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On C^m solutions to systems of linear inequalities

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ABSTRACT

Recent work of C. Fefferman and the first author [8] has demonstrated that the linear system of equations

$$\sum_{j=1}^{M} A_{ij}(x)F_j(x) = f_i(x) \qquad (i = 1, \dots, N),$$

has a C^m solution $F = (F_1, \dots, F_M)$ if and only if $f_1, \dots, f_N$ satisfy a certain finite collection of partial differential equations. Here, the A_{ij} are fixed semialgebraic functions.
In this paper, we consider the analogous problem for systems of linear *inequalities*:

$$\sum_{j=1}^{M} A_{ij}(x)F_j(x) \leq f_i(x) \qquad (i = 1, \dots, N).$$

Our main result is a negative one, demonstrated by counterexample: the existence of a C^m solution F may not, in general, be determined via an analogous finite set of partial differential inequalities in $f_1, \dots, f_N$.

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1. Introduction

Fix $m, M, n, N \in \mathbb{N}$. Consider the system of linear equations given by

$$\sum_{j=1}^{M} A_{ij}(x)F_j(x) = f_i(x) \qquad (i = 1, \ldots, N), \tag{1}$$

where the A_{ij} and f_i are given functions on $\mathbb{R}^n$, while $F_1, \ldots, F_M \in C^m(\mathbb{R}^n)$ are unknown functions to be solved for fixed m.[1] Notice that we do not impose any regularity conditions on A_{ij} and f_i; in fact, they may be discontinuous functions, e.g., indicator functions on closed sets. While elementary linear algebra can be used to find the set of solutions $F_1(x), \ldots, F_M(x)$ at any given $x \in \mathbb{R}^n$, analyzing the set of solutions which vary smoothly in x (in particular, lie in C^m) is much more difficult, with most progress coming only recently [6,8–11,13].

We begin with a review of the literature on this subject before turning to the main object: C^m solutions for systems of linear *inequalities* (4).

Regarding (1), the simplest question to be asked is the following:

Problem 1.1 *(Brenner-Epstein-Hochster-Kollár Problem).* Given A_{ij}, f_i as in (1), determine if there exists a C^m solution $F = (F_1, \ldots, F_M)$.

Problem 1.1 was solved by Fefferman and the first author in [7] (see also [6,9]), which motivated a number of related works [2,9,13,14].

Next, one may try to analyze the set of $f = (f_1, \ldots, f_N) \in C^\infty$ for which there exists a C^m solution F. For various reasons, it is helpful to consider particular cases of A_{ij}, namely *semialgebraic functions*: a function $F : \mathbb{R}^n \to \mathbb{R}$ is semialgebraic if its graph can be represented as the solution set to finitely many polynomial equations and/or inequalities. For instance, rational functions and the indicator function on the circle are semialgebraic while exponential functions are not. (See below for more discussion on the choice of semialgebraic functions for this problem.)

Problem 1.2. Given semialgebraic A_{ij} as in (1), characterize the set of $f \in C^\infty(\mathbb{R}^n, \mathbb{R}^N)$ for which there exists a C^m solution F.

To motivate the solution to Problem 1.2, let us review an example of Epstein and Hochster [4]. Consider the single linear equation

$$x^2 F_1 + y^2 F_2 + xyz^2 F_3 = f(x, y, z). \tag{2}$$

[1] $C^m(\mathbb{R}^n)$ denotes the vector space of m-times continuously differentiable functions $\mathbb{R}^n$, with no growth conditions assumed at infinity. Similarly, $C^m(\mathbb{R}^n, \mathbb{R}^D)$ denotes the space of all such $\mathbb{R}^D$-valued functions on $\mathbb{R}^n$.

There exist continuous F_1, F_2, F_3 satisfying (2) if and only if

$$\begin{cases} f(x,y,z) = \frac{\partial f}{\partial x}(x,y,z) = \frac{\partial f}{\partial y}(x,y,z) = 0 & \text{for } x = y = 0, z \in \mathbb{R} \\ \text{and} & \\ \frac{\partial^2 f}{\partial x \partial y}(x,y,z) = \frac{\partial^3 f}{\partial x \partial y \partial z}(x,y,z) = 0 & \text{at } x = y = z = 0. \end{cases} \tag{3}$$

Note that while no differentiability requirements on the F_j are made, derivatives still show up in conditions on the f_i in (3). This example illustrates the general form of the solution to Problem 1.2 as proven by Fefferman and the first author in [8]:

Theorem 1.3. *Fix $m \geq 0$, and let $(A_{ij}(x))_{1 \leq i \leq N, 1 \leq j \leq M}$ be a matrix of semialgebraic functions on $\mathbb{R}^n$. Then there exist linear partial differential operators $L_1, L_2, \ldots, L_{\nu_{\max}}$, for which the following hold.*

- *Each L_ν acts on vectors $f = (f_1, \ldots, f_N) \in C^\infty(\mathbb{R}^n, \mathbb{R}^N)$, and has the form*

$$L_\nu f(x) = \sum_{i=1}^{N} \sum_{|\alpha| \leq \bar{m}} a_{\nu i \alpha}(x) \partial^\alpha f_i(x),$$

  where the coefficients $a_{\nu i \alpha}$ are semialgebraic. (Perhaps $\bar{m} > m$.)
- *Let $f = (f_1, \ldots, f_N) \in C^\infty(\mathbb{R}^n, \mathbb{R}^N)$. Then the system (1) admits a $C^m(\mathbb{R}^n, \mathbb{R}^M)$ solution $F = (F_1, \ldots, F_M)$ if and only if $L_\nu f = 0$ on $\mathbb{R}^n$ for each $\nu = 1, \ldots, \nu_{\max}$.*

We now turn to the case of inequalities:

$$\sum_{j=1}^{M} A_{ij}(x) F_j(x) \leq f_i(x) \quad (i = 1, \ldots, N) \text{ on } \mathbb{R}^n. \tag{4}$$

Problem 1.4. Given A_{ij}, f_i as in (4), determine if there exists a C^m solution $F = (F_1, \ldots, F_M)$.

Problem 1.4 is the analogue of Problem 1.1 for inequalities and was solved recently by Jiang and the authors in [14].

The focus of this paper is the following analogue of Problem 1.2 for inequalities.

Problem 1.5. Fix $m, M, n, N \in \mathbb{N}$ and let $A_{ij} : \mathbb{R}^n \to \mathbb{R}$ be semialgebraic. Characterize the set of $f = (f_1, \ldots, f_N) \in C^\infty(\mathbb{R}^n, \mathbb{R}^N)$ for which there exists a C^m solution $F = (F_1, \ldots, F_M)$ to (4).

It is well-known that, for fixed x, any linear, convex constraints may be put into the form (4). For our phrasing of Problem 1.5, such equivalence does not hold. For instance, in a linear programming problem, the equality constraint $a \cdot x = b$ is equivalent to requiring both inequalities $a \cdot x \leq b$ and $(-1) \cdot x \leq -b$. However, in the context of Problem 1.5, replacing one constraint with two constraints leads to the presence of an additional f_i and a different problem. A much more general version of Problem 1.5 could be stated, but this would be unnecessary for the purposes of providing a counterexample.

The recent solution [14] to Problem 1.1 for (4) by Jiang and the current authors provides a key step to analyze Problem 1.5. Much like the solution to Problem 1.1 in [7], it solved Problem 1.5 in terms of the "Glaeser refinement technique", which is a higher-dimensional generalization of the divided difference [1,5,12]. This work [14] provides a solution to Problem 1.5 *in principle*, but in practice it is difficult to verify the conditions.

To motivate our expected result on a system of inequalities, let us consider an example. For simplicity, we temporarily ignore the previous discussion and consider systems more general than those described by (4). Suppose $f \in C^\infty(\mathbb{R})$ and consider the following inequalities for $x \in \mathbb{R}$,

$$\begin{cases} x^2 \mathbb{I}_{x \geq 0} F \leq f \leq x \mathbb{I}_{x \geq 0} F \\ x \mathbb{I}_{x \leq 0} F \leq f \leq x^2 \mathbb{I}_{x \leq 0} F \end{cases} \tag{5}$$

for unknown continuous F on $\mathbb{R}$. One checks that a continuous solution F exists if and only if f satisfies

$$\left[\begin{array}{l} f(0) = 0, \\ f'(0) \geq 0. \end{array} \right. \tag{6}$$

Note that the derivative of f enters into (6), even though we are merely looking for continuous solutions F.

This simple example helps us formulate a result similar to Theorem 1.3 for a system of inequalities. At its simplest, it says that the existence of a C^m solution may be determined by a finite set of linear partial differential inequalities in the f_i.

Conjecture 1.6. *Fix $m \geq 0$ and let $(A_{ij}(x))_{1 \leq i \leq N, 1 \leq j \leq M}$ be a matrix of semialgebraic functions on $\mathbb{R}^n$. Then, there exist linear partial differential operators*

$$L_{1,1}, \ldots, L_{1,\nu_1}, \ldots, L_{\mu_{\max},1}, \ldots, L_{\mu_{\max},\nu_{\mu_{\max}}}, L'_{1,1}, \ldots, L'_{1,\nu'_1}, \ldots, L'_{\mu_{\max},1}, \ldots, L'_{\mu_{\max},\nu'_{\mu_{\max}}}$$

for which the following hold:

1. *Each $L_{\mu,\nu}$ acts on vectors $f = (f_1, \ldots, f_N) \in C^\infty(\mathbb{R}^n, \mathbb{R}^N)$ and has the form*

$$L_{\mu,\nu} f(x) = \sum_{i=1}^{N} \sum_{|\alpha| \leq \bar{m}} a_{\mu\nu i\alpha}(x) \partial^\alpha f_i(x),$$

or

$$L'_{\mu,\nu} f(x) = \sum_{i=1}^{N} \sum_{|\alpha| \le \bar{m}} a'_{\mu\nu i\alpha}(x) \partial^\alpha f_i(x),$$

where the coefficients $a_{\mu\nu i\alpha}, a'_{\mu\nu i\alpha}$ *are semialgebraic and* $\bar{m} \ge m$.

2. *Let* $f = (f_1, \ldots, f_N) \in C^\infty(\mathbb{R}^n, \mathbb{R}^N)$. *Then the system* (4) *admits a solution* $F = (F_1, \ldots, F_M) \in C^m(\mathbb{R}^n, \mathbb{R}^M)$ *if and only if there exists* $1 \le \mu \le \mu_{\max}$ *such that* $L_{\mu,\nu} f \ge 0$ *on* $\mathbb{R}^n$ *for each* $1 \le \nu \le \nu_\mu$ *and* $L'_{\mu,\nu} f > 0$ *on* $\mathbb{R}^n$ *for each* $1 \le \nu \le \nu'_\mu$.

It may appear natural to simply replace the condition $L_\nu f = 0$ in Theorem 1.3 with $L_\nu f \ge 0$, or perhaps $L_\nu f > 0$, corresponding to the case $\mu_{\max} = 1$ above. However, an attempt to replicate the proof of Theorem 1.3 with inequalities in place of equations leads naturally to a more general condition. Furthermore, one may interpret the conditions in Conjecture 1.6 as a more general formalization of the idea of "determined by a finite set of linear partial differential inequalities."

The main result of this paper is that Conjecture 1.6 is false for $n \ge 2$. A counterexample is given for the case of $C^0(\mathbb{R}^2, \mathbb{R}^2)$. For $n = 1$, the conjecture remains open.

The starting point for the construction of our counterexample is that a semialgebraic function may have an infinite number of directional limits at a single point. As a result, computing the Glaeser refinement at that point amounts to taking the infinite intersection of polytopes, which may not itself be a polytope. (This problem is avoided in the solution to Problem 1.2 found in [8] since the infinite intersection of affine spaces is itself an affine space.) This motivates the design of the counterexample, which is stated fully in Section 3.

It is natural to ask why, if semialgebraic functions can lead to such problems, one does not simply use polynomials in place of semialgebraic functions in (4), in alignment with the versions stated in [3,4]. The reason is that the difference quotients used in Glaeser refinements are semialgebraic functions, and in following the analysis of say, [8], any attempt to begin with polynomial coefficients A_{ij} leads to the use of semialgebraic functions anyway. While our counterexample requires the greater generality of semialgebraic functions (versus polynomials) it shows that in order to prove a version of Conjecture 1.6 for the polynomial case new techniques would have to be developed. Furthermore, it would be reasonable for this case to simply require an analogous, yet more complicated counterexample.

We begin with a review of our main computational tool, Glaeser refinement, and its importance in Section 2. In Section 4, we compute the Glaeser refinement for our example manually and determine explicit criteria for the existence of C^0 solutions. We use this result to demonstrate the nonexistence of linear criteria in Section 5, officially disproving Conjecture 1.6.

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2. Glaeser refinement

As our counterexample is in the case of continuous functions, we provide the following definition of Glaeser refinement for this special case. (See [5] and [14] for more general definitions of Glaeser refinement.)

Definition 2.1. If $(K(x))_{x\in E}$ is a collection of subsets of $\mathbb{R}^d$, we define the C^0-Glaeser refinement of $(K(x))_{x\in E}$, denoted $(\tilde{K}(x))_{x\in E}$ by

$$\tilde{K}(x) = \{z \in K(x) : \forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } y \in B_\delta(x) \Rightarrow \exists z' \in K(y), |z - z'| < \epsilon\}.$$

Theorem 2.1. *There exists $l^* = l^*(n, d)$ such that the following holds.*

Let $E \subset \mathbb{R}^n$ be compact $(K(x))_{x\in E}$ be a collection of closed, convex sets in $\mathbb{R}^d$. Let $(K'(x))_{x\in E}$ be the l^*-th iterated Glaeser refinement of $(K(x))_{x\in E}$. Then, $(K(x))_{x\in E}$ has a section if and only if $K'(x)$ is nonempty for all $x \in E$.

Theorem 2.1 follows somewhat easily from the Michael selection theorem (see [15]); however, to spare the reader this work, we cite it as a mere special case of Theorem 1.5 in [14].

3. The counterexample

Let $E = [0, 1]^2$ and write elements of E as $x = (x_1, x_2)$. For brevity, we will write 0 in place of $(0, 0)$ when usage is clear by context.

Consider the system of equations

$$\frac{x_1^4}{(x_1^2+x_2^2)^2}F_1(x) + \frac{x_2^4}{(x_1^2+x_2^2)^2}F_2(x) \le f_1(x) \tag{7}$$

$$\frac{x_2^4}{(x_1^2+x_2^2)^2}F_1(x) - \frac{x_1^4}{(x_1^2+x_2^2)^2}F_2(x) \le f_2(x) \tag{8}$$

$$-\frac{x_1^4}{(x_1^2+x_2^2)^2}F_1(x) - \frac{x_2^4}{(x_1^2+x_2^2)^2}F_2(x) \le f_3(x) \tag{9}$$

$$-\frac{x_2^4}{(x_1^2+x_2^2)^2}F_1(x) + \frac{x_1^4}{(x_1^2+x_2^2)^2}F_2(x) \le f_4(x) \tag{10}$$

for $(x_1, x_2) \in E \setminus \{0\}$ and

$$0 \le f_1(x), f_2(x), f_4(x), \qquad -10^6 F_1(x) \le f_3(x) \tag{11}$$

for $x = 0$.

We may summarize this system in the form

$$A(x)F(x) \le f(x), \tag{12}$$

where $f = (f_1, f_2, f_3, f_4)$ and $F = (F_1, F_2)$.

For $x \neq 0$, define

$$B(x) = \begin{bmatrix} \frac{x_1^4}{(x_1^2+x_2^2)^2} & \frac{x_2^4}{(x_1^2+x_2^2)^2} \\ \frac{x_2^4}{(x_1^2+x_2^2)^2} & -\frac{x_1^4}{(x_1^2+x_2^2)^2} \end{bmatrix},$$

so that (7), (8), (9), and (10) together may be rewritten in the form

$$\begin{bmatrix} -f_3(x) \\ -f_4(x) \end{bmatrix} \leq B(x)F(x) \leq \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix}. \tag{13}$$

For later use, we note the trivial fact that for all $x \neq 0$,

$$\|B(x_1, x_2)^{-1}\| \leq 4. \tag{14}$$

Define

$$H_0(x) = \{y \in \mathbb{R}^2 : A(x)y \leq f(x)\}$$

and $H_{k+1}(x) = \tilde{H}_k(x)$ for $k \geq 0$, where $(\tilde{H}(x))_{x \in E}$ is the C^0-Glaeser refinement of the bundle $(H(x))_{x \in E}$.

4. Nonlinear criteria for characterization

Lemma 4.1. *Let $x_0 \in E$. If $x_0 = 0$, then $H_0(x_0)$ is nonempty.*

Let $x_0 \in E \setminus \{0\}$. Then $H_0(x_0)$ is nonempty if and only if

$$-f_3(x_0) \leq f_1(x_0), \qquad -f_4(x_0) \leq f_2(x_0). \tag{15}$$

Proof. By definition and (11), $H_0(0) = \{(y_1, y_2) : -10^6 y_1 \leq f_3(0)\}$. This is nonempty, independent of the choice of $f_1, \ldots, f_4$.

Let $x_0 \in E \setminus \{0\}$. Suppose (15) holds and choose $z = (z_1, z_2)$ such that

$$-f_3(x_0) \leq z_1 \leq f_1(x_0), \qquad -f_4(x_0) \leq z_2 \leq f_2(x_0).$$

Then, by (13), $B^{-1}(x_0)z \in H_0(x_0)$, so $H_0(x_0)$ is nonempty. If (15) fails, then clearly there is no solution to (13) with $x = x_0$ and $H_0(x_0)$ is empty. □

Lemma 4.2. *Suppose $H_0(x)$ is nonempty for all $x \in E \setminus \{0\}$. Then, $H_k(x) = H_0(x)$ for all $x \in E \setminus \{0\}$ and $k \geq 0$ and $H_k(0) = H_1(0)$ for ll $k \geq 1$.*

Proof. Fix $x \in E \setminus \{0\}$ and let $y = (y_1, y_2) \in H_0(x)$. Fix $\epsilon > 0$ Then,

$$\begin{bmatrix} -f_3(x) \\ -f_4(x) \end{bmatrix} \leq B(x) \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} := \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \leq \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix}.$$

Choose $\delta > 0$ such that $x' \in E \cap B_\delta(x)$ implies $|f_i(x') - f_i(x)| < \epsilon/10$ for all i and

$$\|B(x')^{-1} - B(x)^{-1}\| < \epsilon/10 \times \min\{1, 1/(\|z\| + 1)\}, \tag{16}$$

where $z = (z_1, z_2)$.

Thus, for such x',

$$-f_3(x') - \epsilon/10 \le z_1 \le f_1(x') + \epsilon/10 \tag{17}$$

and

$$-f_4(x') - \epsilon/10 \le z_2 \le f_2(x') + \epsilon/10. \tag{18}$$

Since $H_0(x')$ is nonempty (by assumption), $-f_3(x') \le f_1(x')$. Furthermore, by (17), there exists

$$-f_3(x') \le z_1' \le f_1(x')$$

satisfying

$$|z_1' - z_1| \le \epsilon/10. \tag{19}$$

Similarly, by (18) there exists $-f_4(x') \le z_2' \le f_2(x')$ satisfying

$$|z_2' - z_2| \le \epsilon/10. \tag{20}$$

Let $z' = (z_1', z_2')$ and $y' = B(x')^{-1}(z') \in H_0(x')$. Thus, by (14), (16), (19), and (20),

$$\begin{aligned}
|y' - y| &= |B(x')^{-1}(z') - B(x)^{-1}(z)| \\
&\le |B(x')^{-1}(z') - B(x')^{-1}(z)| + |B(x')^{-1}(z) - B(x)^{-1}(z)| \\
&= |B(x')^{-1}(z' - z)| + |((B(x')^{-1} - B(x)^{-1})(z)| \\
&\le \|B(x')^{-1}\|(\epsilon/5) + \epsilon/10 \min\{1, 1/(\|z\| + 1)\}\|z\| \\
&\le 4\epsilon/5 + \epsilon/10 < \epsilon.
\end{aligned}$$

We conclude that $y \in H_1(x)$ since $x' \in E \cap B_\delta(x)$ was arbitrary and $y' \in H_0(x')$ was as desired. Thus, $H_1(x) = H_0(x)$ for all $x \in E \setminus \{0\}$ and $H_1(x)$ is nonempty for all $x \in E \setminus \{0\}$. One may prove $H_{k+1}(x) = H_k(x)$ for $x \in E \setminus \{0\}$ similarly, from which one may conclude $H_k(x) = H_0(x)$ for $x \in E \setminus \{0\}$ and $k \ge 0$.

An element $y \in H_k(0)$ lies in $H_{k+1}(0)$ if and only if it satisfies a certain condition depending on $H_k(x)$ for x in an arbitrarily small neighborhood of the origin. By hypothesis, all such $H_k(x)$ are the same, so further applications of Glaeser refinement make no difference. □

Corollary 4.3. *Let l^* be as in Theorem 2.1. Then, $H_{l^*}(x)$ is nonempty for all $x \in E$ if and only if $H_0(x)$ is nonempty for all $x \in E \setminus \{0\}$ and $H_1(0)$ is nonempty.*

By Lemma 4.1, we may use (15) to categorize when $H_0(x)$ is nonempty. We now move to the case of $H_1(0)$.

Let $y = (y_1, y_2) \in H_0(0)$. We would like to determine if $y \in H_1(0)$.

Write $x \in E \setminus \{0\}$ in polar coordinates as $x = r\theta$. Noting that $B(r\theta)$ depends solely on θ, we introduce the notation

$$B(\theta) = \begin{bmatrix} \cos^4\theta & \sin^4\theta \\ \sin^4\theta & -\cos^4\theta \end{bmatrix},$$

so $B(\theta) = B(x)$ for $x = r\theta$.

Thus, $y \in H_0(r\theta)$ if and only if

$$\begin{bmatrix} -f_3(r\theta) \\ -f_4(r\theta) \end{bmatrix} \le B(\theta) \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \le \begin{bmatrix} f_1(r\theta) \\ f_2(r\theta) \end{bmatrix}. \tag{21}$$

Lemma 4.4. *Suppose $H_0(x)$ is nonempty for all $x \in E$. Then,*

$$H_1(0) = \left\{ y \in \mathbb{R}^2 : \begin{bmatrix} -f_3(0) \\ -f_4(0) \end{bmatrix} \le B(\theta) \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \le \begin{bmatrix} f_1(0) \\ f_2(0) \end{bmatrix} \text{ for all } \theta \in [0, \pi/2] \right\} \cap H_0(0).$$

Proof. First suppose $y \in H_0(0)$ and

$$\begin{bmatrix} -f_3(0) \\ -f_4(0) \end{bmatrix} \le B(\theta) \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \le \begin{bmatrix} f_1(0) \\ f_2(0) \end{bmatrix} \text{ for all } \theta \in [0, \pi/2]. \tag{22}$$

Fix $\epsilon > 0$. Choose $\delta > 0$ such that $x \in E \cap B_\delta(0)$ implies

$$|f_i(x) - f_i(0)| < \epsilon/10 \text{ for all } i. \tag{23}$$

Let $\theta_0 \in [0, \pi/2]$ and $0 < r < \delta$ (that is, $x \in E \cap B_\delta(0)$). By (22),

$$\begin{bmatrix} -f_3(0) \\ -f_4(0) \end{bmatrix} \le B(\theta_0) \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} := \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \le \begin{bmatrix} f_1(0) \\ f_2(0) \end{bmatrix}$$

Since $0 < r < \delta$, we use (23) to obtain

$$-f_3(r\theta_0) - \epsilon/10 \le z_1 \le f_1(r\theta_0) + \epsilon/10$$

and

$$-f_4(r\theta_0) - \epsilon/10 \le z_2 \le f_2(r\theta_0) + \epsilon/10.$$

Since $H_0(r\theta_0)$ is nonempty (by assumption), $-f_3(r\theta_0) \leq f_1(r\theta_0)$. Furthermore, by (23), there exists

$$-f_3(r\theta_0) \leq z_1' \leq f_1(r\theta_0)$$

satisfying

$$|z_1' - z_1| \leq \epsilon/10. \tag{24}$$

Similarly, there exists $-f_4(r\theta_0) \leq z_2' \leq f_2(r\theta_0)$ satisfying

$$|z_2' - z_2| \leq \epsilon/10. \tag{25}$$

Let $y' = B(r\theta_0)^{-1}(z_1', z_2') \in H_0(r\theta_0)$. Thus, by (14), (24), and (25),

$$\begin{aligned}|y' - y| &= |B(\theta_0)^{-1}(z_1', z_2') - B(\theta_0)^{-1}(z_1, z_2)| \\ &\leq \|B(\theta_0)^{-1}\| \cdot \|(z_1', z_2') - (z_1, z_2)|| \\ &\leq 4(\epsilon/5) < \epsilon.\end{aligned}$$

Therefore, by definition, $y \in H_1(0)$.

Now suppose that $y \in H_1(0)$. Then, $y \in H_0(0)$ so we need only check (22).

By definition, for all $\epsilon > 0$, there exists $\delta_0 > 0$ such that (21) has a solution $z = (z_1, z_2)$ satisfying $|z - y| < \epsilon$ whenever $0 < r < \delta_0$ and $\theta \in [0, \pi/2]$. Here, $z \in H_0(r\theta)$.

Let $\theta_0 \in [0, \pi/2]$ and $\epsilon > 0$. Choose δ_0 as above and $0 < \delta < \delta_0$ such that $x \in E \cap B_\delta(0)$ implies

$$|f_i(x) - f_i(0)| < \epsilon \text{ for all } i.$$

Let $0 < r < \delta$. As a particular case of (21) and the fact $|y - z| < \epsilon$,

$$\cos^4\theta y_1 + \sin^4\theta y_2 \leq \cos^4\theta z_1 + \sin^4\theta z_2 + 2\epsilon \leq f_1(0) + 3\epsilon.$$

Since $\epsilon > 0$ was arbitrary,

$$\cos^4\theta y_1 + \sin^4\theta y_2 \leq f_1(0).$$

By similar arguments involving f_2, f_3, and f_4 one at a time, we obtain (22). $\square$

Considering the conclusion of Lemma 4.4 and the fact that $H_1(0) \subset H_0(0)$ trivially, the main question at hand is which (y_1, y_2) satisfy (22), that is,

$$\begin{aligned}\cos^4\theta y_1+\sin^4\theta y_2&\le f_1(0)\\ \sin^4\theta y_1-\cos^4\theta y_2&\le f_2(0)\\ -\cos^4\theta y_1-\sin^4\theta y_2&\le f_3(0)\\ -\sin^4\theta y_1+\cos^4\theta y_2&\le f_4(0)\end{aligned}$$

for all $\theta\in[0,\pi/2]$.

First consider the set

$$R_1:=\{y\in\mathbb{R}^2:\cos^4\theta y_1+\sin^4\theta y_2\le f_1(0)\text{ for all }\theta\in[0,\pi/2]\}.$$

Plugging in $\theta=0$ and $\theta=\pi/2$, we have $y_1\le f_1(0)$ and $y_2\le f_1(0)$ as defining constraints for R_1. Since $f_1(0)\ge 0$ by (11) and $\sin^4\theta+\cos^4\theta\le 1$ for all θ, these two inequalities imply all the rest and

$$R_1=\{(y_1,y_2):y_1\le f_1(0),y_2\le f_1(0)\} \tag{26}$$

Similarly,

$$\begin{aligned}R_2:&=\{y\in\mathbb{R}^2:\sin^4\theta y_1-\cos^4\theta y_2\le f_2(0)\text{ for all }\theta\in[0,\pi/2]\} &(27)\\ &=\{(y_1,y_2):y_1\le f_2(0),y_2\ge -f_2(0)\} &(28)\end{aligned}$$

and

$$\begin{aligned}R_4:&=\{y\in\mathbb{R}^2:-\sin^4\theta y_1+\cos^4\theta y_2\le f_4(0)\text{ for all }\theta\in[0,\pi/2]\} &(29)\\ &=\{(y_1,y_2):y_1\ge -f_4(0),y_2\le f_4(0)\} &(30)\end{aligned}$$

The region

$$R_3:=\{y\in\mathbb{R}^2:-\cos^4\theta y_1-\sin^4\theta y_2\le f_3(0)\text{ for all }\theta\in[0,\pi/2]\}$$

will not be described so easily since the value of $f_3(0)$ is allowed to be negative, that is, R_3 is not determined by $-\cos^4\theta y_1-\sin^4\theta y_2\le f_3(0)$ for just two choices of θ.

So suppose here that $f_3(0)<0$.

Substituting $M=-f_3(0)>0$ and $a=\sin^2\theta$, we have

$$\begin{aligned}R_3&=\{y\in\mathbb{R}^2:(1-a)^2y_1+a^2y_2\ge M\text{ for all }a\in[0,1]\}\\ &=\{y\in\mathbb{R}^2:y_2\ge\frac{M-(1-a)^2y_1}{a^2}\text{ for all }a\in(0,1],y_1\ge M\}\\ &=\bigcup_{y_1>M}\{(y_1,y_2):y_2\ge\frac{M-(1-a)^2y_1}{a^2}\text{ for all }a\in(0,1]\}.\end{aligned}$$

Given y_1, we find the largest value of $\frac{M-(1-a)^2 y_1}{a^2}$ ranging over all $a \in (0,1]$; call this value $W(y_1)$. It follows from the above that

$$\begin{aligned} R_3 &= \bigcup_{y_1 > M} \{(y_1, y_2) : y_2 \geq W(y_1)\} \\ &= \{y \in R^2 : y_1 > M, y_2 \geq W(y_1)\}. \end{aligned}$$

To compute $W(y_1)$, we define

$$V(y_1, a) = \frac{M - (1-a)^2 y_1}{a^2} = \frac{M}{a^2} - \frac{y_1}{a^2} + \frac{2y_1}{a} - y_1$$

and find its maximum in a. By elementary calculus,

$$\frac{\partial V}{\partial a} = \frac{-2M}{a^3} + \frac{2y_1}{a^3} - \frac{2y_1}{a^2}$$

and

$$\frac{\partial^2 V}{(\partial a)^2} = \frac{6M}{a^4} - \frac{6y_1}{a^4} + \frac{4y_1}{a^3}.$$

Solving $\frac{\partial V}{\partial a} = 0$ for a gives $a = 1 - \frac{M}{y_1}$. By simple computation, $\frac{\partial^2 V}{(\partial a)^2} < 0$ at $a = 1 - \frac{M}{y_1}$ so this is indeed a local maximum. As the only critical point, it is the global maximum. (In order for $a = 0$ or $a = 1$ to compete, there would need to be a local minimum between $a = 1 - \frac{M}{y_1}$ and $a = 0$ or $a = 1$, but we have already found all the critical points.)

We find

$$\begin{aligned} W(y_1) &= V\left(y_1, 1 - \frac{M}{y_1}\right) \\ &= \frac{M - (1 - 1 - \frac{M}{y_1})^2 y_1}{(1 - \frac{M}{y_1})^2} \\ &= \frac{M y_1}{y_1 - M} = M + \frac{M^2}{y_1 - M}. \end{aligned}$$

We conclude that

$$R_3 = \{y \in R^2 : y_1 > M, y_2 \geq M + \frac{M^2}{y_1 - M}\}. \tag{31}$$

In words, R_3 is the region contained in the upper-right quadrant of the plane with boundary given by the upper-right component of a hyperbola with asymptotes $y_1 = M$ and $y_2 = M$.

Thus, we have established the following:

Lemma 4.5. *Suppose $H_0(x)$ is nonempty for all $x \in E \setminus \{0\}$ and $f_3(0) < 0$. Then,*

$$H_1(0) = \{y : y_1 \geq -10^{-6} f_3(0)\} \cap (R_1 \cap R_3) \cap (R_2 \cap R_4), \tag{32}$$

where R_1, R_2, R_3, R_4 are explicitly described in (26), (28), (31), and (30), respectively.

Putting together Lemmas 4.1 and 4.5, Corollary 4.3, and Theorem 2.1 we have:

Proposition 4.6. *Let $f_1, f_2, f_3, f_4 \in C^\infty(\mathbb{R}^2, \mathbb{R})$ such that $f_3(0) < 0$. Then, (12) has a C^0 solution if and only if*

$$-f_3(x) \leq f_1(x) \text{ and } -f_4(x) \leq f_2(x) \text{ for all } x \in E \setminus \{0\}; \tag{33}$$

and $H_1(0)$, as specified in (32), is nonempty.

We now restrict to the case where all the f_i are constant. Define

$$K = \{f \in C^\infty(\mathbb{R}^2; \mathbb{R}^4) : f_1, f_2, f_3, f_4 \text{ are constant}\}.$$

Furthermore, define

$$K_0 = \{f \in K : (12) \text{ has a solution}, f_3 \leq -.1\}.$$

By Proposition 4.6,

$$\begin{aligned} K_0 = \{f \in K : f_1, f_2, f_4 \geq 0, f_3 \leq -0.1, f_1 + f_3 \geq 0, f_2 + f_4 \geq 0\} \\ \cap \{f \in K : \{y : y_1 \geq -10^{-6} f_3\} \cap (R_1 \cap R_3) \cap (R_2 \cap R_4)\} \neq \emptyset \end{aligned} \tag{34}$$

The above can be made sense of through the fact that the R_i depend on f in their definitions.

Viewing K as a four-dimensional Hilbert space, we claim K_0 is not a polytope. To see this, restrict further to the affine subspace where $f_3(0) = -1$ and $f_1(0) = 2$. Thus,

$$R_1 = \{(y_1, y_2) : y_1 \leq 2, y_2 \leq 2\}$$

and

$$R_3 = \{y \in \mathbb{R}^2 : y_1 > 1, y_2 \geq 1 + \frac{1}{y_1 - 1}\}.$$

One may readily see that $\{y : y_1 \geq -10^{-6} f_3(0)\} = \{y : y_1 \geq 10^{-6}\}$ contains $R_1 \cap R_3$ so this restriction is superfluous and we need only consider whether $\cap_i R_i$ is nonempty.

Since

$$R_2 \cap R_4 = \{(y_1, y_2) : -f_4(0) \leq y_1 \leq f_2(0), -f_2(0) \leq y_2 \leq f_4(0)\}$$

and $f_2(0), f_4(0) \geq 0$, the question becomes whether the upper right corner $(f_2(0), f_4(0))$ meets $R_1 \cap R_3$. In the range of $1 \leq f_2(0), f_4(0) \leq 2$, this is a nonlinear problem since $R_1 \cap R_3$ has a curved boundary given by $y_2 \geq 1 + \frac{1}{y_1 - 1}$. Thus, K_0 may not be defined by a finite number of linear inequalities.

Lemma 4.7. *The set $\tilde{K}$ of $f \in K$ such that (12) has a C^0 solution may not be defined by finitely many linear inequalities.*

Proof. Suppose $\tilde{K}$ may be defined by finitely many linear inequalities. Then $\tilde{K} \cap \{f : f_3 \leq -0.1\}$ may be defined by finitely many linear inequalities. However, $K_0 = \tilde{K} \cap \{f : f_3 \leq -0.1\}$, contradicting our above reasoning. □

5. Disproof of conjecture

So far, we have found nonlinear criteria on f for the existence of a C^0 solution F to the system (12). However, this does not automatically show that there do not exist linear criteria.

Suppose, for the sake of contradiction, that Conjecture 1.6 holds. That is, there exist linear partial differential operators

$$L_{1,1}, \ldots, L_{1,\nu_1}, \ldots, L_{\mu_{\max},1}, \ldots, L_{\mu_{\max},\nu_{\mu_{\max}}}, L'_{1,1}, \ldots, L'_{1,\nu'_1}, \ldots, L'_{\mu_{\max},1}, \ldots, L'_{\mu_{\max},\nu'_{\mu_{\max}}}$$

for which the following hold:

1. Each $L_{\mu,\nu}$ acts on vectors $f = (f_1, \ldots, f_N) \in C^\infty(\mathbb{R}^n, \mathbb{R}^N)$ and has the form

$$L_{\mu,\nu} f(x) = \sum_{i=1}^{N} \sum_{|\alpha| \leq \bar{m}} a_{\mu\nu i\alpha}(x) \partial^\alpha f_i(x),$$

   or

$$L'_{\mu,\nu} f(x) = \sum_{i=1}^{N} \sum_{|\alpha| \leq \bar{m}} a'_{\mu\nu i\alpha}(x) \partial^\alpha f_i(x),$$

   where the coefficients $a_{\mu\nu i\alpha}, a'_{\mu\nu i\alpha}$ are semialgebraic and $\bar{m} \geq m$.
2. Let $f = (f_1, \ldots, f_N) \in C^\infty(\mathbb{R}^n, \mathbb{R}^N)$. Then the system (4) admits a solution $F = (F_1, \ldots, F_M) \in C^m(\mathbb{R}^n, \mathbb{R}^M)$ if and only if there exists $1 \leq \mu \leq \mu_{\max}$ such that $L_{\mu,\nu} f \geq 0$ on $\mathbb{R}^n$ for each $1 \leq \nu \leq \nu_\mu$ and $L'_{\mu,\nu} f > 0$ on $\mathbb{R}^n$ for each $1 \leq \nu \leq \nu'_\mu$.

We refer to the above as the "Supposed Criteria."

For $f = (f_1, \ldots, f_4) \in K$, there is a solution to our system if and only if there exists $1 \le \mu \le \mu_{\max}$ such that

$$\sum_{i=1}^{N} a_{\mu\nu i}(x) f_i \ge 0 \tag{35}$$

for all $x \in E$ and $1 \le \nu \le \nu_\mu$ and

$$\sum_{i=1}^{N} a'_{\mu\nu i}(x) f_i > 0 \tag{36}$$

for all $x \in E$ and $1 \le \nu \le \nu'_\mu$, where $a_{\mu\nu i} = a_{\mu\nu i0}$ and $a'_{\mu\nu i} = a'_{\mu\nu i0}$.

For $x \in E$, let

$$R_x = \{f \in K : \exists 1 \le \mu \le \mu_{\max} \text{ such that (35) and (36) hold}\}.$$

By definition,

$$\tilde{K} = K \cap (\cap_{x \in E} R_x)). \tag{37}$$

The immediate concern here is that while each R_x may be defined by a finite number of linear inequalities, the infinite intersection found in (37) may give rise to a set which may not be defined by a finite number of linear inequalities. However, the following lemma demonstrates some redundancy in the inequalities defined in (35) and (36).

Lemma 5.1. *For $x \in E \setminus \{0\}$,*

$$R_x \supset \{f \in K : f_1 + f_3 \ge 0, f_2 + f_4 \ge 0\}.$$

In other words, the Supposed Criteria applied away from the origin may be no stricter on the set of constant functions than Proposition 4.6.

Proof. Let $x \in E \setminus \{0\}$ and suppose for the sake of contradiction that there exists $f \in K$ satisfying $f_1 + f_3 \ge 0, f_2 + f_4 \ge 0$ yet not lying in R_x.

Choose $0 < r < |x|$ and $\theta \in C_0^\infty(\mathbb{R}^2; \mathbb{R})$ such that $\theta \ge 0$, $\theta \equiv 1$ on $B_{r/2}(x)$, and the support of θ is contained in $B_r(x)$. The zero function ($f_1 \equiv \ldots \equiv f_4 \equiv 0$) is trivially a solution to (12). By Proposition 4.6, $\theta(f_1, f_2, f_3, f_4)$ is a C^0 solution to (12), as multiplication by nonnegative scalars preserves (33) and the computation of $H_1(0)$ is the same as for the zero function due to the truncated support of θ.

However, $\theta(f_1, f_2, f_3, f_4)$ does not satisfy the Supposed Criteria, at least at the point x. This is a contradiction. □

Corollary 5.2.

$$\tilde{K} = K \cap \{f \in K : f_1 + f_3 \geq 0, f_2 + f_4 \geq 0\} \cap R_0.$$

Proof. By the Proven Criteria,

$$\tilde{K} \subset \{f \in K : f_1 + f_3 \geq 0, f_2 + f_4 \geq 0\},$$

so by (37),

$$\tilde{K} = K \cap \{f \in K : f_1 + f_3 \geq 0, f_2 + f_4 \geq 0\} \cap (\cap_{x \in E} R_x)).$$

Thus, by Lemma 5.1,

$$\tilde{K} = K \cap \{f \in K : f_1 + f_3 \geq 0, f_2 + f_4 \geq 0\} \cap R_0. \quad \square$$

By Corollary 5.2, $\tilde{K}$ may be defined via finitely many linear inequalities. However, by Lemma 4.7 this is a contradiction. Therefore, we must reject the assumption that the Supposed Criteria exist and conclude that the set of C^0 solutions to our system of equations may not be characterized by a finite set of partial differential inequalities. This concludes the proof of our counterexample to Conjecture 1.6.

The extension to the case $n > 2$ is trivial as one may consider $\tilde{f}_i(x_1, \ldots, x_n) = f_i(x_1, x_2)$ in place of $f_i(x_1, x_2)$ in the example and repeat the above analysis. Any C^0 solution $F(x_1, x_2)$ from the fully analyzed case extends naturally to a solution C^0 solution $\tilde{F}(x_1, \ldots, x_n) = F(x_1, x_2)$. Similarly, any C^0 solution to the $n > 2$ case of the form $\tilde{F}(x_1, \ldots, x_n)$ automatically restricts to a C^0 solution $F(x_1, x_2) = \tilde{F}(x_1, x_2, 0, \ldots, 0)$.

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