ON L^∞ ESTIMATES FOR COMPLEX MONGE-AMPÈRE EQUATIONS [1]

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Abstract

A PDE proof is provided for the sharp L^∞ estimates for the complex Monge-Ampère equation which had required pluripotential theory before. The proof covers both cases of fixed background as well as degenerating background metrics. It extends to more general fully non-linear equations satisfying a structural condition, and it also gives estimates of Trudinger type.

1 Introduction

The main goal of this paper is to answer a long-standing question in the theory of complex Monge-Ampère equations and its applications to complex geometry, namely whether sharp L^∞ estimates can be established by PDE methods and without pluripotential theory. We shall see that the answer is affirmative and, as may have been anticipated, the PDE proof also gives new estimates as well as extensions to many other fully non-linear equations.

L^∞ estimates have a uniquely storied history in the theory of complex Monge-Ampère equations. Early on, they were recognized as the defining difficulty in the problem of finding Kähler-Einstein metrics. Yau's introduction in 1978 of Moser iteration was the key step in his solution of the Calabi conjecture, and it ushered in a new era for complex Monge-Ampère equations [24]. Yau's Moser iteration method works for equations whose right hand sides are in L^q for $q > n$, where n is the complex dimension of the underlying space. The next major advance was the 1998 result of Kolodziej [13], which established L^∞ estimates for the solution when the right hand side is in L^q, for $q > 1$. This improvement in the range of q is no mere technicality, and it has a profound geometric significance: that L^∞ estimates fail for $q = 1$ is indicative of the necessity of stability conditions in the Kähler-Ricci flow [15], while the cases $1 < q \leq n$ are needed in a wide range of problems, including singular Kähler-Einstein metrics on manifolds of general type [11, 8], the analytic minimal model program [17], and degenerating Calabi-Yau metrics [20]. Kolodziej's method of proof relied heavily on the pluripotential theory developed in the late 1970's by Bedford and Taylor [2, 3]. Actually, for many applications to geometry, an extension of Kolodziej's results to the more general case of degenerating background metrics is necessary. Such an extension was developed in 2007 independently by Demailly and Pali [8] and Eyssidieux, Guedj, and Zeriahi [11], and pluripotential theory continued to be essential.

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An immediate question for the theory of geometric partial differential equations is whether the above L^∞ estimates can be derived by PDE methods, instead of pluripotential theory which is specific to Monge-Ampère equations. This question gained considerable attention over the years, as more and more fully non-linear equations without corresponding pluripotential theory emerged in complex differential geometry. A completely different proof for right hand sides in L^q with $q > 2$ was found in 2011 by Blocki [4], using the Alexandrov-Bakelman-Pucci (ABP) maximum principle.[1] His methods turn out to be remarkably powerful, and have since been applied successfully to many problems, including subsolutions [18, 16], equations with gradient terms [22], and the constant scalar curvature problem [6]. Even so, extensions to degenerating backgrounds as well as the full range $q > 1$ remained out of reach. In a different direction, a PDE proof of L^∞ estimates for the complex Monge-Ampère equation was obtained by J.X. Wang, X.J. Wang, and B. Zhou [23] for domains in $\mathbf{C}^n$. However, their methods do not appear adaptable to the compact manifold case, even in the simplest case when the background Kähler metric is fixed. Thus a fully effective approach to L^∞ estimates remained an open question.

The PDE approach to L^∞ estimates which we present in this paper combines the methods of Wang, Wang, and Zhou [23] with a fundamental new idea of Chen and Cheng [6] in their recent work on constant scalar curvature Kähler metrics, namely to compare the given equation with an auxiliary complex Monge-Ampère equation. A key novelty in our paper resides in the choice of auxiliary complex Monge-Ampère equation, as well as of the test function Φ for the comparison. We now formulate our main results.

Let X be a compact Kähler manifold without boundary of dimension n, and ω_X its Kähler form. If φ is a real smooth function on X, we let $\omega_\varphi = \omega_X + i\partial\bar\partial\varphi$, and let h_φ be the corresponding endomorphism relative to the metric ω_X. Explicitly, if we write $\omega_X = ig_{\bar k j} dz^j \wedge d\bar z^k$ in local holomorphic coordinates, then $(h_\varphi)^j{}_k = g^{j\bar m}(\omega_\varphi)_{\bar m k}$. Let $\lambda[h_\varphi]$ be the vector of eigenvalues of h_φ, and consider the non-linear partial differential equation

$$f(\lambda[h_\varphi]) = e^F, \quad \sup_X \varphi = 0, \quad \lambda[h_\varphi] \in \Gamma, \tag{1.1}$$

for a given function $f(\lambda)$ and real function F normalized such that $\int_X e^{nF}\omega_X^n = V = \int_X \omega_X^n$. Here the function $f(\lambda)$ is assumed to be invariant under permutations of the components of λ, and defined on an open cone $\Gamma \subset \{\lambda : \lambda_1 + \ldots + \lambda_n > 0\}$ with vertex at the origin and containing the first octant $\{\lambda : \lambda_1 > 0, \ldots, \lambda_n > 0\}$. We assume throughout f is elliptic in the sense that $\frac{\partial f(\lambda)}{\partial \lambda_j} > 0$ for any $\lambda \in \Gamma$, and that

$$f(r\lambda) = r\, f(\lambda), \quad r > 0,\ \lambda \in \Gamma, \tag{1.2}$$

which can be viewed as a normalization, as the same homogeneous equation can be expressed with many different functions $f(\lambda)$. We shall need the $L^1(\log L)^p$ norm of e^{nF} with

[1] The possibility of applying ABP maximum principle to the complex Monge-Ampère equation were suggested a while ago by S.Y. Cheng and S.T. Yau.

respect to the measure ω_X^n, which can be recognized as a generalized entropy $\mathrm{Ent}_p(F)$,

$$\mathrm{Ent}_p(F) = \frac{1}{V}\int_X e^{nF}|F|^p\omega_X^n = \frac{1}{n^pV}\|e^{nF}\|_{L^1(\log L)^p}. \tag{1.3}$$

Theorem 1 *Consider the equation (1.1), and assume that the function $f(\lambda)$ satisfies the following structural condition, namely that there exists a constant $\gamma > 0$ so that*

$$\det(\frac{\partial f(\lambda[h])}{\partial h_{ij}}) \geq \gamma, \quad \textit{for all } \lambda \in \Gamma. \tag{1.4}$$

Fix $p > n$. Then for any solution $\varphi \in C^2(X)$, we have the estimate

$$\sup_X|\varphi| \leq C \tag{1.5}$$

where the constant C depends only on n, p, γ, ω_X, and the entropy $\mathrm{Ent}_p(F)$.

We observe that many equations satisfy the structural condition (1.4). They include the Monge-Ampère equation $f(\lambda) = (\prod_{j=1}^n \lambda_j)^{\frac{1}{n}}$, more generally Hessian equations $f(\lambda) = \sigma_k(\lambda)^{1/k}$, $k = 1, \ldots, n$, where $\sigma_k(\lambda)$ is the k-th symmetric function, and quotient Hessian equations such as $f(\lambda) = (\frac{\sigma_k}{\sigma_l})^{1/(k-l)} + c\sigma_m^{1/m}$ for some $c > 0$. Applying Theorem 1 to the Monge-Ampère equation, we obtain immediately Kolodziej's [13] sharp L^∞ bounds. Applying it to Hessian equations, we obtain the L^∞ bounds of Dinew and Kolodziej [10]. In fact, our result is stronger, as the bound in [10] requires that $\exp(nF)$ be in L^q for $q > 1$, while we only need that $\exp(nF)$ be in $L^1(\log L)^p$ for $p > n$. Beyond these cases, the L^∞ estimates in Theorem 1 are new, and appear to be the first obtained in any generality for fully non-linear equations from Kähler geometry.

We discuss now estimates, particularly important for many geometric applications, where the background metric is allowed to degenerate. It is convenient to set up the equation as follows. Let (X, ω_X) be a compact Kähler manifold of dimension n as before, and let χ be a fixed closed and non-negative $(1,1)$-form. For $t \in (0, 1]$, set

$$\omega_t = \chi + t\omega_X \tag{1.6}$$

and for each $\varphi \in C^2(X)$, let $\omega_{t,\varphi} = \omega_t + i\partial\bar\partial\varphi$, $h_{t,\varphi}$ be the corresponding endomorphism relative to the metric ω_X, and $\lambda[h_{t,\varphi}]$ be the vector of eigenvalues of $h_{t,\varphi}$. Consider the family of non-linear partial differential equations

$$f(\lambda[h_{t,\varphi_t}]) = c_t\, e^{F_t}, \quad \sup_X\varphi_t = 0, \quad \lambda[h_{t,\varphi_t}] \in \Gamma, \quad t \in (0, 1] \tag{1.7}$$

for given function $f(\lambda)$, real functions F_t, and positive constant coefficients c_t. Here the functions F_t are normalized by $\int_X e^{nF_t}\omega_X^n = \int_X \omega_X^n$, so that the constants c_t are determined. Denote by V_t the volume of the metric ω_t, $V_t = \int_X \omega_t^n$, and define the energy $E_t(\varphi)$ by

$$E_t(\varphi) = \frac{1}{V_t}\int_X(-\varphi)f(\lambda([h_{t,\varphi}])^n\omega_X^n. \tag{1.8}$$

For functions φ_t solving the equation (1.7), we obviously have

$$E_t(\varphi_t) = \frac{c_t^n}{V_t}\int_X (-\varphi_t)\exp(nF_t)\omega_X^n. \tag{1.9}$$

Theorem 2 *Consider the family of equations (1.7), and assume that $f(\lambda)$ satisfies the structural condition (1.4). Let φ_t be C^2 functions on X satisfying (1.7). Fix $p > n$. Then for any $t \in (0, 1]$, we have*

$$\sup_X |\varphi_t| \leq C \tag{1.10}$$

where C is a constant depending only on $\omega_X, \chi, p, n, \gamma$, and upper bounds for the following three quantities

$$\frac{c_t^n}{V_t}, \quad E_t(\varphi_t), \quad \mathrm{Ent}_p(F_t). \tag{1.11}$$

All three quantities in (1.11) have attractive interpretations. We have already noted from (1.3) that $\mathrm{Ent}_p(F_t)$ is a generalized entropy. The quantity $\frac{c_t^n}{V_t}$ can be viewed as a relative volume, and a substitute for a cohomological constraint when dealing with general equations $f(\lambda)$. The quantity $E_t(\varphi_t)$ is clearly an energy functional, as it reduces to the Dirichlet integral in the case of the Laplacian on surfaces.

As a special case, Theorem 2 applied to the Monge-Ampère equation gives back immediately the L^∞ estimates of Eyssidieux, Guedj, Zeriahi [11], and Demailly, Pali [8], in the full generality of degenerating background metrics. The point here is that the relative volumes c_t^n/V_t must be $1/V$ because of a cohomological constraint, and it follows easily from Jensen's inequality that the energies E_t are uniformly bounded (see the fuller discussion in Theorem 4 in §4 below). Thus Theorem 2 gives the pure PDE proof of these L^∞ estimates that we sought. In fact, it is particularly simple as the ABP maximum principle is not even needed.

More generally, as long as the admissible cone Γ is the one corresponding to PSH functions, we can obtain uniform bounds for the energies E_t. For example, this applies to the equations $f(\lambda) = (\frac{\sigma_k}{\sigma_l})^{1/(k-l)} + c\,\sigma_n^{1/n}$.

For Hessian equations $f(\lambda) = \sigma_k(\lambda)^{\frac{1}{k}}$, $1 \leq k < n$, nothing was known in the case of degenerating background metrics, and Theorem 2 is now the only available result. As we shall see in Theorem 5 in §5 below, if $\|e^{nF_t}\|_{L^p}$ is uniformly bounded for $p > \frac{n}{k}$, we can bound the energy E_t by a multiple of $c_t^n V_t^{-1}$. In particular, for big cohomology classes $[\omega_t]$, the volumes V_t do not tend to 0, the coefficients c_t^n are bounded by the entropy, and we obtain then uniform L^∞ bounds. Such bounds are completely new for fully non-linear equations, and in particular Theorem 6 is new for Hessian equations.

This paper is organized as follows. In §2 and §3, we give the proofs of Theorem 1 and Theorem 2. We actually begin with the proof of Theorem 2 in §2, as Theorem 1 would

follow from Theorem 2 upon control of the energy term E_t. As in [6], we use a comparison of the given equation to an auxiliary complex Monge-Ampère equation. However, for Theorem 2, it is very important to choose the auxiliary equation so as to avoid having to use the ABP maximum principle, as this maximum principle would be an impediment in the case of degenerating background metrics. The additional step of estimating the energy terms to get Theorem 1 from Theorem 2 is provided by Theorem 3, which is the one requiring the ABP maximum principle. In §4 and §5, we show how our results apply to Monge-Ampère and Hessian equations respectively, and how they improve on many recent results in the literature. Finally, we have discussed exclusively so far L^∞ estimates. But not surprisingly, the same methods apply to L^p and exponential estimates as well. We illustrate this in §6 by some applications to Trudinger exponential inequalities for fully non-linear equations, which are either new or independent proofs of known sharp estimates. We also observe that the assumption (1.2) that $f(\lambda)$ was homomegenous of degree 1 was only for simplicity. The results of this paper still hold with this assumption replaced by the weaker Euler inequality with some fixed positive constant Λ,

$$\sum_{j=1}^n \frac{\partial f}{\partial \lambda_j}\lambda_j \leq \Lambda\, f(\lambda), \quad \lambda \in \Gamma.$$

Furthermore, our methods can be adapted to the setting of families of Kähler manifolds (X_j, ω_j) of the same dimension, as long as the α-invariant estimates hold uniformly.

2 Proof of Theorem 2

Let φ_t solve the equation (1.7). Fix $p > n$, and any upper bound $\overline{E}_t > 0$ for $E_t(\varphi_t)$. We shall actually show that

$$\sup_X |\varphi_t| \leq C_0\Big\{\frac{c_t^n}{V_t}(\mathrm{Ent}_p + 1 + \exp(C_1\overline{E}_t))\Big\}^{\frac{n}{p-n}}\overline{E}_t + C(n,p),$$

for constants C_0, C_1 depending only on $n, \omega_X, \chi, \gamma$, and $C(n,p)$ depending only on n, p. Throughout the proof we will fix $t \in (0,1]$, but the constants will be independent of t, unless stated explicitly otherwise.

For any $s > 0$, we let $\Omega_s := \{\varphi_t \leq -s\}$ be the sub-level set of φ_t.

Lemma 1 *There are constants $C = C(n, \omega_X, \chi, \gamma) > 0$ and $\beta_0 = \beta_0(n, \omega_X, \chi, \gamma) > 0$ such that for any $s > 0$*

$$\int_{\Omega_s} \exp\Big\{\beta_0(\frac{-(\varphi_t + s)}{A_s^{1/(n+1)}})^{\frac{n+1}{n}}\Big\}\omega_X^n \leq C\exp\left(C\overline{E}_t\right), \tag{2.1}$$

where $A_s := \frac{c_t^n}{V_t}\int_{\Omega_s}(-\varphi_t - s)e^{nF_t}\omega_X^n$ is the energy of $(\varphi_t + s)_-$.

Proof. We choose a sequence of smooth positive functions $\tau_k : \mathbf{R} \to \mathbf{R}_+$ such that

$$\tau_k(x) = x + \frac{1}{k}, \quad \text{when } x \geq 0, \tag{2.2}$$

and

$$\tau_k(x) = \frac{1}{2k}, \quad \text{when } x \leq -\frac{1}{k},$$

and $\tau_k(x)$ lies between $1/2k$ and $1/k$ for $x \in [-1/k, 0]$. Clearly τ_k converge pointwise to $\tau_\infty(x) = x \cdot \chi_{\mathbf{R}_+}(x)$ as $k \to \infty$, where $\chi_{\mathbf{R}_+}$ denotes the characteristic function of $\mathbf{R}_+$.

We solve an auxiliary complex Monge-Ampère equation on X

$$(\omega_t + i\partial\bar\partial\psi_{t,k})^n = \frac{\tau_k(-\varphi_t - s)}{A_{s,k}} f(\lambda[h_{\varphi_t}])^n \omega_X^n = \frac{\tau_k(-\varphi_t - s)}{A_{s,k}} c_t^n e^{nF_t} \omega_X^n, \tag{2.3}$$

with $\sup \psi_{t,k} = 0$ where $A_{s,k} := \frac{c_t^n}{V_t} \int_X \tau_k(-\varphi_t - s) e^{nF_t} \omega_X^n$ is chosen so that the integrals of both sides of (2.3) are equal. Note that (2.3) admits a unique smooth solution by Yau's theorem [24]. We also observe that as $k \to \infty$

$$A_{s,k} \to A_s = \frac{c_t^n}{V_t} \int_{\Omega_s} (-\varphi_t - s) e^{nF_t} \omega_X^n, \tag{2.4}$$

which follows from Lebesgue's dominated convergence theorem. The limit A_s satisfies $A_s \leq \overline{E}_t$, the assumed upper bound of $E_t(\varphi_t)$.

Denote Φ to be the smooth function

$$\Phi := -\varepsilon(-\psi_{t,k} + \Lambda)^{\frac{n}{n+1}} - (\varphi_t + s) \tag{2.5}$$

where

$$0 < \varepsilon := (\frac{n+1}{n^2})^{\frac{n}{n+1}} A_{s,k}^{\frac{1}{n+1}} \gamma^{-\frac{1}{n+1}}, \quad 0 < \Lambda := \frac{1}{(n+1)n^{n-1}} \frac{A_{s,k}}{\gamma} \tag{2.6}$$

where $\gamma > 0$ is the constant in the structure condition (1.4) of f.

Since X is compact without boundary, the maximum of Φ must be attained at some point, say, $x_0 \in X$. If $x_0 \in X \backslash \Omega_s^\circ$, then

$$\sup_X \Phi = \Phi(x_0) = -\varepsilon(-\psi_{t,k}(x_0) + \Lambda)^{\frac{n}{n+1}} - (\varphi_t(x_0) + s) < -\varphi_t(x_0) - s \leq 0.$$

Otherwise $x_0 \in \Omega_s^\circ$. Then at x_0, $i\partial\bar\partial\Phi(x_0) \leq 0$, and on the right hand side of (2.3) we have

$$\tau_k(-\varphi_t - s)(x_0) = -(\varphi_t(x_0) + s) + 1/k > 0$$

by the definition of τ_k in (2.2).

We denote by $G^{i\bar{j}} = \frac{\partial \log f(\lambda[h])}{\partial h_{ij}} = \frac{1}{f}\frac{\partial f(\lambda[h])}{\partial h_{ij}}$ the coefficients of the linearization of the operator $\log f(\lambda[h])$ with $h = \omega_X^{-1} \cdot \omega_{t,\varphi_t}$. By the ellipticity assumption of $f(\lambda[\cdot])$, $(G^{i\bar{j}})$ is positive definite. Moreover, by the structure condition (1.4) on f, we have

$$\det G^{i\bar{j}} = f^{-n}\det(\frac{\partial f(\lambda[h])}{\partial h_{ij}}) \geq \frac{\gamma}{f(\lambda)^n}.$$

Recall that the eigenvalues of h are by definition $\lambda = (\lambda_1, \ldots, \lambda_n)$. Working in a basis where h is diagonal and ω_X the identity, we find using the definition of $G^{i\bar{j}}$ that

$$\sum_{i,j} G^{i\bar{j}}(\omega_{t,\varphi_t})_{\bar{j}i} = \frac{1}{f(\lambda)}\sum_j \frac{\partial f(\lambda)}{\partial \lambda_j}\lambda_j = 1$$

where we have used the assumption that f is homogeneous of degree one, so that $\sum_i \lambda_i \frac{\partial f(\lambda)}{\partial \lambda_i} = f(\lambda)$. At the maximum point x_0 of Φ, we can write

$$\begin{aligned}
0 &\geq G^{i\bar{j}}\Phi_{\bar{j}i}(x_0) \\
&= \frac{n\varepsilon}{n+1}(-\psi_{t,k}+\Lambda)^{-\frac{1}{n+1}}G^{i\bar{j}}(\psi_{t,k})_{\bar{j}i} + \frac{n\varepsilon}{(n+1)^2}(-\psi_{t,k}+\Lambda)^{-\frac{n+2}{n+1}}G^{i\bar{j}}(\psi_{t,k})_{\bar{j}}(\psi_{t,k})_i - G^{i\bar{j}}(\varphi_t)_{\bar{j}i} \\
&\geq \frac{n\varepsilon}{n+1}(-\psi_{t,k}+\Lambda)^{-\frac{1}{n+1}}G^{i\bar{j}}(\omega_{t,\psi_{t,k}})_{\bar{j}i} - G^{i\bar{j}}(\omega_{t,\varphi_t})_{\bar{j}i} + (1-\frac{\varepsilon n}{n+1}(-\psi_{t,k}+\Lambda)^{-\frac{1}{n+1}})G^{i\bar{j}}(\omega_t)_{\bar{j}i} \\
&\geq \frac{\varepsilon n}{n+1}(-\psi_{t,k}+\Lambda)^{-\frac{1}{n+1}}n\Big(\det G^{i\bar{j}}\cdot\det(\omega_{t,\psi_{t,k}})_{\bar{j}i}\Big)^{1/n} - 1 + (1-\frac{\varepsilon n}{n+1}(-\psi_{t,k}+\Lambda)^{-\frac{1}{n+1}})G^{i\bar{j}}(\omega_t)_{\bar{j}i} \\
&\geq \frac{\varepsilon n^2}{n+1}(-\psi_{t,k}+\Lambda)^{-\frac{1}{n+1}}\gamma^{1/n}\Big(\frac{\tau_k(-\varphi_t-s)}{A_{s,k}}\Big)^{1/n} - 1 + (1-\frac{\varepsilon n}{n+1}\Lambda^{-\frac{1}{n+1}})G^{i\bar{j}}(\omega_t)_{\bar{j}i} \\
&\geq \frac{\varepsilon n^2\gamma^{1/n}}{n+1}(-\psi_{t,k}+\Lambda)^{-\frac{1}{n+1}}\Big(\frac{-\varphi_t-s+1/k}{A_{s,k}}\Big)^{1/n} - 1
\end{aligned}$$

where in the third inequality we used the arithmetic-geometric inequality and in the last one we used the choice of ε and Λ in (2.6). Thus at x_0 we have

$$-(\varphi_t+s)(x_0) < A_{s,k}\Big(\frac{n+1}{n^2\varepsilon\gamma^{1/n}}\Big)^n(-\psi_{t,k}(x_0)+\Lambda)^{n/(n+1)} = \varepsilon(-\psi_{t,k}(x_0)+\Lambda)^{n/(n+1)}$$

which implies that $\Phi(x_0) \leq 0$. Hence we can conclude that $\sup_X \Phi \leq 0$, that is, on X

$$\frac{-(\varphi_t+s)}{A_{s,k}^{1/(n+1)}} \leq (\frac{n+1}{n^2})^{\frac{n}{n+1}}\gamma^{-\frac{1}{n+1}}(-\psi_{t,k}+\frac{1}{(n+1)n^{n-1}}\frac{A_{s,k}}{\gamma})^{\frac{n}{n+1}} \leq C_n(-\psi_{t,k}+\frac{A_{s,k}}{\gamma})^{\frac{n}{n+1}}, \quad (2.7)$$

for some constant C_n depending only on n and γ. Taking the $(\frac{n+1}{n})$-th power of both sides of the previous equation, multiplying it by some small $\beta_0 > 0$, taking the exponential on both sides and then integrating the resulting inequality over Ω_s, we obtain

$$\int_{\Omega_s}\exp\Big\{\beta_0(\frac{-(\varphi_t+s)}{A_{s,k}^{1/(n+1)}})^{\frac{n+1}{n}}\Big\}\omega_X^n \leq \exp(C_n\beta_0 A_{s,k})\int_{\Omega_s}\exp(-C_n\beta_0\psi_{t,k})\omega_X^n. \quad (2.8)$$

Recall that $\omega_t + i\partial\bar\partial\psi_{t,k} > 0$ and $\omega_t = \chi + t\omega_X$. We may assume $\chi \le (a_0 - 1)\omega_X$ for some $a_0 = a_0(\chi, \omega) > 1$, so $\psi_{t,k}$ is also $(a_0\omega_X)$-plurisubharmonic. Now it is a basic fact in Kähler geometry that, for any Kähler class $\hat\chi$ on X, there is a constant $\alpha = \alpha(X, \hat\chi)$ so that

$$\int_X e^{-\alpha_0\psi}\omega_X^n \le C(\alpha_0, n, \hat\chi, \omega_X) \tag{2.9}$$

for any $\alpha_0 < \alpha$ and any $\hat\chi$-plurisubharmonic function ψ with $\sup_X \psi = 0$. The local version of this statement is in [12], and the above global version in [19]. We apply this statement with $\hat\chi = a_0\omega_X$, and fix α_0 with $0 < \alpha_0 < \alpha(X, \hat\chi)$. Then we choose $\beta_0 = \beta_0(n, \omega_X, \chi, \gamma) > 0$ in (2.8) such that $\beta_0 C_n = \alpha_0$, and from (2.8) we can then deduce that

$$\int_{\Omega_s} \exp\Big\{\beta_0\big(\frac{-(\varphi_t + s)}{A_{s,k}^{1/(n+1)}}\big)^{\frac{n+1}{n}}\Big\}\omega_X^n \le Ce^{CA_{s,k}}, \tag{2.10}$$

for some constant $C = C(n, \omega_X, \chi, \gamma) > 0$. Letting $k \to \infty$ in (2.10) we obtain from (2.4)

$$\int_{\Omega_s} \exp\Big\{\beta_0\big(\frac{-(\varphi_t + s)}{A_s^{1/(n+1)}}\big)^{\frac{n+1}{n}}\Big\}\omega_X^n \le Ce^{CA_s} \le Ce^{C\overline{E}_t}, \tag{2.11}$$

for some constant $C = C(n, \omega_X, \chi, \gamma) > 0$. The proof of Lemma 1 is complete.

We come now to the proof of Theorem 2 proper. Fix $p > n$, and define $\eta : \mathbf{R}_+ \to \mathbf{R}_+$ by $\eta(x) = (\log(1 + x))^p$. Note that η is a strictly increasing function with $\eta(0) = 0$, and let η^{-1} be its inverse function. If we let

$$v := \frac{\beta_0}{2}\big(\frac{-\varphi_t - s}{A_s^{1/(n+1)}}\big)^{(n+1)/n} \tag{2.12}$$

then we have for any $z \in \Omega_s$, by the generalized Young's inequality with respect to η,

$$\begin{aligned} v(z)^p e^{nF_t(z)} &\le \int_0^{\exp(nF_t(z))} \eta(x)dx + \int_0^{v(z)^p} \eta^{-1}(y)dy \\ &\le \exp(nF_t(z))(\log(1 + \exp(nF_t(z))))^p + \int_0^{\exp(v(z)-1)} x\eta'(x)dx \\ &\le \exp(nF_t(z))(1 + n|F_t(z)|)^p + v(z)^p \exp(v(z)) \\ &\le \exp(nF_t(z))(1 + n|F_t(z)|)^p + C(p)\exp(2v(z)) \end{aligned}$$

We integrate both sides in the inequality above over $z \in \Omega_s$, and get by Lemma 1 that

$$\begin{aligned} \int_{\Omega_s} v(z)^p e^{nF_t(z)}\omega_X^n &\le \int_{\Omega_s} e^{nF_t}(1 + n|F_t(z)|)^p\omega_X^n + \int_{\Omega_s} e^{2v(z)}\omega_X^n \\ &\le \|e^{nF_t}\|_{L^1(\log L)^p} + C + Ce^{C\overline{E}_t}, \end{aligned}$$

where the constant $C > 0$ depends only on $n, \omega_X, \chi, \gamma, p$. In view of the definition of v, this implies

$$\int_{\Omega_s} (-\varphi_t - s)^{\frac{(n+1)p}{n}} e^{nF_t(z)} \omega_X^n \leq 2^p \beta_0^{-p} A_s^{\frac{p}{n}} (\|e^{nF_t}\|_{L^1(\log L)^p} + C + Ce^{C\overline{E}_t}). \tag{2.13}$$

From the definition of A_s in (2.4), it follows from Hölder inequality that

$$\begin{aligned} A_s &= \frac{c_t^n}{V_t} \int_{\Omega_s} (-\varphi_t - s) e^{nF_t} \omega_X^n \\ &\leq \Big(\frac{c_t^n}{V_t} \int_{\Omega_s} (-\varphi_t - s)^{\frac{(n+1)p}{n}} e^{nF_t} \omega_X^n\Big)^{\frac{n}{(n+1)p}} \cdot \Big(\frac{c_t^n}{V_t} \int_{\Omega_s} e^{nF_t} \omega_X^n\Big)^{1/q} \\ &\leq A_s^{\frac{1}{n+1}} \Big(\frac{c_t^n}{V_t} 2^p \beta_0^{-p} (\|e^{nF_t}\|_{L^1(\log L)^p} + C + Ce^{C\overline{E}_t})\Big)^{\frac{n}{(n+1)p}} \cdot \Big(\frac{c_t^n}{V_t} \int_{\Omega_s} e^{nF_t} \omega_X^n\Big)^{1/q} \end{aligned}$$

where $q > 1$ satisfies $\frac{n}{p(n+1)} + \frac{1}{q} = 1$, i.e. $q = \frac{p(n+1)}{p(n+1)-n}$. The inequality above yields

$$A_s \leq \Big(\frac{c_t^n}{V_t} 2^p \beta_0^{-p} (\|e^{nF_t}\|_{L^1(\log L)^p} + C + Ce^{C\overline{E}_t})\Big)^{1/p} \cdot \Big(\frac{c_t^n}{V_t} \int_{\Omega_s} e^{nF_t} \omega_X^n\Big)^{\frac{1+n}{qn}}. \tag{2.14}$$

Observe that the exponent of the integral on the right hand of (2.14) satisfies

$$\frac{1+n}{qn} = \frac{pn+p-n}{pn} = 1 + \delta_0 > 1,$$

for $\delta_0 := \frac{p-n}{pn} > 0$. For notation convenience, set

$$B_0 := \Big(\frac{c_t^n}{V_t} 2^p \beta_0^{-p} (\|e^{nF_t}\|_{L^1(\log L)^p} + C + Ce^{C\overline{E}_t})\Big)^{1/p}. \tag{2.15}$$

From (2.14) we then get

$$A_s \leq B_0 \Big(\frac{c_t^n}{V_t} \int_{\Omega_s} e^{nF_t} \omega_X^n\Big)^{1+\delta_0}. \tag{2.16}$$

For any $r \in [0,1]$, we note that $-\varphi_t - s \geq r$ on $\Omega_{s+r} = \{\varphi_t \leq -s - r\}$. Thus

$$A_s = \frac{c_t^n}{V_t} \int_{\Omega_s} (-\varphi_t - s) e^{nF_t} \omega_X^n \geq r \cdot \frac{c_t^n}{V_t} \int_{\Omega_{s+r}} e^{nF_t} \omega_X^n. \tag{2.17}$$

If we define $\phi : \mathbf{R}_+ \to \mathbf{R}_+$ by

$$\phi(s) := \frac{c_t^n}{V_t} \int_{\Omega_s} e^{nF_t} \omega_X^n$$

then (2.16) and (2.17) imply that

$$r\phi(s+r) \leq B_0 \phi(s)^{1+\delta_0}, \quad \forall r \in [0,1] \text{ and } s \geq 0. \tag{2.18}$$

ϕ is clearly nonincreasing and continuous, so the lemma below applies to ϕ. It is a classic lemma due to De Giorgi, which was also used in [13, 11]. We include a sketch of the proof for the readers' convenience, and to exhibit the dependence of $\|\varphi_t\|_{L^\infty}$ on the given data.

Lemma 2 *Let $\phi : \mathbf{R}_+ \to \mathbf{R}_+$ be a decreasing right-continuous function with $\lim_{s\to\infty} \phi(s) = 0$. Assume that $r\phi(s+r) \leq B_0\phi(s)^{1+\delta_0}$ for some constant $B_0 > 0$ and all $s > 0$ and $r \in [0,1]$. Then there exists some $S_\infty = S_\infty(\delta_0, B_0, \phi) > 0$ such that $\phi(s) = 0$ for all $s \geq S_\infty$.*

Proof. Fix an $s_0 > 0$ such that $\phi(s_0)^{\delta_0} < \frac{1}{2B_0}$. This s_0 exists since $\phi(s) \to 0$ as $s \to \infty$. Define an increasing sequence (s_j) of positive real numbers inductively by

$$s_{j+1} := \sup\{s > s_j |\ \phi(s) > \frac{1}{2}\phi(s_j)\}.$$

If at some stage $\phi(s_j) = 0$, we stop there. By the right-continuity of ϕ, it follows that $\phi(s_{j+1}) \leq \frac{\phi(s_j)}{2}$ and $s_{j+1} \leq 1 + s_j$ since $\phi(1+s_j) \leq \frac{1}{2}\phi(s_j)$. It follows from the assumptions on ϕ that $s_{j+1} - s_j \leq 2^{-j\delta_0}$, which implies that the sequence (s_j) converges to

$$S_\infty = s_0 + \sum_{j\geq 0}(s_{j+1} - s_j) \leq s_0 + \frac{1}{1 - 2^{-\delta_0}}.$$

It is then clear that $\phi(s) = 0$ for any $s > S_\infty$. The lemma is proved.

We return now to the proof of Theorem 2. By Chebyshev's inequality, we have

$$\phi(s) = \frac{c_t^n}{V_t}\int_{\Omega_s} e^{nF_t}\omega_X^n \leq \frac{1}{s}\frac{c_t^n}{V_t}\int_{\Omega_s}(-\varphi_t)e^{nF_t}\omega_X^n \leq \frac{\overline{E}_t}{s} \to 0 \text{ as } s \to \infty.$$

Thus we may choose $s_0 = (2B_0)^{1/\delta_0}\overline{E}_t$ in the proof of Lemma 2. By (2.18) and Lemma 2, we deduce that

$$\Omega_{S_\infty} = \{\varphi_t \leq -S_\infty\} = \emptyset,$$

so hence

$$\inf_X \varphi_t \geq -S_\infty = -(2B_0)^{1/\delta_0}\overline{E}_t - \frac{1}{1 - 2^{-\delta_0}}, \tag{2.19}$$

where B_0 is the constant in (2.15) and $\delta_0 := \frac{p-n}{pn} > 0$ depends only on n and p. The proof of Theorem 2 is complete.

3 Proof of Theorem 1

Let φ solve the equation (1.1), where F is a given smooth function and $f(\lambda[\cdot])$ is the nonlinear operator as introduced in Section 1. The setting of Theorem 1 with a fixed background ω_X can be viewed as a special case of the setting of Theorem 2 with $\chi = 0$, and t taken to be 1, $\omega_1 = \omega_X$, and in the notations (1.7) and (1.1) for the two settings,

$$e^F = c_1 e^{F_1}, \quad V = V_1. \tag{3.1}$$

We observe that, in view of the normalization $\frac{1}{V}\int_X e^{nF_1}\omega_X^n = 1$ for F_1,

$$c_1^n = \frac{1}{V}\int_X e^{nF}\omega_X^n \leq \mathrm{Ent}_p(F) + e^n \tag{3.2}$$

for any $p \geq 1$. Thus, applying Theorem 2 and assuming that Ent_p is bounded, we find that L^∞ bounds for φ would follow if we can control the energy $E = E_{t=1}$. However, an easy application of Hölder's inequality gives

$$E(\varphi) = \frac{1}{V}\int_X (-\varphi)e^{nF}\omega_X^n \leq \Big(\frac{1}{V}\int_X (-\varphi)^{\frac{n}{n-1}} e^{nF}\omega_X^n\Big)^{(n-1)/n} \leq C \tag{3.3}$$

so it suffices to control the right hand side. This is done in the following theorem, part (c), which completes the proof of Theorem 1:

Theorem 3 *If $f(\lambda[\cdot])$ satisfies the structure condition (1.4), then the following holds:*

(a) *Assume that $p \in (0, n)$. Then there exist constants c_p, $C_p > 0$ depending only on ω_X, n, p, γ and the generalized entropy $\mathrm{Ent}_p(F)$ such that*

$$\int_X \exp\{c_p(-\varphi)^{\frac{n}{n-p}}\}\omega_X^n \leq C_p. \tag{3.4}$$

(b) *Assume that $p = n$. Then for any $N > 0$, there exists constants $c_N > 0$, $C_N > 0$ depending on n, ω_X, N, γ, and the generalized entropy $\mathrm{Ent}_n(F)$ so that*

$$\int_X \exp\{c_N(-\varphi)^N\}\omega_X^n \leq C_N. \tag{3.5}$$

(c) *We have the energy estimate:*

$$\int_X (-\varphi)^N e^{nF}\omega_X^n \leq C \tag{3.6}$$

for $N = \frac{n}{n-p}$ if $p \in [1, n)$, and for any $N > 0$ if $p = n$, where the constant C on the right hand side of (3.6) depends on n, ω_X, γ, N and the entropy $\mathrm{Ent}_p(F)$.

We observe that, in the special case of the Monge-Ampère equation and when $p = 1$, these estimates have been established in [9], using pluripotential theory. Further if $p > n$, then Theorem 1 implies the solutions are L^∞ bounded. Theorem 3 gives a more complete integral estimate of such solutions for the full range of $p \in (0, \infty)$, using pure PDE methods.

For the proof of Theorem 1, we only need part (c) of Theorem 3, but we give the proofs of the other parts as well, as they are of independent interest. As mentioned in the Introduction, for this we need the ABP method developed in [4] and [6] in order to get better integral bounds for φ as stated in (3.4) and (3.5). In particular, we have fix a background metric ω_X, as this method is not effective for handling degenerating families.

Suppose $p \in (0, n]$ and write

$$\Psi_p := \frac{1}{V}\int_X (F^2+1)^{p/2} e^{nF}\omega_X^n \sim \frac{1}{V}\int_X |F|^p e^{nF}\omega_X^n = \mathrm{Ent}_p(F).$$

We can solve the complex Monge-Ampère equation

$$(\omega_X + i\partial\bar\partial\psi)^n = \frac{(F^2+1)^{p/2}}{\Psi_p} e^{nF}\omega_X^n, \quad \sup_X \psi = 0. \tag{3.7}$$

Lemma 3 *Let φ, ψ be the solutions to the equations (1.1), (3.7), respectively. There exist constants $\Lambda, \lambda, C > 0$ depending on n, ω_X, γ, p, and Ψ_p such that*

$$\sup_X(-(-\psi+\Lambda)^\beta - \lambda\varphi) \le C$$

where $\beta = \frac{n-p}{n} \in (0,1)$ if $p < n$, and if $p = n$, we choose an arbitrary $\beta = N^{-1} \in (0,1)$ and the constants λ, Λ, C depend additionally on N.

In the proof below, for a smooth function u on X, we denote

$$\Box u := G^{i\bar j} u_{\bar j i}, \quad |\nabla u|_G^2 := G^{i\bar j}\nabla_{\bar j} u \nabla_i u \tag{3.8}$$

and

$$\mathrm{tr}_G\alpha := G^{i\bar j}\alpha_{\bar j i}, \quad \text{for a smooth } (1,1)\text{-form } \alpha = \alpha_{\bar j i}\sqrt{-1}dz^i\wedge d\bar z^j,$$

where as in Lemma 1, $G^{i\bar j} = \frac{\partial \log f(\lambda[h])}{\partial h_{ij}}$ is the coefficient matrix of the linearized operator of $\log f(\lambda[\cdot])$ at $h = \omega_X^{-1}\cdot\omega_\varphi$, and by the structure condition (1.4) on f, we have $\det(G^{i\bar j}) \ge \gamma f^{-n}$ and $G^{i\bar j}$ is positive definite.

Proof of Lemma 3. We choose constants as follows:

$$\Lambda = \Big(\frac{4^n}{n^n\gamma}\frac{2^p}{\alpha_0^p}\Psi_p\Big)^{1/n(1-\beta)}, \quad \lambda = 4\beta\Lambda^{-(1-\beta)} \tag{3.9}$$

where as usual $\alpha_0 = \alpha_0(X, \omega_X)$ is a fixed constant smaller than the α-invariant of (X, ω_X). We denote

$$\rho := -(-\psi+\Lambda)^\beta - \lambda\varphi. \tag{3.10}$$

For notation convenience, we set $\phi_\delta(t) = t + \sqrt{t^2+\delta} > 0$, which is a smoothing of $2\max(t, 0)$ and converges to it as $\delta \to 0$. Here we will first fix a $\delta > 0$ small and later on δ will be sent to zero. All constants appearing in the proof are independent of δ, unless stated otherwise. From now on we will consider $\phi_\delta(\rho)$ which is monotone decreasing and converges to $2\rho_+$ as $\delta \to 0$. We will omit the δ in $\phi_\delta(\rho)$ and simply write $\phi(\rho)$.

We define a *smooth* function

$$H = \phi(\rho)^b,$$

where $b = 1 + \frac{1}{4n} > 1$ is constant. Since X is compact, H must achieve its maximum at some point in X, say, x_0, and we denote $\sup_X H =: M > 0$ (if $M = 0$ there is nothing to prove). Let $r = \min\{1, r(X, \omega_X)\}$ where $r(X, \omega_X) > 0$ is the injectivity radius of (X, ω_X) viewed as a compact Riemannian manifold. So we can identify the geodesic ball $B_r(x_0)$ as an open smooth domain in $\mathbf{R}^{2n}$ with Euclidean diameter bounded by $3r$, say. Let $\theta \in (0, 1)$ be a small constant defined by

$$\theta := \min\{\frac{r^2 \beta \Lambda^{-(1-\beta)}}{100 M^{1/b}}, \frac{r^2}{100n}\} < \frac{1}{10}. \tag{3.11}$$

As in [6], we choose an auxiliary smooth function η defined on $B_r(x_0)$ so that $\eta \equiv 1$ on $B_{r/2}(x_0)$ and $\eta \equiv 1 - \theta$ on $X \backslash B_{3r/4}(x_0)$, and η lies between 1 and $1 - \theta$ in the annulus $B_{3r/4}(x_0) \backslash B_{r/2}(x_0)$. Moreover η can be chosen to satisfy

$$|\nabla \eta|_g^2 \leq \frac{10\theta^2}{r^2}, \quad |\nabla^2 \eta|_g \leq \frac{10\theta}{r^2}, \tag{3.12}$$

where we identify ω_X with its associated Riemannian metric g.

We can now calculate,

$$\Box(H\eta) = \eta \Box H + H \Box \eta + 2Re(G^{i\bar{j}} \nabla_{\bar{j}} H \nabla_i \eta). \tag{3.13}$$

We observe that the middle term in (3.13) satisfies

$$H \Box \eta = H \mathrm{tr}_G i\partial\bar{\partial}\eta \geq -H \frac{10\theta}{r^2} \mathrm{tr}_G \omega_X.$$

The last term in (3.13) satisfies

$$\begin{aligned} 2Re(G^{i\bar{j}} \nabla_{\bar{j}} H \nabla_i \eta) &= 2b\phi(\rho)^{b-1} Re(G^{i\bar{j}} \nabla_{\bar{j}} \phi(\rho) \nabla_i \eta) \\ &\geq -\frac{b(b-1)}{2} \phi(\rho)^{b-2} |\nabla \phi(\rho)|_G^2 - \frac{2b}{b-1} \phi(\rho)^b |\nabla \eta|_G^2 \\ &\geq -\frac{b(b-1)}{2} \phi(\rho)^{b-2} |\nabla \phi(\rho)|_G^2 - \frac{2b}{b-1} \phi(\rho)^b \frac{10\theta^2}{r^2} \mathrm{tr}_G \omega_X \end{aligned}$$

where in the first inequality we applied the Cauchy-Schwarz inequality. The first term in (3.13) is

$$\begin{aligned} \eta \Box H &= b\eta\phi(\rho)^{b-1} \Box \phi(\rho) + b(b-1)\eta\phi(\rho)^{b-2} |\nabla \phi(\rho)|_G^2 \\ &= b\eta\phi(\rho)^{b-1} \phi'(\rho) \Box \rho + b\eta\phi(\rho)^{b-1} \phi''(\rho) |\nabla \rho|_G^2 + b(b-1)\eta\phi(\rho)^{b-2} |\nabla \phi(\rho)|_G^2. \end{aligned} \tag{3.14}$$

We note that the middle term in (3.14) is nonnegative due to the fact that

$$\phi''(t) = \frac{1}{\sqrt{t^2 + \delta}} - \frac{t^2}{(\sqrt{t^2 + \delta})^3} = \frac{\delta}{(t^2 + \delta)^{3/2}} > 0.$$

To deal with the first term in (3.14) we note by the homogeneity of degree one assumption on f that

$$\Box\varphi = \mathrm{tr}_G i\partial\bar\partial\varphi = \mathrm{tr}_G\omega_\varphi - \mathrm{tr}_G\omega_X = 1 - \mathrm{tr}_G\omega_X.$$

Then we calculate

$$\begin{aligned}
\Box\rho &= \Box(-(-\psi+\Lambda)^\beta - \lambda\varphi)\\
&= \beta(-\psi+\Lambda)^{\beta-1}\Box\psi + \beta(1-\beta)(-\psi+\Lambda)^{\alpha-2}|\nabla\psi|^2_G - \lambda\Box\varphi\\
&\geq \beta(-\psi+\Lambda)^{\beta-1}\mathrm{tr}_G\omega_\psi - \beta(-\psi+\Lambda)^{\beta-1}\mathrm{tr}_G\omega_X - \lambda + \lambda\mathrm{tr}_G\omega_X\\
&\geq n\gamma^{1/n}\beta(-\psi+\Lambda)^{\beta-1}(\frac{(F^2+1)^{p/2}}{\Psi_p})^{1/n} + (\lambda - \beta(-\psi+\Lambda)^{\beta-1})\mathrm{tr}_G\omega_X - \lambda\\
&\geq n\gamma^{1/n}\beta(-\psi+\Lambda)^{\beta-1}(\frac{|F|^p}{\Psi_p})^{1/n} + (\lambda - \beta\Lambda^{\beta-1})\mathrm{tr}_G\omega_X - \lambda, \qquad (3.15)
\end{aligned}$$

where in the second inequality we used the arithmetic-geometric inequality and the equations for φ and ψ. Plugging these inequalities into (3.13), we obtain

$$\begin{aligned}
\Box(H\eta) \quad &\geq \quad -\frac{10\theta}{r^2}\phi(\rho)^b\mathrm{tr}_G\omega_X - \frac{2b}{b-1}\phi(\rho)^b\frac{10\theta^2}{r^2}\mathrm{tr}_G\omega_X\\
&\quad +b\eta\phi(\rho)^{b-1}\phi'(\rho)\Big(n\gamma^{1/n}\beta(-\psi+\Lambda)^{\beta-1}(\frac{|F|^p}{\Psi_p})^{1/n} + (\lambda-\beta\Lambda^{\beta-1})\mathrm{tr}_G\omega_X - \lambda\Big)\\
&\geq \quad b\phi(\rho)^{b-1}\Big\{\frac{9}{10}(\lambda-\beta\Lambda^{\beta-1})\phi'(\rho)\mathrm{tr}_G\omega_X - \frac{20\theta}{r^2 b}\phi(\rho)\mathrm{tr}_G\omega_X \qquad (3.16)\\
&\quad -\frac{20\theta^2}{(b-1)r^2}\phi(\rho)\mathrm{tr}_G\omega_X + n\gamma^{1/n}\beta(-\psi+\Lambda)^{\beta-1}(\frac{|F|^p}{\Psi_p})^{1/n} - \lambda\phi'(\rho)\Big\},
\end{aligned}$$

where we ignored η in the last inequality since $\eta > 9/10$. To deal with the right hand side in the equation (3.16), we note that on the set $\{\rho \leq 0\}$

$$0 \leq \phi(\rho) = \rho + \sqrt{\rho^2+\delta} = \frac{\delta}{\sqrt{\rho^2+\delta}-\rho} \leq \sqrt{\delta},$$

and on this same set the function $\phi'(\cdot)$ satisfies

$$1 \geq \phi'(\rho) = 1 + \frac{\rho}{\sqrt{\rho^2+\delta}} = \frac{\phi(\rho)}{\sqrt{\rho^2+\delta}} \geq 0.$$

So on the set $\{\rho \leq 0\}$ the right hand side of (3.16) is greater or equal to

$$b\phi(\rho)^{b-1}\Big(-\frac{20\theta}{r^2 b}\sqrt{\delta}\mathrm{tr}_G\omega_X - \frac{20\theta^2}{(b-1)r^2}\sqrt{\delta}\mathrm{tr}_G\omega_X - \lambda\Big)$$

On the other hand, on the set $\{\rho > 0\}$, we know $2 \geq \phi'(\rho) > 1$, so the first three terms on the right hand side of (3.16) are positive due to the choice of θ in (3.11), and Λ, λ in

(3.9) and the fact that $\phi(\rho) \leq M^{1/b}$. So on the set $\{\rho > 0\}$ the right hand side of (3.16) is greater or equal to

$$b\phi(\rho)^{b-1}\Big(n\gamma^{1/n}\beta(-\psi+\Lambda)^{\beta-1}(\frac{|F|^p}{\Psi_p})^{1/n} - \lambda\Big).$$

Combining the above two cases, we obtain

$$\begin{aligned}\Box(H\eta) \quad \geq \quad & b\phi(\rho)^{b-1}\Big(n\gamma^{1/n}\beta(-\psi+\Lambda)^{\beta-1}(\frac{|F|^p}{\Psi_p})^{1/n} - \lambda\Big)\cdot\chi_{\{\rho>0\}} \\ & -b\phi(\rho)^{b-1}\Big(\frac{20\theta}{r^2 b}\sqrt{\delta}\mathrm{tr}_G\omega_X + \frac{20\theta^2}{(b-1)r^2}\sqrt{\delta}\mathrm{tr}_G\omega_X + \lambda\Big)\cdot\chi_{\{\rho\leq 0\}},\end{aligned}$$

where χ_E denotes the characteristic function of a set E.

We now apply the ABP maximum principle to the function $H\eta$ on the domain $B_r(x_0) =: B_0$ in $\mathbf{R}^{2n}$. It follows that

$$\begin{aligned}\sup_{B_0}(H\eta) \quad \leq \quad & \sup_{\partial B_0}(H\eta) + C(n)r\Big\{\int_{B_0\cap\{\rho>0\}}\frac{\phi(\rho)^{2n(b-1)}(n\gamma^{1/n}\beta(-\psi+\Lambda)^{\beta-1}(\frac{|F|^p}{\Psi_p})^{1/n}-\lambda)_-^{2n}}{(\det G^{i\bar{j}})^2}\omega_X^n \\ & + \int_{B_0\cap\{\rho\leq 0\}}\frac{\phi(\rho)^{2n(b-1)}(\frac{20\theta}{r^2 b}\sqrt{\delta}\mathrm{tr}_G\omega_X + \frac{20\theta^2}{(b-1)r^2}\sqrt{\delta}\mathrm{tr}_G\omega_X+\lambda)^{2n}}{(\det G^{i\bar{j}})^2}\omega_X^n\Big\}^{1/2n} \\ \leq \quad & \sup_{\partial B_0}(H\eta) + C(n)\Big\{\int_{B_0\cap\{\rho>0\}}\frac{\sqrt{\phi(\rho)}(n\gamma^{1/n}\beta(-\psi+\Lambda)^{\beta-1}(\frac{|F|^p}{\Psi_p})^{1/n}-\lambda)_-^{2n}}{e^{-2nF}}\omega_X^n \\ & + C'\delta^{n(b-1)}\Big\}^{1/2n} \qquad (3.17)\end{aligned}$$

where the constant $C' = C'(n, F, G, \omega_X)$ in the last term may not be uniformly bounded, but this is not a concern, since later on we will let $\delta \to 0$. We observe that the integral involved in the last inequality is in fact integrated over the set where $n\gamma^{1/n}\beta(-\psi+\Lambda)^{\beta-1}(\frac{|F|^p}{\Psi_p})^{1/n} - \lambda < 0$ and $\rho > 0$, and over this set we have from the choice of constants in (3.9)

$$|F| \leq (\Psi_p)^{1/p}\Big(\frac{\lambda}{n\gamma^{1/n}\beta}\Big)^{n/p}(-\psi+\Lambda)^{(1-\beta)n/p} = \frac{\alpha_0}{2}(-\psi+\Lambda)^{(1-\beta)n/p}.$$

At the same time, on the same set, we have $0 < \rho \leq -\lambda\varphi$ and $\phi(\rho) \leq 2\rho + \sqrt{\delta}$. Therefore, we obtain from (3.17) that

$$\begin{aligned}M \quad \leq \quad & (1-\theta)\sup_{\partial B_0}H + C\Big(\int_{B_0}(-\varphi+\sqrt{\delta})^{1/2}\exp\Big(\frac{\alpha_0}{2}(-\psi+\Lambda)^{(1-\beta)n/p}\Big)\omega_X^n + C'\delta^{n(b-1)}\Big)^{1/2n} \\ \leq \quad & (1-\theta)M + C\Big(\int_{B_0}\Big(-\varphi + \exp\Big(\alpha_0(-\psi+\Lambda)^{(1-\beta)n/p}\Big)\Big)\omega_X^n + C'\delta^{n(b-1)}\Big)^{1/2n} \\ \leq \quad & (1-\theta)M + C_0 + C'\delta^{(b-1)/2} \qquad (3.18)\end{aligned}$$

where $C_0 = C_0(n, \omega_X, \gamma, \Psi_p, p) > 0$ is independent of δ and in the last inequality we have used the following inequalities:

(1) $\int_X (-\varphi)\omega_X^n \leq C(n, \omega_X)$ which follows from the Green's formula, $n + \Delta_{\omega_X}\varphi > 0$ (since $\lambda[h_\varphi] \in \Gamma \subset \{\lambda_1 + \cdots + \lambda_n > 0\}$) and the normalization condition $\sup_X \varphi = 0$.

(2) $\int_X \exp\Big(\alpha_0(-\psi + \Lambda)^{(1-\beta)n/p}\Big)\omega_X^n \leq C(n, \omega_X, \Psi_p, p)$. By the choice of β, if $p < n$, $(1-\beta)n/p = 1$; and if $p = n$, then $(1-\beta)n/p = 1 - N^{-1} < 1$, Young's inequality gives the desired estimate.

Hence we conclude that with the choice of θ in (3.11)

$$\min(M^{1-\frac{1}{b}}, M) \leq C_0 + C'\delta^{(b-1)/2} \ \Rightarrow\ M \leq C_0 + C'\delta^{2b},$$

which implies that

$$\sup_X 2\rho_+ \leq \sup_X \phi(\rho) = M \leq C_0 + C'\delta^{2b}.$$

Finally letting $\delta \to 0$ yields the desired estimate $\sup_X \rho_+ \leq C_0$ for some positive constant $C_0 = C_0(n, \omega_X, \gamma, \Psi_p)$ (which may be different from the C_0 in (3.18)). The proof of Lemma 3 is complete.

Proof of Theorem 3. (a) and (b) follow easily from Lemma 3 and the fact the ω_X-plurisubharmonic function ψ satisfies $\int_X e^{-\alpha_0\psi}\omega_X^n \leq C(n, \omega_X)$.

The inequality (3.6) in (c) is an immediate consequence of the estimates in (a) and (b), and Jensen's inequality. More precisely, we have

$$\frac{1}{V}\int_X e^{-nF + c_0(-\varphi)^N} e^{nF}\omega_X^n = \frac{1}{V}\int_X e^{c_0(-\varphi)^N}\omega_X^n \leq C(n, \omega_X, \gamma, \Psi_p, N).$$

Taking the logarithms of both sides and applying Jensen's inequality yields

$$\frac{1}{V}\int_X (c_0(-\varphi)^N - nF)e^{nF}\omega_X^n \leq C(n, \omega_X, \gamma, \Psi_p, N),$$

from which (3.6) follows after noting that Ψ_p is equivalent to the entropy Ent_p and if $p \geq 1$

$$\frac{1}{V}\int_X Fe^{nF}\omega_X^n \leq \mathrm{Ent}_p.$$

The proof of Theorem 3 is complete.

4 Monge-Ampère equations

In this and the next section, we apply Theorems 1 and 2 to the specific cases of the Monge-Ampère and Hessian equations on a compact Kähler manifold (X, ω_X).

We begin by noting that the structural condition (1.4) holds for many equations and is usually easy to check:

Lemma 4 *Assume that $f : \mathbf{R}^n \to \mathbf{R}_+$ is a concave and homogeneous function of degree one, which satisfies $\frac{\partial f(\lambda)}{\partial \lambda_j} > 0$ for any λ in an admissible cone $\Gamma \subset \mathbf{R}^n$. Assume that there is a $\gamma > 0$ such that*

$$f(\mu) \geq n\gamma^{1/n}(\prod_j \mu_j)^{1/n}, \quad \text{for all } \mu \in \Gamma_n := \{\lambda \in \mathbf{R}^n : \lambda_1 > 0, \ldots, \lambda_n > 0\}. \tag{4.1}$$

Then f satisfies the structural condition (1.4).

Proof. By the concavity of f on Γ, for any $\lambda, \mu \in \Gamma$ we have

$$f(\mu) \leq f(\lambda) + \sum_{j=1}^n (-\lambda_j + \mu_j)\frac{\partial f(\lambda)}{\partial \lambda_j} = \sum_{j=1}^n \mu_j \frac{\partial f(\lambda)}{\partial \lambda_j}, \tag{4.2}$$

where we have used the homogeneity of degree one assumption on f, which implies that $\sum_j \lambda_j \frac{\partial f(\lambda)}{\partial \lambda_j} = f(\lambda)$. Taking the infimum of the right hand side of (4.2) over all $\mu \in \Gamma_n$ with $\prod_{j=1}^n \mu_j = 1$, by the arithmetic-geometric inequality and the assumption (4.1) on f, we get

$$\prod_{j=1}^n \frac{\partial f(\lambda)}{\partial \lambda_j} \geq n^{-n}\{\inf_{\mu \in \Gamma_n, \prod_j \mu_j = 1} f(\mu)\}^n \geq \gamma > 0.$$

The desired inequality (1.4) follows from this inequality upon diagonalizing the matrix h. The proof of Lemma 4 is complete.

It follows immediately from this lemma that the functions $f(\lambda) = (\prod_{j=1}^n \lambda_j)^{\frac{1}{n}}$, $f(\lambda) = \sigma_k(\lambda)^{\frac{1}{k}}$, and $f(\lambda) = (\frac{\sigma_k(\lambda)}{\sigma_\ell(\lambda)})^{\frac{1}{k-\ell}} + c\sigma_p(\lambda)^{\frac{1}{p}}$ for $c > 0$, $n \geq k \geq \ell \geq 1$, $n \geq p \geq 1$, corresponding respectively to the Monge-Ampère, the Hessian, and the quotient Hessian equations, all satisfy the structural condition (1.4), and the constant γ depends only on the given numbers $n, k, \ell, p, c > 0$, and the admissible cone $\Gamma = \Gamma_k = \{\lambda \in \mathbf{R}^n : \sigma_1(\lambda) > 0, \ldots, \sigma_k(\lambda) > 0\}$. The last equation appeared recently in [7].

In this section, we focus on the Monge-Ampère equation. To discuss the underlying geometry, it is convenient to rewrite it in the more usual form

$$(\omega_t + i\partial\bar{\partial}\varphi_t)^n = c_t^n e^{nF_t}\omega_X^n, \quad \lambda[h_{t,\varphi_t}] \in \Gamma, \quad \sup_X \varphi_t = 0, \tag{4.3}$$

where $\omega_t = t\omega_X + \chi$ is the family of degenerating background metrics. To apply Theorem 2, we need to control the ratio c_t^n/V_t and the energy E_t. This can be readily done using the following two easy lemmas:

Lemma 5 *Let $V = \int_X \omega_X^n$ be the volume of (X, ω_X), then*

$$V^{-1} = \frac{c_t^n}{V_t}, \quad \forall t \in (0, 1].$$

Proof. Integrating both sides of (4.3), we get

$$c_t^n V = c_t^n \int_X e^{nF_t}\omega_X^n = \int_X (\omega_t + i\partial\bar\partial\varphi_t)^n = \int_X \omega_t^n = V_t.$$

Lemma 6 *There is a uniform constant $C > 0$ depending only on $n, \|e^{nF_t}\|_{L^1(\log L)^1(\omega_X^n)}$, ω_X, χ such that for all $t \in (0,1]$*

$$E_t(\varphi_t) \le C. \tag{4.4}$$

Proof. Recall that

$$E_t(\varphi_t) = \frac{c_t^n}{V_t}\int_X (-\varphi_t)e^{nF_t}\omega_X^n = \frac{1}{V_t}\int_X (-\varphi_t)\omega_{\varphi_t}^n,$$

so that it suffices to show that $\frac{1}{V_t}\int_X(-\varphi_t)\omega_{\varphi_t}^n \le C$, $\forall t \in (0,1]$. We may assume without loss of generality that $\chi \le \omega_X$, so the ω_t-plurisubharmonic function φ_t is also $2\omega_X$-plurisubharmonic and by the α-invariant estimate, there is an $\alpha_0 = \alpha_0(X,\omega_X) > 0$ such that

$$\frac{1}{V_t}\int_X \exp\Big(-\log\frac{\omega_{\varphi_t}^n}{\omega_X^n} - \alpha_0\varphi_t\Big)\omega_{\varphi_t}^n = \frac{1}{V_t}\int_X e^{-\alpha_0\varphi_t}\omega_X^n \le \frac{C(n,\omega_X)}{V_t}$$

By Jensen's inequality it follows that

$$\frac{1}{V_t}\int_X \Big(-\log\frac{\omega_{\varphi_t}^n}{\omega_X^n} - \alpha_0\varphi_t\Big)\omega_{\varphi_t}^n \le \log C - \log V_t$$

which implies that

$$\begin{aligned}
\frac{1}{V_t}\int_X(-\alpha_0\varphi_t)\omega_{\varphi_t}^n &\le \frac{1}{V_t}\int_X \log\,(e^{nF_t}c_t^n)\omega_{\varphi_t}^n + \log C - \log V_t \\
&= \int_X (nF_t)e^{nF_t}\frac{c_t^n}{V_t}\omega_X^n + \log C + \log\frac{c_t^n}{V_t} \\
&\le C\|e^{nF_t}\|_{L^1(\log L)^1(\omega_X)} + C,
\end{aligned}$$

from which the estimate follows since c_t^n and V_t are uniformly equivalent by Lemma 5.

Theorem 2 together with the preceding lemmas implies at once the following basic estimates of Kolodziej [13], Eyssidieux, Guedj, and Zeriahi [11], and Demailly and Pali [8]:

Theorem 4 *Consider the above family (4.3) of complex Monge-Ampère equations, with respect to the degenerating background metrics ω_t, $t \in (0,1]$. Fix any $q > n$. If φ_t is a family of C^2 solution, normalized by $\sup_X \varphi_t = 0$, and if $\|e^{nF_t}\|_{L^1(\log L)^q(\omega_X^n)}$ is uniformly bounded in t, then $\|\varphi_t\|_{L^\infty(X)}$ is uniformly bounded in t as well.*

In many applications, the $(1,1)$-form χ is chosen to be $\chi = \pi^*\omega_Y$, where $\pi : X \to Y$ is a holomorphic map between Kähler manifolds and ω_Y is a Kähler metric on Y.

5 Fully non-linear and Hessian equations

In this section, we consider applications of Theorem 2 to fully non-linear equations besides Monge-Ampère equations. For these applications, we need to consider the relative volumes $c_t^n V_t^{-1}$ and the energies $E_t(\varphi_t)$. We begin with a simple estimate for $E_t(\varphi_t)$, which generalizes the simple considerations which applied earlier to Monge-Ampère equations and is a straightforward application of the Hölder inequality,

Lemma 7 *Consider the energy $E_t(\varphi_t)$ in the formalism for degenerating background metrics as in the set-up (1.7). Then we have for any $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$,*

$$E_t(\varphi_t) \leq \frac{c_t^n}{V_t} \|e^{nF_t}\|_{L^q} \|\varphi_t\|_{L^p}. \tag{5.1}$$

Next we note the following uniform L^p estimate for general functions u, whose Hessian is in a cone $\Gamma_k = \{\lambda; \sigma_\ell > 0,\ 1 \leq \ell \leq k\}$, which is an analogue of the α-invariant estimate for plurisubharmonic functions. It is well-known to experts, but we supply a statement and proof without pluripotential theory, as we could not find a convenient reference:

Lemma 8 *For any $p \in (0, \frac{n}{n-k})$, there is a uniform constant $C = C(n, p, \omega_X) > 0$ such that*

$$\|u\|_{L^p(\omega_X^n)} \leq C,$$

with $\sup_X u = 0$ and $\lambda[\omega_u] \in \Gamma_k$, $\omega_u = \omega_X + i\partial\bar{\partial}u$.

Proof. We use an idea in [10]. Without loss of generality we may assume $\mathrm{Vol}(X, \omega_X) = 1$.

Fix an $s > 0$ and a small $\epsilon > 0$. Let $K = \{u \leq -s\} \subset X$ be compact sub-level set of u. We choose a sequence of smooth positive functions η_j with $\int_X \eta_j \omega_X^n = 1$ which converge to $\eta_\infty := aV_K^{2\epsilon-1}\chi_K + a \cdot \chi_{X\setminus K}$ in $L^{1+\epsilon}(\omega_X^n)$ and also pointwise, where $V_K = \int_K \omega_X^n$ is the volume of the set K and $a > 0$ is a constant such that

$$\int_X \eta_\infty \omega_X^n = \int_X (aV_K^{2\epsilon-1}\chi_K + a \cdot \chi_{X\setminus K})\omega_X^n = aV_K^{2\epsilon} + a\mathrm{Vol(X\setminus K)} = 1.$$

It is not hard to see that $1/2 \leq a \leq \max(2, 2^{2\epsilon}) = 2$. Hence

$$\int_X \eta_\infty^{1+\epsilon} \omega_X^n = a^{1+\epsilon} V_K^{\epsilon+2\epsilon^2} + a^{1+\epsilon}\mathrm{Vol(X\setminus K)} \leq 4.$$

Thus we may assume $\|\eta_j\|_{L^{1+\epsilon}(\omega_X)} \leq 5$ for j large enough. We solve the complex Monge-Ampère equations

$$(\omega_X + i\partial\bar{\partial}v_j)^n = \eta_j \omega_X^n, \quad \sup_X v_j = 0.$$

By Theorem 1 (or [13]), it holds that $\|v_j\|_{L^\infty} \leq C_0$ for a uniform $C_0 = C_0(n, \omega_X, \epsilon)$.

By integration by parts we have

$$\begin{aligned} & \int_X (-u)(\omega_X + i\partial\bar\partial v_j)^k \wedge \omega_X^{n-k} \qquad (5.2)\\ = & \int_X (-u)\omega_X \wedge \omega_{v_j}^{k-1} \wedge \omega_X^{n-k} + (-v_j)(\omega_u - \omega_X) \wedge \omega_{v_j}^{k-1} \wedge \omega_X^{n-k} \\ \leq & \int_X (-u)\omega_X \wedge \omega_{v_j}^{k-1} \wedge \omega_X^{n-k} + C_0 \int_X \omega_u \wedge \omega_{v_j}^{k-1} \wedge \omega_X^{n-k} \\ = & \int_X (-u)\omega_{v_j}^{k-1} \wedge \omega_X^{n-k+1} + C_0. \end{aligned}$$

Applying (5.2) inductively we get

$$\int_X (-u)(\omega_X + i\partial\bar\partial v_j)^k \wedge \omega_X^{n-k} \leq \int_X (-u)\omega_X^n + kC_0 \leq C, \qquad (5.3)$$

for some uniform constant $C = C(n, k, \omega_X, \epsilon) > 0$. On the other hand, by Newton-Maclaurin inequality that $(\frac{\omega_{v_j}^k \wedge \omega_X^{n-k}}{\omega_X^n})^{1/k} \geq c(n,k)(\frac{\omega_{v_j}^n}{\omega_X^n})^{1/n}$ we derive from (5.3) that

$$\int_X (-u)(\eta_j)^{k/n}\omega_X^n \leq C(n, k, \omega_X, \epsilon).$$

Letting $j \to \infty$ and applying Fatou's lemma we get

$$\int_K (-u)V_K^{(2\epsilon-1)k/n}\omega_X^n \leq C(n, k, \omega_X, \epsilon)$$

from which we obtain that

$$V_K = \mathrm{Vol}(\{u \leq -s\}) \leq C(n, k, \omega_X, \epsilon)s^{-\frac{n}{(2\epsilon-1)k+n}}.$$

For any $p < \frac{n}{n-k}$, we have

$$\int_X (-u)^p \omega_X^n \leq 1 + pC(n, k, \omega_X, \epsilon)\int_1^\infty s^{p-1-\frac{n}{(2\epsilon-1)k+n}} ds \leq C(n, k, \omega_X, p)$$

if $\epsilon = \epsilon(p) > 0$ is chosen small enough so that the integral above is integrable. The proof of Lemma 8 is complete.

Returning to the applications of Theorem 2, we observe that the condition that $p < \frac{n}{n-k}$ is equivalent to the dual exponent q satisfying $q > \frac{n}{k}$. Thus Theorem 2 combined with Lemmas 7 and 8 imply at once:

Theorem 5 *Consider the family of fully non-linear equations (1.7) with respect to the degenerating background metrics ω_t, $t \in (0, 1]$. Assume that we have solutions $\varphi_t \in C^2$, normalized by $\sup_X \varphi_t = 0$. Assume that $\lambda[h_{t,\varphi_t}] \in \Gamma_k$, for some fixed k, $1 \leq k \leq n$. Fix $q > n/k$. Then $\|\varphi_t\|_{L^\infty}$ is uniformly bounded by a constant C depending only on n, k, q ω_X, χ $\|e^{nF_t}\|_{L^q(\omega_X^n)}$ and $\frac{c_t^n}{V_t}$.*

We illustrate this theorem by specializing now to the case of Hessian equations, where $f(\lambda) = \sigma_k(\lambda)^{\frac{1}{k}}$ for some $1 \leq k \leq n$. The more familiar form of this equation is

$$(\omega_t + i\partial\bar{\partial}\varphi_t)^k \wedge \omega_X^{n-k} = c_t^k e^{kF_t}\omega_X^n, \quad \sup_X \varphi_t = 0, \tag{5.4}$$

and the condition $\lambda[\omega_{t,\varphi_t}] \in \Gamma_k$ is part of the equation[2]. Thus Theorem 5 applies and, assuming uniform bounds for $\|e^{nF}\|_{L^q}$ for some $q > n/k$, it reduces the uniform estimates for φ_t to a uniform estimate for the relative volumes c_t^n/V_t. An important geometric case when the relative volumes can be controlled is when the classe χ is big, in the sense that its volume $[\chi^n] = \int_X \chi^n$ is strictly positive. In this case, we obtain

Theorem 6 *Fix $1 \leq k < n$, and consider the family (5.4) of Hessian equations with respect to the degenerating background metrics. Assume that χ is big. Then for any $q > \frac{n}{k}$, $\|\varphi\|_{L^\infty}$ is bounded uniformly by a constant C depending only on n, k, q, ω_X, χ and an upper bound for $\|e^{nF_t}\|_{L^q(\omega_X^n)}$.*

Proof of Theorem 6. In view of Theorem 5, it suffices to show that $c_t^n V_t^{-1}$ is uniformly bounded. Since $V_t \geq [\chi^n]$ for any t, this reduces to showing that c_t are themselves uniformly bounded.

The factors c_t are determined by

$$c_t^k \int_X e^{kF_t}\omega_X^n = \int_X \omega_t^k \wedge \omega_X^{n-k} = O(1). \tag{5.5}$$

To estimate c_t, we still need a uniform lower bound of $\int_X e^{kF_t}\omega_X^n$. We use Hölder's inequality as before. Thus we write

$$\int_X e^{kF_t}\omega_X^n \geq \int_{\{F_t \leq 0\}} e^{nF_t}\omega_X^n + \int_{\{F_t > 0\}} \omega_X^n. \tag{5.6}$$

Recall that by our normalization on F_t

$$V = \int_X e^{nF_t}\omega_X^n = \int_{\{F_t \leq 0\}} e^{nF_t}\omega_X^n + \int_{\{F_t > 0\}} e^{nF_t}\omega_X^n. \tag{5.7}$$

If the first term on the right hand side of (5.7) is greater than $V/2$, then (5.6) shows that $\int_X e^{kF_t}\omega_X^n \geq V/2$; otherwise the second term in (5.7) is greater than $V/2$. Then we have by the Hölder inequality

$$\int_{\{F_t > 0\}} e^{nF_t}\omega_X^n \leq \Big(\int_X e^{qnF_t}\Big)^{1/q}\Big(\int_{\{F_t > 0\}} \omega_X^n\Big)^{\frac{q-1}{q}}$$

which yields a uniform lower bound of the second term in (5.6) depending additionally on the assumed $\|e^{nF_t}\|_{L^q}$. Therefore we conclude from (5.5) that $c_t \leq C$ for a uniform constant $C > 0$. The proof of Theorem 6 is complete.

[2]We remark that usually in the equation (5.4), one normalizes the function F_t such that $\int_X e^{kF_t}\omega_X^n = V$. However, our normalization is that $\int_X e^{nF_t}\omega_X^n = V$.

6 Trudinger Inequalities

In this section, we illustrate the versatility of our approach by establishing also inequalities of Trudinger type for general non-linear energies. Let $f(\lambda)$ be a fully nonlinear operator satisfying the same structural conditions as in Theorem 1, and define for each $p > 0$,

$$E_p(\varphi) = \frac{1}{V}\int_X (-\varphi)^p f^n(\lambda[h_\varphi])\omega_X^n \tag{6.1}$$

(this notation is slightly different from the notation E_t used earlier for degenerating background metrics, but there should be no confusion, as the background metric is here fixed, and the index t can be dropped). Recall that we had established before integral estimates for φ in terms of the entropy Ent_p. Trudinger estimates are also exponential estimates, but in terms of the energy E_p. We have

Theorem 7 *Let φ be a plurisubharmonic function such that $\sup\varphi = -1$. Then*

$$\int_X \exp\Big\{c(n,p,\gamma)\alpha\Big(\frac{-\varphi}{E_p(\varphi)^{\frac{1}{n+p}}}\Big)^{\frac{n+p}{n}}\Big\}\omega_X^n \leq 2C_\alpha \tag{6.2}$$

where α and C_α are the constants coming from the α-invariant estimate of (X,ω_X).

We note that when specialized to the case when $f(\lambda)$ is the Monge-Ampère operator $f(\lambda) = (\prod_{k=1}^n \lambda_k)^{\frac{1}{n}}$, our theorem recovers an inequality proved in [1, 9]. Moreover, our estimate has the major advantage that all constants there depend only on the α-invariant of the underlying manifold, hence is uniform over degenerating families with uniform α-invariants.

Proof of Theorem 7: Let us solve the following auxiliary Monge-Ampère equation with $\sup\psi = 0$, which is solvable due to Yau's theorem [24].

$$(\omega_X + i\partial\bar\partial\psi)^n = \frac{(-\varphi)^p e^{nF}}{E_p(\varphi)}\omega_X^n \tag{6.3}$$

then by the same maximum argument as in Lemma 1, we obtain the inequality

$$c(n,p,\gamma)\Big(\frac{-\varphi}{E_p(\varphi)^{1/(n+p)}}\Big)^{\frac{n+p}{n}} \leq -\psi + C(n,p,\gamma)E_p(\varphi)^{\frac{1}{p}} \tag{6.4}$$

Now we pick $\kappa = 2^{\frac{n}{n+p}}C^{\frac{n}{n+p}}c^{-\frac{n}{n+p}}$, where C and c are constants in estimate above, which only depends on n, p and γ. Now set $U_\kappa = \{-\varphi \leq \kappa E_p(\varphi)^{\frac{1}{p}}\}$. Then on $X \setminus U_\kappa$, we have by our choice of κ,

$$\frac{1}{2}c(n,p,\gamma)\Big(\frac{-\varphi}{E_p(\varphi)^{1/(n+p)}}\Big)^{\frac{n+p}{n}} \leq -\psi \tag{6.5}$$

and on U_κ, we have

$$c(n,p,\gamma)\Big(\frac{-\varphi}{E_p(\varphi)^{1/(n+p)}}\Big)^{\frac{n+p}{n}} \leq -c(n,p,\gamma)\kappa^{\frac{p}{n}}\varphi \tag{6.6}$$

now multiplying by $\min(c^{-1}\kappa^{-\frac{p}{n}}, 1/2)\alpha$ and integrating, we get

$$\int_X \exp\bigg\{c'(n,p,\gamma)\alpha\Big(\frac{-\varphi}{E_p(\varphi)^{1/(n+p)}}\Big)^{\frac{n+p}{n}}\bigg\}\omega_X^n \leq \int_{U_\kappa} e^{-\alpha\psi}\omega_X^n + \int_{X\setminus U_\kappa} e^{-\alpha\varphi}\omega_X^n \leq 2C_\alpha \tag{6.7}$$

which is the desired result.

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