

Additive Energies on Discrete Cubes

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Abstract: We prove that for $d \geq 0$ and $k \geq 2$, for any subset A of a discrete cube $\{0, 1\}^d$, the k -higher energy of A (i.e., the number of $2k$ -tuples $(a_1, a_2, \dots, a_{2k})$ in A^{2k} with $a_1 - a_2 = a_3 - a_4 = \dots = a_{2k-1} - a_{2k}$) is at most $|A|^{\log_2(2^k+2)}$, and $\log_2(2^k+2)$ is the best possible exponent. We also show that if $d \geq 0$ and $2 \leq k \leq 10$, for any subset A of a discrete cube $\{0, 1\}^d$, the k -additive energy of A (i.e., the number of $2k$ -tuples $(a_1, a_2, \dots, a_{2k})$ in A^{2k} with $a_1 + a_2 + \dots + a_k = a_{k+1} + a_{k+2} + \dots + a_{2k}$) is at most $|A|^{\log_2 \binom{2k}{k}}$, and $\log_2 \binom{2k}{k}$ is the best possible exponent. We discuss the analogous problems for the sets $\{0, 1, \dots, n\}^d$ for $n \geq 2$.

1 Introduction

The additive energy $E(A)$ of a finite subset A of an additive group G is defined as the number of quadruples $(a_1, a_2, a_3, a_4) \in A \times A \times A \times A$ such that $a_1 + a_2 = a_3 + a_4$ (see [12]). Observe that for any triple (a_1, a_2, a_3) there is at most one a_4 such that $a_1 + a_2 = a_3 + a_4$, so we have the trivial upper bound $E(A) \leq |A|^3$ (here $|A|$ denotes the cardinality of A). This bound is attained, for example, when A is itself a finite group. Considering the diagonal solutions $a_1 = a_3$ and $a_2 = a_4$ we also observe the trivial lower bound $E(A) \geq |A|^2$.

1.1 Higher energies

We define the k -higher energy of a set $A \subseteq \{0, 1\}^d \subset \mathbb{Z}^d$ by

$$\tilde{E}_k(A) := |\{(a_1, a_2, \dots, a_{2k-1}, a_{2k}) \in A^{2k} : a_1 - a_2 = a_3 - a_4 = \dots = a_{2k-1} - a_{2k}\}|.$$

This has been studied by many authors, see [9], [10]. In this case we have the trivial bounds $|A|^k \leq \tilde{E}_k(A) \leq |A|^{k+1}$.

Theorem 1. *Let $d \geq 0$, $k \geq 2$, and let $A \subset \{0, 1\}^d$. Then $\tilde{E}_k(A) \leq |A|^{q_k}$, where $q_k := \log_2(2^k + 2)$. Furthermore, the exponent q_k cannot be replaced by any smaller quantity.*

Remark 2. This Theorem extends a result obtained by Kane–Tao [5, Theorem 7] for $k = 2$.

The second claim in our Theorem 1 follows considering the case $A = \{0, 1\}^d$, in this case we have $|A| = |\{0, 1\}^d| = 2^d$ and $\tilde{E}_k(\{0, 1\}^d) = (2^k + 2)^d$.

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1.2 k -additive energies

We discuss another generalization of Kane–Tao result [5, Theorem 7]. We define the k -additive energy $E_k(A)$ of a subset A of an additive group G as the number of $2k$ -tuples $(a_1, a_2, \dots, a_{2k})$ in A^{2k} with $a_1 + a_2 + \dots + a_k = a_{k+1} + a_{k+2} + \dots + a_{2k}$. In this case the trivial bounds are $|A|^k \leq E_k(A) \leq |A|^{2k-1}$, and we have the following refinement in the cube $\{0, 1\}^d$.

Theorem 3. *Let $d \geq 0$, $2 \leq k \leq 10$, and let $A \subset \{0, 1\}^d$. Then $E_k(A) \leq |A|^{p_k}$, where $p_k := \log_2 \binom{2k}{k}$. Furthermore, the exponent p_k cannot be replaced by any smaller quantity.*

Remark 4. Theorem 3 also extends a result obtained by Kane–Tao ([5, Theorem 7]).

From the well-known bounds for the central binomial coefficient $\frac{4^k}{2\sqrt{\pi k}} \leq \binom{2k}{k} \leq \frac{4^k}{\sqrt{\pi k}}$, one recovers

$$p_k < 2k - 1. \quad (1.1)$$

As previously, the second claim in our Theorem 3 follows considering the case $A = \{0, 1\}^d$, since in this case we have $|A| = |\{0, 1\}|^d = 2^d$ and $E_k(A) = \left[\sum_{i=0}^k \binom{k}{i}^2 \right]^d = \binom{2k}{k}^d$. We prove this theorem by induction on d together with the following subtle inequality for Legendre polynomials.

Lemma 5. *Let $2 \leq k \leq 10$ and $p_k = \log_2 \binom{2k}{k}$. If $a, b \geq 0$, then*

$$\sum_{j=0}^k \binom{k}{j}^2 a^{p_k \frac{k-j}{k}} b^{p_k \frac{j}{k}} \leq (a+b)^{p_k}. \quad (1.2)$$

The polynomials $Q_k(t)$, $k \geq 0$, defined by

$$Q_k(t) = \frac{1}{2^k k!} \frac{d^k}{dt^k} (t^2 - 1)^k = \frac{1}{2^k} \sum_{j=0}^k \binom{k}{j}^2 (t-1)^{k-j} (t+1)^j$$

are called Legendre polynomials. They are orthogonal with respect to Lebesgue measure on the interval $[-1, 1]$, each $Q_k(t)$ has degree k , and they satisfy normalization constraint $Q_k(1) = 1$. Dividing both sides of (1.2) by a^{p_k} (without loss of generality assume $a \neq 0$), then (1.2) takes the form $(y-1)^k Q_k\left(\frac{y+1}{y-1}\right) \leq (1+y^{k/p_k})^{p_k}$ with $y = (b/a)^{p_k/k} \geq 0$. If we let $t := \frac{y+1}{y-1}$ (without loss of generality assume $y \geq 1$), then (1.2) is the same as

$$Q_k(t) \leq \left(\left(\frac{t-1}{2} \right)^{\frac{k}{p_k}} + \left(\frac{t+1}{2} \right)^{\frac{k}{p_k}} \right)^{p_k} \quad \text{for all } t \geq 1.$$

This explains the reason we call Lemma 5 the inequality for Legendre polynomials.

1.3 More general discrete cubes

Let $d \geq 0$. Let us consider additive energies of subsets of general discrete cubes¹ $\{0, 1, \dots, n\}^d$. Let t_n be the smallest number such that

$$E_2(A) \leq |A|^{t_n}$$

¹A related problem about the lower bound for the size of sumsets of subsets of the general discrete cube was studied, e.g., in [1, Theorem 5].

for all $A \subseteq \{0, 1, \dots, n\}^d$. We have seen that in both Theorem 1 and Theorem 3 we have $q_k = \frac{\log \tilde{E}_k(\{0, 1\}^d)}{\log |\{0, 1\}^d|}$ and $p_k = \frac{\log E_k(\{0, 1\}^d)}{\log |\{0, 1\}^d|}$. Thus, one could a-priori expect a similar phenomenon for the additive energy of $\{0, 1, \dots, n\}^d$. However, it turns out that this is not the case in general, not even for the discrete cube $\{0, 1, 2\}^d$.

Proposition 6. *The following inequality holds*

$$t_2 > \frac{\log E_2(\{0, 1, 2\}^d)}{\log |\{0, 1, 2\}^d|}.$$

Although finding the precise values of the optimal powers t_n for general discrete cubes $\{0, 1, \dots, n\}^d$ seems to be a difficult problem, we obtain some bounds describing the asymptotic behavior of t_n as n goes to infinity.

Proposition 7. *If $n = 2m - 1$, then*

$$3 \geq t_n \geq \log_{2m} \left(\frac{16m^3 + 2m}{3} \right) > 3 - \frac{\log(3/2)}{\log(2m)}.$$

If $n = 2m$, then

$$3 \geq t_n \geq \log_{2m} \left(\frac{16m^3 + 24m^2 + 14m + 3}{3} \right) > 3 - \frac{\log(3/2)}{\log(2m)}.$$

2 Proof of Theorem 1

The proof of Theorem 1 proceeds via induction on d . Observe that the result is trivial for $d = 0$. Assume now that $d \geq 1$ and that the result has been established for $d - 1$. Any set $A \subseteq \{0, 1\}^d$ can be written as

$$A = (A_0 \times \{0\}) \uplus (A_1 \times \{1\})$$

for some $A_0, A_1 \subseteq \{0, 1\}^{d-1}$, where $\uplus$ means disjoint union. Then we have

$$\begin{aligned} \tilde{E}_k(A) &= |\{(a_1, a_2, \dots, a_{2k}) \in (A_0 \times A_1)^k : a_1 - a_2 = a_3 - a_4 = \dots = a_{2k-1} - a_{2k}\}| \\ &\quad + |\{(a_1, a_2, \dots, a_{2k}) \in (A_1 \times A_0)^k : a_1 - a_2 = a_3 - a_4 = \dots = a_{2k-1} - a_{2k}\}| \\ &\quad + \sum_{i=0}^k \binom{k}{i} |\{(a_1, a_2, \dots, a_{2k}) \in (A_0^2)^i \times (A_1^2)^{k-i} \\ &\quad \quad : a_1 - a_2 = a_3 - a_4 = \dots = a_{2k-1} - a_{2k}\}| \\ &=: C_1 + C_2 + \tilde{E}_k(A_0) + \tilde{E}_k(A_1) + \sum_{i=1}^{k-1} \binom{k}{i} C_{i,k}. \end{aligned} \tag{2.1}$$

The next proposition plays a fundamental role in our proof.

Proposition 8. *For all $1 \leq i \leq k - 1$ we have that*

$$C_{i,k} \leq |A_0|^{\frac{i}{k} q_k} |A_1|^{\frac{k-i}{k} q_k}.$$

Moreover, we have that

$$C_1 \leq |A_0|^{\frac{q_k}{2}} |A_1|^{\frac{q_k}{2}} \text{ and } C_2 \leq |A_0|^{\frac{q_k}{2}} |A_1|^{\frac{q_k}{2}}.$$

Proof of Proposition 8. We observe that

$$\tilde{E}_k(A) := \sum_{x \in \mathbb{Z}^d} (\chi_A \star \chi_A)^k(x),$$

where χ_A denotes the characteristic function of the set A , and $f \star g$ denotes the correlation of the functions f and g defined by $f \star g(x) := \sum_{y \in \mathbb{Z}^d} f(y)g(x+y)$ [10, Equation 7]. Moreover, by Hölder's inequality we have

$$\begin{aligned} C_{i,k} &= \sum_{x \in \mathbb{Z}^d} (\chi_{A_0} \star \chi_{A_0})^i(x) (\chi_{A_1} \star \chi_{A_1})^{k-i}(x) \\ &\leq \left(\sum_{x \in \mathbb{Z}^d} (\chi_{A_0} \star \chi_{A_0})^k(x) \right)^{\frac{i}{k}} \left(\sum_{x \in \mathbb{Z}^d} (\chi_{A_1} \star \chi_{A_1})^k(x) \right)^{\frac{k-i}{k}} \\ &= \tilde{E}_k^{\frac{i}{k}}(A_0) \tilde{E}_k^{\frac{k-i}{k}}(A_1) \\ &\leq |A_0|^{\frac{q_k i}{k}} |A_1|^{\frac{q_k(k-i)}{k}}. \end{aligned}$$

The first identity follows from the facts that $\chi_{A_0} \star \chi_{A_0}(x)$ counts the number of pairs $(y, z) \in A_0^2$ such that $z - y = x$, and $\chi_{A_1} \star \chi_{A_1}(x)$ counts the number of pairs $(y, z) \in A_1^2$ such that $z - y = x$. We define

$$f \bullet g := \sum_{\substack{a_1, a_2, \dots, a_k \in \{0,1\}^d \\ b_1, b_2, \dots, b_k \in \{0,1\}^d \\ a_1 - b_1 = a_2 - b_2 = \dots = a_k - b_k}} f(a_1) f(a_2) \dots f(a_k) g(b_1) g(b_2) \dots g(b_k).$$

Then

$$\begin{aligned} f \bullet g &= \sum_{c_2, c_3, \dots, c_k \in \{-1, 0, 1\}^d} \left(\sum_{\substack{a_1 \in \{0,1\}^d \\ a_1 + c_i \in \{0,1\}^d}} f(a_1) f(a_1 + c_2) f(a_1 + c_3) \dots f(a_1 + c_k) \right) \\ &\quad \times \left(\sum_{\substack{b_1 \in \{0,1\}^d \\ b_1 + c_i \in \{0,1\}^d}} g(b_1) g(b_1 + c_2) g(b_1 + c_3) \dots g(b_1 + c_k) \right). \end{aligned}$$

Therefore, by the Cauchy-Schwarz inequality we obtain

$$\begin{aligned} C_1 &= \chi_{A_0} \bullet \chi_{A_1} \leq (\chi_{A_0} \bullet \chi_{A_0})^{1/2} (\chi_{A_1} \bullet \chi_{A_1})^{1/2} \\ &= \tilde{E}_k^{1/2}(A_0) \tilde{E}_k^{1/2}(A_1) \leq |A_0|^{\frac{q_k}{2}} |A_1|^{\frac{q_k}{2}}. \end{aligned}$$

Similarly $C_2 \leq |A_0|^{\frac{q_k}{2}} |A_1|^{\frac{q_k}{2}}$. □

Then, from (2.1), using Proposition 8 we obtain

$$\begin{aligned} \tilde{E}_k(A) &= C_1 + C_2 + \tilde{E}_k(A_0) + \tilde{E}_k(A_1) + \sum_{i=1}^{k-1} \binom{k}{i} C_{i,k} \\ &\leq 2|A_0|^{\frac{q_k}{2}} |A_1|^{\frac{q_k}{2}} + \sum_{i=0}^k \binom{k}{i} |A_0|^{\frac{i}{k} q_k} |A_1|^{\frac{k-i}{k} q_k} \\ &= 2|A_0|^{\frac{q_k}{2}} |A_1|^{\frac{q_k}{2}} + (|A_0|^{\frac{q_k}{k}} + |A_1|^{\frac{q_k}{k}})^k. \end{aligned}$$

Thus, to complete the inductive argument, it is enough to prove that for $x = |A_0|$ and $y = |A_1|$ one has

$$2x^{\frac{q_k}{2}}y^{\frac{q_k}{2}} + (x^{\frac{q_k}{k}} + y^{\frac{q_k}{k}})^k \leq (x+y)^{q_k}. \quad (2.2)$$

Lemma 9. *For all $a \in [0, 1]$ we have*

$$(a^{\frac{q_k}{k}} + (1-a)^{\frac{q_k}{k}})^k + 2a^{\frac{q_k}{2}}(1-a)^{\frac{q_k}{2}} \leq 1. \quad (2.3)$$

Observe that (2.2) follows from (2.3) by taking $a = \frac{x}{x+y}$. A key ingredient in the proof of Lemma 9 is the following result established by Carlen, Frank, Ivanisvili and Lieb [2, Proposition 3.1].

Proposition 10. *For all $a \in [0, 1]$ and $p \in (-\infty, 0] \cup [1, 2]$*

$$(a^p + (1-a)^p) \left(1 + \left(\frac{2a^{\frac{p}{2}}(1-a)^{\frac{p}{2}}}{a^p + (1-a)^p} \right)^{\frac{2}{p}} \right)^{p-1} \leq 1. \quad (2.4)$$

Moreover, the reverse inequality holds if $p \in [0, 1] \cup [2, \infty)$.

Proof of Lemma 9. We observe that (2.3) is equivalent to proving

$$1 + \left(\frac{2^{\frac{1}{k}} a^{\frac{q_k}{2k}} (1-a)^{\frac{q_k}{2k}}}{a^{\frac{q_k}{k}} + (1-a)^{\frac{q_k}{k}}} \right)^k \leq \frac{1}{(a^{\frac{q_k}{k}} + (1-a)^{\frac{q_k}{k}})^k}.$$

Since $k < q_k = \log_2(2^k + 2) < k + 1$ for all $k \geq 2$, by taking $p = \frac{q_k}{k}$ in Proposition 10 we obtain

$$\left(1 + \left(\frac{2a^{\frac{q_k}{2k}}(1-a)^{\frac{q_k}{2k}}}{a^{\frac{q_k}{k}} + (1-a)^{\frac{q_k}{k}}} \right)^{\frac{2k}{q_k}} \right)^{\frac{q_k}{k}-1} \leq \frac{1}{(a^{\frac{q_k}{k}} + (1-a)^{\frac{q_k}{k}})^k}. \quad (2.5)$$

Thus, it is enough to prove

$$1 + \left(\frac{2^{\frac{1}{k}} a^{\frac{q_k}{2k}} (1-a)^{\frac{q_k}{2k}}}{a^{\frac{q_k}{k}} + (1-a)^{\frac{q_k}{k}}} \right)^k \leq \left(1 + \left(\frac{2a^{\frac{q_k}{2k}}(1-a)^{\frac{q_k}{2k}}}{a^{\frac{q_k}{k}} + (1-a)^{\frac{q_k}{k}}} \right)^{\frac{2k}{q_k}} \right)^{q_k-k}.$$

Defining $\mu := \frac{2a^{\frac{q_k}{2k}}(1-a)^{\frac{q_k}{2k}}}{a^{\frac{q_k}{k}} + (1-a)^{\frac{q_k}{k}}}$ (observe that $\mu \in [0, 1]$ by AM-GM inequality), it is enough to prove

$$1 + \frac{\mu^k}{2^{k-1}} \leq (1 + \mu^{\frac{2k}{q_k}})^{q_k-k}$$

for all $\mu \in [0, 1]$. By letting $z := \mu^{\frac{2k}{q_k}}$, we reduce the problem to proving

$$1 + \frac{z^{\frac{q_k}{2}}}{2^{k-1}} \leq (1+z)^{q_k-k} \quad (2.6)$$

for all $z \in [0, 1]$. The equality holds at $z = 0$ and $z = 1$. Moreover, the left hand side of (2.6) is convex in z (as $2 \leq k < q_k$), and the right hand side is concave (as $k < q_k < k + 1$). Therefore (2.6) holds for all $z \in [0, 1]$. $\square$

3 Proof of Theorem 3

In this section we show how to obtain Theorem 3 from Lemma 5, and then we prove this lemma. As before, we proceed via induction. Clearly, the result holds for $d = 0$. Assume now $d \geq 1$, and the result has been established for $d - 1$. Any set $A \subseteq \{0, 1\}^d$ can be written as

$$A = (A_0 \times \{0\}) \uplus (A_1 \times \{1\})$$

for some $A_0, A_1 \subseteq \{0, 1\}^{d-1}$.

We have

$$\begin{aligned} E_k(A) &= E_k(A_0) + E_k(A_1) \\ &\quad + \sum_{i=1}^{k-1} \binom{k}{i}^2 |\{(a_1, a_2, \dots, a_{2k}) \in A_0^i \times A_1^{k-i} \times A_0^i \times A_1^{k-i} \\ &\quad : a_1 + \dots + a_k = a_{k+1} + \dots + a_{2k}\}| \\ &= E_k(A_0) + E_k(A_1) + \sum_{i=1}^{k-1} \binom{k}{i}^2 C_{i,k}. \end{aligned} \tag{3.1}$$

Similarly to Proposition 8, we have

Proposition 11. *For all $1 \leq i \leq k - 1$ the following inequality holds*

$$C_{i,k} \leq |A_0|^{\frac{i}{k}p_k} |A_1|^{\frac{k-i}{k}p_k}. \tag{3.2}$$

Observe that Theorem 3 follows from Proposition 11. Indeed, by (3.1), Proposition 11 and (1.2) we have

$$\begin{aligned} E_k(A) &= E_k(A_0) + E_k(A_1) + \sum_{i=1}^{k-1} \binom{k}{i}^2 C_{i,k} \\ &\leq E_k(A_0) + E_k(A_1) + \sum_{i=1}^{k-1} \binom{k}{i}^2 |A_0|^{\frac{i}{k}p_k} |A_1|^{\frac{k-i}{k}p_k} \\ &\leq (|A_0| + |A_1|)^{p_k} \\ &= |A|^{p_k}. \end{aligned}$$

Proof of Proposition 11. We observe that

$$C_{i,k} = \sum_{x \in \mathbb{Z}^d} |\chi_{A_0} *_{i-1} \chi_{A_0} * \chi_{A_1} *_{k-i-1} \chi_{A_1}(x)|^2,$$

where, for compactly supported f, g , we define $f * g(x) := \sum_{y \in \mathbb{Z}^d} f(y)g(x - y)$ and $*_k := (*_{k-1})$. Indeed, this follows from the fact that

$$\chi_{A_0} *_{i-1} \chi_{A_0} * \chi_{A_1} *_{k-i-1} \chi_{A_1}(x)$$

counts the number of k -tuples $(a_1, a_2, \dots, a_i, a_{i+1}, \dots, a_k) \in A_0^i \times A_1^{k-i}$ such that $a_1 + a_2 + \dots + a_k = x$. Then, by

Plancherel's theorem and Hölder's inequality we obtain

$$\begin{aligned}
 C_{i,k} &= \sum_{x \in \mathbb{Z}^d} |\chi_{A_0} *_{i-1} \chi_{A_0} * \chi_{A_1} *_{k-i-1} \chi_{A_1}(x)|^2 \\
 &= \int_{\mathbb{T}^d} |\hat{\chi}_{A_0}(y)|^{2i} |\hat{\chi}_{A_1}(y)|^{2(k-i)} dm(y) \\
 &\leq \left(\int_{\mathbb{T}^d} |\hat{\chi}_{A_0}(y)|^{2k} dm(y) \right)^{\frac{i}{k}} \left(\int_{\mathbb{T}^d} |\hat{\chi}_{A_1}(y)|^{2k} dm(y) \right)^{\frac{k-i}{k}} \\
 &= \left(\sum_{x \in \mathbb{Z}^d} |\chi_{A_0} *_{k-1} \chi_{A_0}|^2 \right)^{\frac{i}{k}} \left(\sum_{x \in \mathbb{Z}^d} |\chi_{A_1} *_{k-1} \chi_{A_1}|^2 \right)^{\frac{k-i}{k}} \\
 &= E_k^{\frac{i}{k}}(A_0) E_k^{\frac{k-i}{k}}(A_1) \leq |A_0|^{\frac{ip_k}{k}} |A_1|^{\frac{(k-i)p_k}{k}},
 \end{aligned}$$

where m is the Haar measure on $\mathbb{T}^d$ with $m(\mathbb{T}^d) = 1$. □

Proof of Lemma 5. After re-scaling, we observe that to prove (1.2) it is sufficient to show

$$\sum_{i=0}^k \binom{k}{i}^2 x^{ip_k/k} \leq (1+x)^{p_k} \quad (3.3)$$

for all $1 \leq x < \infty$. Moreover, after a change of variable, this is equivalent to proving that

$$g_k(y) := \sum_{i=0}^k \binom{k}{i}^2 y^i \leq (1+y^\alpha)^{\frac{k}{\alpha}} =: h_k(y) \quad (3.4)$$

for all $1 \leq y \leq \infty$, where $\alpha := \frac{k}{p_k} \in (1/2, 1)$. Let $f(y) := \log h_k(y) - \log g_k(y)$. We need to show $f(y) \geq 0$ for all $y \geq 1$. Observe that $f(1) = 0$. Moreover

$$\lim_{y \rightarrow \infty} f(y) = \lim_{y \rightarrow \infty} \log \left(\frac{(\frac{1}{y^\alpha} + 1)^{\frac{k}{\alpha}}}{\sum_{i=0}^k \binom{k}{i}^2 y^{-i}} \right) = 0,$$

and, since

$$\left(\frac{1}{y^\alpha} + 1 \right)^{\frac{k}{\alpha}} \geq 1 + \frac{k}{\alpha y^\alpha} \quad \text{and} \quad \sum_{i=0}^k \binom{k}{i}^2 y^{-i} = 1 + O\left(\frac{1}{y}\right),$$

we have $f(y) > 0$ whenever y is sufficiently large. Thus, it is sufficient to prove that f' changes sign at most once in $(1, \infty)$. Observe that

$$\begin{aligned}
 y f'(y) &= \frac{ky^\alpha}{1+y^\alpha} - \frac{\sum_{i=0}^k \binom{k}{i}^2 i y^i}{\sum_{i=0}^k \binom{k}{i}^2 y^i} \\
 &= \frac{y^\alpha \sum_{i=0}^k \binom{k}{i}^2 y^i (k-i) - \sum_{i=0}^k \binom{k}{i}^2 i y^i}{(1+y^\alpha) \left(\sum_{i=0}^k \binom{k}{i}^2 y^i \right)}.
 \end{aligned}$$

Thus, we need to prove that $y^\alpha \sum_{i=0}^k \binom{k}{i}^2 y^i (k-i) - \sum_{i=0}^k \binom{k}{i}^2 i y^i$ changes sign in $(1, \infty)$ at most once. We define

$$\phi(y) := \log \left(y^\alpha \sum_{i=0}^k \binom{k}{i}^2 y^i (k-i) \right) - \log \left(\sum_{i=0}^k \binom{k}{i}^2 i y^i \right).$$

We then have $\phi(1) = 0$ and

$$\phi(y) = \alpha \log(y) + \log\left(\frac{n^2 y^{n-1} + O(y^{n-2})}{ny^n + O(y^{n-1})}\right) \quad \text{as } y \rightarrow \infty.$$

Hence $\lim_{y \rightarrow \infty} \phi(y) = -\infty^2$. It suffices to show that ϕ' changes sign (from + to -) at most once in $(1, \infty)$. Observe that

$$\begin{aligned} \phi'(y) &= \frac{\alpha}{y} + \frac{\sum_{i=0}^k \binom{k}{i}^2 y^{i-1} (k-i)i}{\sum_{i=0}^k \binom{k}{i}^2 y^i (k-i)} - \frac{\sum_{i=0}^k \binom{k}{i}^2 i^2 y^{i-1}}{\sum_{i=0}^k \binom{k}{i}^2 i y^i} \\ &= \frac{\sum_{i=0}^{2k} C_i y^i}{y \left(\sum_{i=0}^k \binom{k}{i}^2 y^i (k-i) \right) \left(\sum_{i=0}^k \binom{k}{i}^2 i y^i \right)}, \end{aligned}$$

where

$$\begin{aligned} C_i &:= \sum_{\substack{j+l=i \\ 0 \leq j, l \leq k}} \binom{k}{j}^2 \binom{k}{l}^2 [\alpha(k-l)j + (k-l)lj - j^2(k-l)] \\ &= \sum_{\substack{j+l=i \\ 0 \leq j, l \leq k}} \binom{k}{j}^2 \binom{k}{l}^2 j(k-l)(\alpha + l - j) \text{ for all } i, 0 \leq i \leq 2k. \end{aligned}$$

Let $P(y) := \sum_{i=0}^{2k} C_i y^i$. We would like to show that $P(y)$ changes sign at most once from + to - in $(1, \infty)$. First, we claim $P(y)$ is a palindromic polynomial, i.e., $C_i = C_{2k-i}$ for all $i = 0, \dots, k$. Indeed,

$$\begin{aligned} C_{2k-i} &= \sum_{\substack{j+l=2k-i \\ 0 \leq j, l \leq k}} \binom{k}{j}^2 \binom{k}{l}^2 j(k-l)(\alpha + l - j) = \\ &= \sum_{\substack{(k-j)+(k-l)=i \\ 0 \leq j, l \leq k}} \binom{k}{k-j}^2 \binom{k}{k-l}^2 (k-(k-j))(k-l)(\alpha + (k-j) - (k-l)). \end{aligned}$$

If we denote $\tilde{l} = k - j$ and $\tilde{j} = k - l$, then we obtain

$$C_{2k-i} = \sum_{\substack{\tilde{l}+\tilde{j}=i \\ 0 \leq \tilde{j}, \tilde{l} \leq k}} \binom{k}{\tilde{l}}^2 \binom{k}{\tilde{j}}^2 \tilde{j}(k-\tilde{l})(\alpha + \tilde{l} - \tilde{j}),$$

which coincides with C_i . Since P is the palindromic polynomial it follows that y_0 is its positive root if and only if $P(1/y_0) = 0$. Therefore, to show that $P(y)$ changes sign from + to - at most once in $(1, \infty)$, it suffices to verify that $P(y)$ has at most two roots in $(0, \infty)$. By Descartes' rule of sign change $P(y)$ has at most two positive roots if there is at most two sign changes between consecutive (nonzero) coefficients C_i , $0 \leq i \leq 2k$. Since $C_i = C_{2k-i}$ it suffices to show that there is at most one sign change between consecutive (nonzero) coefficients, C_i for $0 \leq i \leq k$. Since $C_0 = 0$ we should consider coefficients C_i with $1 \leq i \leq k$. In the table below $C_i^* := \text{sign}(C_i)$, and $2 \leq k \leq 10$.

²Here we use the notation $V(y) = O(U(y))$ at y_0 to denote that an estimate of the form $|V(y)| \leq C|U(y)|$, with some constant $C > 0$, holds around y_0 .

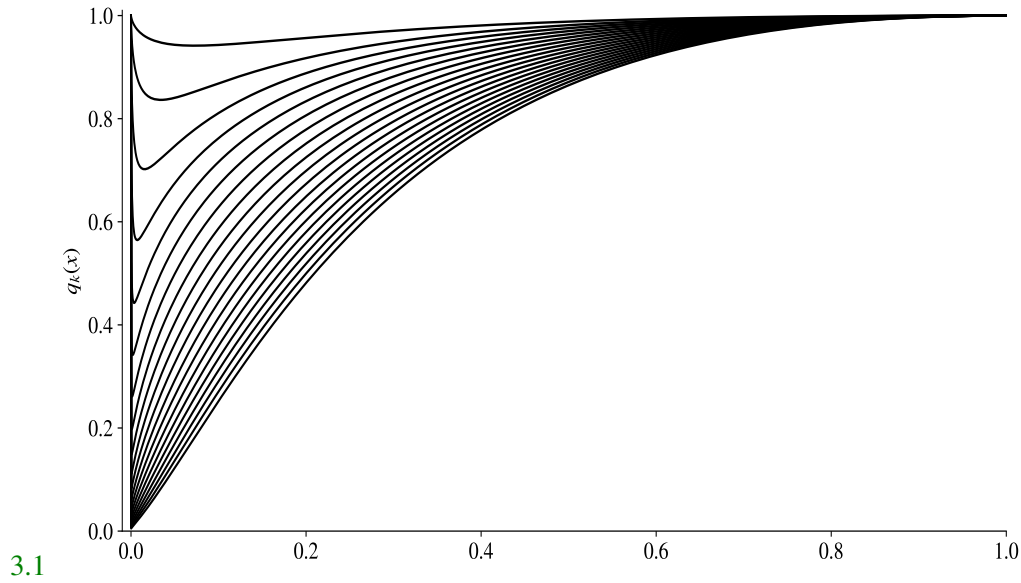


Figure 3.1: Graphs of $q_k(x)$ for $k \in \{2^n; 1 \leq n < 20\}$. The picture suggests that $q_k(x) \leq 1$ for all $x \in [0, 1]$. Lower graphs correspond to larger values of k .

k	C_1^*	C_2^*	C_3^*	C_4^*	C_5^*	C_6^*	C_7^*	C_8^*	C_9^*	C_{10}^*
2	-1	1								
3	-1	1	1							
4	-1	1	1	1						
5	-1	1	1	1	1					
6	-1	1	1	1	1	1				
7	-1	-1	1	1	1	1	1			
8	-1	-1	1	1	1	1	1	1		
9	-1	-1	-1	1	1	1	1	1	1	
10	-1	-1	-1	1	1	1	1	1	1	1

□

Remark 12. It seems to us that Lemma 5 holds for all $k \geq 2$. We have verified at most one sign flip of the numbers C_i , $1 \leq i \leq k$ on a computer for $k \leq 100$. It is an interesting question to verify that there is at most one sign flip in the sequence of C_i for all k .

Note added in proof

Motivated by Remark 12, Vjekoslav Kovač recently proved inequality (1.2) for all $k \geq 2$, see [7].

Remark 13. To prove (3.3) it suffices to show

$$\phi_k(x) := \frac{\sum_{i=0}^k \binom{k}{i}^2 x^{p_k(k-i)/k}}{(1+x)^{p_k}} \leq 1 \quad (3.5)$$

for all $x \in [0, 1]$. The inequality (3.5) can be easily verified around $x = 0$. One can also verify it around $x = 1$. Therefore, to obtain the desired inequality in the whole interval $[0, 1]$ it would be enough to prove that each ϕ_k has

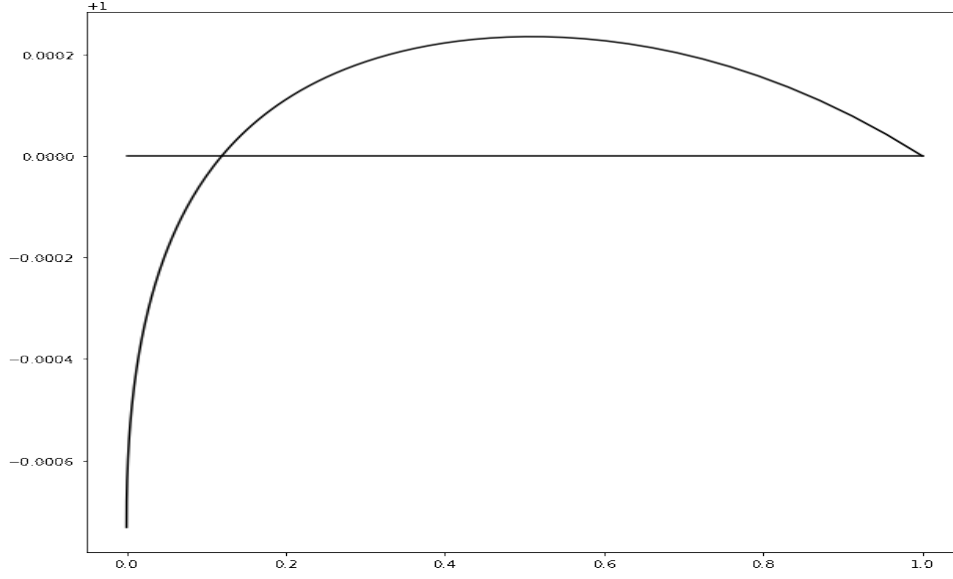


Figure 3.2: Graph of $\psi_3(x)$. We observe that $\psi_3(x)$ is concave, and $\psi_3(x)$ intersects the line $y = 1$ at only one point in $(0, 1)$.

only one critical point in $(0, 1)$. We observe that x is a critical point of ϕ_k if and only if

$$(1+x)^{p_k+1} \phi'_k(x) = \sum_{i=0}^k \binom{k}{i}^2 \left[\frac{p_k(k-i)}{k} x^{p_k \frac{k-i}{k}-1} (1+x) - p_k x^{p_k \frac{k-i}{k}} \right] = 0,$$

or, equivalently

$$\psi_k(x) := \sum_{i=0}^{k-1} \binom{k}{i}^2 \left[\frac{k-i}{k} x^{p_k \frac{k-i}{k}-1} - \frac{i}{k} x^{p_k \frac{k-i}{k}} \right] = 1.$$

Therefore, as $\psi_k(0) = 0$ and $\psi_k(1) = 1$, in order to establish the desired inequality, i.e., $\phi_k(x) \leq 1$ for all $x \in (0, 1)$, it would be enough to prove that $\psi_k(x)$ is concave. For small values of k , one can establish the concavity of ψ_k ; in particular, this is the approach of Kane–Tao [5] for $k = 2$. Figure 3.2 illustrates that ψ_k is concave for $k = 3$. Unfortunately, this is no longer the case if k is large; e.g., Figure 3.3 illustrates the non-concavity of ψ_k for k as small as 7 already. Another approach to prove Lemma 5 would be to show $\phi_{k+1}(x) \leq \phi_k(x)$ which numerically seems correct.

4 Proofs of Propositions 6 and 7

The proof of Kane–Tao [5] of the $\{0, 1\}$ -analogue, as well as the proofs of Theorems 1 and 3 are based on the following two steps:

- *Guessing* the extremizer to the inequality (which, in those cases, happened to be the entire set).
- Showing an inductive bound that allowed us to see that the extremizer candidate is indeed the extremizer.

In the $\{0, 1, 2\}^n$ or more general cases the entire set is not generally the extremizer, and finding the extremizer becomes a key step of the proof:

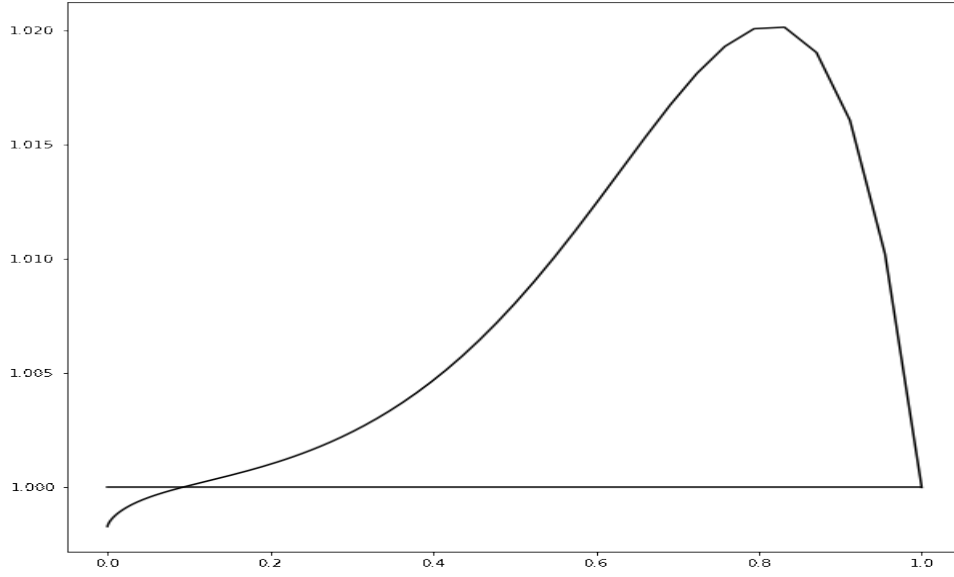


Figure 3.3: Graph of $\psi_7(x)$. We observe that $\psi_7(x)$ is not concave, however $\psi_7(x)$ still intersects the line $y = 1$ at only one point in $(0, 1)$.

- We first construct an auxiliary problem that *inducts*, or, in this case *tensorizes* essentially by construction. Solving this problem is essentially equivalent to *guessing* the extremizers in the previous problems.
- We then show that the solution to this auxiliary problem gives rise to sharp (almost) extremizers of the original problem. This step is new, and necessary due to the fact that the extremizing sets are in general far from being product sets.

4.1 The auxiliary (discrete restriction) problem

For each specific instance of interest (in our case $\{0, 1, 2\}$) the auxiliary problem will then reduce to solving a finite-dimensional optimization problem closely related to the inequalities studied in the previous sections. The way to define these problems will be by defining auxiliary quantities frequently appearing in the discrete restriction theory.

Definition 14 (Discrete extension constants). Given positive integers k, d , and a finite subset $A \subset \mathbb{Z}^d$, we define:

- The discrete extension constant $\text{DE}_{l^q \rightarrow L^{2k}}(A)$ as the smallest constant such that, for any function $f : A \rightarrow \mathbb{R}$ it holds that

$$\left| \sum_{\substack{x_1, \dots, x_k \in A \\ y_1, \dots, y_k \in A \\ \sum x_i = \sum y_i}} f(x_1) \cdots f(x_k) \cdot f(y_1) \cdots f(y_k) \right|^{\frac{1}{2k}} \leq \text{DE}_{l^q \rightarrow L^{2k}}(A) \|f\|_{l^q(A)}. \quad (4.1)$$

- The restricted discrete extension constant $\tilde{\text{DE}}_{l^q \rightarrow L^{2k}}(A)$, which is the best possible constant so that (4.1) holds for all functions $f : A \rightarrow \{0, 1\}$.

The quantities DE , $\tilde{\text{DE}}$ have essentially the same value (Lemma 19), but DE is much easier to work with (Lemma 18). Moreover, understanding for which q we have $\tilde{\text{DE}}_{l^q \rightarrow L^{2k}}(\{0, 1, 2\}^d) \leq 1$ is essentially equivalent to proving Proposition 6.

Lemma 15. *Let A be a finite subset of $\mathbb{Z}^d$. Let $1 \leq p = \frac{2k}{q}$, and $C > 0$. The following statements are equivalent:*

1. *For all subsets $B \subset A$, it holds that*

$$E_k(B) \leq C^{2k} |B|^p.$$

2.

$$\tilde{\text{DE}}_{l^q \rightarrow L^{2k}}(A) \leq C.$$

Proof. Set f in the definition of $\tilde{\text{DE}}$ to be equal to χ_B for B as in part (1). □

The constant DE is called the discrete extension constant because it is, indeed, the operator norm of an extension operator.

Lemma 16 (Fourier transform). *Let A be a finite subset of $\mathbb{Z}^d$. Then $\text{DE}_{l^q \rightarrow L^{2k}}(A)$ is the operator norm of the extension operator³ $\mathcal{E}(f) = \mathcal{F}\{f\}$ from $l^q(A) \subseteq l^q(\mathbb{Z}^d)$ to $L^{2k}(\mathbb{T}^d)$.*

Proof. By definition, $\text{DE}_{l^q \rightarrow L^{2k}}(A)$ is the best constant such that, for any function $f : \mathbb{Z}^d \rightarrow \mathbb{R}$ supported on A , it holds that:

$$\left\| \underbrace{f * f * f \cdots * f}_{k \text{ times}} \right\|_{l^2(\mathbb{Z}^d)}^{1/k} \leq \text{DE}_{l^q \rightarrow L^{2k}}(A) \|f\|_{l^q(\mathbb{Z}^d)}.$$

At the same time, by Plancherel's theorem and the product-convolution rule

$$\|f^{*k}\|_{l^2(\mathbb{Z}^d)} = \|\mathcal{F}\{f^{*k}\}\|_{L^2(\mathbb{T}^d)} = \|\mathcal{F}\{f\}^k\|_{L^2(\mathbb{T}^d)} = \|\mathcal{F}\{f\}\|_{L^{2k}(\mathbb{T}^d)}^k.$$

□

Remark 17. Lemma 16 above shows that the constants $\tilde{\text{DE}}_{l^q \rightarrow L^{2k}}(A)$, $\text{DE}_{l^q \rightarrow L^{2k}}(A)$ make sense for arbitrary $2k \in \mathbb{R}$, and not just even integers.

The following lemma is essentially [3, Proposition 3.3]. For completeness of the argument we include the proof here.

Lemma 18 (Tensorization Lemma). *Let $q \leq 2k$. Then for $A \subseteq \mathbb{Z}^{d_1}$, $B \subseteq \mathbb{Z}^{d_2}$, $A \times B \subseteq \mathbb{Z}^{d_1} \times \mathbb{Z}^{d_2}$ we have*

$$\text{DE}_{l^q \rightarrow L^{2k}}(A \times B) = \text{DE}_{l^q \rightarrow L^{2k}}(A) \text{DE}_{l^q \rightarrow L^{2k}}(B).$$

Proof. The “ $\geq$ ” inequality follows by testing the left hand side operator with the tensor product of (almost) extremizers to the right hand side.

For the opposite direction, let $f : A \times B \rightarrow \mathbb{C}$, and $\hat{f} : \mathbb{T}^{d_1} \times \mathbb{T}^{d_2} \rightarrow \mathbb{C}$ be its Fourier transform. Let $\mathcal{F}_1, \mathcal{F}_2$ be the Fourier transforms on $\mathbb{Z}^{d_1}$ and $\mathbb{Z}^{d_2}$. The goal is to estimate

$$\left\| \mathcal{F}_2\{\mathcal{F}_1 f\}(x_1, x_2) \right\|_{L^{2k}(x_2 \in \mathbb{T}^{d_2})} \left\| \right\|_{L^{2k}(x_1 \in \mathbb{T}^{d_1})}.$$

Fixing x_2 , we apply the DE inequality

$$\left\| \mathcal{F}_2\{\mathcal{F}_1 f\}(x_1, x_2) \right\|_{L^{2k}(x_2 \in \mathbb{T}^{d_2})} \leq \text{DE}_{l^q \rightarrow L^{2k}}(B) \left\| \mathcal{F}_1 f(x_1, b) \right\|_{l^q(b)}.$$

³Here we denote by $\mathcal{F}\{f\}$ the Fourier transform of f , i.e., $\mathcal{F}\{f\}(z) = \sum_{k \in \mathbb{Z}^d} f(k) z^k$.

Now, using the hypothesis that $2k \geq q$, we can reverse the norms

$$\|\|\mathcal{F}_1 f(x_1, b)\|_{l^q(b \in B)}\|_{L^{2k}(x_1 \in \mathbb{T}^{d_1})} \leq \|\|\mathcal{F}_1 f(x_1, b)\|_{L^{2k}(x_1 \in \mathbb{T}^{d_1})}\|_{l^q(b \in B)}.$$

Now the DE inequality can be applied again to $\|\mathcal{F}_1 f(x_1, b)\|_{L^{2k}(x_1 \in \mathbb{T}^{d_1})}$. Joining it all together

$$\begin{aligned} \|\mathcal{F}_2(\mathcal{F}_1 1)(x_1, x_2)\|_{L^{2k}(x_1 \in \mathbb{T}^{d_1})L^{2k}(x_2 \in \mathbb{T}^{d_2})} &\leq \\ &\leq \text{DE}_{l^q \rightarrow L^{2k}}(B) \text{DE}_{l^q \rightarrow L^{2k}}(A) \|f(a, b)\|_{l^q(a \in A)l^q(b \in B)}. \end{aligned}$$

□

4.2 Relating the Discrete extension problem and the original problem

In this section we show that the discrete extension constants $\tilde{\text{DE}}_{l^q \rightarrow L^{2k}}(A^d)$ and $\text{DE}_{l^q \rightarrow L^{2k}}(A^d)$ grow similarly as d goes to infinity. This will allow us to compute the asymptotic behavior of DE in order to find the (much harder) asymptotics for $\tilde{\text{DE}}$. The next lemma is inspired by Bourgain's logarithmic pigeonhole principle (see [11]).

Lemma 19 (Comparison Lemma). *For all $q \geq 1$, $k \geq \frac{1}{2}$, $A \subseteq \mathbb{Z}^d$ it holds that*

$$\tilde{\text{DE}}_{l^q \rightarrow L^{2k}}(A) \leq \text{DE}_{l^q \rightarrow L^{2k}}(A) \leq (2 + \log |A|) \tilde{\text{DE}}_{l^q \rightarrow L^{2k}}(A).$$

Proof. The first inequality follows by the fact that DE is a maximum over a larger class of functions. For the second one, let $f : A \rightarrow \mathbb{R}$. Without loss of generality assume $\|f\|_{l^\infty(A)} = 1$, and that f is nonnegative. We can decompose f as a sum

$$f(x) = \sum_{\substack{i \geq 1 \\ 2^i \leq |A|}} 2^{-i} \varepsilon_i(x) + f_0(x)$$

with the property that $\varepsilon_i : A \rightarrow \{0, 1\}$, and $0 \leq f_0(x) \leq |A|^{-1}$. The value of $\varepsilon_i(x)$ is the i -th digit of the boolean expansion of $f(x)$. Moreover, $\|f_0\|_1 \leq 1$. There are, moreover at most $(\log |A| + 1)$ different ε_i . By the triangle inequality, we have

$$\|\hat{f}\|_{L^{2k}(\mathbb{T}^d)} \leq \sum_{\substack{i \geq 1 \\ 2^i \leq |A|}} 2^{-i} \|\hat{\varepsilon}_i\|_{L^{2k}(\mathbb{T}^d)} + \|\hat{f}_0\|_{L^{2k}(\mathbb{T}^d)}.$$

We bound the sum by the maximum element in the sum (times the number of elements), and the term $\|\hat{f}_0\|_{L^{2k}(\mathbb{T}^d)}$ by 1, to obtain

$$\|\hat{f}\|_{L^{2k}(\mathbb{T}^d)} \leq (1 + \log(|A|)) \max_{i \geq 1} 2^{-i} \|\hat{\varepsilon}_i\|_{L^{2k}(\mathbb{T}^d)} + 1.$$

Now, by applying the $\tilde{\text{DE}}$ bounds on $\hat{\varepsilon}_i$ we get

$$\|\hat{f}\|_{L^{2k}(\mathbb{T}^d)} \leq (1 + \log(|A|)) \tilde{\text{DE}}_{l^q \rightarrow L^{2k}}(A) \max_{i \geq 1} 2^{-i} \|\varepsilon_i\|_{l^q(A)} + 1.$$

By construction $2^{-i} \|\varepsilon_i\|_{l^q(A)} \leq \|f\|_{l^q(A)}$. By checking against a singleton, $\tilde{\text{DE}}$ is always at least 1, and $\|f\|_{l^q(A)} \geq \|f\|_{l^\infty(A)} = 1$. Combining all this, we obtain

$$\|\hat{f}\|_{L^{2k}(\mathbb{T}^d)} \leq (2 + \log(|A|)) \tilde{\text{DE}}_{l^q \rightarrow L^{2k}}(A) \|f\|_{l^q(A)}.$$

□

Remark 20. The exponent of the log in Lemma 19 is probably not sharp (see, for example, the gains in the log-power in [4, Theorem 1.1] or [8, Lemma 2.4]). Finding the sharp exponent is not necessary for our purposes. We thank A. Mudgal for this remark.

The results from this section yield the relationship between Proposition 6 and the discrete extension constant, as follows.

Proposition 21. *Let A be a finite subset of $\mathbb{Z}$. Let $1 \leq p = \frac{2k}{q}$, and $C > 0$. The following are equivalent:*

1. *An inequality of the form*

$$E_k(X) \leq C|X|^p$$

holds for all $X \subseteq A^d$, $d \geq 0$.

2. *An inequality of the form*

$$E_k(X) \leq |X|^p.$$

holds for all $X \subseteq A^d$, $d \geq 0$.

3. $\text{DE}_{l^q \rightarrow L^{2k}}(A) \leq 1$.

Proof. Clearly, (3) $\Rightarrow$ (2) (by Lemma 18 and Lemma 15), and (2) $\Rightarrow$ (1). We show that (1) $\Rightarrow$ (3). By Lemmas 19 and 18 we have

$$\tilde{\text{DE}}_{l^q \rightarrow L^{2k}}(A^d) \leq \text{DE}_{l^q \rightarrow L^{2k}}(A)^d \leq (2 + d \log |A|) \tilde{\text{DE}}_{l^q \rightarrow L^{2k}}(A^d). \quad (4.2)$$

Observe that by Lemma 15, (1) is equivalent to

$$\sup_d \tilde{\text{DE}}_{l^q \rightarrow L^{2k}}(A^d) < \infty. \quad (4.3)$$

By equation (4.2), equation (4.3) is equivalent to

$$\text{DE}_{l^q \rightarrow L^{2k}}(A)^d \leq 1$$

and the result follows. $\square$

Remark 22. The proof of Theorem 21 extends to any finite subset A of an abelian group G without any significant changes, using that the group generated by A inside of G is locally compact and abelian with the discrete topology.

4.3 Concluding the proofs of Propositions 6 and 7

Proof of Proposition 6. Applying Proposition 21 with $A = \{0, 1, 2\} \subseteq \mathbb{Z}$ and $k = 2$ shows that t_2 is equal to the smallest p such that

$$\frac{x^p + y^p + z^p + 4(x^{p/2}y^{p/2} + x^{p/2}z^{p/2} + y^{p/2}z^{p/2}) + 4x^{p/2}y^{p/4}z^{p/4}}{(x + y + z)^p} \leq 1,$$

for all $x, y, z \geq 0$. In particular, taking $x = 1$ and $y = z = 1/2$ we obtain

$$\begin{aligned} t_2 &\geq \inf\{p \in [2, 3]: 4^p - 2^p - 12(2^{p/2}) - 6 \geq 0\} \\ &= \inf\{2 \log_2 w: w \in [2, 2\sqrt{2}], w^4 - w^2 - 12w - 6 \geq 0\} \\ &\geq 2 \log_2(2.5664) > \log_3 19 = \frac{\log E(\{0, 1, 2\}^d)}{\log |\{0, 1, 2\}^d|}. \end{aligned}$$

$\square$

Proof of Proposition 7. The upper bound is trivial, so we focus our attention on the lower bounds. Consider the case $n = 2m - 1$. We prove that $E(\{0, 1, \dots, 2m - 1\}) = \frac{16m^3 + 2m}{3}$. We start observing that for any $a \in \{0, 1, \dots, 2m - 1\}$ the 4-tuple (a, a, a, a) is a solution. Moreover, for all $a, b \in \{0, 1, \dots, 2m\}$ we have that $(a, b, a, b), (a, b, b, a), (b, a, b, a)$ and (b, a, a, b) are also solutions. This gives a total of $2m + 4\binom{2m}{2}$ trivial solutions.

Then, we observe that the m couples $(0, 2m - 1), (2, 2m - 3), (3, 2m - 4), \dots, (m - 1, m)$ add up to $2m - 1$, this gives $8\binom{m}{2}$ nontrivial solutions. Similarly, the couples adding up to $2m - 2$ and $2m$ give $8\binom{m-1}{2} + 4\binom{m-1}{1}$ solutions. More generally, we have that considering the couples adding k or $4m - 2 - k$ we obtain $8\binom{\lceil k/2 \rceil}{2}$ non-trivial solutions if k is odd and $8\binom{k/2}{2} + 4\binom{k/2}{1}$ if k is even. Therefore

$$\begin{aligned} E_2(\{0, 1, \dots, 2m - 1\}) &= 2m + 4\binom{2m}{2} + 8\binom{m}{2} + 4\left(8\sum_{k=2}^{m-1}\binom{k}{2}\right) + 2\left(4\sum_{k=1}^{m-1}k\right) \\ &= \frac{16m^3 + 2m}{3}. \end{aligned}$$

The case $n = 2m$ follows similarly. □

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