dots, v_{j_k}$. For the orthogonal projection of the cube $Q_n$ onto a $k$-dimensional subspace $H$, the vectors $v_j$ can be taken as columns of the $k \times n$ matrix whose rows form an orthonormal basis of $H$, leading to the constraint

$$\sum_{1 \le j_1 < \dots < j_k \le n} \left|\det[v_{j_1} \ \dots \ v_{j_k}]\right|^2 = 1,$$

by the Cauchy–Binnet formula. Then, exactly as in the codimension 1 case, we obtain upper and lower bounds on the volume. This argument goes back to Chakerian and Filliman's work [40].

**Theorem 28** (Chakerian–Filliman [40]). *Fix $1 \le k \le n$. For every $k$-dimensional subspace $H$ in $\mathbb{R}^n$, we have*

$$1 \le \mathrm{vol}_{n-1}(\mathrm{Proj}_H(Q_n)) \le \min\left\{ \sqrt{\binom{n}{k}}, \frac{\mathrm{vol}_{k-1}(B_2^{k-1})^k}{\mathrm{vol}_k(B_2^k)^{k-1}} \left(\frac{n}{k}\right)^{k/2} \right\}.$$

The lower bound is clearly sharp, attained at coordinate subspaces. It also instantly follows from Vaaler's theorem, Theorem 5, upon observing that projections contain sections. The first upper bound $\sqrt{\binom{n}{k}}$ is sharp only when $k = 1, n-1$. The second upper bound is obtained differently, by invoking quermassintegrals (which are additive under Minkowski sums, so they go hand in hand with zonotopes), combined with Urysohn's inequality. A simpler version of the same idea is to note that every $k$-dimensional projection has diameter at most the diameter of the cube $\sqrt{n}$; thus, by the isodiametric inequality, its volume is at most $\mathrm{vol}_k(B_2^k)(\sqrt{n}/2)^k$. All these bounds are of the order $n^{k/2}$ for a fixed $k$ as $n \to \infty$, which is tight. The second of the upper bounds in Theorem 28 is asymptotically better than the first one. Ivanov [64] has developed local conditions for maximizers of $k$-dimensional projections.

In [40], using the isoperimetric inequality for polygons, Chakerian and Filliman additionally obtained a sharp bound for 2-dimensional projections (and thus also $(n-2)$-dimensional ones – see Theorem 31 below).

**Theorem 29** (Chakerian–Filliman [40]). *For $n \ge 2$ and for every 2-dimensional subspace $H$ in $\mathbb{R}^n$, we have*

$$\mathrm{vol}_{n-1}(\mathrm{Proj}_H(Q_n)) \le \frac{1}{\tan(\frac{\pi}{2n})}.$$

Soon after, Filliman [51] discovered a general principle that maximizing volume of the larger class of zonotopes $Z = \sum_{j=1}^n [-\frac{1}{2}v_j, \frac{1}{2}v_j]$ with the constraint $\sum_{j=1}^n |v_j|^2 = n$ on the vectors $v_j$ in $\mathbb{R}^k$ amounts to maximizing it over all zonotopes which are $k$-dimensional projections of the cube. This allowed him to extend the previous estimate to 3-dimensional projections.

**Theorem 30** (Filliman [51]). *For $n \geq 3$ and for every 3-dimensional subspace $H$ in $\mathbb{R}^n$, we have*

$$\mathrm{vol}_3(\mathrm{Proj}_H(Q_n)) \leq \frac{\sqrt{n/3}}{\tan(\frac{\pi}{2n-2})}.$$

This is sharp not only for $n = 3$, but also for $n = 4$ with the extremal projection being the rhombic dodecahedron and for $n = 6$ with the extremal projection being the triacontahedron.

We finish with a striking and remarkable feature of the cube: Its projections onto orthogonal complementary subspaces have the same volume.

**Theorem 31** (McMullen [99], Chakerian–Filliman [40]). *Let $1 \leq k \leq n$. For every $k$-dimensional subspace $H$ in $\mathbb{R}^n$, we have*

$$\mathrm{vol}_k(\mathrm{Proj}_H(Q_n)) = \mathrm{vol}_{n-k}(\mathrm{Proj}_{H^\perp}(Q_n)).$$

This has been found by McMullen and independently by Chakerian and Filliman, using the same approach based on Shephard's formula. In particular, sharp bounds on volumes of $k$-dimensional projections are equivalent to those on $(n-k)$-dimensional ones.

## 3.2 Simplices

Recall that $\Delta_n$ is the $n$-dimensional regular simplex with edge length $\sqrt{2}$, assuming for convenience in this section that $\Delta_n$ is embedded in $\mathbb{R}^n$. The projections of the regular simplex onto certain orthogonal complementary subspaces are conjectured to yield minimal and maximal volume, in huge contrast to the cube, where we have seen in Theorem 31 that such projections always have the *same* volume.

**Conjecture 7** (Filliman [54]). *Fix $1 \leq k \leq n$. Let $H_*$ be a $k$-dimensional subspace in $\mathbb{R}^n$ such that $T_* = \mathrm{Proj}_{H_*}(\Delta_n)$ is a $k$-dimensional simplex, with the vertices of $\Delta_n$ projecting only onto the vertices of $T_*$, as* evenly *as possible: For each $i \leq k+1$, letting $w_i$ be the number of vertices of $\Delta_n$ projecting onto vertex $i$ of $T_*$, we have*

$$w_i = \begin{cases} \ell + 1, & 1 \leq i \leq r, \\ \ell, & r < i \leq k+1, \end{cases}$$

*where we divide $n+1$ by $k+1$ with the remainder $r \in \{0, \ldots, k\}$, $n+1 = (k+1)\ell + r$. Then*

$$\min_{\substack{H \subset \mathbb{R}^n \\ \dim H = k}} \mathrm{vol}_k(\mathrm{Proj}_H(\Delta_n)) = \mathrm{vol}_k(T_*).$$

*Moreover, the polytope* $T^* = \mathrm{Proj}_{H_*^\perp}(\Delta_n)$ *is conjectured to maximize the volume of projections onto the* $(n-k)$*-dimensional subspaces,*

$$\max_{\substack{H\subset\mathbb{R}^n \\ \dim H=n-k}} \mathrm{vol}_k(\mathrm{Proj}_H(\Delta_n)) = \mathrm{vol}_{n-k}(T^*).$$

Filliman developed exterior algebra techniques [53] and used them [54] to confirm this conjecture in the following cases: for the minimum, $k = 1, 2, n-1$ and $n$ is arbitrary or $n \le 6$ and $k$ is arbitrary; for the maximum, $k = 1, 2, n-1$ and $n$ is arbitrary, $k = n-2$ and $n \le 8$, or $k = 3$ and $n = 6$.

## 3.3 Cross-polytopes

In view of Cauchy's formula from Theorem 2, the volume of hyperplane projections of the cross-polytope $B_1^n$ admits a natural probabilistic expression. Since it has $2^n$ congruent (simplicial) facets of $(n-1)$-dimensional volume $\frac{\sqrt{n}}{(n-1)!}$ with outer normals $\frac{1}{\sqrt{n}}(\pm1, \dots, \pm1)$, for every unit vector in $\mathbb{R}^n$, we have

$$\mathrm{vol}_{n-1}(\mathrm{Proj}_{a^\perp}(B_1^n)) = \frac{1}{2(n-1)!}\sum_{\varepsilon\in\{-1,1\}^n} |\langle a, \varepsilon\rangle| = \frac{2^{n-1}}{(n-1)!}\mathbb{E}\left|\sum_{j=1}^n a_j\varepsilon_j\right|,$$

where the expectation is over independent random signs $\varepsilon_j$, $\mathbb{P}(\varepsilon_j = \pm1) = \frac{1}{2}$. Given the constraint $|a| = 1$, the question about extremal volume projections thus becomes that of finding the best constants $c, C$ in the homogeneous inequalities

$$c\Big(\mathbb{E}\Big|\sum a_j\varepsilon_j\Big|^2\Big)^{1/2} \le \mathbb{E}\Big|\sum a_j\varepsilon_j\Big| \le C\Big(\mathbb{E}\Big|\sum a_j\varepsilon_j\Big|^2\Big)^{1/2}. \tag{3.1}$$

Such $L_p$-moment comparison inequalities go back to Khinchin's work [71] on the law of the iterated logarithm. This motivated and should be contrasted with an analogous probabilistic viewpoint on sections from Theorem 27, where instead of the $L_1$-norm, we have the limit of the $L_q$-norm as $q \downarrow -1$. A sharp upper bound follows easily from Jensen's inequality,

$$\mathbb{E}\Big|\sum a_j\varepsilon_j\Big| \le \Big(\mathbb{E}\Big|\sum a_j\varepsilon_j\Big|^2\Big)^{1/2} = |a| = 1,$$

attained if and only if $a = \pm e_i$ for some $i \le n$, that is, the maximum volume projection occurs at precisely coordinate subspaces. The reverse inequality is much deeper: In a different context, a sharp lower bound was conjectured to be attained at vectors $a = \frac{\pm e_i \pm e_j}{\sqrt{2}}$, $i \ne j$, by Littlewood in [93] (cf. Ball's extremizer from Theorem 6), proved much later by Szarek in [130], with subsequently simplified and quite different proofs [60, 88, 131]. We state it here rephrased in terms of volumes of projections, together with the simple upper bound.

**Theorem 32** (Szarek [130]). *Let $n \geq 2$. For every unit vector in $\mathbb{R}^n$, we have*

$$\frac{1}{\sqrt{2}}\operatorname{vol}_{n-1}(B_1^{n-1}) \leq \operatorname{vol}_{n-1}(\operatorname{Proj}_{a^\perp}(B_1^n)) \leq \operatorname{vol}_{n-1}(B_1^{n-1}).$$

*The lower bound is attained if and only if $a = \frac{\pm e_i \pm e_j}{\sqrt{2}}$ for some $i \neq j$, whilst the upper bound is attained if and only if $a = \pm e_i$ for some $i$.*

A stability version has been derived by De, Diakonikolas, and Servedio [44] (see also [100] for a local statement with explicit constants).

Much less is known in higher codimensions. In analogy to Vaaler's theorem, Theorem 5, for the cube, it is natural to conjecture that maximal volume projections of the cross-polytope $B_1^n$ occur at coordinate subspaces.

**Conjecture 8.** *Fix $1 \leq k \leq n$. For every $k$-dimensional subspace $H$ in $\mathbb{R}^n$, we have*

$$\operatorname{vol}_k(\operatorname{Proj}_H(B_1^n)) \leq \operatorname{vol}_k(B_1^k).$$

This conjecture has appeared in this generality in Ivanov's work [65], who has confirmed it for $k = 2, 3$ and arbitrary $n$, using perturbation methods for frames (see also [64]). Earlier, Filliman [52] established the same for $k = 2$, using different, more algebraic methods of his work [53], also reducing the case of $k = 3$ with arbitrary $n$ to $n \leq 42$. For minimal volume projections, we have the following *dual* analog of Conjecture 1 for the cube.

**Conjecture 9** (Ivanov [65]). *Fix $1 \leq k \leq n$. Let $H^*$ be the $k$-dimensional subspace from Conjecture 1. For every $k$-dimensional subspace $H$ in $\mathbb{R}^n$, we have*

$$\operatorname{vol}_k(\operatorname{Proj}_H(B_1^n)) \geq \operatorname{vol}_k(\operatorname{Proj}_{H^*}(B_1^n)).$$

This has been confirmed for $k = 2$ by Ivanov in [65]. As for the cube, the conjectured maximizer is an affine cross-polytope.

## 3.4 Balls of *p*-norms

The sharp results on hyperplane projections of the cross-polytope have been extended by Barthe and Naor in [24] to $\ell_p$-balls (with $p \geq 2$), thereby bringing the knowledge on extremal volume *hyperplane* projections to the same level as for sections (see Tables 1 and 2).

**Theorem 33** (Barthe–Naor [24]). *For every unit vector $a$ in $\mathbb{R}^n$, the function*

$$p \mapsto \frac{\operatorname{vol}_{n-1}(\operatorname{Proj}_{a^\perp}(B_p^n))}{\operatorname{vol}_{n-1}(B_p^{n-1})}$$

*is non-decreasing on $[1, +\infty)$.*

This can be viewed as a counterpart of Meyer and Pajor's theorem, Theorem 15, for *hyperplane* projections. It is an interesting open question to find such a monotonicity result for *all* subspaces. As for the cross-polytope, it is Cauchy's formula that allows to obtain a probabilistic expression for the volume in the hyperplane case. Barthe and Naor's argument goes as follows.

First, the surface area measure is related to the cone volume measure, by a general relation of Naor and Romik [107]. To sketch this, let $\sigma_K$ be the normalized surface area measure on $\partial K$ and let $S$ be the not normalized surface area measure, that is, $S(A) = \mathrm{vol}_{n-1}(\partial K \cap A)/\mathrm{vol}_{n-1}(\partial K)$. Let $\mu_K$ be the normalized cone volume measure, that is, for $A \subseteq \partial K$, let $\mu_K(A) = \mathrm{vol}_n(\mathrm{conv}(\{0\} \cup A))/\mathrm{vol}_n(K)$. Let $C$ denote its not normalized version.

**Lemma 34.** *If $K$ is a symmetric convex body in $\mathbb{R}^n$, then $\sigma_K$ is absolutely continuous with respect to $\mu_K$ and for almost all $x \in \partial K$ one has*

$$\frac{\mathrm{d}\sigma_K}{\mathrm{d}\mu_K}(x) = \frac{n\,\mathrm{vol}_n(K)}{\mathrm{vol}_{n-1}(\partial K)}\big|\nabla(\|\cdot\|_K)(x)\big|.$$

*Sketch of the proof.* For points $x$ such that $x$ is perpendicular to the surface of $K$ one has $|x| \cdot \mathrm{d}S(x) = n\mathrm{d}C(x)$. If the angle between the surface and $x$ is $\alpha$, then $|\cos\alpha| \cdot |x| \cdot \mathrm{d}S(x) = n\mathrm{d}C(x)$. We clearly have $|\cos\alpha| = |\langle n(x), x/|x|\rangle|$. Let $z = \nabla\|\cdot\|_K(x)$. If $x \in \partial K$, then $1+\varepsilon = \|x+\varepsilon x\|_K \approx \|x\|_K + \varepsilon\langle z, x\rangle = 1+\varepsilon\langle z, x\rangle$, which gives $\langle z, x\rangle = 1$. Also, $z$ is a vector perpendicular to $\partial K$. Thus, $n(x) = z/|z|$. We obtain $|\cos\alpha| = \frac{1}{|x|}\cdot|\langle n(x), x\rangle| = \frac{|\langle z,x\rangle|}{|x|\cdot|z|}$. This gives

$$\frac{\mathrm{vol}_{n-1}(\partial K)\mathrm{d}\sigma_K(x)}{|\nabla(\|\cdot\|_K)(x)|} = \frac{\mathrm{d}S(x)}{|\nabla(\|\cdot\|_K)(x)|} = \frac{|\langle z, x\rangle|}{|z|}\mathrm{d}S(x) = n\mathrm{d}C(x) = n\,\mathrm{vol}_n(K)\mathrm{d}\mu_K(x).$$

□

From Lemma 34 we therefore get

$$|\operatorname{Proj}_{a^\perp} K| = \frac{n}{2}\,\mathrm{vol}_n(K)\int_{\partial K}\big|\big\langle(\nabla\|\cdot\|_K)(x), a\big\rangle\big|\mathrm{d}\mu_K(x), \tag{3.2}$$

since $(\nabla\|\cdot\|_K)(x) = n(x)|(\nabla\|\cdot\|_K)(x)|$.

Second, the cone volume measure $\mu_{B_p^n}$ enjoys a probabilistic representation in terms of i. i. d. random variables, discovered by Rachev and Rüschendorf [120] and independently by Schechtman and Zinn [125]. We shall also later need a modification of the representation of the uniform measure on $B_p^n$ obtained in [22] by Barthe, Guédon, Mendelson, and Naor. Let us formulate a generalization of these results discussed in [118].

**Lemma 35.** *Let $K$ be a symmetric convex body and let $Z$ be any random vector in $\mathbb{R}^n$ with density of the form $f(\|x\|_K)$ for some continuous $f : [0,\infty) \to [0,\infty)$. Let $U$ be a random variable uniform in $[0,1]$, independent of $Z$. Then:*

(a) $\frac{Z}{\|Z\|_K}$ *has distribution $\mu_K$ and $U^{1/n}\frac{Z}{\|Z\|_K}$ is uniformly distributed on $K$,*

(b) $\frac{Z}{\|Z\|_K}$ *and $\|Z\|_K$ are independent.*

*In particular, for $K = B_p^n$ one can take $Z = (Y_1, \dots, Y_n)$, where $Y_i$ are i. i. d. random variables having densities $(2\Gamma(1+\frac{1}{p}))^{-1}e^{-|t|^p}$.*

*Proof.* We first claim that for any integrable $h : \mathbb{R}^n \to \mathbb{R}$ the following identity holds:

$$\int h = n|K| \int_0^\infty r^{n-1} \int_{\partial K} h(rz)\mathrm{d}\mu_K(z)\mathrm{d}r. \tag{3.3}$$

To show it one can assume that $h = \mathbf{1}_A$, where $A = [a,b] \cdot A_0$, where $A_0 \subset \partial K$, as these sets generate the $\sigma$-algebra of Borel sets in $\mathbb{R}^n$. For $z \in \partial K$ and $r > 0$ we then have $h(rz) = \mathbf{1}_{[a,b]}(r)\mathbf{1}_{A_0}(z)$. Thus, (3.3) reduces to

$$|A| = |K|\left(\int_a^b nr^{n-1}\mathrm{d}r\right)\mu_K(A_0) = |K|(b^n - a^n)\mu_K(A_0) = \big|[a,b]A_0\big| \tag{3.4}$$

and is therefore true. Now, let us notice that for $\phi : \mathbb{R}^n \to \mathbb{R}$ and $\psi : \mathbb{R} \to \mathbb{R}$ we have

$$\begin{aligned}\mathbb{E}\left[\phi\left(\frac{Z}{\|Z\|_K}\right)\psi(\|Z\|_K)\right] &= \int_{\mathbb{R}^n} \phi\left(\frac{x}{\|x\|_K}\right)\psi(\|x\|_K)f(\|x\|_K)\mathrm{d}x \\ &= n|K| \int_0^\infty \psi(r)f(r)r^{n-1}\mathrm{d}r \int_{\partial K} \phi(z)\mathrm{d}\mu_K(z).\end{aligned}$$

Taking $\phi, \psi \equiv 1$ we learn that $n|K|\int_0^\infty f(r)r^{n-1}\mathrm{d}r = 1$. Thus, taking $\psi \equiv$ and next $\phi \equiv 1$ we arrive at

$$\mathbb{E}\left[\phi\left(\frac{Z}{\|Z\|_K}\right)\right] = \int_{\partial K} \phi(z)\mathrm{d}\mu_K(z), \quad \mathbb{E}[\psi(\|Z\|_K)] = n|K|\int_0^\infty \psi(r)f(r)r^{n-1}\mathrm{d}r.$$

The first equation shows that $\frac{Z}{\|Z\|_K}$ has distribution $\mu_K$. Moreover, we get

$$\mathbb{E}\left[\phi\left(\frac{Z}{\|Z\|_K}\right)\psi(\|Z\|_K)\right] = \mathbb{E}\left[\phi\left(\frac{Z}{\|Z\|_K}\right)\right]\mathbb{E}[\psi(\|Z\|_K)],$$

which shows (b). Finally, (3.4) together with the fact that $U^{1/n}$ has density $nr^{n-1}$ on $[0,1]$ shows that

$$\frac{|A|}{|K|} = \mathbb{P}(U^{1/n} \in [a,b])\mathbb{P}\left(\frac{Z}{\|Z\|_K} \in A_0\right) = \mathbb{P}\left(U^{1/n}\frac{Z}{\|Z\|_K} \in A\right),$$

which shows the second part of point (a). □

We can now prove the probabilistic formula for the volume of the hyperplane projection of $B_p^n$.

**Lemma 36.** *For $p > 1$ and every unit vector $a \in \mathbb{R}^n$, we then have*

$$\mathrm{vol}_{n-1}(\mathrm{Proj}_{a^\perp}(B_p^n)) = \frac{\mathrm{vol}_{n-1}(B_p^{n-1})}{\mathbb{E}|X_1|}\mathbb{E}\left|\sum_{j=1}^n a_j X_j\right|, \tag{3.5}$$

*where $X_1, \dots, X_n$ are i. i. d. random variables with density $f_p(x) = \frac{p}{2(p-1)\Gamma(1/p)}|x|^{\frac{2-p}{p-1}}e^{-|x|^{\frac{p}{p-1}}}$.*

*Proof.* By (3.2) and Lemma 35(a) for some constant $c_{p,n}$ we have

$$\begin{aligned}\mathrm{vol}_{n-1}(\mathrm{Proj}_{a^\perp} B_p^n) &= C(p,n)\mathbb{E}\left|\sum_{i=1}^n a_i\left|\frac{Y_i}{S}\right|^{p-1}\mathrm{sgn}\left(\frac{Y_i}{S}\right)\right| \\ &= C(p,n)\cdot\frac{\mathbb{E}S^{p-1}}{\mathbb{E}S^{p-1}}\cdot\mathbb{E}\left|\sum_{i=1}^n a_i\left|\frac{Y_i}{S}\right|^{p-1}\mathrm{sgn}(Y_i)\right| \\ &= \frac{C(p,n)}{\mathbb{E}S^{p-1}}\cdot\mathbb{E}\left|\sum_{i=1}^n a_i|Y_i|^{p-1}\mathrm{sgn}(Y_i)\right|.\end{aligned}$$

It now suffices to observe that $X_i = |Y_i|^{p-1}\mathrm{sgn}(Y_i)$ for $p > 1$ have densities $f_p$. We then compute $C_{p,n}$ by taking $a = e_1$. □

Next, Meyer and Pajor's arguments involving peakedness are replaced by the stochastic convex (Choquet) ordering, where the independence of $X_j$ is crucial. For $p > 2$, additional structure emerges: $X_j$ are Gaussian mixtures. This leads to an analog of Koldobsky's theorem, Theorem 16, the proof of which was later simplified in [49] by bypassing the Fourier analytic arguments (we shall discuss the arguments in Section 4).

**Theorem 37** (Barthe–Naor [24]). *Let $p > 2$. For every two unit vectors $a$ and $b$ in $\mathbb{R}^n$ such that $(b_1^2, \dots, b_n^2)$ majorizes $(a_1^2, \dots, a_n^2)$, we have*

$$\mathrm{vol}_{n-1}(\mathrm{Proj}_{a^\perp}(B_p^n)) \geq \mathrm{vol}_{n-1}(\mathrm{Proj}_{b^\perp}(B_p^n)).$$

In the range $0 < p < 1$, Cauchy's formula cannot be applied due to the lack of convexity and no non-trivial bounds are known. When $1 < p < 2$, the maximal volume hyperplane projection is onto a coordinate subspace, as follows from Theorem 33, whereas the minimal one is not known. Barthe and Naor [24] have shown that the cross-polytope minimizer $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \dots, 0)^\perp$ is beaten by the diagonal one for every $p > p_0 = \frac{4}{3}$ in large enough dimensions (in particular, as Oleszkiewicz has pointed out in [113], there is no "formal duality" with sections, for there is not such a phase transition at $\frac{p_0}{p_0-1} = 4$).

For higher codimensions than 1, plainly Meyer and Pajor's theorem, Theorem 15, gives a sharp lower bound: For every $p \geq 2$, $1 \leq k \leq n$, and $k$-dimensional subspace in $\mathbb{R}^n$, we have

$$\mathrm{vol}_k(\mathrm{Proj}_H(B_p^n)) \geq \mathrm{vol}_k(B_p^n \cap H) \geq \mathrm{vol}_k(B_p^k),$$

attained at coordinate subspaces. For $0 < p < 2$, using his reverse form of the Brascamp–Lieb inequality from [20], Barthe [21] has established the following lower bound.

**Theorem 38** (Barthe [21]). *Let $0 < p < 2$. Fix $1 \le k \le n$. For every $k$-dimensional subspace $H$ in $\mathbb{R}^n$, we have*

$$\mathrm{vol}_k(\mathrm{Proj}_H(B_p^n)) \ge \left(\frac{k}{n}\right)^{k(1/p-1/2)} \mathrm{vol}_k(B_p^k).$$

This is optimal when $k$ divides $n$ and $p \ge 1$ (attained at subspaces from Conjecture 1).

# 4 Methods

We would like to present and emphasize one particular probabilistic point of view which gathers the major results for both sections and projections under the same umbrella. The point is that as it is very natural to set up hyperplane projection problems as sharp $L_1$–$L_2$ comparison inequalities (thanks to Cauchy's formula; see, e. g., (3.5)), the same probabilistic picture captures sections upon changing the $L_1$-norm to $L_q$-norms with negative exponents $q$.

## 4.1 Sections

This is a straightforward extension to higher codimensions of Kalton and Koldobsky's observation made in [68]; recall (2.8).

**Lemma 39** ([42]). *Let $K$ be a body in $\mathbb{R}^n$ of volume 1, star-shaped with respect to the origin. Let $H$ be a $k$-codimensional subspace in $\mathbb{R}^n$ and let $X$ be a random vector uniform on $K$. Let $\|\cdot\|$ be a norm in $H^\perp$ with the unit ball $B$. Then*

$$\mathrm{vol}_{n-k}(K \cap H) = \lim_{q \to -k+} \frac{k+q}{k\,\mathrm{vol}_k(B)} \mathbb{E}\|\mathrm{Proj}_{H^\perp} X\|^q.$$

*Proof.* If we let $f: H^\perp \to [0, +\infty)$ be the density of $\mathrm{Proj}_{H^\perp} X$, as in (2.2), we have

$$\mathrm{vol}_{n-k}(K \cap H) = f(0).$$

The function $x \mapsto \frac{k-q}{k\,\mathrm{vol}_k(B)}\|x\|^{-q}$ as $q \to -k+$ behaves like the Dirac delta at 0: If $f$ is continuous at 0 and integrable, then

$$\lim_{q \to -k+} \frac{k+q}{k\,\mathrm{vol}_k(B)} \int_{H^\perp} \|x\|^{-q} f(x)\mathrm{d}x = f(0),$$

and the lemma follows. To justify the last identity, for simplicity we identify $H^\perp$ with $\mathbb{R}^k$ and fix $\varepsilon > 0$. The set $\{x, f(x) < f(0) + \varepsilon\}$ contains a neighborhood of 0, say $\delta B$. Then

$$\frac{k+q}{k\,\mathrm{vol}_k(B)} \int_{\mathbb{R}^k} \|x\|^q f(x)\mathrm{d}x \le (f(0)+\varepsilon)\frac{k+q}{k\,\mathrm{vol}_k(B)} \int_{\delta K} \|x\|^q \mathrm{d}x + \frac{k+q}{k\,\mathrm{vol}_k(B)}\delta^q \int_{\mathbb{R}^k} f$$
$$= (f(0)+\varepsilon)\delta^{k+q} + \frac{k+q}{k\,\mathrm{vol}_k(B)}\delta^q \int_{\mathbb{R}^k} f$$

(the last equality follows by the homogeneity of volume and the layer cake representation). Taking lim sup as $q \downarrow -k$ gives an upper bound of $f(0) + \varepsilon$. A lower bound is obtained similarly (the second term above can be dropped). □

For hyperplane sections of the cube, the limit can be evaluated, which leads to a particularly handy expression.

**Lemma 40** (König–Koldobsky [83]). *Let $\xi_1, \xi_2, \ldots$ be i. i. d. random vectors uniform on the sphere $S^2$ in $\mathbb{R}^3$. For a unit vector $a$ in $\mathbb{R}^n$, we have*

$$\mathrm{vol}_{n-1}(Q_n \cap a^\perp) = \mathbb{E}\left|\sum_{j=1}^n a_j\xi_j\right|^{-1}.$$

*Proof.* Lemma 39 yields

$$\mathrm{vol}_{n-1}(Q_n \cap a^\perp) = \lim_{q\to -1+} \frac{1+q}{2}\mathbb{E}\left|\sum_{j=1}^n a_jX_j\right|^q,$$

where $X = (X_1, \ldots, X_n)$ is uniform on $Q_n$, that is, the components $X_j$ are independent uniform on $[-\frac{1}{2}, \frac{1}{2}]$. By Archimedes' hat-box theorem, $\langle \xi_j, e_1\rangle$ has the same distribution as $2X_j$, which allows to get for every fixed $q > -1$

$$\frac{1+q}{2}\mathbb{E}\left|\sum_{j=1}^n a_jX_j\right|^q = 2^{-1-q}\mathbb{E}\left|\sum_{j=1}^n a_j\xi_j\right|^q$$

(see, e. g., [41] for all details). Taking the limit finishes the proof. □

**Remark 41.** Replacing $\xi_j$ by i. i. d. random vectors uniform on a higher-dimensional sphere, say $S^{d+1}$, and the exponent $-1$ by $-d$ results in a formula for sections of balls in $\ell_\infty(\ell_2)$ by block subspaces (see [32, Proposition 3.2]).

To illustrate the applicability of this lemma, we sketch the proof of the lower bound of Theorem 14, the Hadwiger–Hensley bound with an optimal deficit.

*Proof (Sketch).* The key is to write

$$\left|\sum_{j=1}^{n} a_j\xi_j\right|^2 = \sum_{i,j} a_i a_j \langle \xi_i, \xi_j\rangle = 1 + 2\sum_{i<j} a_i a_j \langle \xi_i, \xi_j\rangle.$$

The random variable $R = 2\sum_{i<j} a_i a_j \langle \xi_i, \xi_j\rangle$ has mean 0. Thus, by convexity,

$$\mathbb{E}(1+R)^{-1/2} \geq \mathbb{E}(1-R/2) = 1.$$

To improve upon this, it suffices to use a more precise pointwise inequality, say

$$(1+r)^{-1/2} \geq 1 - \frac{1}{2}r + \frac{1}{3}r^2 - \frac{5}{24}r^3, \quad r > -1,$$

and estimate $\mathbb{E}R^2$ and $\mathbb{E}R^3$, which are explicitly expressed in terms of $a_j$. □

For $B_p^n$-balls, a direct application of Lemma 39 leaves us with a random vector uniform on $B_p^n$ with *mildly* dependent components. This however can be circumvented thanks to the homogeneity of $L_q$-norms.

**Lemma 42** ([42]). *Let $p > 0$ and let $Y_1, Y_2, \ldots$ be i. i. d. random variables with density $e^{-\beta_p^p|x|^p}$, $\beta_p = 2\Gamma(1+1/p)$. Let $H$ be a subspace in $\mathbb{R}^n$ of codimension $k$ such that the rows of a $k\times n$ matrix $U$ form an orthonormal basis of $H^\perp$. Let $v_1, \ldots, v_n \in \mathbb{R}^k$ denote the columns of $U$. Then*

$$\mathrm{vol}_{n-k}(B_p^n \cap H) = \mathrm{vol}_{n-k}(B_p^{n-k}) \lim_{q\to -k+} \frac{k+q}{k\,\mathrm{vol}_k(B_{\|\cdot\|})}\mathbb{E}\left\|\sum_{j=1}^n Y_j v_j\right\|^q,$$

*where $\|\cdot\|$ is a norm on $\mathbb{R}^k$ with unit ball $B_{\|\cdot\|}$.*

*Proof.* Let $X = (X_1, \ldots, X_n)$ be a random vector uniform on $B_p^n$. Lemma 39 then gives the desired formula with $X_j$ in place of $Y_j$ and without the factor $\mathrm{vol}_{n-k}(B_p^{n-k})$. To pass to $Y$ we shall use Lemma 35, which ensures that for $Y = (Y_1, \ldots, Y_n)$ and $S = (\sum_{i=1}^n |Y_i|^p)^{1/p}$ the random vector $\frac{Y}{S}$ is independent of $S$ and moreover $U^{1/n}\frac{Y}{S}$ is uniformly distributed in $B_p^n$ if $U$ is independent of $Y_i$ and uniform on $[0,1]$. Therefore,

$$\mathbb{E}\left\|\sum_{j=1}^n X_j v_j\right\|^q = \mathbb{E}\left\|\sum_{j=1}^n U^{1/n}\frac{Y_j}{S}v_j\right\|^q = \mathbb{E}[U^{q/n}]\cdot\frac{\mathbb{E}[S^q]}{\mathbb{E}[S^q]}\cdot\mathbb{E}\left\|\sum_{j=1}^n \frac{Y_j}{S}v_j\right\|^q = \frac{\mathbb{E}[U^{q/n}]}{\mathbb{E}[S^q]}\cdot\mathbb{E}\left\|\sum_{j=1}^n Y_j v_j\right\|^q.$$

□

This has been instrumental in the proof of Theorem 19. For a simpler application, the Meyer–Pajor monotonicity result from Theorem 15 holds in fact for $L_q$-norms. In view of the previous lemma, this readily implies their theorem.

**Theorem 43.** *For $p > 0$, let $Y_1^{(p)}, Y_2^{(p)}, \dots$ be i. i. d. random variables with density $e^{-\beta_p^p |x|^p}$, $\beta_p = 2\Gamma(1+1/p)$. For every vectors $v_1, \dots, v_n$ in $\mathbb{R}^k$ and $-k < q < 0$, the function*

$$(p_1, \dots, p_n) \mapsto \mathbb{E}\left\| \sum_{j=1}^n Y_j^{(p_j)} v_j \right\|^q$$

*is non-decreasing in each variable.*

*Proof.* Following Kanter [69], we say for two probability measures $\mu$ and $\nu$ on $\mathbb{R}^n$ that $\nu$ is more peaked than $\mu$ if $\nu(K) \geq \mu(K)$ for every symmetric convex set $K$ in $\mathbb{R}^n$. Crucially, this is preserved by taking products and convolutions of even log-concave measures (see [69, Corollaries 3.2 and 3.3]). If $0 < p < p'$, then the density of $Y_1^{(p')}$ intersects the density of $Y_1^{(p)}$ exactly once and dominates it (pointwise) near the origin. Thus, $Y_1^{(p')}$ is more peaked than $Y_1^{(p)}$ and consequently $\sum Y_j^{(p_j')} v_j$ is more peaked than $\sum Y_j^{(p_j)} v_j$ if $p_j \leq p_j'$. In particular, for every $t > 0$,

$$\mathbb{P}\left( \left\| \sum_{j=1}^n Y_j^{(p_j)} v_j \right\| \leq t \right) \leq \mathbb{P}\left( \left\| \sum_{j=1}^n Y_j^{(p_j')} v_j \right\| \leq t \right)$$

and the result follows by integrating in $t$. □

The measure with density $e^{-\beta_p^p |x|^p}$ from Lemma 39 enjoys a *Gaussian mixture* form when $0 < p < 2$. This in turn provides good convolution properties, allowing in particular to evaluate the limit from Lemma 39. We say that a random variable $X$ is a (symmetric) Gaussian mixture if $X$ has the same distribution as $RG$ for some non-negative random variable $R$ and a standard Gaussian random variable $G$, independent of $R$. Gaussian mixtures are continuous, i. e., have densities, and $X$ is a Gaussian mixture if and only if its density $f$ is of the form

$$f(x) = \int_0^\infty e^{-tx^2} \, \mathrm{d}\nu(t)$$

for a Borel measure $\nu$ on $[0, +\infty)$. By Bernstein's theorem, this is equivalent to $g(x) = f(\sqrt{x})$ being completely monotone, that is, $(-1)^n g^{(n)}(x) \geq 0$ for all $n \geq 0$ and $x > 0$, which gives a practical condition. We refer to [49] for further details and more examples. Thus, if $X_1, \dots, X_n$ are independent Gaussian mixtures, say $X_j = R_j G_j$, and $v_1, \dots, v_n$ are vectors in $\mathbb{R}^k$, then, conditioned on the values of $R_j$, $\sum X_j v_j$ is a centered Gaussian random vector in $\mathbb{R}^k$ with covariance matrix $\sum R_j^2 v_j v_j^\top$.

**Lemma 44** ([49, 108]). *Let $0 < p < 2$. There are non-negative i. i. d. random variables $R_1, R_2, \dots$ such that for every subspace $H$ in $\mathbb{R}^n$ of codimension $k$, we have*

$$\mathrm{vol}_{n-k}(B_p^n \cap H) = \mathrm{vol}_{n-k}(B_p^{n-k}) \mathbb{E}\left( \det\left[ \sum_{j=1}^n R_j v_j v_j^\top \right] \right)^{-1/2},$$

*where* $v_1, \ldots, v_n$ *are vectors in* $\mathbb{R}^k$ *such that the rows of the* $k \times n$ *matrix with columns* $v_1, \ldots, v_n$ *form an orthonormal basis of* $H^\perp$.

**Remark 45.** To describe the distribution of $R_j$, for $0 < \alpha < 1$, we let $g_\alpha$ be the density of a standard positive $\alpha$-stable random variable $W_\alpha$, i. e., with the Laplace transform $\mathbb{E}e^{-uW_\alpha} = e^{-u^\alpha}$, $t > 0$, and we let $V_1, V_2, \ldots$ be i. i. d. random variables with density $\frac{\sqrt{\pi}}{2\Gamma(1+1/p)} t^{-3/2} g_{p/2}(t^{-1})$. Then $R_j = (\mathbb{E}V_j^{-1/2})^2 V_j$; see [49].

*Proof of Lemma* 44. $Y_j$ from Lemma 39 are Gaussian mixtures, say $Y_j = T_j G_j$ for some non-negative random variables $T_j$ and standard Gaussians $G_j$, all independent. Then, conditioned on $T_j$, the limit in Lemma 39 gives the density at 0 of the random variable $\sum Y_j v_j$, which, as we said, is centered Gaussian in $\mathbb{R}^k$ with covariance $\sum T_j^2 v_j v_j^\top$; thus, its density at 0 equals $(2\pi)^{-k/2}(\det[\sum_{j=1}^n T_j^2 v_j v_j^\top])^{-1/2}$. □

For hyperplane sections, this formula directly explains Koldobsky's Schur convexity result from Theorem 16.

*Proof of Theorem* 16. We first observe that if $F : \mathbb{R}^n \to \mathbb{R}$ is convex and permutation-symmetric, then $F$ is Schur convex, namely $x \prec y$ implies $F(x) \le F(y)$. Indeed, it is a standard fact (see [25]) that there exist $(\lambda_\sigma)_{\sigma \in S_n}$, where $S_n$ stands for the set of permutations of $\{1, \ldots, n\}$, such that $\lambda_\sigma \ge 0$, $\sum_{\sigma\in S_n} \lambda_\sigma = 1$, and $x = \sum_{\sigma \in S_n} \lambda_\sigma y_\sigma$, where $y_\sigma = (y_{\sigma(1)}, \ldots, y_{\sigma(n)})$. Thus,

$$F(x) = F\Big(\sum_{\sigma\in S_n} \lambda_\sigma y_\sigma\Big) \le \sum_{\sigma \in S_n} \lambda_\sigma F(y_\sigma) = \sum_{\sigma\in S_n} \lambda_\sigma F(y) = F(y).$$

For a unit vector $a$ in $\mathbb{R}^n$, Lemma 44 yields

$$\mathrm{vol}_{n-1}(B_p^n \cap a^\perp) = \mathrm{vol}_{n-1}(B_p^{n-1})\mathbb{E}\Big(\sum_{j=1}^n a_j^2 R_j\Big)^{-1/2}.$$

Since $(\cdot)^{-1/2}$ is convex, the right-hand side is clearly convex and permutation-symmetric ($R_j$ are i. i. d.) as a function of $(a_1^2, \ldots, a_n^2)$ and thus it is also Schur convex. □

## 4.2 Projections

Somewhat analogous to the Fourier analytic approach to sections, there is a formula for the volume of hyperplane projections of a convex body as the Fourier transform of its curvature function, as discovered by Koldobsky, Ryabogin, and Zvavitch [79] (see also their survey [78]). We do *not* touch upon this connection here at all. Instead, we focus on a probabilistic perspective and highlight two approaches to the $L_1$–$L_2$ moment comparison inequalities like (3.1), arising in hyperplane projections.

As we have just seen for sections, for Gaussian mixtures, thanks to their good additive structure, we readily get precise Schur majorization-type results. This proof is from [49].

*Proof of Theorem* 37. Recall formula (3.5) for hyperplane projections. For $p > 2$, the density $f_p(t)$ of $X_i$ is completely monotone; thus, $X_j$ are Gaussian mixtures, say $X_j = R_j G_j$ for some i. i. d. non-negative random variables $R_j$ and standard Gaussians $G_j$, all independent. Then, adding the Gaussians first conditioning on $R_j$ yields

$$\mathbb{E}\left|\sum_{j=1}^{n} a_j X_j\right| = \mathbb{E}\left(\sum_{j=1}^{n} a_j^2 R_j^2\right)^{1/2} \mathbb{E}|G_1|. \tag{4.1}$$

As in the proof for sections, the Schur concavity result follows from the concavity of $(\cdot)^{1/2}$. □

The same argument bluntly extends to arbitrary $L_q$-norms, giving sharp Khinchin inequalities (see [6, 49]).

When $1 \le p < 2$, the density of $X_j$ in (3.5) is bimodal and understanding the $L_1$-norm of their weighted sums is elusive, mainly due to complicated cancelations – a problem which completely disappears in (4.1). For $p = 1$, $X_j$ become discrete (symmetric random signs). We present two completely different Fourier analytic proofs. The first proof, due to Haagerup, is in the same spirit as Ball's proof from [7] for hyperplane cube sections.

*Proof of Theorem* 32 (Haagerup [60]). We want to minimize $\mathbb{E}|\sum a_j \varepsilon_j|$ subject to $\sum a_j^2 = 1$. We can assume that all $a_j$ are positive. If at least one exceeds $\frac{1}{\sqrt{2}}$, say $a_1 > \frac{1}{\sqrt{2}}$, by averaging over the other coefficients we obtain

$$\mathbb{E}\left|\sum a_j \varepsilon_j\right| \ge \mathbb{E}_{\varepsilon_1}\left|a_1 \varepsilon_1 + \mathbb{E}\sum_{j>1} a_j \varepsilon_j\right| = a_1 > \frac{1}{\sqrt{2}},$$

as desired. Now we assume that for all $j$, $a_j \le \frac{1}{\sqrt{2}}$. A starting point is the Fourier analytic formula

$$|x| = \frac{1}{\pi}\int_{\mathbb{R}} (1 - \cos(tx)) t^{-2} \mathrm{d}t, \quad x \in \mathbb{R}.$$

Thus, for an integrable random variable $X$,

$$\mathbb{E}|X| = \frac{1}{\pi}\int_{\mathbb{R}} (1 - \mathrm{Re}(\mathbb{E}e^{itX})) t^{-2} \mathrm{d}t$$

(see also [60, Lemmas 2.3 and 4.2] and [59, Lemma 3]). In particular, using independence and $\mathbb{E}e^{it\varepsilon_j} = \cos t$, we get

$$\mathbb{E}\left|\sum a_j\varepsilon_j\right| = \frac{1}{\pi}\int_{\mathbb{R}}\Big(1 - \prod \cos(ta_j)\Big)t^{-2}\mathrm{d}t.$$

By the AM-GM inequality, this gives the following bound:

$$\mathbb{E}\left|\sum a_j\varepsilon_j\right| \geq \sum a_j^2 F(a_j^{-2})$$

with

$$F(s) = \frac{1}{\pi}\int_{\mathbb{R}}\left(1 - \left|\cos\left(\frac{t}{\sqrt{s}}\right)\right|^s\right)t^{-2}\mathrm{d}t, \quad s > 0.$$

See (2.4) and the ensuing function $\Psi$ in Ball's proof. Here, however, function $F$ can be expressed explicitly. Using $\sum_{n=-\infty}^{\infty}\frac{1}{(t+n\pi)^2} = \frac{1}{\sin^2 t}$, we arrive at

$$\begin{aligned} F(s) = \frac{1}{\pi\sqrt{s}}\int_{\mathbb{R}}(1 - |\cos t|^s)t^{-2}\mathrm{d}t &= \frac{1}{\pi\sqrt{s}}\sum_{n=-\infty}^{\infty}\int_{-\pi/2}^{\pi/2}(1-(\cos t)^s)(t+n\pi)^{-2}\mathrm{d}t \\ &= \frac{1}{\pi\sqrt{s}}\int_{-\pi/2}^{\pi/2}(1-(\cos t)^s)\sin^{-2}t\mathrm{d}t \\ &= \frac{2}{\sqrt{\pi s}}\frac{\Gamma(\frac{s+1}{2})}{\Gamma(\frac{s}{2})}. \end{aligned}$$

**Claim.** $F(s)$ increases on $(0, +\infty)$.

Using this claim and the fact that $a_j \leq \frac{1}{\sqrt{2}}$ for all $j$, we finish the proof,

$$\mathbb{E}\left|\sum a_j\varepsilon_j\right| \geq \sum a_j^2 F(a_j^{-2}) \geq \sum a_j^2 F(2) = F(2) = \frac{1}{\sqrt{2}}.$$

Noteworthily, this is tight when $n = 2$ and $a_1 = a_2 = \frac{1}{\sqrt{2}}$.

To show the claim, we note that $\lim_{s\to\infty} F(s) = \sqrt{\frac{2}{\pi}}$ (e. g., by Stirling's formula) and that

$$F(s+2) = \sqrt{\frac{s}{s+2}}\frac{s+1}{s}F(s) = \left(1 - 1/(s+1)^2\right)^{-1/2}F(s),$$

which iterated yields $F(s+2n) = F(s)\prod_{k=0}^{n-1}(1-1/(s+2k+1)^2)^{-1/2}$, so letting $n \to \infty$,

$$F(s) = \sqrt{\frac{2}{\pi}}\prod_{k=0}^{\infty}\left(1 - 1/(s+2k+1)^2\right)^{1/2}.$$

□

The second proof uses the machinery of Fourier analysis on the discrete cube $\{-1, 1\}^n$. We refer for instance to [111, Chapter 1] for basic background.

*Proof of Theorem* 32 (Kwapień–Latała–Oleszkiewicz [81, 88, 112]).
We work with $L_2(\{-1,1\}^n, \mathbb{R})$ equipped with the product probability measure on the cube $\{-1,1\}^n$, i. e., the distribution of $(\varepsilon_1, \dots, \varepsilon_n)$, and the inner product $\langle f, g\rangle = \mathbb{E}[f(\varepsilon)g(\varepsilon)]$, $f, g\colon \{-1,1\}^n \to \mathbb{R}$. Let

$$f(x) = \left|\sum_{j=1}^{n} a_j x_j\right|, \quad x \in \{-1,1\}^n.$$

We write its discrete Fourier expansion with respect to the orthonormal system of the Walsh functions $w_S(x) = \prod_{j\in S} x_j$ indexed by the subsets $S \subset \{1, \dots, n\}$ with $w_\varnothing(x) \equiv 1$. We have

$$f(x) = \sum_S b_S w_S(x), \quad b_S = \langle f, w_S\rangle.$$

Since $f$ is even, $b_S = 0$ provided $|S|$ is odd. The crux is to consider the Laplace operator $\mathcal{L}$ acting on $L_2(\{-1,1\}^n, \mathbb{R})$,

$$(\mathcal{L}g)(x) = \frac{1}{2}\sum_{y\sim x}(g(y) - g(x)),$$

where the sum is over all neighbors $y$ of $x$, i. e., the points in $\{-1,1\}^n$ differing from $x$ by one component. As can be checked, the Walsh functions are its eigenfunctions, $\mathcal{L}w_S = -|S|w_S$, and for *even* functions $g$, we have the following Poincaré-type inequality:

$$\langle g, -\mathcal{L}g\rangle \geq 2\operatorname{Var}(g).$$

**Claim.** We have $(-\mathcal{L}f)(x) \leq f(x)$ for every $x \in \{-1,1\}^n$.

Using this claim in the Poincaré inequality,

$$2\big(\mathbb{E}f^2 - (\mathbb{E}f)^2\big) \leq \langle f, -\mathcal{L}f\rangle \leq \langle f, f\rangle = \mathbb{E}f^2,$$

which gives $\mathbb{E}f \geq \frac{1}{\sqrt{2}}(\mathbb{E}f)^2$, as desired. The claim follows from rearranging the following consequence of the triangle inequality:

$$|-a_1x_1 + a_2x_2 + \cdots + a_nx_n| + |a_1x_1 - a_2x_2 + \cdots + a_nx_n| + \cdots + |a_1x_1 + a_2x_2 + \cdots - a_nx_n| \\ \geq (n-2)|a_1x_1 + \cdots + a_nx_n|.$$

□

We stress out that this proof is extremely robust: It only uses the triangle inequality and hence extends ad verbatim to the case where the coefficients $a_j$ are vectors in an arbitrary normed vector space.

The history of this argument is a bit convoluted. Latała and Oleszkiewicz's work [88] contains all the crucial ideas of the modern proof presented above; however, it is not written in a Fourier analytic language. The proof presented here was devised by Kwapień and is based on the Walsh functions (the characters of $\{-1,1\}^n$). As we have seen, one of its main components is a strengthened Poincaré-type inequality in the presence of symmetry, the idea of which appeared first in [81] (in the continuous case), extended to the discrete case in [112] (perhaps the first place where this proof appears in print). Oleszkiewicz presented this proof in 1996 at MSRI (during a workshop in harmonic analysis and convex geometry).

We finish with a sketch of the Barthe–Naor proof from [24] of the monotonicity result from Theorem 33 featuring yet another tool, useful in proving Khinchin-type inequalities: the stochastic convex ordering. This circle of ideas was further developed in [50].

In the simplest setting sufficient for our purposes, for two symmetric random variables $X$ and $Y$, we say that $Y$ *dominates* $X$ in the convex (or often called Choquet) stochastic ordering if $\mathbb{E}\phi(X) \le \mathbb{E}\phi(Y)$ for every even convex function $\phi\colon \mathbb{R} \to [0,+\infty]$. It is clear that this tensorizes and is preserved by convolution: If $Y$ dominates $X$ and $Z$ is a symmetric random variable, independent of them, then $Y+Z$ dominates $X+Z$. We will only need the following sufficient condition.

**Lemma 46.** *If random variables $X$ and $Y$ satisfy $\mathbb{E}|X| = \mathbb{E}|Y|$ and have even densities $f$ and $g$, respectively, and there are $0 < x_1 < x_2$ such that $\{t \ge 0,\ g(t) < f(t)\}$ is the interval $(x_1, x_2)$ ($f$ and $g$* intersect *twice), then $Y$ dominates $X$ in the convex stochastic ordering.*

*Proof.* Let $\phi\colon \mathbb{R} \to [0,+\infty]$ be an even convex function. Thanks to the symmetry of $X, Y$ and the constraint $\mathbb{E}|X| = \mathbb{E}|Y|$, the desired inequality $\int \phi f \le \int \phi g$ is equivalent to

$$\int_0^\infty (\phi(x) - \alpha x - \beta)(g(x) - f(x))\mathrm{d}x \ge 0$$

with some (any) $\alpha, \beta \in \mathbb{R}$. We choose $\alpha, \beta$ as the unique parameters such that the convex function $\psi(x) = \phi(x) - \alpha x - \beta$ vanishes at $x_1$ and $x_2$. Then, by convexity, $\psi \le 0$ on $(x_1, x_2)$ and $\psi \ge 0$ outside that interval. Thus, the integrand is pointwise non-negative. □

*Proof of Theorem* 33. In view of (3.5), we aim at showing that the function

$$p \mapsto \frac{1}{\mathbb{E}|X_1^{(p)}|}\mathbb{E}\Big|\sum a_j X_j^{(p)}\Big|$$

is non-decreasing on $[1,+\infty)$, where $X_j^{(p)}$ are i. i. d. random variables with density proportional to $|x|^{\frac{2-p}{p-1}}\exp\{-|x|^{\frac{p}{p-1}}\}$. By the tensorization property, it suffices to prove that for $1 \le p < q$, $X_1^{(q)}/\mathbb{E}|X_1^{(q)}|$ dominates $X_1^{(p)}/\mathbb{E}|X_1^{(p)}|$. This readily follows from Lemma 46. □

# 5 Other connections

We close this survey with two tangential topics related to sections: an application of Ball's cube slicing inequality to entropy power inequalities and a reformulation of the conjectural logarithmic Brunn–Minkowski inequality in terms of sections of the cube.

## 5.1 Entropy power inequalities

Recall (2.2), viz. the volume of a central hyperplane section by $a^\perp$ is the maximum value of the density of the marginal $\langle a, X\rangle = \sum a_j X_j$ (there $f(0) = \|f\|_\infty$ by the symmetry and log-concavity of $X$). The maximum density functional

$$M(X) = \|f\|_\infty$$

of a random vector $X$ in $\mathbb{R}^n$ with density $f$ is closely related to classical topics in probability such as the Lévy concentration function, small ball estimates, and anticoncentration, as well as information theory, particularly the entropy power inequalities. We refer to the comprehensive surveys [96, 110]. The entropy power inequality originated in Shannon's seminal work [128] and asserts that the entropy power

$$N(X) = \exp\left\{\frac{2}{n}h(X)\right\}, \quad h(X) = -\int_{\mathbb{R}^n} f\log f,$$

is superadditive: For *independent* random vectors $X$ and $Y$ in $\mathbb{R}^n$, we have

$$N(X+Y) \geq N(X) + N(Y),$$

and plainly, by induction, the same is true for arbitrarily many independent summands. In analogy, we let

$$N_\infty(X) = \exp\left\{\frac{2}{n}h_\infty(X)\right\} = M(X)^{-2/n}, \quad h_\infty(X) = -\log\|f\|_\infty,$$

be the $\infty$-entropy power of $X$, sometimes called the min-entropy power (because for a fixed distribution, it is the smallest entropy power across the family of all Rényi entropies). The min-entropy power inequality in dimension 1 reads as follows.

**Theorem 47** (Bobkov–Chistyakov [27], Melbourne–Roberto [100]). *Let $X_1, \ldots, X_m$ be independent random variables with bounded densities. Then*

$$N_\infty(X_1 + \cdots + X_m) \geq \frac{1}{2}\sum_{j=1}^m N_\infty(X_j)$$

*with equality if and only if two of these variables are uniform on A and c − A, respectively, for some set A in* $\mathbb{R}$ *of finite measure and some* $c \in \mathbb{R}$, *while the other variables are constant.*

Bobkov and Chistyakov proved this inequality with the sharp constant $\frac{1}{2}$ using Ball's cube slicing inequality, whereas the equality conditions have recently been established by Melbourne and Roberto using their robust version (see Theorem 14).

The argument rests on the following subtle comparison due to Rogozin.

**Theorem 48** (Rogozin [122]). *Let* $X_1, \ldots, X_m$ *be independent random variables with bounded densities and let* $U_1, \ldots, U_m$ *be independent uniform random variables on intervals chosen such that* $M(X_j) = M(U_j)$ *for each j. Then*

$$M(X_1 + \cdots + X_m) \leq M(U_1 + \cdots + U_m).$$

Theorem 47 then follows by invoking Ball's theorem, Theorem 6, which after incorporating the variance constraint amounts to

$$M(U_1 + \cdots + U_m) \leq \sqrt{2}\big(M(U_1)^{-2} + \cdots + M(U_m)^{-2}\big)^{-1/2}.$$

In [97], Madiman, Melbourne, and Xu developed multivariate generalizations of Rogozin's result where the extremal distributions are uniform on the Euclidean ball. They have combined it with Brzeziński's bound (2.7) to obtain an extension of Theorem 47 to $\mathbb{R}^n$-valued random vectors with the sharp constant $\frac{1}{2}$ replaced by $\frac{\Gamma(1+n/2)^{2/n}}{(1+n/2)}$, which is asymptotically sharp (as $m \to \infty$). Previously, using a different argument exploiting Young's inequalities with sharp constants, Bobkov and Chistyakov in [26] obtained such an extension with a slightly worse constant $\frac{1}{e}$ ("attained" as $n \to \infty$), whereas in [121], Ram and Sason obtained constants dependent on the number of summands. Another direction, related to higher-dimensional marginals, has been explored by Livshyts, Paouris, and Pivovarov in [95].

We end this subsection with a conjectural entropic Busemann-type result.

**Conjecture 10** (Ball–Nayar–Tkocz [15]). *Let X be a symmetric log-concave random vector in* $\mathbb{R}^n$. *Then*

$$v \mapsto \sqrt{N(\langle v, X\rangle)} = e^{h(\langle v,X\rangle)}$$

*defines a norm on* $\mathbb{R}^n$.

Note that Busemann's theorem, Theorem 1, is equivalent to this statement with $N_\infty(\cdot)$ in place of the entropy power $N(\cdot)$ (for uniform distributions on symmetric convex bodies which generalizes to all symmetric log-concave distributions by Ball's results from [8]). What supports this conjecture is the fact that $\sqrt{N(\langle v, X\rangle)}$ defines an $e$-quasinorm which is also a $\frac{1}{5}$-seminorm (see [15]), and the conjecture holds for the Rényi entropy power of order 2 (see [92]). For extensions to $\kappa$-concave measures, see [96].

### 5.2 The logarithmic Brunn–Minkowski conjecture

In [37], Böröczky, Lutwak, Yang, and Zhang conjectured a strengthening of the Brunn–Minkowski inequality in the presence of symmetry and convexity, namely

$$\mathrm{vol}_n(M_\lambda(K,L)) \geq \mathrm{vol}_n(K)^\lambda \,\mathrm{vol}_n(L)^{1-\lambda},$$

for all symmetric convex sets $K$ and $L$ in $\mathbb{R}^n$ and every $0 \leq \lambda \leq 1$, where $M_\lambda(K,L)$ is the intersection of the symmetric strips

$$S_\theta = \{x \in \mathbb{R}^n, |\langle x,\theta\rangle| \leq h_K(\theta)^\lambda h_L(\theta)^{1-\lambda}\}$$

over all unit vectors $\theta$ in $S^{n-1}$. Here, as usual $h_K(\theta) \sup_{y\in K}\langle\theta, y\rangle$ denotes the support functional of $K$. Still resisting significant efforts of many researchers over a decade, this far-reaching conjecture stems from the so-called logarithmic Minkowski problem (see [38]); we refer to E. Milman's recent work [103] for further comprehensive background, references, and the best results to date. Relevant to us is an equivalent formulation in terms of a certain convexity property of volumes of sections of rescaled cubes.

**Conjecture 11.** *Let $1 \leq k \leq n$. For every k-dimensional subspace $H$ in $\mathbb{R}^n$, the function*

$$(t_1,\dots,t_n) \mapsto \mathrm{vol}_k(\mathrm{diag}(e^{t_1},\dots,e^{t_1})B_\infty^n \cap H)$$

*is log-concave on $\mathbb{R}^n$.*

For precise statements and explanations of equivalences for this and similar formulations, we refer to [108, 123, 124]. Here, as usual $\mathrm{diag}(e^{t_1},\dots,e^{t_n})$ is the $n \times n$ diagonal matrix with the diagonal entries $e^{t_1},\dots,e^{t_n}$, so that $\mathrm{diag}(e^{t_1},\dots,e^{t_1})B_\infty^n = [-e^{t_1}, e^{t_1}] \times \cdots \times [-e^{t_n}, e^{t_n}]$. In fact, we conjecture that the conjecture remains true with $B_p^n$ in place of the cube $B_\infty^n$ for every $p \geq 1$; we have been able to verify this for $p = 1$ in [108] using Lemma 44.

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