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Ramsey multiplicity and the Turán coloring

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Abstract: Extending an earlier conjecture of Erdős, Burr and Rosta conjectured that among all two-colorings of the edges of a complete graph, the uniformly random coloring asymptotically minimizes the number of monochromatic copies of any fixed graph H. This conjecture was disproved independently by Sidorenko and Thomason. The first author later found quantitatively stronger counterexamples, using the *Turán coloring*, in which one of the two colors spans a balanced complete multipartite graph.

We prove that the Turán coloring is extremal for an infinite family of graphs, and that it is the unique extremal coloring. This yields the first determination of the Ramsey multiplicity constant of a graph for which the Burr–Rosta conjecture fails.

We also prove an analogous three-color result. In this case, our result is conditional on a certain natural conjecture on the behavior of two-color Ramsey numbers.

1 Introduction

1.1 Background

Let H be a fixed graph on t vertices. If $\chi : E(K_n) \to \{\text{red}, \text{blue}\}$ is a two-coloring of the edges of a complete graph K_n, then we denote by $m(H,\chi)$ the number of monochromatic *labeled* copies of H in the coloring χ. We also denote by $m(H,n)$ the minimum of $m(H,\chi)$ over all two-colorings of $E(K_n)$. A simple averaging argument shows that the sequence $c(H,n) := m(H,n)/(n)_t$ is non-decreasing, where $(n)_t = n(n-1)\cdots(n-t+1)$ is the falling factorial. Since this sequence is contained in $[0,1]$, we see that the limit

$$c(H) := \lim_{n\to\infty} c(H,n)$$

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is well defined. This constant, $c(H)$, is called the *Ramsey multiplicity constant* of H. It is often helpful to think of $c(H)$ probabilistically: it equals the asymptotic minimum over all two-colorings of $E(K_n)$ of the probability that a random map $V(H) \to V(K_n)$ yields a monochromatic copy of H. We note that by Ramsey's theorem, $c(H,n)$ is strictly positive for any graph H and any sufficiently large n, which implies that $c(H) > 0$ since the sequence is non-decreasing.

The earliest result about Ramsey multiplicity, preceding the definition, is due to Goodman [24] from 1959, and implies that $c(K_3) = \frac{1}{4}$. Since a random two-coloring of $E(K_n)$ has $\frac{1}{4}(n)_3$ monochromatic labeled copies of K_3 in expectation, the random coloring shows the existence of a coloring matching Goodman's lower bound. A few years later, Erdős [14] conjectured that the random coloring asymptotically minimizes the number of monochromatic copies of K_t for all t, and thus that $c(K_t) = 2^{1-\binom{t}{2}}$. Burr and Rosta [4] formulated a natural generalization of Erdős's conjecture, namely that $c(H) = 2^{1-e(H)}$ for every graph H. Since a random coloring of K_n contains $2^{1-e(H)}(n)_t$ monochromatic labeled copies of H in expectation, we have $c(H) \leq 2^{1-e(H)}$, and the Burr–Rosta conjecture simply says that this upper bound is tight, i.e. that a random coloring asymptotically minimizes the number of monochromatic copies of H.

The Burr–Rosta conjecture was disproved by Sidorenko [44], who showed that it is false when H is a triangle with a pendant edge. At roughly the same time, even the weaker conjecture of Erdős was disproved by Thomason [48], who constructed, for every $t \geq 4$, colorings of K_n with asymptotically fewer than $2^{1-\binom{t}{2}}(n)_t$ monochromatic labeled copies of K_t. A more dramatic counterexample to the Burr–Rosta conjecture was found by the first author [20], who constructed an infinite family of graphs H with m edges and with $c(H) \leq 2^{-\Omega(m \log m)}$, thus showing that the Burr–Rosta conjecture can be very far from true.

The key to the examples from [20] comes from the *Turán coloring* of K_n, namely the coloring where the blue graph is a balanced complete $(k-1)$-partite graph and where the red graph consists of $k-1$ disjoint cliques, each of order $\lfloor \frac{n}{k-1} \rfloor$ or $\lceil \frac{n}{k-1} \rceil$. Let r be the remainder when n is divided by $k-1$, so there are $k-1-r$ cliques of order $\lfloor \frac{n}{k-1} \rfloor$, and further r cliques of order $\lceil \frac{n}{k-1} \rceil$. Since the blue graph has chromatic number $k-1$, there can be no blue copies of any graph H with chromatic number at least k. Moreover, if H is connected, then any red copy of H must appear inside one of the parts. If we let t be the number of vertices in H, then the number of labeled copies of H in the Turán coloring is exactly

$$(k-1-r)\left(\left\lfloor \frac{n}{k-1} \right\rfloor\right)_t + r\left(\left\lceil \frac{n}{k-1} \right\rceil\right)_t \leq (k-1)^{1-t}(n)_t.$$

Said differently, one sees that the probability that a random injection $V(H) \to V(K_n)$ has its image contained entirely within one of the $k-1$ parts is at most $(k-1)^{1-t}$. Thus, the Turán coloring shows that for any connected graph H with t vertices and chromatic number k, we have

$$m(H,n) \leq (k-1)^{1-t}(n)_t \tag{1.1}$$

and therefore

$$c(H) \leq (k-1)^{1-t} = 2^{-\Omega(t \log k)}. \tag{1.2}$$

One way of picking a graph H which optimizes this construction is to start with a clique of order k, and then to add $t-k$ pendant edges to this clique. Thus, H has $\binom{k}{2} + (t-k)$ edges, and (1.2) shows that $c(H)$ is super-exponentially small in the number of edges of H if $k = \omega(1)$ and $t = \omega(k^2/\log k)$.

Graphs H which satisfy the Burr–Rosta conjecture, i.e. those with $c(H) = 2^{1-e(H)}$, are called *common*. Sidorenko's conjecture (see e.g. [43]), a major open problem in extremal graph theory, states that for any bipartite graph H and any $p \in [0,1]$, a random graph with edge density p has in expectation asymptotically the minimum number of copies of H over all graphs of the same order and edge density. It is straightforward to show that H is common if H satisfies Sidorenko's conjecture, so Sidorenko's conjecture implies that all bipartite graphs are common. Sidorenko's conjecture has been verified in many cases (e.g. [12, 26, 30, 11, 8, 47, 34]), yielding a large class of bipartite graphs which are known to be common. Additionally, some non-bipartite graphs are known to be common, such as odd cycles [44], the five-wheel [27], the seven-wheel [25], and certain graphs of arbitrary chromatic number [32]. On the other hand, it is also known that most graphs are uncommon, since Jagger, Šťovíček, and Thomason [28] proved that any graph containing K_4 is uncommon.

In general, we do not understand the Ramsey multiplicity constants of uncommon graphs. For instance, Thomason [48] proved that K_4 is uncommon by exhibiting an explicit coloring witnessing $c(K_4) < \frac{1}{33}$. Using flag algebras, Nieß and Sperfeld [38, 46] independently showed that $c(K_4) > \frac{1}{35}$. There have been very recent improvements [25, 40] to the upper and lower bounds on $c(K_4)$, but it remains unclear what the true value of $c(K_4)$ is, as well as what the asymptotically optimal colorings look like.

1.2 Our results

The discussion above is actually somewhat prototypical of Ramsey theory. Namely, both the random coloring and the Turán coloring arise naturally in many Ramsey-theoretic contexts, and sometimes one of the two is (asymptotically) tight. For instance, the study of Ramsey goodness (see e.g. [39]) asks when the Turán coloring yields optimal bounds for certain off-diagonal Ramsey numbers, and there are diagonal Ramsey problems (see e.g. [7, 9]) for which the random coloring is known to be asymptotically optimal. In instances where neither the random coloring nor the Turán coloring is tight, our knowledge is frequently limited, and there are many problems for which no optimal structure is known or conjectured.

Since common graphs have been extensively studied, and since the field of Ramsey goodness is very rich, it is perhaps surprising that there are no results about when the Turán coloring is optimal for the Ramsey multiplicity problem. In this paper, we study this problem. We make the following definition.

Definition 1.1. A connected graph H on t vertices is called a *bonbon* if, for every sufficiently large n, the Turán coloring of $E(K_n)$ with $\chi(H)-1$ parts is the unique two-edge-coloring of K_n with the minimum number of monochromatic copies of H, up to permuting the colors and the vertices.

In particular, if H is a bonbon, then $c(H) = (\chi(H)-1)^{1-t}$. The word *bonbon* is meant to be reminiscent of the "good" terminology in the study of Ramsey goodness, as well as in the recent work [22, 23] on *Turán goodness*, which studies when the Turán graph is the unique extremizer for a certain extremal problem. Our first main result proves that all graphs in a certain infinite family are bonbons. To define this infinite family, we recall two standard definitions. First, a graph is called *k-critical* if it has chromatic number k, but there is an edge[1] whose deletion lowers the chromatic number. Second, given a

[1] Note that some authors use the term *k-critical* to mean something different, namely that the deletion of *any* edge lowers the chromatic number.

graph H_0, we say that another graph H is *obtained from H_0 by adding pendant edges* if H_0 is an induced subgraph of H, all vertices in $V(H) \setminus V(H_0)$ have degree 1, and their unique neighbor lies in[2] $V(H_0)$.

The graphs that we prove are bonbons are obtained from a k-critical graph by adding sufficiently many pendant edges. A result of Simonovits [45] states that if H is k-critical and if n is sufficiently large, then the Turán graph is the unique n-vertex H-free graph with the maximum number of edges; because of this, k-criticality arises naturally in many questions involving the Turán graph or Turán coloring.

Theorem 1.2. *Let $k \geq 4$, and let H_0 be a k-critical graph with h vertices. Let H be obtained from H_0 by adding $t-h$ pendant edges, where $t \geq (1000kh)^{10}h^{10k}$. Then H is a bonbon. In particular, $c(H) = (k-1)^{1-t}$.*

This gives the first examples of uncommon graphs whose Ramsey multiplicity constants are known exactly. Natural examples of graphs H to which Theorem 1.2 applies are *starbursts* (obtained from K_k by adding an equal number of pendant edges to each vertex of K_k; see [10] for the terminology) and *pineapples* (obtained from K_k by adding pendant edges to a single vertex; see e.g. [49] for the terminology). In both of these examples, we set $H_0 = K_k$, which is a k-critical graph.

All three of the assumptions of Theorem 1.2, namely that H_0 is k-critical, k is at least 4, and t is sufficiently large, are necessary, as we now explain. First, suppose that H_0 is not k-critical. Then if we begin with the Turán coloring and recolor a single edge inside one part from red to blue, then we strictly decrease the number of red copies of H. On the other hand, this new coloring still contains no blue H_0, and thus no blue H, for an embedding of H_0 into the blue graph precisely corresponds to a proper $(k-1)$-coloring of a graph obtained from H_0 by deleting an edge. This shows that if H_0 is not k-critical, then H is not a bonbon.

Next, if $k = 3$, the Turán coloring yields an upper bound $c(H) \leq 2^{1-t}$ as H has t vertices. Since H is a connected graph with at least one cycle, we have that $t \geq e(H)$, and thus this upper bound is no stronger than the upper bound $c(H) \leq 2^{1-e(H)}$ given by the random coloring. If H_0 has at least two cycles, then this shows that H cannot be a bonbon, as the random coloring yields a stronger upper bound than the Turán coloring. Additionally, if H_0 has exactly one cycle, then if H is a bonbon, H is in particular common. However, Jagger, Šťovíček, and Thomason [28, Theorem 4] showed that if H is obtained from a non-bipartite graph by adding sufficiently many pendant edges, then H is uncommon.

Finally, the assumption in Theorem 1.2 that we add sufficiently many pendant edges is also necessary, for a similar reason. For example, if $H_0 = K_k$, and if we add $o(k^2/\log k)$ pendant edges, then the random coloring will again have fewer monochromatic copies of H than the Turán coloring.

1.3 More colors

Our second main result concerns Ramsey multiplicity for more colors, which we now introduce. For any integer $q \geq 3$, let $m_q(H,n)$ denote the minimum number of monochromatic labeled copies of H in any q-coloring of the edges of K_n. As in the case of two colors, a simple averaging argument shows that the limit

$$c_q(H) = \lim_{n\to\infty} \frac{m_q(H,n)}{(n)_t}$$

[2]We stress that we make no assumption about which vertices of H_0 are the neighbors of the new vertices.

exists, and is called the *q-color Ramsey multiplicity constant* of H. As before, $c_q(H)$ is strictly positive by Ramsey's theorem and by the non-decreasing property. For $q \geq 3$, our understanding of q-color Ramsey multiplicity is even more limited than it is for two colors. As in the case of two colors, a random coloring shows that $c_q(H) \leq q^{1-e(H)}$, and graphs H for which this bound is tight are called *q-common*. Sidorenko's conjecture again implies that any bipartite graph is q-common for all q, and a simple product coloring shows that any non-bipartite H is *not* q-common for all sufficiently large q. In the other direction, Král', Noel, Norin, Volec, and Wei [31] recently showed that for every q, there exist non-bipartite q-common graphs. They also showed that if a graph H is q-common for all q, then H must satisfy Sidorenko's conjecture.

Just as the Turán coloring yields a natural upper bound on the two-color Ramsey multiplicity constant for a graph, there is an analogous construction that upper-bounds $c_q(H)$ for $q \geq 3$. Let $r_{q-1}(k)$ denote the $(q-1)$-color Ramsey number of K_k, namely the least r so that any $(q-1)$-coloring of $E(K_r)$ contains a monochromatic copy of K_k. Then we define the *q-color Ramsey-blowup coloring* with parameter k to be the q-coloring of $E(K_n)$ where we partition $V(K_n)$ into $r_{q-1}(k)-1$ equally-sized parts, color the edges between the parts by blowing up a $(q-1)$-coloring of $E(K_{r_{q-1}(k)-1})$ without a monochromatic K_k, and color all edges within the parts with the qth, unused, color. If H is a graph with clique number k, then there can be no monochromatic copy of H in the first $q-1$ colors[3]. Moreover, if H is connected, then every copy of H in color q must lie in one of the $r_{q-1}(k)-1$ parts. Therefore, if H is a t-vertex connected graph with clique number k, the Ramsey-blowup coloring implies that $m_q(H,n) \leq (r_{q-1}(k)-1)^{1-t}(n)_t$ and thus that $c_q(H) \leq (r_{q-1}(k)-1)^{1-t}$. We remark that the Ramsey-blowup coloring exactly generalizes the Turán coloring, because the one-color Ramsey number of K_k is equal to k. Similarly, the following definition generalizes the notion of a bonbon.

Definition 1.3. Let $q,k \geq 2$ be integers. We say that a q-coloring of $E(K_n)$ is *Ramsey-blowup-like* if there is a color so that the graph of edges in that color is a disjoint union of $r_{q-1}(k)-1$ cliques, each of size $\lfloor \frac{n}{r_{q-1}(k)-1} \rfloor$ or $\lceil \frac{n}{r_{q-1}(k)-1} \rceil$, and such that there is no monochromatic K_k in any of the other colors.

Let H be a t-vertex connected graph with clique number k. We say that H is a *q-color bonbon* if, for all sufficiently large n, any q-coloring of $E(K_n)$ with the minimum number of monochromatic copies of H is Ramsey-blowup-like. In particular, if H is a q-color bonbon, then $c_q(H) = (r_{q-1}(k)-1)^{1-t}$.

Note that in the case $q = 2$, this definition of a 2-color bonbon coincides with that of a bonbon from Definition 1.1. This definition might not appear at first to be the natural extension to more than two colors, as we do not fully specify what happens between the parts. In Section 5.1, we discuss why this definition seems to be the right one for this problem.

Our second main theorem is that, conditional on a natural but unproven assumption about Ramsey numbers, a graph obtained from a large clique by adding sufficiently many pendant edges is a three-color bonbon. To state this assumption, we make the following definition, where $r(a,b)$ denotes the off-diagonal two-color Ramsey number, namely the least r so that any red/blue coloring of $E(K_r)$ contains a monochromatic red K_a or a monochromatic blue K_b.

[3]Note that in contrast to the two-color case, here we require H to have clique number k, rather than chromatic number k. The reason is that in a general coloring of $K_{r_{q-1}(k)-1}$ with no monochromatic K_k, there are monochromatic subgraphs with chromatic number larger than k, and thus it is only through the clique number of H that we can ensure that there are no monochromatic copies of H in the first $q-1$ colors.

Definition 1.4. We say that a positive integer k is *polite* if

$$r(k,\lceil k/2\rceil) \le 2^{-31} r(k,k) \tag{1.3}$$

and

$$\frac{r(k,k)-1}{r(k,k-1)} \ge 1+25\left(\frac{r(k,\lceil k/2\rceil)}{r(k,k)}\right)^{1/4}. \tag{1.4}$$

Conjecture 1.5. *Every sufficiently large integer is polite.*

Remark. Erdős [16] wrote in 1981 that "Almost nothing is known about the local growth properties of $r(n,m)$", and not much has changed about our knowledge in the intervening 40 years. Indeed, Conjecture 1.5 would follow from very natural conjectures about the growth of Ramsey numbers which have seen no progress in decades.

First, since it is known that both $r(k,k)$ and $r(k,\lceil k/2\rceil)$ grow exponentially, and since it is natural to expect the latter to grow at a lower exponential rate than the former, it is certainly natural to expect that $r(k,\lceil k/2\rceil) = o(r(k,k))$, which would imply (1.3) for sufficiently large k. Additionally, Burr and Erdős (see [16]) conjectured that $r(k,k) \ge (1+c)r(k,k-1)$ for some absolute constant $c>0$, which would imply (1.4) in conjunction with $r(k,\lceil k/2\rceil) = o(r(k,k))$. However, Conjecture 1.5 seems currently out of reach.

With this definition and conjecture, we can now state our main three-color theorem.

Theorem 1.6. *Let k be a sufficiently large polite integer, and let t be sufficiently large with respect to k. A graph H obtained from K_k by adding $t-k$ pendant edges is a 3-color bonbon.*

Prior to this result, there was only one non-3-common graph whose three-color Ramsey multiplicity constant was known exactly, namely the triangle K_3. In [13], it is proved that $c_3(K_3) = \frac{1}{25}$, and moreover, there is a complete characterization of the extremal colorings. Among them is the Ramsey-blowup coloring, but there are also other extremal colorings which are not Ramsey-blowup-like. Because of this, K_3 is not a three-color bonbon. Assuming Conjecture 1.5, Theorem 1.6 yields an infinite family of three-color bonbons, and in particular yields an infinite family of new examples of graphs which are not 3-common and whose three-color Ramsey multiplicity constant is known exactly.

We remark that much of the proof of Theorem 1.6 mimics that of Theorem 1.2, but several additional complications arise in the three-color setting, and additional ideas are needed to overcome them. Moreover, these complications seem to be inherent to the problem. Indeed, as discussed in Section 5.2, there are major obstructions to extending our proofs to four or more colors, and in fact, there is some reason to believe that no such result is true when the number of colors is at least five: perhaps there do not exist *any* q-color bonbons for $q \ge 5$.

1.4 Ramsey multiplicity upon edge deletion

As an application of Theorem 1.2, we are able to resolve a question of Huang about how edge-deletion affects the Ramsey multiplicity constant. Namely, recall that the Burr–Rosta conjecture asserts that $c(H) = 2^{1-m}$ for any graph H with m edges. If the Burr–Rosta conjecture were true, it would imply that if H' is obtained from H by deleting a single edge, then $c(H') = 2c(H)$.

The Burr–Rosta conjecture is false, but it is natural to wonder whether a weakening of the equality $c(H') = 2c(H)$ is nonetheless true. Note that the Ramsey multiplicity constant is a monotone parameter, so certainly $c(H') \geq c(H)$ if H' is obtained from H by deleting an edge. Huang (private communication) asked whether there is an absolute constant $C > 0$ so that $c(H') \leq C \cdot c(H)$ for all graphs H and all subgraphs H' obtained by deleting an edge. Theorem 1.2 implies that this is false, in a strong form. Namely, let H be obtained from $H_0 = K_5$ by adding $t-5$ pendant edges. If t is sufficiently large, then Theorem 1.2 implies that $c(H) = 4^{1-t}$. Now, let H' be obtained from H by deleting one of the edges of the K_5. Note that $K_5 \setminus e$ is a 4-critical graph, so we may again apply Theorem 1.2 to conclude that $c(H') = 3^{1-t}$ if t is sufficiently large. This shows that $c(H')/c(H)$ cannot be upper-bounded by a constant, and in fact may be exponentially large in the number of vertices of H. We remark that similar questions about the ordinary Ramsey number, rather than the Ramsey multiplicity constant, have been recently studied in [52].

1.5 Outline and notation

The rest of this paper is organized as follows. In Section 2, we prove some general lemmas about Ramsey multiplicity that we will need. In Section 3, we present the proof of Theorem 1.2, which is organized as a series of claims which repeatedly refine the structure of a two-coloring minimizing the number of monochromatic copies of H, to eventually conclude that the coloring is the Turán coloring. In Section 4, we prove Theorem 1.6; the proof bears many similarities to that of Theorem 1.2, so we shorten or omit several of the proofs, choosing to focus on the places where new ideas are needed to handle the added complexity of the three-color case. Finally, we end with some concluding remarks: Section 5.1 discusses why we define q-color bonbons for $q \geq 3$ as we do, Section 5.2 discusses why our techniques seem to fail for $q > 3$ (and why q-color bonbons may not even exist for large q), and Section 5.3 lists some further open problems that arise from this work.

In the two-color case, we use the colors red and blue, which we denote by R and B. In the three-color case, we add the color yellow, denoted by Y. For clarity of presentation, we systematically omit floor and ceiling signs whenever they are not crucial.

2 General lemmas about Ramsey multiplicity

In this section we collect four general results which we use in our proofs. Our first lemma says that in a coloring minimizing the number of monochromatic copies of some graph H, all vertices must lie in roughly the same number of monochromatic copies. For a graph H, a two-coloring χ of $E(K_n)$, and a vertex $v \in V(K_n)$, let us denote by $m_v(H,\chi)$ the number of monochromatic labeled copies of H that contain v.

Lemma 2.1. *For every graph H on t vertices, there exists a constant $C > 0$ such that the following holds. For any two-coloring χ of $E(K_n)$ with the minimum number of monochromatic labeled copies of H, and for any $v \in V(K_n)$,*

$$\left(1 - \frac{C}{n}\right) \frac{t}{n} m(H,n) \leq m_v(H,\chi) \leq \left(1 + \frac{C}{n}\right) \frac{t}{n} m(H,n).$$

Proof. By picking the constant C appropriately, we may assume that n is sufficiently large in terms of H. For simplicity, we abbreviate $m_v(H,\chi)$ as simply m_v. Let u and w be two vertices of K_n such that m_u is minimal and m_w is maximal among all vertices of K_n. We have that

$$t \cdot m(H,n) = \sum_{v \in V(K_n)} m_v,$$

which implies that $m_u \leq \frac{t}{n} m(H,n) \leq m_w$.

We claim that for some constant $C > 0$ depending only on H, we have that $m_u \geq (1 - C/n)m_w$. This immediately implies the desired result (up to changing the constant C), since the fact that the maximum and minimum values of m_v differ by at most a factor of $1 - C/n$ implies that every value of m_v is within a factor of $1 \pm 2C/n$ of the average value.

So suppose for contradiction that $m_u < (1 - C/n)m_w$. Consider the coloring χ' obtained by deleting w and replacing it with a clone u' of u, namely setting $\chi'(u',v) = \chi(u,v)$ for any $v \notin \{u,u'\}$. We also color the edge uu' red. Then we claim that χ' has strictly fewer monochromatic labeled copies of H than χ does, contradicting our assumption on χ. Indeed, in deleting w, we destroy m_w monochromatic labeled copies of H, and when making the clone u', we add m_u monochromatic labeled copies of H containing u' but not u. Additionally, there are at most $(n-2)^{t-2}$ monochromatic labeled copies of H containing both u and u'. In all, we find that

$$m(H,\chi') \leq m(H,\chi) - m_w + m_u + (n-2)^{t-2} < m(H,\chi) + n^{t-2} - \frac{Cm_w}{n}.$$

Since n is sufficiently large, we may assume that $m(H,n) \geq \frac{c(H)}{2} n^t$. Therefore,

$$m_w \geq \frac{t}{n} m(H,n) \geq \frac{t}{n} \frac{c(H)}{2} n^t = \frac{tc(H)}{2} n^{t-1}.$$

Thus, if we let $C = \frac{2}{tc(H)}$ be a constant depending only on H, then we see that

$$\frac{Cm_w}{n} \geq n^{t-2}$$

implying that $m(H,\chi') < m(H,\chi)$, as claimed. Thus, we have our contradiction, and find that $m_u \geq (1 - C/n)m_w$. □

We will need the following lemma (which essentially appears in [14]), a simple and well-known lower bound on the Ramsey multiplicity constants of cliques. We remark that a better lower bound is known [6] for $c(K_k)$ for sufficiently large k, but we stick with the following since it applies for all k.

Lemma 2.2. *Let H_1, H_2 be graphs on h_1, h_2 vertices, respectively, and suppose that $h_1, h_2 \leq h$. For any $n \geq 4^h$, any two-coloring of $E(K_n)$ contains at least $4^{-h^2}(n)_{h_1}$ monochromatic red labeled copies of H_1, or at least $4^{-h^2}(n)_{h_2}$ monochromatic blue labeled copies of H_2.*

Proof. Let $r = r(h_1,h_2)$, and fix a two-coloring of $E(K_n)$ for $n \geq 4^h$. The Erdős–Szekeres [19] bound implies that $r \leq \binom{h_1+h_2-2}{h_1-1} \leq 4^{h-1} \leq n$. By the definition of r, every r-subset of $V(K_n)$ contains a red K_{h_1}

or a blue K_{h_2}. Suppose first that at least half of these subsets contain a red K_{h_1}. We have that every red copy of K_{h_1} appears in exactly $\binom{n-h_1}{r-h_1}$ subsets of size r. Therefore, by double-counting, the total number of red K_{h_1} is at least

$$\frac{\frac{1}{2}\binom{n}{r}}{\binom{n-h_1}{r-h_1}} = \frac{\binom{n}{h_1}}{2\binom{r}{h_1}} \geq \frac{r^{-h_1}}{2}\binom{n}{h_1} \geq 4^{-h^2}\binom{n}{h_1},$$

using the bounds $\binom{r}{h_1} \leq r^{h_1}$, $h_1 \leq h$ and $r \leq 4^{h-1}$. Since every red K_{h_1} contains exactly $h_1!$ red labeled copies of H_1, we find at least $4^{-h^2}(n)_{h_1}$ red labeled copies of H_1 in this case. By interchanging the roles of h_1 and h_2 and blue and red, we get the other desired conclusion in case at least half of the r-subsets of $V(K_n)$ contain a blue K_{h_2}, which completes the proof. □

The next result we need is due to He and the authors [21, Theorem 3.5], though a similar result was proved earlier by Bollobás and Nikiforov [2, Theorem 9]. It can be viewed as a combination of the stability and supersaturation versions of Turán's theorem, and says that if a graph G has minimum degree close to the Turán threshold for containing a copy of K_k, while also containing few copies of K_k, then it must be close to $(k-1)$-partite. For more on this intuition and motivation, see [21, Section 3].

Lemma 2.3 ([21, Theorem 3.5])**.** *For every $\varepsilon > 0$ and every integer $k \geq 3$, there exist $\alpha, \delta > 0$ such that the following holds for all n. Suppose G is a graph on n vertices with minimum degree at least $(1 - \frac{1}{k-1} - \delta)n$ and with at most $\alpha\binom{n}{k}$ copies of K_k. Then $V(G)$ can be partitioned into $V_1 \sqcup \cdots \sqcup V_{k-1}$, such that the total number of internal edges in $V_1, \ldots, V_{k-1}$ is at most $\varepsilon\binom{n}{2}$.*

Moreover, we may take $\delta = \min\{1/(2k^2), \varepsilon/2\}$ and $\alpha = k^{-10k}\varepsilon$.

Finally, we will need the supersaturation version of the Erdős–Stone–Simonovits theorem, due to Erdős and Simonovits [17]; the quantitative estimate we state follows from [5] and a standard proof of the supersaturation theorem, e.g. [29, Lemma 2.1].

Lemma 2.4. *Let H_0 be an h-vertex graph of chromatic number k. For every $\delta > 0$, there exists some $\gamma = \gamma(H_0, \delta) > 0$ such that the following holds for sufficiently large n. If G is an n-vertex graph with at least $(1 - \frac{1}{k-1} + \delta)\binom{n}{2}$ edges, then G contains at least $\gamma(n)_h$ labeled copies of H_0. Moreover, we may take $\gamma = \delta^{1000h^2}$.*

3 Two-color bonbons

In this section we prove Theorem 1.2. From now on, we fix $k \geq 4$ and let H_0 be a fixed k-critical graph on h vertices, and we also treat $t \geq (1000hk)^{10}h^{10k}$ as fixed.

3.1 Choices of parameters

For convenience, we record here all the parameters we will need in our proof of Theorem 1.2. Somewhat unusually, the constraints we have on our parameters do not naturally form a linear order (i.e. it is not the

case that we have parameters $a_1, \dots, a_m$ such that each a_i must be sufficiently small with respect to a_{i+1}), but rather they form a poset structure, generated by the following inequalities:

$$\theta \ll \frac{1}{k}, \qquad \varepsilon \ll \theta, \qquad \varepsilon \ll \frac{1}{h}, \qquad \lambda \ll \varepsilon, \qquad \gamma \ll \lambda.$$

In addition, there is one further parameter τ, which is defined as $(1+\lambda)^{-t}$. In contrast to the other parameters, we cannot simply pick τ to be sufficiently small, as we end up needing both upper and lower bounds on τ. Hence, it is defined as $(1+\lambda)^{-t}$ in order to be sufficiently large with respect to t, and then our lower bound on t ensures that $\tau \ll \gamma$.

The rest of this subsection formalizes the choices of parameters and the inequalities we need them to obey; the reader may safely skip this subsection and simply bear the above qualitative dependencies in mind.

We first pick θ, ε, and λ to be polynomially small in k and h, defined as

$$\theta = \frac{1}{50k} \qquad \varepsilon = \frac{\theta^2}{2h^2k^2} = \frac{1}{5000h^2k^4} \qquad \lambda = \frac{\varepsilon^2}{(200h)^2} = \frac{1}{2 \cdot 10^8 h^4 k^4}.$$

Additionally, we let γ be the parameter from Lemma 2.4, applied to the graph H_0 and $\delta = \lambda$, namely

$$\gamma = \lambda^{1000h^2}.$$

We now define a further parameter depending on both k and t, namely

$$\tau = \frac{1}{(1+\lambda)^t}.$$

The following lemma records the important inequalities that we will need these parameters to satisfy. It is straightforward to check that the choices above guarantee that all these inequalities are satisfied.

Lemma 3.1. *The following inequalities hold.*

(a) $\tau \le \gamma\lambda^h \le 4^{-h^2-1}\lambda^h$.

(b) $2\tau^{h^{-k}} \le k^{-10k}\varepsilon$.

(c) $\theta \ge 2k\tau^{1/h}$.

(d) $e^{8(k-1)\sqrt{\lambda}t} > 4^{h^2}t$.

(e) $\frac{1}{k-1} - \sqrt{2\varepsilon} \ge \frac{1}{k}$.

Proof.

(a) We know that $\gamma = \lambda^{1000h^2} \le 4^{-h^2-1}$, so the second inequality is immediate. For the first, we have that $\gamma\lambda^h \ge \lambda^{5000h^2} \ge 2^{-5000h^2/\lambda} = 2^{-10^{12}h^6k^4}$, using the fact that $x \ge 2^{-1/x}$ for all $x > 0$. Since $1 + x \ge 2^x$ for all $x \in [0,1]$, we also see that $\tau \le 2^{-\lambda t}$. Finally, $t \ge (1000hk)^{10}$, so $\lambda t \ge 10^{12}h^6k^4$, which proves the claim.

(b) Since $h \geq k \geq 4$, it suffices to prove that $\tau \leq 2^{-h^{10k}}\varepsilon$, which itself follows from $\tau \leq 2^{-h^{10k}/\varepsilon}$. We saw in part (a) that $\tau \leq 2^{-\lambda t}$, so it suffices to prove that $\lambda t \geq h^{10k}/\varepsilon$, which is true since $\lambda(1000hk)^{10} \geq 10^{12}h^6k^4 \geq 1/\varepsilon$.

(c) From part (a), we see that $\tau^{1/h} \leq \lambda$, which yields the desired bound since $2k\lambda \leq \theta$.

(d) We first note that by the computation in part (a), we have that $\sqrt{\lambda t} \geq 10^6 h^3 k^2 > h^2$. Since $k \geq 3$, we see that $e^{8(k-1)} \geq 4$. Therefore,

$$e^{8(k-1)\sqrt{\lambda t}} \geq (4^{\sqrt{t}})^{\sqrt{\lambda t}} \geq (4t)^{\sqrt{\lambda t}} \geq t \cdot 4^{\sqrt{\lambda t}} > 4^{h^2}t,$$

where the second inequality uses that $4^{\sqrt{t}} \geq 4t$ for all $t \geq 4$.

(e) We have that $\frac{1}{k-1} - \frac{1}{k} = \frac{1}{k(k-1)} \geq \frac{1}{k^2}$. Since $\sqrt{2\varepsilon} < 1/k^3$, this proves the desired bound. □

3.2 Proof of Theorem 1.2

Recall that we have fixed H_0 and t. We let H be obtained from H_0 by arbitrarily adding $t-h$ pendant edges to H_0, so that H has t vertices. Let the vertices of H_0 be $u_1, \dots, u_h$, and let s_i be the number of pendant edges incident to u_i for $1 \leq i \leq h$, so that $s_1 + \cdots + s_h = t - h$. We assume without loss of generality that $s_1 \geq \cdots \geq s_h$, which in particular implies that $s_1 \geq \frac{t-h}{h} \geq \frac{t}{2h}$.

We recall (1.1), which gives an upper bound on $m(H,n)$ coming from the Turán coloring. Namely, we have that

$$m(H,n) \leq (k-1)^{1-t}(n)_t. \tag{3.1}$$

We also henceforth let n be sufficiently large in terms of H_0 and t, and let χ be an optimal two-coloring of $E(K_n)$, that is a coloring with exactly $m(H,n)$ monochromatic labeled copies of H. Our goal is to show that χ is isomorphic to the Turán coloring. By combining Lemma 2.1 and (3.1), we find that for every vertex $v \in V(K_n)$, we have that

$$m_v(H,\chi) \leq \left(1 + \frac{C}{n}\right)\frac{t}{n}m(H,n) \leq \left(1 + \frac{C}{n}\right)t(k-1)^{1-t}(n-1)_{t-1} \tag{3.2}$$

for some constant C depending only on H.

Let $d = (1+\lambda)n/(k-1)$. Note that d is slightly larger than the red degree of any vertex in the Turán coloring. The following lemma is used several times in the proof of Theorem 1.2. Additionally, its proof exemplifies the kinds of arguments that arise throughout.

Lemma 3.2. *Suppose n is sufficiently large. Let $S \subseteq V(K_n)$ be a set of vertices, each with at least d blue neighbors. Then S contains fewer than $\tau(n)_h$ labeled blue copies of H_0.*

Proof. Any labeled blue copy of H_0 in S extends to at least $(d-h)_{t-h}$ labeled blue copies of H, since every vertex in this H_0 has blue degree at least d. Therefore, if there are $\tau(n)_h$ labeled blue copies of H_0 in S, then the total number of blue labeled copies of H in χ is at least

$$\tau(n)_h \cdot (d-h)_{t-h} = \tau(n)_h \cdot (1-o(1))\left(\frac{1+\lambda}{k-1}\right)^{t-h}(n)_{t-h} \geq (1-o(1))\tau\left(\frac{1+\lambda}{k-1}\right)^{t-h}(n)_t,$$

where the $o(1)$ tends to 0 as $n \to \infty$. We claim that this is more than the upper bound in (3.1), whence a contradiction. To see this, we note that

$$\left(\frac{1+\lambda}{k-1}\right)^{t-h} = \left(\frac{k-1}{1+\lambda}\right)^{h-1}\left(\frac{1+\lambda}{k-1}\right)^{t-1} \geq 8(1+\lambda)^{t-1}(k-1)^{1-t} \geq 4(1+\lambda)^{t}(k-1)^{1-t},$$

using the fact that $h \geq k \geq 4$ and $\lambda \leq \frac{1}{2}$, and hence $(k-1)/(1+\lambda) \geq 2$. Since $(1+\lambda)^t\tau = 1$, this shows that for sufficiently large n, the total number of labeled blue copies of H is at least

$$\tau(n)_h \cdot (d-h)_{t-h} \geq 2(k-1)^{1-t}(n)_t, \tag{3.3}$$

contradicting (3.1). □

We are now ready to begin the proof of Theorem 1.2. The proof is somewhat long, since it proceeds by iteratively improving our understanding of the structure of a coloring which minimizes the number of monochromatic copies of H, until we eventually can conclude that such a coloring is isomorphic to the Turán coloring. In order to keep the logical flow manageable, we split the proof into a number of claims, each of which provides more structural information on the coloring.

Proof of Theorem 1.2. Fix a red/blue coloring χ of $E(K_n)$ with the minimum number of monochromatic copies of H. We partition the vertices of K_n into three subsets. The first, V_R, consists of all vertices with red degree at least $n-d$. Similarly, V_B consists of those vertices with blue degree at least $n-d$. Finally, V_{RB} consists of all remaining vertices, namely those vertices with both red and blue degree at least[4] d. We remark that it is at this partitioning step that our proof fails for $k = 3$. For indeed, if $k = 3$, then $d = \frac{1+\lambda}{2}n$, meaning that V_R and V_B will not in general be disjoint. However, since $k \geq 4$ and $\lambda < 1/2$, we have that $V_R \cap V_B = \varnothing$.

Our first claim shows that V_{RB} must be small.

Claim 3.3. $|V_{RB}| \leq \lambda n$.

Proof. Suppose for contradiction that $|V_{RB}| > \lambda n$. Since n is sufficiently large, we may therefore assume that $|V_{RB}| \geq 4^h$, so we may apply Lemma 2.2 to the induced coloring on V_{RB}, with $H_1 = H_2 = H_0$. We then conclude without loss of generality that V_{RB} contains at least $4^{-h^2}(n)_h$ labeled blue copies of H_0. On the other hand, since every vertex in V_{RB} has blue degree at least d, we may apply Lemma 3.2 and conclude that V_{RB} contains at most $\tau(n)_h$ blue H_0. Combining these two bounds, we see that

$$\tau(n)_h \geq 4^{-h^2}(|V_{RB}|)_h \geq (1-o(1))4^{-h^2}\lambda^h(n)_h.$$

However, Lemma 3.1(a) shows that $\tau \leq 4^{-h^2-1}\lambda^h$, which is a contradiction for n sufficiently large. □

Similarly, our next claim shows that one of V_R and V_B must also be small.

[4]Recall that we are omitting all floor and ceiling signs, and thus are treating d as an integer even though $(1+\lambda)n/(k-1)$ need not be an integer. As such, every vertex has either red degree at least $n-d$ or blue degree at least d, whereas we could have an off-by-one error here if d were not an integer. Such rounding and off-by-one issues occur throughout the paper, but we will not belabor this point further.

Claim 3.4. $\min\{|V_R|,|V_B|\} \le 18\lambda n$.

Proof. Recall that since $k \ge 4$, we have that V_R and V_B are disjoint. Suppose for contradiction that $|V_R|,|V_B| \ge 18\lambda n$. Consider the set of edges between V_R and V_B, and suppose without loss of generality that at most half these edges are blue. Note that since every vertex in V_B has blue degree at least $n-d \ge d$, we must have at most $\tau(n)_h \le \tau\lambda^{-h}(|V_B|)_h$ labeled blue copies of H_0 in V_B, by Lemma 3.2. On the other hand, if the average blue degree inside V_B is at least $(1-\frac{1}{k-1}+\lambda)|V_B|$, then Lemma 2.4 implies that V_B contains at least $\gamma(|V_B|)_h$ labeled blue copies of H_0. But Lemma 3.1(a) shows that $\tau < \gamma\lambda^h$, and we conclude that the average blue degree in V_B is less than $(1-\frac{1}{k-1}+\lambda)|V_B|$.

Now, we add up the blue degrees of all vertices in V_B. On the one hand, this is at least $(n-d)|V_B|$, since every vertex in V_B has blue degree at least $n-d$. On the other hand, in this sum, we count every blue edge in V_B twice and every blue edge between V_B and $V_R \cup V_{RB}$ once. Since we assumed that at most half the edges in $V_B \times V_R$ are blue, we conclude that

$$\begin{aligned}(n-d)|V_B| &\le \left(1-\frac{1}{k-1}+\lambda\right)|V_B|^2 + \frac{1}{2}|V_B||V_R| + |V_B||V_{RB}| \\ &= \left[\left(1-\frac{1}{k-1}+\lambda\right)|V_B| + \frac{1}{2}|V_R| + |V_{RB}|\right]|V_B| \\ &= \left(n - \left(\frac{1}{k-1}-\lambda\right)|V_B| - \frac{1}{2}|V_R|\right)|V_B|,\end{aligned}$$

where in the last step we use that $n = |V_R|+|V_B|+|V_{RB}|$. This implies that

$$d \ge \left(\frac{1}{k-1}-\lambda\right)|V_B| + \frac{1}{2}|V_R| = \frac{1}{k-1}n - \lambda|V_B| - \frac{1}{k-1}|V_{RB}| + \left(\frac{1}{2}-\frac{1}{k-1}\right)|V_R|.$$

Since $d = \frac{1+\lambda}{k-1}n \le \frac{1}{k-1}n + \lambda n$, we conclude that

$$\left(\frac{1}{2}-\frac{1}{k-1}\right)|V_R| \le \lambda n + \lambda|V_B| + \frac{1}{k-1}|V_{RB}| \le 3\lambda n.$$

Finally, since $\frac{1}{2}-\frac{1}{k-1} \ge \frac{1}{6}$ for all $k \ge 4$, we conclude that $|V_R| \le 18\lambda n$, as claimed. □

We henceforth assume without loss of generality that $|V_R| \le 18\lambda n$. Combining this with Claim 3.3, we conclude that $|V_B| \ge (1-19\lambda)n$. Our next claim shows that within V_B, few vertices have blue degree much larger than $n-d$.

Claim 3.5. *Let $V_B' \subseteq V_B$ be the set of vertices in V_B with blue degree at least $(1-\frac{1}{k-1}+3\sqrt{\lambda})n$. Then $|V_B'| \le 21\sqrt{\lambda}|V_B|$.*

Proof. We first recall from the proof of Claim 3.4 that the average blue degree in V_B is at most $(1-\frac{1}{k-1}+\lambda)|V_B|$; indeed, if this were not the case, then Lemma 2.4 would yield many blue copies of H_0 in V_B, contradicting the fact that there cannot be many such copies since every vertex in V_B has blue degree at least $n-d \ge d$.

Additionally, we note that since $|V_B| \geq (1-19\lambda)n$, every vertex in V_B has at least $n-d-19\lambda n$ blue neighbors in V_B, and

$$n-d-19\lambda n = \left(1-\frac{1+\lambda}{k-1}-19\lambda\right)n \geq \left(1-\frac{1}{k-1}-20\lambda\right)n.$$

Similarly, any vertex in V_B' has at least $(1-\frac{1}{k-1}+3\sqrt{\lambda})n-19\lambda n \geq (1-\frac{1}{k-1}+2\sqrt{\lambda})n$ blue neighbors in V_B, since $20\lambda \leq \sqrt{\lambda}$. Now, we sum up over all $v \in V_B$ the number of blue neighbors of v in V_B. On the one hand, this sum is at most $(1-\frac{1}{k-1}+\lambda)|V_B|^2 \leq (1-\frac{1}{k-1}+\lambda)n|V_B|$, by our bound on the average degree in V_B. On the other hand, this sum is at least

$$\begin{aligned}&\sum_{v\in V_B'}\left(1-\frac{1}{k-1}+2\sqrt{\lambda}\right)n+\sum_{v\in V_B\setminus V_B'}\left(1-\frac{1}{k-1}-20\lambda\right)n\\ &\quad=\left(2\sqrt{\lambda}-20\lambda\right)n|V_B'|+\left(1-\frac{1}{k-1}-20\lambda\right)n|V_B|.\end{aligned}$$

Combining this with our upper bound, we find that

$$\left(2\sqrt{\lambda}-20\lambda\right)|V_B'| \leq 21\lambda|V_B|.$$

Since $20\lambda \leq \sqrt{\lambda}$, we conclude that $|V_B'| \leq 21\sqrt{\lambda}|V_B|$. □

Our next claim shows that in fact, there are no vertices of high red degree.

Claim 3.6. *Every vertex of K_n has red degree at most $\frac{1}{k-1}n+65h\sqrt{\lambda}n$.*

Proof. Suppose for contradiction that the red degree of some vertex v is at least $\frac{1}{k-1}n+65h\sqrt{\lambda}n$. Let $T = N_R(v)\cap V_B$. Since there are at most $19\lambda n$ vertices outside of V_B, we see that $|T| \geq \frac{1}{k-1}n+64h\sqrt{\lambda}n$. Recall from Claim 3.5 that V_B' consists of those vertices in V_B with blue degree at least $(1-\frac{1}{k-1}+3\sqrt{\lambda})n$, and that $|V_B'| \leq 21\sqrt{\lambda}|V_B| \leq 21\sqrt{\lambda}n$. Thus, if we let $S = T\setminus V_B'$, we see that $|S| \geq \frac{1}{k-1}n+64h\sqrt{\lambda}n-21\sqrt{\lambda}n \geq \frac{1}{k-1}n+50\sqrt{\lambda}n$, using the fact that $h \geq k \geq 4$.

Recall that $u_1,\dots,u_h$ are the vertices of H_0, and that u_i is incident to s_i pendant edges, with $s_1 \geq \cdots \geq s_h$. Let $H_0' = H_0\setminus\{u_1\}$. We apply Lemma 2.2 to S, with $H_1 = H_0'$ and $H_2 = H_0$. We conclude that S contains at least $4^{-h^2}(|S|)_h$ labeled blue copies of H_0, or at least $4^{-h^2}(|S|)_{h-1}$ labeled red copies of H_0'. If the former happens, then the total number of labeled blue copies of H_0 in S is at least

$$4^{-h^2}(|S|)_h \geq 4^{-h^2}(k-1)^{-h}(n)_h > \tau(n)_h,$$

a contradiction to Lemma 3.2, since every vertex in S has blue degree at least $n-d \geq d$.

Therefore, we may assume that S contains at least $4^{-h^2}(|S|)_{h-1}$ labeled red copies of H_0'. Recall that since $S \subseteq N_R(v)$, every red copy of H_0' in S yields a red copy of H_0 containing v. Note that since $S \subseteq V_B\setminus V_B'$, every vertex in S has red degree at least $(\frac{1}{k-1}-3\sqrt{\lambda})n$. This implies that given a labeled red copy of H_0' in S, we may extend it to a labeled red copy of H in at least

$$\left(\left(\frac{1}{k-1}+65h\sqrt{\lambda}\right)n-t\right)_{s_1}\left(\left(\frac{1}{k-1}-3\sqrt{\lambda}\right)n-t\right)_{s_2}\cdots\left(\left(\frac{1}{k-1}-3\sqrt{\lambda}\right)n-t\right)_{s_h}$$

ways. In this count, we first choose s_1 distinct red neighbors of v, then $s_2,\dots,s_h$ distinct red neighbors of the $h-1$ vertices of the fixed copy of H_0' in S. By subtracting t from each term, we can ensure that all these chosen vertices are distinct, and thus that we are truly embedding a red labeled copy of H. This quantity is at least

$$\begin{aligned}
&\left(\left(\frac{1}{k-1}+64h\sqrt{\lambda}\right)n\right)_{s_1}\left(\left(\frac{1}{k-1}-4\sqrt{\lambda}\right)n\right)_{s_2+\cdots+s_h}\\
&\quad=(1-o(1))\left(\frac{1}{k-1}+64h\sqrt{\lambda}\right)^{s_1}\left(\frac{1}{k-1}-4\sqrt{\lambda}\right)^{s_2+\cdots+s_h}(n)_{t-h}\\
&\quad=(1-o(1))(k-1)^{h-t}(n)_{t-h}(1+64h(k-1)\sqrt{\lambda})^{s_1}(1-4(k-1)\sqrt{\lambda})^{s_2+\cdots+s_h}\\
&\quad\geq(1-o(1))(k-1)^{h-t}(n)_{t-h}\exp\left(32h(k-1)\sqrt{\lambda}s_1-8(k-1)\sqrt{\lambda}(s_2+\cdots+s_h)\right),
\end{aligned}$$

where we use the inequalities $1+x\geq e^{x/2}, 1-x\geq e^{-2x}$, valid for all $x\in[0,\frac{1}{2}]$ (as well as the fact that $32hk\sqrt{\lambda}\leq\frac{1}{2}$, so that we may apply these bounds). Now, we note that $hs_1\geq s_1+\cdots+s_h=t-h$, which implies that $4hs_1-(s_2+\cdots+s_h)\geq 3hs_1\geq t$. Putting this all together, we find that any labeled red copy of H_0' in S extends to at least

$$(1-o(1))(k-1)^{h-t}e^{8(k-1)\sqrt{\lambda}t}(n)_{t-h}$$

labeled red copies of H, where the $o(1)$ term tends to 0 as $n\to\infty$. Recall from above that S contains at least $4^{-h^2}(|S|)_{h-1}$ labeled red copies of H_0'. So the total number of labeled red copies of H containing v is, for n sufficiently large, at least

$$4^{-h^2}(k-1)^{1-h}(n)_{h-1}\cdot(k-1)^{h-t}e^{8(k-1)\sqrt{\lambda}t}(n)_{t-h}\geq 4^{-h^2}e^{8(k-1)\sqrt{\lambda}t}(k-1)^{1-t}(n)_{t-1}.$$

But $e^{8(k-1)\sqrt{\lambda}t}>4^{h^2}t$ by Lemma 3.1(d), which yields a contradiction to the bound (3.2). Thus, there is no vertex v of red degree at least $(\frac{1}{k-1}+65h\sqrt{\lambda})n$. □

Claim 3.6 implies that the blue graph on K_n has minimum degree at least $(1-\frac{1}{k-1}-65h\sqrt{\lambda})n$. Our next claim shows that the coloring has few blue K_k, which will put us into a position to apply Lemma 2.3.

Claim 3.7. *The number of blue K_k is at most $2\tau^{h^{-k}}\binom{n}{k}$.*

Proof. By Claim 3.6, every vertex in K_n has blue degree at least $(1-\frac{1}{k-1}-65h\sqrt{\lambda})n\geq d$, since $65h\sqrt{\lambda}\leq\frac{1}{4}$. This implies, by Lemma 3.2, that the coloring contains fewer than $\tau(n)_h$ labeled blue copies of H_0.

Suppose for contradiction that the coloring contains at least $2\tau^{h^{-k}}\binom{n}{k}$ blue copies of K_k, and let $\mathcal{H}$ be the k-uniform hypergraph on $V(K_n)$ whose edges are these copies of K_k. Fix a proper k-coloring of H_0, let $a_1,\dots,a_k$ be the sizes of the color classes, and let $\mathcal{K}$ be the complete k-partite k-uniform hypergraph with parts of sizes $a_1,\dots,a_k$. An argument of Erdős [15] (see also [50]) implies that there are at least $(1-o(1))(2\tau^{h^{-k}})^{a_1\cdots a_k}n^h\geq(2-o(1))\tau(n)_h$ homomorphisms $\mathcal{K}\to\mathcal{H}$, since $a_1\cdots a_k\leq h^k$. If n is sufficiently large, then at least two-thirds of these homomorphisms are injective, which implies that the coloring contains at least $\tau(n)_h$ labeled blue copies of a complete k-partite graph with parts of sizes $a_1,\dots,a_k$. But each such labeled copy contains a unique blue copy of H_0 with matching labels, a contradiction. □

We now apply Lemma 2.3 to the blue graph on K_n with parameter ε. We note that we may do so, since we proved that the blue graph has minimum degree at least $(1-\frac{1}{k-1}-65h\sqrt{\lambda})n$ and at most $2\tau^{h^{-k}}\binom{n}{k}$ copies of K_k, and we have that $2\tau^{h^{-k}} \le k^{-10k}\varepsilon$ by Lemma 3.1(b), and $65h\sqrt{\lambda} \le \min\{\frac{1}{2k^2}, \frac{\varepsilon}{2}\}$ by our choice of λ. Lemma 2.3 then outputs a partition of $V(K_n)$ into $k-1$ parts $V_1,\dots,V_{k-1}$ with at most $\varepsilon\binom{n}{2}$ internal edges among all the parts. Moreover, we can assume without loss of generality that this partition minimizes the number of internal blue edges among all partitions of $V(K_n)$ into $k-1$ parts; in other words, we assume that $V_1,\dots,V_{k-1}$ is a max $(k-1)$-cut of the blue graph. As a max $(k-1)$-cut, the partition has the following property: every vertex has at most as many blue neighbors in its own part as in any other part. Indeed, if this were not true, we could decrease the number of internal blue edges by moving some vertex to another part, in which it has fewer blue neighbors.

The next claim records some further properties of this partition, namely that it is close to equitable and that most of the edges between V_i and V_j for $i \neq j$ are blue.

Claim 3.8. *The partition $V(K_n) = V_1 \sqcup \cdots \sqcup V_{k-1}$ has the following properties.*

(i) For each $1 \le i \le k-1$, we have that

$$\frac{n}{k-1} - \sqrt{2\varepsilon}n \le |V_i| \le \frac{n}{k-1} + \sqrt{2\varepsilon}n. \tag{3.4}$$

(ii) For each $1 \le i \ne j \le k-1$, we have that

$$e_B(V_i,V_j) \ge (1-k^2\varepsilon)|V_i||V_j|. \tag{3.5}$$

Proof. The blue graph has minimum degree at least $(1-\frac{1}{k-1}-65h\sqrt{\lambda})n \ge (1-\frac{1}{k-1}-\varepsilon)n$, and thus there are at least $(1-\frac{1}{k-1}-\varepsilon)\frac{n^2}{2}$ blue edges. This implies that

$$\left(1-\frac{1}{k-1}-\varepsilon\right)\frac{n^2}{2} \le \sum_{i=1}^{k-1} e_B(V_i) + \sum_{1\le i<j\le k-1} e_B(V_i,V_j) \le \varepsilon\frac{n^2}{2} + \sum_{1\le i<j\le k-1} e_B(V_i,V_j),$$

since there are at most $\varepsilon\binom{n}{2} \le \varepsilon\frac{n^2}{2}$ internal blue edges among $V_1,\dots,V_{k-1}$. Rearranging and multiplying by 2, we find that

$$\begin{aligned} 2\varepsilon n^2 &\ge \left(1-\frac{1}{k-1}\right)n^2 - 2\sum_{1\le i<j\le k-1} e_B(V_i,V_j) \\ &= \sum_{i=1}^{k-1}\left(|V_i| - \frac{n}{k-1}\right)^2 + \sum_{1\le i<j\le k-1} 2\left(|V_i||V_j| - e_B(V_i,V_j)\right), \end{aligned}$$

using the fact that $n^2 = \sum_i |V_i|^2 + 2\sum_{i<j}|V_i||V_j|$. Since each of the summands above is non-negative, we find that $(|V_i| - \frac{n}{k-1})^2 \le 2\varepsilon n^2$ for all i and that $|V_i||V_j| - e_B(V_i,V_j) \le \varepsilon n^2$ for all $i \ne j$. The former is equivalent to (3.4). The latter implies that

$$e_B(V_i,V_j) \ge |V_i||V_j| - \varepsilon n^2 \ge |V_i||V_j| - \varepsilon k^2|V_i||V_j| = (1-k^2\varepsilon)|V_i||V_j|,$$

yielding (3.5), since (3.4) implies that $|V_i| \ge \frac{n}{k-1} - \sqrt{2\varepsilon}n \ge \frac{n}{k}$ by Lemma 3.1(e). □

We will proceed to study the structure of the coloring with respect of $V_1,\dots,V_{k-1}$, to eventually conclude that each V_i is monochromatic red, and that all edges between parts are blue. To begin with, the next claim shows that no vertex in K_n can have considerable blue degree to each V_i.

Recall that H_0 is k-critical, meaning there is an edge $uu' \in E(H_0)$ whose deletion yields a $(k-1)$-colorable graph. Let $\widetilde{H_0} = H_0 \setminus \{u\}$ be obtained from H_0 by deleting one of the endpoints of this edge, so that $\widetilde{H_0}$ is $(k-1)$-colorable. Fix a proper $(k-1)$-coloring of $\widetilde{H_0}$, and let its color classes have sizes $b_1,\dots,b_{k-1}$.

Claim 3.9. *Fix a vertex $v \in V(K_n)$, and let $U_i = N_B(v) \cap V_i$ denote the set of blue neighbors of v inside V_i, for $1 \le i \le k-1$. Then there exists some i such that $|U_i| < \theta |V_i|$.*

Proof. Suppose for contradiction that $|U_i| \ge \theta |V_i|$ for all $1 \le i \le k-1$. By (3.5), we know that $e_B(V_i,V_j) \ge (1-k^2\varepsilon)|V_i||V_j|$ for all $i \ne j$. Therefore, for all $i \ne j$, we have that

$$e_B(U_i,U_j) \ge |U_i||U_j| - k^2\varepsilon |V_i||V_j| \ge \left(1 - \frac{k^2\varepsilon}{\theta^2}\right)|U_i||U_j|. \tag{3.6}$$

Thus, if we pick a random vertex from U_i and a random vertex from U_j, they will be connected by a blue edge with probability at least $1-k^2\varepsilon/\theta^2$. By the union bound, this implies that if we pick b_i random vertices from U_i (with replacement) for each $1 \le i \le k-1$, then these vertices will form a blue homomorphic image of $\widetilde{H_0}$ with probability at least $1 - \binom{h-1}{2}k^2\varepsilon/\theta^2 \ge 1 - h^2k^2\varepsilon/(2\theta^2) = 3/4$, since $\varepsilon = \theta^2/(2h^2k^2)$. Additionally, if n is sufficiently large, then with probability at least $3/4$, these $h-1$ vertices will all be distinct, so we will find a genuine blue copy of $\widetilde{H_0}$. Therefore, the number of blue labeled copies of $\widetilde{H_0}$ in $S \coloneqq U_1 \cup \dots \cup U_{k-1}$ is at least

$$\begin{aligned}
\frac{1}{2}\prod_{i=1}^{k-1}|U_i|^{b_i} &\ge \frac{\theta^{h-1}}{2}\prod_{i=1}^{k-1}|V_i|^{b_i} && [|U_i| \ge \theta|V_i|] \\
&\ge \frac{\theta^{h-1}}{2}(1-2k\sqrt{\varepsilon})^{h-1}\left(\frac{n}{k-1}\right)^{h-1} && [|V_i| \ge \tfrac{n}{k-1} - \sqrt{2\varepsilon}n] \\
&\ge \left(\frac{\theta}{2k}\right)^h (n)_{h-1} && [2k\sqrt{\varepsilon} \le \tfrac{1}{2}] \\
&\ge \tau (n)_{h-1}, && [\theta \ge 2k\tau^{1/h}]
\end{aligned}$$

where the final step uses Lemma 3.1(c). Recall that $S \subseteq N_B(v)$, so every blue copy of $\widetilde{H_0}$ in S yields a blue copy of H_0 containing v. Moreover, by Claim 3.6, we know that v and every vertex of S have at least $(1-\frac{1}{k-1}-65h\sqrt{\lambda})n \ge n/2$ blue neighbors. This shows that every labeled blue copy of $\widetilde{H_0}$ in S extends to at least $(n/2)_{t-h}$ labeled blue copies of H which contain v. Note that

$$\frac{(n/2)_{t-h}}{(d)_{t-h}} = \frac{n/2}{d}\cdot\frac{n/2-1}{d-1}\cdots\frac{n/2-t+h+1}{d-t+h+1} \ge \left(\frac{n/2}{d}\right)^{t-h} \ge \left(\frac{5}{4}\right)^{t-h} \ge \left(\frac{5}{4}\right)^{t/2} \ge t,$$

where the second inequality uses that $d \le \frac{2}{5}n$ since $k \ge 4$, and the final inequality holds since $t \ge 100$.

Combining the computations above, we find that the number of blue copies of H containing v is at least $t\tau(n)_{h-1}(d)_{t-h}$, which is at least $2t(k-1)^{1-t}(n)_{t-1}$ by the same computation as in equation (3.3). This is a contradiction to (3.2) for sufficiently large n. □

Claim 3.9 showed that each vertex v has at least one part V_i to which it has blue degree less than $\theta|V_i|$. The next claim shows that in fact, there must be exactly one such part, in a strong sense: if $|N_B(v)\cap V_i| < \theta|V_i|$, then v must have many blue neighbors in V_j for all $j \neq i$.

Claim 3.10. *Let $v \in V(K_n)$, and let $U_i = N_B(v)\cap V_i$ denote the set of blue neighbors of v in V_i, for $1 \le i \le k-1$. If $|U_i| < \theta|V_i|$ for some i, then $|U_j| \ge (1-2k\theta)|V_j|$ for all $j \neq i$.*

Proof. By Claim 3.6, v has at most $(\frac{1}{k-1} + 65h\sqrt{\lambda})n$ red neighbors. We also know that v has at least $(1-\theta)|V_i| - 1$ red neighbors in V_i, and by (3.4),

$$(1-\theta)|V_i| \ge (1-\theta)\left(\frac{1}{k-1} - \sqrt{2\varepsilon}\right)n \ge \left(\frac{1}{k-1} - \sqrt{2\varepsilon} - \theta\right)n.$$

Hence, v is incident to at most $(65h\sqrt{\lambda} + \sqrt{2\varepsilon} + \theta)n$ red edges with vertices in $V(K_n)\setminus V_i$. Since $65h\sqrt{\lambda} \le \varepsilon$, $\sqrt{\varepsilon} \le \theta/2$, and $\varepsilon \le 1/3$, we have that $\lambda + \sqrt{2\varepsilon} + \theta \le 2\theta$. Since $|V_j| \ge n/k$ for each $j \neq i$, we conclude that v has at most $2k\theta|V_j|$ red neighbors in V_j, as claimed. □

Using the previous two claims, we next show that every vertex has high red degree to its part in the partition, and high blue degree to all other parts.

Claim 3.11. *Let $v \in V_i$. Then $|N_R(v)\cap V_i| \ge (1-\theta)|V_i|$, and $|N_B(v)\cap V_j| \ge (1-2k\theta)|V_j|$ for all $j \neq i$.*

Proof. By Claims 3.9 and 3.10, we know that there exists an index $i' \in [k-1]$ such that $|N_B(v)\cap V_{i'}| \le \theta|V_{i'}|$, and that $|N_B(v)\cap V_j| \ge (1-2k\theta)|V_j|$ for all $j \neq i'$. We claim that $i = i'$.

We recall that since we assumed $V_1,\dots,V_{k-1}$ is a max $(k-1)$-cut of the blue graph, every vertex has at most as many blue neighbors in its part as in any other part. But (3.4) implies that $(1-2k\theta)|V_j| > \theta|V_{i'}|$ for all $j \neq i'$, since $3k\theta < \frac{1}{2}$ and $\sqrt{\varepsilon} < \frac{1}{4k}$. So v has fewer blue neighbors in $V_{i'}$ than in any other part, implying that $i = i'$. □

We now know that the coloring is "almost" a Turán coloring with respect to the partition $V_1 \sqcup \cdots \sqcup V_{k-1}$: most internal edges are red and most other edges are blue. The next claim demonstrates that in fact, all internal edges are red.

Recall that $uu' \in E(H_0)$ is an edge whose deletion yields a $(k-1)$-colorable graph, and let $\widehat{H_0}$ be this graph. Fix a proper $(k-1)$-coloring of $\widehat{H_0}$, and note that in this proper coloring, u and u' must receive the same color, for otherwise we could extend it to a proper $(k-1)$-coloring of H_0. Suppose without loss of generality u and u' receive color 1, and let the sizes of the color classes be $c_1+2, c_2, \dots, c_{k-1}$, so that $\sum c_i = h-2$.

Claim 3.12. *For each $1 \le i \le k-1$, there is no blue edge inside V_i.*

Proof. By relabeling the parts, it suffices to prove this for $i = 1$. So suppose that v, w are two vertices in V_1 such that the edge vw is blue. For each $j > 1$, let W_j denote the common blue neighborhood of v and w in V_j, and let $W_1 = V_1 \setminus \{v,w\}$. By Claim 3.11, v and w are each incident to at most $2k\theta|V_j|$ red edges with vertices in V_j, and hence $|W_j| \ge (1-4k\theta)|V_j| \ge \frac{1}{2}|V_j|$ for all j. Additionally, by the same computation as in (3.6), we see that for every $1 \le i \ne j \le k-1$,

$$e_B(W_i,W_j) \ge |W_i||W_j| - k^2\varepsilon|V_i||V_j| \ge \left(1-4k^2\varepsilon\right)|W_i||W_j|.$$

Therefore, if we pick c_j random vertices (with replacement) from W_j for each $1 \le j \le k-1$, then by the union bound, they will form a blue homomorphic copy of $\widehat{H_0} \setminus \{u,u'\}$ with probability at least $1 - \binom{h-2}{2} 4k^2\varepsilon \ge \frac{3}{4}$. Additionally, if n is sufficiently large, then all these vertices will be distinct with probability at least $\frac{3}{4}$. Thus, the edge vw lies in at least $\frac{1}{2}\prod |W_j|^{c_j}$ labeled blue copies of $\widehat{H_0}$. Each vertex of every such copy of $\widehat{H_0}$ has blue degree at least $(1 - \frac{1}{k-1} - 65h\sqrt{\lambda})n$, so every such blue copy of $\widehat{H_0}$ can be extended to a labeled blue copy of H in at least $((1 - \frac{1}{k-1} - 65h\sqrt{\lambda})n - h)_{t-h}$ ways. So in total, the number of labeled blue copies of H containing the edge vw is at least

$$\left(\left(1 - \frac{1}{k-1} - 65h\sqrt{\lambda}\right)n - h\right)_{t-h} \cdot \frac{1}{2}\prod_{j=2}^{k-1} |W_j|^{c_j} \ge (2k)^{-h}\left(\frac{3}{5}\right)^t n^{t-2},$$

since $(1 - \frac{1}{k-1} - 65h\sqrt{\lambda})n - h > \frac{3}{5}n$ and $|W_j| \ge \frac{1}{2}|V_j| \ge \frac{1}{2k}n$.

Now, suppose that we define a new coloring χ' by recoloring the edge vw red. We claim that doing so decreases the total number of monochromatic copies of H, contradicting our defining assumption on χ. To prove this, we need to upper-bound the number of labeled red copies of H that are created when we recolor vw red. The set of such H consists of those in which vw is a pendant edge and those in which vw is one of the edges of H_0. There are at most $2n^{h-1}(\frac{2}{5}n)^{t-h-1}$ copies of the former type; indeed, we have two choices for which of v and w lies in H_0, at most n^{h-1} choices for the remaining vertices of H_0, and at most $|V_j| + 2k\theta(n - |V_j|) \le \frac{2}{5}n$ choices for each other pendant vertex, since Claim 3.10 implies that every vertex in part V_j has at most $2k\theta(n - |V_j|)$ red neighbors outside V_j. By a similar argument, there are at most $h!n^{h-2}(\frac{2}{5}n)^{t-h}$ copies of the latter type; there are at most n^{h-2} choices for the other vertices of H_0, at most $h!$ automorphisms of H_0, and at most $\frac{2}{5}n$ choices for each pendant vertex.

Therefore, we see that recoloring vw red produces at most

$$2n^{h-1}\left(\frac{2}{5}n\right)^{t-h-1} + h!n^{h-2}\left(\frac{2}{5}n\right)^{t-h} < (2h)^h\left(\frac{2}{5}\right)^t n^{t-2}$$

new labeled red copies of H. We note that since $\frac{3}{2} \ge \sqrt{2}$ and $2^x > x$ for all $x \ge 0$,

$$\left(\frac{3}{2}\right)^t \ge 2^{t/2} \ge 2^{4kh^2} > (4kh)^h.$$

This is equivalent to $(2k)^{-h}(\frac{3}{5})^t n^{t-2} > (2h)^h(\frac{2}{5})^t n^{t-2}$. Therefore, χ' has strictly fewer monochromatic copies of H than χ, a contradiction. □

The previous claim showed that each part V_i contains only red edges. In particular, we see that the blue graph is $(k-1)$-colorable, and thus there are no blue copies of H. The next claim shows that moreover, the edges between V_i and V_j must all be blue.

Claim 3.13. *For every $1 \le i \ne j \le k-1$, all edges between V_i and V_j are blue.*

Proof. Suppose for contradiction that there is a red edge vw, where $v \in V_i$ and $w \in V_j$. Since every edge in V_i is red and $|V_i| > t$ for n sufficiently large, we see that vw must lie in at least one red copy of H. However, if we recolor vw blue, then it will not lie in any blue copy of H, since recoloring it will still maintain the property that the blue graph is $(k-1)$-colorable. This shows that recoloring vw blue must strictly decrease the number of monochromatic copies of H, contradicting our choice of χ. □

At this point, we've found that if the coloring χ minimizes the number of copies of H, then its red graph consists of a disjoint union of $k-1$ red cliques, and its blue graph is complete $(k-1)$-partite. This shows that there are no blue copies of H, and the number of labeled red copies of H equals $\sum_{i=1}^{k-1}(|V_i|)_t$, since each V_i spans a red clique. The function

$$f(x) = \begin{cases} (x)_t & x \geq t \\ 0 & x < t \end{cases}$$

is convex, and agrees with the function $(x)_t$ whenever x is a non-negative integer. Thus, by Jensen's inequality, the quantity $\sum_{i=1}^{k-1}(|V_i|)_t = \sum_{i=1}^{k-1} f(|V_i|)$ is minimized when all the quantities $|V_i|$ are as equal as possible. In other words, the unique coloring χ which minimizes the number of monochromatic copies of H is the Turán coloring, which completes the proof of Theorem 1.2. □

4 Three-color bonbons

In this section, we prove Theorem 1.6. Before proceeding with the proof, we briefly discuss how the three-color case differs from the two-color case, and where new ideas are needed in the proof. The first difference is that in the two-color case, we had three sets V_R, V_B, V_{RB}, and it was fairly easy to show that two of them must be very small (Claims 3.3 and 3.4). In the three-color setting, we start with seven sets, corresponding to the non-empty subsets of $\{R,B,Y\}$. As before, it is straightforward to show that four of these must be small, namely V_R, V_B, V_Y, and V_{RBY}. However, showing that two of the remaining sets are also small requires a new idea, which is where the assumption that k is polite arises: we show that if two of the remaining sets are both large, then we can find many monochromatic K_k by "gluing together" monochromatic $K_{k/2}$ in the two large sets. This step is done in Claim 4.8.

The other big difference is that in the two-color case, we could pick the "error parameter" λ to be very small with respect to $1/k$, and could thus prove Claim 3.6 directly after Claim 3.5. In the three-color setting, there are two such "error parameters": λ, which controls the size of V_{RBY}, and η, which controls the sizes of the remaining small sets. We can again ensure that λ is very small, but η is by necessity reasonably large—much smaller than $1/k$, but larger than $1/r(k,k)$. This means that the sets whose size is controlled by η are not small enough for the argument of Claim 3.6 to work. In order to get around this, we first prove that these sets are actually *empty*, at which point the argument of Claim 3.6 can go through.

We now proceed with the proof of Theorem 1.6. So we fix a sufficiently large polite integer k, a sufficiently large integer t, and let H be obtained from K_k by appending $t-k$ pendant edges. As the overall structure of the proof is broadly similar to that of Theorem 1.2, but with some added difficulties, we omit or shorten the proofs that are essentially identical to those presented in Section 3. Along the same lines, we will keep less careful track of the parameters, only enforcing the hierarchy

$$\eta \ll \frac{1}{k}, \qquad \gamma \ll \frac{1}{k}, \qquad \theta \ll \frac{1}{k}, \qquad \varepsilon \ll \theta, \qquad \lambda \ll \eta, \qquad \lambda \ll \varepsilon,$$

as well as choosing $\tau = (1+\lambda)^{-t}$, and then ensuring that t is sufficiently large so that $\tau \ll \lambda$.

Suppose we have a red/blue/yellow-coloring χ of $E(K_n)$ with the minimum number of monochromatic H. We wish to prove that the coloring is Ramsey-blowup-like, meaning that one of the colors spans

$r(k,k)-1$ disjoint cliques whose sizes differ by at most one, and the remaining two colors contain no monochromatic K_k. Let $r = r(k,k)$, let $\lambda = \lambda(k) > 0$, and let $d = (1+\lambda)n/(r-1)$. We recall that by the Ramsey-blowup coloring, we have

$$m_3(H,n) \le (r-1)^{1-t}(n)_t. \tag{4.1}$$

We now record without proof the three-color analogues of Lemmas 2.1, 2.2 and 3.2. The proofs are identical to those given in the two-color case; the main significant observation is that any upper bound on $c_3(H)$, such as the one given by the Ramsey-blowup coloring in (4.1), yields a version of Lemma 3.2, which is why the parameter d defined above appears in Lemma 4.3.

Lemma 4.1 (Three-color analogue of Lemma 2.1)**.** *For every graph H on t vertices, there exists a constant $C > 0$ such that the following holds. For any three-coloring χ of $E(K_n)$ with the minimum number of monochromatic copies of H, and for any $v \in V(K_n)$,*

$$\left(1-\frac{C}{n}\right)\frac{t}{n}m_3(H,n) \le m_v(H,\chi) \le \left(1+\frac{C}{n}\right)\frac{t}{n}m_3(H,n),$$

where $m_v(H,\chi)$ denotes the number of monochromatic labeled copies of H containing v.

Lemma 4.2 (Three-color analogue of Lemma 2.2)**.** *Let H_1,H_2,H_3 be graphs on h_1,h_2,h_3 vertices, respectively, and suppose that $h_1,h_2,h_3 \le h$. For any $n \ge 27^{h^2}$, any red/blue/yellow-coloring of $E(K_n)$ contains at least $27^{-h^2}(n)_{h_1}$ labeled red copies of H_1, or at least $27^{-h^2}(n)_{h_2}$ labeled blue copies of H_2, or at least $27^{-h^2}(n)_{h_3}$ labeled yellow copies of H_3.*

Remark. The constant 27 appears in this statement because the best known upper bound on the three-color Ramsey number of K_k is $(27-o(1))^k$. In general, the q-color version of such a statement would involve the constant q^q.

Lemma 4.3 (Three-color analogue of Lemma 3.2)**.** *Suppose n is sufficiently large. Let $S \subseteq V(K_n)$ be a set of vertices, each with at least d blue neighbors. Then S contains fewer than $\tau(n)_k$ labeled blue copies of K_k.*

In addition to these three basic lemmas, we also need the following result, which is proved by combining Lemma 2.4 and Ramsey's theorem.

Lemma 4.4. *For all integers k,ℓ, there exists $\gamma = \gamma(k,\ell) > 0$ such that the following holds for all sufficiently large n and any red/blue/yellow-coloring of $E(K_n)$. If there are fewer than $\frac{1}{r(k,\ell)}\binom{n}{2}$ yellow edges, then there are at least $\gamma(n)_k$ labeled blue copies of K_k, or at least $\gamma(n)_\ell$ labeled red copies of K_ℓ.*

Proof. Let G be the n-vertex graph whose edges are the red and blue edges of K_n. By assumption, the number of edges in G is at least

$$\left(1-\frac{1}{r(k,\ell)}\right)\binom{n}{2} \ge \left(1-\frac{1}{r(k,\ell)-1}+\frac{1}{r(k,\ell)^2}\right)\binom{n}{2}.$$

Therefore, by Lemma 2.4, G contains at least $\gamma_0(n)_{r(k,\ell)}$ labeled copies of $K_{r(k,\ell)}$, for some $\gamma_0 > 0$ depending only on k,ℓ. In the original coloring of $E(K_n)$, each such copy of $K_{r(k,\ell)}$ in G contains at

least one blue copy of K_k or at least one red copy of K_ℓ. Suppose first that least half of them contribute a blue copy of K_k. Each labeled blue copy of K_k extends to a labeled copy of $K_{r(k,\ell)}$ in G in at most $\frac{r(k,\ell)!}{k!}(n-k)_{r(k,\ell)-k}$ ways. So in this case, the original coloring of K_n contains at least $\frac{\gamma_0}{2r(k,\ell)!}(n)_k$ labeled blue copies of K_k. Similarly, if at least half the copies of $K_{r(k,\ell)}$ in G contribute a red K_ℓ, we find at least $\frac{\gamma_0}{2r(k,\ell)!}(n)_\ell$ labeled red copies of K_ℓ. In either case, we get the desired result, where $\gamma = \frac{\gamma_0}{2r(k,\ell)!} > 0$ depends only on k and ℓ. □

We also record here a result of Xu, Shao, and Radziszowski [53] on the difference between consecutive Ramsey numbers. This inequality improves by an additive constant of 1 a classical result of Burr, Erdős, Faudree, and Schelp [3].

Lemma 4.5 ([53])**.** *For any $k \geq 5$, we have*

$$r(k,k) \geq r(k,k-1) + 2k - 2.$$

We remark that we will only use the (much) weaker result that $r(k,k) \geq r(k,k-1)+6$ for sufficiently large k. However, we state Lemma 4.5 both because it gives the best known lower bound on $r(k,k) - r(k,k-1)$, and because the weakness of this bound shows how far we are from proving Conjecture 1.5, namely that all sufficiently large integers are polite.

We are now ready to begin the proof of Theorem 1.6.

Proof of Theorem 1.6. We fix a coloring χ of $E(K_n)$ with colors red, blue, and yellow, and assume that χ has the minimum number of monochromatic copies of H among all such colorings. By Lemma 4.1 and (4.1), we have that for any vertex $v \in V(K_n)$,

$$m_v(H,\chi) \leq \left(1+\frac{C}{n}\right)\frac{t}{n}m_3(H,n) \leq \left(1+\frac{C}{n}\right)t(r-1)^{1-t}(n-1)_{t-1}, \tag{4.2}$$

where C is a constant depending only on H.

Let

$$\eta = \frac{8r(k,k/2)}{r-1},$$

and observe that for sufficiently large k,

$$\eta \geq \frac{3k}{r-1} \tag{4.3}$$

since $8r(k,k/2) \geq 3k$ for all large k, as $r(k,k/2)$ grows exponentially. Note too that if k is polite, then

$$\eta \leq 2^{-28} \tag{4.4}$$

and

$$\frac{r-1}{r(k,k-1)} \geq 1 + 25\left(\frac{r(k,k/2)}{r}\right)^{1/4} > 1 + 12\eta^{1/4}. \tag{4.5}$$

For every subset S of $\{R,B,Y\}$, let V_S denote those vertices with degree at least d in each color in S, but no other colors. Note that $V_\varnothing$ is empty, since $3d < n-1$ and thus every vertex has at least d neighbors in at least one color. Our next three claims, which are three-color analogues of Claims 3.3 and 3.4, show that six of the remaining seven sets V_S are small.

Claim 4.6. *If t is sufficiently large in terms of k, then $|V_{RBY}| \leq \lambda n$.*

Proof. This is proved in the same way as Claim 3.3. □

Claim 4.7. $|V_R \cup V_B \cup V_Y| \leq 3\eta n$.

Proof. Suppose for contradiction that this is false, and assume without loss of generality that $|V_R| > \eta n$. Every vertex in V_R has blue and yellow degrees less than d, and therefore has red degree at least $n - 2d = (1 - 2(1+\lambda)/(r-1))n > (1-3/r)n$. In particular, every vertex in V_R has at least

$$|V_R| - \frac{3}{r}n > \left(1 - \frac{3}{\eta r}\right)|V_R| \geq \left(1 - \frac{1}{k}\right)|V_R| \geq \left(1 - \frac{1}{k-1} + \frac{1}{k^2}\right)|V_R|$$

red neighbors in V_R, where the second inequality follows from (4.3). Therefore, by Lemma 2.4, we see that V_R contains at least $\gamma(|V_R|)_k$ red labeled copies of K_k, where γ depends only on k. Any such copy of K_k extends to at least $(n-2d-k)_{t-k}$ red copies of H, since each vertex in V_R has red degree at least $n-2d$, and we subtract k to ensure we don't pick one of the k vertices from the fixed copy of K_k. So in total, the number of red labeled copies of H is at least

$$\gamma(|V_R|)_k(n-2d-k)_{t-k} \geq (1-o(1))\gamma\eta^k\left(1-\frac{3}{r}\right)^{t-k}(n)_t \geq \gamma\eta^k 2^{k-t}(n)_t$$

for n sufficiently large. Since we chose t sufficiently large relative to k (and thus sufficiently large relative to γ and η, which depend only on k), we have that

$$\gamma\eta^k 2^{k-t} > 2t(r-1)^{1-t},$$

as the left-hand side decays exponentially in t with base 2, whereas the right-hand side decays exponentially with base $r-1 > 2$. Thus, the number of red labeled copies of H we've found is strictly more than the upper bound given in (4.1), which is the desired contradiction. □

At this point we've found that $V_\varnothing \cup V_R \cup V_B \cup V_Y \cup V_{RBY}$ is small, so almost all vertices lie in $V_{RB} \cup V_{RY} \cup V_{BY}$. Our next lemma shows that in fact, two of these three sets must also be small.

Claim 4.8. $\min\{|V_{RB}|, |V_{RY}|\} < \eta n$.

Proof. Suppose for contradiction that $|V_{RB}|, |V_{RY}| \geq \eta n$. Note that these two sets are disjoint by definition. Let $c = \frac{1}{8k}$, and note that $\frac{1}{4r(k,k/2)} \leq c$ for sufficiently large k, since $r(k,k/2)$ grows exponentially in k. Since every vertex in V_{RB} has at most d yellow neighbors, the total number of yellow edges between V_{RB} and V_{RY} is at most

$$d|V_{RB}| \leq \frac{d}{\eta n}|V_{RB}||V_{RY}| \leq \frac{2n/(r-1)}{\eta n}|V_{RB}||V_{RY}| = \frac{1}{4r(k,k/2)}|V_{RB}||V_{RY}| \leq c|V_{RB}||V_{RY}|.$$

For the same reason, the number of blue edges between V_{RB} and V_{RY} is at most $d|V_{RY}| \leq c|V_{RB}||V_{RY}|$. Therefore, at least a $1-2c$ fraction of the edges between V_{RB} and V_{RY} are red.

At this point, our goal is to build many red copies of K_k in $V_{RB} \cup V_{RY}$, which will yield a contradiction to Lemma 4.3. To do so, we first build many red copies of $K_{k/2}$ in V_{RB}. Since most of the edges between V_{RB} and V_{RY} are red, each such red $K_{k/2}$ can be glued to many red copies of $K_{k/2}$ in V_{RY}, to form the large collection of red K_k in $V_{RB} \cup V_{RY}$. We now proceed with the technical details of this argument.

Let $V'_{RB} \subseteq V_{RB}$ consist of those vertices with at least $1-4c|V_{RY}|$ red neighbors in V_{RY}, so that $|V'_{RB}| \geq \frac{1}{2}|V_{RB}|$. Recall that every vertex in V'_{RB} has fewer than d yellow neighbors, and thus fewer than $\frac{2d}{\eta n}|V'_{RB}|$ yellow neighbors in V'_{RB}. We also have that

$$\frac{2d}{\eta n} < \frac{4}{\eta r} = \frac{1}{2r(k,k/2)}$$

by our choice of η. We now apply Lemma 4.4 to the induced coloring on V'_{RB} with $\ell = k/2$, which we may do since the above implies that the yellow edge density is less than $\frac{1}{r(k,k/2)}$. We conclude that V'_{RB} contains at least $\gamma(|V'_{RB}|)_k$ labeled blue K_k or at least $\gamma(|V'_{RB}|)_{k/2}$ labeled red $K_{k/2}$, for some $\gamma > 0$ depending only on k. On the other hand, by Lemma 4.3, we see that $|V'_{RB}|$ must contain fewer than $\tau(n)_k$ blue K_k. Recalling that we chose t sufficiently large, and thus τ sufficiently small, we see that $\tau < \gamma$, implying that the former case cannot occur. So V'_{RB} contains at least $\gamma(|V_{RB}|')_{k/2}$ labeled red copies of $K_{k/2}$.

Fix such a red $K_{k/2}$, and call it Q. As every vertex of Q has at least $(1-4c)|V_{RY}|$ red neighbors in V_{RY}, the common red neighborhood of Q in V_{RY} has size at least $(1-4kc)|V_{RY}| = \frac{1}{2}|V_{RY}|$. Let $V'_{RY} \subseteq V_{RY}$ be this common red neighborhood of Q. By exactly the same argument as above (but with the roles of blue and yellow swapped), we see that V'_{RY} must contain at least $\gamma(|V'_{RY}|)_{k/2}$ red $K_{k/2}$. Since this works for every choice of Q, we conclude that $V_{RB} \cup V_{RY}$ contains at least $\gamma^2(|V'_{RB}|)_{k/2}(|V'_{RY}|)_{k/2} \geq \gamma'(n)_k$ red K_k, for some $\gamma' > 0$ depending only on k. As we chose t sufficiently large, we have that $\tau < \gamma'$. This yields a contradiction to Lemma 4.3, since every vertex in $V_{RB} \cup V_{RY}$ has at least d red neighbors. □

By permuting the colors, we may assume without loss of generality that $|V_{RY}|, |V_{BY}| \leq \eta n$. Putting together the previous three claims, we conclude that $|V_{RB}| \geq (1-6\eta)n$.

Before continuing to the next claim, we record some simple bounds for future convenience. First, we note that for $0 \leq x \leq \frac{1}{100}$, we have that

$$\frac{1}{(1+\sqrt{x})(1-6x)} < 1-x.$$

We know that $\eta \leq \frac{1}{100}$ by (4.4), so

$$\frac{1}{(1+\sqrt{\eta})(1-6\eta)} < 1-\eta = 1-\frac{8r(k,k/2)}{r-1} < 1-\frac{1}{r} = \frac{r-1}{r}.$$

By rearranging, and by picking λ sufficiently small with respect to η, we find that

$$\frac{1+\lambda}{(1-6\eta)(r-1)} < \frac{1+\sqrt{\eta}}{r}$$

which implies that

$$d = \frac{1+\lambda}{r-1}n < \frac{1+\sqrt{\eta}}{r}|V_{RB}| \tag{4.6}$$

since $|V_{RB}| \geq (1-6\eta)n$. Our next claim is the three-color analogue of Claim 3.5. It says few vertices in V_{RB} have low yellow degree.

Claim 4.9. *Let $V'_{RB} \subseteq V_{RB}$ denote the set of vertices in V_{RB} with yellow degree less than $\frac{1-\eta^{1/4}}{r}|V_{RB}|$. Then $|V'_{RB}| \leq \eta^{1/4}n$.*

Proof. The proof is very similar to that of Claim 3.5. Suppose first that there are fewer than $\frac{1}{r}\binom{|V_{RB}|}{2}$ yellow edges in V_{RB}. Then by applying Lemma 4.4 with $k = \ell$ to the induced coloring on V_{RB}, we find that V_{RB} must contain at least $\gamma(|V_{RB}|)_k$ labeled red or blue copies of K_k, for some $\gamma > 0$ depending only on k. But this is a contradiction to Lemma 4.3 because every vertex in V_{RB} has red and blue degree at least d, and because we chose t sufficiently large, and thus $\tau < \gamma$.

Therefore, the number of yellow edges in V_{RB} is at least $\frac{1}{r}\binom{|V_{RB}|}{2} = \frac{|V_{RB}|}{2r}(|V_{RB}|-1)$. Every vertex in V_{RB} has at most d yellow neighbors in V_{RB}, and every vertex in V'_{RB} has fewer than $\frac{1-\eta^{1/4}}{r}|V_{RB}|$ yellow neighbors in V_{RB}. Summing this up over all vertices in V_{RB}, we see that the total number of yellow edges in V_{RB} is at most

$$\frac{1}{2}\left(\sum_{v\in V'_{RB}} \frac{1-\eta^{1/4}}{r}|V_{RB}| + \sum_{v\in V_{RB}\setminus V'_{RB}} d\right) < \frac{|V_{RB}|}{2r}\left((1-\eta^{1/4})|V'_{RB}| + (1+\sqrt{\eta})(|V_{RB}|-|V'_{RB}|)\right),$$

where we use (4.6). Comparing our lower and upper bounds for the number of yellow edges in V_{RB}, we find that

$$|V_{RB}| - 1 < (1+\sqrt{\eta})|V_{RB}| - (\eta^{1/4}+\sqrt{\eta})|V'_{RB}| \leq (1+\sqrt{\eta})|V_{RB}| - \eta^{1/4}|V'_{RB}|,$$

which implies that

$$|V'_{RB}| \leq \frac{\sqrt{\eta}}{\eta^{1/4}}|V_{RB}| \leq \eta^{1/4}n. \qquad \square$$

Our next claim does not have a direct analogue in the two-color setting, and this is because of an important difference between the two cases discussed at the beginning of Section 4. Namely, in the case of two colors, we picked λ to be very small with respect to $\frac{1}{k}$, and could thus prove Claim 3.6 directly after Claim 3.5 (whose analogue we have just proven). In the three-color setting, λ is still very small, but η is actually fairly large compared to $\frac{1}{r}$. So if we try to mimic the proof of Claim 3.6 using the information we have so far, we will not be able to deduce a contradiction, because the vertices in $V_R \cup V_B \cup V_Y \cup V_{RY} \cup V_{BY}$ actually may have large yellow degree.

We do eventually prove a three-color analogue of Claim 3.6, in Claim 4.13. But in order for the proof to work, we first show that all the sets $V_R, V_B, V_Y, V_{RY}, V_{BY}$ are empty, which is the content of the next three claims. The first says that no vertex has many yellow neighbors in V_{RB}. Recall that $s_1 \geq \cdots \geq s_k$ denote the number of pendant edges on the vertices of K_k in H.

Claim 4.10. *Every vertex has at most $2\eta^{1/4}n$ yellow neighbors in V_{RB}.*

Proof. Suppose for contradiction that some $v \in V(K_n)$ has at least $2\eta^{1/4}n$ yellow neighbors in V_{RB}, and let $S = N_Y(v) \cap V_{RB}$ be the set of these yellow neighbors. Let $T = S \setminus V'_{RB}$, where V'_{RB} is the set of vertices with fewer than $\eta^{1/4}n$ yellow neighbors, as in Claim 4.9. By Claim 4.9, we see that

$|T| \geq 2\eta^{1/4}n - \eta^{1/4}n = \eta^{1/4}n$. Since $T \subseteq V_{RB}$, Lemma 4.3 implies that T contains fewer than $\tau(n)_k$ red K_k and fewer than $\tau(n)_k$ blue K_k. Since we chose t sufficiently large so that $\tau(n)_k \leq 27^{-k^2}(|T|)_k$, we conclude from Lemma 4.2 that T contains at least $27^{-k^2}(|T|)_{k-1}$ labeled yellow copies of K_{k-1}. Note that $(|T|)_{k-1} \geq (1-o(1))\eta^{(k-1)/4}(n)_{k-1} \geq \eta^{k/4}(n)_{k-1}$ for n sufficiently large. Thus, for sufficiently large n, we find that T contains at least $27^{-k^2}\eta^{k/4}(n)_{k-1}$ labeled yellow copies of K_{k-1}. Fix any such copy of K_{k-1} in T. It can be extended to a labeled yellow copy of H in at least

$$(2\eta^{1/4}n-k)_{s_1}\left(\frac{1-\eta^{1/4}}{r}n-k\right)_{s_2+\cdots+s_k}$$

ways, since v has at least $2\eta^{1/4}n-k$ yellow neighbors besides the k vertices already chosen, and every other vertex of the K_k lies in $V_{RB}\setminus V'_{RB}$, and hence has at least $\frac{1-\eta^{1/4}}{r}n$ yellow neighbors. Note that $2\eta^{1/4} > \frac{1}{r}+\eta^{1/4}$, and so the expression above is at least

$$(1-o(1))r^{k-t}(r\eta^{1/4})^{s_1}(1-\eta^{1/4})^{s_2+\cdots+s_k}(n)_{t-k}.$$

Since $r \geq 4$, we have that $r\eta^{1/4} \geq r(r-1)^{-1/4} \geq \sqrt{r} \geq 2^{k/4}$, using the bound $r \geq 2^{k/2}$ due to Erdős [18]. Additionally, by (4.4), we have that $\eta \leq 2^{-20} = 32^{-4}$, so that $1-\eta^{1/4} \geq 1-\frac{1}{32} \geq 2^{-1/16}$. Putting this all together, we see that

$$(r\eta^{1/4})^{s_1}(1-\eta^{1/4})^{s_2+\cdots+s_k} \geq 2^{\frac{ks_1}{4}}\cdot 2^{-\frac{s_2+\cdots+s_k}{16}} = 2^{\frac{1}{16}(4ks_1-s_2-\cdots s_k)} \geq 2^{t/32},$$

using the fact that $ks_1 \geq s_1+\cdots+s_k = t-k$, so that $4ks_1-(s_2+\cdots+s_k) \geq 3ks_1 \geq t$.

Now, we recall that T contains at least $27^{-k^2}\eta^{k/4}(n)_{k-1}$ labeled yellow copies of K_{k-1}, so in total, the number of yellow labeled copies of H containing v is at least

$$(1-o(1))27^{-k^2}\eta^{k/4}r^{k-t}2^{t/32}(n)_{t-1},$$

which contradicts the bound (4.2) for large n, since we chose t sufficiently large with respect to k. □

As a simple corollary of the previous claim, we can conclude that V_Y is empty.

Claim 4.11. $V_Y = \varnothing$.

Proof. Suppose that there is some vertex $v \in V_Y$. Since v has fewer than d red and fewer than d blue neighbors, and fewer than $2\eta^{1/4}n$ yellow neighbors in V_{RB} by Claim 4.10, we have that

$$2d+2\eta^{1/4}n > |V_{RB}| \geq (1-6\eta)n \geq \frac{n}{2} \geq 2d+2\eta^{1/4}n$$

a contradiction, where we have that $\frac{1+\lambda}{r-1}+\eta^{1/4} < \frac{1}{4}$ for k sufficiently large, because of (4.4). □

Similarly, our next claim shows that every remaining set V_S is empty, with the exception of V_{RB} and V_{RBY}.

Claim 4.12. $V_R = V_B = V_{RY} = V_{BY} = \varnothing$.

Proof. We prove that $V_R \cup V_{RY} = \varnothing$; the exact same proof, by reversing the roles of red and blue, also shows that $V_B \cup V_{BY} = \varnothing$. Suppose that $v \in V_R \cup V_{RY}$. We know that v has at most d blue neighbors, and by Claim 4.10, v also has at most $2\eta^{1/4}n$ yellow neighbors in V_{RB}. So

$$|N_R(v) \cap V_{RB}| \geq |V_{RB}| - d - 2\eta^{1/4}n \geq n - 6\eta n - 3\eta^{1/4}n \geq \left(1 - 10\eta^{1/4}\right)n.$$

Let $T = N_R(v) \cap V_{RB}$. Every vertex in $T \subseteq V_{RB}$ has yellow degree at most

$$d = \frac{d}{|T|}|T| \leq \frac{1+\lambda}{(r-1)(1-10\eta^{1/4})}|T| \leq \frac{(1+\lambda)(1+11\eta^{1/4})}{r-1}|T| \leq \frac{1+12\eta^{1/4}}{r-1}|T|.$$

Here, the second inequality follows since $\frac{1}{1-10x} \leq 1+11x$ for all $0 \leq x \leq 2^{-7}$ and since $\eta^{1/4} \leq 2^{-7}$ by (4.4), and the final inequality holds since we chose λ sufficiently small with respect to η. Note that (4.5) is equivalent to

$$\frac{1+12\eta^{1/4}}{r-1} < \frac{1}{r(k,k-1)}.$$

Therefore, by Lemma 4.4 applied with $\ell = k-1$, we find that T contains at least $\gamma(|T|)_k$ labeled blue copies of K_k or at least $\gamma(|T|)_{k-1}$ labeled red copies of K_{k-1}, for some $\gamma > 0$ depending only on k. The former outcome is impossible by Lemma 4.3 since every vertex in T has blue degree at least d. The latter outcome is also impossible, by the same argument as in Claim 3.9 (which is essentially the same as the argument in Claim 4.10 or Claim 3.6). Namely, any red copy of K_{k-1} in T extends to at least $(d-t)_{t-k-1}$ red copies of H containing v, since both v and every vertex in T have red degree at least d. So if there are $\gamma(|T|)_{k-1}$ labeled red K_{k-1} in T, then we have at least $\gamma(|T|)_{k-1}(d-t)_{t-k-1}$ labeled red copies of H containing v, contradicting the bound (4.2). □

The following is the analogue of Claim 3.6, and is proved in the same way.

Claim 4.13. *Every vertex has yellow degree less than* $(\frac{1}{r-1} + 65k\sqrt{\lambda})n$.

In addition to knowing that no vertex has yellow degree much greater than d, we will also need to know that every vertex has red and blue degree noticeably above d, which is the content of the next claim.

Claim 4.14. *Every vertex has red and blue degree at least* $\frac{2k-6}{r-1}n$.

Proof. Suppose for contradiction that some vertex v has red degree less than $\frac{2k-6}{r-1}n$ (the case of a vertex of low blue degree follows by interchanging the roles of red and blue). By Claim 4.13, v has fewer than $(\frac{1}{r-1} + 65k\sqrt{\lambda})n < \frac{2}{r-1}n$ yellow neighbors, since we chose λ small with respect to k, and thus with respect to r. So we conclude that v has at least $(1 - \frac{2k-4}{r-1})n = \frac{r-2k+3}{r-1}n$ blue neighbors. Let T be the blue neighborhood of v. Every vertex in T has yellow degree less than

$$\left(\frac{1}{r-1} + 65k\sqrt{\lambda}\right)n \leq \left(\frac{1}{r-1} + 65k\sqrt{\lambda}\right)\frac{r-1}{r-2k+3}|T| < \left(\frac{1}{r-2k+3} + 130k\sqrt{\lambda}\right)|T|,$$

since $r-1<2(r-2k+3)$ for sufficiently large k, since r grows exponentially in k. By Lemma 4.5, we know that $r-2k+3\geq r(k,k-1)+1$. Additionally, since we chose λ sufficiently small with respect to k, we see that

$$\frac{1}{r-2k+3}+130k\sqrt{\lambda}\leq\frac{1}{r(k,k-1)+1}+130k\sqrt{\lambda}<\frac{1}{r(k,k-1)}.$$

Therefore, the yellow edge density in T is less than $\frac{1}{r(k,k-1)}$. We now argue identically to the proof of Claim 4.12: by Lemma 4.4, T contains many blue K_k or many red K_{k-1}, both of which are impossible since v and every vertex in T have red and blue degrees at least d. □

For the rest of the proof, let G be the graph of red and blue edges in the coloring, and note that G has minimum degree at least $(1-\frac{1}{r-1}-65k\sqrt{\lambda})n$. In order to apply Lemma 2.3 to G, we need to check that G has few copies of K_r, which is done analogously to Claim 3.7.

Claim 4.15. *G has at most $2\tau(n)_r$ copies of K_r.*

Proof. Every copy of K_r in G yields at least one red or blue copy of K_k in the original coloring. By the same averaging argument as in the proof of Lemma 4.4, if there are at least $2\tau(n)_r$ copies of K_r in G, then there are at least $2\tau(n)_k$ red or blue copies of K_k in the original coloring. Without loss of generality, at least half of these are blue, so there are at least $\tau(n)_k$ labeled blue K_k. However, since every vertex of K_n lies in $V_{RB}\cup V_{RBY}$, and thus has blue degree at least d, we obtain a contradiction to Lemma 4.3. □

We can now apply Lemma 2.3 to G, and find that G is nearly $(r-1)$-partite: it has a partition into parts $V_1,\dots,V_{r-1}$ such that the total number of internal edges in these parts is at most $\varepsilon\binom{n}{2}$. As in Section 3, we can assume that $V_1,\dots,V_{r-1}$ is a max $(r-1)$-cut of G. The following additional properties of the partition are proved identically to Claim 3.8: each part V_i has size $\frac{n}{r-1}\pm\sqrt{2\varepsilon}n$, and $e_{RB}(V_i,V_j)\geq(1-r^2\varepsilon)|V_i||V_j|$ for all $1\leq i<j\leq r-1$, where e_{RB} denotes the number of edges in G, that is, the number of red or blue edges.

The following claim is the three-color analogue of Claim 3.9. The proof proceeds along the same lines as that of Claim 3.9, but in this three-color setting, we need to split into two cases.

Claim 4.16. *Fix a vertex $v\in V(K_n)$, and let $U_i=N_{RB}(v)\cap V_i$ denote the set of red or blue neighbors of v inside V_i, for $1\leq i\leq r-1$. Then there exists some i such that $|U_i|\leq\theta|V_i|$.*

Proof. Suppose for contradiction that $|U_i|\geq\theta|V_i|$ for all $1\leq i\leq r-1$. Then for all $i\neq j$, we have that

$$e_{RB}(U_i,U_j)\geq|U_i||U_j|-r^2\varepsilon|V_i||V_j|\geq\left(1-\frac{r^2\varepsilon}{\theta^2}\right)|U_i||U_j|.$$

This implies that if we pick uniformly random vertices from U_i and U_j, they are joined by a red or blue edge with probability at least $1-r^2\varepsilon/\theta^2$. By the union bound, this implies that if we pick a uniformly random vertex from each U_i, the probability that they form a K_{r-1} with no yellow edges is at least $1-\binom{r-1}{2}r^2\varepsilon/\theta^2\geq\frac{1}{2}$, since we chose $\varepsilon\ll\theta\ll\frac{1}{r}$. Therefore, the number of copies of K_{r-1} in $S:=U_1\cup\dots\cup U_{r-1}$ with no yellow edges is at least

$$\frac{1}{2}\prod_{i=1}^{r-1}|U_i|\geq\frac{\theta^{r-1}}{2}\prod_{i=1}^{r-1}|V_i|\geq\frac{\theta^{r-1}}{2}\left(\frac{n}{r-1}-\sqrt{2\varepsilon}n\right)^{r-1}\geq\beta(n)_{r-1},$$

where $\beta > 0$ is a constant depending only on k (as it depends only on θ, r, and ε, each of which depends only on k). Recall that λ also depends only on k, and that $\tau = (1+\lambda)^{-t}$, which decays exponentially in t for fixed λ. Therefore, since we picked t sufficiently large, we have that $\beta \geq (2t+2)\tau$, as the right-hand side also decays exponentially in t. In all, we conclude that the number of copies of K_{r-1} in S with no yellow edges is at least $(2t+2)\tau(n)_{r-1}$. Since $S \subseteq N_{RB}(v)$, every such K_{r-1} yields a copy of K_r containing v with no yellow edges. Let $\mathcal{Q}$ denote this set of copies of K_r.

For every clique $Q \in \mathcal{Q}$, we fix some red or blue K_k contained in Q, and call it Q'. Let $\mathcal{Q}_1 \subseteq \mathcal{Q}$ be the set of $Q \in \mathcal{Q}$ such that $v \notin Q'$. Note that every red or blue K_k can appear as Q' for at most $(n-k-1)_{r-k-1}$ choices of $Q \in \mathcal{Q}_1$, for having fixed the red or blue K_k, we need to pick $r-k-1$ other vertices besides v to complete to a K_r in $\mathcal{Q}_1$. Therefore, if $|\mathcal{Q}_1| \geq 2\tau(n)_{r-1}$, then we find at least $2\tau(n)_k$ red or blue K_k not containing v. At least half of these are blue (say), which contradicts Lemma 4.3, since each of these blue K_k is contained in $V_{RB} \cup V_{RBY}$, and thus all of their vertices have at least d blue neighbors.

Therefore, if we let $\mathcal{Q}_2 = \mathcal{Q} \setminus \mathcal{Q}_1$, we conclude that $|\mathcal{Q}_2| \geq 2t\tau(n)_{r-1}$. By the same argument as in the last paragraph, this implies that there are at least $2t\tau(n)_{k-1}$ red or blue K_k containing v. Without loss of generality at least half of them are blue. Since every vertex has blue degree at least d, every such blue K_k extends to at least $(d-k)_{t-k}$ labeled blue copies of H, which contradicts the bound (4.2). □

The following claim is the three-color analogue of Claim 3.11.

Claim 4.17. *Let $v \in V(K_n)$, and suppose that v lies in part V_i. Then $|N_Y(v) \cap V_i| \geq (1-\theta)|V_i|$ and $|N_{RB}(v) \cap V_j| \geq (1-2r\theta)|V_j|$ for all $j \neq i$.*

Claim 4.17 is proved by combining the proofs of Claims 3.10 and 3.11, with no new ideas; in particular, one does not need to split into cases as in the proof of Claim 4.16.

We now know that almost all edges inside each part V_i are yellow, and that almost all edges between parts V_i, V_j are red or blue. To conclude the proof, it remains to eliminate these "almost"s. The next claim shows that each part is monochromatic yellow, establishing the three-color analogue of Claim 3.12. The heart of the proof is the same as that of Claim 3.12, but there are a few more cases to consider in this three-color setting.

Claim 4.18. *For every $1 \leq i \leq r-1$, all edges inside V_i are yellow.*

Proof. By relabeling the parts, we may assume that $i = 1$. So suppose for contradiction that there exist $u, v \in V_1$ such that the edge uv is red (the case where it is blue follows identically). For each $j > 1$, let $W_j = N_{RB}(u) \cap N_{RB}(v) \cap V_j$ denote the set of vertices in V_j which are common neighbors of u and v in G. By Claim 4.17, we know that $|W_j| \geq (1-4r\theta)|V_j| \geq \frac{1}{2}|V_j|$ for all $j > 1$, which implies that for all $2 \leq i \neq j \leq k-1$, we have

$$e_{RB}(W_i, W_j) \geq |W_i||W_j| - r^2\varepsilon|V_i||V_j| \geq (1-4r^2\varepsilon)|W_i||W_j|.$$

Therefore, if we pick a random vertex w_j from each W_j, then by the union bound, they will form a copy of K_{r-2} in G with probability at least $1 - \binom{r-2}{2} \cdot 4r^2\varepsilon \geq \frac{1}{2}$. Thus, we find that the number of copies of K_r in G containing the vertices u and v is at least

$$\frac{1}{2}\prod_{j=2}^{r-1}|W_j| \geq \frac{1}{2}(1-4r\theta)^{r-2}\prod_{j=2}^{r-1}|V_j| \geq \frac{1}{2}(1-4r\theta)^{r-2}\left(\frac{n}{r}\right)^{r-2} \geq 4r^{-r}n^{r-2},$$

since we picked θ sufficiently small with respect to k, and thus with respect to r, and since $r \geq 4$. Let $\mathcal{Q}$ denote this collection of K_r.

For every clique $Q \in \mathcal{Q}$, we fix some red or blue K_k contained in Q, and call it Q'. Let $\mathcal{Q}_1 \subseteq \mathcal{Q}$ be the set of $Q \in \mathcal{Q}$ such that Q' does not contain either u or v. Note that every red or blue K_k can appear as Q' for at most n^{r-k-2} choices of $Q \in \mathcal{Q}_1$, for having fixed the vertices of the K_k, we need to pick $r-k-2$ other vertices besides u and v to complete to a K_r in $\mathcal{Q}_1$. Therefore, if $|\mathcal{Q}_1| \geq r^{-r}n^{r-2}$, then we find at least $r^{-r}n^k$ red or blue K_k not containing u or v. At least half of these are blue (say), which contradicts Lemma 4.3, since we have found many blue K_k in $V_{RB} \cup V_{RBY}$.

Next, let $\mathcal{Q}_2$ denote the set of $Q \in \mathcal{Q}$ such that Q' contains u but not v. By the same reasoning, if $|\mathcal{Q}_2| \geq r^{-r}n^{r-2}$, then we can find at least $r^{-r}n^{k-1}$ monochromatic K_k containing u but not v. Say that at least half of these are blue. Each of them extends to at least $(d-t)_{t-k-1}$ blue labeled copies of H containing u, since every vertex has blue degree at least d. As in the proof of Claims 4.12 and 4.13, this is a contradiction to the bound (4.2). By the same reasoning, we see that $|\mathcal{Q}_3| < r^{-r}n^{r-2}$, where $\mathcal{Q}_3$ denotes the set of $Q \in \mathcal{Q}$ such that Q' contains v but not u.

Therefore, if we let $\mathcal{Q}_4$ denote the set of $Q \in \mathcal{Q}$ such that Q' contains both u and v, we find that $|\mathcal{Q}_4| \geq r^{-r}n^{r-2}$. By the same averaging as above, this implies that there are at least $r^{-r}n^{k-2}$ red K_k which contain the edge uv (we know they must be red because the edge uv is red). Moreover, by Claim 4.14, every vertex in such a red K_k has red degree at least $\frac{2k-6}{r-1}n \geq \frac{3}{r-1}n$, so such a red K_k extends to at least $(\frac{3}{r-1}n-k)_{t-k} = (1-o(1))(\frac{3}{r-1})^{t-k}(n)_{t-k}$ labeled red copies of H. So in total, the edge uv lies in at least

$$(1-o(1))r^{-r}\left(\frac{3}{r-1}\right)^{t-k}(n)_{t-2}$$

labeled red copies of H. Now we create a new coloring χ' by recoloring the edge uv yellow, and estimate how many yellow copies of H are produced. There are at most $k!(n)_{k-2}$ labeled yellow K_k containing u and v, and each of these extends to a labeled yellow copy of H in at most $(\frac{2}{r-1}n)_{t-k}$ ways, since every vertex has yellow degree at most $(\frac{1}{r-1}+65k\sqrt{\lambda})n \leq \frac{2}{r-1}n$ by Claim 4.13. Similarly, there are at most $2k!(n)_{k-1}$ labeled yellow K_k containing exactly one of u or v, and each of these extends to a labeled yellow copy of H containing both u and v in at most $(\frac{2}{r-1}n)_{t-k-1}$ ways. In total, by recoloring uv yellow, we create at most

$$k!(n)_{k-2}\left(\frac{2}{r-1}n\right)_{t-k} + 2k!(n)_{k-1}\left(\frac{2}{r-1}n\right)_{t-k-1} \leq (1+o(1))3k!\left(\frac{2}{r-1}\right)^{t-k-1}(n)_{t-2}$$

labeled yellow copies of H. Since we chose t sufficiently large with respect to k, this is less than the number of red copies of H we destroy by recoloring uv yellow, which shows that χ' has strictly fewer monochromatic copies of H than χ, a contradiction. □

We now know that every part V_i is a monochromatic yellow clique. The final claim, analogous to Claim 3.13, says that all the remaining edges are red or blue, and that there are no red or blue K_k.

Claim 4.19. *There is no yellow edge between V_i and V_j for any $1 \leq i \neq j \leq r-1$. Additionally, there is no red or blue copy of K_k.*

Proof. Suppose that there is some yellow edge between V_i and V_j. Since every edge in V_i is yellow and $|V_i| > t$, this edge participates in at least one yellow copy of H. Consider the coloring χ' defined by making all edges within the parts yellow, and making the edges between the parts be a blowup of a Ramsey coloring of K_{r-1}. This coloring has no red or blue H (since there is not even a red or blue K_k), and the number of yellow H is strictly smaller than in χ, since there are exactly as many yellow H within the parts, but none containing vertices from two different parts. This contradicts the minimality of χ.

Now suppose that there is some red or blue K_k in χ. In this case, we again see that χ' has strictly fewer monochromatic copies of H than χ: it has the same number of yellow copies of H, but strictly fewer red or blue copies because we destroyed at least one red or blue K_k. □

Therefore, we have found that each V_i is a monochromatic yellow clique, that all remaining edges are red or blue, and that there is no red or blue K_k. The final step of the proof is the same as that of Theorem 1.2: Jensen's inequality shows that any coloring of this type which minimizes the number of yellow copies of H is one in which the parts are as equally-sized as possible, completing the proof. □

Remark. For two graphs F_1, F_2, let $r(F_1, F_2)$ be the Ramsey number of F_1, F_2, namely the least r so that any red/blue coloring of $E(K_r)$ contains a red copy of F_1 or a blue copy of F_2. Examining the proof of Theorem 1.6, one can see that it holds in greater generality than simply appending pendant edges to cliques, but it still holds in a much more limited setting than Theorem 1.2. Namely, let us say that a graph H_0 is *polite* if there exists a partition of the vertex set of H_0 into two induced subgraphs H_1, H_2 so that for every $i \in \{1,2\}$ and every $v \in V(H_0)$, we have the bounds

$$\frac{r(H_0, H_i)}{r(H_0, H_0)} \leq 2^{-23}$$

and

$$\frac{r(H_0, H_0) - 1}{r(H_0, H_0 \setminus \{v\})} \geq 1 + 25 \left(\frac{r(H, H_i)}{r(H, H)} \right)^{1/4}.$$

Thus, an integer k is polite in the sense of Definition 1.4 if and only if K_k is polite.

Now, suppose that k is sufficiently large, and let H_0 be a polite k-critical graph containing a copy of K_k. Then one can check that the proof of Theorem 1.6 carries through, namely that if one appends sufficiently many pendant edges to H_0, then the resulting graph H is a three-color bonbon. However, we chose not to state and prove the theorem in this generality, simply because we don't expect it to apply to a particularly rich class of graphs.

5 Concluding remarks

5.1 On the definition of three-color bonbons

Although the definition of a three-color bonbon is a natural generalization of that of a two-color bonbon, it is perhaps not the natural generalization one would first guess. As a consequence, there is also a mismatch between the statements of Theorems 1.2 and 1.6. In the case of two colors, we precisely characterized the extremal coloring minimizing the number of monochromatic copies of H: the Turán coloring is the only

such coloring. In the case of three colors, we only (conditionally) proved a result that appears weaker: that any extremal coloring has an equitable vertex partition into $r(k)-1$ yellow cliques and has no red or blue K_k, but we did not prove that the Ramsey-blowup coloring is the unique extremizer. Here, and throughout this section, we use the notation $r(k) \coloneqq r(k,k)$ to denote the diagonal Ramsey number.

In fact, we do not believe that there is a unique extremizer in the three-color case. More generally, we believe that there should exist red/blue colorings of a blowup of $K_{r(k)-1}$ which do not contain a red or a blue K_k, but are not a blowup of a Ramsey coloring, that is, a two-coloring of $E(K_{r(k)-1})$ without a monochromatic K_k. Given a graph G and a positive integer s, we denote by $G[s]$ the s-blowup of G.

Definition 5.1. Let k and s be positive integers. We call a red/blue coloring of $E(K_{r(k)-1}[s])$ a *mixed blowup coloring* if it contains no monochromatic K_k, but is not the blowup of a Ramsey coloring of $E(K_{r(k)-1})$.

If there exist mixed blowup colorings of $K_{r(k)-1}[s]$, then the stronger alternative definition of a three-color bonbon—that the Ramsey-blowup coloring is the unique extremal coloring—is too strong. Indeed, any mixed blowup coloring can be used as the red and blue edges of a three-coloring of K_n, and it will contain the same number of monochromatic H as the Ramsey-blowup coloring, where H is obtained from K_k by appending pendant edges.

As it turns out, there is a simple characterization of when mixed blowup colorings exist.

Lemma 5.2. *Let $k \geq 3$ be an integer and $r \coloneqq r(k)$. The following statements are equivalent.*

(i) There exists a two-coloring of $E(K_{r-2})$ which extends to two distinct two-colorings of $E(K_{r-1})$, neither of which contains a monochromatic K_k.

(ii) For every $s \geq 2$, there exists a mixed blowup coloring of $K_{r-1}[s]$.

(iii) There exists a mixed blowup coloring of $K_{r-1}[2]$.

Proof. First, suppose that (i) holds, and let χ_1, χ_2 be two distinct Ramsey colorings of $E(K_{r-1})$, and let $u \in V(K_{r-1})$ be a vertex such that χ_1 and χ_2 agree on $K_{r-1} \setminus \{u\}$. For any $s \geq 2$, consider the blowup $K_{r-1}[s]$, and let U be the part of $V(K_{r-1}[s])$ corresponding to the vertex u. Between all pairs of vertices of $K_{r-1}[s]$ which are not in U, we color by blowing up χ_1 (or χ_2, since they agree away from u). We then arbitrarily partition U into two non-empty sets U_1, U_2, and color all edges incident to U_1 by blowing up χ_1, and all edges incident to U_2 by blowing up χ_2. The resulting coloring is not a Ramsey-blowup coloring, since χ_1 and χ_2 are distinct. Additionally, it has no monochromatic K_k, since neither χ_1 nor χ_2 has a monochromatic K_k, so it is a mixed blowup coloring. This shows that (i) implies (ii).

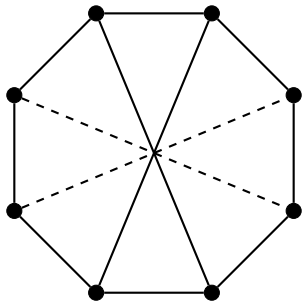
It is immediate that (ii) implies (iii), so assume that (iii) holds. Fix a mixed blowup coloring of $K_{r-1}[2]$. Since this is not a blowup coloring, there must exist two vertices u, u' in one part U whose incident edges are not colored identically. Let v_1 be a vertex in some part U_1 such that the edges uv_1 and $u'v_1$ receive different colors, and let $v_2, \dots, v_{r-2}$ be arbitrary vertices from the $r-3$ parts other than U and U_1. Then $v_1, v_2, \dots, v_{r-2}$ span a coloring of K_{r-2} which extends to two distinct Ramsey colorings of K_{r-1}, by choosing either u or u' as the extension. This shows that (i) holds, and completes the proof. □

We conjecture that mixed blowup colorings exist for infinitely many k.

Conjecture 5.3. *There exist a mixed blowup coloring of $K_{r(k)-1}[2]$ for infinitely many k.*

Thanks to Lemma 5.2, to find a mixed blowup coloring, it suffices to exhibit two Ramsey colorings of $K_{r(k)-1}$ which differ on a single vertex. Unfortunately, we are not able to do this for any value of k. A big part of the problem is doing this requires knowing the value of $r(k)$, and this is only known for $k = 3,4$. Additionally, for both of these values, it is known that there is a unique coloring of $E(K_{r(k)-1})$ with no monochromatic K_k, and in this unique Ramsey coloring, all vertices have the same red and blue degree. This implies that any coloring of $K_{r(k)-2}$ vertices extends to a Ramsey coloring of $K_{r(k)-1}$ in at most one way. For a similar reason, we only conjecture that mixed colorings exist for infinitely many k, as opposed to existing for all sufficiently large k. Indeed, it seems plausible that for infinitely many very special values of k, there is a unique, highly structured Ramsey coloring (e.g. one coming from a Paley graph, as happens for $k = 3$ and $k = 4$). For such k, there should not exist mixed blowup colorings. However, for "most" k, we expect there to be several different Ramsey colorings, among which one can likely find two that differ on a single vertex.

For off-diagonal Ramsey numbers, we can prove the existence of mixed blowup colorings. Indeed, it is known (see e.g. [41]) that $r(3,4) = 9$, and that there are exactly three non-isomorphic colorings of K_8 with no red K_3 and no blue K_4. The following figure shows these three colorings, where the edges correspond to red edges, the non-edges correspond to blue edges, and the dashed edges can be colored either red or blue. In particular, we find that there are distinct colorings that differ only on one vertex, namely any endpoint of one of the dashed edges.



It is known that there are at least 328 colorings of K_{42} without a monochromatic K_5, and in [36, Conjecture 2], it is conjectured that $r(5,5) = 43$ and that these 328 are the only Ramsey colorings for K_5. If this conjecture is true, then it establishes the existence of mixed blowup colorings for $k = 5$. Indeed, several of these colorings of K_{42} contain edges that can be colored in one of two ways. For example, using McKay's list [35, `r55_42some.g6`] of Ramsey (5,5,42) graphs, one can readily check that deleting the edge $(31,39)$ from the first graph yields a graph isomorphic to the 13th graph. In other words, these two colorings differ only on a single edge, and in particular on a single vertex.

For completeness, here are these two 42-vertex graphs, which differ in a single edge and which both have clique and independence numbers 4, presented in the `graph6` format.

- `i?Udjp^j}?W@‘bIRhHgk\SY~ECeQS\CniuKP]RQLdsX~F?b|L?h_SvygSNziSVdZ‘P|CxamFHKax[PhPyVEYxAqkY\_xCfYxNscNtb]k_uFsLruaJwr‘nPMMc]\qGhwyhfLjTELQ}T]h@qtuW`

- `i?Udjp^j}?W@‘bIRhHgk\SY~ECeQS\CniuKP]RQLdsX~F?b|L?h_SvygSNziSVdZ‘P|`
  `CxamFHKax[PhPyVEYxAqkY\_xCfYxNscNtb]k_uFsLruaJwr‘nPMMc]\qGhwyh`
  `dLjTELQ}T]h@qtuW`

The only difference between the strings is the first character of the third row in both cases, corresponding to the fact that the two graphs differ in only one edge.

5.2 Even more colors

Let k be a large integer, and let H be obtained from K_k by adding sufficiently many pendant edges. Given that we proved that the Ramsey multiplicity behavior of H is determined by a blowup coloring of a Ramsey coloring on one fewer color for both two and three colors, it is natural to expect the same behavior to persist for arbitrarily many colors. However, there appear to be major obstacles to proving such a thing, and indeed, it may not be true. Firstly, our proof technique seems to fail right away when dealing with at least four colors: in the step where we partition the vertices according to which colors they have high degree in, it is not clear how to prove that all but one of these sets is small.

To see this, we recall the simple fact, usually attributed to Lefmann [33], that for any integers k, q_1, q_2,

$$r_{q_1+q_2}(k) - 1 \geq (r_{q_1}(k) - 1)(r_{q_2}(k) - 1). \tag{5.1}$$

Indeed, given an optimal q_1-edge coloring of $K_{r_{q_1}(k)-1}$ and an optimal q_2-edge coloring of $K_{r_{q_2}(k)-1}$, we can form a (q_1+q_2)-coloring of $K_{(r_{q_1}(k)-1)(r_{q_2}(k)-1)}$ by taking a lexicographic product, which will have no monochromatic K_k. Equivalently, we can blow up the q_1-coloring to parts of size $r_{q_2}(k)-1$, and then color each part according to the q_2-coloring.

Let us suppose that there exist some k, q_1, q_2 for which inequality (5.1) is actually an equality. In that case, there are many non-isomorphic colorings of K_n with q_1+q_2+1 colors, all of which yield the same bound on the Ramsey multiplicity constant of H. Indeed, we may first equitably partition K_n into $r_{q_1}(k)-1$ parts, and color the edges between these parts according to a blowup of the q_1-coloring. Inside each part, we pick any q_2-subset of the remaining q_2+1 colors, and color according to a blowup of the q_2-coloring. Finally, inside each sub-part, we use the remaining color to form a monochromatic clique of size $n/((r_{q_1}(k)-1)(r_{q_2}(k)-1))$. If we make the same choice inside each top-level part, we get the Ramsey-blowup coloring. However, if we make different choices inside each part, we'll obtain another coloring yielding the same multiplicity bound, which is Ramsey-blowup-like. Indeed, in any Ramsey-blowup-like coloring, all copies of H have the same color, whereas here they may have different colors.

All of this works under the assumption that (5.1) is tight, which may seem like a very strong assumption. Nonetheless, until very recently [51, 42], the best known lower-bound constructions for $r_{q_1+q_2}(k)$ for $q_1+q_2 \geq 5$ were of this product form. Moreover, it is a major open problem (see e.g. [1, 37]) to determine whether $r_q(k)$ grows exponentially or super-exponentially as a function of q (for fixed k); the question of whether (5.1) is tight is a special case of this question.

Thus, it seems as though proving that H is a q-color bonbon for $q \geq 5$ is likely to be very difficult, and it is possible that there do not exist any q-color bonbons for $q \geq 5$.

5.3 Other open problems

Recall that in Theorem 1.2, we prove that we obtain a bonbon if we add $t \geq (1000kh)^{10}h^{10k}$ pendant edges to a k-critical graph H_0 with h vertices. We made no effort to optimize the constants in the lower bound on t, though our proof technique does require t to be at least exponentially large in k. Moreover, as discussed in the introduction, if $H_0 = K_k$, then some lower bound on t is necessary: if we add $o(k^2/\log k)$ pendant edges to K_k, the result will not be a bonbon, since the random coloring yields a stronger upper bound on $c(H)$ than the Turán coloring. It would be interesting to determine what the correct lower bound on t is; for instance, in case $H_0 = K_k$, is H a bonbon even if t is only polynomial in k?

It would also be very interesting to prove that other graphs are bonbons (in any number of colors). A natural place to start is in the family of graphs that we call *generalized lollipops*. We say that a graph H is a (k,t)-generalized lollipop if it has t vertices and contains a K_k whose deletion yields a forest. Equivalently, a (k,t)-generalized lollipop is obtained from K_k by attaching trees comprising $t-k$ total vertices to the vertices of the K_k. For all such graphs, the Turán coloring yields an upper bound on their Ramsey multiplicity constant of $(k-1)^{1-t}$.

Conjecture 5.4. *If $k \geq 4$ and t is sufficiently large in terms of k, then any (k,t)-generalized lollipop is a two-color bonbon.*

A natural special case of this conjecture is interesting in its own right, and may be easier to prove than the full Conjecture 5.4. Namely, the *lollipop* graph $L_{k,t}$ which is obtained from K_k by appending to a single vertex a path with $t-k$ edges.

Conjecture 5.5. *If $k \geq 4$ and t is sufficiently large in terms of k, then $L_{k,t}$ is a two-color bonbon.*

Of course, one could also pose versions of Conjectures 5.4 and 5.5 for three-color bonbons, but we expect such conjectures to be even harder to resolve.

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