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Optimal Poisson kernel regularity for elliptic operators with Hölder continuous coefficients in vanishing chord-arc domains ☆

S. Bortz [a], T. Toro [b,c], Z. Zhao [d,*]

[a] Department of Mathematics, University of Alabama, Tuscaloosa, AL 35487, USA
[b] MSRI / Simons Laufer Mathematical Sciences Institute (SLMath), Berkeley, CA 94720-5070, USA
[c] Department of Mathematics, University of Washington, Seattle, WA 98195, USA
[d] Department of Mathematics, University of Chicago, Chicago, IL 60637, USA

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We show that if Ω is a vanishing chord-arc domain and L is a divergence-form elliptic operator with Hölder-continuous coefficient matrix, then $\log k_L \in VMO$, where k_L is the elliptic Poisson kernel for L in the domain Ω. This extends the previous work of Kenig and Toro in the case of the Laplacian.

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* Corresponding author.
E-mail addresses: sbortz@ua.edu (S. Bortz), toro@uw.edu (T. Toro), zhaozh@uchicago.edu (Z. Zhao).

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Contents

1. Introduction

In this article we study quantitative, asymptotic regularity of the (elliptic-) Poisson kernel for second order divergence form elliptic operators of the form $L = -\operatorname{div} A(X)\nabla$ in (bounded) rough domains, where the coefficient matrix is assumed to be Hölder continuous. This extends the work of Kenig and Toro [33] from the case of $L = -\Delta$, the Laplacian, to this natural class of variable coefficient operators. One may wish to interpret the results here as asymptotic optimality of the solution map for the linear operator L, with some extra consideration. Indeed, the solvability of the L^p-Dirichlet problem, with accompanying non-tangential estimates, is equivalent to the Poisson kernel satisfying a $L^{p'}$-reverse Hölder condition ($p' = p/(p-1)$). Our result here is equivalent to this condition being satisfied for all $p > 1$ and that for fixed p the constant in the reverse Hölder inequality tends to the optimal value, 1, when the balls shrink. In fact, the optimality (in the limit) of this constant for any fixed $p > 1$ implies the optimality for all $p > 1$ and $\log k \in VMO$ (see [32] for a detailed discussion[1]). Here our geometric assumptions on the domain are optimal [35,6], that is, the domains are chord arc with vanishing constant (see Definitions 2.28 and 2.30). These domains can be described as having asymptotic flatness (in the sense of Reifenberg [43]) coupled with surface measure which behaves asymptotically like Lebesgue measure.

Throughout, the ambient space is $\mathbb{R}^{n+1}$, $n \geq 2$ and we often make the identification $\mathbb{R}^{n+1} = \{(x,t) \in \mathbb{R}^n \times \mathbb{R}\}$. We work with divergence form elliptic second order differential equations of the form $L = -\operatorname{div} A(X)\nabla$, where the real, $(n+1)\times(n+1)$ matrix-valued function A satisfies the Λ-ellipticity condition for some $\Lambda \geq 1$, that is, $\|A\|_{L^\infty} \leq \Lambda$ and for almost every $X \in \mathbb{R}^{n+1}$

$$\Lambda^{-1}|\xi|^2 \leq \langle A(X)\xi, \xi\rangle, \quad \forall \xi \in \mathbb{R}^{n+1}.$$

Our main result is the following.

[1] This remarkable theory of 'self-improvement' comes from the study of quasiconformal mappings [23,28].

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded vanishing chord arc domain (see Definitions 2.28 and 2.30) and $L = -\operatorname{div} A(X)\nabla$ be an elliptic operator, with (real, Λ-elliptic) coefficients satisfying the Hölder condition*

$$|A(X) - A(Y)| \leq C_A |X - Y|^{\alpha}, \quad \forall X, Y \in \mathbb{R}^{n+1} \tag{1.2}$$

for some $C_A > 0$ and $\alpha \in (0, 1]$. Then $\log k \in VMO$ (see Definition 2.33), where k is the (elliptic-)Poisson kernel for L on the domain Ω, that is, $k := \frac{d\omega^{X_0}}{d\sigma}$ for some $X_0 \in \Omega$. Here ω^{X_0} is the elliptic measure for L with pole at X_0, and $\sigma = \mathcal{H}^n|_{\partial\Omega}$ is the surface measure for Ω.

Let us lay out the structure of the proof of Theorem 1.1. The restriction to real coefficients is necessary to define the elliptic measure (via the maximum principle and the Riesz representation theorem) as the solution map for the Dirichlet problem for L on the domain Ω. (Note that chord arc domains are Wiener regular and therefore the solution to the L-Dirichlet problem for some $f \in C_c(\partial\Omega)$ is $u_f(X_0) = \int_{\partial\Omega} f(y)\, d\omega^{X_0}(y)$.) On the other hand, some of the solvability results which serve as the starting point for our analysis rely on (complex) analytic perturbation theory [2–4]. Indeed, our study begins with operators on the upper half space, where there is much known for both transversally independent complex L^∞ perturbations of transversally independent operators with constant coefficients and operators which are a Dahlberg-Fefferman-Kenig-Pipher [15,22] type[2] ('transversal') perturbations. For the former we prefer to cite the treatment in [3] and for the latter [1]. These perturbations are maintained under pull-back on small Lipschitz graph domains and due to the Hölder condition we may view our operator (even after pull-back) as a two-fold perturbation of a constant coefficient matrix. Thus, to conclude Theorem 1.1, we employ 'good' approximation schemes developed in [18,44] and [33,45], whereby one approximates a vanishing chord arc domain by domains with small Lipschitz constant and makes the delicate estimates required to show $\log k \in VMO$. Here the former approximation in [18,44] allows us to establish rough 'A_∞ estimates' and the later approximation in [33,45] allows us to establish the refined, asymptotic estimates of Theorem 1.1 (the 'rough' A_∞ estimates are needed to control some errors).

This paper here brings together tools from partial differential equations, harmonic analysis and geometric measure theory developed over the last 40 years. We attempt (perhaps in vain) to give a reasonable account of the relevant results to the current work. For the harmonic measure, in 1976, Dahlberg [14] showed that in a Lipschitz domain the Poisson kernel satisfies an L^2-reverse Hölder condition, which, as we mentioned above, implies the L^2 solvability of the Dirichlet problem. This sparked a deep interest in the study of elliptic operators in rough sets that has persisted for decades. In 1982, Jerison and Kenig [30] showed that in a bounded C^1 domain $\log k \in VMO$. In 1997, Kenig and

[2] The perturbations are a quantitative, 'averaged' refinement of those in [21].

Toro [33] extended the work of Jerison and Kenig to vanishing chord arc domains, that is, Theorem 1.1 with $L = -\Delta$, by using a version of the Semmes decomposition [45].

In this context, for the variable coefficient case there is a good perturbative theory, see [20,22,39]. One can extrapolate optimal Poisson kernel regularity ($\log k \in VMO$) from one operator to another, whenever the discrepancy between the operators is a vanishing perturbation of Dahlberg-Fefferman-Kenig-Pipher type. On the other hand, aside from constant coefficient operators and their (vanishing) perturbations, there appears to be a lack of understanding of the properties that characterize an operator for which $\log k \in VMO$. This paper provides an important example of a natural class of operators for which this is the case. It also addresses a gap in [40]. There the authors used the fact that for a uniformly elliptic operator with Hölder coefficients on Lipschitz domain with small Lipschitz constant $\log k$ has small BMO norm. This result had not been established. In fact attempts to fix this gap by standard methods were unsuccessful. A completely new idea is required to prove this fact, and its introduction is one of the major original contributions of the current paper. There were also small gaps and errors in some 'localization estimates' in [40], so we carefully reprove these results. For the most part the techniques in these 'localization estimates' follow [40,33], but the proof of Theorem 1.1 is completely different and requires a number of new ingredients.

As mentioned above, we leverage powerful, refined theorems in the study of elliptic boundary value problems. We are particularly reliant on the theory built from layer potentials and the operational calculus of first order Dirac operators associated to divergence form elliptic operators with variable coefficients. The modern treatment of these objects was shaped by Auscher, Axelsson, Hofmann, McIntosh and many others [2–5]. We refer the reader to [2] for a relatively comprehensive history, but we remark that these works grew out of the testing conditions ('T1/Tb theory') for singular integrals and Littlewood-Paley type operators [13,12,17,19,36], and the generalizations of this theory. Perhaps the most notable such generalization led to the resolution of the Kato conjecture [5] and served as the basis for the results in [2–4]. These works allow us to treat the L^∞-perturbation, while we use the results in [1] to handle the Dahlberg-Fefferman-Kenig-Pipher type perturbation (the key is that we can locally write our operator as a two-fold perturbation).

Though our results are stated for bounded domains, with suitable modifications analogous local and global results hold for unbounded domains. For instance, an analogue of Theorem 1.1 holds for unbounded domains, if we replace VMO with VMO_{loc}. One can not conclude $\log k_L$ with finite pole is in the space VMO space when the domain has unbounded boundary. Indeed, $\log k_L^X$ is not in VMO, when Ω is the upper half-space, $L = -\Delta$, and $X \neq \infty$. We only treat bounded domains here to simplify the exposition, but we refer interested readers to Section 6.1 in [9], where the necessary local to global argument was presented.

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2. Preliminaries

First we introduce notation that will be standard throughout. For notation specific to chord arc domains and their variants with small constants, see the next subsections (Sections 2.2 and 2.3). Throughout the paper, by allowable constants we always mean the dimension $n \geq 2$, the ellipticity constant $\Lambda \geq 1$ and the Hölder constants $C_A > 0$ and $\alpha \in (0, 1]$.

2.1. Notation

- Given a domain $\Omega \subset \mathbb{R}^{n+1}$ with boundary $\partial\Omega$, for $x \in \partial\Omega$ and $r \in (0, \operatorname{diam} \partial\Omega)$ we let $\Delta(x, r) := B(x, r) \cap \partial\Omega$ denote the **surface ball** of radius r centered at x. We make clear which surface measure we are using any time there is possible ambiguity (e.g. when dealing with multiple domains simultaneously).
- Given $x = (x_1, \dots x_{n+1}) \in \mathbb{R}^{n+1}$ (or $x \in \mathbb{R}^n$ resp.) we define $|x|_\infty = \sup_i\{|x_i|\}$ to be the ℓ^∞ norm of x. Similarly, we let $|x| := |x|_2$ be the standard Euclidean (ℓ^2) distance.
- When working with the upper half space ($\mathbb{R}^{n+1}_+ := \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : t > 0\}$) we use the following notation. Let $y \in \mathbb{R}^n = \mathbb{R}^n \times \{0\}$ and $r > 0$ then we define:
  - The cube $Q(y, r) := \{x \in \mathbb{R}^n \times \{0\} : |x - y|_\infty < r\}$, with side length $2r$ and the (surface) ball $\Delta(y, r) := \{x \in \mathbb{R}^n \times \{0\} : |x - y| < r\}$.
  - Given an n dimensional cube $Q = Q(y, r)$ we let $\ell(Q) := 2r$ be the side length of the cube.
  - Given an n-dimensional cube Q we let R_Q be the Carleson box relative to Q, that is, $R_Q = Q \times (0, \ell(Q))$.
  - The Whitney regions $W(y, r) = \Delta(y, r/2) \times (r/2, 3r/2)$ and
$$D(y, r) := B(y, 10r) \cap \{(x, t) : x \in \mathbb{R}^n, t > r/2\}.$$
- Given a (real) divergence form elliptic operator $L = -\operatorname{div} A\nabla$ we define its transpose (and, in this case, adjoint) $L^T := -\operatorname{div} A^T\nabla$, where A^T is the transpose of A, that is $(A^T)_{i,j} = (A)_{j,i}$.

Definition 2.1 *(Lipschitz domains).* We say a domain (connected open set) $\Omega \subset \mathbb{R}^{n+1}$ is a γ-**Lipschitz domain** if for every $x \in \partial\Omega$ there exists $r > 0$ and an isometric coordinate system with origin $x = \mathcal{O}$ such that

$$\{Y \in \mathbb{R}^{n+1} : |Y - x|_\infty < r\} \cap \Omega = \{Y \in \mathbb{R}^{n+1} : |Y - x|_\infty < r\} \cap \{(y, t) : y \in \mathbb{R}^n, t > \varphi(y)\} \tag{2.2}$$

for some Lipschitz function $\varphi : \mathbb{R}^n \to \mathbb{R}$ with $\varphi(\mathcal{O}) = \mathcal{O}$ and $\|\nabla\varphi\|_\infty \leq \gamma$. We say a domain is a Lipschitz domain if it is a γ-Lipschitz domain for some $\gamma \geq 0$. We call a domain $\Omega \subset \mathbb{R}^{n+1}$ of the form

$$\Omega := \{(y,t) : y \in \mathbb{R}^n, t > \varphi(y)\}$$

for some Lipschitz function $\varphi : \mathbb{R}^n \to \mathbb{R}$ with $\|\nabla\varphi\|_\infty < \infty$ a Lipschitz graph domain (and if $\|\nabla\varphi\|_\infty \le \gamma$, a γ-Lipschitz graph domain).

Remark 2.3. The number of 'charts' needed to cover the boundary in the definition of Lipschitz domain is often important, but if $\partial\Omega$ is bounded the compactness of the boundary ensures that only a finite number of charts is required. If Ω is a Lipschitz graph domain only one chart is needed. When we work with Lipschitz domains the number of charts will always be *uniformly* bounded.

Definition 2.4 *(FKP-Carleson Norm).* Given any matrix $P = P(x,t)$ defined on $\mathbb{R}^{n+1}_+ := \{(x,t) : x \in \mathbb{R}^n, t > 0\}$ we define the Carleson norm of P as

$$\|P\|_{\mathcal{C}} := \sup_Q \left(\frac{1}{|Q|} \iint_{R_Q} \|P\|^2_{L^\infty(W(x,t))}(x,t)\, \frac{dx\,dt}{t} \right)^{1/2},$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$, R_Q is the Carleson box $Q \times (0, \ell(Q))$ and we recall $W(x,t) = \Delta(x, t/2) \times (t/2, 3t/2)$.

Definition 2.5 *(Nontangential Maximal function).* Given any locally L^2-integrable function $F : \mathbb{R}^{n+1}_+ \to \mathbb{R}$ we define the (L^2-modified) non-tangential maximal function $\widetilde{N}_* F : \mathbb{R}^n \to \mathbb{R}$ as

$$\widetilde{N}_* F(x) := \sup_{t>0} \left(\fint\!\!\fint_{W(x,t)} |F(y,s)|^2 \, dy\, ds \right)^{1/2}. \tag{2.6}$$

For $p > 1$ we also define the L^p-modified non-tangential maximal function $\widetilde{N}^p_* F : \mathbb{R}^n \to \mathbb{R}$ as

$$\widetilde{N}^p_*(F(x)) := \sup_{t>0} \left(\fint\!\!\fint_{W(x,t)} |F(y,s)|^p \, dy\, ds \right)^{1/p}.$$

Remark 2.7. We use the notation $\widetilde{N}_* F$ to distinguish it with the standard non-tangential maximal function defined using L^∞ norm.

2.2. PDE estimates in chord arc domains

In this subsection we define chord arc domains and state without proof some well-known results about solutions to elliptic operators as well as elliptic measures on such domains. These two non-tangential maximal functions are equivalent if Moser's estimate holds.

Chord arc domains are domains with scale-invariant connectivity (Harnack chains), interior and exterior openness (corkscrews) and whose boundaries are quantitatively "n-dimensional" (Ahlfors regular boundary).

Definition 2.8 *(Two-sided Corkscrew condition [29]).* We say a domain (an open and connected set) $\Omega \subset \mathbb{R}^{n+1}$ satisfies the two-sided corkscrew condition if there exists a uniform constant $M > 2$ such that for all $x \in \partial\Omega$ and $r \in (0, \operatorname{diam} \partial\Omega)$ there exists $X_1, X_2 \in \mathbb{R}^{n+1}$ such that

$$B(X_1, r/M) \subset B(x, r) \cap \Omega, \quad B(X_2, r/M) \subset B(x, r) \setminus \overline{\Omega}.$$

In the sequel, we write $\mathcal{A}(x, r) := X_1$, for the interior corkscrew point for x at scale r.

Definition 2.9 *(Harnack chain condition [29]).* We say a domain $\Omega \subset \mathbb{R}^{n+1}$ satisfies the Harnack chain condition if there exists a uniform constant $M \geq 2$ such that if $X_1, X_2 \in \Omega$ with $\operatorname{dist}(X_i, \partial\Omega) > \epsilon > 0$ and $|X_1 - X_2| < 2^k\epsilon$ then there exists a 'chain' of open balls $B_1, \dots, B_N$ with $N < Mk$ such that $X_1 \in B_1$, $X_2 \in B_N$, $B_j \cap B_{j+1} \neq \emptyset$ for $j = 1, \dots, N-1$ and $M^{-1} \operatorname{diam} B_j \leq \operatorname{dist}(B_j, \partial\Omega) \leq M \operatorname{diam} B_j$ for $j = 1, \dots, N$.

Definition 2.10 *(Ahlfors regular).* We say a set $E \subset \mathbb{R}^{n+1}$ is Ahlfors regular if E is closed and there exists a uniform constant C such that

$$C^{-1} r^n \leq H^n(B(x, r) \cap E) \leq C r^n, \quad \forall x \in E, \forall r \in (0, \operatorname{diam} E).$$

Definition 2.11 *(NTA and chord arc domains [29]).* We say a domain $\Omega \subset \mathbb{R}^{n+1}$ is and NTA domain if it satisfies the two-sided corkscrew condition and the Harnack chain condition. We say a domain $\Omega \subset \mathbb{R}^{n+1}$ is a **chord arc domain** if it is an NTA domain and $\partial\Omega$ is Ahlfors regular. We refer to the constants M and C in the definitions of the two-sided corkscrew condition, Harnack chain condition and the Ahlfors regularity condition as the 'chord arc constants'.

Remark 2.12. Every bounded (γ-)Lipschitz domain is a chord arc domain as are Lipschitz graph domains. See Remark 2.3.

Now we give several results on the behavior of solutions to real divergence form elliptic equations in chord arc domains. We note that while the original results are stated for harmonic functions their proofs carry over for real divergence form elliptic equations as the primary tools (e.g. the Harnack inequality and Hölder continuity at the boundary [24]) are still available and only introduce dependence on the ellipticity parameter. We also remark that these estimates are suitably local. For instance, if $\Omega \subset \mathbb{R}^{n+1}$ is a γ-

Lipschitz domain and we work 'well-inside'[3] a region as in (2.2) then the estimates on the boundary behavior of solutions depend on dimension, ellipticity and the parameter γ. We refer the reader to [11,31,29]. In the remainder of this section, $L = -\operatorname{div} A\nabla$ is a second order divergence form elliptic operator with real, Λ-elliptic coefficients.

Lemma 2.13 *(Carleson estimate [29]). Let $\Omega \subset \mathbb{R}^{n+1}$ be a chord arc domain, $x \in \partial\Omega$ and $10r \in (0, \operatorname{diam}\partial\Omega)$. If $Lu = 0$, $u \geq 0$ in $\Omega \cap B(x, 2r)$ and u vanishes continuously on $B(x, 2r) \cap \partial\Omega$ then*

$$u(Y) \leq Cu(\mathcal{A}(x, r)), \quad \forall Y \in B(x, r) \cap \Omega.$$

Here $C > 0$ depends on n, Λ and the chord arc constants for Ω.

Lemma 2.14 *(Hölder continuity at the boundary [29,24]). Let $\Omega \subset \mathbb{R}^{n+1}$ be a chord arc domain, $x \in \partial\Omega$ and $10r \in (0, \operatorname{diam}\partial\Omega)$. If $Lu = 0$, $u \geq 0$ in $\Omega \cap B(x, 4r)$ and u vanishes continuously on $B(x, 4r) \cap \partial\Omega$ then*

$$u(Y) \leq C\left(\frac{|Y - x|}{r}\right)^{\mu} \sup\{u(Z) : Z \in \partial B(x, 2r) \cap \Omega\} \leq C\left(\frac{|Y - x|}{r}\right)^{\mu} u(\mathcal{A}(x, r)),$$

for all $Y \in B(x, r) \cap \Omega$. Here $C > 0$ and $\mu \in (0, 1)$ depend on n, Λ and the chord arc constants for Ω.

A simple consequence of this lemma is the following, which is sometimes referred to as Bourgain's estimate.

Lemma 2.15 *(Bourgain's estimate [8,24]). Let $\Omega \subset \mathbb{R}^{n+1}$ be a chord arc domain, $x \in \partial\Omega$ and $10r \in (0, \operatorname{diam}\partial\Omega)$. Then*

$$\omega^{\mathcal{A}(x,r)}(B(x, r)) \gtrsim 1,$$

where the implicit constants depend on n, Λ and the chord arc constants for Ω. Here ω^X is the elliptic measure for the operator L on Ω with pole at X. In particular, by the Harnack inequality, for $x \in \partial\Omega$ and $10r < 10R \in (0, \operatorname{diam}\partial\Omega)$

$$\omega^{\mathcal{A}(x,R)}(B(x, r)) \gtrsim 1,$$

where the implicit constants depend on n, Λ and the chord arc constants for Ω and the ratio R/r.

Lemma 2.16 *(CFMS estimate [11]). Let $\Omega \subset \mathbb{R}^{n+1}$ be a chord arc domain, $x \in \partial\Omega$ and $10r \in (0, \operatorname{diam}\partial\Omega)$. If $X_0 \in \Omega \setminus B(x, 4r)$ then*

[3] Here this means, $\{Y : |Y - x|_\infty \ll r\} \cap \Omega\}$ in Definition 2.1.

$$\frac{\omega^{X_0}(B(x,r))}{r^{n-1}G(X_0,\mathcal{A}(x,r))} \approx 1,$$

where the implicit constants depend on n, Λ and the chord arc constants for Ω. Here ω^X is the elliptic measure for the operator L on Ω with pole at X and $G(X,Y)$ is the L-Green function for Ω with pole at Y.

Lemma 2.17 *(Doubling for elliptic measure). Let $\Omega \subset \mathbb{R}^{n+1}$ be a chord arc domain, $x \in \partial\Omega$ and $10r \in (0, \operatorname{diam} \partial\Omega)$. If $X_0 \in \Omega \setminus B(x,4r)$*

$$\omega^{X_0}(B(x,2r)) \le C\omega^{X_0}(B(x,r)),$$

where C depends on n, Λ and the chord arc constants for Ω.

Lemma 2.18 *(Comparison principle). Let $\Omega \subset \mathbb{R}^{n+1}$ be a chord arc domain, $x \in \partial\Omega$ and $10r \in (0, \operatorname{diam} \partial\Omega)$. If $Lu = Lv = 0$, $u, v \ge 0$ in $\Omega \cap B(x,2r)$, u and v are non-trivial functions which vanish continuously on $B(x,2r) \cap \partial\Omega$ then*

$$\frac{u(X)}{v(X)} \approx \frac{u(\mathcal{A}(x,r))}{v(\mathcal{A}(x,r))}, \quad \forall X \in B(x,r) \cap \Omega,$$

where the implicit constants depend on n, Λ and the chord arc constants for Ω.

Lemma 2.19 *(Quotients of non-negative solutions). Let $\Omega \subset \mathbb{R}^{n+1}$ be a chord arc domain, $x \in \partial\Omega$ and $r \in (0, \operatorname{diam} \partial\Omega/10)$. If $Lu = Lv = 0$, $u, v \ge 0$ in $\Omega \cap B(x,2r)$, and u and v vanish continuously on $B(x,2r) \cap \partial\Omega$, then u/v is Hölder continuous of order $\mu = \mu(n, \Lambda, \text{chord arc constants})$ in $\overline{B(x,r) \cap \Omega}$. In particular, $\lim_{Y \to y}(u/v)(Y)$ exists for $y \in \partial\Omega$[4] and, moreover,*

$$\left|\frac{u(X)}{v(X)} - \frac{u(\mathcal{A}(x,r))}{v(\mathcal{A}(x,r))}\right| \le C\left(\frac{|X-x|}{r}\right)^{\mu} \frac{u(\mathcal{A}(x,r))}{v(\mathcal{A}(x,r))}, \quad \forall X \in B(x,r) \cap \Omega, \tag{2.20}$$

where the constant $C > 0$ and $\mu \in (0,1)$ depend on n, Λ and the chord arc constants for Ω.

Next we define the kernel function. It can be more generally defined (for any $X_0, X_1 \in \Omega$), but we use it only in this specific manner.

Lemma 2.21 *(Kernel function). Let $\Omega \subset \mathbb{R}^{n+1}$ be a chord arc domain, $x \in \partial\Omega$ and $10r \in (0, \operatorname{diam} \partial\Omega)$. If $X_0 \in \Omega \setminus B(x,4r)$ and $X_1 \in B(x,2r) \setminus B(x,r)$ we define for $z \in \partial\Omega$*

[4] Here the limit is taken within Ω.

$$H(z) := H(X_0, X_1, z) := \frac{d\omega^{X_0}}{d\omega^{X_1}}(z).$$

The kernel function has the estimate

$$|H(z)/H(y) - 1| \leq C\zeta^{\mu}, \quad \forall z, y : |z-x|, |y-x| < \zeta r,$$

for all $\zeta \in (0, 1/2)$, *where* $C > 0$ *and* $\mu \in (0,1)$ *depend on* n, Λ *and the chord arc constants for* Ω.

Later, we will 'localize' the coefficients of our operator, the following lemma is useful in this regard.

Lemma 2.22. *Let* $\Omega_i \subset \mathbb{R}^{n+1}$, $i = 1, 2$ *be chord arc domains such that*

$$\Omega_1 \cap B(x, 10r) = \Omega_2 \cap B(x, 10r)$$

where $x \in \partial\Omega_i$ *and* $r \in (0, \operatorname{diam} \partial\Omega_i/20)$. *Suppose further that* $L_i = -\operatorname{div} A_i \nabla$ *are two divergence form* Λ*-elliptic operators with* $A_1 = A_2$ *on* $B(x, 10r)$. *Let* $\omega_i^{X_0}$ *be the* L_i*-elliptic measure for* Ω_i *with pole at* $X_0 \in \Omega_i \cap B(x, 8r) \setminus B(x, 4r)$.[5] *Then* $\omega_1^{X_0}|_{B(x,r)}$ *and* $\omega_2^{X_0}|_{B(x,r)}$ *are mutually absolutely continuous. In particular, if* $\omega_1^{X_0}|_{B(x,r)}$ *and* $\mathcal{H}^n|_{\partial\Omega_1 \cap B(x,r)}$ *are mutually absolutely continuous then so are* $\omega_2^{X_0}|_{B(x,r)}$ *and* $\mathcal{H}^n|_{\partial\Omega_2 \cap B(x,r)}$.

Moreover,

$$\frac{d\omega_1^{X_0}}{d\omega_2^{X_0}}(y) \approx 1, \quad \omega_2^{X_0} - a.e.\ y \in B(x, r/2) \cap \partial\Omega,$$

where the implicit constants depend on n, Λ *and the chord arc constants for* Ω_i, $i = 1, 2$.

Sketch of the proof. Let $y \in B(x,r)$ and $s \in (0,r)$. Then the CFMS estimate (Lemma 2.16) implies that

$$\frac{\omega_1^{X_0}(B(y,s))}{\omega_2^{X_0}(B(y,s))} \approx \frac{G_1(X_0, \mathcal{A}(y,s))}{G_2(X_0, \mathcal{A}(y,s))}, \tag{2.23}$$

where $G_i(X,Y)$ is the L_i-Green function for Ω_i. On the other hand, Bourgain's estimate (Lemma 2.15) and the CFMS estimate imply that

$$G_i(X_0, \mathcal{A}(x, r/50))r^{n-1} \approx \omega_i^{X_0}(B(x, r/50) \approx 1, \quad i = 1, 2.$$

By (2.23) and the comparison principle[6] (Lemma 2.18) then shows

[5] This means the statements on x, r an X_0 are to hold simultaneously for $i = 1, 2$.

[6] Here we need to view $G_i(X_0, Y)$ as a $L_i^T = -\operatorname{div} A_i^T \nabla$ null solution away from X_0. This follows by the fact that if $G(X,Y)$ is the L-Green function then $G_T(X,Y) = G(Y,X)$ is the L^T-Green function. We then apply the comparison principle with L^T in place of L.

$$\frac{\omega_1^{X_0}(B(y,s))}{\omega_2^{X_0}(B(y,s))} \approx \frac{G_1(X_0,\mathcal{A}(y,s))}{G_1(X_0,\mathcal{A}(y,s))} \approx \frac{G_1(X_0,\mathcal{A}(x,r/50))}{G_2(X_0,\mathcal{A}(x,r/50))} \approx 1,$$

where the constants depend on n, Λ and the chord arc constants for Ω_i, $i=1,2$. From this estimate one can deduce all of the properties described in the lemma. $\square$

2.3. Chord arc domains with small constants

We study asymptotic flatness in the form of vanishing chord arc domains. To define these domains, we need to give a few more preliminary definitions.

Definition 2.24 *(Separation property).* Let $\Omega \subset \mathbb{R}^{n+1}$ be a domain. We say that Ω has the **separation property** if for each compact set $K \subset \mathbb{R}^{n+1}$ there exists $R>0$ such that for any $x_0 \in \partial\Omega \cap K$ and $r \in (0,R]$ there exists a vector $\vec{n} \in \mathbb{S}^n$ such that

$$\{X \in B(x_0,r) : \langle X - x_0, \vec{n}\rangle \geq r/4\} \subset \Omega$$

and

$$\{X \in B(x_0,r) : \langle X - x_0, \vec{n}\rangle \leq -r/4\} \subset \Omega^c.$$

Notice that $\vec{n}$ points 'inward'.

Remark 2.25. Provided that $\partial\Omega$ is δ-Reifenberg flat for δ sufficiently small, there is an equivalent definition of the separation property, see [34, Remark 1.1].

In the sequel, $D[\cdot;\cdot]$ will be used to denote the Hausdorff distance between two sets, that is, for $A, B \subset \mathbb{R}^{n+1}$

$$D[A;B] := \sup\{\mathrm{dist}(a,B) : a \in A\} + \sup\{\mathrm{dist}(b,A) : b \in B\}.$$

Definition 2.26 *(Reifenberg flatness).* Given a closed set $\Sigma \in \mathbb{R}^{n+1}$, $x_0 \in \Sigma$ and $r>0$, we define

$$\Theta(x_0,r) := \inf_P \tfrac{1}{r} D[\Sigma \cap B(x_0,r); P \cap B(x_0,r)],$$

where the infimum is taken over all n-planes P through x_0 and

$$\Theta(r) := \sup_{x_0 \in \Sigma} \Theta(x_0,r).$$

For $R>0$ and $\delta \in (0,\delta_n]$, where δ_n is sufficiently small depending only on the dimension (see Remark 2.27) we say Σ is (δ,R)-Reifenberg flat if

$$\sup_{r\in(0,R)} \Theta(r) < \delta.$$

Remark 2.27. The δ_n above is to ensure both that the definition is not vacuous and that if a domain has Reifenberg flat boundary and satisfies the separation property then it is, in fact, an NTA domain. See [33, Lemma 3.1] and Appendix A in [35].

Definition 2.28 *((δ, R)-chord arc domains).* Let $\Omega \subset \mathbb{R}^{n+1}$ be a domain. Given $\delta \in (0, \delta_n]$ and $R > 0$, we say Ω is a (δ, R)**-chord arc domain** if

(1) Ω has the separation property (with parameter R),
(2) $\partial\Omega$ is (δ, R)-Reifenberg flat and
(3) $\mathcal{H}^n(B(x,r) \cap \partial\Omega) \leq (1+\delta)\omega_n r^n$, for all $x \in \partial\Omega$ and $r \in (0, R]$.

Here ω_n is the volume of the n-dimensional unit ball in $\mathbb{R}^n$.

Remarks 2.29.

- Since Hausdorff measure is non-increasing under projections, the δ-Riefenberg flatness of the boundary in the above definition coupled with the upper bound on the surface measure gives the estimate

$$(1-\delta)\omega_n r^n \leq \mathcal{H}^n(B(x,r) \cap \partial\Omega) \leq (1+\delta)\omega_n r^n,$$

  for any $x \in \partial\Omega$ and $r \in (0, R]$. See [33, Remark 2.2].
- There is an equivalent definition of (δ, R)-chord arc domain, which is to replace the third assumption above by $\partial\Omega$ being Ahlfors regular (the Ahlfors regularity constant is not necessarily close to one) and the unit outer normal of Ω has BMO norm bounded above by δ. See for example [35, Definition 1.10]. These two definitions are equivalent (modulo constant) by [33, Theorem 2.1] and [34, Theorem 4.2] respectively. We choose the above definition in this paper for convenience.

Definition 2.30 *(Vanishing chord arc domains).* We say Ω is a **vanishing chord arc domain**

- Ω is a (δ, R)-chord arc domain for some $\delta \in (0, \delta_n]$ and $R > 0$,
- $\limsup_{r \to 0^+} \Theta(r) = 0$ (where Σ in the definition of Θ is $\partial\Omega$) and
- $\lim_{r \to 0^+} \sup_{x \in \partial\Omega} \frac{\mathcal{H}^n(B(x,r) \cap \partial\Omega)}{\omega_n r^n} = 1$.

Note that if Ω is a vanishing chord arc domain then for every $\delta > 0$ there exists $R_\delta > 0$ such that Ω is a (δ, R_δ)-chord arc domain.

To organize our arguments involving these types of domains, we introduce some notation. Given $x_0 \in \mathbb{R}^{n+1}$ and $\vec{n} \in S^n$ we let $P(x_0, \vec{n})$ be the plane through x_0 perpendicular to $\vec{n}$, that is,

$$P(x_0, \vec{n}) := \{X \in \mathbb{R}^{n+1} : \langle X - x_0, \vec{n} \rangle = 0\};$$

and additionally, for $r > 0$ and $\xi \in \mathbb{R}$ we define the shifted cylinder

$$\mathcal{C}(x_0, r, \vec{n}, \xi) := \Big\{ X \in \mathbb{R}^{n+1} : |(X - x_0) - \langle X - x_0, \vec{n}\rangle \vec{n}| \le \frac{r}{\sqrt{n+1}}, \\ |\langle X - x_0, \vec{n}\rangle - \xi| \le \frac{r}{\sqrt{n+1}} \Big\}, \quad (2.31)$$

and if $\xi = 0$, we just write $\mathcal{C}(x_0, r, \vec{n})$. Given $\delta > 0$, we also define the following truncated cylinders

$$\mathcal{C}_\delta^+(x_0, r, \vec{n}) := \Big\{ X \in \mathcal{C}(x_0, r, \vec{n}) : \langle X - x_0, \vec{n}\rangle \ge 2\delta r \Big\}$$

and

$$\mathcal{C}_\delta^-(x_0, r, \vec{n}) := \Big\{ X \in \mathcal{C}(x_0, r, \vec{n}) : \langle X - x_0, \vec{n}\rangle \ge -2\delta r \Big\}.$$

Notice that *both* $\mathcal{C}_\delta^+$ and $\mathcal{C}_\delta^-$ are 'upper' portions of the cylinder $\mathcal{C}$. We also define the 'strip'

$$\mathcal{S}_\delta(x_0, r, \vec{n}) := \Big\{ X \in \mathcal{C}(x_0, r, \vec{n}) : |\langle X - x_0, \vec{n}\rangle| \le 2\delta r \Big\}.$$

We may omit the vector $\vec{n}$ in the above notation when there is no confusion.

Now we connect these objects to our definition of (δ, R)-chord arc domain. Suppose $\Omega \subset \mathbb{R}^{n+1}$ is a (δ, R)-chord arc domain for some $\delta \in (0, \delta_n]$ and $R > 0$. Given $r \in (0, R]$ and $x_0 \in \partial\Omega$ we let $P_{x_0,r}$ be an n-plane such that

$$D[\partial\Omega \cap B(x_0, r); P_{x_0,r} \cap B(x_0, r)] \le \delta r.$$

By the separation property and choice of δ_n sufficiently small we can ensure that for a choice of normal vector to the plane $P_{x_0,r}$, which we label $\vec{n}_{x_0,r}$, we have

$$\mathcal{C}_\delta^+(x_0, r) := \mathcal{C}_\delta^+(x_0, r, \vec{n}_{x_0,r}) \subset \Omega$$

and

$$\mathcal{C}(x_0, r) \setminus C_\delta^-(x_0, r) \subset \Omega^c,$$

where $\mathcal{C}(x_0, r) := \mathcal{C}(x_0, r, \vec{n}_{x_0,r})$ and $C_\delta^-(x_0, r) := \mathcal{C}_\delta^-(x_0, r, \vec{n}_{x_0,r})$. (This means that $\vec{n}_{x_0,r}$ points *inwards*.) We also define

$$\widetilde{\Omega}(x_0, r, \xi) = \Omega \cap \mathcal{C}(x_0, r, \vec{n}_{x_0,r}, \xi) \quad (2.32)$$

for $\xi \in \mathbb{R}$ and drop ξ from the notation when $\xi = 0$. We then note the following inclusions

$$\mathcal{C}_\delta^+(x_0, r) \subset \widetilde{\Omega}(x_0, r) \subset \mathcal{C}_\delta^-(x_0, r)$$

and

$$\mathcal{C}(x_0, r) \cap \partial\Omega \subset \mathcal{S}_\delta(x_0, r),$$

where $\mathcal{S}_\delta(x_0, r) := \mathcal{S}_\delta(x_0, r, \vec{n}_{x_0,r})$. Note that, while $\widetilde{\Omega}(x_0, r)$ may not be an NTA or chord arc domain, the estimates established above hold in a smaller dilate of $\mathcal{C}(x_0, r)$ intersected with $\widetilde{\Omega}(x_0, r)$, when suitably interpreted (for instance, at points on the boundary of $\partial\Omega \cap \mathcal{C}(x_0, (1 - c_n\delta)r)$). See [33, Section 4], in particular the discussion following [33, Remark 4.2].

Finally we recall the definitions of BMO and VMO functions.

Definition 2.33 *(BMO and VMO).* Let μ be a Radon measure in $\mathbb{R}^n$. Then, for all $0 < r < \operatorname{diam}(\operatorname{supp}\mu)$ and all $f \in L^2(\mu)$ we define

$$\|f\|_*(x, r) = \sup_{0<s<r} \left(\fint_{B(x,s)} \left| f(y) - \fint_{B(x,s)} f(z)\, d\mu(z) \right|^2 d\mu(y) \right)^{\frac{1}{2}}. \tag{2.34}$$

We say $f \in BMO(\mu)$ if

$$\|f\|_{BMO(\mu)} := \sup_{0<r<\operatorname{diam}(\operatorname{supp}\mu)} \sup_{x\in\operatorname{supp}\mu} \|f\|_*(x, r) < +\infty.$$

We denote by VMO the closure of uniformly continuous functions on $\operatorname{supp}\mu$ in the BMO-norm. There is also a notion of $\mathrm{VMO}_{\mathrm{loc}}$; $f \in \mathrm{VMO}_{\mathrm{loc}}$ if for every compact set $K \subset \mathbb{R}^n$,

$$\lim_{r\to 0} \sup_{x\in\operatorname{supp}\mu\cap K} \left(\fint_{B(x,s)} \left| f(y) - \fint_{B(x,s)} f(z)\, d\mu(z) \right|^2 d\mu(y) \right)^{\frac{1}{2}} = 0. \tag{2.35}$$

Remark 2.36. In this paper we will work with BMO and VMO functions with respect to the surface measure $\mathcal{H}^n|_{\partial\Omega}$, where Ω is a domain with Ahlfors regular boundary.

3. Perturbations of constant coefficient operators and Poisson kernels

In this section, we restrict our attention to the upper half space $\mathbb{R}^{n+1}_+$, and study the Poisson kernels for elliptic operators that are *perturbations* of constant-coefficient operators. Let A_0 be a real, constant $(n + 1) \times (n + 1)$ matrix satisfying the Λ-ellipticity condition. We say a real $(n+1)\times(n+1)$ matrix satisfying the Λ-ellipticity condition is an **ϵ-perturbation of** A_0, if $A(x,t)$ can be decomposed as

$$A(x,t) = A_1(x) + P(x,t) \tag{3.1}$$

with

$$\|A_0 - A_1\|_{L^\infty} + \|P(x,t)\|_{\mathcal{C}} \le \epsilon, \tag{3.2}$$

see Definition 2.4 for the definition of the Carleson norm $\|\cdot\|_{\mathcal{C}}$.

The goal of this section is to prove the following result:

Theorem 3.3. *For any $\tilde{\beta} \in (0,1)$, there exist $\tilde{\delta} = \tilde{\delta}(\tilde{\beta}, n, \Lambda) > 0$ and $\epsilon = \epsilon(\tilde{\beta}, n, \Lambda)$ such that the following holds. Suppose $A = A(x,t)$ is a real matrix-valued function on $\mathbb{R}^{n+1}_+$ satisfying the Λ-ellipticity condition and moreover, A is an ϵ-perturbation of a constant-coefficient matrix A_0 satisfying the Λ-ellipticity condition. Then the Poisson kernel for $L = -\operatorname{div}(A\nabla)$ in the upper half space, denoted by k_A^X, satisfies*

$$1 \le \frac{\left(\fint_{\Delta(y,\delta' r)} \left[k_A^X(z)\right]^2 \, dz \right)^{1/2}}{\fint_{\Delta(y,\delta' r)} k_A^X(z)\, dz} < 1 + \tilde{\beta} \tag{3.4}$$

for all $y \in \mathbb{R}^n$, $r > 0$, $\delta' < \tilde{\delta}$ and $X \in D(y,r)$. We recall that

$$D(y,r) := B(y, 10r) \cap \left\{ (x,t) \in \mathbb{R}^{n+1}_+ : t > \frac{r}{2} \right\}$$

is the upper cap of the ball $B(y,10r)$.

We will refer to the quotient in (3.4) as the B_2 *constant of k_A^X on* $\Delta(y,\delta' r)$. For a constant-coefficient operator A_0, the Poisson kernel $k_{A_0}^{(0,1)}$ is smooth, so roughly speaking

$$\text{the } B_2 \text{ constant of } k_{A_0}^{(0,1)} \text{ on } \Delta(0,\delta) \ \to 1 \text{ as } \delta \to 0.$$

We expect the same to hold for perturbations of constant-coefficient operators. The above theorem says that, if a matrix A is a sufficiently small perturbation of a constant-coefficient matrix, then its B_2 constant is sufficiently close to 1 on small enough scale (in proportion to the distance of the pole to the boundary).

We first prove a simple estimate on the Poisson kernel of constant-coefficient operators.

Lemma 3.5 *(Non-degeneracy for constant-coefficient operators). There exists a constant $c_1 = c_1(n,\Lambda) > 0$ such that the following holds. If A_0 is a real, constant matrix satisfying the Λ-ellipticity condition, then the Poisson kernel for $L_0 = -\operatorname{div} A_0 \nabla$ in the upper half space, denoted by k_{A_0}, satisfies*

$$\|k^X_{A_0}\|_{L^1(\Delta(y,\delta r))} \geq c_1 \delta^n, \tag{3.6}$$

$$\|k^X_{A_0}\|_{L^2(\Delta(y,\delta r))} \geq c_1 \left(\frac{\delta}{r}\right)^{\frac{n}{2}}, \tag{3.7}$$

for all $y \in \mathbb{R}^n$, $r > 0$, $\delta \in (0,1]$ *and* $X \in D(y,r)$.

Proof. The second inequality follows from combining (3.6) and Hölder's inequality. Since the solutions to constant-coefficient operator are scale invariant, it suffices to show

$$\|k^X_{A_0}\|_{L^1(\Delta(0,\delta))} \geq c_1 \delta^n, \qquad \text{for every } X \in D(0,1). \tag{3.8}$$

We claim that it suffices to consider the case when A_0 is symmetric. If not, let

$$A_{0,s} := \frac{A_0 + A_0^T}{2}$$

be the symmetrization of A_0. Since A_0 is a constant coefficient matrix, the solution $u \in C^2$, and we have

$$\begin{aligned} \operatorname{div}(A_0 \nabla u) = \sum_{i,j} \partial_i (a_{ij} \partial_j u) &= \sum_{i,j} a_{ij}\, \partial_i \partial_j u \\ &= \frac{\sum_{i,j} a_{ij}\, \partial_i \partial_j u + \sum_{i,j} a_{ji}\, \partial_i \partial_j u}{2} = \operatorname{div}(A_{0,s} \nabla u). \end{aligned}$$

Assume the matrix A_0 is symmetric. By the decomposition of symmetric matrices and a change of variable formula, we get an explicit formula for $k^{(0,1)}_{A_0}$:

$$k^{(0,1)}_{A_0}(z) = c_n \det(A_0)^{-\frac{1}{2}} \frac{\left|\sqrt{A_0}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right|}{\left|\sqrt{A_0}^{-1} \begin{pmatrix} -z \\ 1 \end{pmatrix}\right|^{n+1}}, \tag{3.9}$$

where $c_n = \Gamma((n+1)/2)/\pi^{(n+1)/2}$ is the same constant in the Poisson kernel for the Laplacian, and $\sqrt{A_0}^{-1}$ is defined by the diagonalization decomposition of A_0 (the square root matrix and inverse matrix exist, since the eigenvalues of A_0 are all bounded from below by $1/\Lambda$). Then the lower bound (3.8) is obtained easily by (3.9) and the Harnack inequality. $\square$

Next we analyze how the Poisson kernels change under a perturbation of the coefficient matrices.

Proposition 3.10. *There exists* $\epsilon_0 = \epsilon_0(n, \Lambda)$ *such that the following holds for every* $\epsilon < \epsilon_0$. *If* $A = A(x,t)$ *is a real matrix valued function on* $\mathbb{R}^{n+1}_+$ *satisfying the* Λ*-ellipticity* *condition and moreover, A is an ϵ-perturbation of a real, constant-coefficient matrix A_0 satisfying the Λ-ellipticity condition, then*

$$\|k_A^{(x,t)} - k_{A_0}^{(x,t)}\|_{L^2(\mathbb{R}^n)} \le C\epsilon^{\eta'} t^{-n/2},$$

where $C = C(n,\Lambda)$ and $\eta' = \eta'(n,\Lambda) > 0$ and $k_A^{(x,t)}$ and $k_{A_0}^{(x,t)}$ (resp.) are the Poisson kernels for the operators $L = -\operatorname{div} A\nabla$ and $L_0 = -\operatorname{div} A_0\nabla$ (resp.) in the upper half space.

Remark 3.11. The above proposition could be amended, for instance, to include the case of perturbations of real, *t-independent, symmetric* coefficients, but the case of a perturbation of constant coefficients is sufficient for our purposes.

Proof. Without loss of generality assume $x = 0$. By the explicit formula in (3.9), it is easy to see that the kernel $k_{A_0}^{(0,1)}$ for the constant coefficient matrix A_0 has the form $\frac{c}{(1+a_1^2z_1^2+\cdots+a_{n-1}^2z_{n-1}^2)^{\frac{n+1}{2}}}$, and thus $k_{A_0}^{(0,1)} \in L^2(\mathbb{R}^n)$. Fix $t > 0$. We can similarly get that $k_{A_0}^{(0,t)} \in L^2(\mathbb{R}^n)$.

Our first goal is to show

$$\|k_{A_1}^{(0,t)} - k_{A_0}^{(0,t)}\|_{L^2(\mathbb{R}^n)} \le C\epsilon^{\eta'} t^{-n/2}, \tag{3.12}$$

where $A_1(x)$ is the t-independent part in the decomposition (3.1) of A.

To that end we use the L^2-duality:

$$\begin{aligned}\|k_{A_1}^{(0,t)} - k_{A_0}^{(0,t)}\|_{L^2(\mathbb{R}^n)} &= \sup_{\|f\|_{L^2(\mathbb{R}^n)}=1} \left| \int_{\mathbb{R}^n} \left(k_{A_1}^{(0,t)}(y) - k_{A_0}^{(0,t)}(y)\right) f(y)\, dy \right| \\ &= \sup_{\|f\|_{L^2(\mathbb{R}^n)}=1} \left|u_f^1(0,t) - u_f^0(0,t)\right|, \end{aligned} \tag{3.13}$$

where for each $f \in L^2(\mathbb{R}^n)$, we define

$$u_f^i(x,t) = \int_{\mathbb{R}^n} k_{A_i}^{(x,t)}(y) f(y) dy, \quad i = 0,1. \tag{3.14}$$

They are solutions to the Dirichlet problem for the operators $L_i = -\operatorname{div} A_i\nabla$ with data f. Note that when we consider (not necessarily symmetric) elliptic operators in an unbounded domain $\mathbb{R}_+^{n+1}$, there may be several weak solutions to the Dirichlet problem, depending on which function spaces we consider (see [7] for example). Throughout the paper, we will refer to the solution obtained as in (3.14) (i.e. by integrating the boundary data against the elliptic measure) as the *elliptic measure solution*. In particular when $0 \le f \in C_c^\infty(\mathbb{R}^n)$ the elliptic measure always satisfies $0 \le u_f(t,x) \le \sup |f|$.

From the work of Auscher, Axelsson and Hofmann [3, Theorem 1.1] on L^∞ perturbation result of the boundary value problem, the Dirichlet problem to $-\operatorname{div}(A_1(x)\nabla u) = 0$ with L^2 boundary data is well defined: for every $f \in L^2(\mathbb{R}^n)$ there exist solutions $\hat{u}_f^i$, $i = 0, 1$, such that

$$\|\widetilde{N}_* \hat{u}_f^1\|_{L^2(\mathbb{R}^n)}, \|\widetilde{N}_* \hat{u}_f^0\|_{L^2(\mathbb{R}^n)} \le C\|f\|_{L^2(\mathbb{R}^n)}; \tag{3.15}$$

moreover, we have

$$\|\widetilde{N}_*(\hat{u}_f^1 - \hat{u}_f^0)\|_{L^2(\mathbb{R}^n)} \le C\epsilon\|f\|_{L^2(\mathbb{R}^n)}, \tag{3.16}$$

provided that $\epsilon > 0$ in the statement of the theorem is small enough depending on n and Λ. We remark that this requires the use of (complex) analytic perturbations, but we restrict to a "real neighborhood", where the result is (trivially) also true. When the boundary data satisfy $0 \le f \in C_c^\infty(\mathbb{R}^n)$, these two solutions agree, i.e. $u_f^i = \hat{u}_f^i$. We defer the proof to Subsection A.5 of Appendix A.

With $(x, t) = (0, t)$ fixed, we define $W := W(0, t) = \Delta(0, t/2) \times (t/2, 3t/2)$ and for $\gamma \in (0, 1]$ we define $\gamma W = \Delta(0, \gamma t/2) \times (t - \frac{\gamma t}{2}, t + \frac{\gamma t}{2})$. We make the observation that for $\gamma \le 1/2$, $\gamma W \subset W(y, t)$[7] for all $y \in \Delta(x, t/4)$. It follows that for $\gamma \in (0, 1/2]$ and $y \in \Delta(x, t/4)$

$$\begin{aligned}\left(\fint\!\!\!\fint_{\gamma W} |u_f^i(Y)|^2\, dY\right)^{1/2} &\le C\gamma^{-(n+1)/2}\left(\fint\!\!\!\fint_{W(y,t)} |u_f^i(Y)|^2\, dY\right)^{1/2} \\ &\le C\gamma^{-(n+1)/2}\widetilde{N}_* u_f^i(y)\end{aligned}$$

for $i = 0, 1$. Averaging over $y \in \Delta(x, t/4)$ and using (3.15) we see

$$\begin{aligned}\left(\fint\!\!\!\fint_{\gamma W} |u_f^i(Y)|^2\, dY\right)^{1/2} &\le C\gamma^{-(n+1)/2} t^{-n/2}\|\widetilde{N}_* u_f^i\|_{L^2(\Delta(x,t/4))} \\ &\le C\gamma^{-(n+1)/2} t^{-n/2}\|f\|_{L^2(\mathbb{R}^n)}\end{aligned} \tag{3.17}$$

for $i = 0, 1$. Similarly by (3.16), we obtain that for $\gamma \in (0, 1/2]$

$$\left|\fint\!\!\!\fint_{\gamma W} \left(u_f^1(Y) - u_f^0(Y)\right)\, dY\right| \le C\epsilon\gamma^{-(n+1)/2} t^{-n/2}\|f\|_{L^2(\mathbb{R}^n)}. \tag{3.18}$$

[7] See Definition 2.4.

Now we recall that solutions to divergence-form (real-valued) elliptic operators satisfy interior Hölder regularity, by DeGiorgi-Nash-Moser theory [16,42,38]. Thus for $Y, Z \in (1/4)W$ and $i = 0, 1$

$$|u_f^i(Y) - u_f^i(Z)| \le C\left(\frac{|Y-Z|}{t}\right)^{\eta}\left(\fint\!\!\!\fint_{(1/2)W} |u_f^i|^2\right)^{1/2} \le C\left(\frac{|Y-Z|}{t}\right)^{\eta} t^{-n/2}\|f\|_{L^2(\mathbb{R}^n)}, \tag{3.19}$$

where we used (3.17) with $\gamma = 1/2$ in the second inequality, and the constants C and $\eta > 0$ depend only on n and Λ. In particular, if we let $Y = (0, t)$ and average over $Z \in \gamma W \subset (1/4)W$ we obtain

$$\left| u_f^i(0,t) - \fint\!\!\!\fint_{\gamma W} u_f^i(Z)\,dZ \right| \le \gamma^{\eta} t^{-n/2}\|f\|_{L^2(\mathbb{R}^n)} \tag{3.20}$$

for $i = 0, 1$.

Setting $\gamma = \epsilon^{\frac{2}{2\eta+n+1}}$ and then using the triangle inequality, (3.18) and (3.20) we obtain

$$|u_f^1(0,t) - u_f^0(0,t)| \le C(\gamma^{\eta} + \epsilon\gamma^{-(n+1)/2})t^{-n/2}\|f\|_{L^2(\mathbb{R}^n)} \le C\epsilon^{\eta'} t^{-n/2}\|f\|_{L^2(\mathbb{R}^n)}, \tag{3.21}$$

where $\eta' = \frac{2\eta}{2\eta+n+1}$. Then the claim (3.12) follows from combining (3.21) and the L^2 duality (3.13).

Next, we show

$$\|k_{A_1}^{(0,t)} - k_A^{(0,t)}\|_{L^2(\mathbb{R}^n)} \le C\epsilon^{\eta'} t^{-n/2}. \tag{3.22}$$

This will follow exactly as the proof of (3.12), provided we have the analogous estimates to (3.15) and (3.16). These are afforded by the work of Auscher and Axelsson [1] on the boundary value problems for perturbations of t-independent operators.[8] Let $f \in L^2(\mathbb{R}^n)$ and let u_f be the (unique) solution to the L^2-Dirichlet problem for the operator $L = -\operatorname{div} A\nabla$. With u_f^1 as before, we may use [1, Sections 9 and 10]

$$\|\widetilde{N}_* u_f\|_{L^2(\mathbb{R}^n)} \le C\|f\|_{L^2(\mathbb{R}^n)} \tag{3.23}$$

and

$$\|\widetilde{N}_*^{3/2}(u_f - u_f^1)\|_{L^2(\mathbb{R}^n)} \le C\epsilon\|f\|_{L^2(\mathbb{R}^n)}, \tag{3.24}$$

[8] In particular, we must use the representation in [1, Equation (42)] and the bounds established in [1, Theorem 9.2, Lemma 10.2]. Here it should be noted that the functions $\tilde{h}^{+}$ used in the expression [1, Equation (42)] are also in a perturbative regime with linear dependence on the FKP-Carleson norm, see the proof of [1, Corollary 9.5]. See Appendix A.

provided that ϵ is small depending on n and Λ. We discuss the derivation of these two estimates in detail in Appendix A.

Above we have L^p-averages with $p = 3/2$ in the definition of $\widetilde{N}_*^{3/2}$, but we obtain 'comparability' of L^2 and $L^{3/2}$ averages using Moser's local boundedness estimate for solutions since we work with real-valued elliptic operators. To be more precise, by denoting $\widetilde{W}(x,t) := \Delta(x,t) \times (t/4, t) \supset W(x,t)$, Moser type estimate gives

$$\sup_{Y \in W(x,t)} |u_f(Y)| \lesssim \left(\fint_{\widetilde{W}(x,2t)} |u_f|^2 \, dyds \right)^{1/2} \leq \widehat{N}_*(u_f)(x),$$

where $\widehat{N}_*(u_f)$ is the non-tangential maximal function defined as $\widetilde{N}_*(u_f)$ in (2.6), but with a fatter Whitney region $\widetilde{W}(x,t)$ in place of $W(x,t)$. On the other hand, it is a classical result that $\|\widetilde{N}_*(u_f)\|_{L^2(\mathbb{R}^n)} \approx \|\widehat{N}_*(u_f)\|_{L^2(\mathbb{R}^n)}$ (i.e. the L^2-norm of non-tangential maximal function is independent of the opening angle). The same holds for u_f^1. Hence we get

$$\begin{aligned}
\|\widetilde{N}_*(u_f - u_f^1)\|_{L^2(\mathbb{R}^n)} &\lesssim \left\| \left(\widehat{N}_*(u_f)\right)^{1/4} \cdot \left(\widetilde{N}_*^{3/2}(u_f - u_f^1)\right)^{3/4} \right\|_{L^2(\mathbb{R}^n)} \\
&\qquad + \left\| \left(\widehat{N}_*(u_f^1)\right)^{1/4} \cdot \left(\widetilde{N}_*^{3/2}(u_f - u_f^1)\right)^{3/4} \right\|_{L^2(\mathbb{R}^n)} \\
&\lesssim \left(\|\widetilde{N}_*(u_f)\|_{L^2(\mathbb{R}^n)}^{1/4} + \|\widetilde{N}_*(u_f^1)\|_{L^2(\mathbb{R}^n)}^{1/4} \right) \|\widetilde{N}_*^{3/2}(u_f - u_f^1)\|_{L^2(\mathbb{R}^n)}^{3/4} \\
&\leq C\epsilon^{3/4} \|f\|_{L^2(\mathbb{R}^n)}, \qquad (3.25)
\end{aligned}$$

where the last estimate follows by combining (3.23) and (3.24). The inequalities (3.23) and (3.25) are the direct analogue of (3.15) and (3.16). Proceeding exactly as before we obtain (3.22), and combining this with (3.12) we have

$$\|k_A^{(x,t)} - k_{A_0}^{(x,t)}\|_{L^2(\mathbb{R}^n)} \leq C\epsilon^{\eta'} t^{-n/2},$$

as desired. $\square$

Proposition 3.10 has one unfortunate drawback the estimate depends on the placement of the pole. This stops us from working with the small scales without doing some extra work. In order to do work with the small scales, we take three steps:

(1) Use Proposition 3.10 to immediately give us a good 'single-scale B_2 type estimate' for perturbations of constant coefficient operators (Corollary 3.26).
(2) Observe how the change of pole argument interacts with single-scale B_2 estimates to allow us to move down to small scales (Lemma 3.28) by paying a (controllable) penalty.

(3) Combining (1) and (2) we obtain a good estimate for perturbations of constant coefficient operators down to small scales (Corollary 3.30).

Corollary 3.26. *For every $\beta \in (0,1)$ and $\delta \in (0,1)$ there exists $\epsilon_1 = \epsilon_1(\delta, \beta, n, \Lambda) \in (0, \epsilon_0)$ such that the following holds. Suppose $A = A(x,t)$ is a real matrix-valued function on $\mathbb{R}^{n+1}_+$ satisfying the Λ-ellipticity condition and A is an ϵ_1-perturbation of a real, constant-coefficient matrix A_0 satisfying the Λ-ellipticity condition. Then*

$$\frac{\left(\fint_{\Delta(y,\delta r)} \left[k^X_A(z)\right]^2 \, dz\right)^{1/2}}{\fint_{\Delta(y,\delta r)} k^X_A(z)\, dz} \le (1+\beta) \frac{\left(\fint_{\Delta(y,\delta r)} \left[k^X_{A_0}(z)\right]^2 \, dz\right)^{1/2}}{\fint_{\Delta(y,\delta r)} k^X_{A_0}(z)\, dz} \tag{3.27}$$

for all $y \in \mathbb{R}^n$, $r > 0$, $y \in \mathbb{R}^n$ and $X \in D(y,r)$.

Proof. Let $\beta_1 \in (0,1)$ be fixed, whose value will be chosen later. By Proposition 3.10 and Lemma 3.5, if ϵ_1 is chosen so that $C\epsilon_1^{\eta'} \le \beta_1 c_1 \delta^{n/2}$ we have

$$\|k^X_A - k^X_{A_0}\|_{L^2(\Delta(y,\delta r))} \le \beta_1 \|k^X_{A_0}\|_{L^2(\Delta(y,\delta r))}$$

and

$$\|k^X_A - k^X_{A_0}\|_{L^1(\Delta(y,\delta r))} \le \beta_1 \|k^X_{A_0}\|_{L^1(\Delta(y,\delta r))}.$$

Thus, for this choice of ϵ_1 we have

$$\frac{\|k^X_A\|_{L^2(\Delta(y,\delta r))}}{\|k^X_A\|_{L^1(\Delta(y,\delta r))}} \le \frac{1+\beta_1}{1-\beta_1} \frac{\|k^X_{A_0}\|_{L^2(\Delta(y,\delta r))}}{\|k^X_{A_0}\|_{L^1(\Delta(y,\delta r))}},$$

which yields (3.27) upon normalization and appropriate choice of β_1 (so that $\frac{1+\beta_1}{1-\beta_1} \le 1+\beta$). $\square$

The above corollary says that for any $\beta \in (0,1)$, if A is a small perturbation of a constant-coefficient matrix A_0, then the B_2 constant of k^X_A is bounded by $(1+\beta)$ times that of $k^X_{A_0}$. As mentioned above, the caveat is that the smallness also depends on δ, which is the ratio between the radius of the surface ball in consideration and the distance from the pole to the boundary. This prevents us from getting B_2 type estimates for all sufficiently small scales (with fixed pole). We will overcome this by a 'good change of pole'.

Lemma 3.28 *(Change of pole comparison). Let $L = -\operatorname{div} A\nabla$ be a real divergence form operator on $\mathbb{R}^{n+1}_+$, with A satisfying the Λ-ellipticity condition. We also assume the corresponding Poisson kernel k_A exists and is in $L^2_{loc}(\mathbb{R}^n)$. There are constants $\delta_0 = \delta_0(n,\Lambda) \in (0,1)$ and $C = C(n,\Lambda) > 1$ such that for any $\delta \in (0,\delta_0)$ fixed,*

$$
\begin{aligned}
(1-C\delta^\mu)\frac{\left(\fint_{\Delta(y,\delta' r)}\left(k_A^{(y,\frac{\delta'}{\delta}r)}(z)\right)^2\right)^{1/2}}{\fint_{\Delta(y,\delta' r)}k_A^{(y,\frac{\delta'}{\delta}r)}(z)} &\le \frac{\left(\fint_{\Delta(y,\delta' r)}\left(k_A^X(z)\right)^2\right)^{1/2}}{\fint_{\Delta(y,\delta' r)}k_A^X(z)} \\
&\le (1+C\delta^\mu)\frac{\left(\fint_{\Delta(y,\delta' r)}\left(k_A^{(y,\frac{\delta'}{\delta}r)}(z)\right)^2\right)^{1/2}}{\fint_{\Delta(y,\delta' r)}k_A^{(y,\frac{\delta'}{\delta}r)}(z)} \qquad (3.29)
\end{aligned}
$$

for any $y \in \mathbb{R}^n$, $r > 0$, $\delta' \in (0, \delta/10]$ *and* $X \in D(y,r)$.

Proof. We apply Lemma 2.21, which allows us to change poles of the elliptic measure, to the upper half space $\mathbb{R}^{n+1}_+$. To be precise, we consider the radius $s := \frac{2}{3}\frac{\delta'}{\delta}r$, $\zeta = \frac{3}{2}\delta \in (0, 1/2)$ and poles $X_0 = X \in D(y,r)$ and $X_1 = (y, \delta' r/\delta)$. The fact that $X_0 \notin B(y, 4s)$ follows from the assumption $\delta' \le \delta/10$, and clearly $X_1 \in B(y,2s) \setminus B(y,s)$. Then there are constants $C, \mu > 0$ such that the kernel function

$$H(z) = \frac{d\omega^X}{d\omega^{(y,\frac{\delta'}{\delta}r)}}(z)$$

satisfies

$$\left|\frac{H(z)}{H(y)} - 1\right| \le C\delta^\mu, \qquad \text{for every } z \in \Delta(y, \zeta s) = \Delta(y, \delta' r).$$

Hence when δ is sufficiently small, we have for every $z, z' \in \Delta(y, \delta' r)$,

$$\frac{H(z)}{H(z')} \le \frac{H(y)(1+C\delta^\mu)}{H(y)(1-C\delta^\mu)} \le 1 + C'\delta^\mu.$$

On the other hand,

$$k_A^X(z) = \frac{d\omega^X}{d\sigma}(z) = H(z)\,\frac{d\omega^{(y,\frac{\delta'}{\delta}r)}}{d\sigma}(z) = H(z)\,k_A^{(y,\frac{\delta'}{\delta}r)}(z).$$

Therefore

$$\frac{\left(\fint_{\Delta(y,\delta' r)}\left(k_A^X(z)\right)^2\right)^{1/2}}{\fint_{\Delta(y,\delta' r)}k_A^X(z)} \le \sup_{z,z'\in\Delta(y,\delta' r)}\frac{H(z)}{H(z')}\cdot\frac{\left(\fint_{\Delta(y,\delta' r)}\left(k_A^{(y,\frac{\delta'}{\delta}r)}(z)\right)^2\right)^{1/2}}{\fint_{\Delta(y,\delta' r)}k_A^{(y,\frac{\delta'}{\delta}r)}(z)}$$

$$\leq (1 + C'\delta^\mu) \frac{\left(\fint_{\Delta(y,\delta' r)} \left(k_A^{(y, \frac{\delta'}{\delta} r)}(z) \right)^2 \right)^{1/2}}{\fint_{\Delta(y,\delta' r)} k_A^{(y, \frac{\delta'}{\delta} r)}(z)}.$$

This is the second inequality of (3.29). The first inequality is obtained similarly. □

Corollary 3.30. *For every $\beta \in (0,1)$ and $\delta \in (0, \delta_0)$, where δ_0 is from Lemma 3.28, there exists $\epsilon_1 = \epsilon_1(\delta, \beta, n, \Lambda) \in (0, \epsilon_0)$ such that the following holds for every $\epsilon < \epsilon_1$. (Here ϵ_0 is as in Proposition 3.10.)*

If $A = A(x,t)$ is a real matrix valued function on $\mathbb{R}^{n+1}_+$ satisfying the Λ-ellipticity condition and moreover, A is an ϵ-perturbation of a real, constant-coefficient matrix A_0 satisfying the Λ-ellipticity condition, then

$$\frac{\left(\fint_{\Delta(y,\delta' r)} \left[k_A^X(z) \right]^2 dz \right)^{1/2}}{\fint_{\Delta(y,\delta' r)} k_A^X(z)\, dz} \leq (1 + C\delta^\mu)(1+\beta) \frac{\left(\fint_{\Delta(0,\delta)} \left[k_{A_0}^{(0,1)}(z) \right]^2 dz \right)^{1/2}}{\fint_{\Delta(0,\delta)} k_{A_0}^{(0,1)}(z)\, dz} \tag{3.31}$$

for every $y \in \mathbb{R}^n$, $r > 0$, $\delta' \in (0, \delta/10]$ and $X \in D(y,r)$.

Proof. The proof is by combining Corollary 3.26 and the change of pole comparison in Lemma 3.28. Set $s = \frac{\delta'}{\delta} r$, the estimates (3.29) and (3.27) yield

$$\begin{aligned}
\frac{\left(\fint_{\Delta(y,\delta' r)} \left(k_A^X(z) \right)^2 \right)^{1/2}}{\fint_{\Delta(y,\delta' r)} k_A^X(z)} &\leq (1 + C\delta^\mu) \frac{\left(\fint_{\Delta(y,\delta s)} \left(k_A^{(y,s)}(z) \right)^2 \right)^{1/2}}{\fint_{\Delta(y,\delta s)} k_A^{(y,s)}(z)} \\
&\leq (1 + C\delta^\mu)(1+\beta) \frac{\left(\fint_{\Delta(y,\delta s)} \left(k_{A_0}^{(y,s)}(z) \right)^2 \right)^{1/2}}{\fint_{\Delta(y,\delta s)} k_{A_0}^{(y,s)}(z)} \\
&\leq (1 + C\delta^\mu)(1+\beta) \frac{\left(\fint_{\Delta(0,\delta)} \left[k_{A_0}^{(0,1)}(z) \right]^2 dz \right)^{1/2}}{\fint_{\Delta(0,\delta)} k_{A_0}^{(0,1)}(z)\, dz},
\end{aligned}$$

where we use the translation and dilation invariance for k_{A_0}, i.e. Poisson kernel for constant-coefficient operator, in the last estimate. □

We now apply Corollary 3.30 twice. The first application is to get uniform control on the scale at which the B_2 constant becomes 'near optimal' for constant coefficient operators. Surely (3.9) allows us to show the following Lemma, but we prove this via a compactness argument here only using the smoothness of the Poisson kernel for constant coefficient operators and the previous corollary.

Lemma 3.32. *For every $\beta' \in (0,1)$ there exists $\delta_1 = \delta_1(\beta', n, \Lambda) \in (0, \delta_0)$ such that for any real, constant-coefficient elliptic matrix A_0 satisfying the Λ ellipticity condition, we have*

$$\frac{\left(\fint_{\Delta(0,\delta)} \left[k_{A_0}^{(0,1)}(z) \right]^2 \, dz \right)^{1/2}}{\fint_{\Delta(0,\delta)} k_{A_0}^{(0,1)}(z) \, dz} \leq 1 + \beta'$$

for all $\delta \in (0, \delta_1)$.

Proof. Let $\beta' \in (0,1)$ and set $\Lambda' = 2\Lambda$. Let $\beta \in (0,1)$ and $\delta_2 \in (0, \delta_0)$ be such that

$$(1 + C\delta_2^{\mu})(1 + \beta)^2 \leq 1 + \beta',$$

where $\delta_0 = \delta_0(n, \Lambda')$ is as in Lemma 3.28 for Λ'. By (3.9), for every real, constant A_0 satisfying the Λ-ellipticity condition we have that $k_{A_0}^{(0,1)}(z)$ is a smooth function in z. Smoothness, combined with the non-degeneracy (3.6), yields that there exists $\tilde{\delta} = \tilde{\delta}(A_0)$, which we may assume is less than δ_2, such that

$$\frac{\left(\fint_{\Delta(0,\tilde{\delta})} \left[k_{A_0}^{(0,1)}(z) \right]^2 \, dz \right)^{1/2}}{\fint_{\Delta(0,\tilde{\delta})} k_{A_0}^{(0,1)}(z) \, dz} \leq 1 + \beta.$$

Thus, the estimate (3.31) in Corollary 3.30 implies that there exists[9] $\tilde{\epsilon}_1 = \tilde{\epsilon}_1(\tilde{\delta}, \beta, n, \Lambda') < (5\Lambda)^{-1}$ such that if A_0' is a real, constant coefficient matrix the Λ'-ellipticity condition and $\|A_0 - A_0'\|_{L^\infty} < \tilde{\epsilon}_1$

$$\begin{aligned} \frac{\left(\fint_{\Delta(0,\delta)} \left[k_{A_0'}^{(0,1)}(z) \right]^2 \, dz \right)^{1/2}}{\fint_{\Delta(0,\delta)} k_{A_0'}^{(0,1)}(z) \, dz} &\leq (1 + C\tilde{\delta}^{\mu})(1 + \beta) \frac{\left(\fint_{\Delta(0,\tilde{\delta})} \left[k_{A_0}^{(0,1)}(z) \right]^2 \, dz \right)^{1/2}}{\fint_{\Delta(0,\tilde{\delta})} k_{A_0}^{(0,1)}(z) \, dz} \\ &\leq 1 + \beta', \end{aligned}$$

for all $\delta \in (0, \tilde{\delta}/10]$. On the other hand, the collection of balls of the form $B_\infty(A_0, \tilde{\epsilon}_1(A_0)) := \{A_0' : \|A_0 - A_0'\|_{L^\infty} < \tilde{\epsilon}_1(A_0)\}$, as A_0 ranges over all real, constant-coefficient matrices, forms an open[10] cover of the set of all real, constant-coefficient matrices satisfying the Λ-ellipticity condition (in the L^∞-metric), which is a compact set. We may then extract a finite sub-cover $\{B_i\} = \{B(A_0^i, \tilde{\epsilon}_1(A_0^i))\}$ from which the conclusion of the theorem follows by letting $\delta_1 := \min_i \tilde{\delta}(A_0^i)/10$. □

[9] Note that, in particular, $\tilde{\epsilon}_1 = \tilde{\epsilon}_1(A_0)$ since $\tilde{\delta}$ (momentarily) depends on A_0.

[10] This is why we employed the use of Λ' and made the restriction $\tilde{\epsilon}_1 < (5\Lambda)^{-1}$.

Now we are ready to conclude our treatment of perturbations of constant-coefficient operators. We are (finally) able to say, quantitatively, that sufficient proximity to a constant-coefficient operator in the sense of (3.2) controls the B_2 constant at small scales.

Proof of Theorem 3.3. Let $\beta' \in (0,1)$ be such that $(1+\beta')^2 = 1 + \widetilde{\beta}/2$. Next, we choose $\delta \in (0, \delta_1)$, where $\delta_1 = \delta_1(\beta', n, \Lambda)$ is from Lemma 3.32, small enough so that

$$(1 + C\delta^\mu)(1+\beta')^2 \le 1 + \widetilde{\beta}.$$

We apply Corollary 3.30 with $\beta = \beta'$ and δ as above. It follows that if $\epsilon = \epsilon_1(\delta, \beta', n, \Lambda) > 0$ is as in Corollary 3.30 (with $\beta = \beta'$) we have for all A satisfying the hypothesis of the theorem

$$\begin{aligned}\frac{\left(\fint_{\Delta(y,\delta' r)} \left[k_A^X(z)\right]^2 \, dz\right)^{1/2}}{\fint_{\Delta(y,\delta' r)} k_A^X(z)\, dz} &\le (1 + C\delta^\mu)(1+\beta') \frac{\left(\fint_{\Delta(y,\delta)} \left[k_{A_0}^{(0,1)}(z)\right]^2 \, dz\right)^{1/2}}{\fint_{\Delta(0,\delta)} k_{A_0}^{(0,1)}(z)\, dz} \\ &\le 1 + \widetilde{\beta},\end{aligned}$$

for all $y \in \mathbb{R}^n$, $r > 0$, $\delta' \in (0, \delta/10]$ and $X \in D(x,r)$, where we use that $\delta \in (0, \delta_1)$ and Lemma 3.32 in the second line. Setting $\tilde{\delta} = \delta/10$ we obtain the conclusion of the theorem.

4. Operators with Hölder coefficients in Lipschitz domains

In this section, we modify Hölder-continuous coefficient matrices (in Lipschitz domains) so that they are small perturbations of constant coefficient matrices, and this will allow us to use the results of Section 3. We will rely on the "flat" transformation between Lipschitz domains and the upper half space. The relevant computations are standard, so we include them in Appendix C. But we suggest reader to first read the notation and statements in Appendix C, since we rely heavily on them in this section. For instance, we often employ the 'flattening map' Φ defined in Appendix C.

In what follows, if $\tau > 0$ we let Q_τ be the open n-dimensional cube centered at zero with side length 2τ, that is, $Q_\tau = \{x \in \mathbb{R}^n : |x|_\infty < \tau\}$.

Lemma 4.1. *Let $\varphi : \mathbb{R}^n \to \mathbb{R}$, $\varphi(0) = 0$ be a Lipschitz function with Lipschitz constant $\gamma \le \frac{1}{50n}$. Suppose A is a Λ-elliptic matrix satisfying the Hölder condition (1.2). Then for $\tau > 0$ the matrix-valued function*[11]

[11] We remind the reader that $R_{Q_\tau,\varphi}$ is the φ-adapted Carleson box, defined in (C.3).

$$A_{\tau,\varphi}(x,t) := \begin{cases} A(0) & \text{if } x \notin Q_\tau \\ A(x,t) & \text{if } (x,t) \in R_{Q_\tau,\varphi} \\ A(x,\varphi(x)) & \text{otherwise} \end{cases}$$

has the decomposition

$$A_{\tau,\varphi}(x,t) = A_1(x) + P(x,t),$$

with

$$\|A_1 - A(0)\|_{L^\infty(\mathbb{R}^n)} + \|P\|_{\mathcal{C}_\varphi} < C_1\tau^\alpha, \tag{4.2}$$

where C_1 depends on C_A, n and α. Here

$$A_1(x) := A(x,\varphi(x))\mathbb{1}_{Q_\tau}(x) + A(0)(1 - \mathbb{1}_{Q_\tau}(x)), \text{ and } BP(x,t) := A_{\tau,\varphi}(x,t) - A_1(x).$$

In particular, $\widetilde{A} = J_\Phi^T(A_{\tau,\varphi} \circ \Phi^{-1})J_\Phi$ is a real, 4Λ-elliptic matrix with the decomposition

$$\widetilde{A}(x,t) = \widetilde{A}_1(x) + \widetilde{P}(x,t),$$

satisfying

$$\|\widetilde{A}_1 - A_0'\|_{L^\infty(\mathbb{R}^n)} + \|\widetilde{P}\|_{\mathcal{C}} < 2C_1\tau^\alpha + 4\sqrt{n}\gamma\Lambda,$$

where $\widetilde{A}_1 := J_\Phi^T(A_1 \circ \Phi^{-1})J_\Phi = J_\Phi^T(A_1)J_\Phi$, $\widetilde{P} := J_\Phi^T(B \circ \Phi^{-1})J_\Phi$ and $A_0' := J_\Phi^T(0)A(0)J_\Phi(0)$.

Proof. Fix $\tau > 0$. The second statement concerning $\widetilde{A}$, its decomposition and the corresponding bounds follows immediately from Proposition C.6 and (4.2).

To prove the estimate (4.2), we begin with the L^∞ estimate for $A_1(x) - A_0$. For $x \in Q_\tau$

$$|A_1(x) - A(0)| = |A(x,\varphi(x)) - A(0)| \le C_A|(x,\varphi(x))|^\alpha \lesssim \tau^\alpha,$$

where the implicit constant depends on C_A, n and α. Since $A_1(x) = A(0)$ for $x \notin Q_\tau$, we have obtained the estimate $\|A_1 - A(0)\|_{L^\infty(\mathbb{R}^n)} < C\tau^\alpha$.

We are left with estimating the φ-adapted Carleson norm of $P := A_{\tau,\varphi} - A_1$. For any $X \in R_{Q_\tau,\varphi}$, we write $X = (x, \varphi(x) + t) \in \Omega_\varphi$ and $\hat{X} = (x, \varphi(x))$. Hence

$$|P(X)| = |A_{\tau,\varphi}(X) - A_1(X)| = |A(X) - A(\widehat{X})| \le C_A t^\alpha.$$

Since $P(X) \equiv 0$ in $(R_{Q_\tau,\varphi})^c$ it follows that for any cube $Q \subset \mathbb{R}^n$

$$
\begin{aligned}
\|P\|_{\mathcal{C}_\varphi} &= \sup_{Q\subset\mathbb{R}^n}\left(\frac{1}{|Q|}\iint_{R_{Q,\varphi}} \|P\|^2_{L^\infty(W_\varphi(y,s))}\frac{dy\,ds}{s-\varphi(y)}\right)^{1/2} \\
&= \sup_{Q\subset\mathbb{R}^n}\left(\frac{1}{|Q|}\int_0^{\ell(Q)}\int_Q \|P\circ\Phi^{-1}\|^2_{L^\infty(W(x,t))}\frac{dx\,dt}{t}\right)^{1/2} \\
&\le C\sup_{Q\subset\mathbb{R}^n}\left(\int_0^{\min(\ell(Q),4\tau)} t^{2\alpha-1}\,dt\right)^{1/2} \le C\tau^\alpha,
\end{aligned}
$$

where we used the flattening change of variables in the second line. Combining our estimates for $\|A_1 - A(0)\|_{L^\infty(\mathbb{R}^n)}$ and $\|P\|_{\mathcal{C}_\varphi}$ we obtain (4.2). □

The next Proposition says that the modified coefficient matrix enjoys an almost optimal reverse-Hölder estimate.

Proposition 4.3. *Let $\beta > 0$. There exist $\delta_\beta, \gamma_\beta, \tau_\beta > 0$ depending on β and allowable constants such that the following holds.*

Assume A is a real, Λ-elliptic matrix-valued function satisfying the Hölder condition (1.2), and $\varphi : \mathbb{R}^n \to \mathbb{R}$, $\varphi(0) = 0$, is Lipschitz with $\|\nabla\varphi\|_\infty \le \gamma_\beta$. Then for all $\tau \in (0, \tau_\beta)$, the Poisson kernel (denoted by $h^X_{\tau,\varphi}$) for $L_{\tau,\varphi} = -\operatorname{div} A_{\tau,\varphi}\nabla$[12] *and the domain Ω_φ with pole $X \in \Omega_\varphi$ satisfies*

$$
\frac{\left(\fint_{\Delta_\varphi(y,\delta r)}(h^X_{\tau,\varphi})^2\,d\sigma\right)^{1/2}}{\fint_{\Delta_\varphi(y,\delta r)} h^X_{\tau,\varphi}\,d\sigma} \le 1+\beta, \tag{4.4}
$$

for all $y \in \mathbb{R}^n, r > 0, \delta \in (0, \delta_\beta)$ and $X \in \Phi^{-1}(D(y, r))$. Here σ is the surface measure to $\operatorname{Graph}(\varphi) = \{(x, \varphi(x)) : x \in \mathbb{R}^n\}$, $\Delta_\varphi(y, \delta r) := B((y, \varphi(y)), \delta r) \cap \operatorname{Graph}(\varphi)$ is the surface ball, and Φ is the flattening map for φ.

Proof. Let $\widetilde{\beta} = \min\{1, \beta/2\}$ and let $k^Z_{\widetilde{A}}$ be the "pulled back" Poisson kernel, that is, $k^Z_{\widetilde{A}}$ is the Poisson kernel in the upper half space for $\widetilde{L} = -\operatorname{div}\widetilde{A}\nabla$, where $\widetilde{A} = J^T_\Phi(A_{\tau,\varphi}\circ\Phi^{-1})J_\Phi$. Using Lemma 4.1, there exists A_0', a real, constant 4Λ-elliptic matrix such that $\widetilde{A}$ has the decomposition

$$
\widetilde{A}(x,t) = \widetilde{A}_1(x) + \widetilde{P}(x,t)
$$

and

[12] See Lemma 4.1.

$$\|\widetilde{A} - A_0'\|_{L^\infty} + \|\widetilde{P}\|_{\mathcal{C}} \leq 2C_1\tau_\beta^\alpha + 4\sqrt{n}\gamma_\beta\Lambda.$$

In particular we may initially choose τ_β and γ_β small depending on $\tilde{\beta}$ and allowable constants, to guarantee that $\widetilde{A}$ is an $\epsilon(\tilde{\beta})$-perturbation of the constant-coefficient matrix A_0' (as in (3.1), (3.2)), so that by Theorem 3.3 we have

$$\frac{\left(\fint_{\Delta(y,\delta' r)} \left[k_{\widetilde{A}}^Z(x')\right]^2\, dx'\right)^{1/2}}{\fint_{\Delta(y,\delta' r)} k_{\widetilde{A}}^Z(x')\, dx'} < 1 + \widetilde{\beta} \leq 2 \tag{4.5}$$

for any $y \in \mathbb{R}^n, r > 0, \delta' \in (0, \tilde{\delta})$ and $Z \in D(y,r)$. The constant $\tilde{\delta} = \tilde{\delta}(\tilde{\beta}, n, \Lambda)$ was determined in Theorem 3.3. We now fix τ_β and set $\delta_\beta = \tilde{\delta}$, but we will further restrict γ_β in order to control the errors coming from the flattening change of variables.

Fix $r > 0$, $\delta \in (0, \delta_\beta)$, $y \in \mathbb{R}^n$ and $X \in \Phi^{-1}(D(y,r))$ and set $Z = \Phi(X) \in D(y,r)$. Let $P : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be the projection operator, that is, $P(x,t) := x$ for all $(x,t) \in \mathbb{R}^{n+1}$. By Proposition C.1 and (4.5)

$$\frac{\left(\fint_{\Delta_\varphi(y,\delta r)} (h^X_{\tau,\varphi})^2\, d\sigma\right)^{1/2}}{\fint_{\Delta_\varphi(y,\delta r)} (h^X_{\tau,\varphi})^2\, d\sigma} \leq \frac{\mathcal{H}^n(\Delta_\varphi(y,\delta r))}{|P(\Delta_\varphi(y,\delta r))|} \frac{\left(\fint_{P(\Delta_\varphi(y,\delta r))} \left[k_{\widetilde{A}}^Z(x')\right]^2\, dx'\right)^{1/2}}{\fint_{P(\Delta_\varphi(y,\delta r))} k_{\widetilde{A}}^Z(x')\, dx'}, \tag{4.6}$$

where the error term comes from changing the averages. Let us now make the simple observation that for $x_0 \in \mathbb{R}^n$ and $r_0 > 0$

$$\Delta(x_0, r_0(1 + \|\nabla\varphi\|_\infty^2)^{-1/2}) \subseteq P(\Delta_\varphi(x_0, r_0)) \subseteq \Delta(x_0, r_0)), \tag{4.7}$$

where the second inclusion is obvious and the first, similarly, is a consequence of the Pythagorean theorem. Indeed, if $x \in \Delta(x_0, r_0(1 + \|\nabla\varphi\|_\infty^2)^{-1/2})$ then

$$|(x, \varphi(x)) - (x_0, \varphi(x_0))|^2 \leq |x - x_0|^2 + \|\nabla\varphi\|_\infty^2 |x - x_0|^2 < r_0.$$

The inclusions in (4.7) give the estimate

$$\frac{|\Delta(y, \delta r)|}{\mathcal{H}^n(P(\Delta_\varphi(y, \delta r)))} \leq (\delta r)^n \left(\frac{\delta r}{\sqrt{1 + \|\nabla\varphi\|_\infty^2}}\right)^{-n} = (1 + \|\nabla\varphi\|_\infty^2)^{-n/2}.$$

The above estimate and (4.7) yield

$$\left(\fint_{P(\Delta_\varphi(y,\delta r))} \left[k_{\widetilde{A}}^Z(x')\right]^2\, dx'\right)^{1/2} \leq (1 + \|\nabla\varphi\|_\infty^2)^{n/4} \left(\fint_{\Delta(y,\delta r)} \left[k_{\widetilde{A}}^Z(x')\right]^2\, dx'\right)^{1/2}. \tag{4.8}$$

Again using (4.7) we have

$$
\begin{aligned}
\fint_{P(\Delta_\varphi(y,\delta r))} k^Z_{\widetilde{A}}(x')\,dx' &\geq \frac{\omega^Z_{\widetilde{A}}(\Delta(y,\delta r(1+\|\nabla\varphi\|^2_\infty)^{-1/2}))}{|P(\Delta_\varphi(y,\delta r))|} \\
&\geq \frac{\omega^Z_{\widetilde{A}}(\Delta(y,\delta r(1+\|\nabla\varphi\|^2_\infty)^{-1/2}))}{\omega^Z_{\widetilde{A}}(\Delta(y,\delta r))} \fint_{\Delta(y,\delta r)} k^Z_{\widetilde{A}}(x')\,dx',
\end{aligned}
\tag{4.9}
$$

where $\omega^Z_{\widetilde{A}} = k^Z_{\widetilde{A}}\,dx'$ is the elliptic measure for $\widetilde{L}$ on $\mathbb{R}^{n+1}_+$.

Combining (4.6), (4.8) and (4.9) with the estimate

$$
H^n(\Delta_\varphi(y,\delta r)) \leq \sqrt{1+\|\nabla\varphi\|^2_\infty}\,|P(\Delta_\varphi(y,\delta r))|
$$

we have

$$
\frac{\left(\fint_{\Delta_\varphi(y,\delta r)} (h^X_{\tau,\varphi})^2\,d\sigma\right)^{1/2}}{\fint_{\Delta_\varphi(y,\delta r)} (h^X_{\tau,\varphi})^2\,d\sigma} \leq (1+\widetilde{\beta})(1+\|\nabla\varphi\|^2_\infty)^{\frac{n+1}{4}} \frac{\omega^Z_{\widetilde{A}}(\Delta(y,\delta r))}{\omega^Z_{\widetilde{A}}(\Delta(y,\delta r(1+\|\nabla\varphi\|^2_\infty)^{-1/2}))}
\tag{4.10}
$$

where we used (4.5) to control the ratio of the averages of $k^Z_{\widetilde{A}}$ by $(1+\widetilde{\beta})$. Clearly we can make $(1+\|\nabla\varphi\|^2_\infty)^{\frac{n+1}{4}}$ sufficiently close to one by choice of γ_β, so we need to handle the ratios of the elliptic measure. This can be done in a variety of ways, but we choose to do it directly with the following.

Claim 4.11. Let $s > 0$ and suppose $\mu = f\,dx$ for $f \geq 0$, $f \in L^1(\Delta(y,s))$ satisfies

$$
\left(\fint_{\Delta(y,s)} f^2\,dx\right)^{1/2} \leq 2 \fint_{\Delta(y,s)} f\,dx.
\tag{4.12}
$$

Then for $s' \in ([1-(1/4)]^{1/n}s, s)$,

$$
\frac{\mu(\Delta_s)}{\mu(\Delta_{s'})} \leq \frac{1}{1-2\left(\frac{s^n-(s')^n}{s^n}\right)^{1/2}},
$$

where $\Delta_{s'} := \Delta(y,s')$ and $\Delta_s := \Delta(y,s)$.

Proof of Claim 4.11. The proof is a direct consequence of (4.12) and Hölder's inequality. Indeed,

$$
\mu(\Delta_s \setminus \Delta_{s'}) = \int_{\Delta_s\setminus\Delta_{s'}} f\,dx \leq H^n(\Delta_s\setminus\Delta_{s'})^{1/2} \left(\int_{\Delta_s\setminus\Delta_{s'}} f^2\,dx\right)^{1/2}
$$

$$
\begin{aligned}
&\le |\Delta_s \setminus \Delta_{s'}|^{1/2} \left(\int_{\Delta_s} f^2 \, dx \right)^{1/2} \\
&\le 2 \left(\frac{|\Delta_s \setminus \Delta_{s'}|}{|\Delta_s|} \right)^{1/2} \mu(\Delta_s) \\
&\le 2 \left(\frac{s^n - (s')^n}{s^n} \right)^{1/2} \mu(\Delta_s)
\end{aligned}
$$

where we used (4.12) in the second to last line. With this inequality in hand, we easily prove the claim by writing

$$
\frac{\mu(\Delta_s)}{\mu(\Delta_{s'})} = 1 + \frac{\mu(\Delta_s \setminus \Delta_{s'})}{\mu(\Delta_{s'})} \le 1 + 2 \left(\frac{s^n - (s')^n}{s^n} \right)^{1/2} \frac{\mu(\Delta_s)}{\mu(\Delta_{s'})}. \quad \square
$$

The estimate (4.5) allows us to apply Claim 4.11 to the measure $\mu = \omega^Z_{\widetilde{A}}$, $s = \delta r$ and $s' = \delta r(1 + \|\nabla\varphi\|^2_\infty)^{-1/2}$, where s' will satisfy the hypothesis of the claim by the smallness of γ_β. This yields the estimate

$$
\frac{\omega^Z_{\widetilde{A}}(\Delta(y, \delta r))}{\omega^Z_{\widetilde{A}}(\Delta(y, \delta r(1 + \|\nabla\varphi\|^2_\infty)^{-1/2}))} \le \frac{1}{1 - 2\left(1 - (1 + \gamma_\beta^2)^{-n/2}\right)^{1/2}}.
$$

This estimate in concert with (4.10) and choice of γ_β sufficiently small gives

$$
\begin{aligned}
\frac{\left(⨍_{\Delta_\varphi(y,\delta r)} (h^X_{\tau,\varphi})^2 \, d\sigma \right)^{1/2}}{⨍_{\Delta_\varphi(y,\delta r)} (h^X_{\tau,\varphi})^2 \, d\sigma} &\le (1 + \widetilde{\beta}) \frac{(1 + \gamma_\beta^2)^{\frac{n+1}{4}}}{1 - 2\left(1 - (1 + \gamma_\beta^2)^{-n/2}\right)^{1/2}} \\
&\le (1 + \beta),
\end{aligned}
$$

where we recall that $1 + \widetilde{\beta} \le 1 + \beta/2$. $\square$

5. Proof of Theorem 1.1

In this section we present the proof of Theorem 1.1. To begin, we require an ‘almost optimal’ version of Lemma 2.22 (which is a comparison between elliptic measures of two operators who agree locally), after we impose sufficient flatness on the boundary and Hölder regularity on the coefficients. This almost-optimal comparison will allow us to transfer the almost-optimal reverse Hölder estimate in Proposition 4.3 of the modified coefficient matrix to that of the original matrix, in local regions of sufficiently flat Lipschitz domains. We will also use this almost-optimal comparison to relate the Poisson kernel for a (δ, R)-chord arc domain Ω and its localized domain $\widetilde{\Omega}(x_0, r)$, which can be approximated by Lipschitz domains from inside and outside. The notation used here can be found at the end of Section 2.3.

Until we get to the proof of Theorem 1.1 we largely follow [40], tightening the presentation along the way. We begin with a lemma which resembles [40, Lemma 3.7].

Lemma 5.1. *Let Ω be a (δ, R)-chord arc domain with $\delta < \delta_n$. There exists a constant M' depending on the dimension so that the following holds. Let $M \geq M'$, $0 < s < R/M$, $x_0 \in \partial\Omega$ and $\xi \in (-s/100, s/100)$ and set*

$$\mathcal{C} := \mathcal{C}(x_0, Ms, \vec{n}_{x_0,Ms}, \xi),$$

defined in (2.31), and $\Omega_1 := \mathcal{C} \cap \Omega$. Suppose that

- *Ω_2 is a chord arc domain satisfying $\Omega_2 \cap \mathcal{C} = \Omega_1$.*
- *A_1 and A_2 are coefficient matrices defined on Ω_1 and Ω_2, respectively, such that $A_1 = A_2$ on Ω_1.*

For $i = 1, 2$, let ω_i be the elliptic measure for the operator in $L_i := -\operatorname{div} A_i \nabla$ in Ω_i and $G_i(X, Y)$ be the Green function for L_i in Ω_1. If $X \in B(x_0, 10^{-8}(n+1)^{-1/2}Ms)$, it holds

$$\frac{d\omega_1^X}{d\omega_2^X}(y) = \lim_{Y \to y} \frac{G_1(X,Y)}{G_2(X,Y)}, \tag{5.2}$$

for ω_2^X a.e. $y \in B(X, 100\operatorname{dist}(X, \partial\Omega)) \cap \partial\Omega_2$,[13] *where the limit is taken within Ω_1.*

Additionally, if $X_0 = x_0 + tMs\vec{n}_{x_0,Ms}$ for some $t \in \left[\frac{1}{4\sqrt{n+1}}, \frac{3}{4\sqrt{n+1}}\right]$ then

$$\frac{d\omega_1^{X_0}}{d\omega_2^{X_0}}(y) = \lim_{Y \to y} \frac{G_1(X_0,Y)}{G_2(X_0,Y)}, \tag{5.3}$$

for ω_2^X a.e. $y \in B(x_0, 10s) \cap \partial\Omega_2$. Moreover, for every $\epsilon > 0$ there exists $M'' > M'$ depending on ϵ, n and the chord arc constants of Ω_2 such that if $M \geq M''$ it holds that

$$(1+\epsilon)^{-1} \leq \frac{d\omega_1^{X_0}}{d\omega_2^{X_0}}(y) \Big/ \frac{d\omega_1^{X_0}}{d\omega_2^{X_0}}(z) \leq (1+\epsilon) \tag{5.4}$$

for $\omega_2^{X_0}$-a.e. $y, z \in B(x_0, 10s) \cap \partial\Omega_2$.

Proof. Recall that the estimates in Section 2.3 can be applied for the truncated domain Ω_1 as if it were a chord arc domain itself, provided we work away from $\partial\mathcal{C}$. (See the paragraph before Definition 2.33.) The constant M' is simply to overcome the shift of

[13] We remark that by the choice of X, the ball $B(X, 100\operatorname{dist}(X, \partial\Omega))$ is well contained in the cylinder $\mathcal{C}$ and thus $B(X, 100\operatorname{dist}(X, \partial\Omega)) \cap \partial\Omega_2 = B(X, 100\operatorname{dist}(X, \partial\Omega)) \cap \partial\Omega = B(X, 100\operatorname{dist}(X, \partial\Omega)) \cap \partial\Omega_1$.

the cylinder by ξ. First we prove (5.2); the proof for (5.3) is nearly identical and we omit it.

Fix X as in the lemma and set $d_X := \mathrm{dist}(X, \partial\Omega)$. We recall the following Riesz formula (see [27, Lemma 2.25])

$$\int\limits_{\partial\Omega_i} \psi\omega_i^X - \psi(X) = -\iint\limits_{\Omega_i} \langle A_i^T(Y)\nabla_Y G_i(X,Y), \nabla_Y \psi(Y)\rangle \, dY, \tag{5.5}$$

for every $\psi \in C_c^\infty(\mathbb{R}^{n+1})$ and $X \in \Omega_i$, $i = 1, 2$. Applying Lemma 2.19 to the Green's functions $G_1(X, \cdot)$ and $G_2(X, \cdot)$, we have that the limit

$$g(y) := \lim_{Y \to y} \frac{G_1(X,Y)}{G_2(X,Y)} \tag{5.6}$$

exists for ω_2^X-a.e. $y \in B(X, 100d_X) \cap \partial\Omega_2$. On the other hand, similar to Lemma 2.22 (and using the Lebesgue differentiation theorem for Radon measures),

$$\frac{d\omega_1^X}{d\omega_2^X}(y) := \lim_{r \to 0^+} \frac{\omega_1^X(B(y,r))}{\omega_2^X(B(y,r))}$$

exists for ω_2^X-a.e. $y \in B(X, 100d_X) \cap \partial\Omega_2$.

For $y \in B(X, 100d_X) \cap \partial\Omega_2$ and $r \ll d_X$ we let $\psi_{y,r}(X) := \psi(|X - y|/r)$, where $\psi \in C_c^\infty((-2, 2))$ is radially decreasing with $\psi \equiv 1$ on $[-1, 1]$, so that $|\nabla\psi_{y,r}| \leq C/r$. Let $u_{y,r}^i$, $i = 1, 2$, be the variational solution to the Dirichlet problem for L_i in Ω_i with boundary data $\psi_{y,r}|_{\partial\Omega_i}$. A modification of the standard ('harmonic analysis') argument used to prove the Lebesgue differentiation theorem (for doubling measures) also shows for ω_2^X-a.e. $y \in B(X, 100d_X) \cap \partial\Omega_2$

$$\frac{d\omega_1^X}{d\omega_2^X}(y) = \lim_{r \to 0^+} \frac{\int_{\partial\Omega_1} \psi_{y,r}(z)\, d\omega_1^X(z)}{\int_{\partial\Omega_2} \psi_{y,r}(z)\, d\omega_2^X(z)} = \lim_{r \to 0^+} \frac{u_{y,r}^1(X)}{u_{y,r}^2(X)}. \tag{5.7}$$

Indeed, to make such an observation one should appeal to the techniques in [47, Chapter 1] and [46, Chapter 1] by building weighted maximal function out of the radial function $\psi(|X|)$ using the fact that ω_2^X is doubling. More specifically, one can introduce the operator

$$A_{\psi,r}h(x) := \frac{1}{\int_{\partial\Omega_2} \psi_{z,r}(z)\, d\omega_2^X(z)} \int\limits_{\partial\Omega_2} \psi_{z,r}(z)h(z)\, d\omega_2^X(z),$$

and dominate it by an associated (local) maximal operator. Then note that, for sufficiently small r the quotient inside the limit in the middle term of (5.7) is exactly $A_{\psi,r}h(y)$

with $h = \frac{d\omega_1^X}{d\omega_2^X}|_{B(X,200d_X)\cap\partial\Omega_2}$, which is an ω_2^X-integrable function by Lemma 2.22. Finally, use [47, Chapter 1: Theorem 1][14] to provide the (local) weak-type bounds for the maximal operator and follow the ideas in [46, Chapter 1]. We leave the details to interested readers.

Now fix $y \in B(X, 100d_X) \cap \partial\Omega_2$ such that (5.6) and (5.7) hold. Denote $u_r^i := u_{y,r}^i$ and $\psi_r(X) := \psi_{y,r}(X) = \psi(|X-y|/r)$. The fact that $r \ll d_X$ implies that $X \notin B(y, 2r) = \operatorname{supp}\psi_r$ and $\Omega_1 = \Omega_2$ on $\operatorname{supp}\psi_r$. Hence by the Riesz formula (5.5), we have

$$\begin{aligned} u_r^i(X) = \int_{\partial\Omega_i} \psi_r\, \omega_i^X &= -\iint_{\Omega_i} \langle A_i^T(Y)\nabla_Y G_i(X,Y), \nabla_Y\psi_r(Y)\rangle\, dY \\ &= -\iint_{\Omega_1} \langle A_1^T(Y)\nabla_Y G_i(X,Y), \nabla_Y\psi_r(Y)\rangle\, dY, \quad i = 1, 2, \end{aligned} \tag{5.8}$$

where we used that $A_1 = A_2$ on Ω_1 and $\Omega_2 = \Omega_1$ on $\operatorname{supp}\psi_r$.

Notice that by the CFMS estimates and Lemma 2.22, we have $g(y) \approx 1$. We are going to show that

$$\left|u_r^1(X) - g(y)u_r^2(X)\right| \le C\left(\frac{r}{d_X}\right)^\mu g(y)u_r^2(X), \tag{5.9}$$

where μ is the Hölder exponent as in Lemma 2.19. Using that $g(y) \approx 1$ we may divide by the positive constant $u_r^2(X)$ on both sides to obtain

$$\left|\frac{u_r^1(X)}{u_r^2(X)} - g(y)\right| \le C'\left(\frac{r}{d_X}\right)^\mu$$

and letting $r \to 0$ shows

$$\lim_{r\to 0^+} \frac{u_r^1(X)}{u_r^2(X)} = g(y).$$

Then recalling (5.6) and (5.7), the desired equality (5.2) follows. Therefore it suffices to show (5.9) and we do this now.

Using (5.8) and carefully noting that $g(y)$ is a *fixed scalar* since y is fixed, we may use the boundedness of A_1 and the properties of ψ_r to conclude

$$\begin{aligned} &\left|u_r^1(X) - g(y)u_r^2(X)\right| \\ &= \left|\iint_{\Omega_1\cap B(y,2r)} \left\langle A_1^T \nabla_Z\left[G_1(X,Z) - g(y)G_2(X,Z)\right], \nabla_Z\psi_r(Z)\right\rangle dZ\right| \end{aligned}$$

[14] Note that condition (iv) therein is explicitly stated in the proof as not necessary and an analog of the required covering lemma holds in our setting.

$$\lesssim \left(\iint\limits_{\Omega_1 \cap B(y,2r)} \left| \nabla_Z \left[G_1(X,Z) - g(y) G_2(X,Z) \right] \right|^2 dZ \right)^{1/2}$$
$$\times \left(\iint\limits_{\Omega_1 \cap B(y,2r)} |\nabla_Z \psi_r|^2 dZ \right)^{1/2}$$
$$\lesssim r^{\frac{n-1}{2}} \left(\iint\limits_{\Omega_1 \cap B(y,2r)} \left| \nabla_Z \left[G_1(X,Z) - g(y) G_2(X,Z) \right] \right|^2 dZ \right)^{1/2} . \tag{5.10}$$

Again noting that $g(y)$ is a fixed scalar, we may use that $U(Z) = G_1(X,Z) - g(y)G_2(X,Z)$ is a solution to $L_1^T U = 0$ in $\Omega_1 \cap B(y,10r)$ which vanish on $\partial\Omega_1 \cap B(y,10r)$ so that we may apply the boundary Caccioppoli inequality to the function U. It follows that

$$\begin{aligned} \left| u_r^1(X) - g(y) u_r^2(X) \right| &\lesssim r^{n-1} \left(\fint\!\!\fint_{\Omega_1 \cap B(y,4r)} |G_1(X,Z) - g(y) G_2(X,Z)|^2 dZ \right)^{1/2} \\ &\lesssim r^{n-1} \left(\fint\!\!\fint_{\Omega_1 \cap B(y,4r)} \left| \frac{G_1(X,Z)}{G_2(X,Z)} - g(y) \right|^2 |G_2(X,Z)|^2 dZ \right)^{1/2} \\ &\lesssim r^{n-1} \sup_{\Omega_1 \cap B(y,4r)} \left| \frac{G_1(X,\cdot)}{G_2(X,\cdot)} - g(y) \right| \sup_{\Omega_1 \cap B(y,4r)} G_2(X,\cdot). \end{aligned} \tag{5.11}$$

Next, we use Lemma 2.19, the Carleson estimate (Lemma 2.13) to obtain

$$\begin{aligned} \left| u_r^1(X) - g(y) u_r^2(X) \right| &\lesssim \left(\frac{r}{d_X} \right)^{\mu} g(y) r^{n-1} G_2(X, \mathcal{A}((y,r)) \\ &\lesssim \left(\frac{r}{d_X} \right)^{\mu} g(y) \omega_2^X(B(y,r)) \end{aligned}$$

where we used the CFMS estimate (Lemma 2.16) in the second line. Finally, using the local doubling of ω_2^X we see that $u_r^2(X) \approx \omega_2^X(B(y,r))$, which along with the estimate above yields the estimate (5.9). As we had reduced matters to proving (5.9), this shows (5.2).

As remarked above, (5.3) has the same proof as (5.2). To obtain (5.4) we use Lemma 2.19 to deduce that the function

$$g(y) := \lim_{Y \to y} \frac{G_1(X_0,Y)}{G_2(X_0,Y)}$$

satisfies for $y, z \in B(x_0, 10s) \cap \partial\Omega$

$$|g(y) - g(z)| \lesssim \left(\frac{|y-z|}{d_{X_0}}\right)^{\mu} g(x_0) \lesssim \left(\frac{s}{Ms}\right)^{\mu} g(z),$$

where we denote $d_{X_0} := \operatorname{dist}(X_0, \partial\Omega)$. Thus, (5.4) readily follows from (5.3) provided we choose M'' sufficiently large. $\square$

Remark 5.12. Notice that Lemma 5.1, in fact, shows that $\frac{d\omega_1^{X_0}}{d\omega_2^{X_0}}(\cdot)$ is locally Hölder continuous with quantitative estimates. This is simply a consequence of the fact that $\frac{d\omega_1^{X_0}}{d\omega_2^{X_0}}(y) = g(y)$, where $g(y)$ is as in the proof of Lemma 5.1, and g has these estimates by Lemma 2.19.

Combining the Lemma 5.1 with Proposition 4.3 we obtain the following.

Proposition 5.13. *Let $\epsilon \in (0,1)$. There exist positive constants γ, τ and $\widetilde{M} \geq M''$ (M'' is from Lemma 5.1) depending on ϵ and allowable constants such that the following holds.*

Assume $L = -\operatorname{div} A\nabla$ is a divergence form elliptic operator with Λ-elliptic coefficients A satisfying the Hölder condition (1.2), and $\varphi : \mathbb{R}^n \to \mathbb{R}$, $\varphi(0) = 0$, is Lipschitz with $\|\nabla\varphi\|_\infty \leq \gamma$. Then for all $M \geq \widetilde{M}$, $s \in (0, \tau/M)$ and $\xi \in [-s/200, s/200]$, the Poisson kernel, h, for the operator L in the domain

$$\widetilde{\Omega}_{Ms,\xi} = \widetilde{\Omega}_{\varphi,Ms,\xi} := \{(x,t) : x \in \mathbb{R}^n, t > \varphi(x)\} \cap \mathcal{C}(0, Ms, e_{n+1}, \xi)$$

satisfies

$$\frac{\left(\fint_{B(y,r)} (h^X)^2 \, d\sigma(z)\right)^{1/2}}{\fint_{B(y,r)} h^X \, d\sigma(z)} \leq (1+\epsilon)^3, \tag{5.14}$$

for all $y \in B(0, 5s) \cap \operatorname{Graph}(\varphi)$ and $r \in (0, 5s)$. Here $\sigma := \mathcal{H}^n|_{\partial\widetilde{\Omega}_{Ms,\xi}}$ and $X = (\{0\}^n, Mts)$ for some $t \in \left[\frac{1}{4\sqrt{n+1}}, \frac{3}{4\sqrt{n+1}}\right]$.

Proof. We first fix all the constants, the reason for which will become clear shortly. Let $\delta_\epsilon, \gamma_\epsilon, \tau_\epsilon$ be constants from Proposition 4.3 (using $\beta = \epsilon$), and let M'' be the constant from Lemma 5.1. We set $\tau = \tau_\epsilon$ and $\widetilde{M} = 20\sqrt{n+1}\max\{1/\delta_\epsilon, M'', 1\}$. Finally, we set $\gamma = \min\{\gamma_\epsilon, \delta_n, \gamma_n\}$, where γ_n is chosen so that

$$\mathcal{C}(0, s', e_{n+1}, \xi) \cap \Omega_\varphi \subset R_{Q_{s'},\varphi}$$

for all $s' > 0$ and $\xi \leq s'/100$. In particular, the inclusion holds for $\xi \leq s'/(M''100)$. For any Lipschitz function φ whose Lipschitz constant is bounded by γ, the domain

$$\Omega := \Omega_\varphi := \{(x,t) : x \in \mathbb{R}^n, t > \varphi(x)\}$$

clear is a (δ_n, ∞)-chord arc domain. Let $M \geq \widetilde{M}$ and $s \in (0, \tau/M)$ be arbitrary. Then $\widetilde{\Omega}_{Ms,\xi}$ may play the role of Ω_1 in Lemma 5.1 (note $x_0 = 0$ here). Indeed, while $\mathbb{R}^n \times \{0\}$ may not be the plane that minimizes the bilateral distance, the graph of φ is δ_n-flat at scale Ms with respect to this plane.

Let $y \in B(0, 5s) \cap \mathrm{Graph}(\varphi)$ and $r \in (0, 5s)$ be arbitrary. If we write $y = (y', \varphi(y'))$ with $y' \in \mathbb{R}^n$, then $|y'| < 5s$. Notice that Φ, the 'flattening map' for φ, fixes $X = (0, Mts)$ and that

$$|X - y'| \leq \sqrt{|y'|^2 + (Mts)^2} \leq 2Mts$$

by the choice of M, so $X \in \Phi^{-1}(D(y', Mts))$. Moreover, by our choice of $M \geq 20\sqrt{n+1}/\delta_\epsilon$ we have

$$\delta_\epsilon \cdot Mts \geq 5s > r.$$

Thus by Proposition 4.3,

$$\frac{\left(\fint_{B(y,r)} (h_\tau^X)^2 \, d\sigma(z)\right)^{1/2}}{\fint_{B(y,r)} h_\tau^X \, d\sigma(z)} \leq 1 + \epsilon,$$

where h_τ is the Poisson kernel for the operator $L_{\tau,\varphi}$ in $\Omega = \Omega_\varphi$. On the other hand, the elliptic matrices $A_1 = A_{\tau,\varphi}$ and $A_2 = A$ agree on the cylinder $\mathcal{C}(0, Ms, \vec{e}_{n+1}, \xi)$ by the inclusion

$$\mathcal{C}(0, Ms, \vec{e}_{n+1}, \xi) \subset R_{Q_{Ms},\varphi} \subset R_{Q_\tau,\varphi}.$$

Therefore applying (5.4) from Lemma 5.1 we obtain (5.14). $\square$

The following corollary follows immediately by the theory of weights:

Corollary 5.15. *Let $\epsilon \in (0,1)$. There exist constants*[15] *$M^* \geq M'' \geq 1/\epsilon$ and γ'_ϵ, τ'_ϵ depending on ϵ and allowable constants such that the following holds.*

Assume $L = -\operatorname{div} A\nabla$ with Λ-elliptic coefficients satisfying the Hölder condition (1.2), and $\varphi : \mathbb{R}^n \to \mathbb{R}$, $\varphi(0) = 0$, is Lipschitz with $\|\nabla\varphi\|_\infty \leq \gamma'_\epsilon$. Then for all $M \geq M^*$, $s \in (0, \tau'_\epsilon/M)$ and $\xi \in [-s/200, s/200]$ the elliptic measure, $\widetilde{\omega}$, for the operator L in the domain

$$\widetilde{\Omega}_{Ms,\xi} = \widetilde{\Omega}_{\varphi,Ms,\xi} := \{(x,t) : x \in \mathbb{R}^n, t > \varphi(x)\} \cap \mathcal{C}(0, Ms, e_{n+1}, \xi)$$

[15] Here M'' is from Lemma 5.1.

satisfies

$$(1+\epsilon)^{-1}\left(\frac{\sigma(E)}{\sigma(\Delta)}\right)^{1+\epsilon} \leq \frac{\widetilde{\omega}^X(E)}{\widetilde{\omega}^X(\Delta)} \leq (1+\epsilon)\left(\frac{\sigma(E)}{\sigma(\Delta)}\right)^{1-\epsilon}, \quad \forall E \subset \Delta \subset B(0,2s), \tag{5.16}$$

where $\sigma := \mathcal{H}^n|_{\partial\widetilde{\Omega}_{Ms,\xi}}$ *and* $X = (\{0\}^n, Mts)$ *for any* $t \in \left[\frac{1}{4\sqrt{n+1}}, \frac{3}{4\sqrt{n+1}}\right]$ *and* Δ *is an arbitrary surface ball.*

Proof. Proposition 5.13 says that $\widetilde{\omega}^X = h^X d\sigma \in B_2(\sigma)$, and the B_2 constant can be made as close to 1 as possible. For any $\epsilon \in (0,1)$, we choose constants appropriately in the Proposition, so that the B_2 constant (i.e. the right hand side of (5.14)) is bounded above by $\exp((\epsilon/K)^2)$, where K is a dimensional constant as in [32, Corollary 8]. Then by [32, Corollary 8]

$$\widetilde{\omega}^X \in A_{1+\epsilon}(\sigma) \text{ with constant } 1+\epsilon, \quad \widetilde{\omega}^X \in B_{\frac{1}{\epsilon}}(\sigma) \text{ with constant } 1+\epsilon. \tag{5.17}$$

(See [32, Section 3.1] for the definition of $A_p(\sigma)$ weight.) Both estimates in (5.16) then follow easily from (5.17) and Hölder's inequality. □

We are now ready to give the Proof of Theorem 1.1. Here we diverge a bit from the techniques in [40,33], opting for an approach that largely avoids the use of the Poisson kernel and instead works with the elliptic measure more directly. This avoids some of the issues that arise in [40].

Proof of Theorem 1.1. Let Ω be a vanishing chord arc domain. Recall that this means for all $\delta \in (0, \delta_n]$, Ω is a (δ, R_δ) chord arc domain for some $R_\delta > 0$ (see Remarks 2.29). We set $\omega := \omega^{X_0}$ to be the elliptic measure associated to L for the domain Ω with fixed pole $X_0 \in \Omega$.

We first make the following claim which gives a much rougher estimate than what we will produce in the end.

Claim 5.18. The elliptic measure ω is locally an A_∞ weight, that is, there exist τ_0 and constants C_0, θ depending on allowable constants such that

$$\frac{\omega(E)}{\omega(\Delta)} \leq C_0 \left(\frac{\sigma(E)}{\sigma(\Delta)}\right)^{\theta}, \forall E \subset \Delta, \tag{5.19}$$

where Δ is any surface ball with radius less than or equal to τ_0, that is, $\Delta = B(x,r) \cap \partial\Omega$ with $x \in \partial\Omega$ and $r \in (0, \tau_0]$.

To state a more precise estimate, we first fix some constants. For every $\beta \in (0,1)$, we fix a constant $\epsilon \in (0, \beta/2)$ so that

$$(1+\epsilon)\left(\frac{1}{2}+\epsilon\right)^{1-\epsilon} \leq \frac{1}{2}(1+\beta), \tag{5.20}$$

with the intention of using the estimate obtained in Corollary 5.15. Let $M'' > 0$ be the constant found in Lemma 5.1, and $M^*, \gamma'_\epsilon, \tau'_\epsilon > 0$ be constants found in Corollary 5.15, and set $M = \max\{M^*, 1/\beta\}$. Let $\delta \in (0, \delta_n]$ be sufficiently small, depending on $\beta, \epsilon, \gamma'_\epsilon, M^*$ and allowable constants, and recall Ω is a (δ, R_δ)-chord arc domain.

Claim 5.21. For any $x_0 \in \partial\Omega$ and s sufficiently small satisfying

$$sM \leq \min\{\tau'_\epsilon, \tau_0, R_\delta/10, \operatorname{dist}(X_0, \partial\Omega)/5)\},$$

if $E \subset B(x_0, s) \cap \partial\Omega =: \Delta_0$ satisfies

$$\frac{\sigma(E)}{\sigma(\Delta_0)} = \frac{1}{2}$$

then

$$\frac{\widetilde{\omega}(E)}{\widetilde{\omega}(\Delta_0)} = \frac{1}{2} + C\beta, \tag{5.22}$$

where C is a constant depending on the dimension. Here σ is the surface measure for the domain $\widetilde{\Omega}(x_0, Ms)$, which agrees with that of Ω on $B(x_0, s)$; and $\widetilde{\omega}$ is the elliptic measure for $\widetilde{\Omega}(x_0, Ms)$ with pole at $X_1 := x_0 + \frac{1}{2\sqrt{n+1}} Ms\, \vec{n}_{x_0, Ms}$.

Let us take this claim for granted momentarily and see how to conclude the theorem. Assuming (5.22), we want to change to domain from $\widetilde{\Omega}(x_0, Ms)$ to Ω, and change the pole from X_1 to X_0, in the hope to get a similar estimate for $\omega = \omega^{X_0}$. By Lemma 5.1 and the choice of parameters,

$$\frac{\omega^{X_1}(E)}{\omega^{X_1}(\Delta_0)} \leq (1+\epsilon)^2 \frac{\widetilde{\omega}(E)}{\widetilde{\omega}(\Delta_0)} \leq (1+\beta)^2 \left(\frac{1}{2} + C\beta\right).$$

Here we remind the reader that the pole of $\widetilde{\omega}$ is at X_1. To change the pole of ω from X_1 to X_0, we use Lemma 2.21 in a similar manner to Lemma 3.28. Note that the pole X_1 is roughly at distance $Ms \geq s/\beta$ from the center of Δ_0 a surface ball of radius s, and $\operatorname{dist}(X_1, \partial\Omega) \leq Ms \leq \operatorname{dist}(X_0, \partial\Omega)/5$. This allows us to use Lemma 2.21 to say

$$\frac{\omega(E)}{\omega(\Delta_0)} \leq (1 + C\beta^\mu) \frac{\omega^{X_1}(E)}{\omega^{X_1}(\Delta_0)} \leq \frac{1}{2} + C'\beta^\mu. \tag{5.23}$$

To sum up, this combined with Claim 5.21 says that for any $x_0 \in \partial\Omega$ and s sufficiently small (depending on β),

$$\frac{\sigma(E)}{\sigma(\Delta_0)} = \frac{1}{2} \text{ for a Borel set } E \subset \Delta_0 \implies \frac{\omega(E)}{\omega(\Delta_0)} \leq \frac{1}{2} + C'\beta^{\mu}. \tag{5.24}$$

Among other things [32, Theorem 10] establishes the equivalence between the above statement (5.24) (that is, condition (e) in [32, Theorem 10]) and the fact that the B_2 constant of ω is close to 1 (that is, condition (c) in [32, Theorem 10]). We in fact need a quantitative local estimate proved in [32, Theorems 8 and 9], to see that (5.24) implies

$$\left(\fint_{\Delta(x_0,s)} k\, d\sigma \right)^2 \leq (1 + C'\beta^{\mu/2}) \left(\fint_{\Delta(x_0,s)} k^{1/2}\, d\sigma \right), \tag{5.25}$$

where $k = d\omega/d\sigma$ is the Poisson kernel. For the sake of self-containment we include proof of (5.25) in Appendix B, see Lemma B.2. Since β can be chosen arbitrarily small, again by [32, Theorem 8] we conclude that $\log k \in VMO$.

It remains to prove Claims 5.18 and 5.21. We start with Claim 5.21, taking Claim 5.18 for granted.

Proof of Claim 5.21. Without loss of generality we may assume $x_0 = 0$ and $\vec{n}_{x_0,Ms} = \vec{e}_{n+1}$. For notational convenience we set $\widetilde{\Omega} := \widetilde{\Omega}(0, Ms)$.

We make use of the Semmes decomposition to approximate Ω by Lipschitz domains, see [45,33]. To be precise by [33, Lemma 5.1] (replacing δ^4 by δ), when δ is chosen sufficiently small, there exist two Lipschitz functions $\varphi^{\pm} : \mathbb{R}^n \to \mathbb{R}$ whose graphs approximate $\partial\Omega$ from inside(+) and the outside(-). Set $\Gamma_{\pm} := \text{Graph}(\varphi^{\pm})$ and

$$\Omega^{\pm} := \{(x,t) \in \mathbb{R}^{n+1} : x \in \mathbb{R}^n, t > \varphi(x)\}$$

then the following properties hold:

(1) $\|\nabla\varphi^{\pm}\|_{\infty} \leq C_1\delta^{1/4}$
(2) $D[\Gamma^{\pm} \cap \overline{B(0,Ms)}; \partial\Omega \cap \overline{B(0,Ms)}] \leq C_1\delta^{1/4}Ms$
(3) $\mathcal{H}^n\left(B(0,Ms) \cap \partial\Omega \setminus \Gamma_{\pm}\right) \leq c_1 \exp(-c_2\delta^{-1/4})\omega_n(Ms)^n$
(4) $\Omega^+ \cap \mathcal{C}(0,Ms) \subseteq \Omega \cap \mathcal{C}(0,Ms) \subseteq \Omega^- \cap \mathcal{C}(0,Ms)$,

where C_1, c_1, c_2 are all positive constants depending only on the dimension. Now we let $\widetilde{\omega}_{\pm}$ be the elliptic measure (for L) in the domains $\widetilde{\Omega}^{\pm} := \Omega^{\pm} \cap \mathcal{C}(0,Ms)$ with pole at $X_1 := \frac{1}{2\sqrt{n+1}}Mse_n$ and $\sigma_{\pm}$ be the surface measure for $\widetilde{\Omega}^{\pm}$. We choose δ sufficiently small, to ensure that $C_1\delta^{1/4}Ms < s/200$ and $C_1\delta^{1/4} < \gamma'_{\epsilon}$. Thus by (2) $|\varphi^{\pm}(0)| = |\varphi^{\pm}(0) - 0| < s/200$. (This was the reason for the using parameter ξ to shift the cylinder appearing in Corollary 5.15 above.) Applying Corollary 5.15 to Lipschitz domains $\widetilde{\Omega}^{\pm}$, we have

$$(1+\epsilon)^{-1}\left(\frac{\sigma_{\pm}(F)}{\sigma_{\pm}(\Delta)}\right)^{1+\epsilon} \leq \frac{\widetilde{\omega}_{\pm}(F)}{\widetilde{\omega}_{\pm}(\Delta)} \leq (1+\epsilon)\left(\frac{\sigma_{\pm}(F)}{\sigma_{\pm}(\Delta)}\right)^{1-\epsilon}, \quad \forall F \subset \Delta \subset B((0,\varphi^{\pm}(0)), 2s), \tag{5.26}$$

where Δ is a surface ball of the form $B(x,r) \cap \partial\Omega^{\pm}$. (We will not use $\widetilde{\Omega}^{+}$ for the proof of this claim, but it is used in proving Claim 5.18.)

Now let $E \subset \Delta_0 = B(0,s) \cap \partial\Omega$ be such that $\sigma(E) = (1/2)\sigma(\Delta_0)$. Set $\Delta_- := B((0,\varphi^-(0)),s) \cap \Gamma_-$. Then

$$\frac{\sigma_-(E \cap \Gamma_-)}{\sigma_-(\Delta_-)} = \frac{\sigma(E \cap \Gamma_-)}{\sigma_-(\Delta_-)} \leq \frac{\sigma(E)}{\sigma(\Delta_0)} \frac{\sigma(\Delta_0)}{\sigma_-(\Delta_-)} \leq \frac{1}{2} \frac{(1+\delta)\omega_n s^n}{(1+C_1^2\delta^{1/2})^{-n/2}\omega_n s^n} \leq \frac{1}{2} + \epsilon$$

by choice of δ sufficiently small, where we used property (1) of the Semmes decomposition, (4.7) for the lower bound on $\sigma_-(\Delta_-)$ and for the upper bound on $\sigma(\Delta_0)$ we used the definition of (δ, R)-chord arc domain (Definition 2.28). It follows from (5.26) and the choice (5.20) of ϵ that

$$\frac{\widetilde{\omega}_-(E \cap \Gamma_-)}{\widetilde{\omega}_-(\Delta_-)} \leq \frac{1}{2}(1+\beta). \tag{5.27}$$

We now estimate

$$\frac{\widetilde{\omega}(E)}{\widetilde{\omega}(\Delta_0)} = \frac{\widetilde{\omega}(E \cap \Gamma_-)}{\widetilde{\omega}(\Delta_0)} + \frac{\widetilde{\omega}(E \setminus \Gamma_-)}{\widetilde{\omega}(\Delta_0)} =: I + II. \tag{5.28}$$

Notice that

$$\begin{aligned}
\frac{\sigma(\Delta_0 \setminus \Gamma_-)}{\sigma(\Delta_0)} &\leq \frac{\sigma(M\Delta_0 \setminus \Gamma_-)}{\sigma(\Delta_0)} \\
&\leq \frac{c_1 \exp(-c_2\delta^{-1/4})\omega_n (Ms)^n}{(1-\delta)\omega_n s^n} \\
&= \frac{c_1 \exp(-c_2\delta^{-1/4}) M^n}{1-\delta},
\end{aligned} \tag{5.29}$$

by property (3) and the definition of (δ, R)-chord arc domain (Definition 2.28). Therefore by a simple pole change argument (or the shaper version, Lemma 2.21), Lemma 2.22 and the fact that ω is A_∞ (Claim 5.18) we obtain

$$II \leq \frac{\widetilde{\omega}(\Delta_0 \setminus \Gamma_-)}{\widetilde{\omega}(\Delta_0)} \lesssim \frac{\omega(\Delta_0 \setminus \Gamma_-)}{\omega(\Delta_0)} \lesssim \left(\frac{\sigma(\Delta_0 \setminus \Gamma_-)}{\sigma(\Delta_0)}\right)^{\theta} < \beta$$

by choice of δ small. It remains to treat term I, which is a matter of 'removing the minus sign' in (5.27). Since $\widetilde{\Omega} \subset \widetilde{\Omega}^-$, by the maximal principle

$$\frac{\widetilde{\omega}(E \cap \Gamma_-)}{\widetilde{\omega}(\Delta_0)} \leq \frac{\widetilde{\omega}_-(E \cap \Gamma_-)}{\widetilde{\omega}_-(\Delta_-)} \frac{\widetilde{\omega}_-(\Delta_-)}{\widetilde{\omega}(\Delta_0)} \leq \frac{1}{2}(1+\beta) \frac{\widetilde{\omega}_-(\Delta_-)}{\widetilde{\omega}(\Delta_0)}.$$

Thus, we may reduce proving Claim 5.21 to the estimate

$$\frac{\widetilde{\omega}_-(\Delta_-)}{\widetilde{\omega}(\Delta_0)} \leq (1+\beta)^2 \tag{5.30}$$

and we do this now.

Notice $|X_1 - 0| \approx Ms \approx |X_1 - (\varphi^-(0), 0)|$ so that Bourgain's estimate (Lemma 2.15) and the Harnack inequality yield the estimate

$$c \leq \widetilde{\omega}_-(\Delta_-), \widetilde{\omega}(\Delta_0) \leq 1, \tag{5.31}$$

where c is a constant depending on ϵ, by way of M. Additionally, by (5.26) we have that

$$\begin{aligned}
&\frac{\widetilde{\omega}_-(\Delta_- \setminus (1-\delta^{1/16})\Delta_-)}{\widetilde{\omega}_-(\Delta_-)} \\
&\leq (1+\epsilon)\left(\frac{\sigma_-(\Delta_- \setminus (1-\delta^{1/16})\Delta_-)}{\sigma_-(\Delta_-)}\right)^{1-\epsilon} \\
&\leq (1+\epsilon)\left(\frac{1-(1+C_1^2\delta^{1/2})^{-n/2}(1-\delta^{1/16})^n}{(1+C_1^2\delta^{1/2})^{-n/2}}\right)^{1-\epsilon} \\
&\leq \frac{\beta}{1+\beta}
\end{aligned}$$

by choice of δ, where we used the property (1) and the inclusion (4.7). Therefore

$$\widetilde{\omega}_-(\Delta_-) \leq (1+\beta)\widetilde{\omega}_-((1-\delta^{1/16})\Delta_-)$$

and to prove (5.30), it is enough to show

$$\widetilde{\omega}_-((1-\delta^{1/16})\Delta_-) \leq (1+\beta)\widetilde{\omega}(\Delta_0). \tag{5.32}$$

Let us now assume that δ is small enough so that $C_1\delta^{1/4}Ms < \delta^{1/8}s$. Let $x \in \partial\widetilde{\Omega}\setminus\Delta_0 = \partial\widetilde{\Omega} \setminus B(0,s)$, then either $x \in \partial\Omega \cap \overline{B(0,Ms)} \setminus B(0,s)$ or $x \in \partial\mathcal{C}(0,Ms)$. In the first case, the choice of δ and the property (2) guarantee that there exists $\hat{x} \in \Gamma_- \cap \overline{B(0,Ms)}$ such that $|x-\hat{x}| < \delta^{1/8}s$, and thus

$$|\hat{x} - (\varphi^-(0), 0)| \geq |x| - |x-\hat{x}| - |(\varphi^-(0), 0)| \geq (1-2\delta^{1/8})s.$$

In particular,

$$B\left(\hat{x}, \frac{\delta^{1/16}}{2}\right) \cap (1-\delta^{1/16})\Delta_- = \emptyset.$$

This means we may use the Hölder continuity of solutions vanishing at the boundary (Lemma 2.14) to yield the estimate

$$\widetilde{\omega}_-^x((1-\delta^{1/16})\Delta_-) \le C\left(\frac{\delta^{1/8}}{\delta^{1/16}}\right)^{\mu} \le C\delta^{\mu/16},$$

where μ is from Lemma 2.14. In the second case, that is, when $x \in \partial\mathcal{C}(0, Ms) \cap \partial\widetilde{\Omega}$, then $x \in \partial\mathcal{C}(0, Ms) \cap \partial\widetilde{\Omega}^-$ since $\Omega \cap \mathcal{C}(0, Ms) \subset \Omega^- \cap \mathcal{C}(0, Ms)$. Hence $\widetilde{\omega}_-^x((1-\delta^{1/16})\Delta_-) = 0$. In either case, we have that

$$\widetilde{\omega}_-^x((1-\delta^{1/16})\Delta_-) \le C\delta^{\mu/16}$$

whenever $x \in \partial\widetilde{\Omega} \setminus \Delta_0$. Therefore by the maximum principle (and that $\widetilde{\omega}_-^x((1-\delta^{1/16})\Delta_-) \le 1$ for $x \in \Delta_0$) we have

$$\begin{aligned}\widetilde{\omega}_-((1-\delta^{1/16})\Delta_-) &\le \widetilde{\omega}(\Delta_0) + C\delta^{\mu/16}\\ &\le (1 + (C/c)\delta^{\mu/16})\widetilde{\omega}(\Delta_0)\\ &\le (1+\beta)\widetilde{\omega}(\Delta_0)\end{aligned}$$

by choice of δ sufficiently small, where we used (5.31) in the second line. This shows (5.32) and the claim follows. □

We are left with proving Claim 5.18, which will be a consequence of the maximum principle, the Semmes decomposition above and the theory of weights. The proof is essentially contained [18], but we include a proof that tracks the constants carefully.

Proof of Claim 5.18. By the work of Coifman and Ferfferman [10], or rather its (local) generalization to spaces of homogeneous type, to prove the claim it is enough to show the following. For any $\eta \in (0,1)$, there exists $\gamma = \gamma(\eta) \in (0,1)$ such that for every surface ball $\Delta_0 = B(x,s) \cap \partial\Omega$ with $x \in \partial\Omega$ and $s \in (0, \tau_0]$, if $E \subset \Delta_0$ is a Borel set satisfying $\sigma(E) \ge \gamma\sigma(\Delta_0)$, then $\omega(E) \ge \eta\omega(\Delta_0)$. Here $\tau_0 > 0$ is a fixed constant whose value is to be specified later. Without loss of generality we assume $x = 0$.

Similar to the proof of Claim 5.21, the idea here is also to use the elliptic measure of Lipschitz domain (this time we use Ω^+ instead of Ω^-) to estimate ω. For that purpose we just need a crude version of Corollary 5.15. Let $\epsilon = 1/2$ be fixed, thus we fix the constants $M^*, \gamma'_\epsilon, \tau'_\epsilon$ accordingly. Set $M = M^*$. We fix $\delta \in (0, \delta_n]$ satisfying $C_1\delta^{1/4} \le \gamma'_\epsilon$, then there exists $R_\delta > 0$ such that Ω is a (δ, R_δ)-chord arc domain. Let $\tau_0 = \min\{R_\delta, \tau'_\epsilon\}/M$ and $s \in (0, \tau_0]$ be arbitrary. Then Ω has Semmes decomposition in $B(0, Ms)$; moreover by the choice of δ we may apply Corollary 5.15 to $\widetilde{\Omega}^+$. Hence in particular, $\widetilde{\omega}^+ \in A_\infty(\sigma_+)$ on $\Delta_+ := B((0, \varphi^+(0)), s) \cap \Gamma_+$. For $\widetilde{\eta} > 0$ to be determined later, there exists $\gamma = \gamma(\widetilde{\eta}) > 0$ such that

$$\frac{\sigma_+(F)}{\sigma_+(\Delta_+)} \ge \frac{\gamma}{2} \text{ for a Borel set } F \subset \Delta_+ \implies \frac{\widetilde{\omega}_+(F)}{\widetilde{\omega}_+(\Delta_+)} \ge \widetilde{\eta}. \tag{5.33}$$

We then use properties (1) and (3) of Semmes decomposition along with the definition of (δ, R)-chord arc domain to ensure that

$$\sigma(\Delta_+ \cap \Delta_0) \geq (1-\delta)\omega_n s^n - c_1 \exp(-c_2\delta^{-1/4})\omega_n (Ms)^n \geq \left(1 - \frac{\gamma}{3}\right)\sigma(\Delta_0).$$

We may need to choose γ slightly bigger (depending on the value of δ), to make sure the last inequality holds. Suppose that $E \subset \Delta_0$ is a Borel set with $\sigma(E) \geq \gamma\sigma(\Delta_0)$. Then

$$\sigma_+(E \cap \Delta_+) = \sigma(E \cap \Delta_+) \geq \frac{2}{3}\gamma\sigma(\Delta_0) \geq \frac{\gamma}{2}\sigma_+(\Delta_+),$$

where we used the estimates of $\sigma(\Delta_0), \sigma_+(\Delta_+)$ and that δ is chosen sufficiently small. It follows from (5.33) that

$$\frac{\widetilde{\omega}_+(E \cap \Delta_+)}{\widetilde{\omega}_+(\Delta_+)} \geq \widetilde{\eta}.$$

By the maximum principle and Bourgain's estimate

$$\widetilde{\omega}(E) \geq \widetilde{\omega}_+(E \cap \Delta_+) \approx \frac{\widetilde{\omega}_+(E \cap \Delta_+)}{\widetilde{\omega}_+(\Delta_+)} \geq \widetilde{\eta}.$$

By Lemma 2.22 and a simple change of pole argument (or the shaper version, Lemma 2.21) we have

$$\frac{\omega(E)}{\omega(\Delta_0)} \approx \omega^{X_1}(E) \approx \widetilde{\omega}(E).$$

Hence

$$\frac{\omega(E)}{\omega(\Delta_0)} \geq C\widetilde{\eta} = \eta$$

as desired. Here we chose $\widetilde{\eta}$ to account for the constant C. □

As we had reduced the proof of the theorem to Claims 5.18 and 5.21, we have proved the theorem. □

Data availability

No data was used for the research described in the article.

Appendix A. A brief explanation of the first order approach

The goal of this appendix is twofold. Firstly, we justify that the solution to the L^2-Dirichlet problem obtained in [1] satisfies the estimates (3.23) and (3.24). Whereas, modulo verification of the assumptions, (3.23) is explicitly stated in [1, Theorem 2.4(ii)], the estimate (3.24) requires a deeper understanding of [1] and is obtained by combining several lemmas and theorems in various sections. We recommend to the interested reader to read *Section 3, Road map to the proofs* in [1] for the general idea of their first order approach. We give a brief overview here. After the general approach and all relevant notation are laid out, we give a rigorous proof of (3.24) in Subsection A.4. Then, we show that when the boundary data satisfies $0 \le f \in C_c^\infty(\mathbb{R}^n)$, the solution above (namely, the solution to the L^2-Dirichlet problem obtained in [1]) agrees with the classical elliptic measure solution (see (3.14)). This is carried out in Subsection A.5.

In what follows we will adopt the notation of [1] and write[16] (t, x) instead of (x, t) and for a vector $\vec{v} \in \mathbb{C}^{n+1}$ write $v = (v_\perp, v_\parallel)$ with $v_\perp$ being a scalar.[17] This change of basis is also reflected in the definition of the coefficient matrix A in any divergence form elliptic operator. Note also that some of the conditions here are often easier to state in the scalar case, but we maintain the notation in [1] for the ease of identifying estimates therein.

The first-order approach takes its name from the following: Suppose that u is a solution to

$$L_A u := -\operatorname{div} A\nabla u = 0 \text{ in } \mathbb{R}^{n+1}_+ \tag{A.1}$$

with $\nabla u \in L^2_{loc}(\mathbb{R}_+; L^2(\mathbb{R}^n, \mathbb{C}^{n+1}))$, then its conormal gradient

$$f = \nabla_A u := \begin{bmatrix} \partial_{\nu_A} u \\ \nabla_x u \end{bmatrix} \qquad \text{with } \partial_{\nu_A} u = (A\nabla_{t,x} u)_\perp \tag{A.2}$$

solves the first-order equation

$$\partial_t f + DBf = 0, \tag{A.3}$$

where $D = \begin{bmatrix} 0 & \operatorname{div}_x \\ -\nabla_x & 0 \end{bmatrix}$ and B is a matrix defined by A as follows. We first write $A = \begin{bmatrix} A_{\perp\perp} & A_{\perp\parallel} \\ A_{\parallel\perp} & A_{\parallel\parallel} \end{bmatrix}$, where $A_{\perp\perp}$ is a scalar. The strong ellipticity of the matrix A implies $A_{\perp\perp}$ is strictly positive. We thus define

$$B = \hat{A} := \underline{A}(\overline{A})^{-1} = \begin{bmatrix} 1 & 0 \\ A_{\parallel\perp} & A_{\parallel\parallel} \end{bmatrix} \begin{bmatrix} A_{\perp\perp} & A_{\perp\parallel} \\ 0 & I \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ A_{\parallel\perp} & A_{\parallel\parallel} \end{bmatrix} \begin{bmatrix} A_{\perp\perp}^{-1} & -A_{\perp\perp}^{-1} A_{\perp\parallel} \\ 0 & I \end{bmatrix}. \tag{A.4}$$

In fact, if we make the restriction that solutions f to (A.3) belong to the space $L^2_{loc}(\mathbb{R}_+; \mathcal{H})$ where

$$\mathcal{H} := \left\{ g \in L^2(\mathbb{R}^n, \mathbb{C}^{n+1}) : \operatorname{curl}_x g = 0 \right\},$$

[16] This is done through a change of basis, so we keep the notion that $\mathbb{R}^{n+1}_+ = \{(t, x) : t > 0\}$.

[17] The authors work with elliptic systems in [1], but our overview will be for equations for simplicity, i.e. $m = 1$.

we have that the solutions u to divergence form elliptic operator (A.1) (with the 'gradient bound on slices' as above) are in one-to-one correspondence with solutions f to the first-order equation (A.3). See [1, Proposition 4.1], where this is worked out in detail and note that this does not require the operator to be t-independent. Therefore to find solutions to boundary value problems (Neumann BVPs, regularity BVPs, or Dirichlet BVPs) of (A.1) using first order approach, we need to

(1) study the well-posedness of the first-order equation (A.3) in an appropriate functional space;
(2) relate and determine the trace of f at $t = 0$ from the boundary value to (A.1).

Notice that the first task depends on the operator DB and it has nothing to do with what type of BVPs we consider.

A.1. Well-posedness of the first order equation

When the operator $A(t, x) = A_0(x)$ is t-independent and is a small L^∞-perturbation of a constant matrix (or a real symmetric matrix), the well-posedness of the corresponding first order equation

$$\partial_t f + DB_0 f = 0, \qquad \text{with } B_0 = \hat{A}_0 \tag{A.5}$$

has been established in [4]. (This result has been obtained before, in [21,2,3], but we appeal to [4] since the notion of well-posedness there is compatible with [1].) The authors remark that the operator DB_0 is not a sectorial operator, but instead bisectorial, that is, its spectrum is contained in a double sector around the real axis. This means that the natural operator e^{-tDB_0} associated with the free evolution equation (A.5) is not well-defined on all of $\mathcal{H} \subset L^2(\mathbb{R}^n; \mathbb{C}^{1+n})$ for any $t \neq 0$. Thus we need to split $\mathcal{H}$ into the spectral subspace $E_0^+\mathcal{H}$ for the sector in the right half plane and the spectral subspace $E_0^-\mathcal{H}$ for the sector in the left half plane.[18] Moreover, the authors in [4] show that any solution f to (A.5) in the appropriate functional space (such that $f \in L^2_{loc}(\mathbb{R}_+; \mathcal{H})$ and $\widetilde{N}_* f \in L^2$) is given by the *generalized Cauchy reproducing formula*

$$f = C_0^+ f_0 = e^{-tDB_0} E_0^+ f_0, \qquad \text{for some } f_0 \in E_0^+\mathcal{H}. \tag{A.6}$$

Additionally, using the notation $f_t = f(t, \cdot)$ it satisfies

$$\lim_{t\to 0} f_t = f_0 \quad \text{and} \quad \lim_{t\to\infty} f_t = 0$$

[18] $E^{\pm_0} := \chi^\pm(DB_0)$ provided by the bounded holomorphic calculus, where χ^+ is the indicator of the right-half of the complex plane and χ^- is the indicator of the left-half of the complex plane. See [1, Section 4], between Proposition 4.1 and Proposition 4.3.

in the L^2 sense.

Since every solution to (A.5) is given by an explicit formula, to study the well-posedness when the matrix $A(t,x)$ is a (Carleson measure) perturbation of a t-independent operator $A_0(x)$, we rewrite its first order equation (A.3) as

$$\partial_t f + DB_0 f = D\mathcal{E}f, \tag{A.7}$$

where $\mathcal{E}(t,x) = B_0(x) - B(t,x)$ denotes the perturbation. We remind the reader that $B = \hat{A}$ and $B_0 = \hat{A}_0$ are defined as in (A.4). We also remark that if $\|A - A_0\|_{\mathcal{C}} < \infty$ then using the notation in [1, Lemma 5.5] (see [1, Definition 5.4] for the definition of the operator norm $\|\cdot\|_*$)

$$\|\mathcal{E}\|_* \lesssim \|\mathcal{E}\|_{\mathcal{C}} = \|B - B_0\|_{\mathcal{C}} \lesssim \|A - A_0\|_{\mathcal{C}},$$

with implicit constants depending on the dimension and ellipticity constants. This is because

$$B - B_0 = \underline{A}(\overline{A})^{-1} - \underline{A_0}(\overline{A_0})^{-1} = [(\underline{A} - \underline{A}_0)(\overline{A})^{-1}] + [\,\underline{A}_0((\overline{A})^{-1} - (\overline{A}_0)^{-1})]$$

and $\Lambda \lesssim \operatorname{Re} A_{\perp\perp}, \operatorname{Re}(A_0)_{\perp\perp} \le \max\{\|A\|_\infty, \|A_0\|_\infty\}$, which allows us to control both bracketed terms using the Carleson norm of $A - A_0$. Hence under the assumption $\|A - A_0\|_{\mathcal{C}} \ll 1$, the right hand side of (A.7) can be thought of as a small error to the free evolution of $\partial_t f + DB_0 f$. The above discussion formally justifies [1, Theorem 8.2], which says solutions to (A.3), or equivalently (A.7), in the appropriate functional space is of the form

$$f_t = C_0^+ h^+ + S_A f_t = e^{-tDB_0} E_0^+ h^+ + S_A f_t, \qquad \text{for some } h^+ \in E_0^+ \mathcal{H}; \tag{A.8}$$

and formally speaking as $t \to 0$ we have $f_t \to f_0 = h^+ + h^-$, where $h^- \in E_0^- \mathcal{H}$ and is determined explicitly. (Notice that when $h^+ \in E_0^+ \mathcal{H}$, we have $e^{-DB_0} E_0^+ h^+ = e^{-t|DB_0|} h^+$, see for example the discussion in page 62 of [1]. So the above formula is the same as that in [1, Theorem 8.2].) The operator S_A is given formally by

$$S_A f_t := \int_0^t e^{-(t-s)DB_0} E_0^+ D\mathcal{E}_s f_s ds - \int_t^\infty e^{(s-t)DB_0} E_0^- D\mathcal{E}_s f_s ds, \tag{A.9}$$

see [1, Equation (1)]; for a rigorous treatment see [1, Proposition 7.1]. Moreover they also show

$$\|\widetilde{N}_*(S_A f)\|_{L^2(\mathbb{R}^n)} \lesssim \|\mathcal{E}\|_* \|\widetilde{N}_* f\|_{L^2(\mathbb{R}^n)} \lesssim \|A - A_0\|_{\mathcal{C}} \|\widetilde{N}_* f\|_{L^2(\mathbb{R}^n)},$$

whereby one concludes the boundedness of $(1 - S_A)^{-1}$ on the space

$$\mathcal{X} := \{f : \mathbb{R}^{n+1}_+ \to \mathbb{C}^{1+n};\ \widetilde{N}_* f \in L^2\}$$

for $\|A - A_0\|_{\mathcal{C}}$ sufficiently small. Therefore one seeks solutions of the form

$$f = (1 - S_A)^{-1} C_0^+ h^+, \qquad \text{for some } h^+ \in E_0^+ \mathcal{H}. \tag{A.10}$$

We summarize the above discussion. Our goal is to find solutions to divergence form elliptic equations (A.1) with prescribed boundary values (they could be Neumann, regularity, or Dirichlet boundary values). By the one-to-one correspondence between solutions to (A.1) and solutions to the first-order equation (A.3), it suffices to determine h^+ using the prescribed boundary value and then use the ansatz (A.10) to compute f. In other words, we need to determine the trace of f_t using the boundary value of u.

A.2. Determining the trace of f by Neumann or regularity boundary values of u

By looking at (A.2), how f is defined using u, it should be intuitively clear that Neumann BVPs and regularity BVPs are more natural in the first order approach, since the trace of f_t is related naturally to the Neumann or regularity boundary value. To be more precise, when $A = A_0$ is t-independent, a solution $f = C_0^+ h^+$ satisfies the Neumann boundary value $(f_0)_\perp = \partial_{\nu_{A_0}} u = \varphi$ if and only if its trace h^+ solves the equation $\Gamma_{A_0} h^+ = \varphi$, where

$$\begin{aligned} \Gamma_{A_0} :& E_0^+ \mathcal{H} \to L^2(\mathbb{R}^n, \mathbb{C}) \\ & h^+ \mapsto (h^+)_\perp \end{aligned}$$

is the identification map from the trace of f to the Neumann boundary value of the solution u to the divergence form elliptic operator (A.1). In other words the well-posedness of the Neumann BVP is equivalent to Γ_{A_0} being an isomorphism. Similarly the well-posedness of the regularity BVP is equivalent to the identification map (with regularity boundary value)

$$\begin{aligned} \Gamma_{A_0} :& E_0^+ \mathcal{H} \to \{g \in L^2(\mathbb{R}^n, \mathbb{C}^n) : \operatorname{curl}_x g = 0\} \\ & h^+ \mapsto (h^+)_\parallel \end{aligned}$$

being an isomorphism. We remark that even for t-independent operators, these maps are not always invertible; however Γ_{A_0} is invertible if we assume a-priori the well-posedness of BVPs for L_{A_0} (for example see [1, Corollary 8.6]).

Now we begin to consider operators with t-dependent coefficients. Recall in the above discussion on the well-posedness of equation (A.3), or equivalently (A.7), a solution f of the form (A.8) (for some $h^+ \in E_0^+ \mathcal{H}$) has trace

$$f_0 = h^+ + h^- \text{ and } h^- = \int_0^\infty \Lambda e^{-s\Lambda} \widehat{E}_0^- \mathcal{E}_s f_s \, ds \in E_0^- \mathcal{H},$$

where $\Lambda = |DB_0|$ and $\widehat{E}_0^-$ is defined as [1, Equation (22)]. See [1, Theorem 8.2] for the precise statement. Therefore f satisfies the Neumann boundary condition $(f_0)_\perp = \partial_{\nu_A} = \varphi$ if and only if h^+ solves the equation $\Gamma_A h^+ = \varphi$, where Γ_A is the map

$$\begin{aligned}\Gamma_A :& E_0^+\mathcal{H} \to L^2(\mathbb{R}^n, \mathbb{C}) \\ & h^+ \mapsto (f_0)_\perp = \left(h^+ + \int_0^\infty \Lambda e^{-s\Lambda} \widehat{E}_0^- \mathcal{E}_s f_s ds \right)_\perp .\end{aligned}$$

It follows that the well-posedness[19] of Neumann BVPs in appropriate function space $\mathcal{X}$ is equivalent to Γ_A being an isomorphism. On the other hand, one can show[20]

$$\|\Gamma_A - \Gamma_{A_0}\|_{L^2 \to L^2} \lesssim \|\mathcal{E}\|_* \lesssim \|A - A_0\|_{\mathcal{C}}$$

so that one may deduce the invertibility of Γ_A from that of Γ_{A_0}, provided the Carleson norm here is small. Thus, the Neumann BVP is well-posed for L_A, provided the smallness of the Carleson norm and the well-posedness of the Neumann BVP for L_{A_0}. The method for the regularity problem is similar (where we instead wish to invert the tangential trace).

A.3. Dirchlet boundary value problems

Solving Dirichlet BVPs using the above first order approach is more complicated compared to Neumann BVPs or regularity BVPs; and we will give more details in this section since it is directly related to our proof of (3.24). It is not obvious how the first order approach applies, due to the lack of identification between the trace of f and the Dirichlet boundary value of u. Instead of the equation (A.3) for the conormal gradient f, we consider vector-valued solutions to first order equation

$$\partial_t v + BDv = 0. \tag{A.11}$$

Heuristically, applying D to the equation (A.11) gives $(\partial_t + DB)(Dv) = 0$. On the other hand, u solves the divergence form elliptic equation if and only if $(\partial_t + DB)(\nabla_A u) = 0$. By comparing $f = \nabla_A u$ with $f = Dv$ we find that $u = -v_\perp$. More precisely, for coefficients $A(t, x)$ which are (Carleson measure) perturbations of t-independent coefficient $A_0(x)$, we have that solutions to the divergence form elliptic equation (A.1) obeying a certain square function estimate (that is, $\nabla u \in \mathcal{Y}$ defined below in (A.16)) are of the form

$$u = c - v_\perp$$

[19] Here one is seeking solutions in the space $\mathcal{X}$ and the boundedness of $(1 - S_A)^{-1} C_0^+$ as an operator from E_0^+ into $\mathcal{X}$ are provided in [1].

[20] See the proof of [1, Corollary 8.6].

where v solves the first order equation (A.11) and $c \in \mathbb{C}$. Moreover, we have

$$\lim_{t\to 0} u_t = c - (v_0)_\perp \quad \text{ and } \lim_{t\to\infty} u_t = c$$

in the L^2 sense. In particular, if we impose that u has Dirichlet boundary value in $L^2(\mathbb{R}^n, \mathbb{C})$, the constant c is zero. See [1, Theorem 9.3]. We will follow the first order approach as before to study solutions to equation (A.11) and then find an ansatz for solutions to Dirichlet BVPs.

For t-independent operators we note that B_0D is another bisectorial operator, just like DB_0. So we define the spectral projections $\widetilde{E}_0^{\pm} = \chi^{\pm}(B_0D)$ as before, which splits the space $\mathcal{H}$. We make an important remark with regards to DB_0 and B_0D:

$$B_0\, b(DB_0) = b(B_0D)\, B_0 \tag{A.12}$$

where $b(\cdot)$ denotes the functional calculus to an operator on L^2. (See the discussion in [1, Section 7].) This observation allows us to switch between DB_0 and B_0D. Similar to the argument for DB_0, we have that solutions to (A.11) obeying a square function estimate are of the form

$$v = \widetilde{C}_0^{+} v_0 + c = e^{-tB_0D}\widetilde{E}_0^{+} v_0 + c \tag{A.13}$$

for a unique $v_0 \in \widetilde{E}_0^{+}L^2$ and some $c \in \mathbb{C}^{1+n}$. Therefore, for t-independent operators we have the representation formula

$$u = c - \left(\widetilde{C}_0^{+} v_0\right)_\perp, \qquad v_0 \in \widetilde{E}_0^{+}L^2, c \in \mathbb{C} \tag{A.14}$$

for solutions u to Dirichlet BVPs obeying a square function estimate. See [1, Corollary 9.4].

Now we consider perturbations of t-independent operators. Recall that solutions f to (A.3) are of the form

$$f_t = e^{-tDB_0}h^+ + S_A f_t, \qquad \text{or equivalently } f_t = (I - S_A)^{-1}e^{-tDB_0}h^+, \tag{A.15}$$

for some $h^+ \in E_0^+\mathcal{H}$. We remark that to adapt to Dirichlet BVPs, we work with the functional space

$$\mathcal{Y} := \left\{ f : \mathbb{R}^{n+1}_+ \to \mathbb{C}^{1+n}; \int_0^\infty \|f_t\|^2_{L^2(\mathbb{R}^n)} t\,dt < \infty \right\} = L^2(\mathbb{R}_+, t\,dt; L^2(\mathbb{R}^n, \mathbb{C}^{1+n})), \tag{A.16}$$

instead of $\mathcal{X}$. Moreover [1, Proposition 7.1] shows that

$$\|S_A\|_{\mathcal{Y}\to\mathcal{Y}} \lesssim \|A - A_0\|_{\mathcal{C}}.$$

Provided that $\|A - A_0\|_{\mathcal{C}}$ is sufficiently small, $(1 - S_A)^{-1}$ exists as a bounded operator on the space $\mathcal{Y}$.

We want to find v satisfying $f = Dv$, so we basically need to factor out D in (A.15) (even though D is not injective). Indeed by (A.12), imposing the free evolution term $g := e^{-tDB_0}h^+$ (the first term in (A.15)) in $\mathcal{Y}$ allows us to rewrite

$$g = De^{-tB_0D}\widetilde{E}_0^+\widetilde{h}^+ = D\widetilde{C}_0^+\widetilde{h}^+ \tag{A.17}$$

for some $\widetilde{h}^+ \in \widetilde{E}_0^+L^2$ determined by $h^+ = D\widetilde{h}^+$. And formally for S_A, we obtain starting from (A.9) that $S_A = D\widetilde{S}_A$ where

$$\widetilde{S}_A f_t := \int_0^t e^{-(t-s)B_0D}\widetilde{E}_0^+ D\mathcal{E}_s f_s ds - \int_t^\infty e^{(s-t)B_0D}\widetilde{E}_0^- D\mathcal{E}_s f_s ds. \tag{A.18}$$

See [1, Proposition 7.2] for the rigorous treatment of $\widetilde{S}_A$. Putting them together we get

$$f_t = e^{-tDB_0}h^+ + S_A f_t = D\widetilde{C}_0^+\widetilde{h}^+ + D\widetilde{S}_A f_t,$$

and thus we can set

$$v = \widetilde{C}_0^+\widetilde{h}^+ + \widetilde{S}_A f_t = \widetilde{C}_0^+\widetilde{h}^+ + \widetilde{S}_A(I - S_A)^{-1}D\widetilde{C}_0^+\widetilde{h}^+.$$

In the second equality we substitute the expression for f_t in (A.15) and we also use (A.17). Notice that the first term is exactly the same as (A.13), the solution to $\partial_t v + B_0Dv = 0$ (the constant c there is always zero for L^2-Dirichlet BVPs). Therefore, to solve the Dirichlet problem to (A.1), we make the ansatz

$$u = \left(\widetilde{C}_0^+\widetilde{h}^+ + \widetilde{S}_A(I - S_A)^{-1}D\widetilde{C}_0^+\widetilde{h}^+\right)_\perp \tag{A.19}$$

for some $\widetilde{h}^+ \in \widetilde{E}_0^+L^2$. Moreover,

$$\lim_{t\to 0} v_t = v_0 \quad \text{and} \quad \lim_{t\to\infty} v_t = 0$$

in the L^2 sense. The trace v_0 can be written explicitly as

$$v_0 = \widetilde{h}^+ + \widetilde{h}^- \text{ and } \widetilde{h}^- = -\int_0^\infty e^{-s\widetilde{\Lambda}}\widetilde{E}_0^-\mathcal{E}_s f_s\, ds \in \widetilde{E}_0^- L^2, \tag{A.20}$$

with $\widetilde{\Lambda} = |B_0D|$. See [1, Theorems 9.2, 9.3] and the proof of [1, Corollary 9.5].

It then follows that the solution u has Dirichlet boundary value $\varphi \in L^2(\mathbb{R}^n)$ if and only if $\widetilde{h}^+ \in \widetilde{E}_0^+L^2$ satisfies

$$\varphi = \lim_{t\to 0} u_t = -(v_0)_\perp = \left(-\widetilde{h}^+ + \int_0^\infty e^{-s\widetilde{\Lambda}} \widetilde{E}_0^- \mathcal{E}_s f_s \, ds\right)_\perp,$$

where $f = (I - S_A)^{-1} D\widetilde{C}_0^+ \widetilde{h}^+$ by the formula (A.15). In other words, if we define the map

$$\begin{aligned} \widetilde{\Gamma}_A :& \widetilde{E}_0^+ L^2 \to L^2(\mathbb{R}^n, \mathbb{C}) \\ & \widetilde{h}^+ \mapsto \left(-\widetilde{h}^+ + \int_0^\infty e^{-s\widetilde{\Lambda}} \widetilde{E}_0^- \mathcal{E}_s f_s \, ds\right)_\perp, \end{aligned}$$

then $\widetilde{h}^+$ is determined by the equation $\widetilde{\Gamma}_A \widetilde{h}^+ = \varphi$. The well-posedness of Dirichlet BVPs is equivalent to $\widetilde{\Gamma}_A$ being an isomorphism. In particular, when $A = A_0$ is t-independent, this map is just

$$\begin{aligned} \widetilde{\Gamma}_{A_0} :& \widetilde{E}_0^+ L^2 \to L^2(\mathbb{R}^n, \mathbb{C}) \\ & \widetilde{h}^+ \mapsto -\left(\widetilde{h}^+\right)_\perp. \end{aligned}$$

Assuming the Dirichlet problem for L_{A_0} is well-posed, the map $\widetilde{\Gamma}_{A_0}$ is invertible. Since (see the proof of [1, Corollary 9.5])

$$\|\widetilde{\Gamma}_{A_0} - \widetilde{\Gamma}_A\|_{L^2\to L^2} \lesssim \|\mathcal{E}\|_* \lesssim \|A - A_0\|_{\mathcal{C}}, \tag{A.21}$$

it follows that $\widetilde{\Gamma}_A$ is also invertible provided $\|A - A_0\|_{\mathcal{C}} \ll 1$. Therefore the Dirichlet problem for L_A is also well-posed.

A.4. Proof of estimates for non-tangential maximal function

We are now ready to use the results in [1] to prove the desired estimates (3.23) and (3.24) for the non-tangential maximal function. Recall that the elliptic matrix we consider can be written of the form

$$A(t, x) = A_0(x) + B(t, x)$$

where $A_0(x)$ is a small (t-independent) perturbation of a *constant* real elliptic matrix and $B(t, x)$ has small Carleson norm, see (3.1) and (3.2). (Here we denote the t-independent matrix by $A_0(x)$ instead of $A_1(x)$ in (3.1), in order to be consistent with the notation in the rest of the Appendix.) The well-posedness for Dirichlet problems is established in [3] for elliptic operators whose t-independent coefficient matrices are small perturbations of the constant matrix. Thus the Dirichlet problem for L_{A_0} is well-posed if ϵ in (3.2) is sufficiently small. We claim that this solution agrees with the *elliptic measure solution* (as in (3.14)) for smooth, compactly supported data and we postpone the proof of this fact to Subsection A.5.

The estimate (3.23) just follows directly from [1, Theorem 2.4(ii)]. Now we set out to prove (3.24). We write corresponding solutions to L_A and L_{A_0} as $u_{\varphi,A}$ and u_{φ,A_0}, respectively. By the discussion in the above section, we can write

$$\begin{aligned} u_{\varphi,A_0} &= \left(\widetilde{C}_0^+\widetilde{\Gamma}_{A_0}^{-1}\varphi\right)_\perp, \\ u_{\varphi,A} &= \left(\widetilde{C}_0^+\widetilde{\Gamma}_A^{-1}\varphi + \widetilde{S}_A(I-S_A)^{-1}D\widetilde{C}_0^+\widetilde{\Gamma}_A^{-1}\varphi\right)_\perp. \end{aligned}$$

Hence

$$\begin{aligned} u_{\varphi,A} - u_{\varphi,A_0} &= \left(\widetilde{C}_0^+(\widetilde{\Gamma}_A^{-1} - \widetilde{\Gamma}_{A_0}^{-1})\varphi\right)_\perp + \left(\widetilde{S}_A(I-S_A)^{-1}D\widetilde{C}_0^+\widetilde{\Gamma}_A^{-1}\varphi\right)_\perp \\ &= \mathrm{I} + \mathrm{II}. \end{aligned}$$

Notice that I is just $\widetilde{C}_0^+\tilde{h}^+$ with

$$\tilde{h}^+ := (\widetilde{\Gamma}_A^{-1} - \widetilde{\Gamma}_{A_0}^{-1})\varphi \in \widetilde{E}_0^+ L^2.$$

Recall (A.21) and the invertibility of $\widetilde{\Gamma}_{A_0}$, we have

$$\begin{aligned} \|\widetilde{\Gamma}_A^{-1} - \widetilde{\Gamma}_{A_0}^{-1}\|_{L^2\to L^2} &= \left\|\widetilde{\Gamma}_A^{-1}\left(\widetilde{\Gamma}_{A_0} - \widetilde{\Gamma}_A\right)\widetilde{\Gamma}_{A_0}^{-1}\right\|_{L^2\to L^2} \\ &\lesssim \|\widetilde{\Gamma}_A - \widetilde{\Gamma}_{A_0}\|_{L^2\to L^2} \\ &\lesssim \|A - A_0\|_{\mathcal{C}}, \end{aligned}$$

provided $\|A - A_0\|_{\mathcal{C}}$ is sufficiently small. It follows then

$$\|\tilde{h}^+\|_{L^2} \lesssim \|A - A_0\|_{\mathcal{C}}\|\varphi\|_{L^2}.$$

As in the proof of [1, Theorem 10.1], we may write $\tilde{h}^+$ as $\tilde{h}^+ = B_0 h^+$ with $h^+ \in E_0^+\mathcal{H}$ so that

$$\|\widetilde{N}_*(\mathrm{I})\|_{L^2} = \|\widetilde{N}_*(\widetilde{C}_0^+ B_0 h^+)\|_{L^2} = \|\widetilde{N}_*(B_0 C_0 h^+)\|_{L^2} \approx \|C_0 h^+\|_{\mathcal{X}} \lesssim \|h^+\|_{L^2} \approx \|\tilde{h}^+\|_{L^2},$$

where we use (A.12), [1, Theorem 5.2] and the accretivity of B_0. Thus

$$\|\widetilde{N}_*^{3/2}(\mathrm{I})\|_{L^2} \le \|\widetilde{N}_*(\mathrm{I})\|_{L^2} \lesssim \|A - A_0\|_{\mathcal{C}}\|\varphi\|_{L^2},$$

as desired. To handle II we use [1, Lemma 10.2] which says that

$$\|\widetilde{N}_*^{3/2}((\widetilde{S}_A f)_\perp)\|_{L^2} \lesssim \|\mathcal{E}\|_*\|f\|_{\mathcal{Y}} \lesssim \|A - A_0\|_{\mathcal{C}}\|f\|_{\mathcal{Y}}.$$

(This is why we have non-tangential maximal function with power 3/2 in (3.24).) Thus, to obtain a desirable bound for II it is enough to show

$$\|(I - S_A)^{-1} D\widetilde{C}_0^+ \tilde{h}^+\|_{\mathcal{Y}} \lesssim \|\tilde{h}^+\|_{L^2}, \qquad \text{for any } \tilde{h}^+ \in \widetilde{E}_0^+ L^2, \tag{A.22}$$

where we used that $\widetilde{\Gamma}_A^{-1}$ is an isomorphism to exchange $\widetilde{\Gamma}_A^{-1}\varphi$ for $\tilde{h}^+$. Recall (see [1, Proposition 7.1]) that $(I - S_A)^{-1} : \mathcal{Y} \to \mathcal{Y}$ is bounded; moreover by the accretivity of B_0 and the square function estimate for the operator $B_0 D$ we have

$$\|D\widetilde{C}_0^+ \tilde{h}^+\|_{\mathcal{Y}} \approx \|B_0 D e^{-tB_0 D} \tilde{h}^+\|_{\mathcal{Y}} \approx \|\tilde{h}^+\|_{L^2}, \quad \forall \tilde{h}^+ \in \widetilde{E}_0^+ L^2.$$

This finishes the proof of (A.22) and therefore

$$\begin{aligned} \|\widetilde{N}_*^{3/2}(\mathrm{II})\|_{L^2} &\lesssim \|A - A_0\|_{\mathcal{C}} \|(I - S_A)^{-1} D\widetilde{C}_0^+ \widetilde{\Gamma}_A^{-1}\varphi\|_{\mathcal{Y}} \\ &\lesssim \|A - A_0\|_{\mathcal{C}} \|\widetilde{\Gamma}_A^{-1}\varphi\|_{L^2} \\ &\lesssim \|A - A_0\|_{\mathcal{C}} \|\varphi\|_{L^2}. \end{aligned}$$

This concludes the proof of (3.24) modulo showing that the solution here agrees with the elliptic measure solution, when the boundary data is smooth.

A.5. For smooth data the solution in [1] agrees with the elliptic measure solution

It remains to show that given $0 \le f \in C_c^\infty(\mathbb{R}^n)$ the unique L^2-solution u in the sense of [1] agrees with the elliptic measure solution u_f as in (3.14).[21] (It suffices to consider functions $0 \le f \in C_c^\infty(\mathbb{R}^n)$, since this is enough to characterize the elliptic measure, or equivalently the Poisson kernel. Assuming that $u \in C(\overline{\mathbb{R}^{n+1}_+})$, by the maximum principle it suffices to show $u - u_f(X) \to 0$ as $|X| \to \infty$. Since we know that $u_f(X) \to 0$ as $|X| \to \infty$,[22] to prove $u \equiv u_f$ it suffices to show u is continuous all the way to the boundary and $u(X) \to 0$ as $|X| \to \infty$. In fact, we will show in the following lemma that $u \in \dot{C}^\beta(\overline{\mathbb{R}^{n+1}_+})$, where $\dot{C}^\beta$ is the homogeneous Hölder space:

$$\|v\|_{\dot{C}^\beta(E)} := \sup_{\substack{x,y \in E \\ x \ne y}} \frac{|v(x) - v(y)|}{|x - y|^\beta},$$

for a function $v : E \to \mathbb{R}$.

[21] This is not without cause, since in general (for example, when the coefficient matrix is non-symmetric), even with smooth boundary data different notions of solutions may not agree, see the example in [7]. Even for the Laplacian in the upper half space, it is well known that solutions to the Dirichlet problem are not unique.

[22] This can be proven by comparing it with the elliptic measure of a compact set. Let $K = \operatorname{supp} f$ and assume $K \subset B_R$ for some $R > 0$. Since $0 \le f \in C_c^\infty(\mathbb{R}^n)$, we have $u_f(X) \le \omega^X(\Delta_R) \cdot \sup f$. On the other hand, let A_R denote the interior corkscrew point for the ball B_R. By Lemma 2.16 and the estimate of the Green's function, when $X \in \mathbb{R}^{n+1}_+ \setminus B_{4R}$ we have $\omega^X(\Delta_R) \approx G(X, A_R) \cdot R^{n-1} \lesssim \frac{R^{n-1}}{|X - A_R|^{n-1}}$. The right hand side converges to zero as $|X| \to \infty$. Therefore $\omega^X(\Delta_R) \to 0$, and thus $u_f(X) \to 0$ as $|X| \to \infty$.

Lemma A.23 *([26,25]). Let*

$$A(t,x) = A_0(x) + B(t,x)$$

be a real matrix such that $A_0(x) = A_c + \widetilde{B}(x)$, A_c *is a real constant elliptic matrix,* $\|\widetilde{B}\|_\infty < \epsilon_0$ *and* $\|B(t,x)\|_{\mathcal{C}} < \epsilon_0$. *If* ϵ_0 *is small depending on the ellipticity of* A_c *and dimension then there exists* $\beta \in [0,1)$ *depending on ellipticity, the dimension and* ϵ_0 *such that the following holds. If* $f \in C_c^\infty(\mathbb{R}^n)$ *and* u *is the (unique) solution to the* L^2*-Dirichlet problem for* $L = -\operatorname{div} A\nabla$ *on* $\mathbb{R}^{n+1}_+$ *produced above, then*

$$\|u\|_{\dot{C}^\beta(\overline{\mathbb{R}^{n+1}_+})} \lesssim \|f\|_{\dot{C}^\beta(\mathbb{R}^n)}, \tag{A.24}$$

where the implicit constant depends on dimension and ellipticity of A_c.

Proof. Here we appeal to the 'second order methods' noting that the L^2 solution to the Dirichlet problem above is unique so that we are working with the same solution. The lemma is, in fact, a 'direct' result of [26, Theorem 1.4] and [25, Theorem 1.35]. First note that A_c is constant so that we may assume A_c is symmetric as demonstrated in Lemma 3.5 above. Thus, [26, Theorem 1.4] gives the solvability of the $\dot{C}^{\beta'}$ Dirichlet problem for coefficients A_0 with $\beta' \in (0, \beta_0')$ depending on dimension, ellipticity of A_c and ϵ_0, provided ϵ_0 is small enough. Similarly, (at the cost of a smaller β) [25, Theorem 1.35] can be applied to perturb from the coefficients A_0 to A, giving the solvability of the $\dot{C}^\beta$ Dirichlet problem with $\beta \in (0, \beta_0'/2)$ for $L = -\operatorname{div} A\nabla$ provided ϵ_0 is small enough. Here is where one should be careful: We need to check that the $\dot{C}^\beta$ and L^2 solutions agree when the data is in $\dot{C}^\beta(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, we will call this 'compatibility'. Note that the $\dot{C}^\beta$ and L^2 solutions agree for the operator with coefficients $L_c = -\operatorname{div} A_c\nabla$ because they are both given by convolution with a elliptic-Poisson kernel (see Theorems 1.4 and 3.3 in [37]). The interested reader can carefully check[23] that the perturbations [26, Theorem 1.4] and [25, Theorem 1.35] preserve this compatibility. □

With the lemma in hand, we are going to prove $u(X) \to 0$ as $|X| \to \infty$. By definition it holds that

$$\|\widetilde{N}_*(u)\|_{L^2(\mathbb{R}^n)} \lesssim \|f\|_{L^2(\mathbb{R}^n)}$$

which (by interior estimates) implies that

$$\sup_{t>0} \|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \|f\|_{L^2(\mathbb{R}^n)}.$$

[23] This will be a result of the boundary trace of the layer potentials being perturbative in the norms $\dot{C}^\alpha$ and L^2: In the case of [26] this is done in [26, Section 4] and in the case of [25] in [25, Proposition 7.21].

Next, we see for $s \in \mathbb{R}_+$, by breaking up $\mathbb{R}^n$ into cubes of side length roughly s and using Caccioppoli's inequality, that

$$\int_s^{2s} \int_{\mathbb{R}^n} |\nabla u|^2 \, dx \, dt \lesssim \frac{1}{s^2} \int_{s/2}^{5s/2} \int_{\mathbb{R}^n} |u|^2 \, dx \, dt \lesssim \frac{\|f\|^2_{L^2(\mathbb{R}^n)}}{s},$$

where we used $\sup_{t>0} \|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \|f\|_{L^2(\mathbb{R}^n)}$. Thus,

$$\int_s^{\infty} \int_{\mathbb{R}^n} |\nabla u|^2 \, dx \, dt = \sum_{k=0}^{\infty} \int_{2^k s}^{2^{k+1} s} \int_{\mathbb{R}^n} |\nabla u|^2 \, dx \, dt \lesssim \sum_{k=0}^{\infty} \frac{\|f\|^2_{L^2(\mathbb{R}^n)}}{2^k s} \lesssim \frac{\|f\|^2_{L^2(\mathbb{R}^n)}}{s},$$

written compactly $\|\nabla u\|_{L^2(\mathbb{R}^{n+1}_s)} \lesssim \frac{\|f\|_{L^2(\mathbb{R}^n)}}{\sqrt{s}}$, where $\mathbb{R}^{n+1}_s = \{(t,x) : t > s, x \in \mathbb{R}^n\}$. It is a fact[24] that there exists a constant c such that $u - c \in Y^{1,2}(\mathbb{R}^{n+1}_s) := \{v \in L^{\frac{2n}{n-1}}(\mathbb{R}^{n+1}_s) : \nabla v \in L^2(\mathbb{R}^{n+1}_s)\}$, but since $\sup_{t>0} \|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} < +\infty$ it must be the case that $c = 0$ and hence $u \in Y^{1,2}(\mathbb{R}^{n+1}_s)$. Moreover, by the Poincaré Sobolev inequality

$$\|u\|_{L^{\frac{2n}{n-1}}(\mathbb{R}^{n+1}_s)} \lesssim \|\nabla u\|_{L^2(\mathbb{R}^{n+1}_s)} \lesssim \|f\|_{L^2(\mathbb{R}^n)}/\sqrt{s}, \tag{A.25}$$

where we used our estimate established above.

Now fix $\gamma > 0$ and let $C_1 := \|u\|_{\dot{C}^\beta(\overline{\mathbb{R}^{n+1}_+})} < \infty$. Our goal is to show that there exists M_γ so that if $|X| > M_\gamma$ then $|u(X)| \le \gamma$. Note that if there exists $X = (t,x)$ such that $|u(X)| > \gamma$ then we have that $|u(Y)| > \gamma/2$ for all $Y \in \mathbb{R}^{n+1}_+$ such that $|X - Y| < (\gamma/2C_1)^{1/\beta}$. In particular, if $s \in \mathbb{R}_+$ and there exists $X = (t,x)$ such that $|u(X)| > \gamma$ with $t \ge s$ then

$$\|u\|_{L^{\frac{2n}{n-1}}(\mathbb{R}^{n+1}_s)} \gtrsim \gamma^{\frac{n^2-1}{\beta 2n}+1}$$

where we used that $|u(Y)| > \gamma/2$ in $B(X, (\gamma/2C_1)^{1/\beta}) \cap \mathbb{R}^{n+1}_s$. Thus, choosing s_0 large enough so that

$$\frac{\|f\|_{L^2(\mathbb{R}^n)}}{\sqrt{s_0}} \ll \gamma^{\frac{n^2-1}{\beta 2n}+1}$$

we have from (A.25) that $\|u\|_{L^\infty(\mathbb{R}^{n+1}_{s_0})} \le \gamma$.

Having established the bound for $t \ge s_0$, it suffices to show that there exists A so that if $|x| > A$ and $t \in [0, s_0]$ then $|u(t,x)| \le \gamma$. To do this, we use that $\sup_{t>0} \|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} < \|f\|_{L^2(\mathbb{R}^n)}$ and hence

[24] See [41, Theorem 1.78]. One can adapt the proof there using nested cubes (this time not concentric dilates though) which exhaust $\mathbb{R}^{n+1}_s$.

$$\|u\|_{L^2([0,s_0]\times\mathbb{R}^n)} \lesssim \sqrt{s_0}\|f\|_{L^2(\mathbb{R}^n)}.$$

Arguing as above, it would be impossible to have $X_k = (x_k, t_k)$ with $|x_k| \to \infty$ and $t_k \in [0, s_0]$ such that $|u(X_k)| \geq \gamma$. Indeed, for this would give that $\|u\|_{L^2([0,s_0]\times\mathbb{R}^n)} = \infty$. This concludes the proof that $u(X) \to 0$ as $|X| \to \infty$.

Appendix B. From (5.24) to (5.25)

The goal of this section is to prove Lemma B.2, which will immediately show (5.24) implies (5.25). We first make an observation that allows us to use the work of Korey [32] with impunity in the setting of this work. Specifically, we would like to use the consequences of [32, Theorem 6] and the portion of [32, Theorem 10] unique to the work there.

Lemma B.1. *Let $\Gamma \subset \mathbb{R}^{n+1}$ be a closed set with locally finite $\mathcal{H}^n$ measure, that is $\mathcal{H}^n(\Gamma \cap B(0,r)) < \infty$ for every $r > 0$. Set $\sigma = \mathcal{H}^n|_\Gamma$. Then for every Borel set of $E \subseteq \Gamma$ and $\tau \in [0,1]$ there exists a Borel F with $\sigma(F) = \tau\sigma(E)$.*

Proof. Fix E as above. Clearly, we only need to show result for $\tau = 1/2$. We see by monotonicity of measure $\sigma(B(0,R)\cap E) > (1/2)\sigma(E)$ for some R large enough. Let $r_0 := \sup\{r : \sigma(B(0,r) \cap E) < (1/2)\sigma(E)\}$. By monotonicity of measure $\sigma(E \cap \partial B(0,r_0)) + \sigma(B(0,r_0)\cap E) \geq (1/2)\sigma(E)$ and $\sigma(B(0,r_0)\cap E) \leq (1/2)\sigma(E)$. Thus, $\tau'\sigma(E\cap\partial B(0,r_0)) + \sigma(B(0,r_0) \cap E) = (1/2)\sigma(E)$ for some $\tau' \in [0,1]$. Next we note that $\widetilde{\sigma} := H^n|_{\partial B(0,r_0)}$ already has the diffusivity property, that is, for every $E' \subset \partial B(0,r_0)$ and $\tau' \in [0,1]$ there exists $F' \subseteq E'$ with $\widetilde{\sigma}(F') = \tau'\widetilde{\sigma}(E')$ so that we may take $E' = E \cap \partial B(0,r_0)$ and find $F' \subseteq E \cap \partial B(0,r_0)$ so that $\sigma(F \cap \partial B(0,r_0)) = \tau'\sigma(E \cap \partial B(0,r_0))$. Setting $F = F' \cup (B(0,r_0) \cap E)$ we have that $\sigma(F) = (1/2)\sigma(E)$. $\square$

Lemma B.2. *Let $\Gamma \subset \mathbb{R}^{n+1}$ be a closed set with locally finite $\mathcal{H}^n$ measure and set $\sigma = \mathcal{H}^n|_\Gamma$. Suppose $x_0 \in \Gamma$ and $r_0 > 0$ are such that $\sigma(\Delta_0) > 0$, where $\Delta_0 = B(x_0,r_0) \cap \Gamma$. Suppose $k \in L^1_{loc}(d\sigma)$ with $k \geq 0$ and set $w = k\,d\sigma$. There exists ϵ_0 and c, absolute constants, so that the following holds. If there exists $\epsilon > 0$ such that for every $F \subset \Delta_0$ with $\sigma(F)/\sigma(\Delta_0) = 1/2$ it holds that $w(F)/w(\Delta_0) \leq 1/2 + \epsilon$ for some $\epsilon \in (0, \epsilon_0)$ then*

$$\left(\fint_{\Delta(x_0,s)} k\,d\sigma\right)^2 \leq (1 + c\epsilon)\left(\fint_{\Delta(x_0,s)} k^{1/2}\,d\sigma\right).$$

Proof. We start the proof exactly as in the proof (d) to (c) in [32, Theorem 10]. Set $f := \sqrt{k}$ an $L^2_{loc}(d\sigma)$ function. Set m_{k,Δ_0} to be the **median** of the function k on Δ_0, that is

$$\sigma(\{x \in \Delta_0 : k > m_{k,\Delta_0}\}), \sigma(\{x \in \Delta_0 : k < m_{k,\Delta_0}\}) \leq (1/2)\sigma(\Delta_0)$$

Set $E' = \{x \in \Delta_0 : k > m_{k,\Delta_0}\}$ and $F' = \{x \in \Delta_0 : k < m_{k,\Delta_0}\}$. Then by Lemma B.1 there exists $G \subseteq \Delta_0 \setminus (E' \cup F')$ such that $\sigma(E' \cup G) = (1/2)\sigma(\Delta_0)$. Setting $E := E' \cup G$ and $F := \Delta_0 \setminus E$ we have that $\sigma(E) = \sigma(F) = (1/2)\sigma(\Delta_0)$,

$$k(x) \le m_{k,\Delta_0}, \quad \sigma - a.e. x \in F$$

and

$$k(x) \ge m_{k,\Delta_0}, \quad \sigma - a.e. x \in E.$$

By hypothesis

$$\frac{w(F)}{w(\Delta_0)} = 1 - \frac{w(E)}{w(\Delta_0)} \ge \frac{1}{2} - \epsilon.$$

Then

$$\begin{aligned}
\fint_{\Delta_0} f^2 \, d\sigma &= \frac{1}{\sigma(\Delta_0)} \int_{\Delta_0} k \, d\sigma = \frac{w(\Delta_0)}{\sigma(\Delta_0)} \\
&\le (1-2\epsilon)^{-1} \frac{w(F)}{\sigma(F)} = (1-2\epsilon)^{-1} \frac{1}{\sigma(F)} \int_F k \, d\sigma \\
&= (1-2\epsilon)^{-1} \frac{1}{\sigma(F)} \int_F f^2 \, d\sigma \le (1-2\epsilon)^{-1} \frac{1}{\sigma(F)} \int_F f \, d\sigma \sqrt{m_{k,\Delta_0}},
\end{aligned}$$

where we used $f \le \sqrt{m_{k,\Delta_0}}$ σ-a.e. on F. On the other hand,

$$m_{k,\Delta_0} \le \fint_E k \, d\sigma = \frac{w(E)}{\sigma(E)} \le \frac{\frac{1}{2} + \epsilon}{\frac{1}{2}} \frac{w(\Delta_0)}{\sigma(\Delta_0)} = (1+2\epsilon) \fint_{\Delta_0} k \, d\sigma,$$

where we used $\sigma(E)/\sigma(\Delta_0) = 1/2$ and the hypothesis of the lemma. Combining these two inequalities we have

$$\begin{aligned}
\fint_{\Delta_0} f^2 \, d\sigma &\le (1-2\epsilon)^{-1} \left(\fint_F f \, d\sigma \right) \sqrt{m_{k,\Delta_0}} \\
&\le (1+2\epsilon)^{1/2} (1-2\epsilon)^{-1} \left(\fint_{\Delta_0} f \, d\sigma \right) \left(\fint_{\Delta_0} k \, d\sigma \right)^{1/2} \\
&= (1+2\epsilon)^{1/2} (1-2\epsilon)^{-1} \left(\fint_{\Delta_0} f \, d\sigma \right) \left(\fint_{\Delta_0} f^2 \, d\sigma \right)^{1/2},
\end{aligned}$$

where we used that the average over F of k is less than the average over Δ_0 of k, by the definition of F and $E = \Delta_0 \setminus F$. Thus, we have

$$\left(\fint_{\Delta_0} f^2 \, d\sigma \right)^{1/2} \leq (1 + c\epsilon) \fint_{\Delta_0} f \, d\sigma, \tag{B.3}$$

provided that ϵ_0 is sufficiently small. □

To show (5.24) implies (5.25), we apply the previous lemma with $\epsilon = C'(\beta')^\mu$. We also remark here that in order to conclude that $\log k$ in VMO from (5.25) one can use the methods in [32] (armed with Lemma B.1).

Appendix C. Pullbacks and pushforwards for Lipschitz domains

In this appendix, we recall the well-known applications of arguments concerning the "flat" transformation of solutions to (second order divergence form) elliptic equations on Lipschitz domains to solutions of a transformed elliptic operator on the upper half space. This transformation is well adapted to the perturbations under consideration in this work as many have noted. We continue to work in $\mathbb{R}^{n+1}$, identified as $\mathbb{R}^{n+1} = \{(x,t) \in \mathbb{R}^n \times \mathbb{R}\}$. For a Lipschitz function $\varphi : \mathbb{R}^n \to \mathbb{R}$, with $\varphi(0) = 0$[25] we set

$$\Omega_\varphi := \{(x,t) \in \mathbb{R}^{n+1} : t > \varphi(x)\}.$$

We will often just write Ω except when the dependence on φ is important. We also define

$$\mathrm{Graph}(\varphi) := \{(x,t) \in \mathbb{R}^{n+1} : t = \varphi(x)\}.$$

The following proposition follows by a simple change of variables.

Proposition C.1. *Let $\varphi : \mathbb{R}^n \to \mathbb{R}$, $\varphi(0) = 0$ be a Lipschitz function and $\Omega = \Omega_\varphi$ be as above. Let $L = -\operatorname{div} A\nabla$ be an elliptic operator with real coefficient matrix A. Let $\Phi(x,t)$ be the* ***flattening map for*** *φ, $\Phi(x,t) := (x, t - \varphi(x))$, so that $\Phi(\Omega_\varphi) = \mathbb{R}^{n+1}_+$ and $\Phi(\mathrm{Graph}(\varphi)) = \mathbb{R}^n \times \{0\}$. Then $u : \Omega \to \mathbb{R}$ solves the differential equation*

$$(D)_{L,\Omega} \begin{cases} Lu = 0 \in \Omega \\ u = f \in C_c(\partial\Omega) \end{cases}$$

if and only if $\tilde{u} : \mathbb{R}^{n+1}_+ \to \mathbb{R}$ solves the differential equation

[25] We can always arrange for this by shifting all of the objects under consideration.

$$(D)_{\widetilde{L},\mathbb{R}^{n+1}_+} \begin{cases} \widetilde{L}\tilde{u} = 0 \in \mathbb{R}^{n+1}_+ \\ \tilde{u} = \tilde{f}, \end{cases}$$

where $\tilde{f}(x) = f(x, \varphi(x)) \in C_c(\mathbb{R}^n)$, $\tilde{u}(X) = (u \circ \Phi^{-1})(X)$ *and* $\widetilde{L} = -\operatorname{div} \widetilde{A}\nabla$ *with* $\widetilde{A} = J_\Phi^T(A \circ \Phi^{-1})J_\Phi$. *Here* J_Φ *is the Jacobian matrix of* Φ, *given by*

$$J_\Phi(X) = J_\Phi(x,t) = \left[\begin{array}{c|c} I_{n\times n} & -\nabla\varphi(x) \\ \hline 0 & 1 \end{array}\right],$$

so that, in fact, J_Φ *is a function of* x. *In particular, if* A *is* t*-independent so is* $\tilde{A}$.

Moreover, if $h_A^X = \frac{d\omega^X}{d\sigma}$ *is the Poisson kernel for* L *in* Ω *with pole at* X *exists then*

$$h_A^X(y, \varphi(y)) = \frac{1}{\sqrt{1 + |\nabla\varphi(y)|^2}}\, k_{\tilde{A}}^{\Phi(X)}(y),$$

where $k_{\tilde{A}}^{\Phi(X)}(y)$ *is the Poisson kernel for* $\widetilde{L}$ *in* $\mathbb{R}^{n+1}_+$ *with pole at* $\Phi(X)$.

To see the last fact, note that for $X \in \Omega$ and $f \in C_c(\partial\Omega)$

$$u(X) = \int_{\partial\Omega} h_A^X(Z)f(Z)d\sigma(Z) = \int_{\mathbb{R}^n} h_A^X(y, \varphi(y))f(y, \varphi(y))\sqrt{1 + |\nabla\varphi(y)|^2}dy;$$

and on the other hand,

$$u(X) = \tilde{u}(\Phi(X)) = \int_{\mathbb{R}^n} k_{\tilde{A}}^{\Phi(X)}(y)\tilde{f}(y)\, dy = \int_{\mathbb{R}^n} k_{\tilde{A}}^{\Phi(X)} f(y, \varphi(y))\, dy.$$

From the above we can see that when $\|\nabla\varphi\|_\infty \ll 1$ the Poisson kernels h_A and $k_{\widetilde{A}}$ are very similar. We would like to say that perturbed operators (in the analogous sense of that in Proposition 3.10) remain so under pullback. This amounts to look at how J_Φ acts on vectors and matrices. The following lemma can be directly verified via computation. We provide some brief details.

Lemma C.2. *Let* $\varphi : \mathbb{R}^n \to \mathbb{R}$, $\varphi(0) = 0$ *be a Lipschitz function with Lipschitz constant* $\gamma := \|\nabla\varphi\|_{L^\infty}$. *For almost every* $x \in \mathbb{R}^n$, $J_\Phi = J_\Phi(x)$ *has the following properties.*

(a) *For any* $\xi \in \mathbb{R}^{n+1}$,

$$\begin{aligned} &|J_\Phi\xi - \xi|_2 \le \sqrt{n}\gamma|\xi|_2, && |J_\Phi\xi - \xi|_\infty \le \gamma|\xi|_\infty, \\ &|J_\Phi^T\xi - \xi|_2 \le \gamma|\xi|_2, && |J_\Phi^T\xi - \xi|_\infty \le \sqrt{n}\gamma|\xi|_\infty. \end{aligned}$$

(b) *For any* $(n+1)\times(n+1)$ *matrix* A,

$$|J_\Phi^T AJ_\Phi - A|_\infty \le (\sqrt{n}\gamma + n\gamma^2)|A|_\infty.$$

(c) *For any* $\xi \in \mathbb{R}^{n+1}$,

$$|J_\Phi \xi|_2 \geq \min\left\{\frac{1}{2}, (1+4n\gamma^2)^{-\frac{1}{2}}\right\}|\xi|_2.$$

Here $|\cdot|_2$ and $|\cdot|_\infty$ are the ℓ^2 and ℓ^∞ norms, respectively.

Proof. For $\xi \in \mathbb{R}^{n+1}$, $\xi = (\xi_1, \dots, \xi_n, \xi_{n+1})$ we write $\xi_\| = (\xi_1, \dots, \xi_n)$ and $\xi_\perp = \xi_{n+1}$. To prove the assertions of (a) concerning J_Φ, we write $J_\Phi \xi = (\xi_\| - \xi_\perp \nabla\varphi(x), \xi_\perp)$, so that $J_\Phi\xi - \xi = (-\xi_\perp \nabla\varphi(x), 0)$. Similarly, to treat the estimates involving J_Φ^T we write $J_\Phi^T\xi = (\xi_\|, -\nabla\varphi(x)\cdot\xi_\| + \xi_\perp)$, so that $J_\Phi^T\xi - \xi = (0, -\nabla\varphi(x)\cdot\xi_\|)$. Property (b) follows from property (a) after writing

$$J_\Phi^T A J_\Phi - A = (J_\Phi^T A J_\Phi - J_\Phi^T A) + (J_\Phi^T A - A)$$

so that

$$\begin{aligned}
|J_\Phi^T A J_\Phi - A|_\infty &\leq |J_\Phi^T A J_\Phi - J_\Phi^T A|_\infty + |J_\Phi^T A - A|_\infty \\
&\leq \sqrt{n}\gamma|AJ_\Phi - A|_\infty + \sqrt{n}\gamma|A|_\infty \\
&\leq (\sqrt{n}\gamma + n\gamma^2)|A|_\infty.
\end{aligned}$$

Finally, to see (c), we again write $J_\Phi\xi = (\xi_\| - \xi_\perp\nabla\varphi(x), \xi_\perp)$ so that $|J_\Phi\xi|_2 \geq \max(|\xi_\| - \xi_\perp\nabla\varphi|_2, |\xi_\perp|_2)$. If $|\xi_\perp\nabla\varphi|_2 < \frac{1}{2}|\xi_\||_2$ then the estimate in (c) follows readily. If not, then $|\xi_\perp\nabla\varphi|_2 \geq \frac{1}{2}|\xi_\||_2$ and hence by the Lipschitz condition $|\xi_\perp|_2 \geq \frac{1}{2\sqrt{n}\gamma}|\xi_\||_2$ so that

$$|J_\Phi\xi|_2 \geq |\xi_\perp|_2 \geq \frac{1}{\sqrt{1+4n\gamma^2}}|\xi|_2,$$

from which the estimate in (c) again follows readily. □

Next, we define Whitney and Carleson-type regions, which are well-suited for our purposes. For φ and Ω_φ as above, and $X = (x, \varphi(x) + t) \in \Omega_\varphi$ we define the **φ-adapted Whitney region**

$$W_\varphi(X) := \{(y,s) : |y-x| < t,\ \varphi(y) + t/2 < s < \varphi(y) + 3t/2\}.$$

Note that $\Phi(W_\varphi) = W(x,t)$, the Whitney region in $\mathbb{R}^{n+1}_+$. Next, for a cube $Q \subset \mathbb{R}^n$ we define the **φ-adapted Carleson box**

$$R_{Q,\varphi} = \{(y,s) : y \in Q,\ \varphi(y) < s < \varphi(y) + \ell(Q)\}. \tag{C.3}$$

Finally, for a measurable $(n+1)\times(n+1)$ matrix-valued function P, we define the (FKP) **φ-adapted Carleson norm** of P as

$$\begin{aligned}\|P\|_{\mathcal{C}_\varphi} &:= \sup_{Q\subset\mathbb{R}^n}\left(\frac{1}{|Q|}\iint\limits_{R_{Q,\varphi}} \|P\|^2_{L^\infty(W_\varphi(y,s))}\,\frac{dy\,ds}{s-\varphi(y)}\right)^{1/2} \\ &= \sup_{Q\subset\mathbb{R}^n}\left(\frac{1}{|Q|}\iint\limits_{R_Q} \|P'\|^2_{L^\infty(W(x,t))}\,\frac{dx\,dt}{t}\right)^{1/2} = \|P'\|_{\mathcal{C}},\end{aligned} \tag{C.4}$$

where $P' = P\circ\Phi^{-1}$ and we used the flattening change of variables in the second line.

The following lemma is a direct consequence of the definitions above and Lemma C.2.

Lemma C.5. *Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz function with Lipschitz constant γ and $\varphi(0) = 0$ and suppose that P is a $(n+1)\times(n+1)$ matrix-valued function on Ω_φ with $\|P\|_{\mathcal{C}_\varphi} < \infty$. Then the matrix $\widetilde{P} = J_\Phi^T(P\circ\Phi^{-1})J_\Phi$ satisfies*

$$\|\widetilde{P}\|_{\mathcal{C}} \le (1+\sqrt{n}\gamma + n\gamma^2)\|P\|_{\mathcal{C}_\varphi}.$$

Proof. Recall that $J_\Phi(x,t) = J_\Phi(x)$ so that (C.4) with $P' = \widetilde{P}$ (which means $J_\Phi^T P J_\Phi$ is in place of P)

$$\|\widetilde{P}\|_{\mathcal{C}} \le \|J_\Phi^T P J_\Phi\|_{\mathcal{C}_\varphi} \le (1+\sqrt{n}\gamma+n\gamma^2)\|P\|_{\mathcal{C}_\varphi},$$

where we used Lemma C.2(b) in the second inequality. □

Combining the previous Lemma with Lemma C.2, one easily obtains the following.

Proposition C.6. *Let $\Lambda \ge 1$. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz function with Lipschitz constant $\gamma \le \frac{1}{50n}$ and $\varphi(0) = 0$, and A_0 a real, constant Λ-elliptic $(n+1)\times(n+1)$ matrix. Suppose $A(X)$ is a real, Λ-elliptic, matrix-valued function on $\mathbb{R}^{n+1}$ with the decomposition*

$$A(x,t) = A_1(x) + P(x,t)$$

satisfying

$$\|A_1 - A_0\|_{L^\infty(\mathbb{R}^n)} + \|P\|_{\mathcal{C}_\varphi} < \kappa$$

for some $\kappa \ge 0$. Then $\widetilde{A} = J_\Phi^T(A\circ\Phi^{-1})J_\Phi$ is a real, 8Λ-elliptic matrix with the decomposition

$$\widetilde{A}(x,t) = \widetilde{A}_1(x) + \widetilde{P}(x,t),$$

satisfying

$$\|\widetilde{A}_1 - A_0'\|_{L^\infty(\mathbb{R}^n)} + \|\widetilde{P}\|_{\mathcal{C}} < 2\kappa + 4\sqrt{n}\gamma\Lambda,$$

where $\widetilde{A}_1 := J_\Phi^T(A_1 \circ \Phi^{-1})J_\Phi = J_\Phi^T(A_1)J_\Phi$, $\widetilde{P} := J_\Phi^T(P \circ \Phi^{-1})J_\Phi$ *and*[26] $A_0' := J_\Phi^T(0)A_0J_\Phi(0)$.

Proof. The form of the decomposition $\widetilde{A}(x,t) = \widetilde{A}_1(x) + \widetilde{P}(x,t)$ is immediate from the form of A. In particular, notice that $A_1 \circ \Phi^{-1} = A_1$ since $A_1 = A_1(x)$. The 8Λ-ellipticity of $\widetilde{A}$ is a consequence of Lemma C.2(b) and (c). Indeed, the boundedness of $\tilde{A}$ follows from Lemma C.2(b), here one recalls that $\gamma < 1/(50n)$, so that $\sqrt{n}\gamma + n\gamma^2 \leq 1$ (we will use this several times). To see the lower ellipticity bound, we use the Λ-ellipticity of A and Lemma C.2(c) to obtain

$$\langle \widetilde{A}\xi, \xi\rangle \geq \Lambda^{-1}|J_\Phi \xi|_2^2 \geq \frac{1}{4}\Lambda^{-1}|\xi|_2^2,$$

for almost every x and all $\xi \in \mathbb{R}^{n+1}$.

To obtain the desired estimate for $\|\widetilde{A}_1 - A_0'\|_{L^\infty(\mathbb{R}^n)}$, we write

$$\widetilde{A}_1 - A_0' = (\widetilde{A}_1 - \widetilde{A}_0) + (\widetilde{A}_0 - A_0'), \tag{C.7}$$

where $\widetilde{A}_0$ is the (variable) matrix-valued function $J_\Phi^T(x)A_0J_\Phi(x)$. Using Lemma C.2(b), and the triangle inequality

$$\|\widetilde{A}_1 - \widetilde{A}_0\|_{L^\infty} \leq (1 + \sqrt{n}\gamma + n\gamma^2)\|A_1 - A_0\|_{L^\infty} \leq 2\|A_1 - A_0\|_{L^\infty} \tag{C.8}$$

To handle the second term, we again use Lemma C.2(b) to obtain

$$\begin{aligned}\|\widetilde{A}_0 - A_0'\|_{L^\infty} &\leq \|\widetilde{A}_0 - A_0\|_{L^\infty} + \|A_0 - A_0'\|_{L^\infty} \\ &\leq 2(\sqrt{n}\gamma + n\gamma^2)\|A_0\|_{L^\infty} \leq 4\sqrt{n}\gamma\Lambda.\end{aligned} \tag{C.9}$$

Combining (C.7), (C.8) and (C.9) yields the desirable estimate

$$\|\widetilde{A}_1 - A_0'\|_{L^\infty} \leq 2\|A_1 - A_0\|_{L^\infty} + 4\sqrt{n}\gamma\Lambda.$$

Since Lemma C.5 gives

$$\|\widetilde{P}\|_{\mathcal{C}} \leq (1 + \sqrt{n}\gamma + n\gamma^2)\|P\|_{\mathcal{C}_\varphi} \leq 2\|P\|_{\mathcal{C}_\varphi}$$

we obtain

$$\|\widetilde{A}_1 - A_0'\|_{L^\infty(\mathbb{R}^n)} + \|\widetilde{P}\|_{\mathcal{C}} < 2\kappa + 4\sqrt{n}\gamma\Lambda,$$

as desired. □

[26] Note that A_0' is a constant matrix and one can show that $\widetilde{A}_0$ is 8Λ elliptic in the same manner as $\widetilde{A}$.

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