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Discrete Mechanics and Optimal Control for Passive Walking with Foot Slippage

Alexandre Anahory Simoes[1], Asier López-Gordón[2], Anthony Bloch[3] and Leonardo Colombo[4]

***Abstract*— Forced variational integrators are given by the discretization of the Lagrange-d'Alembert principle for systems subject to external forces, and have proved useful for numerical simulation studies of complex dynamical systems. In this paper we model a passive walker with foot slip by using techniques of geometric mechanics, and we construct forced variational integrators for the system. Moreover, we present a methodology for generating (locally) optimal control policies for simple hybrid holonomically constrained forced Lagrangian systems, based on discrete mechanics, applied to a controlled walker with foot slip in a trajectory tracking problem.**

I. Introduction

Passive-dynamic walkers [1], [2], [3], are templates for human-like walking, describing its biomechanical and energetic aspects. They are uncontrolled and unpowered mechanisms that balance themselves as they walk, similarly to how people walk down a slope. They are modeled as dissipative dynamical systems since energy is lost when collisions are made with the ground. Mastering passive dynamics helps to understand the mechanics of walking. Passive dynamic walkers are piecewise holonomic systems [4], [5], that is, mechanical systems that change at each transition of the dynamics, although they are holonomic within each stride before impacts occur [5].

An implicit assumption for passive-dynamic walkers is that the feet do not slip on contact with the ground. In this paper, we model passive walkers with foot slippage over a flat ground inspired by [6]. In comparison with the model proposed by [6], our model avoids incrementing the dimension of the configuration space to include Lagrange multipliers. In our approach, we reduce the dynamics of the system to the constraint submanifold generated by the constraints associated with the proposed model for walking with foot slip, giving rise to motion equations with fewer degrees of freedom than [6].

Trajectory optimization algorithms aim to find an input trajectory that minimizes a cost function subject to a set of constraints on the system's states and inputs. Trajectory optimization has been implemented extensively for systems with continuous-time dynamics, but many applications in control theory and robotics include impacts and friction contacts making the dynamics non-smooth. In this paper, we develop a trajectory optimization policy for a passive walker experiencing foot slip by introducing controls into the passive walker and by defining geometric integrators for a class of hybrid mechanical systems- that is, (smooth) dynamical systems together with a discrete transition (impact map)- by using discrete (geometric) mechanics techniques.

Variational integrators has been applied to a wide range of problems including optimal control [7], [8], constrained systems [9], [10], nonholonomic systems [11], [12], [13], multi-agent systems [14], [15], [16], etc. Variational integrators for hybrid mechanical systems were used in [17] and [18]. However, these works do not consider the problem of trajectory generation. Such a problem is considered in [19] but for the compass gait biped, while in this work we consider passive walkers under foot slippage.

The remainder of the paper is structured as follows. In Section II the constrained Lagrange-d'Alembert principle is used to derive forced Euler-Lagrage equations for mechanical systems subject to holonomic constraints. After defining simple hybrid holonomically constrained forced Lagrangian system in II-A, we construct a model for a passive walker with foot slip in Section II-B. In Section III, we derive the corresponding variational integrators for this model. Finally, in Section IV we introduce controls into the previous model and together with a suitable discretization of the cost function associated to an optimal control problem, we are able to derive optimal control policies for trajectory generation in a tracking problem.

II. A Passive walker with foot slip

In this section we will examine a simple case of a passive walker where the base is allowed to slide and we will formulate this system as a simple hybrid Lagrangian system. This model is inspired by [6]. In comparison with that model ours avoids incrementing the dimension of the configuration space to include Lagrange multipliers. In our approch we reduce the dynamics of the system to the constraint submanifold N.

[1] A. Anahory Simoes (alexandre.anahory@ie.edu) is with School of Sciences and Technology, IE University, P. de la Castellana 259, 28046 Madrid, Spain.

[2]A. López-Gordón (asier.lopez@icmat.es) is with Instituto de Ciencias Matemáticas (ICMAT-CSIC), C/ Nicolás Cabrera, 13-15, 28049 Madrid, Spain.

[3]A. Bloch (abloch@umich.edu) is with Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA.

[4]L. Colombo (leonardo.colombo@car.upm-csic.es) is with Centro de Automática y Robótica (CAR-CSIC), Ctra. M300 Campo Real, Km 0, 200, Arganda del Rey - 28500 Madrid, Spain.

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A. Simple hybrid holonomically constrained forced Lagrangian systems

Simple hybrid systems [20] (see also [21]) are characterized by the tuple $\mathbf{H} = (D, f, \mathcal{S}, \Delta)$, where D is a smooth manifold, the *domain*, f is a smooth *vector field* on D, $\mathcal{S}$ an embedded submanifold of D with co-dimension 1 called *switching surface*, and $\Delta : \mathcal{S} \to D$ a smooth embedding called the *impact map*. The submanifold $\mathcal{S}$ and the map Δ are also referred to as the *guard* and *reset map*, respectively, in [22]-[23]

The dynamics associated with a hybrid systems corresponds to an autonomous system with impulse effects. We denote by $\Sigma_{\mathbf{H}}$ the *simple hybrid dynamical system* generated by $\mathbf{H}$, that is,

$$\Sigma_{\mathbf{H}} : \begin{cases} \dot{x}(t) = f(x(t)), & x^-(t) \notin \mathcal{S} \\ x^+(t) = \Delta(x^-(t)) & x^-(t) \in \mathcal{S} \end{cases} \tag{1}$$

with $x : I \subset \mathbb{R} \to D$ and x^-, x^+ the states just before and after the moments when integral curves of f intersects $\mathcal{S}$.

Remark 1: A solution of a simple hybrid system may experience a Zeno state if infinitely many impacts occur in a finite amount of time [22], [24], [25], [26]. However, by considering the class of hybrid systems given by mechanical systems with impulsive effects as in [21], we exclude such behavior by considering that the set of impact times is closed and discrete, meaning that there is no chatering about an impact point and therefore excluding Zeno behavior. Necessary and sufficient conditions for the existence of Zeno behavior in the class of simple hybrid Lagrangian systems have been explored in [27] and [25]. $\diamond$

Let Q be an n-dimensional differentiable manifold with local coordinates (q^A), $1 \leq A \leq n$, the configuration space of a mechanical system. Denote by TQ its tangent bundle with induced local coordinates $(q^A, \dot{q}^A)$. Consider $D = TQ$ and a hyper-regular Lagrangian $L : TQ \to \mathbb{R}$. Associated with the dynamics generated by L, there exists a Lagrangian vector field f_L. Note that $\Delta : \mathcal{S} \to TQ$ is continuous. If we denote the closure of $\Delta(\mathcal{S})$ by $\overline{\Delta(\mathcal{S})}$, then we must assume $\overline{\Delta(\mathcal{S})} \cap \mathcal{S} = \emptyset$ and, therefore, an impact does not lead immediately to another impact (see Section 4.1 [21] for more details).

We further assume that $\mathcal{S} \neq \emptyset$ and there exists an open subset $U \subset TQ$ and a differentiable function $h : U \to \mathbb{R}$ such that $\mathcal{S} = \{x \in U \mid h(x) = 0\}$ with $\frac{\partial h}{\partial x}(s) \neq 0$ for all $s \in \mathcal{S}$ (that is, $\mathcal{S}$ is an embedded submanifold of TQ with co-dimension 1) and the Lie derivative of the vector field f_L with respect to h does not vanish on TQ, that is $\mathcal{L}_{f_L} h(w) \neq 0$, $\forall w \in TQ$. A trajectory $\gamma : [0, T] \to TQ$ crosses the switching surface $\mathcal{S}$ at $t_i^- = \inf\{t > 0 | \gamma(t) \in \mathcal{S}\}$. We allow the trajectory $\gamma(t)$ to be continuous but nonsmooth at t_i^-. That is, the velocity before the impact $\dot{q}^-$ is different from the velocity q^+ after the impact at $\mathcal{S}$, namely, $\dot{q}(t_i^-) \neq \dot{q}(t_i^+)$.

Definition 2: A simple hybrid system $\mathbf{H} = (D, f, \mathcal{S}, \Delta)$ is said to be a *simple hybrid holonomically constrained forced Lagrangian system* if it is determined by $\mathbf{H}^{L_N} := (TN, f_N, \mathcal{S}_N, \Delta_N)$, where $f_N : TN \to T(TN)$ is the flow for the holonomically constrained forced Lagrangian system as described in Section II-A, and $\mathcal{S}_N$ and Δ_N are the switching surface and impact maps as described above restricted to submanifolds N and TN, respectively.

The *simple hybrid Lagrangian dynamical system* generated by $\mathbf{H}^{L_N}$ is given by

$$\Sigma_{\mathbf{H}^{L_N}} : \begin{cases} \dot{x}(t) = f_N(x(t)), \text{ if } x^-(t) \notin \mathcal{S}_N, \\ x^+(t) = \Delta_N(x^-(t)), \text{ if } x^-(t) \in \mathcal{S}_N, \end{cases}$$

where $x(t) = (q(t), \dot{q}(t)) \in TN$.

That is, a *trajectory* of a simple hybrid holonomically constrained forced Lagrangian system is determined by the restricted forced Lagrangian dynamics until the instant when the state attains the switching surface $\mathcal{S}_N$. We refer to such an instant as the *impact time*. The impact map Δ_N gives new initial conditions from which f_N evolves until the next impact occurs. Solutions for the simple hybrid holonomically constrained forced Lagrangian system $\mathbf{H}^{L_N}$, are considered right continuous and with finite left and right limits at each impact with $\mathcal{S}_N$.

B. Modeling passive walking with foot slip

We model a passive walker as a two-masses inverted pendulum. The mass of the foot is denoted by m_1 and the hip by m_2. The length of the leg is given by ℓ. The angles of the leg are restricted to $\theta \in [-a, a] \subset \mathbb{R}$ (when θ hits the boundary, $-a$, a new step is taken and θ is reset to a). The coordinates of the center of mass will be given by (x, y) and the coordinates of the foot are $(\overline{x}, \overline{y})$ (see Figure 1).

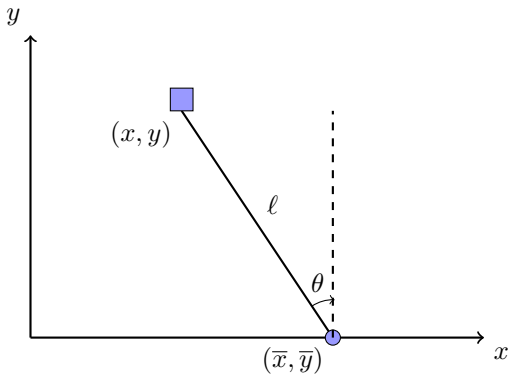

Fig. 1: Leg and foot: The coordinates of the foot are given by $(\overline{x}, \overline{y})$, the center of mass are (x, y) and θ is the angle between the leg of length ℓ and the vertical axis.

Let us denote by $m = m_1 + m_2$, $I = \frac{\ell^2 m_1 m_2}{m}$, $r = \frac{\ell m_2}{m}$, where I is the moment of inertia about the center of mass and r is the distance from the foot to the center of mass, which is kept constant along the motion. Also, note that the coordinates of the center of the mass satisfy $x = \overline{x} + r \sin\theta$, $y = \overline{y} + r\cos\theta$, so that the center of the mass is located along the leg at some point between the foot and the hip. In addition, we impose the constraint $\overline{y} = 0$ which means that the foot will not leave the floor. With this notation, the constraint implies that $y - r\cos\theta = 0$.

Note that for this model, θ is constrained to be in $[-a, a]$. If θ crosses the negative boundary, we say a new step occurs and θ is reset to a. If θ crosses the positive boundary, specifically if $\theta = \frac{\pi}{2}$ (i.e., $x = \bar{x}$), we say that a crash has occurred. In this case, the model stops walking and we report a failure. This also implies that falling forwards is not permitted; the only way to crash is by falling backwards.

Before deriving the hybrid dynamics, we first need to determine the switching surface $\mathcal{S}$. Assume that the leg takes symmetric steps of angle a, that is, $\theta \in [a, -a]$. When $\theta = -a$, the angle is reset to a (corresponding to a new step taking place and the swing legs switching). Therefore, we will take the switching surface $\mathcal{S}$, to be $\mathcal{S} = \{\theta = -a\}$.

The continuous dynamics is determined by a Lagrangian function $L : TQ \to \mathbb{R}$ corresponding to a planar rigid body, where $Q = \mathbb{R}^2 \times \mathbb{S}^1$ is the configuration space locally described by the coordinates $q = (x, y, \theta)$, and $L(q, \dot{q}) = K(q, \dot{q}) - V(q)$, where

$$K(q, \dot{q}) = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{I\dot{\theta}^2}{2},\ V(q) = mgr\cos\theta,$$

together with the (holonomic) constraint $y - r\cos\theta = 0$ defining the submanifold N which may be seen as diffeomorphic to $\mathbb{R} \times \mathbb{S}^1$. The restricted Lagrangian L_N defined on coordinates $(x, \theta, \dot{x}, \dot{\theta})$ is given by the restricted kinetic energy

$$K_N = \frac{m}{2}(\dot{x}^2 + r^2\sin^2\theta\dot{\theta}^2) + \frac{I\dot{\theta}^2}{2}$$

minus the restricted potential function which remains the same under the restriction to N.

We assume that the friction forces of the foot with the ground are non-conservative forces (conservative forces might be included into the potential energy V), which are determined by a fibered map $F : TQ \to T^*Q$. The forces exerted from the friction on the foot in the configurations $q = (x, y, \theta)$ are given by

$$F_x = -\kappa\dot{\bar{x}} = -\kappa(\dot{x} + r\dot{\theta}\cos\theta), \quad F_y = 0,$$
$$F_\theta = -\kappa\dot{\bar{x}}(r\cos\theta) = -\kappa r\cos\theta(\dot{x} + r\dot{\theta}\cos\theta).$$

This force is well-defined on the restriction to N.

At a given position and velocity, the force will act against variations of the position (virtual displacements) and the dynamics should also satisfy the holonomic constraint $\Phi(q) = y - r\cos\theta = 0$.

Euler–Lagrange equations for the restricted Lagrangian L_N and forces F_N are given by

$$m\ddot{x} = -\kappa(\dot{x} + r\dot{\theta}\cos\theta) \tag{2}$$
$$\ddot{\theta}(I + mr^2\sin^2\theta) = -\kappa r\cos\theta(\dot{x} + r\dot{\theta}\cos\theta) + rm\sin\theta(g - r\dot{\theta}^2\cos\theta) \tag{3}$$

on the submanifold N.

The last step to describe the hybrid dynamics for the simple hybrid holonomically constrained forced Lagrangian system is to find the impact map Δ_N. We assume as in [28] a rigid hip, that is, the horizontal position and velocity of the foot do not change at impacts (see Figure 2), namely $\bar{x}^+ = \bar{x}^-$ and $\dot{\bar{x}}^+ = \dot{\bar{x}}^-$. Additionally, we assume that the angular momentum is conserved in the impact. Under these assumptions (see [28], [3] for the case without foot slip and horizontal ground), the impact map is defined as the map $\Delta_N : \mathcal{S} \to TN \subseteq TQ$, where $\mathcal{S}_N = \{\theta = -a\}$, with $\Delta_N(x^-, -a, \dot{x}^-, \dot{\theta}^-) = (x^+, \theta^+, \dot{x}^+, \dot{\theta}^+)$ given by

$$\begin{aligned} x^+ - r\sin\theta^+ &= x^- - r\sin(-a) \\ \theta^+ &= \theta^- + 2a \\ \dot{x}^+ - r\dot{\theta}^+\cos\theta^+ &= \dot{x}^- - r\dot{\theta}^-\cos(-a) \\ \dot{\theta}^+ &= \cos(2a)\dot{\theta}^-. \end{aligned} \tag{4}$$

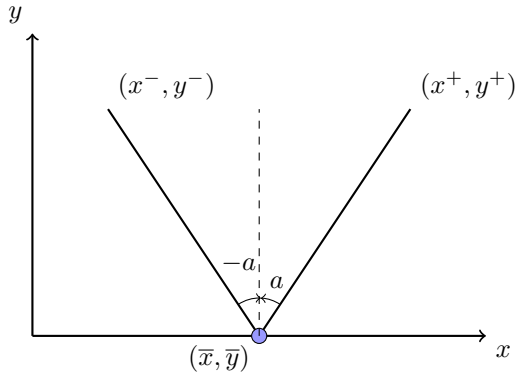

Fig. 2: Depiction of the impact. The position of the foot is continuous at the impact. Resetting the angle forces a reset on the position of the center of mass.

III. Forced variational integrator for a passive walker experiencing foot slip

A *discrete Lagrangian* is a differentiable function $L_d \colon Q \times Q \to \mathbb{R}$, which may be considered as an approximation of the action integral defined by a continuous regular Lagrangian $L \colon TQ \to \mathbb{R}$. That is, given a time step $h > 0$ small enough,

$$L_d(q_0, q_1) \approx \int_0^h L(q(t), \dot{q}(t))\ \mathrm{d}t,$$

where $q(t)$ is the unique solution of the Euler–Lagrange equations with boundary conditions $q(0) = q_0$, $q(h) = q_1$.

Construct the grid $\mathcal{T} = \{t_k = kh \mid k = 0, \ldots, N\}$, with $Nh = T$ and define the discrete path space $\mathcal{P}_d(Q) \coloneqq \{q_d : \{t_k\}_{k=0}^N \to Q\}$. We identify a discrete trajectory $q_d \in \mathcal{P}_d(Q)$ with its image $q_d = \{q_k\}_{k=0}^N$, where $q_k \coloneqq q_d(t_k)$. The discrete action $\mathcal{A}_d : \mathcal{P}_d(Q) \to \mathbb{R}$ for this sequence of discrete paths is calculated by summing the discrete Lagrangian on each adjacent pair, and it is defined by

$$\mathcal{A}_d(q_d) = \mathcal{A}_d(q_0, ..., q_N) \coloneqq \sum_{k=0}^{N-1} L_d(q_k, q_{k+1}). \tag{5}$$

The discrete variational principle [29], states that the solutions of the discrete system determined by L_d must extremize the action sum given fixed points q_0 and q_N. Extremizing $\mathcal{A}_d$ over q_k with $1 \le k \le N-1$, we obtain the following system of difference equations

$$D_1L_d(q_k, q_{k+1}) + D_2L_d(q_{k-1}, q_k) = 0. \tag{6}$$

These equations are usually called *the discrete Euler–Lagrange equations*. Given a solution $\{q_k^*\}_{k\in\mathbb{N}}$ of Eq. (6) and assuming the regularity hypothesis, i.e., the matrix $(D_{12}L_d(q_k, q_{k+1}))$ is regular, it is possible to define implicitly a (local) discrete flow $\Upsilon_{L_d}\colon \mathcal{U}_k \subset Q \times Q \to Q \times Q$ by $\Upsilon_{L_d}(q_{k-1}, q_k) = (q_k, q_{k+1})$ from (6), where $\mathcal{U}_k$ is a neighborhood of the point (q_{k-1}^*, q_k^*).

A. Forced variational integrators for holonomically constrained forced Lagrangian systems

The key idea of variational integrators is that the variational principle is discretized rather than the resulting equations of motion. As we explained before, we discretize the state space TQ as $Q \times Q$ and consider a discrete Lagrangian $L_d : Q \times Q \to \mathbb{R}$ and, in addition, we consider discrete "external forces" $F_d^{\pm} : Q \times Q \to T^*Q$ approximating the continuous-time action and non-conservative external forces given by

$$\int_{t_k}^{t_{k+1}} L(q(t), \dot{q}(t))\, dt \simeq L^d(q_k, q_{k+1}) \tag{7}$$

$$\int_{t_k}^{t_{k+1}} F_i(q(t), \dot{q}(t))\delta q\, dt \simeq F_d^-(q_k, q_{k+1})\delta q_k + F_d^+(q_k, q_{k+1})\delta q_{k+1}. \tag{8}$$

Alternatively, we can directly work with a discretized version of the submanifold N. Here, the restricted discrete Lagrangian $L_N^d : N \times N \to \mathbb{R}$ and discrete "external forces" $F_{N,d}^{\pm} : N \times N \to T^*N$ are approximating the continuous time restricted Lagrangian and force map, respectively.

The discrete-time forced Euler–Lagrange equations on the submanifold N are

$$0 = D_1 L_N^d(q_k, q_{k+1}) + D_2 L_N^d(q_{k-1}, q_k) \tag{9}$$

$$+ F_{N,d}^-(q_k, q_{k+1}) + F_{N,d}^+(q_{k-1}, q_k). \tag{10}$$

B. Constrained forced variational integrator for a passive walker under foot slip

Next, consider the midpoint (second order) discretization rule, that is, $q(t) \simeq \frac{q_k+q_{k+1}}{2}$, $\dot{q}(t) \simeq \frac{q_{k+1}-q_k}{h}$ and define the discrete Lagrangian $L_d : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ as

$$L_d(q_k, q_{k+1}) = hL\left(\frac{q_k + q_{k+1}}{2}, \frac{q_{k+1} - q_k}{h}\right),$$

with $h > 0$ denoting the time step and $q_k = (x_k, y_k, \theta_k)$ for $k = 0, \ldots, N$.

In our model, the discrete Lagrangian $L_d : (\mathbb{R} \times \mathbb{S}) \times (\mathbb{R} \times \mathbb{S}) \to \mathbb{R}$ is given by

$$L_d = \frac{m}{2h}(x_{k+1} - x_k)^2 + \frac{1}{2h}\left(I + mr^2 \sin^2 \frac{\theta_{k+1} + \theta_k}{2}\right) \times (\theta_{k+1} - \theta_k)^2 - hmgr \cos\left(\frac{\theta_k + \theta_{k+1}}{2}\right).$$

The discrete external forces are given by

$$F_d^+ = \frac{h}{2} F\left(q = \frac{q_{k-1} + q_k}{2}, \dot{q} = \frac{q_k - q_{k-1}}{h}\right), \tag{11}$$

$$F_d^- = \frac{h}{2} F\left(q = \frac{q_k + q_{k+1}}{2}, \dot{q} = \frac{q_{k+1} - q_k}{h}\right). \tag{12}$$

where F_d^+ is evaluated in (q_{k-1}, q_k) and F_d^- is evaluated in (q_k, q_{k+1}). Note that the restricted discrete force maps have the same expression.

The discrete Euler–Lagrange equations with forces are then

$$\begin{aligned} 0 =& \frac{m}{h}(2x_k - x_{k+1} - x_{k-1}) + hmg \sin\alpha + F_{d,x}^- + F_{d,x}^+, \\ 0 =& \frac{I}{h}(2\theta_k - \theta_{k-1} - \theta_{k+1}) \\ &+ \frac{mr^2}{2h} \sin\frac{\theta_k + \theta_{k-1}}{2} \cos\frac{\theta_k + \theta_{k-1}}{2}(\theta_k - \theta_{k-1})^2 \\ &+ \frac{mr^2}{2h} \sin\frac{\theta_{k+1} + \theta_k}{2} \cos\frac{\theta_{k+1} + \theta_k}{2}(\theta_{k+1} - \theta_k)^2 + F_{d,\theta}^+ \\ &+ F_{d,\theta}^- - \frac{mghr}{2}\left(\sin(\frac{\theta_k + \theta_{k-1}}{2}) - \sin(\alpha - \frac{\theta_k + \theta_{k+1}}{2})\right). \end{aligned}$$

IV. Discrete Mechanics and Optimal Control for a Controlled Walker under Foot Slip

Next, we add control forces to the previous formalism. The equations of motion are now given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_N}{\partial \dot{q}^A}\right) - \frac{\partial L_N}{\partial q^A} = u_a \overline{Y}_A^a + (F_N)_A, \tag{13}$$

where $\overline{Y}^a = \overline{Y}_A^a(q)dq^A$, $1 \le a \le m < n$ are the control forces, $u(t) = (u_1(t), ..., u_m(t)) \in U$ are the control inputs, and U is an open subset of $\mathbb{R}^m$, the set of admissible controls.

Let us suppose now that the control force is given by $\bar{Y}^1 = \mathrm{d}x$ and $\bar{Y}^2 = \mathrm{d}\theta$. Hence, we have the following controlled equations of motion on N

$$m\ddot{x} = -\kappa(\dot{x} + r\dot{\theta}\cos\theta) + u_x, \tag{14}$$

$$mr^2(\ddot{\theta}\sin^2\theta + \dot{\theta}^2\cos\theta\sin\theta) + I\ddot{\theta} = -\kappa r\cos\theta(\dot{x} + r\dot{\theta}\cos\theta) + u_\theta, \tag{15}$$

as long as $\theta \neq -a$.

Remark 3: Note that in the restricted configuration space $\mathbb{R} \times \mathbb{S}^1$ the system is fully actuated but in the ambient space $\mathbb{R}^2 \times \mathbb{S}^1$ the system is underactuated. ⋄

Suppose that we would like to follow a known reference trajectory $\gamma_r : [0, T] \to Q$ denoted by $\gamma_r(t) = (x_r(t), \theta_r(t))$. We want to find a control strategy minimizing the cost functional

$$\mathcal{J}(q, u) = \frac{1}{2}\int \varepsilon\|u\|^2 + \eta\|\gamma_r - \gamma\|^2 + \rho\|\dot{\gamma}_r - \dot{\gamma}\|^2\ \mathrm{d}t,$$

with γ satisfying the control equations (14) and (15). The parameters ε, η and ρ are the weights of the control inputs, the trajectory-tracking and the velocity-tracking terms, respectively,

We may transpose the optimal control problem to a nonlinear constrained optimization problem using a discretization of the principle above. Indeed, after applying the discretization procedure we come down to the problem of minimizing

$$J_d(q_d, u_d) = \sum_{k=0}^{N-1} C_d(q_k, q_{k+1}, u_k, u_{k+1}),$$

subject to the discrete dynamics

$$\begin{aligned} 0 =& D_1 L_N^d(q_k, q_{k+1}) + D_2 L_N^d(q_{k-1}, q_k) \\ &+ F_{N,d}^-(q_k, q_{k+1}) + F_{N,d}^+(q_{k-1}, q_k) + u_{k-1} + u_k, \end{aligned} \tag{16}$$

with the boundary values q_0, q_N given. Notice that Eq. (16) correspond to the forced discrete Euler–Lagrange equations (10) with $F_{N,d,u}^{\pm}(q_k, q_{k+1}, u_k) = F_{N,d}^{\pm}(q_k, q_{k+1}) + u_k$.

Next, we discretize the optimal control problem. Fixing a time step $h > 0$, we discretize the cost function so that

$$C_d(q_k, q_{k+1}, u_k, u_{k+1}) \approx \int_{kh}^{(k+1)h} C\left(q(t), \dot{q}(t), u(t)\right)\, \mathrm{d}t.$$

Thus we set

$$C_d(q_k, q_{k+1}, u_k, u_{k+1}) = hC\left(\frac{q_k + q_{k+1}}{2}, \frac{q_{k+1} - q_k}{h}, u_k\right),$$

where $u_k = u\left(\frac{t_k + t_{k+1}}{2}\right)$.

The problem is subjected to the discrete dynamics

$$\begin{aligned} 0 =& \frac{m}{h}(2x_k - x_{k+1} - x_{k-1}) + F_{d,x}^- + F_{d,x}^+ \\ &+ u_{x,k} + u_{x,k-1}, \\ 0 =& mr\left\{2gh\left(\sin\left(\frac{\theta_{k-1} + \theta_k}{2}\right) + \sin\left(\frac{\theta_k + \theta_{k+1}}{2}\right)\right)\right. \\ &+ \frac{r}{h}\left[(\theta_{k-1} - \theta_k)^2 \sin(\theta_{k-1} + \theta_k)\right. \\ &+ 2(\theta_{k-1} - \theta_k)\cos(\theta_{k-1} + \theta_k) \\ &+ (\theta_k - \theta_{k+1})^2 \sin(\theta_k + \theta_{k+1}) \\ &\left.\left. + 2(\theta_{k+1} - \theta_k)\cos(\theta_k + \theta_{k+1})\right]\right\} \\ &- \frac{2}{h}(\theta_{k-1} - 2\theta_k + \theta_{k+1})\left(2I + mr^2\right) \\ &+ 4(F_{d,\theta}^- + F_{d,\theta}^+ + u_{\theta,k-1} + u_{\theta,k}), \end{aligned} \tag{17}$$

and to the boundary conditions $q_0 = (x_0, \theta_0)$ and $q_N = (x_N, \theta_N)$ fixed. In addition, we have the following conditions on the initial and final velocities:

$$\begin{aligned} \mathbb{F}L(q_0, \dot{q}_0) &= \mathbb{F}^{F_{N,d,u}^-} L_d(q_0, q_1, u_0), \\ \mathbb{F}L(q_N, \dot{q}_N) &= \mathbb{F}^{F_{N,d,u}^+} L_d(q_{N-1}, q_{N-1}, u_{N-1}), \end{aligned} \tag{18}$$

where $\mathbb{F}L$ denotes the continuous Legendre transform and $\mathbb{F}^{F_{N,d,u}^{\pm}}$ denotes the forced discrete Legendre transform, i.e.,

$$D_2 L(q_0, \dot{q}_0) + D_1 L_d(q_0, q_1) + F_d^-(q_0, q_1) + u_0 = 0, \tag{19}$$

$$\begin{aligned} &D_2 L(q_N, \dot{q}_N) - D_2 L_d(q_{N-1}, q_N) \\ &\quad - F_d^+(q_{N-1}, q_N) + u_{N-1} = 0. \end{aligned} \tag{20}$$

Remark 4: If we discretize the reference trajectory by evaluating it at discrete time $\gamma_r\left(h\frac{2k+1}{2}\right) = \left(x_r\left(h\frac{2k+1}{2}\right), \theta_r\left(h\frac{2k+1}{2}\right)\right) \equiv (x_{r,k}, \theta_{r,k})$, then the midpoint discrete cost function reads

$$\begin{aligned} C_d(q_k, q_{k+1}, u_k, u_{k+1}) = \frac{h}{2}\Bigg[&\varepsilon u_k^2 + \eta\left(\frac{x_{k+1} + x_k}{2} - x_{r,k}\right)^2 \\ &+ \eta\left(\frac{\theta_{k+1} + \theta_k}{2} - \theta_{r,k}\right)^2 + \rho\left(\frac{x_{k+1} - x_k}{h} - \dot{x}_{r,k}\right)^2 \\ &+ \rho\left(\frac{\theta_{k+1} - \theta_k}{h} - \dot{\theta}_{r,k}\right)^2\Bigg]. \end{aligned}$$

The discrete optimal control problem consists on finding a discrete trajectory $\{(x_k, \theta_k, u_k)\}$ solution of the problem

$$\begin{aligned} &\min \sum_{k=0}^{N-1} C_d(q_k, q_{k+1}, u_k, u_{k+1}) \\ &\quad \text{discrete equations (17)} \\ &\quad \text{boundary conditions (18)} \end{aligned} \tag{21}$$

Next, we incorporate impacts in the variational setting by finding a discretization of the impact set $S_d \subseteq Q \times Q$ and of the impact map $\Delta_d : S_d \to Q \times Q$. Let

$$S_d = \{(x_0, \theta_0, x_1, \theta_1) | \theta_0 = a\}$$

and $\Delta_d(x_0^-, -a, x_1^-, \theta_1^-)$ is given by the discretization of Eqs. (4) via the midpoint rule:

$$\begin{aligned} &\frac{x_0^+ + x_1^+}{2} = \frac{x_0^- + x_1^-}{2} - r\sin(-a) + r\sin\left(\frac{\theta_0^+ + \theta_1^+}{2}\right), \\ &\theta_0^+ = 2a + \theta_0^-, \\ &x_1^+ - x_0^+ - r(\theta_1^+ - \theta_0^+)\cos\left(\frac{\theta_0^+ + \theta_1^+}{2}\right) = x_1^- - x_0^- \\ &-r(\theta_1^- - \theta_0^-)\cos(-a), \\ &\theta_1^+ - \theta_0^+ = \cos(2a)(\theta_1^- - \theta_0^-), \end{aligned}$$

that is,

$$\begin{aligned} x_0^+ &= x_0^- - \frac{1}{2}r(\theta_0^- - \theta_1^-)(\cos a - \cos(2a)\cos\psi) \\ &\quad + r(\sin a + \sin\psi), \\ x_1^+ &= x_1^- + \frac{1}{2}r(\theta_0^- - \theta_1^-)(\cos a - \cos(2a)\cos\psi) \\ &\quad + r(\sin a + \sin\psi), \\ \theta_0^+ &= 2a + \theta_0^-, \\ \theta_1^+ &= \cos(2a)(\theta_1^- - \theta_0^-) + a, \end{aligned}$$

where $\psi = a + \frac{1}{2}\cos(2a)(\theta_1^- - \theta_0^-)$.

Note that the energy of the system is not preserved between impacts. Indeed,

$$E_L = \frac{m}{2}(\dot{x}^2 + r^2\sin^2\theta\dot{\theta}^2) + \frac{I\dot{\theta}^2}{2} + mgr\cos\theta,$$

and then,

$$E_L \circ \Delta = \cdots + \frac{I(\dot{\theta}^+)^2}{2} = \cdots + \frac{I\dot{\theta}^2}{2}\cos^2(2a) \neq E_L.$$

We have performed a Python numerical simulation with $N = 80$ steps, time step $h = 0.1$, parameters $g = 9.8$, $\alpha = 0$, $\kappa = 0.2$, $r = 1$, $m = 1$, $I = 0.5$, $a = \frac{\pi}{6}$, $\varepsilon = 0.1$, $\eta = 100$, $\rho = 1$; initial values $x_0 = 0$, $\theta_0 = \frac{\pi}{6}$, $\dot{x}_0 = 1$, and $\dot{\theta}_0 = 0.1$. The reference trajectory is given by $\gamma_r(t) = (\bar{x}_r(t) + r\cos(\theta_r(t)), \theta_r(t))$ for $t_{i-1} < t < t_i$, where $\bar{x}_r(t) = \bar{x}_{r,i-1} + \dot{\bar{x}}_{r,i-1}t$ and $\theta_r(t) = a + \dot{\theta}_{r,i-1}(t - t_{i-1})$. The values of the parameters are $t_0 = 0$, $\bar{x}_{r,0} = 0$, $\theta_{r,0} = a$, $\dot{\bar{x}}_{r,0} = 1$, $\dot{\theta}_{r,0} = -0.08$ and t_i for $i \geq 1$ is the instant of the i-th impact (determined by the equation $\theta(t_i) = -a$). The parameters $x_{r,i}$, $\theta_{r,i}$, $\dot{x}_{r,i}$, $\dot{\theta}_{r,i}$ are defined by (4).

The evolution of the x- and θ-coordinates of the center of mass are plotted in Figs. 3; comparing them with the

reference trajectory. The curves described by the center of mass, foot, leg and reference trajectory on the xy-plane are represented in Fig. 4 (left) and the control inputs' evolution is represented in Fig. 4 (right). Note that the foot's trajectory approaches the reference one due to the activation of control inputs responsible for tracking and maintaining speed.

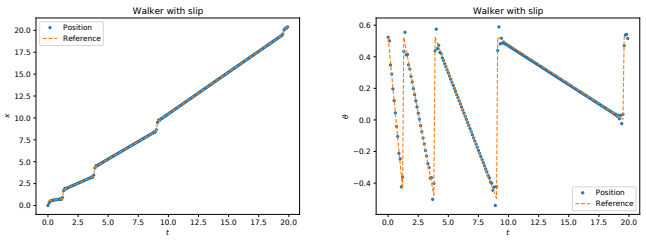

Fig. 3: Left: Horizontal components of the position of the center of mass and the reference trajectory as functions of time. Right: Angular components of the position of the center of mass and the reference trajectory as functions of time.

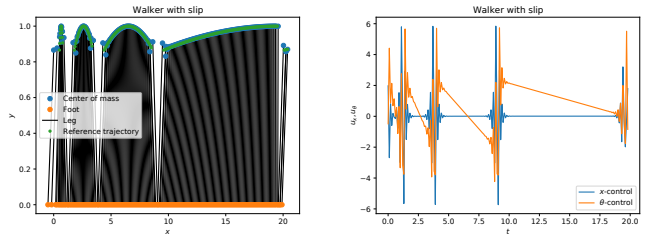

Fig. 4: Left: Trajectories of the center of mass and the foot compared with the reference trajectory. Right: Horizontal and angular components of the control inputs as functions of time.

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