

# On $L^\infty$ estimates for Monge-Ampère and Hessian equations on nef classes<sup>1</sup>

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## Abstract

The PDE approach developed earlier by the first three authors for  $L^\infty$  estimates for fully non-linear equations on Kähler manifolds is shown to apply as well to Monge-Ampère and Hessian equations on nef classes. In particular, one obtains a new proof of the estimates of Boucksom-Eyssidieux-Guedj-Zeriahi and Fu-Guo-Song for the Monge-Ampère equation, together with their generalization to Hessian equations.

## 1 Introduction

The goal of this short note is to show that the PDE approach introduced in [12, 13] for  $L^\infty$  and Trudinger-type estimates for general classes fully non-linear equations on a compact Kähler manifold applies as well to Monge-Ampère and Hessian equations on nef classes.

The key to the approach in [12, 13] is an estimate of Trudinger-type, obtained by comparing the solution  $\varphi$  of the given equation to the solution of an auxiliary Monge-Ampère equation with the energy of the sublevel set function  $-\varphi + s$  on the right hand side. We shall see that, in the present case of nef classes, the argument can still be made to work by replacing  $\varphi$  by  $\varphi - V$ , where  $V$  is the envelope of the nef class. Applied to the Monge-Ampère equation, this gives a PDE proof of the estimates obtained earlier for nef classes by Boucksom-Eyssidieux-Guedj-Zeriahi [2] and Fu-Guo-Song [9]. The estimates which we obtain with this method applied to Hessian equations seem new.

We note that the use of an auxiliary Monge-Ampère equation had been instrumental in the recent progress of Chen and Cheng [3] on the constant scalar curvature Kähler metrics problem. There the auxiliary equation involved the entropy, and not the energy of sublevel set functions as in our case. More generally, auxiliary equations have often been used in the theory of partial differential equations, notably by De Giorgi [5], and more recently by Dinew and Kolodziej [7, 6] in their approach to Hölder estimates for the complex Monge-Ampère equation.

## 2 The Monge-Ampère equation

We begin with the Monge-Ampère equation. Let  $(X, \omega)$  be a compact Kähler manifold and  $\chi$  be a closed  $(1, 1)$ -form on  $X$ . We assume the cohomology class  $[\chi]$  is nef and let

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$\nu \in \{0, 1, \dots, n\}$  be the numerical dimension of  $[\chi]$ , i.e.

$$\nu = \max\{k \mid [\chi]^k \neq 0 \text{ in } H^{k,k}(X, \mathbf{C})\}.$$

When  $\nu = n$  we say the class  $[\chi]$  is *big*.

Let  $\hat{\omega}_t = \chi + t\omega$  for  $t \in (0, 1]$ . The form  $\hat{\omega}_t$  may not be positive but its class is Kähler. We consider the following family of complex Monge-Ampère equations

$$(\hat{\omega}_t + i\partial\bar{\partial}\varphi_t)^n = c_t e^F \omega^n, \quad \sup_X \varphi_t = 0 \quad (2.1)$$

where  $c_t = [\hat{\omega}_t]^n = O(t^{n-\nu})$  is a normalizing constant and  $F \in C^\infty(X)$  satisfies  $\int_X e^F \omega^n = \int_X \omega^n$ . This equation admits a unique smooth solution  $\varphi_t$  by Yau's theorem [17].

The form  $\chi$  is not assumed to be semipositive, so the usual  $L^\infty$  estimate of  $\varphi_t$  may not hold [15]. As in [2, 9], we need to modify the solution  $\varphi_t$  by an envelope  $V_t$  of the class  $[\hat{\omega}_t]$ , defined as follows,

$$V_t = \sup\{v \mid v \in PSH(X, \hat{\omega}_t), v \leq 0\}.$$

Then we have:

**Theorem 1** *Consider the equation (2.1), and assume that the cohomology class of  $\chi$  is nef. For any  $s > 0$ , let  $\Omega_s = \{\varphi_t - V_t \leq -s\}$  be the sub-level set of  $\varphi_t - V_t$ .*

(a) *Then there are constants  $C = C(n, \omega, \chi) > 0$  and  $\alpha_0 = \alpha_0(n, \omega, \chi) > 0$  such that*

$$\int_{\Omega_s} \exp\left\{\alpha_0\left(\frac{-(\varphi_t - V_t + s)}{A_s^{1/(1+n)}}\right)^{\frac{n+1}{n}}\right\} \omega^n \leq C \exp(CE_t), \quad (2.2)$$

where  $A_s = \int_{\Omega_s} (-\varphi_t + V_t - s) e^F \omega^n$  and  $E_t = \int_X (-\varphi_t + V_t) e^F \omega^n$ .

(b) *Fix  $p > n$ . There is a constant  $C(n, p, \omega, \chi, \|e^F\|_{L^1(\log L)^p})$  so that for all  $t \in (0, 1]$ , we have*

$$0 \leq -\varphi_t + V_t \leq C(n, p, \omega, \chi, \|e^F\|_{L^1(\log L)^p}). \quad (2.3)$$

We remark that the estimates in Theorem 1 continue to hold for a family of Kähler metrics (maybe with distinct complex structures) which satisfy a uniform  $\alpha$ -invariant type estimate.

*Proof of Theorem 1*

We would like to find an auxiliary equation with smooth coefficients, so that its solvability can be guaranteed by Yau's theorem. For this, we need a lemma due to Berman [1] on a smooth approximation for  $V_t$  (see also Lemma 2 below). Fix a time  $t \in (0, 1]$ .

**Lemma 1** *Let  $u_\beta$  be the smooth solution to the complex Monge-Ampère equation*

$$(\hat{\omega}_t + i\partial\bar{\partial}u_\beta)^n = e^{\beta u_\beta} \omega^n.$$

*Then  $u_\beta$  converges uniformly to  $V_t$  as  $\beta \rightarrow \infty$ .*

We remark that by [4],  $V_t$  is a  $C^{1,1}$  function on  $X$ , although this fact is not used in this note. We now return to the proof of Part (a) of Theorem 1.

We choose a sequence of positive functions  $\tau_k : \mathbf{R} \rightarrow \mathbf{R}_+$  such that  $\tau_k(x)$  decreases to  $x \cdot \chi_{\mathbf{R}_+}(x)$  as  $k \rightarrow \infty$ . Fix a smooth function  $u_\beta$  as in Lemma 1. The  $u_\beta$  depends on  $t$ , but for simplicity we omit the subscript  $t$ . We solve the following auxiliary Monge-Ampère equation on  $X$

$$(\hat{\omega}_t + i\partial\bar{\partial}\psi_{t,k})^n = c_t \frac{\tau_k(-\varphi_t + u_\beta - s)}{A_{s,k,\beta}} e^F \omega^n, \quad \sup_X \psi_{t,k} = 0, \quad (2.4)$$

where  $A_{s,k,\beta} = \int_X \tau_k(-\varphi_t + u_\beta - s) e^F \omega^n$ . Since  $\psi_{t,k} \leq V_t$  and  $u_\beta$  converges uniformly to  $V_t$ . By taking  $\beta$  large enough, we may assume  $\psi_{t,k} < u_\beta + 1$ .

Define a function

$$\Phi = -\varepsilon(-\psi_{t,k} + u_\beta + 1 + \Lambda)^{n/(n+1)} - (\varphi_t - u_\beta + s),$$

with the constants

$$\varepsilon^{n+1} = A_{s,k,\beta} n^{-n} (n+1)^n, \quad \Lambda = n^{n+1} (n+1)^{-n-1} \varepsilon^{n+1}. \quad (2.5)$$

As a smooth function on the compact manifold  $X$ ,  $\Phi$  must achieve its maximum at some  $x_0 \in X$ . If  $x_0 \in X \setminus \Omega_s^\circ$ , then

$$\Phi(x_0) \leq -(\varphi_t - u_\beta + s) \leq -V_t + u_\beta \leq \epsilon_\beta$$

where  $\epsilon_\beta \rightarrow 0$  as  $\beta \rightarrow \infty$ . On the other hand, if  $x_0 \in \Omega_s^\circ$  we calculate ( $\Delta_t$  denotes the Laplacian with respect to the metric  $\omega_t = \hat{\omega}_t + i\partial\bar{\partial}\varphi_t$ )

$$\begin{aligned} 0 &\geq \Delta_t \Phi(x_0) \\ &= -\varepsilon \frac{n}{n+1} (-\psi_{t,k} + u_\beta + \Lambda + 1)^{-\frac{1}{n+1}} \text{tr}_{\omega_t} (-i\partial\bar{\partial}\psi_{t,k} + i\partial\bar{\partial}u_\beta) - \text{tr}_{\omega_t} (i\partial\bar{\partial}\varphi_t - i\partial\bar{\partial}u_\beta) \\ &\quad + \frac{n\varepsilon}{(n+1)^2} (-\psi_{t,k} + u_\beta + 1 + \Lambda)^{-\frac{n+2}{n+1}} \text{tr}_{\omega_t} i\partial(\psi_{t,k} - u_\beta) \wedge \bar{\partial}(\psi_{t,k} - u_\beta) \\ &\geq \frac{n\varepsilon}{n+1} (-\psi_{t,k} + u_\beta + \Lambda + 1)^{-\frac{1}{n+1}} \text{tr}_{\omega_t} (\hat{\omega}_{t,\psi_{t,k}} - \hat{\omega}_{t,u_\beta}) - n + \text{tr}_{\omega_t} \hat{\omega}_{t,u_\beta} \\ &\geq \frac{n\varepsilon}{n+1} (-\psi_{t,k} + u_\beta + \Lambda + 1)^{-\frac{1}{n+1}} n \left( \frac{\hat{\omega}_{t,\psi_{t,k}}^n}{\omega_t^n} \right)^{1/n} - n + \left( 1 - \frac{n\varepsilon}{n+1} (-\psi_{t,k} + u_\beta + \Lambda + 1)^{-\frac{1}{n+1}} \right) \text{tr}_{\omega_t} \hat{\omega}_{t,u_\beta} \\ &\geq \frac{n^2\varepsilon}{n+1} (-\psi_{t,k} + u_\beta + \Lambda + 1)^{-\frac{1}{n+1}} \left( \tau_k(-\varphi_t + u_\beta - s) A_{s,k,\beta}^{-1} \right)^{1/n} - n + \left( 1 - \frac{n\varepsilon}{n+1} \Lambda^{-\frac{1}{n+1}} \right) \text{tr}_{\omega_t} \hat{\omega}_{t,u_\beta} \\ &\geq \frac{n^2\varepsilon}{n+1} (-\psi_{t,k} + u_\beta + \Lambda + 1)^{-\frac{1}{n+1}} (-\varphi_t + u_\beta - s)^{1/n} A_{s,k,\beta}^{-1/n} - n. \end{aligned}$$

Therefore, at  $x_0 \in \Omega_s^\circ$

$$-(\varphi_t - u_\beta + s) \leq \left( \frac{n\varepsilon}{n+1} \right)^n A_{s,k,\beta} (-\psi_{t,k} + u_\beta + \Lambda + 1)^{\frac{n}{n+1}} = \varepsilon (-\psi_{t,k} + u_\beta + \Lambda + 1)^{\frac{n}{n+1}},$$

i.e.  $\Phi(x_0) \leq 0$ . Combining the two cases, we conclude that  $\sup_X \Phi \leq \epsilon_\beta \rightarrow 0$  as  $\beta \rightarrow \infty$ . It then follows that on  $\Omega_s$

$$(-\varphi_t + u_\beta - s)^{\frac{n+1}{n}} \leq C_n A_{s,k,\beta}^{1/n} (-\psi_{t,k} + u_\beta + 1 + A_{s,k,\beta}) + \epsilon_\beta^{(n+1)/n}$$

Letting  $\beta \rightarrow \infty$  we have

$$(-\varphi_t + V_t - s)^{\frac{n+1}{n}} \leq C_n A_{s,k}^{1/n} (-\psi_{t,k} + V_t + 1 + A_{s,k}),$$

where  $A_{s,k} = \int_X \tau_k(-\varphi_t + V_t + s) e^F \omega^n$ . Observe that by definition  $V_t \leq 0$  and by the  $\alpha$ -invariant estimate [14, 16], there exists an  $\alpha_0(n, \omega, \chi)$  such that

$$\int_{\Omega_s} \exp\left(\alpha_0 \frac{(-\varphi_t + V_t - s)^{\frac{n+1}{n}}}{A_{s,k}^{1/n}}\right) \omega^n \leq \int_{\Omega_s} \exp\left(\alpha_0 C_n (-\psi_{t,k} + 1 + A_{s,k})\right) \omega^n \leq C e^{C A_{s,k}}. \quad (2.6)$$

Letting  $k \rightarrow \infty$ , we obtain

$$\int_{\Omega_s} \exp\left(\alpha_0 \frac{(-\varphi_t + V_t - s)^{\frac{n+1}{n}}}{A_s^{1/n}}\right) \omega^n \leq C e^{C A_s}.$$

Part (a) of Theorem 1 is proved by noting that  $A_s \leq E_t$  for any  $s > 0$ .

Once Part (a) of Theorem 1 has been proved, Part (b) can be proved by following closely the arguments in [12].

Fix  $p > n$ , and define  $\eta : \mathbf{R}_+ \rightarrow \mathbf{R}_+$  by  $\eta(x) = (\log(1+x))^p$ . Note that  $\eta$  is a strictly increasing function with  $\eta(0) = 0$ , and let  $\eta^{-1}$  be its inverse function. Denote

$$v := \frac{\alpha_0}{2} \left( \frac{-\varphi_t + V_t - s}{A_s^{1/(n+1)}} \right)^{(n+1)/n} \quad (2.7)$$

then by the generalized Young's inequality with respect to  $\eta$ , for any  $z \in \Omega_s$ ,

$$\begin{aligned} v(z)^p e^{F(z)} &\leq \int_0^{\exp(F(z))} \eta(x) dx + \int_0^{v(z)^p} \eta^{-1}(y) dy \\ &\leq \exp(F(z))(1 + |F(z)|)^p + C(p) \exp(2v(z)) \end{aligned}$$

We integrate both sides in the inequality above over  $z \in \Omega_s$ , and get by Part (a), Theorem 1 that

$$\begin{aligned} \int_{\Omega_s} v(z)^p e^{F(z)} \omega^n &\leq \int_{\Omega_s} e^F (1 + |F(z)|)^p \omega^n + \int_{\Omega_s} e^{2v(z)} \omega^n \\ &\leq \|e^F\|_{L^1(\log L)^p} + C + C e^{C E_t}, \end{aligned}$$

where the constant  $C > 0$  depends only on  $n, \omega_X, \chi$ . In view of the definition of  $v$ , this implies

$$\int_{\Omega_s} (-\varphi_t + V_t - s)^{\frac{(n+1)p}{n}} e^{F(z)} \omega^n \leq 2^p \alpha_0^{-p} A_s^{\frac{p}{n}} (\|e^F\|_{L^1(\log L)^p} + C + C e^{C E_t}). \quad (2.8)$$

From the definition of  $A_s$ , it follows from Hölder inequality that

$$\begin{aligned} A_s &= \int_{\Omega_s} (-\varphi_t + V_t - s) e^F \omega^n \\ &\leq \left( \int_{\Omega_s} (-\varphi_t + V_t - s)^{\frac{(n+1)p}{n}} e^F \omega^n \right)^{\frac{n}{(n+1)p}} \cdot \left( \int_{\Omega_s} e^F \omega^n \right)^{1/q} \\ &\leq A_s^{\frac{1}{n+1}} \left( 2^p \alpha_0^{-p} (\|e^F\|_{L^1(\log L)^p} + C + C e^{CE_t}) \right)^{\frac{n}{(n+1)p}} \cdot \left( \int_{\Omega_s} e^F \omega^n \right)^{1/q} \end{aligned}$$

where  $q > 1$  satisfies  $\frac{n}{p(n+1)} + \frac{1}{q} = 1$ , i.e.  $q = \frac{p(n+1)}{p(n+1)-n}$ . The inequality above yields

$$A_s \leq \left( 2^p \beta_0^{-p} (\|e^F\|_{L^1(\log L)^p} + C + C e^{CE_t}) \right)^{1/p} \cdot \left( \int_{\Omega_s} e^F \omega^n \right)^{\frac{1+n}{qn}}. \quad (2.9)$$

Observe that the exponent of the integral on the right hand of (2.9) satisfies

$$\frac{1+n}{qn} = \frac{pn + p - n}{pn} = 1 + \delta_0 > 1,$$

for  $\delta_0 := \frac{p-n}{pn} > 0$ . For notation convenience, set

$$B_0 := \left( 2^p \alpha_0^{-p} (\|e^F\|_{L^1(\log L)^p} + C + C e^{CE_t}) \right)^{1/p}. \quad (2.10)$$

From ((2.9)) we then get

$$A_s \leq B_0 \left( \int_{\Omega_s} e^F \omega^n \right)^{1+\delta_0}. \quad (2.11)$$

If we define  $\phi : \mathbf{R}_+ \rightarrow \mathbf{R}_+$  by  $\phi(s) := \int_{\Omega_s} e^F \omega^n$  then ((2.11)) and the definition of  $A_s$  imply that

$$r\phi(s+r) \leq B_0\phi(s)^{1+\delta_0}, \quad \forall r \in [0, 1] \text{ and } s \geq 0. \quad (2.12)$$

$\phi$  is clearly nonincreasing and continuous, so a De Giorgi type iteration argument shows that there is some  $S_\infty$  such that  $\phi(s) = 0$  for any  $s \geq S_\infty$ . This finishes the proof of the  $L^\infty$  estimate of  $\varphi_t - V_t$ , combining with a bound on  $E_t$  by  $\|e^F\|_{L^1(\log L)^1}$  which follows from Jensen's inequality (c.f. Lemma 6 in [12]). The proof of Theorem 1 is complete.

Finally, we note the recent advances in the theory of envelopes in [10] and [11], which can provide an approach to  $L^\infty$  estimates for Monge-Ampère equations on Hermitian manifolds.

### 3 Complex Hessian equations

We explain in this section how the proof of Theorem 1 can be modified to give a similar result for a degenerate family of complex Hessian equations. With the same notations as above, we consider the  $\sigma_k$ -equations

$$(\hat{\omega}_t + i\partial\bar{\partial}\varphi_t)^k \wedge \omega^{n-k} = c_t e^F \omega^n, \quad \sup_X \varphi_t = 0. \quad (3.1)$$

Define the envelope corresponding to the  $\Gamma_k$ -cone

$$\tilde{V}_{t,k} = \sup\{v \mid v \in SH_k(X, \omega, \hat{\omega}_t) \cap C^2, v \leq 0\}$$

where  $v \in SH_k(X, \omega, \hat{\omega}_t) \cap C^2$  means that the eigenvalue vector of the linear transformation  $\omega^{-1} \cdot (\hat{\omega}_t + i\partial\bar{\partial}v)$  lies in the  $\Gamma_k$ -cone.

Let

$$E_t(\varphi_t) = \int_X (-\varphi_t + \tilde{V}_{t,k}) e^{nF/k} \omega^n$$

be the entropy associated to the equation (3.1) as in [12] and let  $\bar{E}_t$  be an upper bound of  $E_t(\varphi_t)$ . Then the following  $L^\infty$ -estimate holds for the solution  $\varphi_t$  to (3.1).

**Theorem 2** *Let  $\varphi_t$  be the solution to (3.1), then there exists a constant depending on  $\bar{E}_t$ ,  $\|e^{\frac{n}{k}F}\|_{L^1(\log L)^p}$ ,  $\frac{c_t}{[\hat{\omega}_t]^k [\omega]^{n-k}}$ ,  $p > n$  such that*

$$0 \leq -\varphi_t + \tilde{V}_{t,k} \leq C.$$

This theorem can be derived using a similar argument as in Section 2 with suitable modifications for  $\sigma_k$  equations, c.f. [12], so we omit the details. The only novel ingredient is the smooth approximation of  $\tilde{V}_{t,k}$ , as in Lemma 1. One can adapt the method in [1] to derive this required approximation. For the convenience of the reader, we present a sketch of the proof.

**Lemma 2** *Fix  $t \in (0, 1]$ . There exists a sequence of smooth functions  $u_\beta \in SH_k(X, \omega, \hat{\omega}_t)$  converging uniformly to  $\tilde{V}_{t,k}$  as  $\beta \rightarrow \infty$ .*

*Proof.* Let  $u_\beta \in SH_k(X, \omega, \hat{\omega}_t)$  be the solution to the  $\sigma_k$ -equations

$$(\hat{\omega}_t + i\partial\bar{\partial}u_\beta)^k \wedge \omega^{n-k} = c_t e^{\beta u_\beta} \omega^n, \quad (3.2)$$

which admits a unique smooth solution by [8]. We claim that there is a constant  $C_t > 0$  such that

$$\sup_X |u_\beta - \tilde{V}_{t,k}| \leq \frac{C_t \log \beta}{\beta},$$

from which the lemma follows.

By the maximum principle, at a maximum point of  $u_\beta$ ,  $i\partial\bar{\partial}u_\beta \leq 0$ , so  $\beta u_\beta \leq \log \frac{\hat{\omega}_t^k \wedge \omega^{n-k}}{c_t \omega^n} \leq C_t$ , that is  $u_\beta - \frac{C_t}{\beta} \leq 0$ . By the definition of  $\tilde{V}_{t,k}$ , it follows that

$$u_\beta - \frac{C_t}{\beta} \leq \tilde{V}_{t,k}. \quad (3.3)$$

On the other hand, we fix a smooth  $u \leq 0$  such that  $\hat{\omega}_t + i\partial\bar{\partial}u > 0$ . Such a  $u$  exists because  $[\hat{\omega}_t]$  is a Kähler class by assumption. For any  $v \in SH_k(X, \omega, \hat{\omega}_t) \cap C^2$  with  $v \leq 0$ , we consider the barrier function

$$\tilde{u} = \frac{1}{\beta}u + (1 - \frac{1}{\beta})v - \frac{C'_t \log \beta}{\beta}$$

where  $C'_t > 0$  is a large constant to be determined. By direct calculation, we have

$$(\hat{\omega}_t + i\partial\bar{\partial}\tilde{u})^k \wedge \omega^{n-k} \geq \frac{1}{\beta^n} (\hat{\omega}_t + i\partial\bar{\partial}u)^k \wedge \omega^{n-k} \geq e^{\beta\tilde{u}} \omega^n$$

where the last inequality holds if we choose  $C'_t$  large enough so that

$$e^{-C'_t \log \beta} \leq \frac{1}{\beta^k} \min_X \frac{(\hat{\omega}_t + i\partial\bar{\partial}u)^k \wedge \omega^{n-k}}{\omega^n}.$$

Therefore we get

$$(\hat{\omega}_t + i\partial\bar{\partial}\tilde{u})^k \wedge \omega^{n-k} \geq e^{\beta(\tilde{u} - u_\beta)} (\hat{\omega}_t + i\partial\bar{\partial}u_\beta)^k \wedge \omega^{n-k}.$$

At the maximum point of  $\tilde{u} - u_\beta$ ,  $(\hat{\omega}_t + i\partial\bar{\partial}\tilde{u})^k \wedge \omega^{n-k} \leq (\hat{\omega}_t + i\partial\bar{\partial}u_\beta)^k \wedge \omega^{n-k}$ . This shows that  $\tilde{u} - u_\beta \leq 0$  on  $X$ . Taking supremum over all such  $v$ 's in  $\tilde{u}$ , it follows that

$$(1 - \frac{1}{\beta}) \tilde{V}_{t,k} \leq u_\beta + \frac{C_t \log \beta}{\beta}.$$

The lemma follows from this and (3.3).

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