Open system dynamics in interacting quantum field theories

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A quantum system that interacts with an environment generally undergoes non-unitary evolution described by a non-Markovian or Markovian master equation. In this paper, we construct the non-Markovian Redfield master equation for a quantum scalar field that interacts with a second field through a bilinear or nonlinear interaction on a Minkowski background. We use the resulting master equation to set up coupled differential equations that can be solved to obtain the equal-time two-point function of the system field. We show how the equations simplify under various approximations including the Markovian limit, and argue that the Redfield equation-based solution provides a perturbative resummation to the standard second order Dyson series result. For the bilinear interaction, we explicitly show that the Redfield solution is closer to the exact solution compared to the perturbation theory-based one. Further, the environment correlation function is oscillatory and non-decaying in this case, making the Markovian master equation a poor approximation. For the nonlinear interaction, on the other hand, the environment correlation function is sharply peaked and the Redfield solution matches that obtained using a Markovian master equation in the late-time limit.

I. INTRODUCTION

Most physical systems that we encounter, classical or quantum, are *open*, in the sense that they interact with some environment that is either not of interest or unknown to us. In the case of quantum systems, the open system approach is natural when observables of interest act only on a subspace of the full Hilbert space. Equations of motion for reduced system objects, such as the density operator, can be obtained by systematically *tracing out* environment degrees of freedom, converting environment operators into correlation functions. A common technique that is used to describe the non-unitary/non-Hamiltonian dynamics of an open quantum system is the master equation approach. In particular, for open systems where the environment consists of many degrees of freedom, the reduced dynamics are often well-described by a quantum Markov process. For such systems, the time evolution of the reduced density operator is described by the celebrated Gorini-Kossakowski-Lindblad-Sudarshan (GKLS) master equation [1, 2],

$$\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = -i\big[\hat{H},\hat{\rho}\big] + \sum_{\alpha}\gamma_\alpha\left[\hat{L}_\alpha\hat{\rho}\hat{L}_\alpha^\dagger - \frac{1}{2}\big\{\hat{L}_\alpha^\dagger\hat{L}_\alpha,\hat{\rho}\big\}\right], \quad (1)$$

where $\hat{\rho}=\hat{\rho}(t)$ is the reduced density operator in the Schrödinger picture. The operators $\hat{L}_\alpha$ are referred to as Lindblad or jump operators, and the coefficients γ_α are relaxation rates for different decay channels of the system. More generally, one can carry out a microscopic derivation of the master equation, starting from the von Neumann equation for a composite system and explicitly tracing out environment degrees of freedom under a series of approximations. This approach has the advantage that it leads to an intermediate *non-Markovian* master equation called the Redfield equation [3], which will be our prime focus in this work.

The Redfield equation is similar to the GKLS equation in that it is time-local but differs from it as the γ_α coefficients are time-dependent. This is important when coherent feedback from the environment is non-negligible and induces *memory* in the reduced system. It is worth noting, however, that the Redfield equation is not guaranteed to generate a completely positive map, unlike the GKLS equation. In fact, it does not necessarily generate a completely positive map even when environment correlation functions decay rapidly enough to justify taking a Markovian limit, wherein time-dependent rates are replaced by their late-time values. Instead, one has to additionally make the *rotating wave approximation* (RWA), which drops rapidly oscillating terms in the Redfield equation, to obtain the completely positive Davies equation [4, 5] that can then be put in GKLS form. It is also worth noting that there have been many significant efforts to restore positivity in the Redfield equation [6–14], but each solution comes with limited validity. Despite its limitations, the Redfield equation is a tractable non-Markovian master equation and continues to be an important tool in the study of open quantum systems, notably for studies of quantum transport [15].

Recently, there has been much interest in applying the open system approach to relativistic settings to gain insights into, for example, interacting quantum field theories [16–23], primordial perturbation theory [24–35], dark matter [36], gravitational decoherence [37, 38], gravitational waves [39], construction of particle detectors in curved spacetime [40–42], and holography [43–45]. Our interest in this paper is to revisit the open quantum field theory (QFT) paradigm and to use it to calculate correlation functions of a system field of interest. Specif-

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ically, we demonstrate how the open system approach allows for approximations that prioritize the dynamics of the reduced system, which can lead to more accurate solutions than standard loop corrections or Dyson series perturbation theory results.

First, we provide a careful derivation of the Redfield equation for two interacting scalar QFTs on a Minkowski background. We consider two forms of interaction: one, bilinear in the system and environment fields, and another where the system field remains linear but the environment field is nonlinear. We then use the resulting Redfield equation to calculate the equal-time two-point function of the system field and compare the result to a standard loop correction. For the bilinear interaction, we find that the Redfield equation-based solution is closer to the exact solution than a loop correction, suggesting that the Redfield equation provides a *perturbative resummation* to the standard loop correction. We also find that both the Markovian limit and RWA fail to capture the exact dynamics in this case, as the environment correlation function that enters the Redfield equation is oscillatory and non-decaying. For the nonlinear interaction, on the other hand, the environment correlation function is sharply peaked, and the late-time behavior of the Redfield solution matches that obtained in the Markovian limit.

The remainder of this paper is organized as follows. In section II, we introduce the Lagrangian for a composite system of two scalar fields and write the quantized Hamiltonian in terms of their Fourier modes. We next provide a comprehensive derivation of the Redfield equation in section III, making a special effort to explicate the various approximations and their regimes of validity, and then discuss the Markovian limit and RWA. In section IV, we outline a method to calculate the equal-time two-point function from the Redfield equation. We also discuss how to obtain limiting cases such as the Markovian limit and the standard loop correction, relegating details of the solution in the Markovian limit to appendix A. In section V, we apply this method to calculate the two-point function in the case of a bilinear interaction and compare it to the exact result. We present the details of the exact solution in appendix B and show the various coefficients that enter the Redfield equation in appendix C. We next apply the Redfield equation-based method to the case of a nonlinear environment in section VI and show the various coefficients that enter the Redfield equation in appendix D. We finally discuss a few subtleties of the calculation that deserve further understanding but are outside the scope of this work in section VII and end with a discussion in section VIII.

II. SETUP

We consider a composite system of two real, interacting Klein-Gordon fields, $\phi(\vec{x},t)$ and $\chi(\vec{x},t)$, with masses m and M, respectively, in $3+1$D. We restrict our interest to observables of the ϕ field, treating the χ field as an environment, and specialize to a single interaction term that is linear in ϕ and either linear or quadratic in χ. The Lagrangian density that we consider is then

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{2}(\partial\chi)^2 - \frac{1}{2}M^2\chi^2 - \frac{\lambda}{n!}\phi\chi^n, \tag{2}$$

where $(\partial\phi)^2 = -\dot{\phi}^2 + \left(\vec{\nabla}\phi\right)^2$, the dot denoting a derivative with time, and λ is a coupling constant with mass dimension $3-n$, where $n \in \{1,2\}$. We have adopted the mostly plus metric signature $(-,+,+,+)$ and work in units where $\hbar = c = 1$. Identifying the canonically conjugate momenta $\pi_\phi = \dot{\phi}$ and $\pi_\chi = \dot{\chi}$, we perform the usual Legendre transformation and integrate over $\mathbb{R}^3$ to obtain the Hamiltonian. We then impose that the only non-vanishing field commutation relations are $\left[\hat{\phi}(\vec{x}), \hat{\pi}_\phi(\vec{y})\right] = \left[\hat{\chi}(\vec{x}), \hat{\pi}_\chi(\vec{y})\right] = i\delta^3\left(\vec{x}-\vec{y}\right)$, and write the quantized Hamiltonian in the Schrödinger picture as

$$\hat{H} = \hat{H}_{0,\mathrm{S}} + \hat{H}_{0,\mathrm{E}} + \hat{H}_{\mathrm{int}}\,, \tag{3}$$

where

$$\hat{H}_{0,\mathrm{S}} = \frac{1}{2}\int \mathrm{d}^3x \left[\hat{\pi}_\phi^2 + \left(\vec{\nabla}\hat{\phi}\right)^2 + m^2\hat{\phi}^2\right], \tag{4}$$

$$\hat{H}_{0,\mathrm{E}} = \frac{1}{2}\int \mathrm{d}^3x \left[\hat{\pi}_\chi^2 + \left(\vec{\nabla}\hat{\chi}\right)^2 + M^2\hat{\chi}^2\right], \tag{5}$$

are the system and environment free Hamiltonians, respectively, and

$$\hat{H}_{\mathrm{int}} = \frac{\lambda}{n!}\int \mathrm{d}^3x\hat{\phi}\hat{\chi}^n \tag{6}$$

is the interaction Hamiltonian. We remark here that this interaction is not bounded from below due to the odd power of $\hat{\phi}$. However, this instability in the system dynamics can be easily removed by including suitable self-interactions for $\hat{\phi}$ and $\hat{\chi}$. We ignore such additional terms for simplicity and also note that the instability does not cause issues in a perturbative treatment for $m \neq 0$, $M \neq 0$, and with the $\hat{\phi}$ field initialized in the vacuum [46].

Since it will be convenient to write the master equation in terms of eigenoperators of the system's free Hamiltonian, $\hat{H}_{0,\mathrm{S}}$, we choose to work in momentum space where the creation and annihilation operators are eigenoperators of the free Hamiltonian. We use the Fourier expansions

$$\hat{\phi}(\vec{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\hat{\phi}_{\vec{k}}e^{i\vec{k}\cdot\vec{x}}\,, \tag{7}$$

$$\hat{\chi}(\vec{x}) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\hat{\chi}_{\vec{p}}e^{i\vec{p}\cdot\vec{x}}\,, \tag{8}$$

where we have indicated the arguments of the Fourier modes with subscripts such that $\hat{\phi}_{\vec{k}} \equiv \hat{\phi}(\vec{k})$ and $\hat{\chi}_{\vec{p}} \equiv$

$\hat{\chi}(\vec{p})$. We further write the field operators $\hat{\phi}_{\vec{k}}$ and $\hat{\chi}_{\vec{p}}$ in terms of their creation and annihilation operators,

$$\hat{\phi}_{\vec{k}} = \frac{1}{\sqrt{2\omega_k}}\left(\hat{a}_{\vec{k}} + \hat{a}^\dagger_{-\vec{k}}\right), \tag{9}$$

$$\hat{\chi}_{\vec{p}} = \frac{1}{\sqrt{2\Omega_p}}\left(\hat{b}_{\vec{p}} + \hat{b}^\dagger_{-\vec{p}}\right), \tag{10}$$

where we have defined the frequencies $\omega_k^2 \equiv k^2 + m^2$ and $\Omega_p^2 \equiv p^2 + M^2$, that inherit the commutation relations

$$\left[\hat{a}_{\vec{k}}, \hat{a}^\dagger_{\vec{k}'}\right] = (2\pi)^3\delta^3\left(\vec{k} - \vec{k}'\right), \tag{11}$$

$$\left[\hat{b}_{\vec{p}}, \hat{b}^\dagger_{\vec{p}'}\right] = (2\pi)^3\delta^3\left(\vec{p} - \vec{p}'\right). \tag{12}$$

We can then write the free Hamiltonian, $\hat{H}_0 = \hat{H}_{0,\mathrm{S}} + \hat{H}_{0,\mathrm{E}}$, in the form

$$\hat{H}_0 = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\omega_k \hat{a}^\dagger_{\vec{k}}\hat{a}_{\vec{k}} + \int \frac{\mathrm{d}^3p}{(2\pi)^3}\Omega_p \hat{b}^\dagger_{\vec{p}}\hat{b}_{\vec{p}}, \tag{13}$$

where we have dropped the (infinite) zero-point energy term and note that it can be removed systematically by shifting the free Lagrangian density by a constant. For the interaction Hamiltonian, we choose to explicitly write the system field in terms of creation and annihilation operators. As shown in section III C, this allows us to easily transition from the Redfield master equation to the Davies master equation by isolating the time dependence of system operators in the interaction picture. Thus we write $\hat{H}_{\mathrm{int}}$ as

$$\hat{H}_{\mathrm{int}} = \lambda \sum_{\alpha=1}^{2} \int_{\vec{k}} \hat{L}_{\vec{k},\alpha}\hat{\mathcal{O}}_{-\vec{k}}, \tag{14}$$

where we have defined the dimensionless system integral $\int_{\vec{k}} \equiv \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\omega_k^3}$, the dimensionless system operators

$$\hat{L}_{\vec{k},1} = \omega_k^{3/2}\hat{a}_{\vec{k}}, \quad \hat{L}_{\vec{k},2} = \omega_k^{3/2}\hat{a}^\dagger_{-\vec{k}}, \tag{15}$$

and the environment operator

$$\hat{\mathcal{O}}_{-\vec{k}} \equiv \frac{\omega_k}{\sqrt{2}n!}\int \mathrm{d}^3x \hat{\chi}^n(\vec{x})e^{i\vec{k}\cdot\vec{x}} \tag{16}$$

corresponding to the system mode $\vec{k}$. In the interaction picture, the $\hat{L}_{\vec{k},\alpha}$ inherit the time-dependence of the creation and annihilation operators,

$$\hat{L}_{\vec{k},\alpha}(t) = e^{i(-1)^\alpha\omega_k(t-t_0)}\hat{L}_{\vec{k},\alpha}, \tag{17}$$

where t_0 is the initial time at which all pictures coincide. We simply used the t-dependence to distinguish operators in the interaction picture from those in the Schrödinger picture above and use the same simplified notation below as well. We find later that the $\hat{L}$ operators appear in the master equation in the same way as the Lindblad operators of a GKLS equation. Since it is not always possible to write the master equation in GKLS form, however, we refer to the $\hat{L}$ operators as the dimensionless eigenoperators of the system.

III. MASTER EQUATION CONSTRUCTION

We assume that the system and environment are initially uncorrelated and that the interaction is turned on at the time t_0. The initial state of the composite system can then be written without loss of generality as a tensor product,

$$\hat{\sigma}(t_0) = \hat{\rho}(t_0) \otimes \hat{\rho}_{\mathrm{E}}, \tag{18}$$

where $\hat{\rho}_{\mathrm{E}} \equiv \hat{\rho}_{\mathrm{E}}(t_0)$. This can be time-evolved in the standard way, using the unitary time-evolution operator $\hat{U}(t, t_0)$,

$$\hat{\sigma}(t) = \hat{U}(t, t_0)\hat{\sigma}(t_0)\hat{U}^\dagger(t, t_0). \tag{19}$$

For the composite system Hamiltonian $\hat{H} = \hat{H}_0 + \hat{H}_{\mathrm{int}}$, $\hat{U}(t, t_0)$ is in turn given by

$$\hat{U}(t, t_0) = e^{-i\hat{H}_0(t-t_0)}Te^{-i\int_{t_0}^{t}\hat{H}_{\mathrm{I}}(t_1)\mathrm{d}t_1}, \tag{20}$$

where T indicates time-ordering and we have defined the interaction Hamiltonian in the interaction picture $\hat{H}_{\mathrm{I}}(t) \equiv e^{i\hat{H}_0(t-t_0)}\hat{H}_{\mathrm{int}}(t_0)e^{-i\hat{H}_0(t-t_0)}$, where $\hat{H}_{\mathrm{int}}(t_0)$ is simply $\hat{H}_{\mathrm{int}}$. Now, taking a time-derivative of eq. (19) yields the von Neumann equation, which we write in the interaction picture as

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\sigma}_{\mathrm{I}}(t) = -i\left[\hat{H}_{\mathrm{I}}(t), \hat{\sigma}_{\mathrm{I}}(t)\right]. \tag{21}$$

We are interested in writing dynamical expressions that are valid to second order in the coupling without treating the density operator itself as a perturbative object. To this end, we first integrate eq. (21) from t_0 to t to find the formal solution for the density operator,

$$\hat{\sigma}_{\mathrm{I}}(t) = \hat{\sigma}(t_0) - i\int_{t_0}^{t}\mathrm{d}t_1\left[\hat{H}_{\mathrm{I}}(t_1), \hat{\sigma}_{\mathrm{I}}(t_1)\right]. \tag{22}$$

We then substitute this result for $\hat{\sigma}_{\mathrm{I}}(t)$ on the right-hand side of eq. (21) to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\sigma}_{\mathrm{I}}(t) = -i\left[\hat{H}_{\mathrm{I}}(t), \hat{\sigma}(t_0)\right] - \int_{t_0}^{t}\mathrm{d}t_1\left[\hat{H}_{\mathrm{I}}(t), \left[\hat{H}_{\mathrm{I}}(t_1), \hat{\sigma}_{\mathrm{I}}(t_1)\right]\right]. \tag{23}$$

Note that although the second term contains two factors of the interaction Hamiltonian, we have not made any approximation; hence, the expression above is valid for *any* interaction strength at *all* orders of λ. It is easily verified that replacing $\hat{\sigma}_{\mathrm{I}}(t_1)$ on the right-hand side with $\hat{\sigma}(t_0)$ would exactly recover the standard second-order perturbation theory expression, but we will not make this approximation here.

We now perform a partial trace over the environment degrees of freedom in eq. (23) to obtain the master equation for the reduced density operator $\hat{\rho}_{\rm I}(t) \equiv {\rm Tr}_{\rm E}\, \hat{\sigma}_{\rm I}(t)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}_{\rm I}(t) = -i\left[\hat{H}_{\rm eff}(t), \hat{\rho}(t_0)\right] - \int_{t_0}^{t} \mathrm{d}t_1\, {\rm Tr}_{\rm E}\left[\hat{H}_{\rm I}(t), \left[\hat{H}_{\rm I}(t_1), \hat{\sigma}_{\rm I}(t_1)\right]\right], \quad (24)$$

where we have used eq. (18) to rewrite the first commutator on the right-hand side in terms of an effective Hamiltonian $\hat{H}_{\rm eff}(t) \equiv {\rm Tr}_{\rm E}\left\{\hat{H}_{\rm I}(t)\hat{\rho}_{\rm E}\right\}$. It is common to assume that the environment operators coupling to the system have zero mean in the state $\hat{\rho}_{\rm E}$, so that $\hat{H}_{\rm eff}(t) = 0$. Although this is not always true and depends on the choice of interaction and initial state, it can generally be imposed by including a mean environment field term in the system Hamiltonian [47]. We choose to explicitly keep the effective Hamiltonian term but find that it does not contribute to the equal-time two-point function of the reduced system in any of the cases considered in this work.

A. Redfield master equation

As mentioned earlier, eq. (24) still describes the exact dynamics of the composite system; however, to make further progress, we now restrict to the perturbative regime and make two standard approximations, namely, the Born and Markov approximations. We would first like to reduce eq. (24) to an integrodifferential equation for the reduced density operator, $\hat{\rho}_{\rm I}(t)$. For this, it is useful to define a Zwanzig projection operator $\hat{\mathcal{P}}$ on the space of density operators for the composite system such that

$$\begin{aligned}\hat{\mathcal{P}}\hat{\sigma}_{\rm I}(t) &= {\rm Tr}_{\rm E}\left\{\hat{\sigma}_{\rm I}(t)\right\} \otimes \hat{\rho}_{\rm R} \\ &= \hat{\rho}_{\rm I}(t) \otimes \hat{\rho}_{\rm R}\,,\end{aligned} \quad (25)$$

where $\hat{\rho}_{\rm R}$ is a time-independent reference state for the environment. We construct a complementary operator $\hat{\mathcal{Q}}$ through the relationship $\hat{\mathbb{1}} = \hat{\mathcal{P}} + \hat{\mathcal{Q}}$, which allows us to decompose the full density operator as

$$\begin{aligned}\hat{\sigma}_{\rm I}(t) &= \hat{\mathcal{P}}\hat{\sigma}_{\rm I}(t) + \hat{\mathcal{Q}}\hat{\sigma}_{\rm I}(t) \\ &= \hat{\rho}_{\rm I}(t) \otimes \hat{\rho}_{\rm R} + \hat{\mathcal{Q}}\hat{\sigma}_{\rm I}(t)\,.\end{aligned} \quad (26)$$

In the literature on projection operators, $\hat{\rho}_{\rm I}(t) \otimes \hat{\rho}_{\rm R}$ and $\hat{\mathcal{Q}}\hat{\sigma}_{\rm I}(t)$ are referred to as the *relevant* and *irrelevant* parts, respectively [48, 49].

Using eq. (26) in eq. (24) and choosing the reference state to be the initial state of the environment gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}_{\rm I}(t) = -i\left[\hat{H}_{\rm eff}(t), \hat{\rho}(t_0)\right] - \int_{t_0}^{t} \mathrm{d}t_1\, {\rm Tr}_{\rm E}\left\{\left[\hat{H}_{\rm I}(t), \left[\hat{H}_{\rm I}(t_1), \hat{\rho}_{\rm I}(t_1)\otimes\hat{\rho}_{\rm E}\right]\right] + \left[\hat{H}_{\rm I}(t), \left[\hat{H}_{\rm I}(t_1), \hat{\mathcal{Q}}\hat{\sigma}_{\rm I}(t_1)\right]\right]\right\}. \quad (27)$$

We have come a bit closer to the goal of reducing all instances of $\hat{\sigma}_{\rm I}(t_1)$ to $\hat{\rho}_{\rm I}(t_1)$ and $\hat{\rho}_{\rm E}$, with the exception of the irrelevant part of $\hat{\sigma}_{\rm I}(t_1)$, which brings us to our first approximation.

We now restrict to the weak-coupling regime and expand the term containing the irrelevant part of $\hat{\sigma}_{\rm I}(t_1)$ to second order in λ. Since there are already two factors of λ (one from each $\hat{H}_{\rm I}$), we only need to make the expansion $\hat{\mathcal{Q}}\hat{\sigma}_{\rm I}(t_1) = \hat{\mathcal{Q}}\hat{\sigma}_{\rm I}(t_0) + \mathcal{O}(\lambda)$. It is clear from the definitions in eqs. (18) and (25), however, that the projection operator leaves the initial density operator unchanged, from which we conclude that $\hat{\mathcal{Q}}\hat{\sigma}_{\rm I}(t_0) = 0$. Therefore, in the weak-coupling regime, we can write the master equation as

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}_{\rm I}(t) = -i\left[\hat{H}_{\rm eff}(t), \hat{\rho}(t_0)\right] - \int_{t_0}^{t} \mathrm{d}t_1\, {\rm Tr}_{\rm E}\left[\hat{H}_{\rm I}(t), \left[\hat{H}_{\rm I}(t_1), \hat{\rho}_{\rm I}(t_1)\otimes\hat{\rho}_{\rm E}\right]\right]. \quad (28)$$

This approximation is called the Born approximation and amounts to factorizing the composite system density operator as

$$\hat{\sigma}_{\rm I}(t_1) \approx \hat{\rho}_{\rm I}(t_1) \otimes \hat{\rho}_{\rm E}\,, \quad (29)$$

so that the density operator of the environment is almost stationary in the interaction picture. The Born approximation is also often presented as the assumption that the environment is sufficiently large and the interaction sufficiently weak so that environment excitations induced by the system are negligible compared to its free dynamics. Since weak coupling was sufficient in the above analysis, however, we make the Born approximation without any requirement on the number of degrees of freedom in the environment. We also note that although eq. (28) is only valid to second order in perturbation theory, the Born approximation is not equivalent to truncating the Dyson series expansion since $\hat{\rho}_{\rm I}(t_1)$ itself has not been expanded in a perturbation series. In fact, as shown in section IV, leaving $\hat{\rho}_{\rm I}(t_1)$ intact allows one to obtain a second-order resummation for the equal-time two-point function of the reduced system.

We can write the master equation in eq. (28) in a more convenient form by expanding out the nested commutator on the right-hand side and using the explicit form of the interaction Hamiltonian from eq. (14). This gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}_{\mathrm{I}}(t) = -i\left[\hat{H}_{\mathrm{eff}}(t), \hat{\rho}(t_0)\right] - \lambda^2 \sum_{\alpha,\beta}\int_{\vec{k}} \frac{1}{2\omega_k}\int_{t_0}^{t} \mathrm{d}t_1 \left\{\mathcal{C}_k(t,t_1)\left[\hat{L}^{\dagger}_{\vec{k},\alpha}(t), \hat{L}_{\vec{k},\beta}(t_1)\hat{\rho}_{\mathrm{I}}(t_1)\right] + \mathrm{h.c.}\right\}, \tag{30}$$

where h.c. indicates the Hermitian conjugate, and we have moved all system operators out of the environment trace, defining the environment correlation function $\mathcal{C}_k(t,t_1)$ through the relation [65]

$$\mathrm{Tr}_{\mathrm{E}}\left\{\hat{\mathcal{O}}_{-\vec{k}'}(t)\hat{\mathcal{O}}_{-\vec{k}}(t_1)\hat{\rho}_{\mathrm{E}}\right\} = \frac{\omega_k^2}{2}\mathcal{C}_k(t,t_1)(2\pi)^3\delta^3\left(\vec{k}+\vec{k}'\right), \tag{31}$$

with the $\omega_k^2/2$ separated out for later convenience. We can see from eq. (30) that the Born approximation not only allows us to write an integrodifferential equation for $\hat{\rho}_{\mathrm{I}}$ but also to collate the effect of the environment into time-dependent correlation functions that appear as coefficients on system operators.

Although we have made tidy progress by making the Born approximation, the time integral over $\hat{\rho}_{\mathrm{I}}(t_1)$ in eq. (30) implies that we are still accounting for correlations that are not time-local due to the interaction with the environment. In the standard picture of open systems, the environment is typically a reservoir with correlation functions that decay rapidly compared to the timescale over which the system evolves appreciably, naturally suppressing time-nonlocal system correlations [50] [66]. This lack of *memory* in the environment justifies the Markov approximation, which amounts to the replacement $\hat{\rho}_{\mathrm{I}}(t_1) \to \hat{\rho}_{\mathrm{I}}(t)$ and is interpreted as neglecting coherent feedback from the environment. We will first argue that the Markov approximation is also justified when the environment correlation function $\mathcal{C}_k(t,t_1)$ is purely oscillatory, as long as the characteristic timescale, τ_{E}, of $\mathcal{C}_k(t,t_1)$ is much shorter than the characteristic timescale of the system, τ_{S}. We can see this by considering the t_1 integral in eq. (30),

$$\int_{t_0}^{t} \mathrm{d}t_1 \mathcal{C}_k(t,t_1) e^{i(-1)^{\beta}\omega_k(t_1-t_0)}\hat{\rho}_{\mathrm{I}}(t_1), \tag{32}$$

the exponential factor being the time-dependence of $\hat{L}_{\vec{k},\beta}(t_1)$. We integrate by parts to find that [67]

$$\begin{aligned}\int_{t_0}^{t} &\mathrm{d}t_1 \mathcal{C}_k(t,t_1) e^{i(-1)^{\beta}\omega_k(t_1-t_0)}\hat{\rho}_{\mathrm{I}}(t_1) \\ &= \int_{t_0}^{t} \mathrm{d}t_1 \mathcal{C}_k(t,t_1) e^{i(-1)^{\beta}\omega_k(t_1-t_0)}\hat{\rho}_{\mathrm{I}}(t) \\ &\quad - \int_{t_0}^{t} \mathrm{d}t_1 \mathcal{K}_{k,\beta}(t,t_1)\frac{\mathrm{d}}{\mathrm{d}t_1}\hat{\rho}_{\mathrm{I}}(t_1),\end{aligned} \tag{33}$$

where we have defined

$$\mathcal{K}_{k,\beta}(t,t_1) = \int_{t_0}^{t_1} \mathrm{d}t_2 \mathcal{C}_k(t,t_2) e^{i(-1)^{\beta}\omega_k(t_2-t_0)}. \tag{34}$$

The first term on the right-hand side of eq. (33) is precisely the Markov approximation, so we are left with the task of showing that the second term is negligible. If $\mathcal{K}_{k,\beta}(t,t_1)$ is a purely sinusoidal function with a period of $\tau_\star$, then we partition the domain of integration as

$$(t_0, t) = \bigcup_{n=0}^{N-1}(t_0+n\tau_\star, t_0+(n+1)\tau_\star) \cup (t_0+N\tau_\star, t), \tag{35}$$

where $N = \lfloor (t-t_0)/\tau_\star \rfloor$, and rewrite the integral in the second term on the right-hand side of eq. (33) as $\int_{t_0}^{t}\mathrm{d}t_1 = \sum_{n=0}^{N-1}\int_{t_0+n\tau_\star}^{t_0+(n+1)\tau_\star}\mathrm{d}t_1 + \int_{t_0+N\tau_\star}^{t}\mathrm{d}t_1$. Now if $\hat{\rho}_{\mathrm{I}}(t_1)$ varies over a characteristic timescale $\tau_{\mathrm{S}} \gg \tau_\star$, then we can move the derivative of $\hat{\rho}_{\mathrm{I}}(t_1)$ outside of the integrals as

$$\begin{aligned}&\int_{t_0}^{t} \mathrm{d}t_1 \mathcal{K}_{k,\beta}(t,t_1)\frac{\mathrm{d}}{\mathrm{d}t_1}\hat{\rho}_{\mathrm{I}}(t_1) \\ &= \sum_{n=0}^{N-1}\frac{\mathrm{d}}{\mathrm{d}t_1}\hat{\rho}_{\mathrm{I}}(t_1)\Big|_{t_0+(n+1)\tau_\star}\int_{t_0+n\tau_\star}^{t_0+(n+1)\tau_\star}\mathrm{d}t_1 \mathcal{K}_{k,\beta}(t,t_1) \\ &\quad + \frac{\mathrm{d}}{\mathrm{d}t_1}\hat{\rho}_{\mathrm{I}}(t_1)\Big|_{t}\int_{t_0+N\tau_\star}^{t}\mathrm{d}t_1 \mathcal{K}_{k,\beta}(t,t_1).\end{aligned} \tag{36}$$

The integrals under the summation now vanish identically, and we are only left with the second term on the right-hand side. Furthermore, since $N\tau_\star \to t - t_0$ as $\tau_\star \to 0$, we find that $t - (t_0 + N\tau_\star) \approx 0$ for sufficiently small $\tau_\star$, and so the second term is also negligible if the environment timescale is much shorter than the system timescale. We can, therefore, make the Markov approximation, replacing $\hat{\rho}_{\mathrm{I}}(t_1) \to \hat{\rho}_{\mathrm{I}}(t)$ in eq. (32).

In either of the two cases discussed above, whether the environment correlation function is purely oscillatory with a short enough timescale (which we will find for $\lambda\hat{\phi}\hat{\chi}$) or decays sufficiently fast (for $\lambda\hat{\phi}\hat{\chi}^2$), the replacement $\hat{\rho}_{\mathrm{I}}(t_1) \to \hat{\rho}_{\mathrm{I}}(t)$ is justified and leaves us with a time-local master equation called the Redfield equation,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}_{\mathrm{I}}(t) &= -i\left[\hat{H}_{\mathrm{eff}}(t), \hat{\rho}(t_0)\right] \\ &\quad - \int_{t_0}^{t}\mathrm{d}t_1\, \mathrm{Tr}_{\mathrm{E}}\left[\hat{H}_{\mathrm{I}}(t), \hat{H}_{\mathrm{I}}(t_1)\hat{\rho}_{\mathrm{I}}(t)\hat{\rho}_{\mathrm{E}}\right] + \mathrm{h.c.},\end{aligned} \tag{37}$$

where we have suppressed the $\otimes$ between the density operators and restored the interaction Hamiltonian to highlight the general form. As discussed in the introduction, it is well-known that the Redfield equation cannot be put into GKLS form in general and is thus not guaranteed to yield a completely positive map. The standard procedure to ensure positivity is to further take the Markovian limit and make the RWA, and the resulting equation is called the Davies equation. In the following two subsections, we rewrite the Redfield equation in a form that allows for a straightforward transition to the Davies equation so that we can easily compare our results in section IV under different approximations.

B. Markovian limit

When the justification for the Markov approximation is the rapid decay of environment correlation functions, we can make a further simplification to the Redfield equation. Specifically, we can remove the initial state dependence by taking the Markovian limit [49], the general procedure for which is as follows. Suppose we are interested in the integral of some function $f(t_1)$ from the initial time t_0 to a later time t. We can make a change of variables such that the integration is over the time elapsed since t_0 as

$$I(t) = \int_{t_0}^{t} \mathrm{d}t_1 f(t_1) \overset{t_1 \to t-t'}{=} \int_0^{t-t_0} \mathrm{d}t' f(t-t') \,. \tag{38}$$

Now, if the function $f(t-t')$ decays to zero over the domain of integration, the value of $I(t)$ does not change on increasing the upper limit of the integral. The initial time can, therefore, be removed by letting $t - t_0 \to \infty$, yielding the Markovian limit of $I(t)$,

$$I^M(t) = \int_0^{\infty} \mathrm{d}t_1 f(t-t_1) \,, \tag{39}$$

and removing the memory of the system from the dynamics. Environments for which the Markovian limit is permissible are referred to as Markovian environments, and the reduced dynamics are said to be Markovian.

Rather than taking the Markovian limit at this stage, we will continue with the upper limit on the integral as $t-t_0$ so that the Markovian limit can be obtained simply by letting $t-t_0 \to \infty$ in the t_1 integral. We thus rewrite the Redfield equation in eq. (37) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}_{\mathrm{I}}(t) = -i\left[\hat{H}_{\mathrm{eff}}(t), \hat{\rho}(t_0)\right] - \int_{t_1} \mathrm{Tr}_{\mathrm{E}}\left[\hat{H}_{\mathrm{I}}(t), \hat{H}_{\mathrm{I}}(t-t_1)\hat{\rho}_{\mathrm{I}}(t)\hat{\rho}_{\mathrm{E}}\right] + \mathrm{h.c.}\,, \tag{40}$$

where $\int_{t_1} \equiv \int_0^{t-t_0} \mathrm{d}t_1$.

C. Standard form of the master equation

We now specialize to the case that the environment is initialized in its vacuum state and expand environment correlation functions using Wick's contractions. Since the resulting two-point correlators conserve momentum and are invariant under time translations, we can write eq. (40) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}_{\mathrm{I}}(t) = -i\left[\hat{H}_{\mathrm{eff}}(t), \hat{\rho}(t_0)\right] - \sum_{\alpha,\beta} \int_{\vec{k}} F_{k,\alpha\beta}(t)\Gamma_{k,\beta}(t)\left[\hat{L}^{\dagger}_{\vec{k},\alpha}, \hat{L}_{\vec{k},\beta}\hat{\rho}_{\mathrm{I}}(t)\right] + \mathrm{h.c.}\,, \tag{41}$$

with the coefficients

$$F_{k,\alpha\beta}(t) \equiv e^{-i\left((-1)^{\alpha}-(-1)^{\beta}\right)\omega_k(t-t_0)}\,, \tag{42}$$

$$\Gamma_{k,\beta}(t) \equiv \lambda^2 \int_{t_1} \frac{e^{-i(-1)^{\beta}\omega_k t_1}}{2\omega_k}\mathcal{C}_k(t_1)\,, \tag{43}$$

where $F_{k,\alpha\beta}$ is Hermitian in the sense that it satisfies the relation $F_{k,\alpha\beta} = F^*_{k,\beta\alpha}$. The environment correlation function takes the explicit form [68]

$$\mathcal{C}_k(t_1) = \frac{8\pi^3}{n!}\int \delta^3\left(\vec{k} + \textstyle\sum_i \vec{p}_i\right)\prod_{j=1}^{n}\frac{\mathrm{d}^3p_j}{(2\pi)^3}\frac{e^{-i\Omega_{p_j}t_1}}{2\Omega_{p_j}}\,, \tag{44}$$

where we have dropped equal-time bubble diagrams since they can be canceled by adding appropriate renormalization counterterms, as we show explicitly for the $n=2$ case in section VI, and used time translation invariance to write $\mathcal{C}_k$ with a single time argument. As noted before, $\mathcal{C}_k$ only depends on the magnitude of k since the $\vec{p}_j$ integrals can always be evaluated relative to the direction of $\vec{k}$. It is also worth noting that everything under the summation in eq. (41) is dimensionless except for $\Gamma_{k,\beta}$, which is a rate with mass dimension one. Finally, including the Hermitian conjugate term explicitly, we can write eq. (41) in the standard form

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}_{\mathrm{I}}(t) = -i\left[\hat{H}_{\mathrm{eff}}(t), \hat{\rho}(t_0)\right] - i\sum_{\alpha,\beta}\int_{\vec{k}} F_{k,\alpha\beta}(t) S_{k,\alpha\beta}(t)\left[\hat{L}^{\dagger}_{\vec{k},\alpha}\hat{L}_{\vec{k},\beta}, \hat{\rho}_{\mathrm{I}}(t)\right]$$
$$+ \sum_{\alpha,\beta}\int_{\vec{k}} F_{k,\alpha\beta}(t)\gamma_{k,\alpha\beta}(t)\left[\hat{L}_{\vec{k},\beta}\hat{\rho}_{\mathrm{I}}(t)\hat{L}^{\dagger}_{\vec{k},\alpha} - \frac{1}{2}\left\{\hat{L}^{\dagger}_{\vec{k},\alpha}\hat{L}_{\vec{k},\beta}, \hat{\rho}_{\mathrm{I}}(t)\right\}\right], \tag{45}$$

where we have defined

$$\gamma_{k,\alpha\beta}(t) \equiv \Gamma_{k,\beta}(t) + \Gamma^*_{k,\alpha}(t), \tag{46}$$

$$S_{k,\alpha\beta}(t) \equiv \frac{1}{2i}\left[\Gamma_{k,\beta}(t) - \Gamma^*_{k,\alpha}(t)\right], \tag{47}$$

both of which are also Hermitian matrices that satisfy $\gamma_{k,\alpha\beta} = \gamma^*_{k,\beta\alpha}$ and $S_{k,\alpha\beta} = S^*_{k,\beta\alpha}$.

In eqs. (41) and (45), $F_{k,\alpha\beta}$ is the interaction picture time-dependence of the dimensionless system operators, $\hat{L}_{\vec{k},\alpha}$, which we have separated out so that we may readily make the RWA. The motivation for the RWA is more evident for a generic master equation where this time-dependence is $\exp[-i(\omega' - \omega)t]$ and there are summations over both Bohr frequencies ω and ω', which can be positive or negative. When the difference of the Bohr frequencies is sufficiently large, this exponential rapidly oscillates on timescales over which the system dissipates, and the corresponding terms average to zero. The RWA neglects these *off-diagonal* contributions, leaving only terms with $\omega' = \omega$. Of course, this is not the case for systems with dense level spacing, further implying that the RWA is a poor approximation for systems with a continuous spectrum [11, 14]. This observation naturally signals that the RWA is not likely to be valid for a field theory where the ω_k represent a continuum of Bohr frequencies. One subtlety in the present case is that momentum conservation has modified the above argument so that the magnitudes of ω and ω' are the same. Indeed, making the RWA amounts to replacing $F_{k,\alpha\beta} \to \delta_{\alpha\beta}$ which neglects all terms proportional to $\exp[\pm i2\omega_k(t - t_0)]$, requiring $2\omega_k$ to be large for all k. This requirement is distinctly problematic in the massless limit, where $\omega_k = k$ can be arbitrarily small, and is not clearly resolved in the massive case. We will, therefore, not make this approximation in general but consider how it affects our results only from a pedagogical standpoint.

While we could continue to work in the interaction picture, we now choose to transform to the Schrödinger picture. This introduces a factor of $F^*_{k,\alpha\beta}(t)$ which multiplies $F_{k,\alpha\beta}(t)$ to yield unity, and a commutator of $\hat{\rho}(t)$ with the free Hamiltonian of the system. This commutator can be combined with the second term of eq. (45) to define a modified system Hamiltonian,

$$\hat{H}_{\mathrm{S}}(t) \equiv \hat{H}_{0,\mathrm{S}} + \sum_{\alpha,\beta}\int_{\vec{k}} S_{k,\alpha\beta}(t)\hat{L}^{\dagger}_{\vec{k},\alpha}\hat{L}_{\vec{k},\beta}, \tag{48}$$

where, for $\alpha = \beta$, the second term on the right-hand side is identified as an environment-induced shift in energy levels of the system and is then referred to as the Lamb shift. Transforming back to the Schrödinger picture turns out to be mathematically convenient as it circumvents the need to handle time-dependent coefficients appearing in the Markovian limit, but we comment on the interaction picture formulation in the next section.

Using the fundamental relationship $\hat{\rho}_{\mathrm{I}}(t) = e^{i\hat{H}_{0,\mathrm{S}}(t-t_0)}\hat{\rho}(t)e^{-i\hat{H}_{0,\mathrm{S}}(t-t_0)}$, we now write the standard form of the Redfield equation in the Schrödinger picture as

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}(t) = -i\left[\hat{H}_{\mathrm{eff}}, \hat{\rho}_0(t)\right] - i\left[\hat{H}_{\mathrm{S}}(t), \hat{\rho}(t)\right] + \sum_{\alpha,\beta}\int_{\vec{k}}\gamma_{k,\alpha\beta}(t)\left[\hat{L}_{\vec{k},\beta}\hat{\rho}(t)\hat{L}^{\dagger}_{\vec{k},\alpha} - \frac{1}{2}\left\{\hat{L}^{\dagger}_{\vec{k},\alpha}\hat{L}_{\vec{k},\beta}, \hat{\rho}(t)\right\}\right], \tag{49}$$

where we have defined $\hat{\rho}_0(t) \equiv e^{-i\hat{H}_{0,\mathrm{S}}(t-t_0)}\hat{\rho}(t_0)e^{i\hat{H}_{0,\mathrm{S}}(t-t_0)}$ and used $\hat{\rho}_{\mathrm{E}} = |0_{\mathrm{E}}\rangle\langle 0_{\mathrm{E}}|$ to reduce the Schrödinger picture effective Hamiltonian to $\hat{H}_{\mathrm{eff}} = \langle 0_{\mathrm{E}}|\hat{H}_{\mathrm{int}}|0_{\mathrm{E}}\rangle$. The last term in eq. (49) is non-Hamiltonian and describes the decoherence of the system due to the environment. Eq. (49) is the final master equation we will use in the remainder of the paper. We note that the procedure to make the RWA is not evident anymore since the time-dependence of the rapidly oscillating terms has been absorbed into the states. The approximation can, nevertheless, be made by simply dropping the off-diagonal ($\alpha \neq \beta$) terms.

IV. EQUAL-TIME TWO-POINT FUNCTION

In principle, one can solve the master equation in eq. (49) for $\hat{\rho}$ and use it to investigate the dynamics of the reduced system. Especially for systems with a large Hilbert space, however, this is challenging as it amounts to determining all matrix elements of $\hat{\rho}$. Since we are typically interested in system observables, we can instead use the master equation to directly find how *correlation functions* of the reduced system evolve in time. In this paper, in particular, we are interested in calculating the equal-time two-point correlation function of $\hat{\phi}_{\vec{k}}$, which is easily related to the eigenoperators $\hat{a}_{\vec{k}}$ and $\hat{a}^\dagger_{\vec{k}}$ of the system's free Hamiltonian. In this section, we use eq. (49) to write down coupled differential equations in correlation functions of $\hat{a}_{\vec{k}}$ and $\hat{a}^\dagger_{\vec{k}}$ and discuss how the various approximations introduced earlier are manifested. We solve the resulting equations for the two specific system-environment interactions in the next two sections.

We start with the equal-time two-point correlation function for $\hat{\phi}_{\vec{k}}$, written in the Schrödinger picture as

$$\mathrm{Tr}_{\mathrm{S}}\left\{\hat{\phi}_{\vec{k}}\hat{\phi}_{\vec{k}'}\hat{\rho}(t)\right\} = \mathcal{G}_k(t)(2\pi)^3\delta^3\left(\vec{k}+\vec{k}'\right), \tag{50}$$

and expand the left-hand side in terms of correlation functions of $\hat{a}_{\vec{k}}$ and $\hat{a}^\dagger_{\vec{k}}$. Since each correlation function must conserve momentum, we define the functions $\xi_k = \xi_{k,1} + i\xi_{k,2}$ and $\xi_{k,3}$ such that

$$\mathrm{Tr}_{\mathrm{S}}\left\{\hat{a}_{\vec{k}}\hat{a}_{\vec{k}'}\hat{\rho}(t)\right\} = \xi_k(t)(2\pi)^3\delta^3\left(\vec{k}+\vec{k}'\right), \tag{51}$$

$$\mathrm{Tr}_{\mathrm{S}}\left\{\left[\hat{a}_{\vec{k}}\hat{a}^\dagger_{-\vec{k}'} + \hat{a}^\dagger_{-\vec{k}}\hat{a}_{\vec{k}'}\right]\hat{\rho}(t)\right\} = \xi_{k,3}(t)(2\pi)^3\delta^3\left(\vec{k}+\vec{k}'\right), \tag{52}$$

in terms of which $\mathcal{G}_k$ is given by

$$\mathcal{G}_k(t) = \frac{1}{2\omega_k}\left[2\xi_{k,1}(t) + \xi_{k,3}(t)\right]. \tag{53}$$

We now multiply the master equation in eq. (49) by appropriate combinations of creation and annihilation operators and obtain a set of coupled first-order differential equations for ξ_k and $\xi_{k,3}$. The contribution of the $\hat{H}_{\mathrm{eff}}$ term vanishes identically since it can always be written as the expectation of three system operators. Thus, the coefficients that appear in the coupled equations are combinations of the functions $S_{k,\alpha\beta}$ and $\gamma_{k,\alpha\beta}$. Suppressing the k subscripts on all variables, we find that

$$\dot{\xi}(t) = -\left[\gamma_-(t) + i2\omega_{\mathrm{S}}(t)\right]\xi(t) - i2S_{\mathrm{o}}(t)\xi_3(t) - \gamma_{\mathrm{o}}(t), \tag{54}$$

$$\dot{\xi}_3(t) = -\gamma_-(t)\xi_3(t) + 8\,\mathrm{Im}\left[S_{\mathrm{o}}(t)\xi^*(t)\right] + \gamma_+(t), \tag{55}$$

where we have defined the functions $\gamma_{\mathrm{o}}(t) \equiv \gamma_{k,12}(t)$, $S_{\mathrm{o}}(t) \equiv S_{k,12}(t)$, $\gamma_\pm(t) \equiv \gamma_{k,11}(t) \pm \gamma_{k,22}(t)$, and $\omega_{\mathrm{S}}(t) \equiv \omega_k + S_{k,11}(t) + S_{k,22}(t)$, with the subscript 'o' denoting 'off-diagonal'. Once an initial density operator is chosen, eqs. (54) and (55) can be solved using initial conditions obtained from eqs. (51) and (52). We denote the resulting two-point function obtained by using these solutions for $\xi(t)$ and $\xi_3(t)$ in eq. (53) with $\mathcal{G}^{\mathrm{R}}_k(t)$.

We note that although we have chosen to formulate differential equations for the system creation and annihilation operators in the Schrödinger picture, this is not necessary [69]. Similar strategies have been employed in the interaction picture, where operators are time-dependent, and using the field basis, where the $\hat{L}_{\vec{k}}$ are not eigenoperators of the free Hamiltonian [19, 32, 51]. In these cases, the time dependence requires additional care – for example, if eq. (45) is used in place of eq. (49), then the ξ variables become *mixed picture correlators* of Schrödinger picture operators with the interaction picture density operator. The two-point function can then be constructed by transferring the free time-evolution from $\hat{\rho}(t)$ to the $\hat{\phi}_{\vec{k}}$ in eq. (50) and expanding the two-point function in terms of the mixed picture correlators. The main differences are that $\omega_{\mathrm{S}}(t) \to S_{k,11}(t) + S_{k,22}(t)$ and the off-diagonal coefficients are multiplied by a factor of $F_{k,12}(t)$.

Approximations

The coupled differential eqs. (54) and (55) obtained from the Redfield equation are well-suited for numerical solutions. Finding analytical solutions, on the other hand, is not feasible in general but may be possible after making further simplifications. We consider four simplified cases below: the Markovian limit, the RWA, the Markovian limit under the RWA, and the standard Dyson series perturbation theory.

1. *Markovian limit*

Under the Markovian limit, we let $t - t_0 \to \infty$ in $\Gamma_{k,\beta}$, removing the time-dependence from $\gamma_{k,\alpha\beta}$ and $S_{k,\alpha\beta}$. Since there is no scope for ambiguity, we will indicate this limit by simply dropping the time argument. In this case, the coupled eqs. (54) and (55) become

$$\dot{\xi}(t) = -\left(\gamma_- + i2\omega_{\mathrm{S}}\right)\xi(t) - i2S_{\mathrm{o}}\xi_3(t) - \gamma_{\mathrm{o}}, \tag{56}$$

$$\dot{\xi}_3(t) = -\gamma_-\xi_3(t) + 8\,\mathrm{Im}\left[S_{\mathrm{o}}\xi^*(t)\right] + \gamma_+. \tag{57}$$

The Markovian limit is a substantial simplification and leads to a set of coupled, first-order autonomous differential equations that can be solved by using, for example, the Laplace transform, as detailed in appendix A. We denote the resulting two-point function in the Markovian limit, obtained by using these solutions for $\xi(t)$ and $\xi_3(t)$ in eq. (53), with $\mathcal{G}^{\mathrm{M}}_k(t)$.

2. RWA

Under the RWA, we drop the off-diagonal terms in eqs. (54) and (55), resulting in the set of *decoupled* equations,

$$\dot{\xi}(t) = -\left[\gamma_-(t) + i2\omega_\mathrm{S}(t)\right]\xi(t)\,, \tag{58}$$

$$\dot{\xi}_3(t) = -\gamma_-(t)\xi_3(t) + \gamma_+(t)\,. \tag{59}$$

The ξ equation is solved by simple separation of variables and the ξ_3 equation can be solved with an integrating factor; the solutions are

$$\xi(t) = \xi(t_0)e^{-\int_{t_0}^{t}\mathrm{d}t_1\left[\gamma_-(t_1)+i2\omega_\mathrm{S}(t_1)\right]}\,, \tag{60}$$

$$\xi_3(t) = \xi_3(t_0)e^{-\int_{t_0}^{t}\mathrm{d}t_1\gamma_-(t_1)} + e^{-\int_{t_0}^{t}\mathrm{d}t_1\gamma_-(t_1)}\int_{t_0}^{t}\mathrm{d}t_1 e^{\int_{t_0}^{t_1}\mathrm{d}t_2\gamma_-(t_2)}\gamma_+(t_1)\,. \tag{61}$$

We denote the resulting two-point function under the RWA, obtained by using these solutions for $\xi(t)$ and $\xi_3(t)$ in eq. (53), with $\mathcal{G}_k^{\mathrm{RWA}}(t)$.

3. RWA and Markovian limit

Performing the RWA and taking the Markovian limit together effectively reduces the Redfield equation to the Davies equation. In this case, the coefficients in eqs. (58) and (59) become time-independent and their solutions reduce to

$$\xi(t) = \xi(t_0)e^{-(\gamma_-+i2\omega_\mathrm{S})(t-t_0)}\,, \tag{62}$$

$$\xi_3(t) = \xi_3(t_0)e^{-\gamma_-(t-t_0)} + \frac{\gamma_+}{\gamma_-}\left[1-e^{-\gamma_-(t-t_0)}\right]. \tag{63}$$

In both examples considered in the following sections, where we assume that the system starts in the vacuum of the free theory, we find that $\gamma_+ = \gamma_-$ and that these solutions reduce to their initial values: $\xi(t) = 0$ and $\xi_3(t) = 1$. Therefore, the Davies master equation does not capture the open system dynamics that we are interested in, and we do not consider it further in the paper.

4. Dyson series perturbation theory

As discussed in section III A, the Redfield equation matches the standard Dyson series at second order in λ when $\hat{\rho}_\mathrm{I}(t_1)$ is expanded to zeroth order on the right-hand side of eq. (28), replacing $\hat{\rho}_\mathrm{I}(t_1)$ there with $\hat{\rho}(t_0)$. This also decouples eqs. (54) and (55), resulting in the equations

$$\dot{\xi}(t) = -i2\omega_k\xi(t) - \left[\gamma_-(t) + i2\tilde{\omega}_\mathrm{S}(t)\right]\xi(t_0) - i2S_\mathrm{o}(t)\xi_3(t_0) - \gamma_\mathrm{o}(t)\,, \tag{64}$$

$$\dot{\xi}_3(t) = -\gamma_-(t)\xi_3(t_0) + 8\,\mathrm{Im}\left[S_\mathrm{o}(t)\xi^*(t_0)\right] + \gamma_+(t)\,, \tag{65}$$

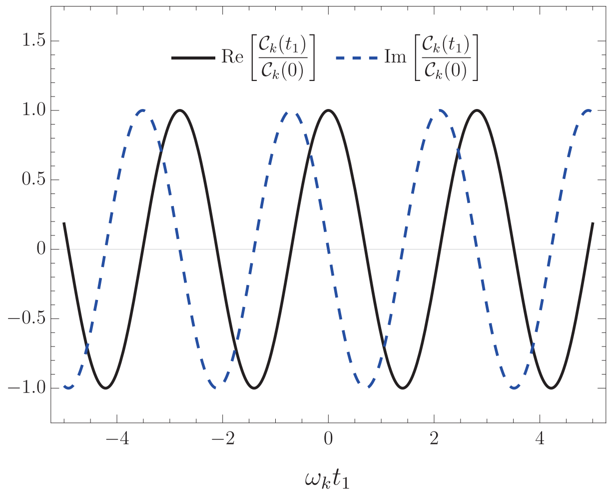

FIG. 1: The environment correlation as a function of elapsed time, t_1, in a $\lambda\phi\chi$ interacting theory, computed for $M = 3m$ and $k = m$, and normalized to unity at $t_1 = 0$.

where $\tilde{\omega}_\mathrm{S}(t) = S_{k,11}(t) + S_{k,22}(t)$. These expressions can be obtained directly from eqs. (54) and (55) by replacing all $\xi(t)$ and $\xi_3(t)$ on the right-hand side with their respective initial conditions, except for the $\omega_k\xi(t)$ term that originates from transforming the derivative on the left-hand side of eq. (45) back to the Schrödinger picture and should, therefore, not be replaced. We denote the resulting two-point function in perturbation theory, obtained by using the solutions for $\xi(t)$ and $\xi_3(t)$ in eq. (53), with $\mathcal{G}_k^{\mathrm{PT}}(t)$. Note that $\mathcal{G}_k^{\mathrm{PT}}(t)$ is essentially the second-order loop correction to the two-point function.

V. $\lambda\phi\chi$ INTERACTION

The simplest way to couple the system and environment fields is through a bilinear coupling. This corresponds to setting $n = 1$ in eqs. (2) and (6), so that the interaction Hamiltonian is given by

$$\hat{H}_\mathrm{int} = \lambda\int \mathrm{d}^3x\,\hat{\phi}(\vec{x})\hat{\chi}(\vec{x})\,. \tag{66}$$

The resulting theory is *exactly* solvable since the full Hamiltonian remains quadratic. The $\lambda\phi\chi$ interaction thus provides an excellent benchmark for the master equation result under various approximations. The exact calculation is detailed in appendix B and, in particular, the exact solution for the two-point function, denoted $\mathcal{G}_k^{\mathrm{exact}}(t)$, is shown in eq. (B10). We note that the exact solution breaks down for a large enough λ because the potential is unbounded from below and that the constraint on λ can be avoided by instead choosing the interaction to be $\frac{1}{2}\lambda\left(\phi-\chi\right)^2$. We continue to work with the $\lambda\phi\chi$ interaction, however, as it remains well-behaved in the perturbative regime that we are interested in here.

We now discuss the master equation solution for the $\lambda\phi\chi$ interaction. Consider first the correlation function $\mathcal{C}_k(t_1)$, given by eq. (44), that appears in the $\Gamma_{k,\beta}(t)$ coefficients. With $n = 1$, it becomes

$$\mathcal{C}_k(t_1) = \frac{e^{-i\Omega_k t_1}}{2\Omega_k}\,, \tag{67}$$

where evaluating the single $\vec{p}$ integral over the momentum-conserving δ-function forces the system and environment momenta to be equal. The Redfield equation coefficients are now constructed by using the above expression for $\mathcal{C}_k(t_1)$ in eq. (43), using the resulting $\Gamma_{k,\beta}$ in eqs. (46) and (47), and finally using the resulting $\gamma_{k,\alpha\beta}$ and $S_{k,\alpha\beta}$ in the definitions of $\gamma_{\rm o}$, $S_{\rm o}$, $\gamma_\pm$, and $\omega_{\rm S}$ after eq. (55). We show the coefficients explicitly in appendix C for completeness.

It is clear from eq. (67) that the environment correlation $\mathcal{C}_k(t_1)$ for this interaction is purely oscillatory, as shown in fig. 1 for clarity. Given our discussion in section III B, we should thus not expect the Markovian limit to capture the dynamics in this case accurately. Another way to see that the dynamics here must be non-Markovian is to notice that each mode in the system is coupled to exactly one mode in the environment; thus, the composite system behaves like a collection of pairwise coupled oscillators. Consequently, there will be substantial memory in the evolution of the system density operator that should not be neglected. The Markovian limit can, nevertheless, be imposed by setting the upper limit of the time integral in $\Gamma_{k,\beta}(t)$ to ∞, yielding the time-independent function of k,

$$\Gamma_{k,\beta} = \lambda^2 \int_0^\infty {\rm d}t_1 \frac{e^{-i(-1)^\beta \omega_k t_1}}{4\omega_k\Omega_k} e^{-i\Omega_k t_1}\,, \tag{68}$$

which can be simplified by resolving the right-hand side into a sum of Fourier cosine and sine transforms.

Although we do not expect the Markovian limit to be valid, the Markov approximation used in constructing our master equation can still hold for this interaction. In section III A, we concluded that the Markov approximation is justifiable for rapidly oscillating environment correlators so long as the characteristic timescale $\tau_{\rm S}$ of the system is much longer than the period $\tau_\star$ of the kernel $\mathcal{K}_{k,\beta}$. While $\tau_{\rm S} \sim \omega_k^{-1}$, we can obtain $\tau_\star$ by setting $\beta = 1$ in eq. (34), so as to obtain the longest relevant timescale for this interaction, which gives $\tau_\star \sim |\Omega_k - \omega_k|^{-1}$. We thus require that $|\Omega_k - \omega_k| \gg \omega_k$. While this condition breaks down for $k \to \infty$, the effect of the interaction is suppressed in this limit since $\Gamma_{k,\beta}(t)$ is itself suppressed. For $k \to 0$, on the other hand, this condition translates into $|M - m| \gg m$. We next show that the Redfield equation-based solution is more accurate than the perturbation theory-based solution for $M \gg m$, suggesting that the Markov approximation is indeed a good approximation in this limit.

To distinguish the validity of the Markov approximation from that of perturbation theory, we consider the relative error of the solution obtained using the Redfield equation,

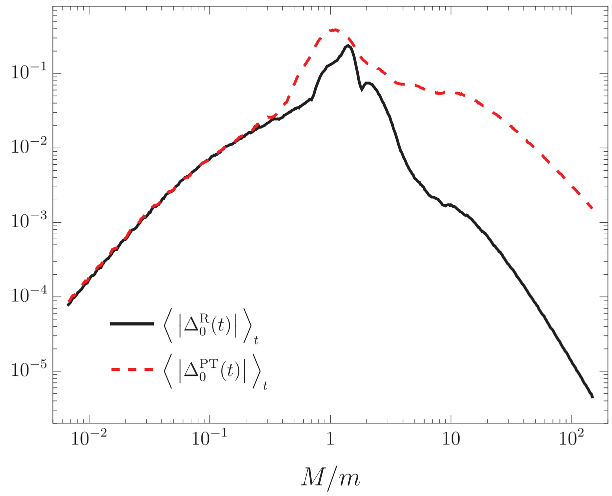

FIG. 2: The time-averaged absolute error in the two-point function calculated using the Redfield equation and perturbation theory as a function of M/m in a $\lambda\phi\chi$ interacting theory, for $\lambda = Mm/2$ and $k = 0$, and averaged over the time interval $\sqrt{2\lambda}t \in [0, 10]$.

$$\Delta_k^{\rm R}(t) = \frac{\mathcal{G}_k^{\rm R}(t) - \mathcal{G}_k^{\rm exact}(t)}{\mathcal{G}_k^{\rm exact}(t)}\,, \tag{69}$$

and that obtained using Dyson series perturbation theory, $\Delta_k^{\rm PT}(t)$. We solve for $\Delta_k^{\rm R}(t)$ and $\Delta_k^{\rm PT}(t)$ numerically, using eq. (B10) for $\mathcal{G}_k^{\rm exact}(t)$. In fig. 2, we plot the time-averaged zero-mode absolute errors $\langle|\Delta_0^{\rm R}(t)|\rangle_t$ and $\langle|\Delta_0^{\rm PT}(t)|\rangle_t$ as a function of M/m, upon fixing the interaction strength, $\lambda = Mm/2 = \text{constant}$ [70], and averaging over the time interval $\sqrt{2\lambda}t \in [0, 10]$. As expected, the absolute error decreases as the mass scales separate for the Redfield and perturbation theory solutions due to a corresponding decrease in $\Gamma_{k,\beta}$. For $M > m$, however, the error in the Redfield solution decreases more rapidly than in perturbation theory, indicating that the Redfield solution offers an improvement over perturbation theory in this regime due to the improving validity of the Markov approximation.

In fig. 3, we compare the relative errors of the two solutions further for a given $M > m$ and for different choices of interaction strength λ. We find that the relative error is smaller for the Redfield solution and, while both solutions break down for large interaction strength and at late times as expected, the breakdown of the Redfield solution is *slower* than the perturbation theory one. This suggests that solving the coupled differential eqs. (54) and (55) obtained from the Redfield equation provides a perturbative resummation that offers an improvement over the Dyson series perturbation theory result or, equivalently, the second-order loop correction. Understanding the set of diagrams that the Redfield solution resums – for example, whether it resums all 1PI diagrams at second-order

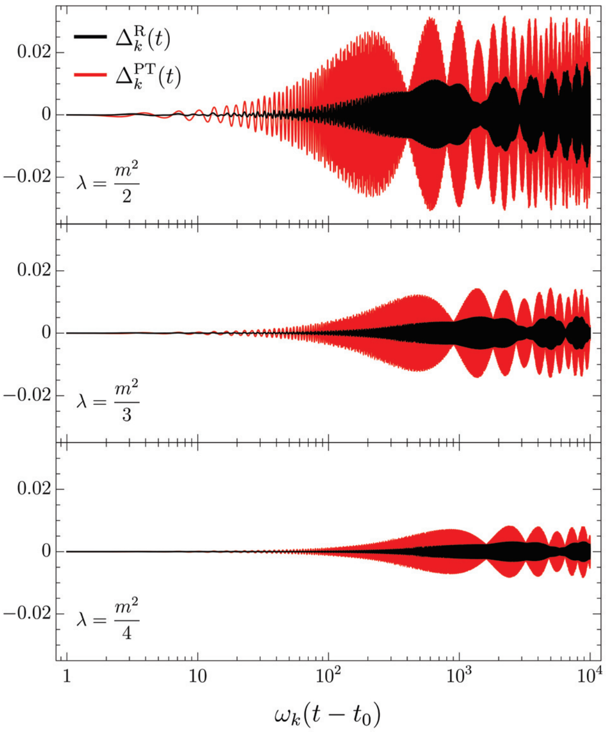

FIG. 3: The relative error in the two-point function calculated using the Redfield equation and perturbation theory as a function of time in a $\lambda\phi\chi$ interacting theory, for three relatively large values of λ, $k = m$, and $M = 3m$. Expectedly, the error is suppressed for smaller values of λ.

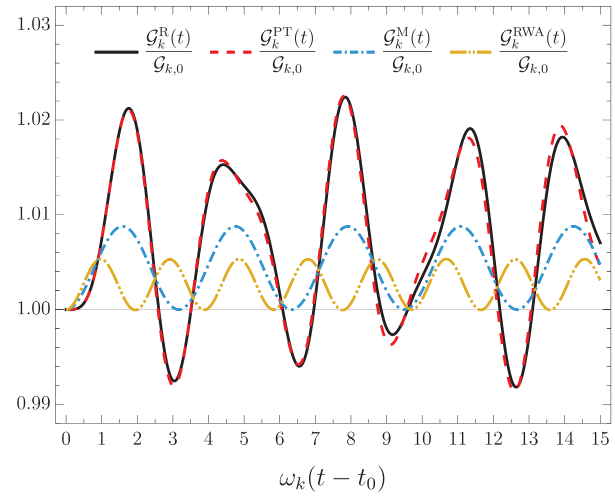

FIG. 4: The two-point function calculated using the Redfield equation, perturbation theory, the Redfield equation in the Markovian limit, and the Redfield equation under the RWA as a function of time in a $\lambda\phi\chi$ interacting theory, for $\lambda = m^2/2$, $k = m$, and $M = 3m$, and normalized by the free theory solution, $\mathcal{G}_{k,0} = (2\omega_k)^{-1}$.

in λ or other diagrams – would be an interesting direction to pursue in future work.

Lastly, we compare the Redfield and perturbation theory solutions to those obtained in the Markovian limit and under the RWA in fig. 4. As expected, neither approximation coincides with the Redfield dynamics, which capture the exact dynamics on these timescales, in a meaningful way. As mentioned earlier, we do not expect the Markovian limit to be valid since the environment correlation is purely oscillatory and we do not expect the RWA to be a good approximation for systems with a continuous spectrum.

VI. $\lambda\phi\chi^2$ INTERACTION

We next couple the system and environment fields nonlinearly, specifically setting $n = 2$ in eq. (2), so that the interaction is given by $\lambda\phi\chi^2$. Note that we now need to renormalize the theory since loop corrections to, for example, $\hat{\phi}$ correlations will contain UV-divergent (environment) momentum integrals. It turns out that the master equation approach is compatible with standard methods of renormalization, and we thus proceed by shifting and rescaling the fields in eq. (2) with $n = 2$. Ignoring any vertex corrections since they do not contribute at $O(\lambda^2)$, we then add the following counterterm Lagrangian density to the system-environment Lagrangian density in eq. (2),

$$\mathcal{L}_{\mathrm{ct}} = -\frac{1}{2}\delta_\phi(\partial\phi)^2 - \frac{1}{2}\delta_m\phi^2 + Y_\phi\phi - \frac{1}{2}\delta_\chi(\partial\chi)^2 - \frac{1}{2}\delta_M\chi^2 + Y_\chi\chi\,, \tag{70}$$

where δ_ϕ, δ_χ, δ_m, δ_M, Y_ϕ, and Y_χ are the usual counterterms required to cancel any UV divergences. Note that, in general, we may need to add counterterms in the initial state as well [52–57], but they are not needed for the interaction that we consider here.

In fact, we can set all χ-counterterms in eq. (70) to zero: δ_χ and δ_M can be set to zero since environment correlation functions are calculated at tree level and Y_χ can be set to zero since the interaction is quadratic in χ and does not generate a one-point expectation value. Additionally, since this interaction is linear in ϕ, the divergence of $\langle\hat{\phi}(\vec{x},t)\rangle$ is at first order and results from the contraction of the two factors of $\hat{\chi}$ in the interaction Hamiltonian. Therefore, for this interaction and choice of environment state, we can drop the Y_ϕ counterterm as well and, equivalently, normal order the environment operator in the interaction Hamiltonian. This, in fact, sets $\hat{H}_{\mathrm{eff}}(t) = 0$, as discussed after eq. (24), and is also consistent with dropping equal-time bubble diagrams in the environment as discussed after eq. (44). With these simplifications, the quantized interaction Hamiltonian that we consider in this section becomes

$$\hat{H}_{\text{int}} = \int \mathrm{d}^3x \bigg[\frac{\delta_\phi}{2}\left(\hat{\pi}_\phi^2(\vec{x}) + (\vec{\nabla}\hat{\phi}(\vec{x}))^2\right) + \frac{\delta_m}{2}\hat{\phi}^2(\vec{x}) + \frac{\lambda}{2}\hat{\phi}(\vec{x}) : \hat{\chi}^2(\vec{x}) : \bigg] . \tag{71}$$

We now write the counterterm contribution in terms of the dimensionless eigenoperators of the system introduced in section II. This leads to a correction in the system's unitary dynamics, such that $S_{k,\alpha\beta}(t)$ in eq. (47) becomes

$$S_{k,\alpha\beta}(t) \equiv \frac{1}{2i}\left[\Gamma_{k,\beta}(t) - \Gamma^*_{k,\alpha}(t)\right] + \frac{1}{4\omega_k}\left[\delta_\phi\left(k^2 + (-1)^{\alpha+\beta}\omega_k^2\right) + \delta_m\right] , \tag{72}$$

which enters the modified system Hamiltonian in eq. (48), which in turn enters the master equation in eq. (49).

Let us now consider the environment correlation function for $\lambda\phi\chi^2$. For $n = 2$, eq. (44) retains one momentum integral, with the momentum-conserving delta function restricting one of the environment modes to be the sum of the other two modes in the interaction, hence

$$\mathcal{C}_k(t_1) = \frac{1}{8}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{e^{-i\left(\Omega_p + \Omega_{|\vec{k}+\vec{p}|}\right)t_1}}{\Omega_p\Omega_{|\vec{k}+\vec{p}|}} . \tag{73}$$

We set $M = 0$ to simplify the integral over $\vec{p}$ and find, as usual, that it is UV-divergent. On regulating the integral by introducing a small imaginary piece to the time parameter such that $t_1 \to t_1 - i\epsilon$ with $\epsilon \in \mathbb{R}^+$, we obtain the ϵ-dependent environment correlation function,

$$\mathcal{C}_k(t_1) = -\frac{i}{32\pi^2}\frac{e^{-ikt_1}}{t_1 - i\epsilon} . \tag{74}$$

Although $\mathcal{C}_k(t_1)$ does not diverge as $\epsilon \to 0$, the regularization scheme simply converts the momentum divergence to an initial time one, which becomes apparent when computing the t_1 integral. We can, nevertheless, infer a couple of important points from eqs. (73) and (74), which we highlight before continuing with our renormalization procedure.

First, we note that the momentum integral in eq. (73) is a direct result of the nonlinearity of the interaction. In contrast to the $\lambda\phi\chi$ theory, where a given system mode only couples to a single environment mode, in the $\lambda\phi\chi^2$ theory, a given system mode couples to *all* environment modes. In this sense, the environment can be considered *large*, analogous to the famous (Markovian) Caldeira-Leggett model of quantum Brownian motion [58, 59]. Second, from eq. (74) and fig. 5, we see that $\mathcal{C}_k(t_1)$ decays rapidly away from $t_1 = 0$, suppressing memory in the system density operator and suggesting that the dynamics are Markovian. We, therefore, expect the Markovian limit of the Redfield equation to capture the late-time behavior of the system, which is what we find below.

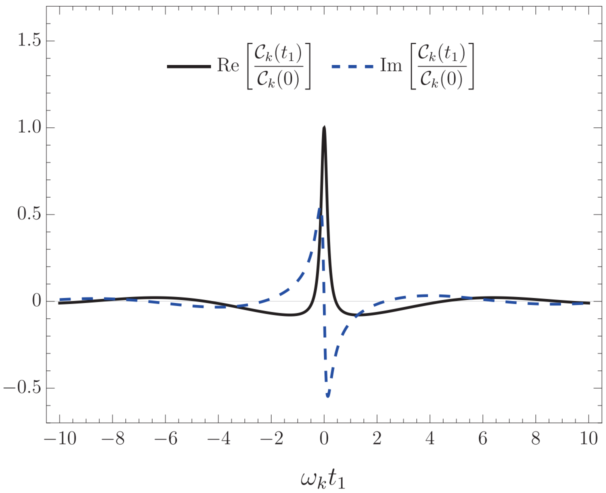

FIG. 5: The environment correlation as a function of elapsed time, t_1, in a $\lambda\phi\chi^2$ interacting theory, computed for $\epsilon = 1/10$ and $k = m$, and normalized to unity at $t_1 = 0$.

Let us now return to the renormalization procedure. Using eq. (74) in eq. (43), we find that the time integral leads to a $\ln\epsilon$ divergence that persists in $S_{k,\alpha\beta}(t)$ and can be canceled by choosing the counterterms

$$\delta_\phi = 0 , \quad \delta_m = -\frac{\lambda^2}{16\pi^2}\ln\left(\epsilon\mu_{\mathrm{r}}\right) , \tag{75}$$

where we have introduced the renormalization scale μ_{r}. We show the final coefficients that enter the master equation for this interaction and with the above choice of counterterms in appendix D. We note that while the coefficients in eqs. (D2) and (D5) diverge at the initial time, since $\mathrm{Ci}(x)$ diverges at zero, correlation functions must remain finite at the initial time. A simple argument for this is that the perturbation theory solution continuously matches the initial conditions at the initial time, and the Redfield solution reduces to the perturbation theory one at early times. Also, as expected, the coefficients depend on μ_{r}, which is typically chosen in a fixed loop order calculation such that the result has reduced dependence on it at the energy scales of interest. We discuss some subtleties related to choosing μ_{r} in the next section and simply choose a reasonable value for it in the analysis that follows.

We now compare the Redfield equation-based solution to the solutions obtained using perturbation theory, in the Markovian limit, and under the RWA in fig. 6. From the left panel of fig. 6, we see that the Redfield and perturbation theory solutions agree on short timescales. The Markovian limit and RWA, on the other hand, are both poor approximations at short timescales, although the solution in the Markovian limit retains more of the phase information. From the right panel of fig. 6, we see that the Redfield solution relaxes at late times, whereas the perturbation theory one continues to oscillate. This qualitative difference can be attributed to the Redfield equa-

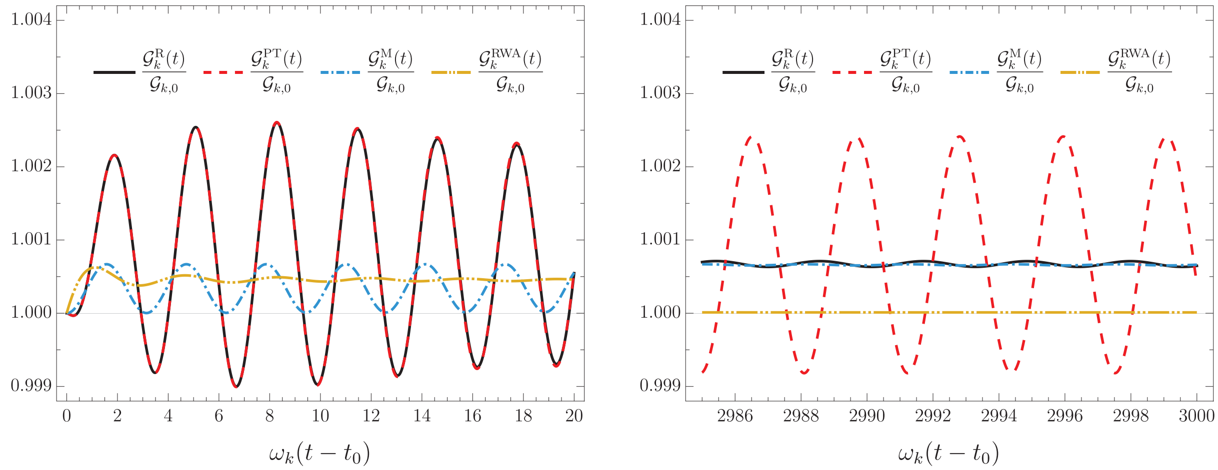

FIG. 6: The two-point function calculated using the Redfield equation, perturbation theory, the Redfield equation in the Markovian limit, and the Redfield equation under the RWA as a function of time in a $\lambda\phi\chi^2$ interacting theory, for $\lambda = m/2$, $k = m$, and $\mu_{\mathrm{r}} = 10m$, and normalized by the free theory solution, $\mathcal{G}_{k,0} = (2\omega_k)^{-1}$. (Left) The two-point function at early times. (Right) The two-point function at sufficiently late times showing a relaxation of the Redfield solution as well as in the Markovian limit. We note that for the numerical Redfield solution, we impose initial conditions at $\omega_k(t-t_0) = 10^{-10}$ since some of the coefficients diverge as $t \to t_0$.

tion representing a resummation of second-order dynamics. We further see that the Markovian limit is a good approximation at late times, since the solution relaxes to the same value as that obtained using the Redfield equation. This supports our expectation that the environment in the $\lambda\phi\chi^2$ theory is approximately Markovian. Lastly, we see that the RWA is a poor approximation at late times as well, as the solution relaxes to the free theory one.

VII. COMMENTS ON THE RENORMALIZATION SCALE

We next show that the renormalization scale, μ_{r}, encountered in the previous section, can have a significant effect on the late-time solution obtained using the master equation, and certain choices of μ_{r} can lead to physically inconsistent results. For the $\lambda\phi\chi^2$ theory, we can write an analytical expression for the late-time correlation in the Markovian limit where, as shown in the previous section, the solution agrees with the Redfield solution,

$$\mathcal{G}_{k,0}^{-1}\mathcal{G}_k^{\mathrm{M}}(t\to\infty) = \frac{\frac{\lambda^2}{16\pi^2\omega_k^2}\ln\left(\frac{\omega_k+k}{m}\right)-1}{\frac{\lambda^2}{16\pi^2\omega_k^2}\ln\left(\frac{\mu_{\mathrm{r}}}{me^{\gamma}}\right)-1}\,, \tag{76}$$

where γ is the Euler-Mascheroni constant. Using the Markovian limit solutions for $\xi(t)$ and $\xi_3(t)$, we can further construct the two-point function for the conjugate momentum and find that $\lim_{t\to\infty}\langle\hat{\pi}_{\phi,\vec{k}}(t)\hat{\pi}_{\phi,\vec{k}'}(t)\rangle = \langle\hat{\pi}_{\phi,\vec{k}}(t_0)\hat{\pi}_{\phi,\vec{k}'}(t_0)\rangle$. The uncertainty principle in the late-time limit, therefore, reduces to

$$\lim_{t\to\infty}\Delta\phi_{\vec{k}}(t)\Delta\pi_{\phi,\vec{k}}(t) = \frac{1}{2}\sqrt{\mathcal{G}_{k,0}^{-1}\mathcal{G}_k^{\mathrm{M}}(t\to\infty)} \geq \frac{1}{2}\,. \tag{77}$$

Imposing the inequality in eq. (77) then leads to the following condition on μ_{r} for the theory to be physical,

$$\mu_{\mathrm{r}} > (\omega_k + k)\,e^{\gamma}\,. \tag{78}$$

The k-dependence of μ_{r} implied by the above equation is not implausible as we may need to choose μ_{r} in such a way to remove any dependence on it at a given energy scale k. Nevertheless, the breakdown of uncertainty for an *incorrect choice* of μ_{r} is not encountered in perturbation theory. In fact, the perturbation theory solution does not break the uncertainty principle for either $\lambda\phi\chi$ or $\lambda\phi\chi^2$ and, in particular, for any $\mu_{\mathrm{r}} > 0$ for the latter. This suggests that the breakdown of uncertainty here is *not* a feature of renormalization in QFT but is rather inherited from the Redfield equation construction. Interestingly, whereas the pathology of the Redfield description is typically diagnosed via positivity violation of the dynamical state of the system, here we diagnose this by demanding physical bounds on the relevant field correlators. While we leave a detailed study of this issue and potential connections between the unphysical predictions to future work, we briefly examine the Hamiltonian corrections, or the $S_{k,\alpha\beta}(t)$ coefficients, in the master equations for both interactions, as any μ_{r}-dependence only appears in these terms.

We first consider the $S_{k,\alpha\beta}(t)$, which can be interpreted as a Lamb-type shift in the system frequency spectrum, with negative eigenvalues [58, 59]. For the $\lambda\phi\chi$ interaction and considering the simpler case of the Davies equation, the eigenvalues can be calculated exactly and are given by

$$S_{k,11} = -\frac{\lambda^2}{4\omega_k \Omega_k \left(\Omega_k - \omega_k\right)}, \quad (79)$$

$$S_{k,22} = -\frac{\lambda^2}{4\omega_k \Omega_k \left(\Omega_k + \omega_k\right)}. \quad (80)$$

While the second eigenvalue is always negative, the first one is negative only for $\Omega_k > \omega_k$, which is consistent with the requirement of $M > m$ for the Markov approximation to be valid. Violating this mass hierarchy, therefore, leads to an unphysical frequency shift and can lead to a broken uncertainty principle. For the more general case of the Redfield equation, on the other hand, we find that one eigenvalue of $S_{k,\alpha\beta}$ is positive and the other is negative. For parameter values that preserve the uncertainty principle, however, we find that the *trace* of $S_{k,\alpha\beta}$ is negative, at least at late times. We also find that the uncertainty principle is preserved when *all* unitary corrections, in both $S_{k,\alpha\beta}$ as well as $\gamma_{k,\alpha\beta}$, are ignored.

We next consider the $S_{k,\alpha\beta}(t)$ coefficients for the $\lambda\phi\chi^2$ interaction. For the Davies equation, its eigenvalues are given by

$$S_{k,11} = -\frac{\lambda^2}{64\pi^2\omega_k} \ln\left[\frac{\mu_{\rm r}}{\left(\omega_k - k\right) e^{\gamma}}\right], \quad (81)$$

$$S_{k,22} = -\frac{\lambda^2}{64\pi^2\omega_k} \ln\left[\frac{\mu_{\rm r}}{\left(\omega_k + k\right) e^{\gamma}}\right]. \quad (82)$$

Demanding that both eigenvalues are negative gives us two conditions, with the second one being more restrictive and exactly matching the condition obtained from the uncertainty principle in eq. (78). For the Redfield equation, on the other hand, we find the same qualitative behavior as mentioned in the previous paragraph for $\lambda\phi\chi$. We note that dropping all unitary corrections in $S_{k,\alpha\beta}$ and $\gamma_{k,\alpha\beta}$ would now also drop all instances of $\mu_{\rm r}$. These observations suggest that the broken uncertainty principle originates from the Lamb-type frequency shift terms, which are known to be problematic in general [60]. Our procedure to set the counterterm appropriately to rectify the Lamb shift-induced violation of uncertainty is reminiscent of recent results that have shown how, by making suitable modifications to the system Hamiltonian, *fitter* Redfield descriptions can be obtained both in terms of accuracy of steady state predictions as well as describing transient dynamics [61].

VIII. DISCUSSION

Our goal in this paper was to use the quantum master equation approach to study the dynamics of a system scalar field ϕ coupled to an environment scalar field χ through a $\lambda\phi\chi^n$ interaction, with $n \in \{1, 2\}$. We were specifically interested in understanding whether the master equation approach provides an improvement upon standard Dyson series results and whether the dynamics of ϕ, upon tracing out χ, are described by a non-Markovian or Markovian master equation. For this purpose, we first derived the non-Markovian Redfield master equation for the reduced density operator of ϕ by tracing out the χ field at second order in λ and under two standard approximations: the Born approximation that restricts to weak coupling and the Markov approximation that assumes a separation of timescales between the system and environment. We then used the master equation to set up coupled differential equations in two-point correlations of the creation and annihilation operators of $\hat{\phi}_{\vec{k}}(t)$, and discussed how the coupled equations simplify in various limits including the Markovian limit, RWA, and standard perturbation theory expansion. The resulting coupled equations can finally be solved to obtain the equal-time two-point function of $\hat{\phi}_{\vec{k}}(t)$ in any of these limits.

As a first explicit example, we considered a $\lambda\phi\chi$ interaction. We argued that the Markovian limit must be invalid for this interaction since (i) the environment correlation function does not decay in time and (ii) a single system mode only couples to a single environment mode. However, we showed that the Markov approximation, and thus the Redfield master equation construction, remains valid as long as the χ field is much heavier than the ϕ field. We next used the resulting Redfield equation to solve for the two-point function analytically/numerically in various limits, and showed that the Markovian limit indeed fails to capture the late-time behavior. Finally, we showed that the Redfield equation-based solution offers a substantial improvement over the standard perturbation theory-based one, suggesting that the Redfield equation provides a *perturbative resummation* for the calculation of observables.

As a second explicit example, we considered a $\lambda\phi\chi^2$ interaction. An immediate consequence of the nonlinear environment was to make the environment correlation function UV-divergent. We found, however, that the master equation approach admits standard renormalization techniques, such that adding appropriate counterterms in the unitary dynamics allowed us to cancel all UV-divergences, while introducing a renormalization scale $\mu_{\rm r}$ in the master equation. We next argued that the Markovian limit must be valid for this interaction since (i) the (regularized) environment correlation function decays rapidly in time and (ii) a single system mode couples to all environment modes. Choosing an arbitrary value of $\mu_{\rm r} = 10m$, we then solved the resulting Redfield equation for the two-point function analytically/numerically in various limits, and showed that the Markovian limit indeed captures the late-time behavior of the full Redfield solution. Finally, we showed that the perturbation theory solution significantly differs from the Redfield solution at late times, again suggesting that the Redfield equation provides a perturbative resummation for the calculation of observables.

We also highlighted a couple of open questions that we have left to future work. First, while our results suggested that the Redfield solution provides a perturbative resummation, it is unclear which diagrams it resums – for example, whether it resums all 1PI diagrams at second-order in λ or other diagrams as well. Second, for the $\lambda\phi\chi^2$ interaction, we found that resummed correlations can break the uncertainty principle at late times for certain values of $\mu_{\rm r}$. We argued that this may be an artifact of an unphysical Lamb-type correction in the master equation construction but it deserves further investigation.

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[65] We will see in sections V and VI that for both interactions we consider, this environment correlator only depends on the magnitude k.
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[67] Since we have used definite integrals in the integration by parts, there are terms that depend on the initial time t_0. It is straightforward to show, however, that these terms cancel identically.
[68] Since we are working at $O(\lambda^2)$, only tree-level diagrams contribute to the environment correlation function. This will be convenient when we renormalize the $\lambda\phi\chi^2$ theory in section VI.
[69] This qualitative equivalence between quantum master equations expressed in Schrödinger and interaction pictures does not hold when free evolution of the system itself is non-unitary [64].
[70] We never have to explicitly choose a value for Mm since the explicit dependence on M and m appears as their ratio for this choice of parameters.

Appendix A: Solving the coupled differential equations in the Markovian limit

In this appendix, we solve for the functions $\xi(t)$ and $\xi_3(t)$, defined in eqs. (51) and (52), in the Markovian limit. We start with the coupled eqs. (56) and (57), that we write again for convenience,

$$\dot{\xi}(t) = -\left(\gamma_- + i2\omega_{\rm S}\right)\xi(t) - i2S_{\rm o}\xi_3(t) - \gamma_{\rm o}\,, \tag{A1}$$

$$\dot{\xi}_3(t) = -\gamma_-\xi_3(t) + 8\,\mathrm{Im}\left[S_{\rm o}\xi^*(t)\right] + \gamma_+\,, \tag{A2}$$

where γ_o, S_o, $\gamma_\pm$, and ω_S are all time-independent. In terms of the real and imaginary parts of $\xi = \xi_1 + i\xi_2$ and the time parameter $T = t - t_0$, these can be written as the three equations,

$$\frac{\mathrm{d}\xi_1}{\mathrm{d}T} = -\gamma_-\xi_1 + 2\omega_{\rm S}\xi_2 + 2\,\mathrm{Im}\left[S_{\rm o}\right]\xi_3 - \mathrm{Re}\left[\gamma_{\rm o}\right]\,, \tag{A3}$$

$$\frac{\mathrm{d}\xi_2}{\mathrm{d}T} = -\gamma_-\xi_2 - 2\omega_{\rm S}\xi_1 - 2\,\mathrm{Re}\left[S_{\rm o}\right]\xi_3 - \mathrm{Im}\left[\gamma_{\rm o}\right]\,, \tag{A4}$$

$$\frac{\mathrm{d}\xi_3}{\mathrm{d}T} = -\gamma_-\xi_3 + 8\,\mathrm{Im}\left[S_{\rm o}\right]\xi_1 - 8\,\mathrm{Re}\left[S_{\rm o}\right]\xi_2 + \gamma_+\,. \tag{A5}$$

We now simplify the above equations by making the change of variables $\xi_i(T) = \eta_i(T)e^{-\gamma_- T}$, which gives

$$\frac{\mathrm{d}\eta_1}{\mathrm{d}T} - 2\omega_{\rm S}\eta_2 - 2\,\mathrm{Im}\left[S_{\rm o}\right]\eta_3 = -\,\mathrm{Re}\left[\gamma_{\rm o}\right]e^{\gamma_- T}\,, \tag{A6}$$

$$2\omega_{\rm S}\eta_1 + \frac{\mathrm{d}\eta_2}{\mathrm{d}T} + 2\,\mathrm{Re}\left[S_{\rm o}\right]\eta_3 = -\,\mathrm{Im}\left[\gamma_{\rm o}\right]e^{\gamma_- T}\,, \tag{A7}$$

$$-8\,\mathrm{Im}\left[S_{\rm o}\right]\eta_1 + 8\,\mathrm{Re}\left[S_{\rm o}\right]\eta_2 + \frac{\mathrm{d}\eta_3}{\mathrm{d}T} = \gamma_+ e^{\gamma_- T}\,. \tag{A8}$$

We next take the Laplace transform to move from the T domain to the ε domain, making use of the property

$$\mathcal{L}_T\left\{\frac{\mathrm{d}}{\mathrm{d}T}f(T)\right\} = \int_0^\infty \mathrm{d}T\,\frac{\mathrm{d}}{\mathrm{d}T}f(T)e^{-\varepsilon T} = \lim_{b\to\infty}\left[f(T)e^{-\varepsilon T}\right]_0^b + \varepsilon\int_0^\infty \mathrm{d}T\left[f(T)e^{-\varepsilon T}\right] = \varepsilon f(\varepsilon) - f(0)\,. \tag{A9}$$

Then the resulting equations can be written in matrix form as

$$\begin{bmatrix} \varepsilon & -2\omega_{\mathrm{S}} & -2\,\mathrm{Im}\,[S_{\mathrm{o}}] \\ 2\omega_{\mathrm{S}} & \varepsilon & 2\,\mathrm{Re}\,[S_{\mathrm{o}}] \\ -8\,\mathrm{Im}\,[S_{\mathrm{o}}] & 8\,\mathrm{Re}\,[S_{\mathrm{o}}] & \varepsilon \end{bmatrix} \begin{bmatrix} \eta_1(\varepsilon) \\ \eta_2(\varepsilon) \\ \eta_3(\varepsilon) \end{bmatrix} = \begin{bmatrix} \eta_1(0) - \frac{\mathrm{Re}[\gamma_{\mathrm{o}}]}{\varepsilon-\gamma_-} \\ \eta_2(0) - \frac{\mathrm{Im}[\gamma_{\mathrm{o}}]}{\varepsilon-\gamma_-} \\ \eta_3(0) + \frac{\gamma_+}{\varepsilon-\gamma_-} \end{bmatrix} , \tag{A10}$$

and solutions for $\xi_i(t)$ are then finally given by

$$\begin{bmatrix} \xi_1(t) \\ \xi_2(t) \\ \xi_3(t) \end{bmatrix} = \mathcal{L}^{-1}_{t-t_0} \left\{ \begin{bmatrix} \varepsilon & -2\omega_{\mathrm{S}} & -2\,\mathrm{Im}\,[S_{\mathrm{o}}] \\ 2\omega_{\mathrm{S}} & \varepsilon & 2\,\mathrm{Re}\,[S_{\mathrm{o}}] \\ -8\,\mathrm{Im}\,[S_{\mathrm{o}}] & 8\,\mathrm{Re}\,[S_{\mathrm{o}}] & \varepsilon \end{bmatrix}^{-1} \begin{bmatrix} \eta_1(0) - \frac{\mathrm{Re}[\gamma_{\mathrm{o}}]}{\varepsilon-\gamma_-} \\ \eta_2(0) - \frac{\mathrm{Im}[\gamma_{\mathrm{o}}]}{\varepsilon-\gamma_-} \\ \eta_3(0) + \frac{\gamma_+}{\varepsilon-\gamma_-} \end{bmatrix} \right\} e^{-\gamma_-(t-t_0)} , \tag{A11}$$

where $\mathcal{L}^{-1}_{t-t_0}$ denotes the inverse Laplace transform to the $t-t_0$ domain.

Appendix B: Exact solution for the equal-time two-point function in the $\lambda\phi\chi$ theory

In this appendix, we find the exact solution for the $\lambda\phi\chi$ interacting theory. We start with the Heisenberg equations of motion for the system and environment annihilation operators, $\hat{a}_{\vec{k}}$ and $\hat{b}_{\vec{k}}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{a}_{\vec{k}}(t) = -i\omega_k \hat{a}_{\vec{k}}(t) - i\frac{\lambda}{2\sqrt{\omega_k\Omega_k}}\left[\hat{b}_{\vec{k}}(t) + \hat{b}^{\dagger}_{-\vec{k}}(t)\right] , \tag{B1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{b}_{\vec{k}}(t) = -i\Omega_k \hat{b}_{\vec{k}}(t) - i\frac{\lambda}{2\sqrt{\omega_k\Omega_k}}\left[\hat{a}_{\vec{k}}(t) + \hat{a}^{\dagger}_{-\vec{k}}(t)\right] , \tag{B2}$$

and assume a general solution of the form

$$\hat{a}_{\vec{k}}(t) = f_0(t)\hat{a}_{\vec{k}} + f_1^*(t)\hat{a}^{\dagger}_{-\vec{k}} + f_2(t)\hat{b}_{\vec{k}} + f_3^*(t)\hat{b}^{\dagger}_{-\vec{k}} , \tag{B3}$$

$$\hat{b}_{\vec{k}}(t) = h_0(t)\hat{b}_{\vec{k}} + h_1^*(t)\hat{b}^{\dagger}_{-\vec{k}} + h_2(t)\hat{a}_{\vec{k}} + h_3^*(t)\hat{a}^{\dagger}_{-\vec{k}} , \tag{B4}$$

where operators on the right-hand side are Schrödinger picture operators and the functions f_i and h_i are Bogoliubov coefficients that are yet to be determined. Substituting these equations into eqs. (B1) and (B2) yields the set of coupled differential equations

$$\dot{f}_0(t) + i\omega_k f_0(t) + i\frac{\lambda}{2\sqrt{\omega_k\Omega_k}}\left[h_2(t) + h_3(t)\right] = 0 , \tag{B5}$$

$$\dot{f}_1(t) - i\omega_k f_1(t) - i\frac{\lambda}{2\sqrt{\omega_k\Omega_k}}\left[h_2(t) + h_3(t)\right] = 0 , \tag{B6}$$

$$\dot{h}_2(t) + i\Omega_k h_2(t) + i\frac{\lambda}{2\sqrt{\omega_k\Omega_k}}\left[f_0(t) + f_1(t)\right] = 0 , \tag{B7}$$

$$\dot{h}_3(t) - i\Omega_k h_3(t) - i\frac{\lambda}{2\sqrt{\omega_k\Omega_k}}\left[f_0(t) + f_1(t)\right] = 0 , \tag{B8}$$

and a second set that can be obtained by making the two exchanges $f_i \leftrightarrows h_i$ and $\omega_k \leftrightarrows \Omega_k$ above. Eqs. (B5) to (B8) can now be solved exactly using the same Laplace transform technique as in appendix A. The equal-time two-point correlator for the system can then be written in terms of the functions f_i as

$$\mathcal{G}^{\mathrm{exact}}_k(t) = \frac{1}{2\omega_k}\left[|f_0(t)|^2 + |f_1(t)|^2 + |f_2(t)|^2 + |f_3(t)|^2 + f_0(t)f_1^*(t) + f_0^*(t)f_1(t) + f_2(t)f_3^*(t) + f_2^*(t)f_3(t)\right] , \tag{B9}$$

and is given by

$$\begin{aligned} \mathcal{G}^{\mathrm{exact}}_k(t) = \frac{1}{2\omega_k}\Bigg\{1 + \frac{\lambda^2(\Omega_k-\omega_k)}{2\Omega_k\left(\omega_+^2-\omega_-^2\right)^2}\Bigg[&\frac{2\lambda^2(3\Omega_k-\omega_k) + \Omega_k^2(\Omega_k-\omega_k)^2(\Omega_k+3\omega_k)}{(\Omega_k^2\omega_k^2-\lambda^2)(\Omega_k-\omega_k)} \\ &+ \frac{\Omega_k+\omega_k}{\Omega_k-\omega_k}\left(\frac{\omega_+^2-\Omega_k^2}{\omega_+^2}\cos\left[2\omega_+(t-t_0)\right] - \frac{\omega_+^2-\omega_k^2}{\omega_-^2}\cos\left[2\omega_-(t-t_0)\right]\right) \\ &+ 2\left(1+\frac{\Omega_k\omega_k}{\omega_+\omega_-}\right)\cos\left[(\omega_++\omega_-)(t-t_0)\right] + 2\left(1-\frac{\Omega_k\omega_k}{\omega_+\omega_-}\right)\cos\left[(\omega_+-\omega_-)(t-t_0)\right]\Bigg]\Bigg\} , \end{aligned} \tag{B10}$$

where we have defined the frequencies

$$\omega_\pm = \sqrt{\frac{\Omega_k^2+\omega_k^2}{2} \pm \sqrt{\lambda^2 + \left(\frac{\Omega_k^2-\omega_k^2}{2}\right)^2}}\,. \tag{B11}$$

We note that the form of the frequencies $\omega_\pm$ above suggests that the perturbative regime for this theory is characterized by the condition $|\lambda| \ll |\Omega_k^2 - \omega_k^2| = |M^2 - m^2|$. Perturbation theory must, therefore, not be a good approximation when m and M are comparable, consistent with fig. 2.

Appendix C: Redfield equation coefficients for $\lambda\phi\chi$ theory

In this appendix, we display the coefficients of the Redfield equation for the $\lambda\phi\chi$ interaction described in section V,

$$\gamma_{\rm o}(t) = \frac{i\lambda^2}{2\omega_k\Omega_k\left(\Omega_k^2-\omega_k^2\right)}\left[\omega_k - \Big(\omega_k\cos\left[\Omega_k\left(t-t_0\right)\right] + i\Omega_k\sin\left[\Omega_k\left(t-t_0\right)\right]\Big)e^{-i\omega_k(t-t_0)}\right], \tag{C1}$$

$$S_{\rm o}(t) = -\frac{\lambda^2}{4\omega_k\Omega_k\left(\Omega_k^2-\omega_k^2\right)}\left[\Omega_k - \Big(\Omega_k\cos\left[\Omega_k\left(t-t_0\right)\right] + i\omega_k\sin\left[\Omega_k\left(t-t_0\right)\right]\Big)e^{-i\omega_k(t-t_0)}\right], \tag{C2}$$

$$\gamma_{+}(t) = \frac{\lambda^2}{\omega_k\Omega_k\left(\Omega_k^2-\omega_k^2\right)}\Big(\Omega_k\cos\left[\omega_k\left(t-t_0\right)\right]\sin\left[\Omega_k\left(t-t_0\right)\right] - \omega_k\cos\left[\Omega_k\left(t-t_0\right)\right]\sin\left[\omega_k\left(t-t_0\right)\right]\Big), \tag{C3}$$

$$\gamma_{-}(t) = \frac{\lambda^2}{\omega_k\Omega_k\left(\Omega_k^2-\omega_k^2\right)}\Big(\omega_k\cos\left[\omega_k\left(t-t_0\right)\right]\sin\left[\Omega_k\left(t-t_0\right)\right] - \Omega_k\cos\left[\Omega_k\left(t-t_0\right)\right]\sin\left[\omega_k\left(t-t_0\right)\right]\Big), \tag{C4}$$

$$\omega_{\rm S}(t) = \omega_k - \frac{\lambda^2}{2\omega_k\Omega_k\left(\Omega_k^2-\omega_k^2\right)}\Big(\Omega_k - \Omega_k\cos\left[\omega_k\left(t-t_0\right)\right]\cos\left[\Omega_k\left(t-t_0\right)\right] - \omega_k\sin\left[\Omega_k\left(t-t_0\right)\right]\sin\left[\omega_k\left(t-t_0\right)\right]\Big). \tag{C5}$$

For this interaction, the Markovian limit is only well-defined *before* evaluating the time integral, as in eq. (68), so that the time integral can be written as a sum of Fourier cosine and sine transforms. The resulting coefficients can, however, also be obtained by dropping the oscillating terms in the expressions above.

Appendix D: Redfield equation coefficients for $\lambda\phi\chi^2$ theory

In this appendix, we display the coefficients of the Redfield equation for the $\lambda\phi\chi^2$ interaction described in section VI,

$$\begin{aligned}\gamma_{\rm o}(t) = {} & \frac{\lambda^2}{64\pi^2\omega_k}\Big(\pi + \mathrm{Si}\left[\left(\omega_k-k\right)\left(t-t_0\right)\right] - \mathrm{Si}\left[\left(\omega_k+k\right)\left(t-t_0\right)\right]\Big) \\ & + i\frac{\lambda^2}{64\pi^2\omega_k}\left(\mathrm{Ci}\left[\left(\omega_k-k\right)\left(t-t_0\right)\right] - \mathrm{Ci}\left[\left(\omega_k+k\right)\left(t-t_0\right)\right] + \ln\left[\frac{\omega_k+k}{\omega_k-k}\right]\right),\end{aligned} \tag{D1}$$

$$\begin{aligned}S_{\rm o}(t) = {} & \frac{\lambda^2}{128\pi^2\omega_k}\left(2\gamma - 2\ln\left[\frac{\mu_{\rm r}}{m}\right] - \mathrm{Ci}\left[\left(\omega_k-k\right)\left(t-t_0\right)\right] - \mathrm{Ci}\left[\left(\omega_k+k\right)\left(t-t_0\right)\right]\right) \\ & + i\frac{\lambda^2}{128\pi^2\omega_k}\Big(\mathrm{Si}\left[\left(\omega_k-k\right)\left(t-t_0\right)\right] + \mathrm{Si}\left[\left(\omega_k+k\right)\left(t-t_0\right)\right]\Big),\end{aligned} \tag{D2}$$

$$\gamma_{+}(t) = \frac{\lambda^2}{32\pi^2\omega_k}\Big(\pi + \mathrm{Si}\left[\left(\omega_k-k\right)\left(t-t_0\right)\right] - \mathrm{Si}\left[\left(\omega_k+k\right)\left(t-t_0\right)\right]\Big), \tag{D3}$$

$$\gamma_{-}(t) = \frac{\lambda^2}{32\pi^2\omega_k}\Big(\mathrm{Si}\left[\left(\omega_k-k\right)\left(t-t_0\right)\right] + \mathrm{Si}\left[\left(\omega_k+k\right)\left(t-t_0\right)\right]\Big), \tag{D4}$$

$$\omega_{\rm S}(t) = \omega_k + \frac{\lambda^2}{64\pi^2\omega_k}\left(2\gamma - 2\ln\left[\frac{\mu_{\rm r}}{m}\right] - \mathrm{Ci}\left[\left(\omega_k-k\right)\left(t-t_0\right)\right] - \mathrm{Ci}\left[\left(\omega_k+k\right)\left(t-t_0\right)\right]\right). \tag{D5}$$

where we have used the standard definition for the sine and cosine integrals,

$$\mathrm{Si}(z) = \int_0^z \mathrm{d}t\frac{\sin(t)}{t}\,, \quad \mathrm{Ci}(z) = -\int_z^\infty \mathrm{d}t\frac{\cos(t)}{t}\,, \tag{D6}$$

and the renomalization scale μ_{r} indicates where divergences were removed by choosing the counterterms in eq. (75). The Markovian limit is straightforward to take as each function has a well-defined limit as $t - t_0 \to \infty$.