Physica D 450 (2023) 133766

Contents lists available at ScienceDirect

Physica D

journal homepage: www.elsevier.com/locate/physd

Symmetric Toda, gradient flows, and tridiagonalization

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ARTICLE INFO

Article history:
Received 31 January 2023
Received in revised form 19 April 2023
Accepted 20 April 2023
Available online 29 April 2023

In honor of Hermann Flaschka

Keywords:
Toda flows
Integrability
Gradient flows

ABSTRACT

The Toda lattice (1967) is a Hamiltonian system given by n points on a line governed by an exponential potential. Flaschka (1974) showed that the Toda lattice is integrable by interpreting it as a flow on the space of symmetric tridiagonal $n \times n$ matrices, while Moser (1975) showed that it is a gradient flow on a projective space. The symmetric Toda flow of Deift, Li, Nanda, and Tomei (1986) generalizes the Toda lattice flow from tridiagonal to all symmetric matrices. They showed the flow is integrable, in the classical sense of having d integrals in involution on its $2d$-dimensional phase space. The system may be viewed as integrable in other ways as well. Firstly, Symes (1980, 1982) solved it explicitly via QR-factorization and conjugation. Secondly, Deift, Li, Nanda, and Tomei (1986) 'tridiagonalized' the system into a family of tridiagonal Toda lattices which are solvable and integrable. In this paper we derive their tridiagonalization procedure in a natural way using the fact that the symmetric Toda flow is diffeomorphic to a twisted gradient flow on a flag variety, which may then be decomposed into flows on a product of Grassmannians. These flows may in turn be embedded into projective spaces via Plücker embeddings, and mapped back to tridiagonal Toda lattice flows using Moser's construction. In addition, we study the tridiagonalized flows projected onto a product of permutohedra, using the twisted moment map of Bloch, Flaschka, and Ratiu (1990). These ideas are facilitated in a natural way by the theory of total positivity, building on our previous work (2023).
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1. Introduction

This paper concerns the symmetric Toda flows and their connections with the classical Toda lattice, gradient flows on adjoint orbits, and flows on moment polytopes. The (finite nonperiodic) *Toda lattice* [1] (cf. [2]) is a Hamiltonian system of n points on a line of unit mass governed by an exponential potential, with Hamiltonian

$$\frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n-1} e^{q_i - q_{i+1}}.$$

Following Flaschka [3], we make the change of variables

$$a_i := \tfrac{1}{2}e^{\frac{q_i - q_{i+1}}{2}} \text{ for } 1 \le i \le n-1 \quad \text{and} \quad b_i := -\frac{1}{2}p_i \text{ for } 1 \le i \le n,$$

which we arrange into the symmetric tridiagonal matrix

$$M := \begin{bmatrix} b_1 & a_1 & 0 & \cdots & 0 \\ a_1 & b_2 & a_2 & \cdots & 0 \\ 0 & a_2 & b_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_n \end{bmatrix}; \quad \text{we also let}$$

$$\pi_{\mathrm{u}}(M) := \begin{bmatrix} 0 & -a_1 & 0 & \cdots & 0 \\ a_1 & 0 & -a_2 & \cdots & 0 \\ 0 & a_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

be the skew-symmetric part of M. Then the Hamiltonian equations can be written in Lax form as

$$\dot{M} = [M, \pi_{\mathrm{u}}(M)], \tag{1.1}$$

where $\dot{M}$ denotes the derivative of M with respect to time t. Then the eigenvalues of M are preserved along the flow, and allow one to define n integrals in involution [4–6], showing that the Toda lattice is integrable.

Moreover, Moser [4] expressed the Toda lattice flow as a gradient flow on a projective space. Namely, let $\lambda_1 > \cdots > \lambda_n$ denote the eigenvalues of M (they are necessarily distinct), and

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1 A.M.B. was partially supported by NSF grants DMS-1613819 and DMS-2103026, and AFOSR, United States of America grant FA 9550-22-1-0215.

https://doi.org/10.1016/j.physd.2023.133766
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for $1 \leq i \leq n$, let $v^{(i)} \in \mathbb{R}^n$ be the eigenvector of M with eigenvalue λ_i such that $\|v^{(i)}\| = 1$ and its first entry $v_1^{(i)}$ is positive ($v_1^{(i)}$ is necessarily nonzero). Setting $x := (v_1^{(1)} : \cdots : v_1^{(n)}) \in \mathbb{P}^{n-1}(\mathbb{R})$, we can write the Hamiltonian equations (1.1) as

$$\dot{x}_i = \lambda_i x_i \quad \text{for all } 1 \leq i \leq n. \tag{1.2}$$

Conversely, given $x \in \mathbb{P}^{n-1}(\mathbb{R})$ with positive entries, Moser constructed a unique corresponding symmetric tridiagonal matrix M with eigenvalues $\lambda_1 > \cdots > \lambda_n$ and positive entries immediately above and below the diagonal, giving a diffeomorphism

$$x \mapsto M. \tag{1.3}$$

More generally, Deift, Li, Nanda, and Tomei [7] considered the flow (1.1) for arbitrary (not necessarily tridiagonal) $n \times n$ symmetric matrices M, called the *(full) symmetric Toda flow*. They constructed $\lfloor \frac{n^2}{4} \rfloor$ integrals in involution, showing that the flow is integrable.

This system may be seen to be integrable in other ways as well. Firstly, Symes [5,6] found an explicit solution using factorization and conjugation. Namely, given an $n \times n$ matrix g, let $\pi_{\mathrm{U}}(g)$ denote the unitary matrix obtained by applying Gram–Schmidt orthonormalization to the columns of g; equivalently, $\pi_{\mathrm{U}}(g)$ is the Q-term in the QR-factorization of g. Then if $M(t)$ denotes the solution to (1.1) beginning at the symmetric matrix M_0, we have

$$M(t) = \pi_{\mathrm{U}}(\exp(tM_0))^{-1} M_0\, \pi_{\mathrm{U}}(\exp(tM_0)) \quad \text{for all } t \in \mathbb{R}.$$

Secondly, Deift, Li, Nanda, and Tomei [7, Section 7] 'tridiagonalized' the symmetric Toda flow into a family of $n-1$ Toda lattices which are solvable and integrable. The main goal of this paper is to derive this tridiagonalization procedure in a natural and geometric way.

In order to state our results, we introduce some notation. Let U_n denote the group of $n \times n$ unitary matrices, and let $\mathfrak{u}_n$ denote its Lie algebra, consisting of all $n \times n$ skew-Hermitian matrices. Given a symmetric (or more generally, Hermitian) $n \times n$ matrix M, we associate to it the skew-Hermitian matrix $L := \mathrm{i}M$ (where $\mathrm{i} = \sqrt{-1}$). If M has eigenvalues λ, then L lies in the *adjoint orbit* $\mathcal{O}_\lambda$ of $\mathfrak{u}_n$ consisting of all skew-Hermitian matrices with eigenvalues $\mathrm{i}\lambda_1, \dots, \mathrm{i}\lambda_n$. Then we can write the symmetric Toda flow as the flow

$$\dot{L} = [L, \pi_{\mathfrak{u}}(-\mathrm{i}L)] \quad \text{on } \mathcal{O}_\lambda. \tag{1.4}$$

Above, the projection $\pi_{\mathfrak{u}}(\cdot)$ onto $\mathfrak{u}_n$ is defined such that $N - \pi_{\mathfrak{u}}(N)$ is upper-triangular with real diagonal entries.

In the case that the eigenvalues $\lambda_1, \dots, \lambda_n$ are distinct, we construct a piecewise-smooth involution ϑ_λ on $\mathcal{O}_\lambda$, called the *twist map*. It sends $L = g(\mathrm{i}\,\mathrm{Diag}(\lambda))g^{-1}$ to $g^{-1}(\mathrm{i}\,\mathrm{Diag}(\lambda))g$, where for a given L the unitary matrix $g \in \mathrm{U}_n$ of eigenvectors is chosen according to a certain normalization condition (3.2), coming from the Bruhat decomposition. We then use ϑ_λ to show that the symmetric Toda flow (1.4) is a gradient flow (see Theorem 4.6):

Theorem 1.1. *Let $\lambda = (\lambda_1 > \cdots > \lambda_n)$. Then the symmetric Toda flow (1.4) on $\mathcal{O}_\lambda$ is, upon applying the twist map ϑ_λ, the gradient flow in the Kähler metric with respect to $\mathrm{Diag}(-\mathrm{i}\lambda_1, \dots, -\mathrm{i}\lambda_n) \in \mathfrak{u}_n$.*

Now let $\mathrm{Fl}_n(\mathbb{C})$ denote the *complete flag variety*, consisting of all chains of subspaces $(V_1, \dots, V_{n-1})$ of $\mathbb{C}^n$ such that

$$V_1 \subset \cdots \subset V_{n-1} \quad \text{and} \quad \dim(V_k) = k \text{ for all } 1 \leq k \leq n-1.$$

Given $L \in \mathcal{O}_\lambda$, let V_k (for $1 \leq k \leq n-1$) denote the subspace of $\mathbb{C}^n$ spanned by the eigenvectors corresponding to the eigenvalues $\mathrm{i}\lambda_1, \dots, \mathrm{i}\lambda_k$. It is well-known (and one can verify) that the map

$$\mathcal{O}_\lambda \xrightarrow{\cong} \mathrm{Fl}_n(\mathbb{C}), \quad L \mapsto (V_1, \dots, V_{n-1}) \tag{1.5}$$

is an isomorphism. Also, $\mathrm{Fl}_n(\mathbb{C})$ has the structure of a projective variety, given by the *Plücker embedding*

$$\mathrm{Fl}_n(\mathbb{C}) \hookrightarrow \prod_{k=1}^{n-1} \mathbb{P}^{\binom{n}{k}-1}(\mathbb{C}) \tag{1.6}$$

(see (2.3)). Then by Theorem 1.1, we can embed the symmetric Toda flow (1.4) on $\mathcal{O}_\lambda$ as a gradient flow on a product of projective spaces. The gradient flow on each projective space can be written in the form (1.2), which by Moser's map (1.3) is equivalent to a Toda lattice flow; in general, for the kth projective space, the associated tridiagonal matrix has size $\binom{n}{k} \times \binom{n}{k}$. We may summarize this as follows (see Theorem 6.13 for a precise statement):

Theorem 1.2. *The tridiagonalization procedure of Deift, Li, Nanda, and Tomei [7, Section 7] of the symmetric Toda flow (1.1) on $n \times n$ symmetric matrices M is given as follows. By multiplying by i and applying the maps ϑ_λ, (1.5), and (1.6), we can embed the symmetric Toda flow as a gradient flow on the product of projective spaces*

$$\prod_{k=1}^{n-1} \mathbb{P}^{\binom{n}{k}-1}(\mathbb{R}).$$

For $(p_1, \dots, p_{n-1}) \in \prod_{k=1}^{n-1} \mathbb{P}^{\binom{n}{k}-1}(\mathbb{R})$, we disregard the zero coordinates of every p_i (for $1 \leq i \leq n-1$) and replace each nonzero coordinate of p_i with its absolute value. Then applying Moser's map (1.3) embeds the symmetric Toda flow into a family of $n-1$ tridiagonal Toda lattice flows.

As an application, we apply a construction of Bloch, Flaschka, and Ratiu [8] which maps the tridiagonal Toda lattice flows homeomorphically onto the associated moment polytope (a *permutohedron*). Theorem 1.2 allows us to embed the symmetric Toda flow as a flow on a product of $n-1$ permutohedra (see Proposition 7.3). We also consider a closely related construction, which maps the symmetric Toda flow to a flow on a product of $n-1$ *hypersimplices* (moment polytopes for Grassmannians). We pose the problem of whether this map is an embedding (see Problem 7.6).

We mention that one of our motivations for studying the symmetric Toda flow is its relationship with the theory of *total positivity* for flag varieties, introduced by Lusztig [9]. While total positivity is not part of the statements of our main results, it was key to our preceding work [10] and provided the impetus for this work. For example, Theorem 1.1 is inspired by [10, Theorem 8.6]. We refer to [10] for further details on total positivity, as well as for references to related works in the literature.

Outline

In Section 2 we recall some background. In Section 3 we introduce the twist map ϑ_λ. In Section 4 we show that the symmetric Toda flow is a twisted gradient flow. In Section 5 we discuss the tridiagonal Toda lattice and Moser's map (1.3). In Section 6 we recall the tridiagonalization construction of [7], and give our new interpretation of it. In Section 7 we apply this construction to study Toda flows on moment polytopes.

We remark that the Toda flows are traditionally studied over the real numbers, and our statement of Theorem 1.2 follows this tradition. However, all of our arguments and results hold over the complex numbers, so we work over $\mathbb{C}$ (for example, Theorem 1.2 follows by specializing Theorem 6.13 over $\mathbb{R}$). This is for the sake both of generality, and for consistency with our Lie-algebraic setup. On the other hand, none of our arguments require working over $\mathbb{C}$ (rather than $\mathbb{R}$).

2. Background

We recall some important background on flag varieties, adjoint orbits, and total positivity. Our notation throughout this paper is consistent with our previous work [10], to which we refer for further details and examples.

Throughout the paper, we fix a strictly decreasing vector $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$. We point out that in [10], we more generally consider λ which are weakly decreasing; we will not need to do so here.

Let $\mathbb{N} := \{0, 1, 2, \dots, \}$, and for $n \in \mathbb{N}$, define $[n] := \{1, 2, \dots, n\}$. For $k \in \mathbb{N}$, we let $\binom{[n]}{k}$ denote the set of k-element subsets of $[n]$. We let $e_1, \dots, e_n$ denote the unit vectors of $\mathbb{C}^n$. Given an $m \times n$ matrix A and subsets $I \subseteq [m]$ and $J \subseteq [n]$, we let $A_{I,J}$ denote the submatrix of A using rows I and columns J. If $|I| = |J|$, we let $\Delta_{I,J}(A)$ denote $\det(A_{I,J})$, called a *minor* of A. If $J = [k]$, where $k = |I|$, we call $\Delta_{I,[k]}(A)$ a *left-justified minor* of A, which we denote by $\Delta_I(A)$.

We let $\mathbb{P}^{n-1}(\mathbb{C})$ denote the *projective space* of all nonzero vectors $(x_1 : \cdots : x_n)$ modulo rescaling. We let $\mathfrak{gl}_n(\mathbb{C})$ denote the Lie algebra of $n \times n$ matrices over $\mathbb{C}$, with Lie bracket

$$[L, M] := LM - ML \quad \text{for all } L, M \in \mathfrak{gl}_n(\mathbb{C}).$$

We let $\mathrm{ad}_L := [L, \cdot]$ denote the *adjoint operator* of $L \in \mathfrak{gl}_n(\mathbb{C})$. We let $\mathrm{diag}(L) \in \mathbb{C}^n$ denote the vector of diagonal entries of $L \in \mathfrak{gl}_n(\mathbb{C})$. Finally, we let $\mathrm{Diag}(c_1, \dots, c_n) \in \mathfrak{gl}_n(\mathbb{C})$ denote the $n \times n$ diagonal matrix with diagonal entries $c_1, \dots, c_n$.

We recall the *Cauchy–Binet identity* (see e.g. [11, I.(14)]): if A is an $m \times n$ matrix, B is an $n \times p$ matrix, and $1 \le k \le m, p$, then

$$\Delta_{I,J}(AB) = \sum_{K \in \binom{[n]}{k}} \Delta_{I,K}(A)\Delta_{K,J}(B) \quad \text{for all } I \in \binom{[m]}{k} \text{ and } J \in \binom{[p]}{k}. \tag{2.1}$$

2.1. Flag varieties and adjoint orbits

We introduce Grassmannians and complete flag varieties, which will play an important role in the paper.

Definition 2.1. Let $0 \le k \le n$. We define the *Grassmannian* $\mathrm{Gr}_{k,n}(\mathbb{C})$ as the set of all k-dimensional linear subspaces of $\mathbb{C}^n$. Given $V \in \mathrm{Gr}_{k,n}(\mathbb{C})$, we say that an $n \times k$ matrix A *represents* V if its columns form a basis of V. We have the *Plücker embedding*

$$\mathrm{Gr}_{k,n}(\mathbb{C}) \hookrightarrow \mathbb{P}^{\binom{n}{k}-1}(\mathbb{C}), \quad V \mapsto \left(\Delta_I(A) : I \in \binom{[n]}{k}\right), \tag{2.2}$$

which does not depend on the choice of A. We call the projective coordinates $\Delta_I(\cdot)$ on $\mathrm{Gr}_{k,n}(\mathbb{C})$ *Plücker coordinates*.

Definition 2.2. Let $\mathrm{GL}_n(\mathbb{C})$ denote the general linear group of all $n \times n$ invertible matrices over $\mathbb{C}$, and let $\mathrm{B}_n(\mathbb{C})$ denote the Borel subgroup of $\mathrm{GL}_n(\mathbb{C})$ of all upper-triangular matrices. We define the *complete flag variety* as the quotient

$$\mathrm{Fl}_n(\mathbb{C}) := \mathrm{GL}_n(\mathbb{C})/\,\mathrm{B}_n(\mathbb{C}),$$

which we may identify with the variety of complete flags of linear subspaces of $\mathbb{C}^n$

$$\{V = (V_1, \dots, V_{n-1}) : 0 \subset V_1 \subset \cdots \subset V_{n-1} \subset \mathbb{C}^n \text{ and } \dim(V_k) = k \text{ for } 1 \le k \le n-1\}.$$

This identification sends $g \in \mathrm{GL}_n(\mathbb{C})/\,\mathrm{B}_n(\mathbb{C})$ to the tuple $(V_1, \dots, V_{n-1})$, where each V_k is the span of the first k columns of g. We will freely alternate between regarding complete flags as elements $g \in \mathrm{GL}_n(\mathbb{C})/\,\mathrm{B}_n(\mathbb{C})$ or as tuples $(V_1, \dots, V_{n-1})$.

We have the *Plücker embedding*

$$\mathrm{Fl}_n(\mathbb{C}) \hookrightarrow \prod_{k=1}^{n-1} \mathbb{P}^{\binom{n}{k}-1}(\mathbb{C}), \quad g \mapsto \left(\Delta_I(g) : I \in \binom{[n]}{k}\right)_{1 \le k \le n-1}, \tag{2.3}$$

which is given by the embedding

$$\mathrm{Fl}_n(\mathbb{C}) \hookrightarrow \prod_{k=1}^{n-1} \mathrm{Gr}_{k,n}(\mathbb{C}), \quad (V_1, \dots, V_{n-1}) \mapsto (V_1, \dots, V_{n-1}), \tag{2.4}$$

and then applying the Plücker embedding (2.2) to each term $\mathrm{Gr}_{k,n}(\mathbb{C})$. We call the left-justified minors $\Delta_I(g)$ appearing above the *Plücker coordinates* of $g \in \mathrm{Fl}_n(\mathbb{C})$ (also known as *flag minors*).

Example 2.3. We can write a generic element of the complete flag variety $\mathrm{Fl}_3(\mathbb{C})$ as

$$g := \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} \in \mathrm{GL}_3(\mathbb{C})/\,\mathrm{B}_3(\mathbb{C}), \quad \text{where } a, b, c \in \mathbb{C}.$$

The Plücker embedding (2.3) takes g to

$$\big((\Delta_1(g) : \Delta_2(g) : \Delta_3(g)), (\Delta_{12}(g) : \Delta_{13}(g) : \Delta_{23}(g))\big) = \big((1 : a : b), (1 : c : ac - b)\big) \in \mathbb{P}^3(\mathbb{C}) \times \mathbb{P}^3(\mathbb{C}). \quad \diamond$$

We recall that $\mathcal{O}_\lambda$ denotes the adjoint orbit of $\mathfrak{u}_n$ consisting of all skew-Hermitian matrices with eigenvalues $\mathrm{i}\lambda_1, \dots, \mathrm{i}\lambda_n$:

$$\mathcal{O}_\lambda = \{g(\mathrm{i}\,\mathrm{Diag}(\lambda))g^{-1} : g \in \mathrm{U}_n\}.$$

We observe that we can write the isomorphism $\mathcal{O}_\lambda \xrightarrow{\cong} \mathrm{Fl}_n(\mathbb{C})$ from (1.5) equivalently as

$$\mathcal{O}_\lambda \xrightarrow{\cong} \mathrm{Fl}_n(\mathbb{C}), \quad g(\mathrm{i}\,\mathrm{Diag}(\lambda))g^{-1} \mapsto g.$$

Remark 2.4. The embedding (2.4) has a natural interpretation in $\mathcal{O}_\lambda$. Namely, given $(V_1, \dots, V_{n-1}) \in \mathrm{Fl}_n(\mathbb{C})$, let $\mathrm{i}M \in \mathcal{O}_\lambda$ be the corresponding element under the isomorphism (1.5). Then

$$M = \left(\sum_{k=1}^{n-1} (\lambda_k - \lambda_{k+1}) P_k\right) + \lambda_n I_n, \tag{2.5}$$

where $P_k \in \mathfrak{gl}_n(\mathbb{C})$ is orthogonal projection onto V_k [10, Lemma 4.16].

2.2. Total positivity

We recall the notion of total positivity for $\mathrm{Fl}_n(\mathbb{C})$.

Definition 2.5 (*[9,12]*). Let $0 \le k \le n$. We say that an element of $\mathrm{Fl}_n(\mathbb{C})$ is *totally positive* (respectively, *totally nonnegative*) if all its Plücker coordinates are real and positive (respectively, nonnegative), up to rescaling. This defines the totally positive part $\mathrm{Fl}_n^{>0}$ and the totally nonnegative part $\mathrm{Fl}_n^{\ge 0}$. (This definition is different from, but equivalent to, the original definition of Lusztig [9,12]; see [13, Section 1.4] for references and a history of this equivalence.)

We define the *totally positive* part $\mathcal{O}_\lambda^{>0}$ to be the inverse image of $\mathrm{Fl}_n^{>0}$ under the isomorphism (1.5). We similarly define the *totally nonnegative* part $\mathcal{O}_\lambda^{\ge 0}$.

Example 2.6. The element $g \in \mathrm{Fl}_3(\mathbb{C})$ from Example 2.3 is totally positive if and only if $a, b, c, ac - b > 0$. $\diamond$

3. The general twist map

In this section we construct an involution ϑ on $\mathrm{Fl}_n(\mathbb{C})$, which we call the *twist map*. This generalizes the totally nonnegative twist map we defined on $\mathrm{Fl}_n^{\geq 0}$ in [10]; see Remark 3.7. We refer to [10, Section 3.3] for further motivation and a discussion of related work.

We begin by recalling the Bruhat decomposition of $\mathrm{Fl}_n(\mathbb{C})$; for further details, see, e.g., [14, Section 1.2].

Definition 3.1. Given $n \in \mathbb{N}$, let $\mathfrak{S}_n$ denote the symmetric group of all permutations of $[n]$. For $w \in \mathfrak{S}_n$, we define the *(signed) permutation matrix* $\mathring{w} \in \mathrm{GL}_n(\mathbb{C})$ by

$$\mathring{w}_{i,j} := \begin{cases} \pm 1, & \text{if } i = w(j); \\ 0, & \text{otherwise}, \end{cases} \qquad \text{for } 1 \leq i,j \leq n,$$

where the signs are chosen so that all left-justified minors of $\mathring{w}$ are nonnegative. Note that

$$(\mathring{w^{-1}}) = \delta_n(\mathring{w})^{-1}\delta_n, \qquad \text{where } \delta_n := \mathrm{Diag}(1, -1, 1, \dots, (-1)^{n-1}). \tag{3.1}$$

We also regard $\mathring{w}$ as an element of $\mathrm{Fl}_n(\mathbb{C})$, and define the *Schubert cell*

$$\mathring{X}^w := \mathrm{B}_n(\mathbb{C}) \cdot \mathring{w} \subseteq \mathrm{Fl}_n(\mathbb{C}),$$

which consists of all $V \in \mathrm{Fl}_n(\mathbb{C})$ such that for all $1 \leq k \leq n-1$, the lexicographically maximal $I \in \binom{[n]}{k}$ such that $\Delta_I(V) \neq 0$ is $I = w([k])$. We have the *Bruhat decomposition*

$$\mathrm{Fl}_n(\mathbb{C}) = \bigsqcup_{w \in \mathfrak{S}_n} \mathring{X}^w.$$

We now define the general twist map.

Definition 3.2. Given $n \in \mathbb{N}$, define the involution $\iota : \mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$ (called the *positive inverse*) by

$$\iota(g) := \delta_n g^{-1} \delta_n.$$

In other words, $\iota(g)_{i,j} = (-1)^{i+j}(g^{-1})_{i,j}$ for $1 \leq i,j \leq n$.

Given $V \in \mathrm{Fl}_n(\mathbb{C})$, we define a canonical representative $g \in \mathrm{U}_n$ of V as follows: if $V \in \mathring{X}^w$ (where $w \in \mathfrak{S}_n$), then

$$\begin{aligned} &\Delta_{w[k]}(g) \in \mathbb{R}_{>0} \quad \text{and} \quad \Delta_I(g) = 0 \text{ for all} \\ &\quad I \in \tbinom{[n]}{k} \text{ with } I >_{\mathrm{lex}} w([k]) \end{aligned} \tag{3.2}$$

for all $1 \leq k \leq n$. We let $\vartheta(V) \in \mathrm{Fl}_n(\mathbb{C})$ denote the complete flag represented by $\iota(g)$. This defines the *(Iwasawa) twist map* $\vartheta : \mathrm{Fl}_n(\mathbb{C}) \to \mathrm{Fl}_n(\mathbb{C})$.

Remark 3.3. The name *twist map* is motivated by the twist maps defined by Berenstein, Fomin, and Zelevinsky on $\mathrm{N}_n(\mathbb{C})$ and $\mathrm{GL}_n(\mathbb{C})$ [15,16]. The key difference is that our map ϑ is based on the Iwasawa (or *QR*-) decomposition of $\mathrm{GL}_n(\mathbb{C})$, rather than the Bruhat decomposition. A different twist map was defined on $\mathrm{Fl}_n(\mathbb{C})$ in the latter sense by Galashin and Lam [17].

Example 3.4. Let

$$g := \frac{1}{2}\begin{bmatrix} \sqrt{2} & -1 & 1 \\ \sqrt{2} & 1 & -1 \\ 0 & \sqrt{2} & \sqrt{2} \end{bmatrix} \in \mathrm{U}_3, \qquad \text{whence}$$

$$\iota(g) = \delta_3 g^{-1} \delta_3 = \frac{1}{2}\begin{bmatrix} \sqrt{2} & -\sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} \\ 1 & 1 & \sqrt{2} \end{bmatrix}.$$

We can verify that g satisfies (3.2) with $w := 231 \in \mathfrak{S}_3$. Therefore $\vartheta : \mathrm{Fl}_3(\mathbb{C}) \to \mathrm{Fl}_3(\mathbb{C})$ takes the complete flag represented by g to the complete flag represented by $\iota(g)$. Note that $\iota(g)$ satisfies (3.2) for the permutation $w^{-1} = 312$, in agreement with Proposition 3.5. $\diamondsuit$

Proposition 3.5. *The twist map ϑ on $\mathrm{Fl}_n(\mathbb{C})$ is an involution. For each $w \in \mathfrak{S}_n$, it restricts to a diffeomorphism $\mathring{X}^w \to \mathring{X}^{w^{-1}}$.*

Proof. Let $w \in \mathfrak{S}_n$. The twist map restricted to $\mathring{X}^w$ is smooth, so it suffices to show that given $V \in \mathring{X}^w$, we have $\vartheta(V) \in \mathring{X}^{w^{-1}}$ and that $\vartheta(\vartheta(V)) = V$.

Let $g \in \mathrm{U}_n$ be the canonical representative of V as in (3.2), and write

$$g = b\mathring{w}b', \qquad \text{where } b, b' \in \mathrm{B}_n(\mathbb{C}).$$

By the Cauchy–Binet identity (2.1) and since b' is upper-triangular, we have

$$\begin{aligned} \Delta_{w([k])}(g) &= \sum_{I,J \in \binom{[n]}{k}} \Delta_{w([k]),I}(b)\Delta_{I,J}(\mathring{w})\Delta_J(b') \\ &= \Delta_{w([k]),w([k])}(b)\Delta_{[k]}(b') \end{aligned} \tag{3.3}$$

for all $0 \leq k \leq n$. Then by (3.2), we obtain

$$b_{w(k),w(k)}b'_{k,k} = \frac{\Delta_{w([k])}(g)}{\Delta_{w([k-1])}(g)} > 0 \quad \text{for all } 1 \leq k \leq n. \tag{3.4}$$

By (3.1), we have

$$\iota(g) = \delta_n g^{-1}\delta_n = (\delta_n b'^{-1}\delta_n)(\mathring{w^{-1}})(\delta_n b^{-1}\delta_n),$$

so $\vartheta(V) \in \mathring{X}^{w^{-1}}$. Also, for all $1 \leq k \leq n$, applying (3.3) to $\Delta_{w^{-1}([k])}(\iota(g))$ gives

$$\begin{aligned} \Delta_{w^{-1}([k])}(\iota(g)) &= \Delta_{w^{-1}([k]),w^{-1}([k])}(\delta_n b'^{-1}\delta_n)\Delta_{[k]}(\delta_n b^{-1}\delta_n) \\ &= \prod_{i=1}^{k}(b'_{w^{-1}(i),w^{-1}(i)}b_{i,i})^{-1} > 0, \end{aligned}$$

using (3.4). Hence $\iota(g)$ is the canonical representative of $\vartheta(V)$ as in (3.2), and since ι is an involution, we obtain $\vartheta(\vartheta(V)) = V$. $\square$

Remark 3.6. We could just as well have defined the twist map ϑ using the decomposition of $\mathrm{Fl}_n(\mathbb{C})$ into *opposite Schubert cells*

$$\mathring{X}_w := \mathrm{B}_n^-(\mathbb{C}) \cdot \mathring{w} \subseteq \mathrm{Fl}_n(\mathbb{C}) \qquad \text{for } w \in \mathfrak{S}_n,$$

rather than Schubert cells. The resulting twist map would be different from the one in Definition 3.2; the relationship between the two maps can be derived from the fact that $\mathrm{B}_n^-(\mathbb{C})$ is equal to $\mathrm{B}_n(\mathbb{C})$ conjugated by $\mathring{w}_0$, where $w_0 := (i \mapsto n+1-i) \in \mathfrak{S}_n$. These conventions are ultimately not important for our purposes, because we take absolute values in Definition 6.8.

Remark 3.7. In Definition 3.2, we have defined the twist map ϑ on $\mathrm{Fl}_n(\mathbb{C})$ in a piecewise manner, based on the Bruhat decomposition. While ϑ defines a smooth map on each Schubert cell $\mathring{X}^w \subseteq \mathrm{Fl}_n(\mathbb{C})$, in general ϑ is not continuous when passing between cells. However, ϑ displays remarkable positivity properties, as we explored in [10]; in particular, it restricts to an involution on the totally nonnegative part $\mathrm{Fl}_n^{\geq 0}$, which extends to a smooth map in an open neighborhood inside $\mathrm{Fl}_n(\mathbb{R})$ [10, Definition 3.21]. We emphasize, however, that such a smooth extension differs from the general twist map ϑ outside of $\mathrm{Fl}_n^{\geq 0}$, since ϑ is not necessarily continuous on the boundary of $\mathrm{Fl}_n^{\geq 0}$. For example, let $g(t) \in \mathrm{Fl}_3(\mathbb{C})$ be represented by the matrix g from Example 3.4, with the (3, 1)-entry replaced by $t \in \mathbb{R}$. Then $g(0) \in \mathrm{Fl}_3^{\geq 0}$, and $g(t) \notin \mathrm{Fl}_3^{\geq 0}$ for all

$t < 0$. We have $\lim_{t\to 0} g(t) = g(0)$, but we can verify that

$$\lim_{t\to 0,\, t<0} \vartheta(g(t)) = \frac{1}{2}\begin{bmatrix} -\sqrt{2} & \sqrt{2} & 0 \\ -1 & -1 & \sqrt{2} \\ 1 & 1 & \sqrt{2} \end{bmatrix} \neq \vartheta(g(0)) \in \mathrm{Fl}_3(\mathbb{C}).$$

Finally, we observe that ϑ defines an involution on $\mathcal{O}_\lambda \cong \mathrm{Fl}_n(\mathbb{C})$ under the isomorphism (1.5).

Definition 3.8. We define $\vartheta_\lambda : \mathcal{O}_\lambda \to \mathcal{O}_\lambda$ as the involution on $\mathcal{O}_\lambda$ induced by the involution ϑ on $\mathrm{Fl}_n(\mathbb{C})$, via the isomorphism (1.5). Explicitly,

$$\vartheta_\lambda(g(\mathrm{i}\,\mathrm{Diag}(\lambda))g^{-1}) := \iota(g)(\mathrm{i}\,\mathrm{Diag}(\lambda))(\iota(g))^{-1}$$
$$= \delta_n g^{-1}(\mathrm{i}\,\mathrm{Diag}(\lambda))g\delta_n$$

for all $g \in \mathrm{U}_n$ satisfying (3.2) (for some $w \in \mathfrak{S}_n$ and all $1 \le k \le n$).

4. The symmetric Toda flow as a twisted gradient flow

In this section we use the twist map ϑ_λ to show that the symmetric Toda flow is a twisted gradient flow on $\mathcal{O}_\lambda$ in the Kähler metric (see Theorem 4.6). This generalizes [10, Theorem 8.6(ii)], where we proved the same result restricted to the totally nonnegative part $\mathcal{O}_\lambda^{\ge 0}$; we refer to [10, Section 8] for further discussion and context.

4.1. Background on the Kähler metric and gradient flows

We begin by recalling background on the Kähler metric on $\mathcal{O}_\lambda$ and gradient flows, following [10, Section 5].

Definition 4.1. Let ν denote the *Killing form* on $\mathfrak{gl}_n(\mathbb{C})$, given by

$$\nu(L, M) := 2n\,\mathrm{tr}(LM) - 2\,\mathrm{tr}(L)\,\mathrm{tr}(M) \quad \text{for all } L, M \in \mathfrak{gl}_n(\mathbb{C}).$$

Then $-\nu(\cdot,\cdot)$ defines a $[\cdot,\cdot]$-invariant pairing (i.e. $\nu(\mathrm{ad}_L(M), N) = -\nu(M, \mathrm{ad}_L(N))$) which is positive semidefinite on $\mathfrak{u}_n$.

Now let $L \in \mathcal{O}_\lambda$. For $X \in \mathfrak{u}_n$, we define X^L and X_L by the (unique) decomposition

$$X = X^L + X_L, \quad \text{where } X^L \in \mathrm{im}(\mathrm{ad}_L) \text{ and } X_L \in \ker(\mathrm{ad}_L).$$

The *normal metric* on $\mathcal{O}_\lambda$ is given at $L \in \mathcal{O}_\lambda$ by

$$\langle [L, X], [L, Y]\rangle_{\mathrm{normal}} := -\nu(X^L, Y^L)$$

for all tangent vectors $[L, X]$ and $[L, Y]$ at L. Then the *Kähler metric* on $\mathcal{O}_\lambda$ is given at $L \in \mathcal{O}_\lambda$ by

$$\langle [L, X], [L, Y]\rangle_{\text{Kähler}} := \langle \sqrt{-\mathrm{ad}_L^2}([L, X]), [L, Y]\rangle_{\mathrm{normal}},$$

where $\sqrt{-\mathrm{ad}_L^2}$ denote the positive square root of the positive semidefinite operator $-\mathrm{ad}_L^2$.

Definition 4.2. Given $N \in \mathfrak{u}_n$, we define the *gradient flow on* $\mathcal{O}_\lambda$ *with respect to* N (in a particular Riemannian metric) as the flow given by

$$\dot{L}(t) = \mathrm{grad}(H)(L(t)), \quad \text{where } H(M) := \nu(M, N) \text{ for all } M \in \mathcal{O}_\lambda.$$

We have the following explicit description of gradient flows on $\mathcal{O}_\lambda$ in the Kähler metric:

Proposition 4.3 (*[18, Section 3]; [19, Appendix]*)**.** *Let $L(t)$ evolve according to the gradient flow on $\mathcal{O}_\lambda$ beginning at L_0 with respect to $N \in \mathfrak{u}_n$ in the Kähler metric, and let $V(t) \in \mathrm{Fl}_n(\mathbb{C})$ be the corresponding complete flag under the isomorphism (1.5), with $V_0 := V(0)$. Then*

$$V(t) = \exp(tiN)V_0 \quad \text{for all } t \in \mathbb{R}. \tag{4.1}$$

Letting $g(t) \in \mathrm{U}_n$ be any representative of $V(t)$, we have $L(t) = g(t)(\mathrm{i}\,\mathrm{Diag}(\lambda))g(t)^{-1}$. Explicitly, we can take $g_0 \in \mathrm{U}_n$ representing V_0, and then take

$$g(t) = \pi_{\mathrm{U}}(\exp(tiN)g_0) \quad \text{for all } t \in \mathbb{R}. \tag{4.2}$$

We observe that (as will be useful later) in the case that N is a diagonal matrix, we can explicitly describe the Plücker coordinates of the element $V(t)$ in (4.1) in terms of those of V_0.

Lemma 4.4. *Let $N := -\mathrm{i}\,\mathrm{Diag}(c_1, \dots, c_n) \in \mathfrak{u}_n$, and suppose that $V(t)$ evolves according to (4.1). Then*

$$\Delta_I(V(t)) = e^{(\sum_{i\in I} c_i)t}\Delta_I(V_0) \quad \text{for all } I \subseteq [n].$$

Proof. This follows by a direct calculation, since $\exp(tiN) = \mathrm{Diag}(e^{c_1 t}, \dots, e^{c_n t})$. □

Remark 4.5. We mention that in addition to (4.2), there is another way to obtain an explicit solution to $L(t)$. Namely, let $V(t) = (V_1(t), \dots, V_{n-1}(t))$ be as in Proposition 4.3. Then as in (2.5), we write

$$-\mathrm{i}L(t) = \left(\sum_{k=1}^{n-1}(\lambda_k - \lambda_{k+1})P_k(t)\right) + \lambda_n I_n, \tag{4.3}$$

where $P_k(t) \in \mathfrak{gl}_n(\mathbb{C})$ is orthogonal projection onto $V_k(t)$. Regarding elements of $\mathrm{Gr}_{k,n}(\mathbb{C})$ as $n \times k$ matrices (as in Definition 2.1), we have the formula

$$P_k(t) = V_k(t)(V_k(t)^*V_k(t))^{-1}V_k(t)^*$$
$$= \exp(tiN)(V_0)_k((V_0)_k^*\exp(2tiN)(V_0)_k)^{-1}(V_0)_k^*\exp(tiN).$$

This leads (via (4.3)) to an explicit expression for $L(t)$ which does not require computing a QR-decomposition, as in (4.2).

4.2. The symmetric Toda flow

We now show that the symmetric Toda flow (1.4) on $\mathcal{O}_\lambda$ is a twisted gradient flow. This generalizes [10, Theorem 8.6(ii)], where we proved the same result restricted to the totally nonnegative part $\mathcal{O}_\lambda^{\ge 0}$; essentially the same proof applies, using the general twist map ϑ_λ defined in Section 3.

Theorem 4.6. *Set $N := -\mathrm{i}\,\mathrm{Diag}(\lambda) \in \mathfrak{u}_n$. The symmetric Toda flow (1.4) restricted to $\mathcal{O}_\lambda$ is the twisted gradient flow with respect to N in the Kähler metric. That is, if $L(t)$ evolves according to (1.4) beginning at $L_0 \in \mathcal{O}_\lambda$, then $\vartheta_\lambda(L(t))$ is the gradient flow with respect to N in the Kähler metric beginning at $\vartheta_\lambda(L_0) \in \mathcal{O}_\lambda$ (cf. Definition 4.2 and Proposition 4.3). In the notation of Proposition 4.3, we have*

$$V(t) = \vartheta(\mathrm{Diag}(e^{\lambda_1 t}, \dots, e^{\lambda_n t}) \cdot \vartheta(V_0)) \quad \text{for all } t \in \mathbb{R}. \tag{4.4}$$

Proof. The proof is the same as in [10, Proof of Theorem 8.6]. We only need to observe that if $g_0 \in \mathrm{U}_n$ satisfies (3.2) for some $w \in \mathfrak{S}_n$ and all $1 \le k \le n$, then so does $\pi_{\mathrm{U}}(\exp(tiN)g_0)$. □

Remark 4.7. We note that Bloch [20, Section 6] showed that the tridiagonal Toda lattice flow (1.4) (with L tridiagonal) can be written as

$$\dot{L} = [L, [L, N]], \quad \text{where } N := -\mathrm{i}\,\mathrm{Diag}(n-1, \dots, 1, 0) \in \mathfrak{u}_n,$$

In particular, by [21,22], the tridiagonal Toda flow restricted to $\mathcal{O}_\lambda$ is the gradient flow with respect to N in the *normal* metric. (However, this result does not directly extend to the full symmetric Toda flow; cf. [23].) It is curious that the Toda lattice flow can be written as a gradient flow in two different metrics in rather different ways.

5. Tridiagonal matrices and the Moser map

In this section, we explicitly describe Moser's map (1.3) and its connection to the tridiagonal Toda lattice flow. We closely follow [10, Section 4.4]. In order to use the framework of adjoint orbits, we work with skew-Hermitian matrices $\mathrm{i}M$ rather than symmetric (or Hermitian) matrices M.

Definition 5.1. We define $\mathcal{J}_\lambda^{>0}$ (respectively, $\mathcal{J}_\lambda^{\geq 0}$) to be the set of elements $\mathrm{i}M$ of $\mathcal{O}_\lambda$ such that M is a real tridiagonal matrix with positive (respectively, nonnegative) entries immediately above and below the diagonal. Equivalently, $\mathcal{J}_\lambda^{>0}$ (respectively, $\mathcal{J}_\lambda^{\geq 0}$) is the set of tridiagonal matrices in $\mathcal{O}_\lambda^{>0}$ (respectively, $\mathcal{O}_\lambda^{\geq 0}$) [10, Proposition 4.18]. The space $\mathcal{J}_\lambda^{>0}$ is known as an *isospectral manifold of Jacobi matrices*.

Definition 5.2. Let $\mathbb{P}_{>0}^{n-1}$ (respectively, $\mathbb{P}_{\geq 0}^{n-1}$) denote the subset of $\mathbb{P}^{n-1}(\mathbb{C})$ where all coordinates are real and positive (respectively, nonnegative), up to rescaling. Given $x \in \mathbb{P}_{>0}^{n-1}$, we define the *Vandermonde flag* $\mathsf{Vand}(\lambda, x) \in \mathrm{Fl}_n(\mathbb{C})$ as the complete flag $(V_1, \dots, V_{n-1})$, where

$$V_k := \mathrm{span}(x, \mathsf{Diag}(\lambda)x, \dots, \mathsf{Diag}(\lambda)^{k-1}x) \quad \text{for } 1 \leq k \leq n-1.$$

Equivalently, $\mathsf{Vand}(\lambda, x)$ is represented by the rescaled Vandermonde matrix $(\lambda_i^{j-1}x_i)_{1\leq i,j\leq n} \in \mathrm{GL}_n(\mathbb{C})$. We let $\mathsf{Mos}_\lambda(x) \in \mathcal{O}_\lambda$ denote element corresponding to $\mathsf{Vand}(\lambda, x) \in \mathrm{Fl}_n(\mathbb{C})$ under the isomorphism (1.5). We call Mos_λ the *Moser map*, since it essentially appeared (with a different, but equivalent, definition) in [4].

Theorem 5.3 (*Moser [4, Section 3]; Bloch and Karp [10, Corollary 4.24]*)**.** *The Moser map* $\mathsf{Mos}_\lambda : \mathbb{P}_{>0}^{n-1} \xrightarrow{\cong} \mathcal{J}_\lambda^{>0}$ *is a diffeomorphism.*

Example 5.4. Let $\lambda := (1, 0, -1)$. Then the Moser map Mos_λ sends $(x_1 : x_2 : x_3) \in \mathbb{P}_{>0}^2$ to

$$\mathrm{i}\begin{bmatrix} \frac{x_1^2-x_3^2}{x_1^2+x_2^2+x_3^2} & \frac{\sqrt{x_1^2x_2^2+4x_1^2x_3^2+x_2^2x_3^2}}{x_1^2+x_2^2+x_3^2} & 0 \\ \frac{\sqrt{x_1^2x_2^2+4x_1^2x_3^2+x_2^2x_3^2}}{x_1^2+x_2^2+x_3^2} & \frac{(x_1^2-x_3^2)(x_2^4-4x_1^2x_3^2)}{(x_1^2+x_2^2+x_3^2)(x_1^2x_2^2+4x_1^2x_3^2+x_2^2x_3^2)} & \frac{2x_1x_2x_3\sqrt{x_1^2+x_2^2+x_3^2}}{x_1^2x_2^2+4x_1^2x_3^2+x_2^2x_3^2} \\ 0 & \frac{2x_1x_2x_3\sqrt{x_1^2+x_2^2+x_3^2}}{x_1^2x_2^2+4x_1^2x_3^2+x_2^2x_3^2} & \frac{x_2^2(x_3^2-x_1^2)}{x_1^2x_2^2+4x_1^2x_3^2+x_2^2x_3^2} \end{bmatrix}$$

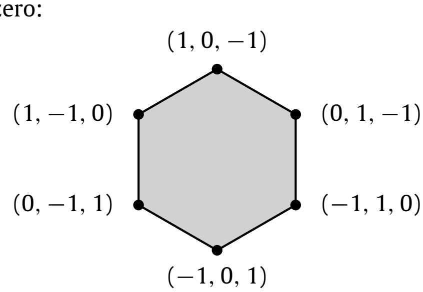
$\in \mathcal{J}_\lambda^{>0}$. ◊

We now discuss the topology of the compact isospectral manifold $\mathcal{J}_\lambda^{\geq 0}$.

Definition 5.5. Let $\mathsf{Perm}(\lambda) \subseteq \mathbb{R}^n$ denote the polytope whose vertices are all $n!$ permutations of $\lambda = (\lambda_1, \dots, \lambda_n)$. We call $\mathsf{Perm}(\lambda)$ a *permutohedron*. We also define the *moment map*

$$\mu : \mathfrak{u}_n \to \mathbb{R}^n, \quad \mathrm{i}M \mapsto \mathrm{diag}(M).$$

Example 5.6. Let $\lambda := (1, 0, -1)$. Then $\mathsf{Perm}(\lambda)$ is a hexagon in $\mathbb{R}^3$, contained in the hyperplane where all coordinates sum to zero:



. ◊

Tomei [24, Section 4] showed that $\mathcal{J}_\lambda^{\geq 0}$ is homeomorphic to $\mathsf{Perm}(\lambda)$ (in fact, they are isomorphic as regular CW complexes). We will need the following explicit version of this result due to Bloch, Flashcka, and Ratiu [8]. It may be regarded as an analogue of the *Schur–Horn theorem* [25,26], which states that μ maps $\mathcal{O}_\lambda$ surjectively onto $\mathsf{Perm}(\lambda)$.

Theorem 5.7 (*Bloch, Flaschka, and Ratiu [8, Theorem p. 60]; cf. [10, Remark 8.8]*)**.** *The map*

$$\mathcal{J}_\lambda^{\geq 0} \to \mathsf{Perm}(\lambda), \quad L \mapsto \mu(\vartheta_\lambda(L))$$

is a homeomorphism which restricts to a diffeomorphism from $\mathcal{J}_\lambda^{>0}$ *to the interior of* $\mathsf{Perm}(\lambda)$*.*

We emphasize that in general, the map

$$\mathcal{J}_\lambda^{\geq 0} \to \mathsf{Perm}(\lambda), \quad L \mapsto \mu(L)$$

(not involving the twist map) is neither injective nor surjective.

6. Tridiagonalization of the symmetric Toda flow

6.1. The construction of Deift, Li, Nanda, and Tomei

We recall the tridiagonalization of the symmetric Toda flow constructed by Deift, Li, Nanda, and Tomei [7, Section 7]. While we maintain the notation of [7] where possible, we give an exposition tailored to our perspective. One key notational difference is that we reverse the order of the ground set $[n]$ from its use in [7] (e.g. in Definition 6.2, we use e_1 rather than e_n); our convention is consistent with the rest of this paper and with [4,10].

Definition 6.1. Let $A \in \mathfrak{gl}_n(\mathbb{C})$ and $x \in \mathbb{C}^n$. We define the *cyclic subspace*

$$\mathsf{cyc}(A; x) := \mathrm{span}(A^j x : j \in \mathbb{N}) \subseteq \mathbb{C}^n.$$

Equivalently, $\mathsf{cyc}(A; x)$ is the minimal A-invariant subspace of $\mathbb{C}^n$ containing x.

Definition 6.2. Let M be an $n \times n$ Hermitian matrix. Let $m := \dim(\mathsf{cyc}(M; e_1))$, and let A be the $n \times m$ matrix with orthonormal columns obtained by applying the Gram–Schmidt orthonormalization process to the matrix

$$\begin{bmatrix} | & | & & | \\ e_1 & Me_1 & \cdots & M^{m-1}e_1 \\ | & | & & | \end{bmatrix}$$

(so in particular, $A^*A = I_m$). We define the $m \times m$ Hermitian matrix $M_T := A^*MA$, which represents the endomorphism M restricted to $\mathsf{cyc}(M; e_1)$. In light of Proposition 6.3(i), we call M_T the *tridiagonalization* of M.

Proposition 6.3. *Let M be an $n \times n$ Hermitian matrix, and write $M = g\,\mathsf{Diag}(\nu)g^{-1}$, where $\nu_1 \geq \cdots \geq \nu_n$ and $g \in \mathrm{U}_n$.*

(i) The matrix M_T is tridiagonal with positive real entries immediately above and below the diagonal.

(ii) The eigenvalues of M_T are distinct, and its set of eigenvalues is

$$\{\nu_i : 1 \leq i \leq n \text{ and } g_{1,i} \neq 0\}.$$

(iii) Let ν_i (where $1 \leq i \leq n$) be a simple eigenvalue of M which is also an eigenvalue of M_T. Then the corresponding normalized eigenvectors of M and M_T have the same first entry. That is, if $x \in \mathbb{C}^m$ has norm 1 such that $M_T x = \nu_i x$, then $|x_1| = |g_{1,i}|$.

Proof. (i) Let m and A be as in Definition 6.2. We show, equivalently, that the $m \times m$ matrix

$$\begin{bmatrix} | & | & & | \\ e_1 & M_T e_1 & \cdots & (M_T)^{m-1} e_1 \\ | & | & & | \end{bmatrix}$$

is upper-triangular with positive diagonal entries. To this end, let $1 \le j \le m$. By the definition of A, we can write

$$M^{j-1} e_1 = \sum_{i=1}^{j} c_i A e_i, \quad \text{where } c_1, \dots, c_j \in \mathbb{C} \text{ and } c_j > 0.$$

Recall that M_T represents the endomorphism M restricted to $\mathsf{cyc}(M; e_1)$; similarly, $(M_T)^{j-1} = A^* M^{j-1} A$ represents M^{j-1} restricted to $\mathsf{cyc}(M; e_1)$. We obtain

$$(M_T)^{j-1} e_1 = A^* M^{j-1} A e_1 = A^* M^{j-1} e_1 = \sum_{i=1}^{j} c_i A^* A e_i = \sum_{i=1}^{j} c_i e_i.$$

(ii) The matrix M_T represents the endomorphism M restricted to $\mathsf{cyc}(M; e_1)$, which in turn (conjugating by g^{-1}) is similar to the endomorphism $\mathsf{Diag}(\nu)$ restricted to $\mathsf{cyc}(\mathsf{Diag}(\nu); g^{-1}e_1)$. Given $1 \le i \le n$, consider the vector $y \in \mathbb{C}^n$ defined by

$$y_j := \begin{cases} (g^{-1}e_1)_j = \overline{g_{1,j}}, & \text{if } \nu_j = \nu_i; \\ 0, & \text{otherwise} \end{cases} \quad \text{for } 1 \le j \le n.$$

Then by Vandermonde's identity, the distinct nonzero y's form a basis for $\mathsf{cyc}(\mathsf{Diag}(\nu); g^{-1}e_1)$.

(iii) Since ν_i is a simple eigenvalue of both M and M_T, the corresponding eigenvector ge_i of M lies in $\mathsf{cyc}(M; e_1)$. Therefore $ge_i = Ax$ for some $x \in \mathbb{C}^m$, and $\|x\| = \|e_i\| = 1$ because g and A both have orthonormal columns. We have

$$M_T x = A^* M A x = A^* M g e_i = \nu_i A^* g e_i = \nu_i A^* A x = \nu_i x,$$

and $|x_1| = |(A^* g e_i)_1| = |(g e_i)_1| = |g_{1,i}|$. $\square$

Example 6.4. Let

$$M := \frac{1}{4}\begin{bmatrix} -2 & 3\sqrt{2} & 3\sqrt{2} \\ 3\sqrt{2} & -1 & -1 \\ 3\sqrt{2} & -1 & -1 \end{bmatrix} = g\,\mathsf{Diag}(1, 0, -2) g^{-1}, \quad \text{where}$$

$$g := \frac{1}{2}\begin{bmatrix} \sqrt{2} & 0 & \sqrt{2} \\ 1 & -\sqrt{2} & -1 \\ 1 & \sqrt{2} & -1 \end{bmatrix} \in \mathrm{U}_3 .$$

Since $M^2 e_1 = 2e_1 - Me_1$, the cyclic subspace $\mathsf{cyc}(M; e_1)$ is 2-dimensional. Taking the matrix

$$\begin{bmatrix} | & | \\ e_1 & Me_1 \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{3}{2\sqrt{2}} \\ 0 & \frac{3}{2\sqrt{2}} \end{bmatrix}$$

and applying the Gram–Schmidt orthonormalization process gives $A := \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$. Then

$$M_T = A^* M A = \frac{1}{2}\begin{bmatrix} -1 & 3 \\ 3 & -1 \end{bmatrix} = h\,\mathsf{Diag}(1, -2) h^{-1},$$

$$\text{where } h := \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \in \mathrm{U}_2 .$$

We can verify that the properties of M_T in Proposition 6.3 hold in this case. $\Diamond$

We now consider actions of $\mathrm{GL}_n(\mathbb{C})$ and $\mathfrak{gl}_n(\mathbb{C})$ on the exterior power $\bigwedge^k(\mathbb{C}^n)$.

Definition 6.5. For $0 \le k \le n$, let $\bigwedge^k(\mathbb{C}^n)$ denote the kth exterior power of $\mathbb{C}^n$, which we identify with $\mathbb{C}^{\binom{n}{k}}$ by fixing the basis

$$\{e_{i_1} \wedge \cdots \wedge e_{i_k} : 1 \le i_1 < \cdots < i_k \le n\} \tag{6.1}$$

ordered lexicographically. Given $g \in \mathrm{GL}_n(\mathbb{C})$, we let $g^{(k)} \in \mathrm{GL}(\bigwedge^k(\mathbb{C}^n)) \cong \mathrm{GL}_{\binom{n}{k}}(\mathbb{C})$ denote the map induced by g, i.e.,

$$g^{(k)}(x_1 \wedge \cdots \wedge x_k) := gx_1 \wedge \cdots \wedge gx_k \quad \text{for all } x_1, \dots, x_k \in \mathbb{C}^n.$$

Given $M \in \mathfrak{gl}_n(\mathbb{C})$, we let $M_{(k)} \in \mathfrak{gl}(\bigwedge^k(\mathbb{C}^n)) \cong \mathfrak{gl}_{\binom{n}{k}}(\mathbb{C})$ denote the vector field induced by M, i.e.,

$$M_{(k)}(x_1 \wedge \cdots \wedge x_k) := \sum_{i=1}^{k} x_1 \wedge \cdots \wedge x_{i-1} \wedge Mx_i \wedge x_{i+1} \wedge \cdots \wedge x_k$$
$$\text{for all } x_1, \dots, x_k \in \mathbb{C}^n.$$

Equivalently, $M_{(k)} = \frac{d}{dt}\big|_{t=0} \exp(tM)^{(k)}$.

Deift, Li, Nanda, and Tomei [7] observe that the operations $M \mapsto M_T$ and $M \mapsto M_{(k)}$ are both compatible with the symmetric Toda flow (the proofs in [7] are over $\mathbb{R}$, but easily extend over $\mathbb{C}$). In the statements below, given a Hermitian matrix N, we let $N(t)$ denote the symmetric Toda flow (1.1) beginning at N.

Proposition 6.6 (*Deift, Li, Nanda, and Tomei [7, Propositions 7.2 and 7.3]*)**.** *Let M be an $n \times n$ Hermitian matrix, let $0 \le k \le n$, and let $t \in \mathbb{R}$. Then the following two diagrams commute:*

$$\begin{tikzcd} M \arrow[r, mapsto, "\text{Toda}"] \arrow[d, "\cdot_T"] & M(t) \arrow[d, "\cdot_T"] \\ M_T \arrow[r, mapsto, "\text{Toda}"] & M_T(t) = M(t)_T \end{tikzcd} \quad \textit{and}$$

$$\begin{tikzcd} M \arrow[r, mapsto, "\text{Toda}"] \arrow[d, "\cdot_{(k)}"] & M(t) \arrow[d, "\cdot_{(k)}"] \\ M_{(k)} \arrow[r, mapsto, "\text{Toda}"] & M_{(k)}(t) = M(t)_{(k)} \end{tikzcd} .$$

Deift, Li, Nanda, and Tomei [7] then apply Proposition 6.6 to embed the symmetric Toda flow into a product of tridiagonal Toda flows:

Theorem 6.7 (*Deift, Li, Nanda, and Tomei [7, Theorem p. 230]*)**.** *Let M be an $n \times n$ Hermitian matrix such that for all $1 \le k \le n-1$, the $\binom{n}{k}$ sums of k distinct eigenvalues of M are pairwise distinct. Then for all $t \in \mathbb{R}$, the following diagram commutes:*

$$\begin{tikzcd} M \arrow[r, mapsto, "\text{Toda}"] \arrow[d, "\left((\cdot_{(k)})_T\right)_{1 \le k \le n-1}"] & M(t) \arrow[d, "\left((\cdot_{(k)})_T\right)_{1 \le k \le n-1}"] \\ \left((M_{(k)})_T\right)_{1 \le k \le n-1} \arrow[r, mapsto, "\text{Toda}"] & \left((M_{(k)})_T(t)\right)_{1 \le k \le n-1} = \left((M(t)_{(k)})_T\right)_{1 \le k \le n-1} \end{tikzcd} . \tag{6.2}$$

Moreover, the map

$$t \mapsto \left((M_{(k)})_T(t)\right)_{1 \le k \le n-1} = \left((M(t)_{(k)})_T\right)_{1 \le k \le n-1} \quad \text{for } t \in \mathbb{R} \tag{6.3}$$

is injective. That is, a generic trajectory of the symmetric Toda flow embeds into a product of trajectories of tridiagonal Toda flows.

6.2. Tridiagonalization via the twist map, Plücker embedding, and Moser map

We now rephrase Theorem 6.7 (and give a new proof) using three maps we introduced earlier: the twist map ϑ, the Plücker embedding Δ, and the Moser map Mos_λ. For convenience, we extend the domain of Mos_λ from $\mathbb{P}^{n-1}_{>0}$ to $\mathbb{P}^{n-1}(\mathbb{C})$:

Definition 6.8. Recall the Moser map $\mathrm{Mos}_\lambda : \mathbb{P}^{n-1}_{>0} \to \mathcal{O}_\lambda$ defined in Definition 5.2. We define the *extended Moser map*

$$\widetilde{\mathrm{Mos}}_\lambda : \mathbb{P}^{n-1}(\mathbb{C}) \to \bigcup_\nu \mathcal{O}_\nu$$

as follows, where the union above is over all nonempty subsequences ν of λ. Given $x \in \mathbb{P}^{n-1}(\mathbb{C})$, we set $I := \{i \in [n] : x_i \neq 0\}$, and define

$$y := (|x_i| : i \in I) \in \mathbb{P}^{|I|-1}_{>0} \quad \text{and} \quad \nu := (\lambda_i : i \in I).$$

Then we define $\widetilde{\mathrm{Mos}}_\lambda(x) := \mathrm{Mos}_\nu(y)$.

Example 6.9. We have $\widetilde{\mathrm{Mos}}_{(\lambda_1,\lambda_2,\lambda_3)}(-1 : 0 : 3i + 4) = \mathrm{Mos}_{(\lambda_1,\lambda_3)}(1 : 5)$. $\diamondsuit$

We can write the tridiagonal Toda lattice flow in terms of the Moser map, as follows.

Proposition 6.10 (Moser [4, (1.4)]). *Let $L \in \mathcal{J}^{>0}_\lambda$, and let $L(t)$ denote the symmetric Toda flow (1.4) beginning at L. Take $x \in \mathbb{P}^{n-1}_{>0}$ as in Theorem 5.3 such that $\mathrm{Mos}_\lambda(x) = L$. Then*

$$L(t) = \mathrm{Mos}_\lambda(e^{\lambda_1 t}x_1 : \cdots : e^{\lambda_n t}x_n) \quad \textit{for all } t \in \mathbb{R}.$$

Proof. This is a restatement of (1.2). Alternatively, we can apply Theorem 4.6 and (4.1). $\square$

We recall that we are working with both Hermitian and skew-Hermitian matrices; if M is an $n\times n$ Hermitian matrix with distinct eigenvalues λ, then we have a corresponding skew-Hermitian matrix $L = \mathrm{i}M \in \mathcal{O}_\lambda$.

Proposition 6.11. *Let M be an $n\times n$ Hermitian matrix with distinct eigenvalues λ, and write $M = g\,\mathrm{Diag}(\lambda)g^{-1}$, where $g \in \mathrm{U}_n$. Then*

$$\widetilde{\mathrm{Mos}}_\lambda(g_{1,1} : \cdots : g_{1,n}) = \mathrm{i}M_T.$$

Proof. This follows from Proposition 6.3 and Theorem 5.3. $\square$

We now generalize Proposition 6.11 to give an interpretation of $(M_{(k)})_T$. For $0 \le k \le n$, we define

$$\lambda^{(k)} := \left(\sum_{i\in I}\lambda_i : I \in \tbinom{[n]}{k}\right) \in \mathbb{R}^{\binom{n}{k}}.$$

Moreover, we reorder the elements of $\binom{[n]}{k}$ (from the lexicographic order) so that the entries of $\lambda^{(k)}$ are weakly decreasing. Importantly, in both orders, the first element of $\binom{[n]}{k}$ is $[k]$.

Corollary 6.12. *Let M be an $n \times n$ Hermitian matrix with distinct eigenvalues λ, and let $V \in \mathrm{Fl}_n(\mathbb{C})$ denote the complete flag corresponding to $\mathrm{i}M$ under the isomorphism (1.5). Let $0 \le k \le n$ be such that the entries of $\lambda^{(k)}$ are distinct. Then*

$$\widetilde{\mathrm{Mos}}_{\lambda^{(k)}}\left(\Delta_I(\vartheta(V)) : I \in \tbinom{[n]}{k}\right) = \mathrm{i}(M_{(k)})_T.$$

Proof. Let $g \in \mathrm{U}_n$ be the canonical representative of V as in (3.2), so that $M = g\,\mathrm{Diag}(\lambda)g^{-1}$. Then the eigenvalues of M are $\lambda^{(k)}_I$ for $I \in \binom{[n]}{k}$, with corresponding eigenvectors $\bigwedge_{i\in I} ge_i$. The first entry of $\bigwedge_{i\in I} ge_i$ is the coefficient of $e_1 \wedge \cdots \wedge e_k$, namely, $\Delta_{[k],I}(g)$. We have

$$|\Delta_{[k],I}(g)| = |\Delta_I(\iota(g))| = |\Delta_I(\vartheta(V))|,$$

so the result follows from Proposition 6.11. $\square$

We now state our main result:

Theorem 6.13. *Let M be a Hermitian matrix with distinct eigenvalues λ such that for all $1 \le k \le n-1$, the entries of $\lambda^{(k)}$ are distinct. Let $V \in \mathrm{Fl}_n(\mathbb{C})$ denote the complete flag corresponding to $\mathrm{i}M$ under the isomorphism (1.5). Then under the Toda flow, we have*

$$(M_{(k)})_T(t) = (M(t)_{(k)})_T = -\mathrm{i}\widetilde{\mathrm{Mos}}_{\lambda^{(k)}}\left(e^{\lambda^{(k)}_I t}\Delta_I(\vartheta(V)) : I \in \tbinom{[n]}{k}\right) \tag{6.4}$$

for all $t \in \mathbb{R}$ and $1 \le k \le n-1$, which we see explicitly is a twisted gradient flow. In particular, the diagram (6.2) commutes, and we can write the embedding (6.3) as

$$t \mapsto \left(-\mathrm{i}\widetilde{\mathrm{Mos}}_{\lambda^{(k)}}\left(e^{\lambda^{(k)}_I t}\Delta_I(\vartheta(V)) : I \in \tbinom{[n]}{k}\right)\right)_{1\le k\le n-1} \quad \textit{for } t \in \mathbb{R}. \tag{6.5}$$

This embeds a generic trajectory of the symmetric Toda flow into a product of trajectories of tridiagonal Toda flows.

Proof. The fact that $(M_{(k)})_T(t)$ equals the right-hand side of (6.4) follows from Corollary 6.12 and Proposition 6.10. The fact that $(M(t)_{(k)})_T$ equals the right-hand side of (6.4) follows similarly, where instead of Proposition 6.10, we use (4.4) and Lemma 4.4. The fact that (6.5) is an embedding follows from Theorem 5.3. $\square$

Remark 6.14. Suppose λ is such that for all $1 \le k \le n-1$, the entries of $\lambda^{(k)}$ are distinct. While (6.3) (equivalently, (6.5)) is an embedding, the map

$$\begin{aligned} M \mapsto{}& \left((M_{(k)})_T\right)_{1\le k\le n-1} \\ ={}& \left(-\mathrm{i}\widetilde{\mathrm{Mos}}_{\lambda^{(k)}}\left(\Delta_I(\vartheta(V)) : I \in \tbinom{[n]}{k}\right)\right)_{1\le k\le n-1} \end{aligned} \tag{6.6}$$

(where $V \in \mathrm{Fl}_n(\mathbb{C})$ denotes the complete flag corresponding to $\mathrm{i}M$ under the isomorphism (1.5)) is not injective on the set of $n \times n$ Hermitian matrices M with eigenvalues λ. Indeed, we can see from the expression on the right-hand side above that for each $1 \le k \le n-1$, we only recover the vector $\left(|\Delta_I(\vartheta(V))| : I \in \binom{[n]}{k}\right)$ of absolute values of the Plücker coordinates of $\vartheta(V)_k$, and not necessarily the complex argument of each Plücker coordinate. If in addition we record the complex argument of each $\Delta_I(\vartheta(V))$ (which, up to a change in notation, is the element $\tau_I(M)$ of [7, Section 7]), then we obtain an injection. For example, the map (6.6) gives an embedding when restricted to $\mathrm{i}M \in \mathcal{O}^{>0}_\lambda$, since in this case every Plücker coordinate $\Delta_I(\vartheta(V))$ is positive.

Remark 6.15. Deift, Li, Nanda, and Tomei [7, Theorem 3.3] also show that the symmetric Toda flow is Liouville integrable, which depends on finding sufficient independent integrals in involution with respect to the appropriate Poisson bracket. As they discuss, the explicit embedding constructed in this section gives an alternative demonstration of integrability, which is independent of Liouville integrability. It would be interesting to further study the connection between these two approaches.

7. Toda flows on moment polytopes

We recall the discussion of the moment map μ from Section 5, and in particular Theorem 5.7, which implies that the twisted moment map $\mu \circ \vartheta_\lambda$ gives a diffeomorphism from $\mathcal{J}_\lambda^{>0}$ to the interior of $\mathsf{Perm}(\lambda)$. In this section, we study variations of this map applied not just to tridiagonal matrices $\mathcal{J}_\lambda^{\geq 0}$, but to a general adjoint orbit $\mathcal{O}_\lambda$. One such variation will be based on the embedding of Section 6.

We begin by studying the twisted moment map $\mu \circ \vartheta_\lambda$ on $\mathcal{O}_\lambda$. The following result is closely related to work of Kodama and Williams [27, Section 6].

Proposition 7.1. *The twisted moment map*

$$\mathcal{O}_\lambda \to \mathsf{Perm}(\lambda), \quad L \mapsto \mu(\vartheta_\lambda(L)) \tag{7.1}$$

is an embedding when restricted to a generic trajectory of the symmetric Toda flow (1.4).

Proof. Given $L \in \mathcal{O}_\lambda$, let $L(t)$ ($t \in \mathbb{R}$) denote the symmetric Toda flow beginning at L, and let $V(t) \in \mathrm{Fl}_n(\mathbb{C})$ denote the complete flag corresponding to $L(t)$ under the isomorphism (1.5). Then by (4.4), we have

$$\vartheta(V(t)) = \mathsf{Diag}(e^{\lambda_1 t}, \dots, e^{\lambda_n t}) \cdot \vartheta(V_0). \tag{7.2}$$

Now let $\mathrm{T}_n(\mathbb{C})$ denote the subset of diagonal matrices in $\mathrm{GL}_n(\mathbb{C})$, and let $\mathrm{T}_n^{>0}$ denote its positive part, consisting of diagonal matrices with positive diagonal entries. Then $\mathrm{T}_n(\mathbb{C})$ acts on $\mathrm{Fl}_n(\mathbb{C})$ by left multiplication. We may regard μ as a moment map on $\mathrm{Fl}_n(\mathbb{C})$, and therefore it maps a generic $\mathrm{T}_n^{>0}$-orbit homeomorphically onto the interior of the moment polytope $\mathsf{Perm}(\lambda)$ (see, e.g., [28, Section 4.2]). Since $\mathsf{Diag}(e^{\lambda_1 t}, \dots, e^{\lambda_n t}) \in \mathrm{T}_n^{>0}$ for all $t \in \mathbb{R}$, the result follows. □

Remark 7.2. We note that one can write down the moment map μ explicitly in terms of Plücker coordinates, as follows. Let $\mathsf{HS}_{k,n}$ denote the convex hull in $\mathbb{R}^n$ of all $\binom{n}{k}$ vectors with k 1's and $n-k$ 0's, called a *hypersimplex*. Then we have the *Grassmannian moment map*

$$\mu_{k,n} : \mathrm{Gr}_{k,n}(\mathbb{C}) \to \mathbb{R}^n, \quad V \mapsto \left(\frac{\sum_{I \in \binom{[n]}{k},\, I \ni i} |\Delta_I(V)|^2}{\sum_{I \in \binom{[n]}{k}} |\Delta_I(V)|^2} \right)_{1 \leq i \leq n},$$

whose image is $\mathsf{HS}_{k,n}$ (cf. [29, Section 2]). One can verify that if $P \in \mathfrak{gl}_n(\mathbb{C})$ is orthogonal projection onto $V \in \mathrm{Gr}_{k,n}(\mathbb{C})$, then $\mathrm{diag}(P) = \mu_{k,n}(V)$.

Now let $L \in \mathcal{O}_\lambda$, and let $V = (V_1, \dots, V_{n-1}) \in \mathrm{Fl}_n(\mathbb{C})$ denote the complete flag corresponding to L under the isomorphism (1.5). Recall from (2.5) that we may write

$$-\mathrm{i}L = \left(\sum_{k=1}^{n-1} (\lambda_k - \lambda_{k+1}) P_k \right) + \lambda_n I_n,$$

where $P_k \in \mathfrak{gl}_n(\mathbb{C})$ is orthogonal projection onto V_k. Then

$$\begin{aligned} \mu(L) &= \left(\sum_{k=1}^{n-1} (\lambda_k - \lambda_{k+1})\, \mathrm{diag}(P_k) \right) + \lambda_n\, \mathrm{diag}(I_n) \\ &= \left(\sum_{k=1}^{n-1} (\lambda_k - \lambda_{k+1}) \mu_{k,n}(V_k) \right) + \lambda_n(1, \dots, 1). \end{aligned} \tag{7.3}$$

We note that (7.3) provides a convenient way to calculate (7.1) for a trajectory $L(t)$ of the Toda flow, using the formula (7.2). Namely, we have

$$\mu(\vartheta_\lambda(L(t))) = \left(\sum_{k=1}^{n-1} (\lambda_k - \lambda_{k+1}) \mu_{k,n}\big(\mathsf{Diag}(e^{\lambda_1 t}, \dots, e^{\lambda_n t}) \cdot \vartheta(V_k)\big) \right) + \lambda_n(1, \dots, 1).$$

We now apply the moment map μ to the image of the embedding used in Theorem 6.13.

Proposition 7.3. *Suppose $\lambda \in \mathbb{R}^n$ is such that for all $1 \leq k \leq n-1$, the entries of $\lambda^{(k)}$ are distinct. Given $L \in \mathcal{O}_\lambda$, let $V \in \mathrm{Fl}_n(\mathbb{C})$ denote the complete flag corresponding to L under the isomorphism (1.5). Let $\mathcal{O}_\lambda'$ denote the subset of $\mathcal{O}_\lambda$ of all L such that $\Delta_I(\vartheta(V)) \neq 0$ for all $I \subseteq [n]$. Then we have the continuous map*

$$\begin{aligned} &\mathcal{O}_\lambda' \to \prod_{k=1}^{n-1} \mathsf{Perm}(\lambda^{(k)}), \\ &L \mapsto \left(\mu\Big(\widetilde{\mathsf{Mos}}_{\lambda^{(k)}}\big(\Delta_I(\vartheta(V)) : I \in \tbinom{[n]}{k}\big)\Big) \right)_{1 \leq k \leq n-1}. \end{aligned} \tag{7.4}$$

Moreover, when (7.4) is restricted to any subset of $\mathcal{O}_\lambda'$ which fixes the complex argument of $\Delta_I(\vartheta(V))$ for each nonempty $I \subsetneq [n]$, then it is a diffeomorphism onto its image. In particular, (7.4) restricted to $\mathcal{O}_\lambda^{>0}$ is a diffeomorphism onto its image.

Note that (7.4) is obtain by applying (6.6) (to $L = \mathrm{i}M$, rather than M), and then applying μ to each of the $n-1$ factors.

Proof. This follows from Theorem 5.7, using the discussion in Remark 6.14. □

Remark 7.4. It is not clear how to extend Proposition 7.3 from $\mathcal{O}_\lambda^{>0}$ to $\mathcal{O}_\lambda^{\geq 0}$. This is because the Moser map Mos_λ is only defined on $\mathbb{P}_{>0}^{n-1}$, not $\mathbb{P}_{\geq 0}^{n-1}$.

Remark 7.5. We note that in (7.4), the codomain has dimension much greater than that of the domain ($\prod_{k=1}^{n-1}(\binom{n}{k} - 1)$ versus $n!$). This comes from the same property of the Plücker embedding (2.3), whose image in $\prod_{k=1}^{n-1} \mathbb{P}^{\binom{n}{k}-1}(\mathbb{C})$ is cut out by *Grassmann–Plücker* and *incidence-Plücker relations*. It may be interesting to study the image of this subset in $\prod_{k=1}^{n-1} \mathsf{Perm}(\lambda^{(k)})$.

The formula (7.3) suggests an interpolation between the two maps considered in Propositions 7.1 and 7.3, namely,

$$\mathcal{O}_\lambda \to \prod_{k=1}^{n-1} \mathsf{HS}_{k,n}, \quad L \mapsto \big(\mu_{k,n}(\vartheta(V_k))\big)_{1 \leq k \leq n-1} \tag{7.5}$$

(with notation as in Remark 7.2). We do not expect (7.5) to be injective on $\mathcal{O}_\lambda$, for similar reasons as discussed in Remark 6.14. However, we do not know whether it is injective when we require all Plücker coordinates to be nonnegative:

Problem 7.6. Is the map (7.5) injective when restricted to $\mathcal{O}_\lambda^{\geq 0}$?

CRediT authorship contribution statement

Anthony M. Bloch: Contributed to the theory and exposition in this work. **Steven N. Karp:** Contributed to the theory and exposition in this work.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgments

We thank the anonymous reviewers for helpful comments.

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