Metric and Path-Connectedness Properties of the Fréchet Distance for Paths and Graphs

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Abstract

The Fréchet distance is often used to measure distances between paths, with applications in areas ranging from map matching to GPS trajectory analysis to handwriting recognition. More recently, the Fréchet distance has been generalized to a distance between two copies of the same graph embedded or immersed in a metric space; this more general setting opens up a wide range of more complex applications in graph analysis. In this paper, we initiate a study of some of the fundamental topological properties of spaces of paths and of graphs mapped to $\mathbb{R}^n$ under the Fréchet distance, in an effort to lay the theoretical groundwork for understanding how these distances can be used in practice. In particular, we prove whether or not these spaces, and the metric balls therein, are path-connected.

1 Introduction

One-dimensional data in a Euclidean ambient space is heavily studied in the computational geometry literature, and is central to applications in GPS trajectory and road network analysis [2,10,12,22]. One widely used distance measure on one-dimensional data is the Fréchet distance, which accounts for both geometric closeness as well as the connectivity of the paths or graphs being compared [1, 3–7, 9–16, 18–20]. We build a theoretical foundation for these application areas by investigating spaces of paths and graphs in $\mathbb{R}^n$, including their metric and topological properties, under the Fréchet distance. The motivation for this work is simple: as practical approaches to compute the Fréchet distance between paths [5,13] and between graphs [9,16,18] grow in popularity, it is natural to inquire about the fundamental properties of such distances, in an effort to better understand exactly what they are capturing.

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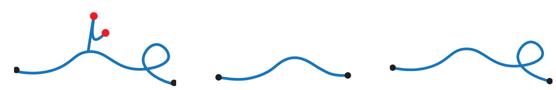
We begin by defining the Fréchet distance between paths and graphs. Using open balls under the Fréchet distance to generate a topology, we study the metric and topological properties of the induced spaces. In particular, we work with three classes of paths: the space $\Pi_\mathcal{C}$ of all paths in $\mathbb{R}^n$, the space $\Pi_\mathcal{E}$ of all paths in $\mathbb{R}^n$ that are embeddings (i.e., maps that are homeomorphisms onto the image), and the space $\Pi_\mathcal{I}$ of all paths in $\mathbb{R}^n$ that are immersions (local embeddings). See Figure 1 for examples of paths in $\mathbb{R}^2$. In addition, we study the three analogous spaces of graphs: the sets $\mathcal{G}_\mathcal{C}$, $\mathcal{G}_\mathcal{I}$, and $\mathcal{G}_\mathcal{E}$ of continuous maps, immersions, and embeddings of graphs, respectively. This paper establishes the core metric and topological properties of the Fréchet distance on graphs and paths in Euclidean space.



(a) Continuous (b) Embedding (c) Immersion

Figure 1: Example of paths continuously mapped, embedded, and immersed in $\mathbb{R}^2$. The space of continuous maps allows arbitrary self-intersection on a path including backtracking (which occurs at the two red points); embeddings must induce homeomorphisms onto their image; and immersions are locally embeddings.

2 Background

In this section, we establish the definitions and notation from geometry and topology used throughout. We assume basic knowledge of concepts in topology. For common definitions central to this paper, we refer readers to Appendix A, or for greater detail, to [8,17].

Definition 1 (Types of Maps) *Let $\mathbb{X}$ and $\mathbb{Y}$ be topological spaces. A map $\alpha\colon \mathbb{X} \to \mathbb{Y}$ is called* continuous *if for each open set $U \subset \mathbb{Y}$, $\alpha^{-1}(U)$ is open in $\mathbb{X}$. We call α an* embedding *if α is injective. Equivalently, an* embedding *is a continuous map that is homeomorphic onto its image. If α is locally an embedding, then we say that α is an* immersion.

In particular, a continuous map $\gamma\colon [0,1] \to \mathbb{R}^n$ is called a *path* in $\mathbb{R}^n$. We call a path $\gamma\colon [0,1] \to \mathbb{R}^n$

rectifiable if γ has finite *length* (see Definition 32 in Appendix A.3). Moreover, we call a graph G rectifiable if there exists a finite cover of G such that every element in the cover is a rectifiable path.

Paths in $\mathbb{R}^n$ Letting $\widetilde{\Pi_{\mathcal{C}}}$ denote the set of all rectifiable paths in $\mathbb{R}^n$, we now define the path Fréchet distance.

Definition 2 (The Path Fréchet Distance [3]) *The* Fréchet distance $d_{FP}\colon \widetilde{\Pi_{\mathcal{C}}} \times \widetilde{\Pi_{\mathcal{C}}} \to \bar{\mathbb{R}}_{\geq 0}$ *between* $\gamma_1, \gamma_2 \in \widetilde{\Pi_{\mathcal{C}}}$ *is defined as:*

$$d_{FP}(\gamma_1, \gamma_2) := \inf_{r\colon [0,1]\to[0,1]} \max_{t\in[0,1]} ||\gamma_1(t) - \gamma_2(r(t))||_2,$$

where r ranges over all homeomorphisms such that $r(0) = 0$, and $||\cdot||_2$ denotes the Euclidean norm.

Graphs Mapped to $\mathbb{R}^n$ We define a *graph* $G = (V, E)$ as a finite set of vertices V and a finite set of edges E. Self-loops and multiple edges between a pair of vertices are allowed.[1] We topologize a graph by thinking of it as a CW complex; see Appendix A.1. If $\phi\colon G \to \mathbb{R}^d$ is a map, then we call (G, ϕ) a graph-map pair. We extend the path Fréchet distance to the Fréchet distance between graphs continuously mapped into $\mathbb{R}^n$:

Definition 3 (Graph Fréchet Distance) *Let $(G,\phi), (H,\psi)$ be continuous, rectifiable graph-map pairs. We define the Fréchet distance between (G,ϕ) and (H,ψ) by minimizing over all homeomorphisms:*[2]

$$d_{FG}\left((G,\phi),(H,\psi)\right) := \begin{cases} \inf_h ||\phi - \psi\circ h||_\infty & G \cong H. \\ \infty & \textit{otherwise.} \end{cases}$$

For simplicity of exposition, when $G \cong H$, we write the LHS of this equation as $d_{FG}(\phi,\psi)$. Furthermore, defining the infimum over an emptyset to be ∞, the graph Fréchet distance is given by the following equation:

$$d_{FG}\left((G,\phi),(H,\psi)\right) := \inf_h ||\phi - \psi\circ h||_\infty,$$

where the infimum is taken over all homeomorphisms $h\colon G \to H$.

Note that if $G = H$ and ϕ is a reparameterization of ψ, then $d_{FG}(\phi,\psi) = 0$.

Observation 1 (Paths as Graphs) *If $G = [0,1]$ and $\alpha,\beta\colon [0,1] \to \mathbb{R}^n$ are paths, then the relationship between path and graph Fréchet distances is as follows:*

$$d_{FG}(\alpha,\beta) = \min\left\{d_{FP}(\alpha,\beta), d_{FP}(\alpha,\beta^{-1})\right\},$$

where $\beta^{-1}\colon I \to \mathbb{R}^n$ is defined by $\beta^{-1}(t) = \beta(1-t)$.

[1] Some references would call this a *multi-graph*, but for simplicity, we just use the term *graph*.

[2] Other generalizations of the Fréchet distance minimize over all "orientation-preserving" homeomorphisms, which can be defined in several ways for stratified spaces, and sometimes adding an orientation is not natural. Thus, we drop this requirement in our definition.

3 Metric Properties

We now address the question: Is this distance a metric? If not, can it be metrized? A well-known known property of the path Fréchet distance is that it is a pseudometric [3, 19]. That is, it satisfies all metric properties except for separability. We proof this property for d_{FG}.

Theorem 4 (Metric Properties of d_{FG}) *d_{FG} is an extended pseudo-metric that does not satisfy separability. When restricted to a homeomorphism class of graphs, d_{FG} is a pseudo-metric.*

Proof. We first prove that d_{FG} is an extended pseudometric (see Definition 27 in Appendix A.3).

Identity: Taking h to be the identity map in Definition 3, we find $d_{FG}((G,\phi_1),(G,\phi_1)) = 0$.

Symmetry: Consider $d_{FG}(\phi_1,\phi_2)$. If $G \ncong H$, then no homeomorphism $h\colon G \to H$ exists. Likewise, no homeomorphism $h'\colon H \to G$ exists. And, so,

$$d_{FG}((G,\phi_1),(H,\phi_2))) = \infty = d_{FG}((H,\phi_2),(G,\phi_1))).$$

Otherwise, since h is a homeomorphism, it is invertible. Thus, we can rewrite this as:

$$d_{FG}(\phi_1,\phi_2) = \inf_{h^{-1}} ||\phi_1\circ h^{-1} - \phi_2||_\infty = d_{FG}(\phi_2,\phi_1).$$

Subadditivity (the triangle inequality): Consider $d_{FG}((G_1,\phi_1),(G_2,\phi_2)) + d_{FG}((G_2,\phi_2),(G_3,\phi_3))$. If $G_1 \ncong G_2$, then $d_{FG}((G_1,\phi_1),(G_2,\phi_2)) = \infty$, and we are done. A symmetric argument follows for $G_2 \ncong G_3$. Thus, we assume $G_1 \cong G_2 \cong G_3$. Using the definition of Fréchet distance and the fact that the infimum is taken over homeomorphisms, we obtain:

$$\begin{aligned} &d_{FG}(\phi_1,\phi_2) + d_{FG}(\phi_2,\phi_3) \\ &\quad= \inf_{h'} ||\phi_1 - \phi_2\circ h'||_\infty + \inf_{h''} ||\phi_2 - \phi_3\circ h''||_\infty. \\ &\quad\geq \inf_{h'} ||\phi_1 - \phi_2\circ h'||_\infty + \inf_{h,h'} ||\phi_2\circ h' - \phi_3\circ h||_\infty \\ &\quad= \inf_{h,h'} ||\phi_1 + (\phi_2\circ h' - \phi_2\circ h') - \phi_3\circ h||_\infty \\ &\quad= \inf_{h} ||\phi_1 - \phi_3\circ h||_\infty \\ &\quad= d_{FG}(\phi_1,\phi_3). \end{aligned}$$

And so, we conclude that d_{FG} satisfies subadditivity.

Noting that if $G \ncong H$ that $d_{FG}\left((G,\phi),(H,\psi)\right) = \infty$, we conclude that d_{FG} is an extended pseudo-metric. However, the graph Fréchet distance between homeomorphic graphs is at most the Hausdorff distance between the images of the two maps. Thus, when restricted to a homeomorphism class of graphs, d_{FG} is a pseudo-metric. □

The only metric property not satisfied is separability. In order to metrize this pseudo-metric, we define $\mathcal{G}_{\mathcal{C}}(G)$ to be the the set of equivalence classes of continuous, rectifiable maps $G \to \mathbb{R}^n$, where two maps, ϕ_1

and ϕ_2, are equivalent if and only if $d_{FG}(\phi_1, \phi_2) = 0$. We write $[\phi_i]$ to denote the equivalence class of maps containing ϕ_i. We define two subspaces of $\mathcal{G}_{\mathcal{C}}(G)$: those representing immersions and embeddings, denoted $\mathcal{G}_{\mathcal{I}}(G)$ and $\mathcal{G}_{\mathcal{E}}(G)$, respectively. Note that $\mathcal{G}_{\mathcal{E}}(G) \subsetneq \mathcal{G}_{\mathcal{I}}(G) \subsetneq \mathcal{G}_{\mathcal{C}}(G)$. Let $\mathcal{G}_{\mathcal{C}}$ denote the induced set of equivalence classes of all graph-map pairs $(G, [\phi])$ such that $[\phi] \in \mathcal{G}_{\mathcal{C}}(G)$. Similiarly, we define $\mathcal{G}_{\mathcal{I}}$ and $\mathcal{G}_{\mathcal{E}}$, and note $\mathcal{G}_{\mathcal{E}} \subsetneq \mathcal{G}_{\mathcal{I}} \subsetneq \mathcal{G}_{\mathcal{C}}$. Hence,

Corollary 5 (Metric Extension for Graphs) *For every graph G, the graph Fréchet distance is a metric on the quotient spaces $\mathcal{G}_{\mathcal{C}}(G)$, $\mathcal{G}_{\mathcal{I}}(G)$, and $\mathcal{G}_{\mathcal{E}}(G)$. Moreover, the graph Fréchet distance is an extended metric on $\mathcal{G}_{\mathcal{C}}$, $\mathcal{G}_{\mathcal{I}}$, and $\mathcal{G}_{\mathcal{E}}$.*

Similarly, we consider paths in $\mathbb{R}^n$: in particular, $\Pi_{\mathcal{C}}$ is the set of equivalences classes of $\widetilde{\Pi_{\mathcal{C}}}$ up to orientation-preserving reparameterization. Equivalently, for $\gamma_1, \gamma_2 \in \widetilde{\Pi_{\mathcal{C}}}$, γ_1 is equivalent to γ_2 iff $d_{FP}(\gamma_1, \gamma_2) = 0$. Likewise, $\Pi_{\mathcal{E}}$ and $\Pi_{\mathcal{I}}$ are the subspaces of embeddings and immersions. Note that $\Pi_{\mathcal{E}} \subsetneq \Pi_{\mathcal{I}} \subsetneq \Pi_{\mathcal{C}}$. We topologize these spaces using the open ball topology (Appendix A.3). Again, by construction, we obtain:

Corollary 6 (Metric Properties of d_{FP}) *The path Fréchet distance is a metric on $\Pi_{\mathcal{C}}, \Pi_{\mathcal{I}}$ and $\Pi_{\mathcal{E}}$.*

4 Path-Connectedness Property

We now examine path-connectedness properties. See Definition 30 and Definition 31 of Appendix A.4 for definitions of path-connectivity.

4.1 Continuous Mappings

We start with the most general spaces of paths and graphs: the continuous, rectifiable maps into $\mathbb{R}^n$. In Euclidean spaces, linear interpolation is a useful tool because it defines the shortest paths between two points. In function spaces, linear interpolation is also nice:

Definition 7 (Linear Interpolation) *Let G be a graph and $\phi_0, \phi_1 \colon G \to \mathbb{R}^n$ be continuous, rectifiable maps. The* linear interpolation *from ϕ_0 to ϕ_1 is the map $\Gamma\colon [0,1] \to \mathcal{G}_{\mathcal{C}}(G)$ sending $t \in [0,1]$ to (G, ϕ_t), where:*

$$\phi_t := (1-t)\phi_0 + t(\phi_1 \circ h_*). \tag{1}$$

For ease of notation, we sometimes write $\Gamma_t := \Gamma(t)$.

Note that $(1-t)\phi_0 + t\phi_1$ is a linear combination of ϕ_0 and ϕ_1 (using $c_0 = 1-t$ and $c_1 = t$ in Definition 34). Thus, Γ is a continuous family of linear combinations of the maps ϕ_0 and ϕ_1; we show Γ is continuous in Lemma 35. in Appendix B.1. If $G = [0,1]$, the linear interpolation between graphs is simply linear interpolation between paths. For an example of linear interpolation between graphs, see Figure 4 in Appendix B.1.

However, linear interpolation is not well-defined in $\mathcal{G}_{\mathcal{C}}$, as we could have $\phi_1, \phi_2 \in [\phi] \in \mathcal{G}_{\mathcal{C}}(G)$. In fact, $\Gamma(t; \phi_1, \phi_2) = \Gamma(t; \phi_1, \phi_3)$ if and only if $\phi_1 = \phi_2$.

Definition 8 (Family of Interpolations) *Let G be a graph and $[\phi_0], [\phi_1] \in \mathcal{G}_{\mathcal{C}}(G)$. We define $\mathcal{C}([\phi_0], [\phi_1])$ to be the set of all linear interpolations between elements of $[\phi_0]$ and of $[\phi_1]$.*

We now demonstrate the existence of a family of interpolations between any two equivalence classes within $(\mathcal{G}_{\mathcal{C}}(G), d_{FG})$, proving path-connectivity.

Theorem 9 (Continuous Maps of Graphs) *For every graph G, the extended metric space $(\mathcal{G}_{\mathcal{C}}(G), d_{FG})$ is path-connected. Moreover, the connected components of $(\mathcal{G}_{\mathcal{C}}, d_{FG})$ are in one-to-one correspondence with the homeomorphism classes of graphs.*

Proof. Let $[\phi_0], [\phi_1] \in \mathcal{G}_{\mathcal{C}}(G)$. Let $\Gamma \in \mathcal{C}([\phi_0], [\phi_1])$. By Lemma 35 in Appendix B.1, Γ is continuous, and so $(\mathcal{G}_{\mathcal{C}}(G), d_{FG})$ is path-connected.

Moreover, suppose $(G, [\phi_0]), (H, [\phi_1]) \in \mathcal{G}_{\mathcal{C}}$ for the graphs G, H which are not homeomorphic. Then, $d_{FG}((G, [\phi_0]), (H, [\phi_1])) = \infty$, and connected components of the extended metric space $\mathcal{G}_{\mathcal{C}}$ are vacuously in one-to-one correspondence with homeomorphism classes of graphs. $\square$

Setting $G = [0,1]$, an identical proof holds for paths.

Corollary 10 (Continuous Maps of Paths) *The space $\Pi_{\mathcal{C}}$ is path-connected.*

We now demonstrate the stricter property of the path-connectivity of open distance balls:

Lemma 11 (Metric Balls in $(\mathcal{G}_{\mathcal{C}}, d_{FP})$) *Metric balls with finite radius in $(\mathcal{G}_{\mathcal{C}}, d_{FP})$ are path-connected.*

Proof. Let $\delta \in \mathbb{R}$ such that $\delta > 0$. Let $(G, [\phi_0]) \in \mathcal{G}_{\mathcal{C}}$.

Consider the metric ball $\mathbb{B} := \mathbb{B}_{d_{FG}}([\phi_0], \delta)$ in $\mathcal{G}_{\mathcal{C}}$. Let $[\phi_1], [\phi_2] \in \mathbb{B}$. We wish to find a path from $[\phi_1]$ to $[\phi_2]$. We first find a path in $\mathbb{B}_{d_{FG}}([\phi_0], \delta)$ from $[\phi_0]$ to $[\phi_2]$, as follows. Set

$$\varepsilon = \delta - d_{FG}([\phi_0], [\phi_2]).$$

By Lemma 25, we know that there exists a homeomorphism $h_* : G \to G$ such that the following inequality holds: $||\phi_0 - \phi_2 \circ h_*||_\infty < d_{FG}([\phi_0], [\phi_2]) + \varepsilon/2$.

Let $\Gamma \in \mathcal{C}([\phi_0], [\phi_2])$. Then, for all $t \in (0, 1)$,

$$\begin{aligned} &d_{FG}(\Gamma_t, \phi_0) \\ &\quad = \inf_h ||((1-t)\phi_0 + t(\phi_2 \circ h_*)) - \phi_0 \circ h||_\infty \\ &\quad \leq ||((1-t)\phi_0 + t(\phi_2 \circ h_*)) - \phi_0 \circ h_*||_\infty \\ &\quad < d_{FG}([\phi_0], [\phi_2]) + \varepsilon/2 \\ &\quad < \delta. \end{aligned}$$

Thus, $\Gamma_t \in \mathbb{B}_{d_{FG}}([\phi_1], \delta)$, which means there exists a path from ϕ_0 to ϕ_2. Similarly, we find a path Γ' from ϕ_1 to ϕ_0. Concatentating the two paths, $\Gamma' \# \Gamma$ we have a path in $\mathbb{B}_{d_{FG}}([\phi_0], \delta)$ from $[\phi_1]$ to $[\phi_2]$. Hence, metric balls with finite radius in $\mathcal{G}_{\mathcal{C}}$ are path-connected. $\square$

Setting $G = [0, 1]$, we obtain:

Corollary 12 (Metric Balls in $(\Pi_{\mathcal{C}}, d_{FP})$) *Balls in the extended metric space $(\Pi_{\mathcal{C}}, d_{FP})$ are path-connected.*

4.2 Immersions

An immersion is a map that is locally injectivite. Thus, self-intersections are allowed, but a map pausing or backtracking is not. Next, we define these notions, and give examples in Figure 2.

Definition 13 (Pausing) *We say that a path γ* pauses *in an interval $I \subset [0, 1]$ if $\gamma(x) = \gamma(y)$ for every $x, y \in I$. In this case, $[\gamma] \notin \Pi_{\mathcal{I}}$.*

Another possible violation of local injectivity is *backtracking* on a path.

Definition 14 (Backtracking) *We say that a path γ is* backtracking *at a point $x \in [0, 1]$ if there exists $\delta > 0$ such that for every $\epsilon \in (0, \delta)$, either $\gamma|_{(x-\epsilon,x)} \subset \gamma_{(x,x+\epsilon)}$ or $\gamma|_{(x,x+\epsilon)} \subset \gamma|_{(x-\epsilon,x)}$.*

To show the path-connectivity of spaces of immersions, the proof in Theorem 9 for continuous mappings is *almost* sufficient, but these added violations must be addressed. Thus, we introduce additional maneuvers to avoid pauses and backtracking.

Lemma 15 (Rerouting Pauses) *Let $\gamma_0, \gamma_1 \in \widetilde{\Pi_{\mathcal{I}}}$, and let $\Gamma : [0, 1] \to \widetilde{\Pi_{\mathcal{C}}}$ be a path in $\widetilde{\Pi_{\mathcal{C}}}$ from γ_0 to γ_1. Suppose there exists an interval $[t_1, t_2]$ such that for all $t \in [0, 1] \setminus (t_1, t_2)$, Γ_t is an immersion. But, for all $t \in (0, 1)$, Γ_t has a single pause. Then, there exists a different path $\Gamma^* \colon [0, 1] \to \Pi_{\mathcal{I}}$ that avoids the pause.*

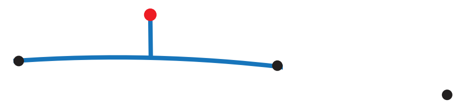
Proof. Let $t \in [t_1, t_2]$. Let the pause in Γ_t be over the interval $(a_t, b_t) \subset [0, 1]$. Let $\varepsilon_t := \min(t_2 - t, t - t_1)$. We stretch the paused interval (a_t, b_t) in Γ_t by defining a map $\Gamma_t^* : [0, 1] \to \widetilde{\Pi_{\mathcal{I}}}$ as follows:

- $\Gamma_t^*(-\infty, a_t]$ is an oriented reparameterization of $\Gamma_t(-\infty, a_t - \varepsilon_t]$.
- $\Gamma_t^*(a_t, b_t)$ is an oriented reparameterization of $\Gamma_t(a_t - \varepsilon_t, a_t] \# \Gamma_t[b_t, b_t + \varepsilon)$
- $\Gamma_t^*[b_t, \infty)$ is an oriented reparameterization of $\Gamma_t[b_t + \varepsilon_t, \infty)]$.

By construction, Γ_t^* has removed the pause between a_t and b_t; hence, $\Gamma_s \in \widetilde{\Pi_{\mathcal{I}}}$. Putting these maps together, we obtain a map $\Gamma^* \colon [0, 1] \to \widetilde{\Pi_{\mathcal{C}}}$, where

$$\Gamma(t) := \begin{cases} \Gamma_t & \text{if } t \notin (t_1, t_2) \\ \Gamma_t^* & \text{if } t \in (t_1, t_2). \end{cases} \tag{2}$$

Moreover, Γ is continuous in $\Pi_{\mathcal{I}}$. $\square$



(a) Forced Backtracking (b) Constant Map

Figure 2: Examples of paths in $\Pi_{\mathcal{C}}$ but not $\Pi_{\mathcal{I}}$. Figure 2a demonstrates a path with necessary backtracking at the red point. Figure 2b demonstrates a constant path which (vacuously) must pause. For a nontrivial example of a path with pauses, consider any parameterization of a path sending an open interval to a point.

Direct linear interpolation can also yield degeneracies by creating a singleton in specific circumstances, or by creating a backtracking point. Each are addressed in the following theorem, and a path is constructed.

Theorem 16 (Path Immersions) *The extended metric space $(\Pi_{\mathcal{I}}, d_{FP})$ of paths immersed in $\mathbb{R}^n$ is path-connected iff $n > 1$.*

Proof. If $n = 1$, it is easy to see that $\Pi_{\mathcal{I}}$ is not path-connected by examining intervals with reversed orientation which trivially degenerate to a point when constructing a path, violating local injectivity.

Now, consider $n > 1$. Let $[\gamma_0], [\gamma_1] \in \Pi_{\mathcal{I}}$. Using Definition 7, let $\Gamma : [0, 1] \to \Pi_{\mathcal{C}}$ be the linear interpolation from γ_0 to γ_1. This interpolation is in $\Pi_{\mathcal{C}}$, not $\Pi_{\mathcal{I}}$, so we explain how to edit Γ so that it stays in $\Pi_{\mathcal{I}}$. If $\Gamma(t) \in \widetilde{\Pi_{\mathcal{I}}}$ for each $t \in [0, 1]$, we are done. Otherwise, let $T \subset I$ be the set of times that introduce a non-immersion (i.e., $t \in T$ iff $\Gamma(t) \notin \widetilde{\Pi_{\mathcal{I}}}$, but $\Gamma(t - \epsilon) \in \widetilde{\Pi_{\mathcal{I}}}$ for all ϵ small enough). There are two things that might have happened at t: either an interval collapsed to a point (a pause) or backtracking was introduced in $\Gamma(t)$.

1. Suppose there exists $t \in T$ where an interval pauses as in Definition 13 and Figure 2b. Note that a pausing event occurs either if an interval of Γ_t becomes degenerate, or Γ_t collapses to a point.

If pausing occurs only on an open interval $(a,b) \subset [0,1]$ of $\gamma_t \in \Gamma_t$, it can be avoided using Lemma 15. If pausing occurs on a closed interval $[a,b] \subset [0,1]$, we convert it to the open set $(a-\varepsilon, b+\varepsilon)$ for small ε, and use Lemma 15. If either $a=0$ or $b=1$, we simply redefine γ_t to start at b or to end at a, respectively, using Lemma 37. The pausing event is guaranteed to conclude at some $t+\delta$ for $\delta \geq 0$ since $[\gamma_1] \in \Pi_{\mathcal{I}}$, and Γ must attain $\gamma_1 \in [\gamma_1]$.

If a pausing event stems from a full collapse to a singleton (i.e. interpolation occurs between two colinear segments with reverse orientation, and consequently degenerate to a point), the collapse can be circumvented by rotating the path defining Γ_t, which is done in Lemma 38.

2. Alternatively, suppose there exists $t \in T$ which corresponds to backtracking at a point in a path Γ_t according to Definition 14 and Figure 2a. Here, Γ_t can remain in $\widetilde{\Pi_{\mathcal{I}}}$ by inflating a ball of radius ϵ for sufficiently small $\epsilon > 0$ about the backtracking point before it is created. This is included in Lemma 39, and shown in Figure 6b.

For all $t \in T$, the described moves can be used to subvert lapses in local injectivity along Γ. Hence, we construct a path Γ by interpolating from γ_0 to γ_1, and applying the required move at each $t \in T$ to handle pauses or backtracking. By the arbitrariness of Γ, we have given a class of continuous paths from any element $\gamma_0 \in [\gamma_0]$ to any $\gamma_1 \in [\gamma_1]$. □

Theorem 17 (Metric Balls in $(\Pi_{\mathcal{I}}, d_{FP})$) *If $n > 1$, then balls in the extended metric space $(\Pi_{\mathcal{I}}, d_{FP})$ are path-connected.*

Proof. Let $[\gamma_0], [\gamma_1] \in \Pi_{\mathcal{I}}$, and let $\delta > 0$. Let $\Gamma \in \Pi_{\mathcal{I}}$ be the map Γ in the proof of Theorem 16. By Lemma 11, linear interpolation does not increase the Fréchet distance. By design, avoiding singleton degeneracies by way of Lemma 38 also does not increase the Fréchet distance. Moreover, by construction, the map Γ^* of Lemma 15 preserves the Fréchet distance. The maneuver in Lemma 39 could potentially increase $d_{FP}(\Gamma_t, \gamma_1)$ at some time $t \in [0,1]$, but in this case any critical backtracking points can be perturbed slightly in order to no longer define the $d_{FP}(\Gamma_t, \gamma_1)$. Hence, these moves need not result in $d_{FP}(\Gamma_t, \gamma_1) > \delta$, meaning that $\Gamma_t \in \mathbb{B}_{d_{FP}}(\phi_1, \delta)$, and balls in $\Pi_{\mathcal{I}}$ are path-connected. □

We use the same maneuvers from Theorem 16 in the context for graphs under d_{FG}.

Theorem 18 (Graph Immersions) *For every graph G, the extended metric space $(\mathcal{G}_{\mathcal{I}}(G), d_{FG})$ is path-connected. Connected components of the extended metric space $(\mathcal{G}_{\mathcal{I}}, d_{FG})$ are in one-to-one correspondence with the homeomorphism classes of graphs.*

Proof. We construct Γ identically to Theorem 16, but interpolation occurs among each edge of G in $\mathcal{G}_{\mathcal{I}}(G)$ rather than between individual segments. As in Theorem 16, local injectivity can only be violated by pauses and backtracking on edges, which are handled using Lemma 15, Lemma 38, and Lemma 39 on each edge. If $(G,[\phi_0]), (H,[\phi_1]) \in \mathcal{G}_{\mathcal{I}}$ for G, H which are not homeomorphic, then $d_{FG}((G,[\phi_0]),(H,[\phi_1])) = \infty$. □

Similarly, we can adopt Theorem 17 for each edge in a graph to show path-connectivity of balls in $\mathcal{G}_{\mathcal{I}}$.

Theorem 19 (Metric Balls in $(\mathcal{G}_{\mathcal{I}}(G), d_{FG})$) *For every graph G, the balls in the extended metric space $(\mathcal{G}_{\mathcal{I}}(G), d_{FG})$ are path-connected.*

Proof. Let $[\phi] \in \mathcal{G}_{\mathcal{I}}$, and let $\delta > 0$. Let $\mathbb{B}$ be the intersection $\mathbb{B}_{d_{FG}}(\phi, \delta) \cap \mathcal{G}_{\mathcal{I}}(G)$. Let $[\phi_0], [\phi_1] \in \mathbb{B}$. Construct the path $\Gamma : [0,1] \to \mathbb{B}$ from $[\phi_0]$ to $[\phi_1]$ in the same way as Theorem 18. Just as in Theorem 17, linear interpolation and the moves in Lemma 38, Lemma 39, and Lemma 15 mandate that $\Gamma(t) \in \mathbb{B}$ for every $t \in (0,1)$ identically to the path Fréchet distance. □

4.3 Embeddings

Lastly, we examine the path-connectedness property of the analogous spaces of embeddings.

Theorem 20 (Path Embeddings) *The extended metric space $(\Pi_{\mathcal{E}}, d_{FP})$ is path-connected in $\mathbb{R}^n$ if and only if $n > 1$.*

Proof. If $n = 1$, two paths with reverse orientations are not path-connected.

Now, let $n > 1$, and let $[\gamma_0], [\gamma_1] \in \Pi_{\mathcal{E}}$. By Alexander's trick,[3] there exists $s_0 \in [0,1]$ such that $\gamma_0' := \gamma_0|_{[s,1]}$ and $s_1 \in [0,1]$ such that $\gamma_1' := \gamma_1|_{[s_1,1]}$, where s_0 and s_1 are nearly straight. Let $\angle$ be the angle between the segments γ_0' and γ_1'. Let $S\colon [\frac{1}{4}, \frac{2}{4}] \to \Pi_{\mathcal{E}}$ be the map rotating γ_0' by $\angle$ to become parallel with γ_1'. Finally, let Γ be the interpolation from $\gamma_0' \circ S$ to γ_1'.

Define $P\colon [0,1] \to \Pi_{\mathcal{E}}$ as the resulting composition:

$$P(t) = \begin{cases} \gamma_0 \mid_{[(1-t)s,1]}, & t \in [0, \frac{1}{4}] \\ S(t), & t \in [\frac{1}{4}, \frac{2}{4}] \\ \Gamma(t) & t \in [\frac{2}{4}, \frac{3}{4}] \\ \gamma_1 \mid_{[(1-t)s,1]}, & t \in [\frac{3}{4}, 1]. \end{cases}$$

The steps attaining γ_0' and γ_1', as nothing else than a restriction of γ_0 and γ_1, are continuous. Moreover, S is continuous as the rotation of γ_0', and Γ is continuous by Lemma 35. By the arbitrariness of the constructed path and γ_0, γ_1, there is a family of continuous paths for any $\gamma_0 \in [\gamma_0], \gamma_1 \in [\gamma_1]$, and $\Pi_{\mathcal{E}}$ is path-connected. □

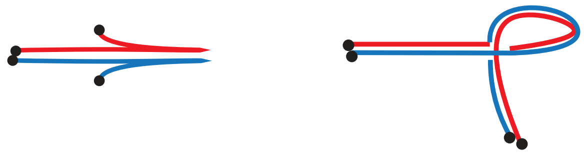


Figure 3: Two embedded paths γ_0, γ_1 in $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively, for which constructing a path $\Gamma : [0,1] \to \Pi_\mathcal{E}, \Gamma(0) = \gamma_0, \Gamma(1) = \gamma_1$ is not possible without having $\Gamma(t) \notin \mathbb{B}_{d_{FP}}(\gamma_1, d_{FP}(\gamma_0, \gamma_1))$ for some $t \in [0,1]$.

Moreover, in high dimensions we can construct a path in $\Pi_\mathcal{E}$ not increasing the Fréchet distance.

Theorem 21 (Metric Balls in $(\Pi_\mathcal{E}, d_{FP})$) *If $n \geq 4$, then balls with finite radius in the extended metric space $(\Pi_\mathcal{E}, d_{FP})$ are path-connected in $\mathbb{R}^n$.*

Proof. If $n \geq 4$, the same map Γ given in Theorem 16 is sufficient, except that self-crossings must be avoided. At each $s \in [0,1]$ where a self-crossing would occur, we perturb Γ by a sufficiently small amount in order to avoid a self-crossing without increasing the Fréchet distance using the maneuver in Lemma 41. □

A simple examination shows that metric balls are not path-connected in low dimensions.

Theorem 22 (Metric Balls in $(\Pi_\mathcal{E}, d_{FP})$) *If $n \in \{1,2,3\}$, then balls with finite radius in the extended metric space $(\Pi_\mathcal{E}, d_{FP})$ are not path-connected in $\mathbb{R}^n$.*

Proof. For $n = 1$, let $[\gamma] \in \Pi_\mathcal{E}$. Let $\gamma^{-1} := \gamma(1-t)$, and note that $\gamma^{-1} \in \Pi_\mathcal{E}$. If $n = 2$, consider two paths within a fixed Fréchet ball that are much wider than their Fréchet distance. If $n = 3$, consider two paths with small Fréchet distance that form a loop, with one section passing under the other. If these loops have reversed orientation between the two paths, the Fréchet distance must increase. See Figure 3 for examples. □

In the setting for graphs, the path-connectedness property reduces to a knot theory problem if $n \leq 3$, and is not maintained. For $n \geq 4$, we use the existence of a sequence of Reidemeister moves [**?**] from any tame knot to another to construct paths in $\mathcal{G}_\mathcal{E}$.

Theorem 23 (Path-Connectivity of $(\mathcal{G}_\mathcal{E}, d_{FG})$, $n \geq 4$) *For all graphs G and $n \geq 4$, the extended metric space $(\mathcal{G}_\mathcal{E}(G), d_{FG})$ is path-connected. Moreover, connected components of the extended metric space $(\mathcal{G}_\mathcal{E}, d_{FG})$ are in one-to-one correspondence with homeomorphism classes of graphs.*

[3]Two embeddings of the n-ball are isotopic, first proven by Alexander [**?**]; see also [**?**, §4].

Proof. Let G be a graph, and $\phi_0, \phi_1 \in \mathcal{G}_\mathcal{E}(G)$. If $n \geq 4$, any tame knot can be unwound by a sequence of Reidemeister moves into the unknot. Construct $\Gamma : [0,1] \to \mathcal{G}_\mathcal{E}(G)$ by linear interpolating until some $t \in (0,1)$ causes Γ_t to self-intersect. At t, there exists a Reidemeister move allowing the crossing event to occur. Hence, any sequence of knots and free edges comprising ϕ_0 and ϕ_1 can be unwound to a sequence of unknots and straight edges, and then interpolated accordingly. Consequently, there exists a path from ϕ_0 to ϕ_1 in $(\mathcal{G}_\mathcal{E}(G), d_{FG})$. Note that we require ϕ_0, ϕ_1 are rectifiable in Section 2. □

In dimension 4 or higher, the path-connectivity of balls in $\mathcal{G}_\mathcal{E}(G)$ is shown in the same way as for paths.

Theorem 24 (Metric Balls in $(\mathcal{G}_\mathcal{E}(G), d_{FG})$) *For all graphs G and $n \geq 4$, metric balls in the space $(\mathcal{G}_\mathcal{E}(G), d_{FG})$ are path-connected.*

Proof. The proof is identical to that in Lemma 41, but Reidemeister moves are used for each edge in a graph rather than a single segment. □

5 Conclusion

In this paper, we studied some fundamental topological properties of spaces of paths and graphs in Euclidean space under the Fréchet distance. In particular, we investigated metric properties of the Fréchet distance on paths and graphs, as well as studying the path-connectedness of metric balls in the space of such graphs. While this work is theoretical and mathematical in nature, we feel that establishing the underlying properties of the topological spaces it can define provides an important theoretical backdrop, which is especially critical due to the widespread popularity of the Fréchet distance in computational geometry, and the growing popularity of its extension for graphs. Our contribution begins a careful study of the Fréchet distance and its topological properties. Extensions to this work abound, and include examining core topological properties of other distance measures in computational geometry, as well as other important properties of the Fréchet distance.

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A Distances and Topology

Let $\bar{\mathbb{R}}$ denote the extended real line: $\bar{\mathbb{R}} = \mathbb{R} \cup \pm\infty$. We now provide the basic definitions relating to distances and topology used throughout this paper.

A.1 Graphs

Graphs are a central object studied in this paper.

Throughout this paper, we use the term *graph* to mean a *multi-graph*. A multi-graph $G = (V, E)$ is a finite set of vertices V and edges E. Self-loops and multiple edges between two vertices are allowed in this setting. A graph is an example of a more general structure called a CW complex, which we topologize as follows: (1) the topology on G restricted to V is the discrete topology; (2) for a edge e, the open sets restricted to is closure $\bar{e}$ are those induced by the subspace topology on $[0,1] \subset \mathbb{R}$ and a homeomorphism $[0,1] \to \bar{e}$; (3) we take the quotient topology on $(\cup_{v \in V}) \cup (\cup_{e \in E} \bar{e})$.

A.2 Fréchet Distance

We defined the path and graph Fréchet distances in Section 2. The path Fréchet distance is well-studied [1,3–7, 9–16, 18–20]. The graph Fréchet distance has been less studied, but many results for paths transfer to graphs.

The proof of the following lemma follows from the definition of Fréchet distance and the definition of infimum.

Lemma 25 (Approximator) *For all graphs G, if $[\phi_0], [\phi_1] \in \Pi_{\mathcal{C}}(G)$, then for every $\varepsilon > 0$, there exists a homeomorphism $h_* \colon G \to G$ such that*

$$||\phi_0 - \phi_1 \circ h||_\infty < d_{FG}(\phi_0, \phi_1) + \epsilon.$$

Proof. By Definition 3,

$$d_{FG}([\phi_1], [\phi_2]) = \inf_h ||\phi_1 - \phi_2 \circ h||_\infty.$$

Then, by the definition of infimum, for every $\varepsilon > 0$, there exists $h_* : G \to G$ such that

$$\begin{aligned} ||\phi_1 - \phi_2 \circ h_*||_\infty &< \inf_h ||\phi_1 - \phi_2 \circ h||_\infty + \varepsilon/2 \\ &= d_{FG}([\phi_1], [\phi_2]) + \varepsilon/2, \end{aligned}$$

as was to be shown. □

A.3 Defining Spaces from Distances and Metrics

Given a set $\mathbb{X}$ and a $d \colon \mathbb{X} \times \mathbb{X} \to \bar{\mathbb{R}}_{\geq 0}$, we topologize $\mathbb{X}$ as follows:

Definition 26 (The Open Ball Topology) *Let $\mathbb{X}$ be a set and $d \colon \mathbb{X} \times \mathbb{X} \to \bar{\mathbb{R}}_{\geq 0}$ a distance function. For each $r > 0$ and $x \in \mathbb{X}$, let $\mathbb{B}_d(x, r) := \{y \in \mathbb{X} \mid d(x, y) < r\}$. The open ball topology on $\mathbb{X}$ with respect to d is the topology generated by $\{\mathbb{B}_d(x, r) \mid x \in \mathbb{X}, r > 0\}$. We call $(\mathbb{X}, d)$ a* distance space*.*

In words, $\mathbb{B}_d(x, r)$ denotes the open ball of radius r centered at x with respect to d. We use these open balls to generate a topology on $\mathbb{X}$, allowing x to range over $\mathbb{X}$ and r to range over all positive real numbers.

We are particularly interested in distance functions that are either a pseudo-metric or a metric. These are defined as follows.

Definition 27 (Pseudo-Metric) *Let $\mathbb{X}$ be a set and let $d \colon \mathbb{X} \times \mathbb{X} \to \bar{\mathbb{R}}_{\geq 0}$ be a distance function. We call d a* pseudo-metric *on $\mathbb{X}$ if d satisfies the following:*

- *Finiteness: $d(x, y) < \infty$ for all $x, y \in \mathbb{X}$.*
- *Identity: $d(x, x) = 0$ for all $x \in \mathbb{X}$.*
- *Symmetry: $d(x, y) = d(y, x)$ for all $x, y \in \mathbb{X}$.*
- *Subadditivity (the triangle inequality): $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in \mathbb{X}$*

If d satisfies everything except finiteness, then we call d an extended pseudo-metric*.*

In order to be a metric, d must fulfill stricter criteria:

Definition 28 (Metric) *Let $\mathbb{X}$ be a set and let the function $d \colon \mathbb{X} \times \mathbb{X} \to \bar{\mathbb{R}}_{\geq 0}$ be a pseudo-metric. We say that d is a* metric *if d also satisfies:*

- *Seperability: for any $x, y \in \mathbb{X}$, if $x \neq y$, then $d(x, y) > 0$.*

Often, if $(\mathbb{X}, d)$ is a pseudo-metric space, a standard procedure is to define an equivalence class for $x, y \in \mathbb{X}$ where $x \sim y$ if $d(x, y) = 0$. Then, the quotient space $\mathbb{X}/_\sim$ is a metric space.

Common examples of metrics on function spaces are those induced by L_p-norms. For example, let $(\mathbb{Y}, d_\mathbb{Y})$ be distance space, let $\mathbb{X}$ be any topological space, and let $f, g \colon \mathbb{X} \to \mathbb{Y}$. Then, the distance induced by the L_∞-norm between f and g is:

$$||f - g||_\infty = \max_{x \in \mathbb{X}} d_\mathbb{Y}(f(x), f(y)).$$

A.4 Paths and Maps

With the basic definitions from topology in hand, we are equipped to define a property of fundamental interest in topology: path-connectedness.

Definition 29 (Path) *A* path *in a topological space $\mathbb{X}$ between two elements $a, b \in \mathbb{X}$, is defined to be a continuous map $\gamma : [0,1] \to \mathbb{X}$ where $\gamma(0) = a$, and $\gamma(1) = b$.*

Given two paths $\gamma_1, \gamma_2 \colon [0,1] \to \mathbb{X}$ such that $\gamma_1(1) = \gamma_2(0)$, we combine them by taking both at double-speed. This is called the concatenation of paths. In particular, $\gamma_1 \# \gamma_2 \colon [0,1] \to \mathbb{R}^n$ is defined by:

$$\gamma_1 \# \gamma_2(t) := \begin{cases} \gamma_1(2t) & t \in [0, 0.5]. \\ \gamma_2(2t-1) & \text{otherwise.} \end{cases}$$

Given one path $\gamma \colon [0,1] \to \mathbb{X}$ and an interval $[a,b] \subseteq [0,1]$, the restriction of γ to $[a,b]$ is also a path, given by:

$$\gamma|_{[a,b]}(t) := \gamma(a + t(b-a)).$$

With the definition of paths, we define a primary property of interest in this paper: path-connectivity.

Definition 30 (Path-Connectivity) *A topological space $\mathbb{X}$ is called* path-connected *if there exists a* path *between any two elements in $\mathbb{X}$.*

We also define the path-connectedness property specifically for distance balls:

Definition 31 (Path-Connectivity of Balls) *Let $(\mathbb{X}, d)$ be a topological space, let $x \in \mathbb{X}$. We say that the distance balls in $(\mathbb{X}, d)$ are* path-connected *if for every $x \in \mathbb{X}$ and $r \in \mathbb{R}_{\geq 0}$, the distance ball $\mathbb{B}_d(y, r)$ is path-connected.*

And, the *length* of a path in a distance space is given by:

Definition 32 (Length) *Let $(\mathbb{X}, d)$ be a distance space and let γ be a* path *in $(\mathbb{X}, d)$. Let $\mathcal{P}$ be the set of all finite subsets $P = \{t_i\}$ of $[0,1]$ such that such that $0 = t_0 < t_1 < \ldots < t_n = 1$. The* length *$L_d(\gamma)$ of γ is:*

$$L_d(\gamma) := \sup_{P \in \mathcal{P}} \sum_{i=1}^{n} d(\gamma(t_i), \gamma(x_{i-1})).$$

Additionally, it is often useful in our setting to reparameterize paths, both to define the Fréchet distance and to maintain properties such as injectivity in a map.

Definition 33 (Reparameterization) *Let $\mathbb{X}, \mathbb{Y}$ be a topological spaces, $\phi \colon \mathbb{X} \to \mathbb{Y}$, and $h \colon \mathbb{X} \to \mathbb{X}$ is a homeomorphism. Then, we call $\phi \circ h$ a reparameterization of ϕ. In the setting where $\mathbb{X} = [0,1]$ and $h(0) = 0$, we call $\phi \circ h$ an* orientation-preserving *reparameterization.*

B Omitted Details for Path-Connectivity

In this appendix, we provide additional context for the proofs of path-connectivity in Section 4.

B.1 Additional Details on Interpolation

Given two continuous maps of the same graph into $\mathbb{R}^n$, we define the interpolation between them. First, we need to define linear combinations of graphs (and paths).

Definition 34 (Linear Combination of Graphs) *Let G be a graph, let $\phi_1, \phi_2 : [0,1] \to \mathbb{R}^n$ be continuous, rectifiable maps, and $c_1, c_2 \in \mathbb{R}$. Then, the linear combination $\phi = c_1\phi_1 + c_2\phi_2$ is defined as follows: the map $\phi \colon G \to \mathbb{R}^n$ is defined by $\phi(x) := c_2\phi_1(x) + c_2\phi_2(x)$. In this case, we may also say (G, ϕ) is a linear combination of graph-map pairs (G, ϕ_1) and (G, ϕ_2).*

In this definition, we observe that ϕ is continuous (since ϕ_0 and ϕ_1 are continuous), which means that linear combinations are well-defined in the set of all continuous, rectifiable maps. It is not well defined in the space $\mathcal{G}_{\mathcal{C}}(G)$ overall.

Lemma 35 (Linear Interpolation is Continuous) *For all graphs G, linear interpolation between graphs in $\mathcal{G}_{\mathcal{C}}(G)$ (and hence between homeomorphic graphs in $\mathcal{G}_{\mathcal{C}}$) is a continuous function.*

Proof. Let $[\phi_0], [\phi_1] \in \mathcal{G}_{\mathcal{C}}(G)$. Let $\Gamma \colon [0,1] \to \mathcal{G}_{\mathcal{C}}(G)$ be the linear interpolation from ϕ_0 to ϕ_1.

We prove that Γ satisfies the ε-δ definition of continuity. Let $\varepsilon > 0$. Set $\delta = \frac{\varepsilon}{d_{FG}([\phi_0],[\phi_1])}$. Let $s, t \in [0,1]$ such that $|s - t| < \delta$. Then, we have

$$\begin{aligned} & d_{FG}([\Gamma_t], [\Gamma_s]) \\ & = d_{FG}([(1-t)\phi_0 + t\phi_1], [(1-s)\phi_0 + s\phi_1]) \\ & = \inf_h ||((1-t)\phi_0 + t\phi_1) - ((1-s)\phi_0 + s\phi_1) \circ h||_\infty, \end{aligned}$$

where h ranges over all reparameterizations of $[0,1]$. Continuing, we find:

$$\begin{aligned} & d_{FG}([\Gamma_t], [\Gamma_s]) \\ & = \inf_h ||(s-t)\phi_0 + (t-s)\phi_1 \circ h||_\infty \\ & = |t-s| \inf_h ||-\phi_0 + \phi_1 \circ h||_\infty \\ & < \delta \cdot \inf_h ||-\phi_0 + \phi_1 \circ h||_\infty \\ & = \varepsilon, \end{aligned}$$

by definition of δ.

And so, we have shown that Γ satisfies the the ε-δ definition of continuity. In extended metric spaces, the ε-δ definition of continuity is equivalent to topological continuity (e.g., see proof in [21, Lemma 7.5.7] for metric spaces). Thus, we conclude that linear interpolation between graphs in $\mathcal{G}_{\mathcal{C}}G$ is continuous. □

Setting $G = [0,1]$, an identical argument shows that linear interpolation between paths in $\Pi_{\mathcal{C}}$ is also continuous.

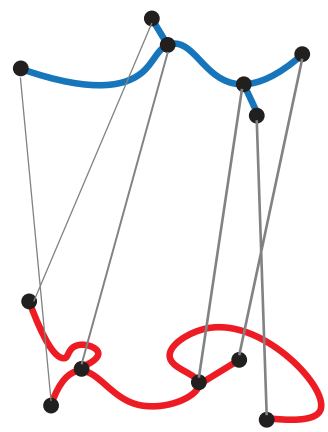


Figure 4: The interpolation between two embeddings of a graph in $\mathbb{R}^n$. For simplicity, we show the interpolation between the vertices in the embeddings, and the interpolation between edges is inferred accordingly.

Corollary 36 (Linear Interpolation between Paths) *For all $[\gamma_0], [\gamma_1] \in \Pi_{\mathcal{C}}$, the linear interpolation from γ_0 to γ_1 is continuous.*

B.2 Paths Between Immersions in Greater Detail

This subsection includes additional details to maintain local injectivity for an arbitrary path $\Gamma : [0,1] \to \Pi_{\mathcal{I}}$. We begin by examining the case where pausing occurs on the closed interval $[a,b] \subset [0,1]$ in the domain of an immersed path $\gamma_t \in \Gamma_t$, and the interval includes either 0 or 1. We subvert this by finding an alternate (but 'close') path Γ^*.

Lemma 37 (Pausing at Endpoints) *Let $[\gamma_0], [\gamma_1] \in \Pi_{\mathcal{I}}$, and let $\Gamma : [0,1] \to \Pi_{\mathcal{C}}$ be a path in $\Pi_{\mathcal{C}}$ starting at γ_0 and ending at γ_1. Suppose that there exists an interval $[t_1, t_2] \subset [0,1]$ such that for $t \in [0,1] \backslash (t_1, t_2)$, Γ_t is an immersion and, for $t \in (0,1)$, Γ_t has a single pause (and no other violations of local injectivity). Then, there exists an alternate path Γ^* in $\Pi_{\mathcal{C}}$ starting at γ_0 and ending at γ_1 such that Γ^* is a path in $\Pi_{\mathcal{I}}$.*

Proof. We use the same idea as in Lemma 15, but instead stretch the unit interval into only one side of the original domain of the path Γ_t. That is:

$$\Gamma_t^*(x) := \begin{cases} \Gamma_t(x \cdot a) & \text{if } b = 1 \\ \Gamma_t((x-b) \cdot (1-b) + b) & \text{if } a = 0 \end{cases} \tag{3}$$

If Γ_t pauses on $[a,1]$, we know that the we can focus on the image of $[0,a]$. Likewise, if Γ_t pauses on $[0,b]$, we turn to the image of $[b,1]$. Then, replace Γ_t that pauses with the newly defined Γ_t^*. And so, we define a new map $\Gamma^* \colon [0,1] \to \Pi_{\mathcal{C}}$ as follows:

$$\Gamma^*(t) := \begin{cases} \Gamma_t & \text{if } t \notin (t-\varepsilon, t+\delta) \\ \Gamma_t^* & \text{if } t \in (t-\varepsilon, t+\delta) \end{cases} \tag{4}$$

Indeed, it is easy to verify that each Γ_t^* preserves local injectivity so $\Gamma_t^* \in \Pi_{\mathcal{I}}$. Moreover, Γ^* is continuous. □

We now examine the case when linear interpolation results in a singleton, which causes a degeneracy in spaces of immersions. We give a maneuver to subvert this for paths.

Lemma 38 (Dodging Singletons) *Let $[\gamma_0], [\gamma_1] \in \Pi_{\mathcal{I}}$, and let $\Gamma : [0,1] \to \Pi_{\mathcal{C}}$ be a linear interpolation from γ_0 to γ_1. Let $t \in [0,1]$ such that $\Gamma(t)$ is a constant map, forcing $\Gamma(t) \notin \Pi_{\mathcal{I}}$. We can avoid this total degeneracy by rotating $\Gamma(t)$.*

Proof. Linear interpolation of γ_0 to γ_1 produces a singleton if the two equivalence classes of paths are colinear with reversed orientation. Hence, if Γ_t degenerates to a constant map, there exists sufficiently small $\epsilon > 0$ to continuously rotate $\Gamma(t-\epsilon)$ by π without forcing $d_{FP}(\Gamma(t), \gamma_1) > d_{FP}(\gamma_0, \gamma_1)$. Thereby reversing the orientation of $\Gamma(t+\epsilon)$, and avoiding the constant map for any $\gamma_t \in \Gamma_t$. See Figure 5 for an example. □

We now consider the case of backtracking during linear interpolation, which violates local injectivity. We introduce a maneuver to solve this potential degeneracy in spaces of immersions.

Lemma 39 (The Q-Tip Maneuver) *Let $[\gamma_0], [\gamma_1] \in \Pi_{\mathcal{I}}$, and let $\Gamma : [0,1] \to \Pi_{\mathcal{C}}$ be a linear interpolation from γ_0 to γ_1. Let $t \in [0,1]$ such that $\Gamma(t)$ creates backtracking for some $\Gamma(t)$. Inflating a ball about the critical backtracking point corrects this violation of injectivity.*

Proof. In the scenario of a backtracking event, local injectivity is only violated at the exact critical point $\Gamma_t(x)$ for $x \in [0,1]$ where backtracking occurs. For sufficiently small $\varepsilon, \delta > 0$, continuously inflate a ball of radius δ about $\Gamma_{t-\varepsilon}(x)$ such that $d_{FP}([\Gamma t - \epsilon], [\gamma_1])$ remains fixed, creating the path Γ_t^* with a ball replacing the critical point, so that $\Gamma_t^* \in \widetilde{\Pi_{\mathcal{I}}}$. Then, replace any backtracking Γ_t with the corresponding Γ_t^*. For every $t \in [0,1]$ it holds that $\Gamma_t \in \Pi_{\mathcal{I}}$, and by the continuity of the inflation, Γ remains continuous. For an example of this maneuver, see Figure 6b. □

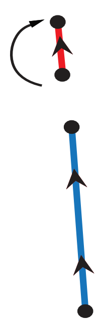
B.3 Balls of Path Embeddings in Greater Detail

In what follows, we elaborate on counterexamples for the path-connectivity of balls in $\Pi_{\mathcal{E}}$ and $\mathcal{G}_{\mathcal{E}}$. We begin with a counterexample for path embeddings in $\mathbb{R}^2$.

We continue with a brief description of counterexamples for the path-connectivity of embedded paths in $\mathbb{R}^3$.

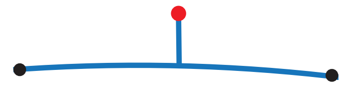
(a) Paths with reversed orientation

(b) Interpolate

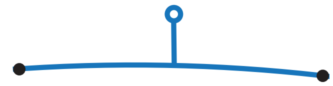


(c) Rotate when sufficiently close

Figure 5: For colinear paths with opposing orientation, rotating by π avoids degenerating to the constant map, keeping Γ in $\Pi_{\mathcal{I}}$. Moreover, rotation with sufficiently small Fréchet distance maintains the path-connectivity of balls.



(a) Example path with backtracking



(b) Inflate the critical backtracking point

Figure 6: Reconcile forced backtracking along a path by inflating a ball about the critical backtracking point, thereby maintaining local injectivity.

Lemma 40 (3d Balls in $\Pi_{\mathcal{E}}$) *If $n = 3$, metric balls in the space $(\Pi_{\mathcal{E}}, d_{FP})$ of embedded paths in $\mathbb{R}^n$ are not path-connected.*

Proof. Metric balls are not in general path-connected in three dimensions. For a simple counterexample, suppose γ_0 comprises a loop in $\mathbb{R}^3$, where a segment crossed on top of itself, avoiding self-intersection by some small distance δ, with long tails at either end of the crossing of length 2δ. Suppose also that γ_1 comprises the mirror image of γ_0. Then, $d_{FP} = \delta$, but it is not possible to construct a path from γ_0 to γ_1 without increasing the Fréchet distance between the two, since γ_0 must conduct a self-crossing, which increases the Fréchet distance by at least 2δ. Again, see Figure 3 $\square$

We conclude with additional details demonstrating the path-connectivity of balls for embedded paths in $\mathbb{R}^4$ or higher.

Lemma 41 (Balls in $(\Pi_{\mathcal{E}}, d_{FP})$, $n \geq 4$) *If $n \geq 4$, balls in the metric space of embedded paths $(\Pi_{\mathcal{E}}, d_{FP})$ in $\mathbb{R}^n$ are path-connected.*

Proof. Let $[\gamma_0], [\gamma_1], [\gamma_2] \in \Pi_{\mathcal{E}}$ in the ambient space $\mathbb{R}^n$, for $n \geq 4$. Let $\delta > 0$, and $\mathbb{B} := \mathbb{B}_{d_{FG}}([\gamma_0], \delta) \subset \Pi_{\mathcal{E}}$. Since all topological knots are represented equivalently in only three dimensions, without loss of generality, we consider the projections of every $\gamma_0 \in [\gamma_0], \gamma_1 \in [\gamma_1]$, and $\gamma_2 \in [\gamma_2]$ in $\mathbb{R}^3$. Construct a continuous $\Gamma : [0,1] \to \Pi_{\mathcal{E}}$ by the linear interpolation from $\Gamma(0) = \gamma_1$ to $\Gamma(1) = \gamma_2$. By the rectifiability of the embeddings γ_1 and γ_2, the interpolation must reduce $d_{FP}(\gamma_1, \gamma_2)$ by some $\epsilon > 0$ before a self-crossing is required in the image of Γ_t at some $t \in [0,1]$.

At t, conduct a self-crossing by perturbing Γ_t in the fourth dimension by no more than $\epsilon/2$. This increases

$d_{FP}(\Gamma_t, \gamma_2)$ by no more than $\epsilon/2$. Hence, $d_{FP}(\Gamma_t, \gamma_2)$ is either strictly decreasing as $t \to 1$, or necessarily satisfies $d_{FP}(\Gamma_t, \gamma_2) \leq \delta - \epsilon/2$ for $\epsilon > 0$. This is to say, for all $t \in [0, 1]$, $d_{FP}(\Gamma_t, \gamma_2) \leq \delta$, and $\Gamma_t \in \mathbb{B}$. Hence, metric balls in the space are path-connected. $\square$