Reasoning students employed when mathematizing during a predator-prey modelling task.

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Abstract. In this chapter, we address the problem of why blockages occur during mathematization by introducing a method for studying *mathematizing* based in quantitative reasoning. We report on interview data with six tertiary STEM majors as they developed models of the population dynamics of cats and birds in a backyard habitat. Our analysis focused on real-world relationships participants tried to express when using a given arithmetic operation in a predator-prey modelling task. Our results reveal the conceptions of $\times$ participants used to justify their models when constructing an expression for the decrease in the bird population. We conclude by discussing the method's utility for studying mathematization and with conjectures on how instructors might leverage participants' justifications to scaffold their emergent models towards a conventionally correct model.

1 Introduction

Mathematizing (the process of transforming a real-world situation into a mathematical representation) receives a lot of attention in modelling literature because it is difficult for students (Brahmia, 2014; Galbraith & Stillman, 2006; Jankvist & Niss, 2020; Stillman & Brown, 2014). Typically, when studies note students' difficulty with mathematization, they describe the difficulty in terms of "blockages" that occur (Galbraith & Stillman, 2006; Klock & Siller, 2020; Maaß, 2005; Schaap et al., 2011). Some of these difficulties were: fail to define variables, fail to use appropriate methods to mathematise, fail to understand mathematical content, and fail to realize dependencies between variables (Klock and Stiller, 2020). Describing the difficulties students encounter while mathematizing has been a critical step in fully conveying the complexity of mathematizing. The next step for the field is to understand the genesis of these difficulties with mathematizing. Recently, Cevikbas et al. (2022) conducted a systematic literature review on modelling competencies. They suggested conducting more theoretical research to find new ways to conceptualize modelling competencies. This chapter approaches studying mathematization through a lens that can reveal the root causes of the difficulties students encounter while positioning students' work from an anti-deficit perspective (see Adireda, 2019).

One blockage to mathematization reported in the literature is "representing elements mathematically so formulae can be applied", and an example of this difficulty was "expressing total length in terms of edge distances along the field" (Stillman et al., 2010). We started building our lens by asking ourselves why might tertiary STEM students have difficulty "representing elements mathematically so formulae can be applied"? Interpreted differently, this question could be said as "why might students choose to represent real-world relationships (like the interactions between a predator and a prey population) with a given arithmetic symbol (like $\times$)"? This led us to our research question: What conceptions of +, −, ×, ÷ do STEM undergraduates use to justify their choices of arithmetic symbols while mathematizing during a predator-prey modelling task? To answer our research question, we started by creating a catalogue of the elements students put into a model and the conceptions of $+, -, \times, \div$ students used when combining elements with these arithmetic symbols. We anticipate this work will support educators' efforts to leverage students' conceptions of +, −, ×, ÷ and aid students in connecting their spontaneous models to a normatively correct model. For this chapter, we report on participants' conceptions of the symbol $\times$ as they modeled the population dynamics in a predatory-prey scenario.

2 Theoretical Perspective

We use Lesh and Doerr (2003)'s definition of mathematical model: "Models are conceptual systems (consisting of elements, relations, operations, and rules governing interactions) that are expressed using external notation systems, and that are used to construct, describe, or explain the behaviors of other system(s) - perhaps so that the other system can be manipulated or predicted intelligently. A mathematical model focuses on structural characteristics of the relevant system" (Lesh & Doerr, 2003, p. 10). Our theoretical and methodological approach is to unpack this definition of mathematical modelling in terms of ideas from quantitative reasoning (QR) and covariational reasoning (CR), which will aid in describing the *elements* of a student's model and her *conception* of $+, -, \times, \div$ that allow her to combine the elements. We selected QR and CR as a theoretical/methodological lens because existing QR literature suggested that the operations students use on quantities reflect the quantitative relationships perceived by the student, that quantitative reasoning plays a key role in the refinement of the model (Larson, 2013), and that the models that students could potentially make during a modelling task depend on (and are constrained by) the quantities the student imposes onto the situation (Czocher & Hardison, 2021).

Quantitative reasoning is the conceptualization of a situation into a network of quantities and quantitative relationships (Thompson, 2011). A quantity is not the same thing as a variable. It is a triple of an object, attribute, and quantification (Thompson, 2011). Quantification means to conceptualize an object with a measurable attribute so that the measure is proportional to its unit (Thompson, 2011). A quantity is made from an individual's conceptions of objects within the situation. Two individuals may quantify an attribute differently, thus a quantity is idiosyncratic to the individual (Ellis, 2007). An example of a quantity in a predator-prey context is the number of prey animals at time t. The object is the prey population, the attribute is amount, and evidence of quantification could be a student explaining they could feasibly measure the number of prey animals on one day by counting them. A quantitative operation is a conceptual operation where an individual creates a new quantity in relation to one (or more) already created quantities (Ellis, 2007; Thompson, 2011). When a student envisions two quantities varying together and thinks about the ways they change in relationship to each other, and how they vary simultaneously, this reasoning is called this covariational reasoning (CR) (Carlson et al., 2002).

We operationalize Lesh and Doerr (2003)'s definition in terms of QR/CR as follows: a mathematical model is a conceptual system that encompasses all of a student's ideas and concepts regarding a relevant real-world system (for example, a relevant system could be a fish tank or an island habitat). The elements of the model are the quantities students impose onto the task scenario. The relationships between elements, operations, and rules governing interactions are determined by students' QR/CR.

From elementary and secondary research, repeated addition is one conception of × and creating parts from a whole is one conception of ÷ (Nunes & Bryant, 2021). According to Brahmia (2014) and Schwartz (1988) students need conceptions of × and ÷ that differ from repeated addition and creating parts from a whole to depict relationships involving ratios/rates, that are commonly needed for modeling dynamic situations with differential equations. For example, Schwartz (1988) pointed out that the notion of multiplication being repeated addition does not work for cases such as ((miles ÷ hours) × hours = miles) because iterating the relationship between miles and hours "number of hours times" cannot be done.

3 Methods

This study draws data from a larger study of facilitator scaffolding moves that foster undergraduates' modelling competencies. Data were collected via individual cognitive task-based interviews with 23 participants. Participants saw at least six tasks over ten 1-hour sessions. The tasks were designed to scaffold participants' modelling activities by attending to quantitative reasoning and appealing to similarities in mathematical structure across real-world contexts. In this chapter, we report on six participants' (described below) work on the task called *Cats and Birds.* We chose the *Cats and Birds* task because predator-prey scenarios provide the opportunity for students to quantify and combine distinct types of quantities such as amounts of quantity, amounts of change, rates of change, and per-capita rates using various quantitative combinations and arithmetic symbols. The goal of this task was for participants to write a system of differential equations modelling the interdependent dynamics of a bird and cat population. The task is set up with 12 sub-questions to scaffold their quantitative reasoning about the task scenario, culminating in a version of the Lotka-Volterra equations. Here, we focus on the first 4 sub-questions that guided the participant in constructing a model for the decrease in magnitude of the bird population due only to cats during an arbitrary segment of time Δt. The first 4-sub-questions are in Table 1. We focus on this part of the task because sub-question #4 was pivotal in students' reasoning for later subtasks; we wished to observe students' use of × during the pivotal point in the task. Interview protocols included asking participants their meaning for symbols they write and why their choice of $+, -, \times, \div$ is appropriate (e.g., "what does α mean?", "why did you decide to multiply $B(t)$ and $C(t)$?").

Table 1 The first 4-subquestions of the Cats and Birds Task

1	Consider a backyard habitat, where cats are the natural predators of birds. Let $B(t)$ be the number of birds and $C(t)$ be the number of cats at time t. How many cat-bird interactions would be possible at time t? [Call this model Eqn 1]
2	Not every cat and every bird encounter each other. Only some percentage of potential catbird encounters are realized per unit time, α. How would you adapt your model above to incorporate that fact? [Call this model Eqn 2]
3	Cats are very good hunters, but they aren't perfect. Sometimes the bird gets away, and so only some percentage of actual encounters end with a cat killing a bird. How would you adapt your model above [Eqn 2] to incorporate that fact? [Call this model Eqn 3]
4	Consider the decrease in magnitude of bird population due only to cat predation during a short interval of time, Δt. Write an expression modeling this decrease, in terms of the size of cat and bird populations present at time t. [Call this model Eqn 4]

We recruited participants who stated they had some familiarity with mathematical concepts like instantaneous rate of change with respect to time to increase the likelihood that participants would be able to productively discuss their mathematical reasoning as well as have avenues for entry into engaging with the tasks. In this chapter, we report results from six STEM majors (2 physics, 3 electrical engineers, and 1 civil engineer) who already completed differential equations, and who self-reported their mathematics grades as between A's and C's. We report results from these six participants to showcase differing conceptions of $\times$.

Data analysis started by identifying the quantities the participant imposed onto the task scenario by describing the object, attribute, and how the participant exhibited quantification for that attribute according to the quantification criteria developed by Czocher and Hardison (2021). We then noted instances where $\times$ were used on the quantities. We then documented the participants' conception (or inferred conception) of $\times$. We took the participants' use of an arithmetic symbol as evidence that they performed an arithmetic operation. This is reasonable because we worked with participants in advanced mathematics.

4 Results

Across the six participants, two distinct quantities tended to emerge from work on subtasks one through three. These two quantities were *dead birds at time* t and *dead birds per unit time*. The mathematical representations of these two quantities look identical but represent two distinct attributes of the bird population. The results characterize the quantities participants imposed onto the task scenario and the conception for $\times$ participants employed when combining those quantities to yield the final quantity to uncover the conditions that lead to some participants creating one quantity over the other. As a direct consequence of examining participants' reasoning, we report on the conceptions for × the participants employed, regardless of mathematical correctness. Because participants chose their own symbols to represent variables, we present a "standardized" version of the participants' work.

4.1 Dead Birds at Time t

In standardized notation, our participants constructed the quantity *dead birds at time t*, typically depicted as:

$$\delta \times \left(\alpha \times \left(B(t) \times C(t)\right)\right)$$

Participants combined *number of birds at time* t (represented by $B(t)$) and *number of cats at time* t (represented by $C(t)$) with × to create a new quantity: *total possible encounters between cats and birds at time* t. Neturo (physics) explained that he chose × because:

Neutro: One cat will interact with bird one, bird two, bird three, and bird four. That's the most bird interactions that bird, or that cat one can have. And then cat two and three, each are the same thing. Each bird or each cat has four interactions, one for each bird. That's why you have the number of cats times the number of birds because each bird interacts with each or each cat interacts with each bird one time in the maximum possible number of bird-cat encounters.

The other participants gave similar explanations for choosing ×. We infer the participants were iterating *number of birds at time t*, *number of cats at time t* many times, indicating the participants reasoning in this way conceptualized × as repeated addition.

To account for the fact that not all birds encounter all cats, some participants created a quantity called *percentage of encounters that are realized* (represented by α). They combined *total possible encounters between cats and birds at time t* (represented by $C(t) \times B(t)$) with *percentage of encounters that are realized* with × to create the new quantity *number of actual encounters at time* t. Pattern (civil engineering) explained that he chose × because:

Pattern: Okay. So now that we have a percentage, then you just do $C(t)$ times $B(t)$ times α equals encounters. Because you're going to take… So this (pointing to $C(t) \times B(t)$) is the total possible encounters that could possibly happen if perfect conditions are met for each cat to meet each bird, and then you're going to take a percentage of that total, and that would be your total there.

Pattern, and some other participants, were not iterating one quantity by the magnitude of the other. Instead, participants with this way of reasoning were taking *number of total possible encounters between cats and birds at time t* and finding a subset of those encounters. That subset represented the *number of actual encounters at time t*. This conception of × is similar to what Thompson (1990) called "comparing quantities multiplicatively", in which he gave the example "This is (multiplicatively) what part of that?" (p.10). However, our participants depicted this quantitative relationship using $\times$ instead of $\div$.

To account for the fact that not all encounters result in a bird's death, some participants constructed another quantity, similar to the one represented by α, called *percentage of actual encounters that resulted in a birds' death* (represented by δ). Participants combined *percentage of actual encounters that resulted in a bird's death* (represented by δ) with the quantity called *number of actual encounters at time t* ($\alpha \times \left(C(t) \times B(t)\right)$) with $\times$ to create the new quantity *number of encounters that result in a bird's death at time* t. Participants were taking *number of actual encounters at time t* and finding a subset of those encounters. The resulting subset represented the *number of encounters that result in a bird's death at time* t. Participants then re-interpreted the meaning of this quantity so that the object was the bird population rather than the set of encounters between birds and cats. In doing so, the participants constructed the quantity *dead birds at time* t, where the object is the bird population, and the attribute is amount of dead birds. We note that this re-interpretation was not trivial, and some participants needed assistance from the facilitator.

4.2 Dead Birds per unit time

In this section, we describe how our participants constructed the quantity *dead birds per standardized time unit*, typically depicted as

$$\delta \times \left(\alpha \times \left(B(t) \times C(t)\right)\right)$$

There were multiple combinations of quantities and different explanations for the participants' use of $\times$ that resulted in the quantity *dead birds per unit time*. To contrast this quantity with the one described in the previous section, we characterize how participants used $\times$ to combine two quantities to result in a new quantity whose attribute is a frequency.

Some participants combined *number of birds at time* t (represented by $B(t)$) and *number of cats at time* t (represented by $C(t)$) with $\times$ to create a new quantity called *frequency of total possible encounters between cats and birds per unit time*. Peet (electrical engineering) explained that he chose $\times$ because:

Peet: $B(t)$ is counting the number of birds at time t. So at whatever time we decide, and $C(t)$ is counting the number of cats.
Int: So then what is $B(t)$ times $C(t)$ counting?
Peet: The number of cat and bird interactions over time.
Int: You had said something about the units being the encounters, the interactions over time. What did you mean by over time, as the units?
Peet: Because in the $B(t)$ and $C(t)$ compares the number of birds to number of cats at time t, so I guess I would change this time to delta time, so in whichever span of

time that you decide you want to observe the relationship with these cats and birds, then you'll see the interactions that happen in that time.

Peet was not iterating *number of birds at time t*, *number of cats at time t* many times. We infer Peet was attending to the notion that both quantities *number of birds at time t* and *number of cats at time t* varied with time. Because both quantities covaried with time, combining them with × created a frequency for Peet. Said differently, Peet combined two quantities corresponding to amount attributes with ×, producing a quantity whose attribute, for him, was a frequency.

Alternatively, some participants created a quantity called *percentage of encounters that are realized per unit time* (represented by α). Participants combined *percentage of encounters that are realized per unit time* (represented by α) with *number of total possible encounters between cats and birds at time t* (represented by $C(t) \times B(t)$) with × to create a new quantity *actual encounters per unit time*. Khriss (physics) explained that he chose × because:

Khriss: Just taking a percentage of the total possibility.
Int: OK. Is that α? What is α meaning? Like, what does that represent?
Khriss: Just rate.
Int: Rate of, like, what kind of units might you give it?
Khriss: I'd say encounters per time interval.

We infer Khriss, and other participants who reasoned this way, were taking *number of total possible encounters between cats and birds at time t* and finding a subset of those encounters and transforming that subset so that it represents the rate of encounters per unit time. This is similar to what Thompson (1990) called "instantiating a rate", which he gave the example "Travel 5 hours per mile for 6 miles" (p.10). However, we infer our participants were finding a subset and instantiating a rate simultaneously. We now unpack this conception of × using dimensional units. The dimensional units of *percentage of encounters that are realized per unit time* (represented by α) were "actual encounters per total encounters per unit time" and the dimensional units of *number of total possible encounters between cats and birds at time t* were "total encounters". Combining *percentage of encounters that are realized per unit time* and *number of total possible encounters between cats and birds at time t* with × resulted in dimensional units of "actual encounters per unit time." Similarly, some participants then constructed another quantity called *percentage of encounters that resulted in a bird's death* (represented by δ). *Percentage of encounters that resulted in a bird's death* was combined with other quantities with × to create the quantity *encounters that resulted in a bird's death per unit time*. We infer participants held the same conception of × to create *encounters that resulted in a bird's death per unit time* that was subsequently used to create *dead birds per unit time.* We note here that the two quantities we have reported on are quantitatively distinct from each other even though they appear to result from the same calculation.

5 Discussion

We have reported four conceptions of × participants used when combining quantities with arithmetic symbols in a predator-prey task. Two of those conception of the symbol × , *repeated addition* and *instantiating a rate*, were present in other studies (e.g., Nunes & Bryant, 2021; Thompson, 2011). The other two conceptions of the symbol ×, *creation of a frequency* and *subsetting*, have not been previously reported in literature. We postulate that these newly observed conceptions of × were observable because our participants worked on a modeling task focused on a predator-prey relationship. This conjecture is based on findings that the models participants make depend on (and are constrained by), the quantities the participant imposes onto the situation (Czocher & Hardison, 2019), which inherently impacts the types of relationships the participants were able to express using arithmetic symbols. We speculate additional conceptions of arithmetic symbols could be observed in other task scenarios that call for advanced mathematics. For example, Sherin (2001) found several different meanings for arithmetic symbols physics students used to understand kinematic equations such as $-$ indicating either opposition in influence (i.e., inflow-outflow) or taking some part of a whole. Further research is needed to document additional conceptions of × as well as $+, -, \div$ that arise from modeling other real-world scenarios.

Research has shown that focusing on quantitative reasoning can improve students' ability to apply their mathematical knowledge to biology tasks (Hester et al. 2014). We expand on this in one way, framing mathematization through a quantitative reasoning lens helped us uncover the real-world relationships participants were trying to express when they used a given arithmetic operation in a predator-prey modelling task. In particular, the symbols $\delta \times \left(\alpha \times \left(B(t) \times C(t)\right)\right)$ represented both *dead birds at time* t (an amount) and *dead birds per unit time* (a frequency). This ties to our overarching objective to study mathematization from the quantitative reasoning lens to reveal root causes of the difficulties students face during mathematization. We noticed that participants who constructed the quantity *dead birds per unit time* more easily transitioned from subtask three to subtask four. We conjecture the reason was because participants could use a well-known formula like "rate times time equals amount". Said differently, "representing elements mathematically so formulae can be applied" was not a blockage for participants who created the quantity *dead birds per unit time* while it was a blockage for participants who created the quantity *dead birds at time t*. The participants' difficulties in subtask four arose from a discrepancy between their conception of × and the real-world relationship they were trying to represent. Specifically, it did not make sense to them to combine an amount quantity like *dead birds at time* t with an elapsed time quantity like *duration of time* with × to create a quantity called *dead birds during* Δt.

We hypothesize that one way an instructor could leverage students' conceptions of +, −, ×, ÷ to help students' connect their spontaneous models to a normatively correct model might look like guiding them to re-quantify a quantity in a way that compatible with the student's in-the-moment conception of +, −, ×, ÷. The first step would be to uncover the student's in-the-moment conceptions.

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