Drawing Reeb Graphs

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1 Introduction

Reeb Graphs. Reeb graphs have become an important tool in computational topology for the purpose of visualizing continuous functions on complex spaces as a simplified discrete structure. Essentially, these graphs capture how level sets of a function evolve and behave in a topological space; the components of each level set become a vertex, and edges connect vertices on adjacent level sets that are connected.[1] To define the Reeb graph more precisely, let $\mathcal{M}$ be a compact, connected, and orientable manifold of dimension n, and let $f : \mathcal{M} \to \mathbb{R}$ be a smooth function. Define an equivalence relation on the points of M by setting $y \sim y'$ if $y, y' \in g^{-1}(a)$, and y and y' are in the same path component of $g^{-1}(a)$. Then the Reeb graph is the resulting quotient space $G_f = M/\sim$. Originally introduced in [11], recent work on computing Reeb graph efficiently [7, 10] has led to their increasing use in areas such as graphics and shape analysis; see [2, 13] for recent surveys on some of these applications, particularly in graphics and visualization.

Despite how prevalent the use of Reeb graphs is, surprisingly little work has been done on generating nice drawings of these structures. The only work in this area we are aware of considers book embeddings of these graphs [9]; while of combinatorial interest, the algorithms seem less practical for easy viewing of larger Reeb graphs. So, this leads to a very natural question, both theoretical and practical: how difficult is it to draw Reeb graphs?

[1] We direct the reader to the poster for illustrations of Reeb graph and other concepts throughout this abstract.

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Given the structural properties of Reeb graphs, there is an obvious connection to level drawings of graphs, since Reeb functions are simply real-valued labels on the vertices.

Connection to Layered Graph Drawing. *Layered graph drawing*, also known as *hierarchical* or *Sugiyama-style* graph drawing is a type of graph drawing in which the vertices of a directed graph are drawn in horizontal rows, or *layers*, and in which edges are y-monotone curves [1, 3, 12]; depending on the application, vertices may or may not be pre-assigned to levels. As with many drawing styles, layered drawings tend to be more readable when the number of crossings is low; hence a significant amount of effort has been directed towards crossing minimization in layered drawings of graphs. When a drawing without any crossings is possible, the graph is called *upward planar* or *level planar*. While it is possible to test whether a given directed acyclic graph with a single source and a single sink admits an upward planar drawing [5], the problem becomes NP-complete when there can be multiple sources or sinks [6]. When the layers are preassigned, the problem is easier, and testing whether a graph admits a *level planar* embedding is possible in linear time [8]. However, when a planar embedding is not possible, the problem of minimizing crossings is still of interest, and this problem is NP-complete, even when there are only two levels [4].

In Reeb graphs, vertices are preassigned to layers, since the y-coordinate carries information that should not be lost.

Challenge. Reeb graphs may be seen as a special case of layered graph drawing with preassigned layers; hence, existing algorithms can be used. Unfortunately, because the problem is NP-hard, existing approaches are mostly heuristic in nature. On the other hand, Reeb graphs have specific structural properties, which may make drawing them easier than the more general layered graph drawing problem. For example, the hardness proof by Garey and Johnson [4] crucially uses high-degree vertices, while Reeb graphs of *generic* (i.e. tame and constructible) manifolds only have vertices of degree 1 or 3. In addition, Reeb graphs have a graph-theoretical *genus* that is directly related to the topological genus of the underlying manifold, and thus it is of interest to design algorithms that can exploit a small number of (undirected) cycles in the graph.

Contribution. We present the following results.

– The problem of deciding whether a given (generic) Reeb graph admits a drawing with at most k crossings, is NP-complete. Our proof uses only vertices of constant degree, but does require a large number of cycles.
– We conjecture that the problem of drawing Reeb graphs may be fixed-parameter tractable (FPT) in the number of (undirected) cycles in the graph.
– As a first step towards establishing our conjecture, we show that when the graph *is* a cycle, we can find a crossing-minimal embedding in polynomial time.

References

1. Bastert, O., Matuszewski, C.: Layered drawings of digraphs. In: Kaufmann, M., Wagner, D. (eds.) Drawing Graphs. LNCS, vol. 2025, pp. 87–120. Springer, Heidelberg (2001). https://doi.org/10.1007/3-540-44969-8_5
2. Biasotti, S., Giorgi, D., Spagnuolo, M., Falcidieno, B.: Reeb graphs for shape analysis and applications. Theoret. Comput. Sci. Comput. Algebraic Geom. Appl. **392**(13), 5–22 (2008)
3. Di Battista, G., Eades, P., Tamassia, R., Tollis, I.G.: Layered Drawings of Digraphs, pp. 265–302. Prentice Hall, Hoboken (1998)
4. Garey, M.R., Johnson, D.S.: Crossing number is NP-complete. SIAM J. Algebraic Discre. Methods **4**(3), 312–316 (1983)
5. Garg, A., Tamassia, R.: Upward planarity testing. Order **12**(2), 109–133 (1995)
6. Garg, A., Tamassia, R.: On the computational complexity of upward and rectilinear planarity testing. SIAM J. Comput. **31**(2), 601–625 (2001)
7. Harvey, W., Wang, Y., Wenger, R.: A randomized $O(m \log m)$ time algorithm for computing Reeb graphs of arbitrary simplicial complexes. In: Proceedings of the 2010 Annual Symposium on Computational Geometry, SoCG '10, pp. 267–276, New York, NY, USA. ACM (2010)
8. Jünger, M., Leipert, S.: Level planar embedding in linear time. In: Kratochvíyl, J. (ed.) GD 1999. LNCS, vol. 1731, pp. 72–81. Springer, Heidelberg (1999). https://doi.org/10.1007/3-540-46648-7_7
9. Kurlin, V.: Book embeddings of Reeb graphs (2013)
10. Parsa, S.: A deterministic $O(m \log m)$ time algorithm for the Reeb graph. In: Proceedings of the 28th Annual ACM Symposium on Computational Geometry, SoCG '12. ACM (2012)
11. Reeb, G.: Sur les points singuliers d'une forme de pfaff completement integrable ou d'une fonction numerique [on the singular points of a completely integrable pfaff form or of a numerical function]. Comptes Rendus Acad. Sci. Paris **222**, 847–849 (1946)
12. Sugiyama, K., Tagawa, S., Toda, M.: Methods for visual understanding of hierarchical system structures. IEEE Trans. Syst. Man Cybernet. SMC **11**(2), 109–125 (1981)
13. Yan, L., et al.: Scalar field comparison with topological descriptors: properties and applications for scientific visualization (2021)