How is Reasoning with Quantities Limited in Mathematical Modelling?

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Abstract. One reason mathematical modelling remains highly challenging for students is because it requires knowledge about both mathematics and the real-world. Recent work suggests promoting the learning of mathematical modelling as conceiving quantities and establishing relationships among quantities could help students overcome the challenges they experience. While promising, this approach may be oversimplistic in its claims. Through analyzing data collected via a teaching experiment methodology, we present one student's (Szeth's) work on two tasks to illustrate how Szeth's reasoning with quantities was limited during his model construction process in the following ways: Szeth (i) used already constructed mathematical expressions to reason about how quantities vary, and (ii) did not construct a mathematically correct expression despite having reasoned with quantities.

1 Introduction

Scholars have advocated for the importance of including mathematical modelling (hereafter, modelling) into the mathematics curriculum because it motivates the use of mathematics in the world outside of the classroom (e.g., Blum & Niss, 1991; Zbiek & Connor, 2006). However, researchers in the field collectively agree that modelling is challenging for students (e.g., Stillman et al., 2010; Jankvist & Niss, 2020). Therefore, researchers have focused on finding ways to reduce, mitigate, or overcome challenges faced by students while they engage in modelling. For example, scholars have investigated the ways of improving the learning of modelling through supporting *mathematical modelling competencies* directly and developing the appropriate learning enviroments, through designing appropriate tasks and task sequences to deliver that support (e.g., Anhalt et al., 2018; Durandt & Lautenbach, 2020). Despite progress in these areas, students' modelling skills remains difficult to cultivate (Cevikbas, Kaiser, & Schukajlow, 2022). As a potential solution, Cevikbas et al. (2022) call for new theoretical work on the conceptualization of modeling competencies.

Some studies have addressed this call by operationalizing modelling by using theories from quantitative reasoning (Thompson, 2011). Through characterizing students' mathematical models through quantities and quantitative relationships, Larsen (2013) made the case that quantitative reasoning is a central mechanism in model development because *products* of one stage at model development become the *objects* at the next stage. Czocher & Hardison (2021) underwent methodological work to propose the indicators of students' conceiving quantities and developed the notion of a *modelling space* as the set of mathematical relationships on conceived quantities. Other scholars have have investgated how students learn mathematical concepts through engaging in quantitative reasoning while modeling real-world contexts (e.g., Ellis, 2007). The collective work that marries quantititaive reasoning and modelling points towards the idea that *reasoning with quantities* affords students' model construction process, supporting Thompson's (2011) statement that "modelling is simply mathematics in the context of quantitative reasoning" (p. 52). As the field moves towards developing instructional materials that draw on theories from quantitative reasoning, an understanding of the ways in which *reasoning with quantities* is limited during students' modelling activities is needed to give a satisfactory portrayal of both the strengths and the limitations of this approach. In this chapter, we address the question: *In what ways is students' reasoning with quantities limited during their model construction process?*

2 Theoretical Perspective

Quantitative reasoning refers to the mental operations involved in conceiving a situation entailing quantities and relationships among quantities (Thompson, 1990). Quantities are conceptual entities that exist in the mind of an individual. They consist of three interdependent components: an object, a measurable attribute, and a quantification. *Quantification* involves conceiving a measurable attribute of an object and a unit of measure and forming a proportional relationship between the attribute's measure and the unit of measure (Thompson, 2011). *Quantitative operation* "is the conception of two quantities being taken to produce a new quantity" (p.10). As a result of a quantitative operation a *quantitative relationship* is created: the quantities operated upon along with the quantitative operation are in relation to the result of operating (Thompson, 1994, p.14). Examples of quantitative operations include *combining two quantities additively* and *comparing two quantities additively*. For example, the amount by which the mass of cancerous cells grew during an hour is a quantity that may be constructed by *additively comparing* the masses of the cancerous cells at the beginning and end of that hour. Reasoning about quantities may also entail reasoning about how quantities' values vary or not, in relation to each other, termed *co-variational reasoning* (Thompson & Carlson, 2017). Examples of mental operations involved in covariational reasoning include *gross coordination of values* and *coordination of values*.

We define *operation on quantities* to include Thompson's quantitative operations, mental operations involved in (co)variational reasoning, and other operations on quantities that yield a new quantity (Kularajan & Czocher, 2022). We define *relationship among quantities* to include all mental relationships that were constructed as a result of operating on quantities. We define *reasoning with quantities* as the mental operations involved in conceiving a situation consisting of measurable attributes and relations among those measurable attributes AND the mental operations involved in reasoning about varying quantities.

3 Methods

3.1 Data Collection

Data for the study was collected via 3 individual 10-hour teaching experiments (Steffe & Thompson, 2000) conducted with undergraduate STEM majors. The overall goal of the teaching experiment was to examine how modelers construct quantitative relationships and generalize (or not) the quantitative relationships to novel contexts. Students worked on 8-11 modelling tasks from a variety of real-world contexts (e.g., bank account, predator-prey, disease transmission). The tasks were sequenced to encourage the students to use the *models of* past situations as *models for* present situations (Gravemeijer, 1999). The interviewer's questioning during the teaching experiment often took the form of asking for clarification, probing, asking for explanation, suggesting alternative situations in which modelers' particular lines of reasoning may or may not work, and providing modifications to the task that may (or not) have perturbed the modeler's thinking. To simplify presentation of results, we illustrate our findings using one student's (Szeth's) work on two modelling tasks, described below. Szeth double majored in mathematics and physics at a public university in the USA. The cancerous mass task presented below is abridged; the full version included a table containing measurements of the mass at each hour during a 24-hour period.

The Cancerous Mass Task (abridged version)
Cancer cells can be grown in a lab, for study in their own right and also as a basis for further medical research. The HeLa cell line has a seemingly unique ability to continuously grow and divide in the laboratory. Samples of HeLa are measured as a mass and the 24-hour propagation rate is anticipated to be 69% of its current mass. Create an expression that would model how quickly the sample is growing.

The Disease Transmission Task
Suppose a disease is spread by contact between sick and well members of the community. If members of the community move about freely among each other, develop a mathematical model that informs us about the dynamics of how the disease would spread through the population.

3.2 Data Analysis

Consistent with the teaching experiment methodology (Steffe & Thompson, 2000), the entire data corpus was analyzed in two phases. In the first phase, we produced narrative accounts of each modeler's modelling activities including *what* the modeler did, *how* they accomplished it, and our accounts—informed by the modelers' explanations—of *why* they did so. Next, we refined our accounts of the modelers' mathematics by paying explicit attention to the modelers' quantification (Czocher & Hardison, 2021), quantitative operations (Thompson, 1990) and (co)variational reasoning (Thompson & Carlson, 2017). Finally, we sought connections and distinctions across modelers' mental operations through comparing their mental operations over the course of the teaching experiment study (within modelers, across tasks). To do this, we paid attention to their consistencies or inconsistencies in reasoning, their conceptual development, the cognitive obstacles they experienced, and how they overcame these obstacles. In the second phase, we asked *how reasoning with quantities manifested during modelers' construction of models for real-world scenarios?* To answer this question, we attended to how the mental operations involved in reasoning with quantities presented themselves in modelers' modeling activities. Examples of these modelling activities include validating models, constructing expressions or graphs, and constructing measurable attributes of objects. We inferred the goals the modelers set, the mathematical concepts they used, and traced the evolution of their mathematical models. We triangulated our narrative accounts with the videos, generating a list of ways modelers reasoned with quantities during the tasks. Finally, we sorted instances where *reasoning with quantities* was limited in modelers' model construction process. We operationalized modelers' limited *reasoning with quantities* to mean any constraints that may be present in their reasoning with quantities or also to mean only so much of reasoning with quantities was present in modelers' modelling activities. In the next section, we share descriptions of two such instances.

4 Results

In our data, we found two ways in which students' reasoning with quantities was limited during their model construction process. In particular we found instances where Szeth (i) leveraged mathematical expressions to reason about how quantities vary with each other, and (ii) did not construct a mathematically correct expression compatible with the situation. In the first case, we show how Szeth's reasoning with quantities was *limited to* the mathematical expression he constructed; in the second case we show how Szeth's reasoning with quantities was *limiting* for producing an expression that is mathematically correct. We illustrate these instances below.

4.1 Leveraging Mathematical Expressions to Reason about how Quantities Vary with Each Other

In the first half of the *The Cancerous Mass Task,* Szeth was prompted to explore how the cancerous cells were growing during the 24-hour period. In particular, he was asked to evaluate the percent change in mass during a 3-hour, 2-hour, and 1-hour period. Szeth constructed expressions 1 and 2 for the rate at which the mass was changing with respect to time.

$$m' = 0.0293 \cdot m \tag{1}$$

$$m' = 0.703 \cdot m \tag{2}$$

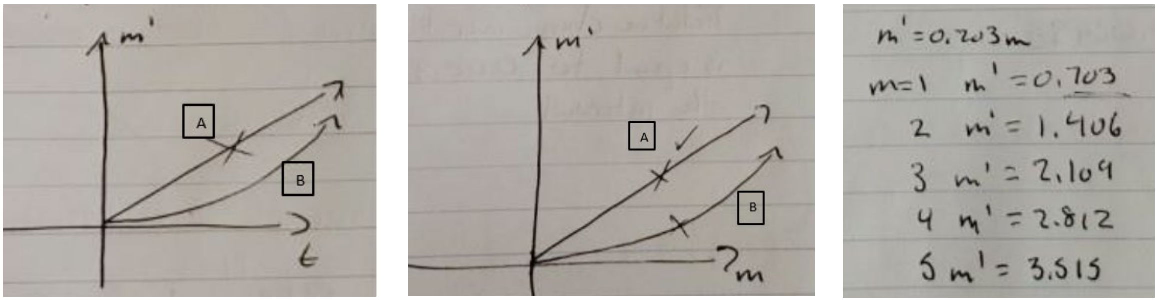
In expressions 1 and 2, Szeth defined m' as the rate at which the mass of the cancerous cells was changing, 0.0293 as the hourly percent change in mass, 0.703 as the daily percent change in mass, and m as the current mass. Szeth solved expression 1 and arrived at expression 3.

$$m = e^{0.0293 \cdot t} \tag{3}$$

We asked Szeth to produce graphs to explain how (i) the rate at which mass changes with respect to time varies with time, and (ii) the rate at which the mass changes with respect to time varies with the current mass. Szeth gave the explanation in the excerpt below.

Szeth: So it looks like m' should grow over time (t), because over time the mass of the sample will be changing. And I guess this number [*referring to 0.703 in* *expression 1]* wouldn't really change our slope here. So, it should be fairly linear line [*draws a straight line for* m' *vs* t *(See Fig. 1.1(a), A*)]. So then mass versus rate. Mass is increasing; the rate should also increase. It's increasing constantly [*referring to* m'], the mass is not a constant increase. I want to say it's linear, too [*draws a straight-line graph for* m'*vs* m *Fig. 1.1(b), A*]. Yeah, because at each interval of mass [*gesturing over Fig 1.1(a), A]* it will correspond to a proportionate rate [*pointing to 0.0293 in expression 1*].

Szeth drew both the graphs in Fig. 1.1, by referring to expressions 1 and 3. While Szeth was drawing Fig. 1.1 (a), when he said that "m' should grow over time, because over time the mass of the sample will be changing," Szeth was *grossly coordinating* the values of m' and t. This *gross coordination of values* was afforded through expressions 1 and 3. Using expression 1, Szeth *grossly coordinated the values* of m' and m. Using expression 3, Szeth *grossly coordinated the values* of m and t. However, he referred to 0.703 as a constant slope, and translated the *gross coordination of values* of m' and t to an increasing straight line as shown in Fig. 1.1(a), A. At the same time, Szeth used expression 1 to produce a linear graph for m' varying with m, as shown in Fig. 1.1(b), A.



(a) (b) (c)

Fig. 1.1(**a**) Szeth's Graph for m'vs t, (**b**) Szeth's Graph for m'vs m, and (**c**) Szeth Coordinates Numerical Values for m and m'

Szeth gave further evidence he used expression 1 to *grossly coordinate* the values of m' and t when the interviewer asked him to explain the graph in Fig. 1.1(a) for m' vs t. He replied:

Szeth: I was thinking of arbitrary points along the m' growth, and then what a corresponding rate should be. And then I thought of our... the equation up here [*referring to expression 1*]. And so, I decided that it should grow linearly like this because of this equation [*pointing to expression 1*] – It was a similar thought process [the production of his graph for m' vs t], using this equation [*pointing to expression 1*]. But rather than considering a mass to a rate, I thought of, how does the mass change over time [*gesturing over expression 3*] and that affect the rate? And so, I came to the conclusion that this number, 0.703, wouldn't be changing over time. That's a constant value. But the m would be. So overall your rate would be increasing.

To produce a graph for how m' changes with t, Szeth first *coordinated the values* of m and m'. However, Szeth used expression 1, as an aid to decide how the values of m and m' change in relation to each other. As a result, Szeth produced the graph in Fig. 1.1(b), A. Next

Szeth engaged in the *gross coordination of values* of m and t, using expression 3. Finally, Szeth engaged in the *nested coordination* of m and t, and m' and m to *grossly coordinate* m' and t. This was evident when he said, "I thought of how does the mass change over time and that affect the rate." Szeth decided that since 0.703 would not be changing and m would be increasing with time, then m' would also be increasing with time. He translated this increase in m' with time to a linear increase. For both the graphs in Fig. 1.1, Szeth leveraged expressions 1 and 3 to establish how m' varies with m and t.

We asked Szeth to explain how expression 1 informed him that m' would increase linearly with time. Szeth indicated that he overlooked that m is increasing exponentially with time. Szeth substituted for m in expression 1 as below.

$$m' = 0.703 \cdot e^{0.0293t} \tag{4}$$

Based on expression 4, he explained: since m is growing exponentially with time, then m'also should grow exponentially with time. Subsequently, Szeth changed his m' vs t graph to be an exponential curve (Fig. 1.1(a), B). However, he then decided that the graph for m' vs m should also be an exponentially increasing curve (Fig. 1.1(b), B). When we asked him why he changed his m' vs m graph, Szeth explained "so now that I'm thinking of it [m] as exponentially growing, the rate [m'] should also be growing exponentially with the mass." Szeth deduced that m' grows exponentially with m because the m grows exponentially with t. To validate his graph for m' vs m, (Fig. 1.1, B), Szeth *coordinated the values* of m' and m, once again using expression 1. His work is shown in Fig. 1.1(c). After computing the values for m' as shown in Fig. 1.1(c), Szeth said that the m' vs m graph should be linear. Szeth reasoned that "Yeah, because it's growing by that much [*pointing at 0.703*] each interval" it should be a linearly increasing graph. He crossed off his exponential curve in Fig. 1.1(b), B and marked a check sign against his linear graph in Fig. 1.1(b), A.

This vignette shows how Szeth leveraged the mathematical expressions he had already constructed to support reasoning with quantities for both constructing and validating a graph. He used expression 1 to construct a graph representing the covariation of the rate of change of mass with respect to time and time. He used expression 2 to validate a graph representing how rate of change of mass with respect to time varies with mass. Here, Szeth's reasoning with quantities was limited to the expressions he constructed.

4.2 Reasoning with Quantities Does Not Always Lead to the Construction of Mathematically Correct Expressions

Prior to working on *The Disease Transmission Task*, Szeth had worked on *The Cats and Birds Task* (predator-prey context), where he was introduced to the concept that the number of interactions between two species can be constructed through multiplicatively combining the amount of each specie. In *The Disease Transmission Task*, Szeth constructed expression 5 for the rate at which the disease spreads, assuming that not all healthy persons, $H(t)$, who come in contact with a sick person, $S(t)$, fall sick.

$$S'(t) = \alpha \cdot S(t) \cdot H(t) \tag{5}$$

Szeth defined α as the "percentage of interactions that lead to people getting sick." We asked Szeth to construct an expression for $H'(t)$. Szeth nominalized $H'(t)$ as "how quickly people would get healthy or recover from being sick." He explained that for sick people to become healthy "they just need to take time." Szeth mathematized this reasoning as below.

$$H'(t) = S(t) \cdot r \quad (6)$$

He defined r as the number of days it would take to "rest and recover." According to expression 6, for Szeth, the change in healthy people is equal to the number of sick people who "rest and recover" from the disease. The following conversation was exchanged:

Interviewer:	Okay, so what if I do get sick though and I'm not recovered?
Szeth:	Oh, okay. It's like you just never recover, I guess?
Interviewer:	Or maybe not never recover but I'm just... for right now, I'm a sick person.
Szeth:	Okay.
Interviewer:	How would that be in your model? Or should it be in your model?
Szeth:	Right. That's what I'm debating in my head because in a way it sounds like the same as if you're just healthy, you're not contributing to an increase or decrease. So I'm wondering if you're just sick, it's the same feel or if you're just sick, would that maybe then somehow lead to a decrease in like how quickly people get healthy again? So, in which case I'm thinking what if you just like subtract $S(t)$ maybe. That seems kind of weird to do because saying this is just sick people just like, "This is just sick people," but then these people also then recovering, right? I guess these don't [*referring to* $S(t)$*]* which is what we're saying.

The intent behind the interviewer querying how Szeth would account for a person who does get sick but does not recover was to encourage Szeth to consider how the number of sick people impacts the change in healthy people. Szeth said that he would subtract the sick people from $S(t) \cdot r$ to indicate the decrease in the amount of healthy people, writing expression 7. Szeth was *grossly coordinating* the values of $S(t)$ and $H'(t)$.

$$H'(t) = S(t) \cdot r \ - S(t) \quad (7)$$

In expression 7, when the interviewer asked him if he was taking away the number of sick people from the number of healthy people, Szeth said that he viewed $S(t) \cdot r$ as "the number of people who would be recovered over time" and not the people who already recovered.

Szeth accounted for the rate of change of healthy people with respect to time by considering how the number of healthy people (an amount) would increase in amount by a sick person taking time to recover from the disease and become healthy (or decrease in amount by a sick person never recovering from the disease). Szeth was mathematizing his quantitative reasonings using arithmetic operations that are normatively used to indicate an increase and decrease, even though the result of those operations produced a quantity different from what was asked from Szeth. Szeth's approach was to replace each quantitative relation with an arithmetic operation. In this approach, he did not attend to the quantitative meaning of the outcome of the arithmetic operations. Although Szeth showed evidence of conceiving quantities and reasoning about quantities, he produced a mathematical expression for $H'(t)$ that is mathematically incorrect.

5 Discussion

Previous research on quantitative reasoning posits that students' construction of robust quantitative relationships can be supportive of creating mathematical representations (such as formulas and graphs) that are mathematically correct and compatible with the real-world situation (e.g., Moore & Carlson, 2012). At the same time, our analysis found that students' quantitative reasoning may be limited during their model construction process in two ways. First, students may use already constructed mathematical expressions to reason about how quantities vary. To illustrate this, we showed how Szeth used expressions 1 and 3 to construct graphs for how m' varies with m and t, as in section 4.1. In this case, Szeth's reasoning with quantities was *limited to* the mathematical expressions he had already constructed. This finding demonstrates that modelers' reasoning with quantities can be governed by the mathematical expressions they construct within a given scenario. Second, reasoning with quantities does not always lead to the construction of mathematically correct expressions. To illustrate this, we showed how Szeth did not produce a normatively correct mathematical expression for the rate of change of healthy people with respect to time, despite having reasoned how the number of sick people, and the number of people who recover from the disease, impact the number of healthy people. In the second case, Szeth's reasoning with quantities was *limiting* for producing an expression that is mathematically correct. This finding demonstrates that productive quantitative reasoning could occur without resulting in effective modelling; simply replacing the mental operations involved in reasoning with quantities with arithmetic operations is not sufficient for producing a valid mathematical model. The fact that Szeth's mental operations for reasoning with quantities does not coincide with the mathematical expressions he produced raises questions about how to guide students towards successful mathematization in the event that the students' constructed mathematical expressions, through reasoning with quantities, diverge from the desired learning outcome we have for the students.

While one goal of incorporating modelling in mathematics classrooms is for students to obtain normatively correct expressions, students' learning is optimized when they construct mathematical expressions that align with their reasoning of and about quantities. Extensive research has supported the stance that incorporating *reasoning with quantities* into learning environments to teach mathematical concepts bears productive learning outcomes. While we agree with and stand by this stance, through this chapter we document borderline instances that may not contribute to this narrative in hopes to inform the field about the intricacies that need to be taken into consideration for curricula development. In order to move forward with the theoretical stance of promoting the learning of mathematical modelling as conceiving quantities and establishing relationships among quantities, future research should investigate the characteristics of the modelling tasks and nature of the scaffolding moves, that enable the learning of modelling in this way, and modelers' response to such interventions.

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