

CURVATURE AND SHARP GROWTH RATES OF LOG-QUASIMODES ON COMPACT MANIFOLDS

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ABSTRACT. We obtain new optimal estimates for the $L^2(M) \rightarrow L^q(M)$, $q \in (2, q_c]$, $q_c = 2(n+1)/(n-1)$, operator norms of spectral projection operators associated with spectral windows $[\lambda, \lambda + \delta(\lambda)]$, with $\delta(\lambda) = O((\log \lambda)^{-1})$ on compact Riemannian manifolds (M, g) of dimension $n \geq 2$ all of whose sectional curvatures are nonpositive or negative. We show that these two different types of estimates are saturated on flat manifolds or manifolds all of whose sectional curvatures are negative. This allows us to classify compact space forms in terms of the size of L^q -norms of quasimodes for each Lebesgue exponent $q \in (2, q_c]$, even though it is impossible to distinguish between ones of negative or zero curvature sectional curvature for any $q > q_c$.

1. Introduction and main results. This paper addresses the question of whether one can “hear” the “shape” of a connected compact manifold, if “shape” refers to the sign of its sectional curvatures. We answer this question in the affirmative if the manifold is of constant sectional curvature and also if one uses the correct type of “radio”. Specifically, we shall show that there is a classification of the three genres of manifolds of constant sectional curvature (positive, zero and negative) using the (sharp) growth rate of L^q -norms of log-quasimodes if q is any critical or subcritical exponent for the universal bounds of Sogge [26], even though, as we shall review, norms involving any supercritical exponent cannot distinguish between manifolds of negative curvature and flat manifolds. So, “radios” involving critical or subcritical exponents must be used for this problem. Similarly, as we shall demonstrate, the different geometries exhibit different types of concentration of quasimodes near periodic geodesics, which turns out to be key to answering to this question. The positive estimates that we obtain for manifolds all of whose sectional curvatures are nonpositive or negative, though, do not require the assumption of constant curvature.

The main step to accomplish this classification is to obtain sharp critical and subcritical L^q -estimates for log-quasimodes on compact manifolds of negative and nonpositive sectional curvatures. We improve the earlier estimates in [3] and [7] and obtain, for the first time, different but sharp estimates for both types of geometries. We also are able to characterize compact connected space forms in terms of the size of quasimodes measured by any critical or subcritical $L^q(M)$ -norm. As we shall indicate, this is impossible to do

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for *any* exponent in the supercritical range $q > q_c$, with, here and in what follows,

$$(1.1) \quad q_c = 2(n+1)/(n-1),$$

denoting the critical exponent for our compact n -dimensional Riemannian manifold (M, g) .

Before stating our main results, let us review the local universal estimates of Sogge [26]. These concern the Lebesgue norms of eigenfunctions and quasimodes with spectrum in unit intervals.

If Δ_g is the Laplace-Beltrami operator associated with the metric g on M , we let $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$ denote the eigenvalues labeled with respect to multiplicity of the first order operator $P = \sqrt{-\Delta_g}$ and e_{λ_j} the associated L^2 -normalized eigenfunctions. So,

$$(1.2) \quad (Pe_{\lambda_j})(x) = \lambda_j e_{\lambda_j}(x), \quad \text{and} \quad \int_M |e_{\lambda_j}(x)|^2 dx = 1, \quad \text{if } P = \sqrt{-\Delta_g}.$$

Here, abusing notation a bit, dx denotes the Riemannian volume element, dV_g , and λ_j also denotes the frequency of the Laplace eigenfunctions, which means that $-\Delta_g e_{\lambda_j} = \lambda_j^2 e_{\lambda_j}$. If $I \subset [0, \infty)$ is an interval, we shall be concerned with the associated spectral projection operators

$$(1.3) \quad (\chi_I f)(x) = \sum_{\lambda_j \in I} E_j f(x), \quad \text{with} \quad E_j f(x) = \left(\int_M f(y) \overline{e_{\lambda_j}(y)} dy \right) \cdot e_{\lambda_j}(x).$$

Also, we shall say that the spectrum of f is in I and write $\text{Spec } f \subset I$ if $E_j f = 0$ for $\lambda_j \notin I$.

In [26], the universal bounds for $q > 2$

$$(1.4) \quad \|\chi_{[\lambda, \lambda+1]} f\|_{L^q(M)} \lesssim \lambda^{\mu(q)} \|f\|_{L^2(M)}, \quad \lambda \geq 1,$$

$$\text{with } \mu(q) = \begin{cases} n(\frac{1}{2} - \frac{1}{q}) - \frac{1}{2}, & q_c \leq q \leq \infty \\ \frac{n-1}{2}(\frac{1}{2} - \frac{1}{q}), & 2 < q \leq q_c, \end{cases}$$

were obtained. Note that when $q = q_c$ the above exponent is just

$$(1.5) \quad \mu(q_c) = 1/q_c.$$

Here, and in what follows, $\lesssim$ refers to an inequality with an implicit, but unspecified constant. As was shown in [29] the unit-band spectral estimates (1.4) are sharp on *any* compact manifold (M, g) , regardless of the geometry.

So, as in what seems to be the first paper in the program, [31], in order to obtain improvements over the bounds in (1.4) under certain geometric assumptions, one must replace the unit intervals $[\lambda, \lambda+1]$ by ones of the form $[\lambda, \lambda + \delta(\lambda)]$ with $\delta(\lambda) \searrow 0$ as $\lambda \rightarrow \infty$. For the most part, we shall take

$$(1.6) \quad \delta(\lambda) = (\log \lambda)^{-1}, \quad \lambda \gg 1.$$

Note that if

$$(1.7) \quad V_{[\lambda, \lambda + \delta(\lambda)]} = \{\Phi_\lambda : \text{Spec } \Phi_\lambda \subset [\lambda, \lambda + \delta(\lambda)]\}$$

is the space of $\delta(\lambda)$ -quasimodes, we of course have for $q > 2$

$$(1.8) \quad \sup_{\Phi_\lambda \in V_{[\lambda, \lambda + \delta(\lambda)]}} \frac{\|\Phi_\lambda\|_{L^q(M)}}{\|\Phi_\lambda\|_{L^2(M)}} = \|\chi_{[\lambda, \lambda + \delta(\lambda)]}\|_{2 \rightarrow q}, \quad \text{if } V_{[\lambda, \lambda + \delta(\lambda)]} \neq \emptyset.$$

Here, and in what follows, $\|\cdot\|_{p \rightarrow q}$ denotes the $L^p(M) \rightarrow L^q(M)$ operator norm.

As in many earlier results, our attempts to relate properties of high energy eigenfunctions e_λ in terms of the geodesic flow $\Phi_t : T^*M \setminus 0 \rightarrow T^*M \setminus 0$ or the half-wave operators $U_t = e^{itP}$ is limited by the role of the Ehrenfest time. Recall that if $a(x, \xi) \in S_{1,0}^0$ is a zero-order symbol, and if $Op(a)$ is the associated zero-order pseudo-differential operator, then Egorov's theorem from microlocal analysis says that $U_{-t}Op(a)U_t - Op(a \circ \Phi_t)$ is a pseudo-differential operator of order -1 . Thus, if, as above, e_λ is an L^2 -normalized eigenfunction of P with eigenvalue λ , we have for small $|t| > 0$ that $\|[U_{-t}Op(a)U_t - Op(a \circ \Phi_t)]e_\lambda\|_2 = O(\lambda^{-1})$. If the sectional curvatures of M are negative, though, this estimate breaks down as t approaches a multiple of $\log \lambda$. Indeed there exists an optimal C_M so that for $t > 0$

$$\|[U_{-t}Op(a)U_t - Op(a \circ \Phi_t)]e_\lambda\|_2 \lesssim \lambda^{-1} \exp(C_M t).$$

So, these improved estimates coming from Egorov's theorem only persist for $|t|$ smaller than the Ehrenfest time

$$T_E = T_E(\lambda) = \frac{\log \lambda}{C_M}.$$

Since we need to use this and related tools from microlocal analysis, we are naturally limited to studying spectral projection operators $\chi_{[\lambda, \lambda + \delta(\lambda)]}$ with $\delta(\lambda)$ as in (1.6). We refer the reader to the excellent exposition in Zelditch [39] for a more thorough discussion of the role of the Ehrenfest time in settings where the geodesic flow is ergodic or chaotic, such as when (M, g) is of negative curvature, which is a main focus of this paper.

Keeping this in mind, let us state our main result.

Theorem 1.1. *Let (M, g) be an n -dimensional connected compact Riemannian manifold. Then, if all the sectional curvatures are nonpositive, for $\lambda \gg 1$ we have the uniform bounds*

$$(1.9) \quad \|\chi_{[\lambda, \lambda + (\log \lambda)^{-1}]} f\|_{L^q(M)} \leq C (\lambda (\log \lambda)^{-1})^{\mu(q)} \|f\|_{L^2(M)}, \quad 2 < q \leq q_c,$$

with q_c and $\mu(q)$ as in (1.1) and (1.4), respectively. Moreover, if all the sectional curvatures of M are negative, for $\lambda \gg 1$ we have the uniform bounds

$$(1.10) \quad \|\chi_{[\lambda, \lambda + (\log \lambda)^{-1}]} f\|_{L^q(M)} \leq C_q \lambda^{\mu(q)} (\log \lambda)^{-1/2} \|f\|_{L^2(M)}, \quad 2 < q \leq q_c,$$

with the constant C_q in (1.10) depending on q .

As we shall see, the bounds in (1.9) are always sharp when M is flat, and, as was shown in Blair, Huang and Sogge [3], one cannot have stronger bounds if M is a product manifold with S^1 as a factor. Moreover, Germain and Myerson [15] proved negative results for all tori, including ones that are not such a product manifold. See also Hickman [18] who proved bounds with numerology essentially as in (1.9) on $\mathbb{T}^n$ for $q = q_c$.

The estimates in (1.10) are stronger than those in (1.9) since $\mu(q) < 1/2$ for $2 < q \leq q_c$, and, hence $(\log \lambda)^{-1/2} \ll (\log \lambda)^{-\mu(q)}$ if $\lambda \gg 1$.

We point out that Bérard [1] and Hassell and Tacy [16] had shown earlier that if M is any compact manifold with nonpositive sectional curvature, then the analog of (1.10) is

valid for all $q > q_c$. By well known arguments that we review in §3 below, the $(\log \lambda)^{-1/2}$ gains that they obtained for log-quasimodes never can be improved. Thus, unlike the case in Theorem 1.1 which concerns critical and subcritical exponents, the bounds for supercritical exponents $q > q_c$ do *not* distinguish between the two geometries considered above, i.e., compact manifolds of nonpositive or strictly negative sectional curvatures.

We would also like to point out that the analog of the bounds in (1.10) do not hold in the Euclidean case when $2 < q < q_c$. These involve the Euclidean spectral projection operators for $\sqrt{-\Delta_{\mathbb{R}^n}}$,

$$\chi_{[\lambda, \lambda + \delta(\lambda)]} f(x) = (2\pi)^{-n} \int_{\{\xi \in \mathbb{R}^n: |\xi| \in [\lambda, \lambda + \delta(\lambda)]\}} e^{ix \cdot \xi} \hat{f}(\xi) d\xi,$$

with

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) dx$$

denoting the Euclidean Fourier transform. It is an easy exercise to see that the Stein-Tomas [37] restriction theorem is equivalent to the estimates saying that for $\lambda \gg 1$ we have the uniform bounds

$$(1.11) \quad \|\chi_{[\lambda, \lambda + (\log \lambda)^{-1}]} f\|_{L^q(\mathbb{R}^n)} \leq C \lambda^{\mu(q)} (\log \lambda)^{-1/2} \|f\|_{L^2(\mathbb{R}^n)}, \quad q \in [q_c, \infty],$$

with again q_c and $\mu(q)$ as in (1.1) and (1.4), respectively. Conversely, unlike in (1.10), the analog of (1.11) cannot hold *for any* $q \in (2, q_c)$, since, this would imply that there must be a $L^{q'}(\mathbb{R}^n) \rightarrow L^2(S^{n-1})$ Fourier restriction theorem with q' being the conjugate exponent for q . This is impossible by the standard Knapp example since the sharp range for the $L^p(\mathbb{R}^n) \rightarrow L^2(S^{n-1})$ restriction bounds is the one treated in [37], that is $p \in [1, \frac{2(n+1)}{n+3}]$, which are the exponents that are dual to the exponents $q \in [q_c, \infty]$. Moreover, the standard Knapp construction for $\mathbb{R}^n$ shows that the bounds coming from interpolating between the estimate in (1.11) for $q = q_c$ and the trivial $L^2 \rightarrow L^2$ estimates, i.e.,

$$\|\chi_{[\lambda, \lambda + (\log \lambda)^{-1}]} f\|_{2 \rightarrow q} = O(\lambda^{\mu(q)} (\log \lambda)^{-\frac{n+1}{2}(\frac{1}{2} - \frac{1}{q})}), \quad q \in (2, q_c),$$

are sharp, and these are less favorable than the bounds in (1.10) since $\frac{n+1}{2}(\frac{1}{2} - \frac{1}{q}) < \frac{1}{2}$ for $2 < q < q_c$.

It seems remarkable that in (1.10) we are able to obtain the optimal $(\delta(\lambda))^{1/2}$, $\delta(\lambda) = (\log \lambda)^{-1}$, improvements over the $\delta(\lambda) \equiv 1$ bounds, since such improvements in the Euclidean case break down for any *subcritical* exponent $q \in (2, q_c)$. The improved bounds in (1.10) for these exponents is due to the chaotic nature of the geodesic flow in manifolds with negative sectional curvature, which results in much more favorable dispersive bounds for solutions of the wave in the universal cover that we shall exploit.

The optimal and perhaps unexpected improvements in (1.10) can also perhaps be framed as an analog for compact manifolds of negative curvature of the classical and celebrated Kunze-Stein [22] phenomenon. Chen and Hassell [13] established related results in the noncompact case in which they showed that the analog of (1.11) for hyperbolic space $\mathbb{H}^n$ is valid for all exponents $q \in (2, \infty]$ (see also [21]). Thus, in this setting (as in (1.10)) there is a natural analog of the Stein-Tomas extension theorem *for all exponents* $q > 2$. It is notable that, although the sharp critical and subcritical log-quasimode estimates for flat compact manifolds are weaker than the ones for Euclidean space, the results

that we obtain for compact manifolds all of whose sectional curvatures are negative agree with the Chen and Hassell bounds for such quasimodes in $\mathbb{H}^n$.

Using the bounds in Theorem 1.1 along with the fact that the unit-band spectral projection bounds in (1.4) are always sharp (see [29]) and a “Knapp example” that we present in §4 for connected compact flat manifolds, we are able to characterize all compact connected space forms in terms of the size of log-quasimodes as measured by *critical and subcritical* Lebesgue exponents, which is a natural affirmation of Bohr’s quantum correspondence principle (cf. [40, §1.5]).

To state this result we recall that if $f, g \geq 0$ then $f(\lambda) = \Theta(g(\lambda))$ if $\limsup_{\lambda \rightarrow \infty} \frac{f(\lambda)}{g(\lambda)} \in (0, \infty)$, i.e., $f(\lambda) = O(g(\lambda))$ and also $f(\lambda) = \Omega(g(\lambda))$ (the negation of $f(\lambda) = o(g(\lambda))$). Our other main result then is the following result which says that compact manifolds of constant sectional curvature are characterized by the growth L^q -norms of log-quasimodes, if $q \in (2, q_c]$.

Theorem 1.2. *Assume that (M, g) is a connected compact manifold of constant sectional curvature K and fix any exponent $q \in (2, q_c]$. Then, if $\mu(q)$ is as in (1.4),*

$$(1.12) \quad \sup\{\|\Phi_\lambda\|_{L^q(M)} : \Phi_\lambda \in V_{[\lambda, \lambda + (\log \lambda)^{-1}]}, \|\Phi_\lambda\|_{L^2(M)} = 1\} \\ = \begin{cases} \Theta(\lambda^{\mu(q)} (\log \lambda)^{-1/2}) & \iff K < 0 \\ \Theta(\lambda^{\mu(q)} (\log \lambda)^{-\mu(q)}) & \iff K = 0 \\ \Theta(\lambda^{\mu(q)}) & \iff K > 0. \end{cases}$$

Also, if $(\log \lambda)^{-1} \leq \delta(\lambda) \searrow 0$ as $\lambda \rightarrow \infty$ and $\lambda \rightarrow \lambda \delta(\lambda)$ is non-decreasing for $\lambda \geq 2$,

$$(1.13) \quad \sup\{\|\Phi_\lambda\|_{L^q(M)} : \Phi_\lambda \in V_{[\lambda, \lambda + \delta(\lambda)]}, \|\Phi_\lambda\|_{L^2(M)} = 1\} \\ = \begin{cases} \Theta(\lambda^{\mu(q)} (\delta(\lambda))^{1/2}) & \iff K < 0 \\ \Theta(\lambda^{\mu(q)} (\delta(\lambda))^{\mu(q)}) & \iff K = 0 \\ \Theta(\lambda^{\mu(q)}) & \iff K > 0. \end{cases}$$

Here we take the left sides of (1.12) and (1.13) to be zero if $V_{[\lambda, \lambda + \delta(\lambda)]} = \emptyset$.

As we mentioned before, Bérard [1] and Hassell and Tacy [16] obtained the sharp bounds $\|\chi_{[\lambda, \lambda + (\log \lambda)^{-1}]}\|_{2 \rightarrow q} = O(\lambda^{\mu(q)} (\log \lambda)^{-1/2})$ for all *supercritical exponents* $q > q_c$ whenever all the sectional curvatures of (M, g) are *nonpositive*. Thus, in Theorem 1.2 we must consider the range $q \in (2, q_c]$ of subcritical or critical exponents in order to distinguish between flat manifolds and ones with negative sectional curvatures. As we shall see in the proof of Theorem 1.2, the range $q \in (2, q_c]$ is very sensitive to different types of concentration near periodic geodesics. On the other hand, the types of quasimodes saturating the estimates of Bérard [1] and Hassell and Tacy [16] concentrate near points in a manner that is agnostic to the presence of nonpositive versus negative sectional curvatures.

Note that the universal bounds (1.4) as well as (1.9) and (1.10) along with (1.8) imply that the left side of (1.12) is $O(\lambda^{\mu(q)} (\log \lambda)^{-1/2})$ if $K < 0$, $O(\lambda^{\mu(q)} (\log \lambda)^{-\mu(q)})$ if $K = 0$ and $O(\lambda^{\mu(q)})$ if $K > 0$. So, to prove the first assertion in Theorem 1.2 we need to show that we can replace each of these “ O ” bounds by “ Ω ” lower bounds under the appropriate curvature assumption. Obtaining such results when $K > 0$ is relatively easy since any

compact connected space form of positive curvature is the quotient of a round sphere, which allows us to use results from [25]. Proving the Ω -lower bounds for (1.12) when $K < 0$ is also relatively easy since, as we mentioned before, it is straightforward to see that the $(\delta(\lambda))^{1/2}$ improvement in (1.10) cannot be strengthened. Establishing the Ω -lower bounds for flat manifolds is more difficult. We do this by using the fact that a flat connected compact manifold must be of the form $\mathbb{R}^n/\Gamma$ where the deck transformations, Γ , must be a Bieberbach subgroup of the Euclidean group, $E(n)$, of rigid motions in $\mathbb{R}^n$ (see, e.g., [2], [12] and [38]). This fact allows us to construct “Knapp examples” for flat manifolds using arguments from Brooks [9] and Sogge and Zelditch [33] that show that the bounds in (1.9) are always sharp for flat compact manifolds, which means that the left side of (1.12) cannot be $o(\lambda^{\mu(q)}(\log \lambda)^{-\mu(q)})$ when $K = 0$. Similar arguments will yield the second assertion (1.13) in the theorem.

We should also point out that, although $q_c = \frac{2(n+1)}{n-1}$ is the critical exponent for the universal bounds (1.11) for unit-band spectral projection operators, surprisingly, by Theorem 1.2, unless the sectional curvatures of (M, g) are positive it is not the “critical exponent” for projecting onto $(\log \lambda)^{-1}$ or $\delta(\lambda)$ bands as in (1.13), if, as is customary, a “critical exponent” is one for which the bounds for other exponents $q_c \neq q \in (2, \infty]$ follow from an interpolation argument using the trivial L^2 -bounds and dyadic Sobolev estimates for the ranges $(2, q_c)$ and $(q_c, \infty]$, respectively.

Our paper is organized as follows. In the next section we shall present the proof of Theorem 1.1 which requires global estimates that exploit the curvature assumptions as well as local harmonic analysis estimates. The global estimates come from ones in Bérard [1], Hassell and Tacy [16] and our earlier works Blair and Sogge [7], Blair, Huang and Sogge [3] and Sogge [30]. The local harmonic analysis estimates that we require will be proved in an appendix. The ones that we need for the critical exponent were obtained earlier; however, the ones needed for the subcritical bounds in (1.10) require modifications of the earlier arguments that also give a simplified approach for handling those used for $q = q_c$ both for manifolds of nonpositive and negative sectional curvatures. In the third section we prove Theorem 1.2. As we indicated the main step will be to construct Knapp examples for compact flat manifolds which involve quasimodes concentrating near a given periodic geodesic. In §4, we go over some new results that are a consequence of Theorem 1.1 and measure concentration properties of quasimodes, such as lower bounds for L^1 -norms originally studied by Sogge and Zelditch [32] and later by Hezari and Sogge [17] as a tool to analyze properties of nodal sets of eigenfunctions. In §4 we also state a couple of problems about possible generalizations of our results.

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2. Proving log-quasimode estimates.

We shall now focus on proving the estimates in Theorem 1.1 for the critical exponent q_c defined in (1.1). After we do this we shall give the proof of the subcritical estimates which is slightly easier.

In order to exploit calculations involving the half-wave operators we shall consider smoothed out spectral projection operators of the form

$$(2.1) \quad \rho_\lambda = \rho(T(\lambda - P)), \quad T = c_0 \log \lambda, \quad P = \sqrt{-\Delta_g},$$

where

$$(2.2) \quad \rho \in \mathcal{S}(\mathbb{R}), \quad \rho(0) = 1 \quad \text{and} \quad \text{supp } \hat{\rho} \subset \delta \cdot [1 - \delta_0, 1 + \delta_0] = [\delta - \delta_0\delta, \delta + \delta_0\delta],$$

with $0 < \delta, \delta_0 < 1/8$ (depending on M) to be specified later. We shall need that δ is smaller than the injectivity radius of M and it also must be chosen small enough so that the phase function $d_g(x, y)$ satisfies the Carleson-Sjölin condition when $d_g(x, y) \approx \delta$, with $d_g(\cdot, \cdot)$ denoting the Riemannian distance function. Similarly we shall choose δ_0 so that we can use the bilinear oscillatory integral estimates that arise. Lastly, in view of the earlier discussion of the Ehrenfest time, we shall take with $c_0 > 0$ in (2.1) to be eventually specified small enough depending on (M, g) .

By a simple orthogonality argument, we would obtain the bounds in (1.9) if we could show that, for T as in (2.2) and if the sectional curvatures of (M, g) are nonpositive, then we have for $\lambda \gg 1$ the uniform bounds

$$(2.3) \quad \|\rho_\lambda f\|_{L^{q_c}(M)} \leq C(\lambda(\log \lambda)^{-1})^{1/q_c} \|f\|_{L^2(M)},$$

since, as noted in (1.5), $\mu(q_c) = 1/q_c$. Similarly, we would obtain the bounds in (1.10) for $q = q_c$ if we could show that when the sectional curvatures of (M, g) are negative we have for $\lambda \gg 1$

$$(2.4) \quad \|\rho_\lambda f\|_{L^{q_c}(M)} \leq C\lambda^{1/q_c}(\log \lambda)^{-1/2} \|f\|_{L^2(M)}.$$

To prove these estimates we shall need to use local harmonic analysis involving the “local operators”

$$(2.5) \quad \sigma_\lambda = \rho(\lambda - P).$$

It will be convenient to localize a bit more using microlocal cutoffs. Specifically, let us write

$$(2.6) \quad I = \sum_{j=1}^N B_j(x, D)$$

where each $B_j \in S_{1,0}^0(M)$ is a zero order pseudo-differential operator with symbol supported in a small neighborhood of some $(x_j, \xi_j) \in S^*M$. The size of the support will be described shortly; however, we point out now that these operators will not depend on the spectral parameter $\lambda \gg 1$.

For present and future use, let us choose a Littlewood-Paley bump function satisfying

$$(2.7) \quad \beta \in C_0^\infty((1/2, 2)), \quad \beta(\tau) = 1 \quad \text{for } \tau \text{ near } 1, \quad \text{and} \quad \sum_{j=-\infty}^{\infty} \beta(2^{-j}\tau) = 1, \quad \tau > 0.$$

Then the dyadic operators

$$(2.8) \quad B = B_{j,\lambda} = B_j \circ \beta(P/\lambda)$$

are uniformly bounded on $L^p(M)$, i.e.,

$$(2.9) \quad \|B\|_{p \rightarrow p} = O(1) \quad \text{for } 1 \leq p \leq \infty.$$

We use these dyadic microlocal cutoffs to further localize σ_λ as follows

$$(2.10) \quad \tilde{\sigma}_\lambda = B \circ \sigma_\lambda,$$

where B is one of the N operators coming from (2.6) and (2.8). We also shall make use of the “semi-global” operators

$$(2.11) \quad \tilde{\rho}_\lambda = \tilde{\sigma}_\lambda \circ \rho_\lambda.$$

The σ_λ are smoothed out versions of the operators in (1.4). They satisfy the same operator norms, and the two sets of estimates are easily seen to be equivalent. Similarly, it is easy to use orthogonality to see that the following uniform bounds are valid for $q \in (2, \infty]$ and $\lambda \gg 1$

$$(2.12) \quad \|(I - \sigma_\lambda) \circ \rho_\lambda\|_{2 \rightarrow q} \leq CT^{-1} \lambda^{\mu(q)}, \text{ if } T \geq 1$$

$$\text{and } \|\sigma_\lambda - \beta(P/\lambda) \circ \sigma_\lambda\|_{2 \rightarrow q} = O(\lambda^{-N}), \forall N.$$

Consequently, by (2.6) and (2.12), in order to prove (2.3) and (2.4) it suffices to show that for $\lambda \gg 1$ we have

$$(2.13) \quad \|\tilde{\rho}_\lambda f\|_{L^{q_c}(M)} \leq C(\lambda(\log \lambda)^{-1})^{1/q_c} \|f\|_{L^2(M)}$$

under the assumption that all of the sectional curvatures of (M, g) are nonpositive, as well as

$$(2.14) \quad \|\tilde{\rho}_\lambda f\|_{L^{q_c}(M)} \leq C\lambda^{1/q_c} (\log \lambda)^{-1/2} \|f\|_{L^2(M)}$$

when the sectional curvatures of (M, g) are all negative.

Recall (see e.g., [25]) that on the standard round sphere S^n the improved bounds in Theorem 1.1 cannot hold for $q = q_c$. Indeed, the L^2 -normalized zonal functions Z_λ and the Gaussian beams G_λ (highest weight spherical harmonics) each have $L^{q_c}(S^n)$ norms which are comparable to λ^{1/q_c} if $\lambda = \sqrt{k(k+n-1)}$, $k \in \mathbb{N}$, is a nonzero eigenvalue of $\sqrt{-\Delta_{S^n}}$. Thus, in order to prove Theorem 1.1 we need to rule out the existence of log-quasimodes under our geometric assumptions that behave like the Z_λ or G_λ . The G_λ satisfy $\|G_\lambda\|_{L^\infty(S^n)} \approx \lambda^{\frac{n-1}{4}}$ and have negligible mass outside of a $\lambda^{-1/2+}$ tube about the equator. Here, and in what follows, $\lambda^{\sigma+}$ refers to quantities involving $\lambda^{\sigma+\varepsilon}$ with $\varepsilon > 0$ arbitrary, but with implicit constants of course depending on ε .

Motivated by this, and to be able to use the kernel estimates that Bérard [1] and Hassell and Tacy [16] used to prove their bounds for supercritical exponents and to also be able to utilize bilinear oscillatory integral estimates from harmonic analysis, it is natural to make a height decomposition using the semi-global operators $\tilde{\rho}_\lambda$ that essentially corresponds to the “height” of the aforementioned Gaussian beams, G_λ . We shall always assume, as we may, that the function f in (2.13) or (2.14) is L^2 -normalized:

$$(2.15) \quad \|f\|_{L^2(M)} = 1.$$

We then split our tasks (2.13) and (2.14) into estimating the L^{q_c} -norms of $\tilde{\rho}_\lambda f$ over the two regions

$$(2.16) \quad A_+ = \{x \in M : |\tilde{\rho}_\lambda f(x)| \geq \lambda^{\frac{n-1}{4} + \frac{1}{8}}\}$$

and

$$(2.17) \quad A_- = \{x \in M : |\tilde{\rho}_\lambda f(x)| < \lambda^{\frac{n-1}{4} + \frac{1}{8}}\}.$$

Of course $M = A_+ \cup A_-$. Also, we point out that there is nothing particularly special about the exponent $1/8$ in the above definitions. It could be replaced by any sufficiently small positive exponent in what follows. We choose the exponent $1/8$ to hopefully help the reader going through the numerology in some of the calculations that ensue.

Favorable $L^{q_c}(A_+)$ norms rule out quasimodes having large sup-norms as the zonal functions Z_λ do on S^n ; however, they do not rule out the existence of modes behaving like the G_λ since $\|G_\lambda\|_\infty \ll \lambda^{\frac{n-1}{4} + \frac{1}{8}}$ for $\lambda \gg 1$. On the other hand, favorable $L^{q_c}(A_-)$ bounds do rule out log-quasimodes tightly concentrating near periodic geodesics as the G_λ do on S^n . Also, the bounds for the region A_- are much harder than the ones for A_+ and require bilinear techniques from harmonic analysis and the use of more refined microlocal cutoffs. We also remark that when we turn to the improved $L^{q_c}(M)$ -norms for $q \in (2, q_c)$ in (1.10), we shall not have to make use of the splitting in (2.16)–(2.17) since modes potentially behaving like the Z_λ cannot saturate subcritical norms.

2.1. High floor estimates.

Let us now show that for any (M, g) all of whose sectional curvatures are nonpositive we have for T as in (2.1) with $c_0 > 0$ small enough

$$(2.18) \quad \|\tilde{\rho}_\lambda f\|_{L^{q_c}(A_+)} \leq C\lambda^{1/q_c} T^{-1/2}.$$

Although this estimate is not taken over all of M , we note that the $T^{-1/2} \approx (\log \lambda)^{-1/2}$ improvement matches up with that in (2.14) and is stronger than the $T^{-1/q_c} \approx (\log \lambda)^{-1/q_c}$ required for (2.13). We shall present the simple proof of (2.18) for the sake of completeness even though this bound was obtained in the earlier work [3], which in turn followed arguments in [7] and [30].

In order to prove (2.18) we need a global estimate for a kernel that will arise in a natural “ TT^* ” argument. Specifically, if $a \in C_0^\infty((-1, 1))$ equals one on $(-1/2, 1/2)$ we require the pointwise “global” kernel bound

$$(2.19) \quad G_\lambda(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (1 - a(t)) T^{-1} \hat{\Psi}(t/T) e^{it\lambda} (e^{-itP})(x, y) dt \\ = O(\lambda^{\frac{n-1}{2}} \exp(C_0 T)), \quad \Psi = |\rho|^2, \quad 1 \leq T \lesssim \log \lambda.$$

This “global” estimate is valid whenever the sectional curvatures of (M, g) are nonpositive (see e.g. [1], [16], [27], [30]). One proves the bound by standard arguments after lifting the calculation up to the universal cover and then using the Hadamard parametrix, just as was done by Bérard [1]. Since $\hat{\Psi}$ is compactly supported, the number of terms in the sum that arises grows exponentially if (M, g) has negative curvature. This accounts for the exponential factor in the right side of (2.19) (in the spirit of our earlier discussion of the Ehrenfest time). We also remark that such a bound cannot hold on S^n due to the fact that the half-wave operators there are essentially periodic.

To use the estimate (2.19) to prove (2.18) we first note that by (2.9), (2.12) and (2.15)

$$\|\tilde{\rho}_\lambda f\|_{L^{q_c}(A_+)} \leq \|B\rho_\lambda f\|_{L^{q_c}(A_+)} + C\lambda^{1/q_c} / \log \lambda.$$

Consequently, we would have (2.18) if we could show that

$$(2.20) \quad \|B\rho_\lambda f\|_{L^{q_c}(A_+)} \leq C\lambda^{1/q_c}(\log \lambda)^{-1/2} + \frac{1}{2}\|\tilde{\rho}_\lambda f\|_{L^{q_c}(A_+)}.$$

To prove this we shall adapt an argument of Bourgain [8] (see also [3] and [30]). So, we choose g satisfying

$$\|g\|_{L^{q'_c}(A_+)} = 1 \quad \text{and} \quad \|B\rho_\lambda f\|_{L^{q_c}(A_+)} = \int B\rho_\lambda f \cdot (\mathbf{1}_{A_+} \cdot g) dx.$$

Then since $\Psi(T(\lambda - P)) = \rho_\lambda \circ \rho_\lambda^*$ for Ψ as in (2.19), by (2.15) and the Schwarz inequality

$$(2.21) \quad \begin{aligned} \|B\rho_\lambda f\|_{L^{q_c}(A_+)}^2 &= \left(\int f \cdot (\rho_\lambda^* B^*)(\mathbf{1}_{A_+} \cdot g)(x) dx \right)^2 \\ &\leq \int |\rho_\lambda^* B^*(\mathbf{1}_{A_+} \cdot g)(x)|^2 dx \\ &= \int (B \circ \Psi(T(\lambda - P)) \circ B^*)(\mathbf{1}_{A_+} \cdot g)(x) \cdot \overline{\mathbf{1}_{A_+}(x)g(x)} dx \\ &= \int (B \circ L_\lambda \circ B^*)(\mathbf{1}_{A_+} \cdot g)(x) \cdot \overline{\mathbf{1}_{A_+}(x)g(x)} dx \\ &\quad + \int (B \circ G_\lambda \circ B^*)(\mathbf{1}_{A_+} \cdot g)(x) \cdot \overline{\mathbf{1}_{A_+}(x)g(x)} dx \\ &= I + II. \end{aligned}$$

Here G_λ is the operator whose kernel is in (2.19) and so

$$L_\lambda = (2\pi T)^{-1} \int a(t) \hat{\Psi}(t/T) e^{it\lambda} e^{-itP} dt.$$

Consequently, $L_\lambda h = T^{-1} \sum_j m(\lambda; \lambda_j) E_j h$, where E_j denotes the projection onto the eigenspace of P with eigenvalue λ_j and the spectral multiplier satisfies

$$m(\lambda; \lambda_j) = O((1 + |\lambda - \lambda_j|)^{-N}), \quad \forall N.$$

Consequently, by (1.4),

$$\|L_\lambda\|_{q'_c \rightarrow q_c} \lesssim T^{-1} \lambda^{2/q_c}.$$

Since $T = c_0 \log \lambda$, if we use Hölder's inequality and (2.9) we conclude that

$$(2.22) \quad \begin{aligned} |I| &\leq \|BL_\lambda B^*(\mathbf{1}_{A_+} \cdot g)\|_{q_c} \cdot \|\mathbf{1}_{A_+} \cdot g\|_{q'_c} \\ &\lesssim \|L_\lambda B^*(\mathbf{1}_{A_+} \cdot g)\|_{q_c} \cdot \|\mathbf{1}_{A_+} \cdot g\|_{q'_c} \\ &\lesssim \lambda^{2/q_c} (\log \lambda)^{-1} \|B^*(\mathbf{1}_{A_+} \cdot g)\|_{q'_c} \cdot \|\mathbf{1}_{A_+} \cdot g\|_{q'_c} \\ &\lesssim \lambda^{2/q_c} (\log \lambda)^{-1} \|g\|_{L^{q'_c}(A_+)}^2 \\ &= \lambda^{2/q_c} (\log \lambda)^{-1}. \end{aligned}$$

To estimate II , we choose $c_0 > 0$ small enough so that if C_0 is the constant in (2.19)

$$\exp(C_0 T) \leq \lambda^{1/8} \quad \text{if } T = c_0 \log \lambda \text{ and } \lambda \gg 1.$$

As a result

$$\|G_\lambda\|_{1 \rightarrow \infty} \lesssim \lambda^{\frac{n-1}{2} + \frac{1}{8}}.$$

Consequently, since the dyadic operators B are uniformly bounded on L^1 and L^∞ , if we repeat the preceding argument we obtain

$$|II| \leq C\lambda^{\frac{n-1}{2}+\frac{1}{8}} \|\mathbf{1}_{A_+} \cdot g\|_1^2 \leq C\lambda^{\frac{n-1}{2}+\frac{1}{8}} \|g\|_{L^{q'_c}(A_+)}^2 \cdot \|\mathbf{1}_{A_+}\|_{q_c}^2 = C\lambda^{\frac{n-1}{2}+\frac{1}{8}} \|\mathbf{1}_{A_+}\|_{q_c}^2.$$

After recalling the definition (2.16) we can estimate the last factor as follows

$$\|\mathbf{1}_{A_+}\|_{q_c}^2 \leq \left(\lambda^{\frac{n-1}{4}+\frac{1}{8}}\right)^{-2} \|\tilde{\rho}_\lambda f\|_{L^{q_c}(A_+)}^2,$$

which yields

$$|II| \lesssim \lambda^{-1/8} \|\tilde{\rho}_\lambda f\|_{L^{q_c}(A_+)}^2 \leq \left(\frac{1}{2} \|\tilde{\rho}_\lambda f\|_{L^{q_c}(A_+)}\right)^2,$$

assuming, as we may, that λ is large enough.

If we combine this bound with the earlier one (2.22) for I , we conclude that (2.20) is valid, which concludes the proof of our estimates (2.18) involving the large-height region A_+ . $\square$

2.2. High ceiling estimates.

To finish the proof of the estimates in Theorem 1.1 for $q = q_c$, in view of (2.18), it suffices to prove the following

Proposition 2.1. *Suppose that all of the sectional curvatures of (M, g) are nonpositive. Then for $\lambda \gg 1$ and T as in (2.1) we have*

$$(2.23) \quad \|\tilde{\rho}_\lambda f\|_{L^{q_c}(A_-)} \leq C(\lambda T^{-1})^{1/q_c},$$

and if all the sectional curvatures are negative

$$(2.24) \quad \|\tilde{\rho}_\lambda f\|_{L^{q_c}(A_-)} \leq C\lambda^{1/q_c} T^{-1/2}.$$

To prove this proposition we need to borrow and adapt results from the bilinear harmonic analysis in [23] and [35].

We shall utilize a microlocal decomposition which we shall now describe. We first recall that the symbol $B(x, \xi)$ of B in (2.8) is supported in a small conic neighborhood of some $(x_0, \xi_0) \in S^*M$. We may assume that its symbol has small enough support so that we may work in a coordinate chart Ω and that $x_0 = 0$, $\xi_0 = (0, \dots, 0, 1)$ and $g_{jk}(0) = \delta_k^j$ in the local coordinates. So, we shall assume that $B(x, \xi) = 0$ when x is outside a small relatively compact neighborhood of the origin or ξ is outside of a small conic neighborhood of $(0, \dots, 0, 1)$. These reductions and those that follow will contribute to the number of terms in (2.6); however, it will be clear that the N there will be independent of $\lambda \gg 1$. Similarly, the positive numbers δ and δ_0 in (2.2) may depend on the summand in (2.6), but, at the end we can just take each to be the minimum of what is required for each $j = 1, \dots, N$.

Next, let us define the microlocal cutoffs that we shall use. We fix a function $a \in C_0^\infty(\mathbb{R}^{2(n-1)})$ supported in $\{z : |z_k| \leq 1, 1 \leq k \leq 2(n-1)\}$ which satisfies

$$(2.25) \quad \sum_{j \in \mathbb{Z}^{2(n-1)}} a(z - j) \equiv 1.$$

We shall use this function to build our microlocal cutoffs. By the above, we shall focus on defining them for $(y, \eta) \in S^*\Omega$ with y near the origin and η in a small conic neighborhood

of $(0, \dots, 0, 1)$. We shall let

$$\Pi = \{y : y_n = 0\}$$

be the points in Ω whose last coordinate vanishes. Let $y' = (y_1, \dots, y_{n-1})$ and $\eta' = (\eta_1, \dots, \eta_{n-1})$ denote the first $n-1$ coordinates of y and η , respectively. For $y \in \Pi$ near 0 and η near $(0, \dots, 0, 1)$ we can just use the functions $a(\theta^{-1}(y', \eta') - j)$, $j \in \mathbb{Z}^{2(n-1)}$ to obtain cutoffs of scale θ . We will always have $\theta \in [\lambda^{-1/8}, 1]$.

We can then extend the definition to a neighborhood of $(0, (0, \dots, 0, 1))$ by setting for $(x, \xi) \in S^*\Omega$ in this neighborhood

$$(2.26) \quad a_j^\theta(x, \xi) = a(\theta^{-1}(y', \eta') - j) \quad \text{if } \Phi_s(x, \xi) = (y', 0, \eta', \eta_n) \text{ with } s = d_g(x, \Pi).$$

Here Φ_s denotes geodesic flow in $S^*\Omega$. Thus, $a_j^\theta(x, \xi)$ is constant on all geodesics $(x(s), \xi(s)) \in S^*\Omega$ with $x(0) \in \Pi$ near 0 and $\xi(0)$ near $(0, \dots, 0, 1)$. As a result,

$$(2.27) \quad a_j^\theta(\Phi_s(x, \xi)) = a_j^\theta(x, \xi)$$

for s near 0 and $(x, \xi) \in S^*\Omega$ near $(0, (0, \dots, 0, 1))$.

We then extend the definition of the cutoffs to a conic neighborhood of $(0, (0, \dots, 0, 1))$ in $T^*\Omega \setminus 0$ by setting

$$(2.28) \quad a_j^\theta(x, \xi) = a_j^\theta(x, \xi/p(x, \xi)).$$

Notice that if $(y'_\nu, \eta'_\nu) = \theta j = \nu$ and γ_ν is the geodesic in $S^*\Omega$ passing through $(y'_\nu, 0, \eta_\nu) \in S^*\Omega$ with $\eta_\nu \in S^*_{(y'_\nu, 0)}\Omega$ having η'_ν as its first $(n-1)$ coordinates then

$$(2.29) \quad a_j^\theta(x, \xi) = 0 \quad \text{if } \text{dist}((x, \xi), \gamma_\nu) \geq C_0\theta, \quad \nu = \theta j,$$

for some fixed constant $C_0 > 0$. Also, a_j^θ satisfies the estimates

$$(2.30) \quad |\partial_x^\sigma \partial_\xi^\gamma a_j^\theta(x, \xi)| \lesssim \theta^{-|\sigma| - |\gamma|}, \quad (x, \xi) \in S^*\Omega$$

related to this support property.

Finally, if $\psi \in C_0^\infty(\Omega)$ equals one in a neighborhood of the x -support of $B(x, \xi)$, and if $\tilde{\beta} \in C_0^\infty((0, \infty))$ equals one in a neighborhood of the support of the Littlewood-Paley bump function in (2.7) we define

$$(2.31) \quad A_\nu^\theta(x, \xi) = \psi(x) a_j^\theta(x, \xi) \tilde{\beta}(p(x, \xi)/\lambda), \quad \nu = \theta j \in \theta \cdot \mathbb{Z}^{2(n-1)}.$$

It then follows that the pseudo-differential operators $A_\nu^\theta(x, D)$ with these symbols belong to a bounded subset of $S_{7/8, 1/8}^0(M)$, due to our assumption that $\theta \in [\lambda^{-1/8}, 1]$. We have constructed these operators so that for small enough $\delta > 0$ we have

$$(2.32) \quad A_\nu^\theta(x, \xi) = A_\nu^\theta(\Phi_t(x, \xi)) \quad \text{on } \text{supp } B(x, \xi) \text{ if } |t| \leq 2\delta.$$

We shall need a few simple but very useful facts about these operators:

Lemma 2.2. *Let $\theta_0 = \lambda^{-1/8}$. Then*

$$(2.33) \quad \|A_\nu^{\theta_0} h\|_{\ell_\nu^q L^q(M)} \lesssim \|h\|_{L^q(M)}, \quad 2 \leq q \leq \infty,$$

$$(2.34) \quad \left\| \sum_{\nu'} (A_{\nu'}^{\theta_0})^* H(\nu', \cdot) \right\|_{L^p(M)} \lesssim \|H\|_{\ell_{\nu'}^p L^p(M)}, \quad 1 \leq p \leq 2.$$

Also, if $\delta > 0$ in (2.2) is small enough and $\mu(q)$ is as in (1.4)

$$(2.35) \quad \|B\sigma_\lambda A_\nu^{\theta_0} - BA_\nu^{\theta_0} \sigma_\lambda\|_{2 \rightarrow q} = O(\lambda^{\mu(q) - \frac{1}{4}}), \quad q \in (2, q_c].$$

Proof. To prove (2.33) we note that, by interpolation, it suffices to prove the inequality for $q = 2$ and $q = \infty$. The estimate for $q = 2$ just follows from the fact that the $S_{7/8, 1/8}^0$ operators $\{A_\nu^{\theta_0}\}$ are almost orthogonal due to (2.25). The estimate for $q = \infty$ follows from the fact that the kernels satisfy

$$(2.36) \quad \sup_x \int |A_\nu^{\theta_0}(x, y)| dy \leq C.$$

To see (2.36), we note that in addition to (2.30) we have that $\partial_r^k a(x, r\omega) = \lambda^{-k}$, if $\omega \in S^{n-1}$, by (2.31) since $a_\nu^\theta(x, \xi)$ is homogeneous of degree zero in ξ . So, if, as we shall shortly do, we work in local Fermi normal coordinates so that the projection of γ_ν onto M is the n th coordinate axis, using these estimates for radial derivatives, (2.30) and a simple integration by parts argument yields that

$$A_\nu^{\theta_0}(x, y) = O(\lambda^{\frac{7(n-1)}{8} + 1} (1 + \lambda^{7/8} |(x' - y')|)^{-N} (1 + \lambda |x_n - y_n|)^{-N}),$$

for all N which of course yields (2.36).

Since (2.34) follows via duality from (2.33) we are just left with proving (2.35). To do this we recall that by (2.8) the symbol $B(x, \xi) = B_\lambda(x, \xi) \in S_{1,0}^0$ vanishes when $|\xi|$ is not comparable to λ . In particular, it vanishes if $|\xi|$ is larger than a fixed multiple of λ , and it belongs to a bounded subset of $S_{1,0}^0$. Furthermore, if $a_\nu^{\theta_0}(x, \xi)$ is the principal symbol of our zero-order dyadic microlocal operators, we recall that by (2.32) we have that for $\delta > 0$ small enough

$$(2.37) \quad a_\nu^{\theta_0}(x, \xi) = a_\nu^{\theta_0}(\Phi_t(x, \xi)) \quad \text{on supp } B_\lambda \text{ if } |t| \leq 2\delta,$$

where $\Phi_t : T^*M \setminus 0 \rightarrow T^*M \setminus 0$ denotes geodesic flow in the cotangent bundle.

By Sobolev estimates for M , in order to prove (2.35), it suffices to show that for $q \in (2, q_c]$

$$(2.38) \quad \|(\sqrt{I + P^2})^{n(\frac{1}{2} - \frac{1}{q})} [B_\lambda \sigma_\lambda A_\nu^{\theta_0} - B_\lambda A_\nu^{\theta_0} \sigma_\lambda]\|_{2 \rightarrow 2} = O(\lambda^{\mu(q) - \frac{1}{4}}).$$

To prove this we recall that by (2.2) and (2.5)

$$\sigma_\lambda = (2\pi)^{-1} \int_{-2\delta}^{2\delta} \hat{\rho}(t) e^{it\lambda} e^{-itP} dt.$$

Therefore by Minkowski's integral inequality, we would have (2.38) if

$$(2.39) \quad \sup_{|t| \leq 2\delta} \|(\sqrt{I + P^2})^{n(\frac{1}{2} - \frac{1}{q})} [B_\lambda e^{-itP} A_\nu^{\theta_0} - B_\lambda A_\nu^{\theta_0} e^{-itP}]\|_{2 \rightarrow 2} = O(\lambda^{\mu(q) - \frac{1}{4}}).$$

Next, to be able to use Egorov's theorem, we write

$$[B_\lambda e^{-itP} A_\nu^{\theta_0} - B_\lambda A_\nu^{\theta_0} e^{-itP}] = B_\lambda [(e^{-itP} A_\nu^{\theta_0} e^{itP}) - B_\lambda A_\nu^{\theta_0}] \circ e^{-itP}.$$

Since e^{-itP} also has L^2 -operator norm one, we would obtain (2.39) from

$$(2.40) \quad \sup_{|t| \leq 2\delta} \|(\sqrt{I + P^2})^{n(\frac{1}{2} - \frac{1}{q})} B_\lambda [(e^{-itP} A_\nu^{\theta_0} e^{itP}) - A_\nu^{\theta_0}]\|_{2 \rightarrow 2} = O(\lambda^{\mu(q) - \frac{1}{4}}).$$

By Egorov's theorem (see e.g. Taylor [36, §VIII.1])

$$A_{\nu,t}^{\theta_0}(x,D) = e^{-itP} A_{\nu}^{\theta_0} e^{itP}$$

is a one-parameter family of zero-order pseudo-differential operators, depending on the parameter t , whose principal symbol is $a_{\nu}^{\theta_0}(\Phi_{-t}(x,\xi))$. By (2.37) and the composition calculus of pseudo-differential operators the principal symbol of $B_{\lambda}A_{\nu,t}^{\theta_0}$ and $B_{\lambda}A_{\nu}^{\theta_0}$ both equal $B_{\lambda}(x,\xi)a_{\nu}^{\theta_0}(x,\xi)$ if $|t| \leq 2\delta$. If $\theta = 1$ then $A_{\nu}^{\theta} \in S_{1,0}^0$, and, so, in this case we would have that $B_{\lambda}(e^{-itP}A_{\nu}^{\theta}e^{itP}) - B_{\lambda}A_{\nu}^{\theta}$ would be a pseudo-differential operator of order -1 with symbol vanishing for $|\xi|$ larger than a fixed multiple of λ (see e.g., [27, Theorem 4.3.6]). Since we are assuming that $\theta_0 = \lambda^{-1/8}$, by the way they were constructed, the symbols $A_{\nu}^{\theta_0}$ belong to a bounded subset of $S_{7/8,1/8}^0$. So, by [36, p. 147], for $|t| \leq 2\delta$, $B_{\lambda}(e^{-itP}A_{\nu}^{\theta_0}e^{itP}) - B_{\lambda}A_{\nu}^{\theta_0}$ belong to a bounded subset of $S_{7/8,1/8}^{-3/4}$ with symbols vanishing for $|\xi|$ larger than a fixed multiple of λ due to the fact that the symbol $B_{\lambda}(x,\xi)$ has this property (see e.g., [36, p. 46]).

We also need to take into account the other operator inside the norm in (2.40). Since $(\sqrt{I+P^2})^{n(\frac{1}{2}-\frac{1}{q})}$ is a standard pseudo-differential operator of order $n(\frac{1}{2}-\frac{1}{q})$ the operators in the left of (2.40) belong to a bounded subset of $S_{7/8,1/8}^{n(\frac{1}{2}-\frac{1}{q})-\frac{3}{4}}(M)$ with symbols vanishing for $|\xi|$ larger than a fixed multiple of λ . Consequently, the left side of (2.40) is $O(\lambda^{n(\frac{1}{2}-\frac{1}{q})-\frac{3}{4}})$. Since $\mu(q) = \frac{n-1}{2}(\frac{1}{2}-\frac{1}{q})$ for $q \in (2, q_c]$, a simple calculation shows that $n(\frac{1}{2}-\frac{1}{q}) - \frac{3}{4} \leq \mu(q) - \frac{1}{4}$ for such q , which yields (2.35) and completes the proof of the lemma. $\square$

Next we note that by (2.25), (2.26) and (2.31), we have that, as operators between any $L^p(M) \rightarrow L^q(M)$ spaces, $1 \leq p, q \leq \infty$, for $\theta \geq \lambda^{-1/8}$

$$(2.41) \quad \tilde{\sigma}_{\lambda} = \sum_{\nu} \tilde{\sigma}_{\lambda} A_{\nu}^{\theta} + O(\lambda^{-N}), \forall N.$$

This just follows from the fact that $R(x,D) = I - \sum_{\nu} A_{\nu}^{\theta} \in S_{7/8,1/8}^0$ has symbol supported outside of a neighborhood of $B(x,\xi)$, if, as we may, we assume that the latter is small.

In view of (2.41) we have for $\theta_0 = \lambda^{-1/8}$

$$(2.42) \quad (\tilde{\sigma}_{\lambda} h)^2 = \sum_{\nu, \nu'} (\tilde{\sigma}_{\lambda} A_{\nu}^{\theta_0} h) (\tilde{\sigma}_{\lambda} A_{\nu'}^{\theta_0} h) + O(\lambda^{-N} \|h\|_2^2).$$

If $\theta_0 = \lambda^{-1/8}$ then the $\nu = \theta_0 \cdot \mathbb{Z}^{2(n-1)}$ index a $\lambda^{-1/8}$ -separated set in $\mathbb{R}^{2(n-1)}$. We need to organize the pairs of indices ν, ν' in (2.42) as in many earlier works (see [23] and [35]). We consider dyadic cubes τ_{μ}^{θ} in $\mathbb{R}^{2(n-1)}$ of side length $\theta = 2^k \theta_0 = 2^k \lambda^{-1/8}$, $k = 0, 1, \dots$, with τ_{μ}^{θ} denoting translations of the cube $[0, \theta)^{2(n-1)}$ by $\mu = \theta \mathbb{Z}^{2(n-1)}$. Then two such dyadic cubes of side length θ are said to be *close* if they are not adjacent but have adjacent parents of side length 2θ , and, in that case, we write $\tau_{\mu}^{\theta} \sim \tau_{\mu'}^{\theta}$. Note that close cubes satisfy $\text{dist}(\tau_{\mu}^{\theta}, \tau_{\mu'}^{\theta}) \approx \theta$ and so each fixed cube has $O(1)$ cubes which are “close” to it. Moreover, as noted in [35], any distinct points $\nu, \nu' \in \mathbb{R}^{2(n-1)}$ must lie in a unique pair of close cubes in this Whitney decomposition of $\mathbb{R}^{2(n-1)}$. Consequently, there

must be a unique triple $(\theta = \theta_0 2^k, \mu, \mu')$ such that $(\nu, \nu') \in \tau_\mu^\theta \times \tau_{\mu'}^\theta$ and $\tau_\mu^\theta \sim \tau_{\mu'}^\theta$. We remark that by choosing B to have small support we need only consider $\theta = 2^k \theta_0 \ll 1$.

Taking these observations into account implies that that the bilinear sum in (2.42) can be organized as follows:

$$(2.43) \quad \sum_{\{k \in \mathbb{N}: k \geq 10 \text{ and } \theta = 2^k \theta_0 \ll 1\}} \sum_{\{(\mu, \mu'): \tau_\mu^\theta \sim \tau_{\mu'}^\theta\}} \sum_{\{(\nu, \nu') \in \tau_\mu^\theta \times \tau_{\mu'}^\theta\}} (\tilde{\sigma}_\lambda A_\nu^{\theta_0} h) \cdot (\tilde{\sigma}_\lambda A_{\nu'}^{\theta_0} h) \\ + \sum_{(\nu, \nu') \in \Xi_{\theta_0}} (\tilde{\sigma}_\lambda A_\nu^{\theta_0} h) \cdot (\tilde{\sigma}_\lambda A_{\nu'}^{\theta_0} h),$$

where Ξ_{θ_0} indexes the remaining pairs such that $|\nu - \nu'| \lesssim \theta_0 = \lambda^{-1/8}$, including the diagonal ones where $\nu = \nu'$.

As above, let $\mu(q) = \frac{n-1}{2}(\frac{1}{2} - \frac{1}{q})$, $q \in (2, q_c]$ be the exponent in the universal bounds (1.4). Then the key estimate that we shall use and follows from variable coefficient bilinear harmonic analysis arguments then is the following.

Proposition 2.3. *If $n \geq 2$ and $\theta_0 = \lambda^{-1/8}$, $\lambda \gg 1$ and if (2.15) is valid*

$$(2.44) \quad \|\tilde{\sigma}_\lambda h\|_{L^{q_c}(A_-)} \lesssim \left(\sum_\nu \|\tilde{\sigma}_\lambda A_\nu^{\theta_0} h\|_{L^{q_c}(M)}^{q_c} \right)^{1/q_c} + \lambda^{\frac{1}{q_c}-}, \quad \text{if } h = \rho_\lambda f,$$

assuming that the conic support of $B(x, \xi)$ in (2.8) is small and that δ and δ_0 in (2.2) are also small. Also, if $2 < q \leq \frac{2(n+2)}{n}$,

$$(2.45) \quad \|\tilde{\sigma}_\lambda h\|_{L^q(M)} \lesssim \left(\sum_\nu \|\tilde{\sigma}_\lambda A_\nu^{\theta_0} h\|_{L^q(M)}^q \right)^{1/q} + \lambda^{\mu(q)-} \|h\|_{L^2(M)}.$$

Here, $\lambda^{\mu-}$ means a factor involving an unspecified exponent smaller than μ . Note that λ -power gains are much better than the log λ -power gains in Theorem 1.1.

The first estimate, (2.44), occurred in earlier works ([4], [5] and [7]) and requires that the norms in the left be taken over A_- and that $h = \rho_\lambda$ so that $\tilde{\sigma}_\lambda h = \tilde{\rho}_\lambda f$. The other estimate involving smaller Lebesgue exponents is new and does not require these restrictions. For the sake of completeness, we shall present the proofs in an appendix. Our proofs which are based on a more direct application of Lee's [23] bilinear inequality and the above Whitney decomposition not only yield (2.45), but also give a simpler and self-contained proof of (2.44) which we shall present in this appendix.

We need to assemble two more ingredients which, along with (2.44) and Lemma 2.2, will easily allow us to prove Proposition 2.1. This will give us the bounds for $q = q_c$ in Theorem 1.1, and, as we shall see, we will be able to easily obtain the nontrivial subcritical estimates in this theorem using (2.45) and these methods.

The first of these ingredients involves a dyadic decomposition of the “global” operator G_λ defined by (2.19), which involves the Littlewood-Paley $\beta \in C_0^\infty((1/2, 2))$ described in (2.7). As in (2.19) we let $\Psi = |\rho|^2$. Its Fourier transform $\hat{\Psi}$ then is compactly supported. By (2.2), we may assume that $\hat{\Psi}(t) = 0$, $|t| > 1/2$. Also, let $\beta_0(s) = 1 - \sum_{j=1}^\infty \beta(s/2^j)$, $s > 0$ and $\beta_0(0) = 1$ so that $\beta_0(|s|)$ equals one near the origin and is in $C_0^\infty(\mathbb{R})$.

If we then let

$$(2.46) \quad L_{\lambda,T} = (2\pi T)^{-1} \int \beta_0(|t|) \hat{\Psi}(t/T) e^{it\lambda} e^{-itP} dt,$$

and

$$(2.47) \quad G_{\lambda,T,N} = (2\pi T)^{-1} \int \beta(|t|/N) \hat{\Psi}(t/T) e^{it\lambda} e^{-itP} dt, \quad N = 2^j, \quad j \in \mathbb{N}.$$

It then follows that $G_{\lambda,N} = 0$ if $N > T$, and, moreover,

$$(2.48) \quad \rho_\lambda = L_{\lambda,T} + \sum_{2 \leq N=2^j \leq T} G_{\lambda,T,N}.$$

By the universal bounds (1.4), we of course have

$$(2.49) \quad \|L_{\lambda,T}\|_{q' \rightarrow q} = O(T^{-1} \lambda^{2\mu(q)}), \quad q > 2,$$

as we essentially used in the proof of (2.18).

Then, in addition to Lemma 2.2 and Proposition 2.3, the next key ingredient needed to prove Theorem 1.1 is the following.

Proposition 2.4. *Let $\theta_0 = \lambda^{-1/8}$ and assume that for $T = c_0 \log \lambda$ as in (2.1) we have the following bounds for the microlocalized kernels*

$$(2.50) \quad |(A_{\nu}^{\theta_0} G_{\lambda,T,N} (A_{\nu'}^{\theta_0})^*)(x, y)| \leq CT^{-1} \lambda^{\frac{n-1}{2}} N^{1-\alpha}, \quad N = 2^j, \quad j \in \mathbb{N}.$$

We then have

$$(2.51) \quad \|A_{\nu}^{\theta_0} \rho_\lambda f\|_{\ell_v^{q_c} L^{q_c}(M)} \leq C \lambda^{\frac{1}{q_c}} \|f\|_{L^2(M)} \cdot \begin{cases} T^{-\frac{\alpha}{n+1}}, & \text{if } \alpha < \frac{n+1}{2} \\ T^{-\frac{1}{2}}, & \text{if } \alpha > \frac{n+1}{2}. \end{cases}$$

Also, if $q \in (2, q_c)$ we have

$$(2.52) \quad \|A_{\nu}^{\theta_0} \rho_\lambda f\|_{\ell_v^q L^q(M)} \leq C \lambda^{\mu(q)} T^{-1/2} \|f\|_{L^2(M)}, \quad \text{if } \alpha > \frac{q}{q-2}.$$

We shall momentarily postpone the simple proof of this Proposition and record one last result that we need to prove our main estimates.

Lemma 2.5. *Fix a compact manifold (M, g) all of whose sectional curvatures are non-positive. Then if $T = c_0 \log \lambda$ is as in (2.1) with $c_0 > 0$ small enough we have for $\lambda \gg 1$*

$$(2.53) \quad |(A_{\nu}^{\theta_0} G_{\lambda,T,N} (A_{\nu'}^{\theta_0})^*)(x, y)| \leq CT^{-1} \lambda^{\frac{n-1}{2}} N^{1-\frac{n-1}{2}}, \quad N \in \mathbb{N}.$$

Moreover, if all of the sectional curvatures of (M, g) are negative we have for such $c_0 > 0$

$$(2.54) \quad |(A_{\nu}^{\theta_0} G_{\lambda,T,N} (A_{\nu'}^{\theta_0})^*)(x, y)| \leq C_m T^{-1} \lambda^{\frac{n-1}{2}} N^{1-m}, \quad N \in \mathbb{N}, \quad \text{for each } m = 1, 2, \dots$$

At the end of the section we shall recall the proof of (2.53) and (2.54) which were obtained in the earlier works [3] and [7].

Having assembled all the necessary ingredients, let us now prove Proposition 2.1 which, as we noted before, would complete the proof of the estimates for $q = q_c$ in Theorem 1.1.

Proof of Proposition 2.1. We first note that by (2.11) and (2.44) we have

$$(2.55) \quad \|\tilde{\rho}_\lambda f\|_{L^{q_c}(A_-)} \lesssim \left(\sum_\nu \|\tilde{\sigma}_\lambda A_\nu^{\theta_0} \rho_\lambda f\|_{L^{q_c}(M)}^{q_c} \right)^{1/q_c} + \lambda^{\frac{1}{q_c}-} \|f\|_2,$$

since $\|\rho_\lambda\|_{2 \rightarrow 2} = O(1)$. Consequently, to prove the bounds in it suffices to show that the first term in the right is dominated by the right side of (2.23) when all of the sectional curvatures of M are nonpositive and by the right side of (2.24) if they all are negative. We recall that we are assuming as in (2.15) that f is L^2 -normalized.

We first note that, since $\mu(q_c) = 1/q_c$, by using (2.10), (2.35) and (2.33), we obtain

$$(2.56) \quad \begin{aligned} \sum_\nu \|\tilde{\sigma}_\lambda A_\nu^{\theta_0} \rho_\lambda f\|_{L^{q_c}(M)}^{q_c} &= \sum_\nu \|\tilde{\sigma}_\lambda A_\nu^{\theta_0} \rho_\lambda f\|_{q_c}^2 \cdot \|\tilde{\sigma}_\lambda A_\nu^{\theta_0} \rho_\lambda f\|_{q_c}^{q_c-2} \\ &\leq \sum_\nu \|\tilde{\sigma}_\lambda A_\nu^{\theta_0} \rho_\lambda f\|_{q_c}^2 \cdot \|BA_\nu^{\theta_0} \sigma_\lambda \rho_\lambda f\|_{q_c}^{q_c-2} \\ &\quad + \sum_\nu \|\tilde{\sigma}_\lambda A_\nu^{\theta_0} \rho_\lambda f\|_{q_c}^2 \cdot \|(BA_\nu^{\theta_0} \sigma_\lambda - B\sigma_\lambda A_\nu^{\theta_0}) \rho_\lambda f\|_{q_c}^{q_c-2} \\ &\lesssim \sum_\nu \|\tilde{\sigma}_\lambda A_\nu^{\theta_0} \rho_\lambda f\|_{q_c}^2 \cdot \|BA_\nu^{\theta_0} \sigma_\lambda \rho_\lambda f\|_{q_c}^{q_c-2} + \sum_\nu \lambda^{\frac{2}{q_c}} \|A_\nu^{\theta_0} \rho_\lambda f\|_2^2 \cdot \lambda^{(\frac{1}{q_c}-\frac{1}{4})(q_c-2)} \|\rho_\lambda f\|_2^{q_c-2} \\ &\lesssim \sum_\nu \|\tilde{\sigma}_\lambda A_\nu^{\theta_0} \rho_\lambda f\|_{q_c}^2 \cdot \|BA_\nu^{\theta_0} \sigma_\lambda \rho_\lambda f\|_{q_c}^{q_c-2} + \lambda^{1-\frac{1}{4}(q_c-2)} \\ &\leq C \left(\sum_\nu \|\tilde{\sigma}_\lambda A_\nu^{\theta_0} \rho_\lambda f\|_{q_c}^{q_c} \right)^{\frac{2}{q_c}} \left(\sum_\nu \|BA_\nu^{\theta_0} \sigma_\lambda \rho_\lambda f\|_{q_c}^{q_c} \right)^{\frac{q_c-2}{q_c}} + C \lambda^{1-\frac{1}{4}(q_c-2)}, \end{aligned}$$

using also Hölder's inequality in the last line.

By Young's inequality, we can bound the second to last term as follows

$$\begin{aligned} C \left(\sum_\nu \|\tilde{\sigma}_\lambda A_\nu^{\theta_0} \rho_\lambda f\|_{q_c}^{q_c} \right)^{\frac{2}{q_c}} \left(\sum_\nu \|BA_\nu^{\theta_0} \sigma_\lambda \rho_\lambda f\|_{q_c}^{q_c} \right)^{\frac{q_c-2}{q_c}} \\ = C \delta \left(\sum_\nu \|\tilde{\sigma}_\lambda A_\nu^{\theta_0} \rho_\lambda f\|_{q_c}^{q_c} \right)^{\frac{2}{q_c}} \cdot \delta^{-1} \left(\sum_\nu \|BA_\nu^{\theta_0} \sigma_\lambda \rho_\lambda f\|_{q_c}^{q_c} \right)^{\frac{q_c-2}{q_c}} \\ \leq C \left[\frac{2}{q_c} \delta^{\frac{q_c}{2}} \sum_\nu \|\tilde{\sigma}_\lambda A_\nu^{\theta_0} \rho_\lambda f\|_{q_c}^{q_c} + \frac{q_c-2}{q_c} \delta^{-\frac{q_c}{q_c-2}} \sum_\nu \|BA_\nu^{\theta_0} \sigma_\lambda \rho_\lambda f\|_{q_c}^{q_c} \right]. \end{aligned}$$

If $\delta > 0$ is small enough so that $C \frac{2}{q_c} \delta^{\frac{q_c}{2}}$ is smaller than $1/2$, we can absorb the contribution of the first term in the right side of the preceding inequality into the left side of (2.56) and conclude that

$$(2.57) \quad \begin{aligned} \sum_\nu \|\tilde{\sigma}_\lambda A_\nu^{\theta_0} \rho_\lambda f\|_{L^{q_c}(M)}^{q_c} &\lesssim \sum_\nu \|BA_\nu^{\theta_0} \sigma_\lambda \rho_\lambda f\|_{q_c}^{q_c} + \lambda^{1-\frac{1}{4}(q_c-2)} \\ &\lesssim \sum_\nu \|A_\nu^{\theta_0} \sigma_\lambda \rho_\lambda f\|_{q_c}^{q_c} + \lambda^{1-\frac{1}{4}(q_c-2)}, \end{aligned}$$

using (2.9) in the last line. Next, if we use (2.12) along with the L^{q_c} almost orthogonality bounds in (2.33) we can control the nontrivial term on the right as follows

$$(2.58) \quad \sum_\nu \|A_\nu^{\theta_0} \sigma_\lambda \rho_\lambda f\|_{q_c}^{q_c} \lesssim \sum_\nu \|A_\nu^{\theta_0} \rho_\lambda f\|_{q_c}^{q_c} + T^{-q_c} \lambda.$$

If we combine (2.55)–(2.58), we conclude that

$$(2.59) \quad \|\tilde{\rho}_\lambda f\|_{L^{q_c}(A_-)} \lesssim \|A_\nu^{\theta_0} \rho_\lambda f\|_{\ell_\nu^{q_c} L^{q_c}(M)} + O(\lambda^{\frac{1}{q_c}-} + T^{-1} \lambda^{\frac{1}{q_c}}).$$

If the sectional curvatures of M are all nonpositive we conclude from Lemma 2.5 that (2.50) is valid for $\alpha = \frac{n-1}{2}$, and so, by (2.51) in Proposition 2.4 we obtain in this case

$$(2.60) \quad \begin{aligned} \|\tilde{\rho}_\lambda f\|_{L^{q_c}(A_-)} &\lesssim \|A_\nu^{\theta_0} \rho_\lambda f\|_{\ell_\nu^{q_c} L^{q_c}(M)} + O(\lambda^{\frac{1}{q_c}-} + T^{-1} \lambda^{\frac{1}{q_c}}) \\ &\lesssim T^{-\frac{1}{n+1} \cdot \frac{n-1}{2}} \lambda^{\frac{1}{q_c}} = (T^{-1} \lambda)^{\frac{1}{q_c}} \approx (\lambda(\log \lambda)^{-1})^{\mu(q_c)}, \end{aligned}$$

which along with the earlier bound (2.18) yields (1.9) for $q = q_c = \frac{2(n+1)}{n-1}$.

If all of the sectional curvatures of M are negative, then Lemma 2.5 says that (2.50) is valid for any $\alpha \in \mathbb{N}$ and so we can use the more favorable case of (2.51) involving $T^{-1/2}$. As a result, if we repeat the arguments leading to (2.60) we conclude that if all the sectional curvatures of M are negative we have

$$(2.61) \quad \|\tilde{\rho}_\lambda f\|_{L^{q_c}(A_-)} \lesssim \lambda^{\mu(q_c)} (\log \lambda)^{-1/2},$$

which along with (2.18) yields (1.10) for the critical index $q = q_c$. $\square$

Let us also now handle the subcritical bounds in Theorem 1.1.

Proof of subcritical estimates in Theorem 1.1. Inequality (1.9) for $q \in (2, q_c)$ just follows via interpolation from the $q = q_c$ estimate we just obtained and the fact that the projection operators are bounded on $L^2(M)$ with norm one.

Thus, we only have to prove the subcritical bounds in (1.10) for $q \in (2, q_c)$, assuming as there that all of the sectional curvatures of M are negative. By interpolation with the trivial L^2 estimate just mentioned and the case $q = q_c$ that we just obtained, we see that it suffices to prove the estimates for $q \in (2, \frac{2(n+2)}{n}]$. We make this reduction in order to use (2.45), which as we noted before is an estimate over all of M (unlike (2.44)). As before, in order to prove (1.10) for q in the above range, it suffices to show that when f is L^2 -normalized as in (2.15) we have

$$(2.62) \quad \|\tilde{\rho}_\lambda f\|_{L^q(M)} \lesssim T^{-1/2} \lambda^{\mu(q)}, \quad q \in (2, \frac{2(n+2)}{n}],$$

for T as in (2.1) with $c_0 > 0$ sufficiently small depending on our manifold M of negative curvature.

If we use (2.45) in place of (2.44), and repeat the proof of (2.59) we obtain

$$(2.63) \quad \|\tilde{\rho}_\lambda f\|_{L^q(M)} \lesssim \|A_\nu^{\theta_0} \rho_\lambda f\|_{\ell_\nu^q L^q(M)} + O(\lambda^{\mu(q)-} + T^{-1} \lambda^{\mu(q)}), \quad q \in (2, \frac{2(n+2)}{n}].$$

Since, as we just exploited, (2.50) is valid for all $\alpha \in \mathbb{N}$ under our curvature assumption, by (2.52) the first term in the right hand side is $O(T^{-1/2} \lambda^{\mu(q)})$, which yields (2.62) and completes the proof. $\square$

Let us now prove Proposition 2.4.

Proof of Proposition 2.4. If $Uf(x, \nu) = A_\nu \rho_\lambda f(x)$, then (2.51) is equivalent to

$$(2.64) \quad \|UU^*\|_{\ell^{q'_c} L^{q'_c} \rightarrow \ell^{q_c} L^{q_c}} \lesssim \begin{cases} T^{-\frac{2\alpha}{n+1}} \lambda^{\frac{2}{q_c}}, & \text{if } \alpha < \frac{n+1}{2}, \\ T^{-1} \lambda^{\frac{2}{q_c}}, & \text{if } \alpha > \frac{n+1}{2}, \end{cases}$$

with

$$(2.65) \quad \begin{aligned} (UU^*F)(x, \nu) &= \sum_{\nu'} \left((A_{\nu'}^{\theta_0} \circ \rho^2(T(\lambda - P)) \circ (A_{\nu'}^{\theta_0})^*) F(\cdot, \nu') \right)(x) \\ &= \sum_{\nu'} \left((A_{\nu'}^{\theta_0} \circ L_{\lambda, T} \circ (A_{\nu'}^{\theta_0})^*) F(\cdot, \nu') \right)(x) \\ &\quad + \sum_{2 \leq N=2^j \leq T} \left[\sum_{\nu'} \left((A_{\nu'}^{\theta_0} \circ G_{\lambda, T, N} \circ (A_{\nu'}^{\theta_0})^*) F(\cdot, \nu') \right)(x) \right]. \end{aligned}$$

If we use (2.33) and (2.34) along with (2.49) we obtain

$$(2.66) \quad \begin{aligned} \left\| \sum_{\nu'} \left((A_{\nu'}^{\theta_0} \circ L_{\lambda, T} \circ (A_{\nu'}^{\theta_0})^*) F(\cdot, \nu') \right)(\cdot) \right\|_{\ell^{q_c} L^{q_c}} \\ \leq \left\| \sum_{\nu'} (L_{\lambda, T} \circ (A_{\nu'}^{\theta_0})^*) F(\cdot, \nu') \right\|_{L^{q_c}} \\ \leq T^{-1} \lambda^{\frac{2}{q_c}} \left\| \sum_{\nu'} (A_{\nu'}^{\theta_0})^* F(\cdot, \nu') \right\|_{L^{q'_c}} \\ \leq T^{-1} \lambda^{\frac{2}{q_c}} \|F\|_{\ell^{q'_c} L^{q'_c}} \end{aligned}$$

which is better than the bounds in (2.64) if $\alpha < \frac{n+1}{2}$ and agrees with them for $\alpha > \frac{n+1}{2}$.

To finish the proof of (2.64), we also need to estimate the N -summands in (2.65),

$$(2.67) \quad W_N F = \sum_{\nu'} \left((A_{\nu'}^{\theta_0} \circ G_{\lambda, T, N} \circ (A_{\nu'}^{\theta_0})^*) F(\cdot, \nu') \right)(x), \quad N = 2^j, j \in \mathbb{N}.$$

By (2.47) we clearly have

$$\|G_{\lambda, T, N}\|_{L^2(M) \rightarrow L^2(M)} = O(T^{-1}N).$$

So, if we use (2.33) and (2.34) for $q = 2$ the preceding argument yield for $2 \leq 2^j = N$

$$(2.68) \quad \|W_N\|_{\ell^2 L^2 \rightarrow \ell^2 L^2} = O(T^{-1}N).$$

We also obtain from (2.50)

$$(2.69) \quad \|W_N\|_{\ell^1 L^1 \rightarrow \ell^\infty L^\infty} = O(T^{-1} \lambda^{\frac{n-1}{2}} N^{1-\alpha}).$$

If we interpolate between these two estimates we obtain

$$(2.70) \quad \|W_N\|_{\ell^{q'_c} L^{q'_c} \rightarrow \ell^{q_c} L^{q_c}} = O(T^{-1} \lambda^{\frac{2}{q_c}} N^{1-\frac{2\alpha}{n+1}}).$$

Whence,

$$(2.71) \quad \sum_{2 \leq 2^j = N \leq T} \|W_N\|_{\ell^{q'_c} L^{q'_c} \rightarrow \ell^{q_c} L^{q_c}} \lesssim \begin{cases} T^{-\frac{2\alpha}{n+1}} \lambda^{\frac{2}{q_c}}, & \text{if } \alpha < \frac{n+1}{2}, \\ T^{-1} \lambda^{\frac{2}{q_c}}, & \text{if } \alpha > \frac{n+1}{2}. \end{cases}$$

If we combine (2.65), (2.66) and (2.71), we obtain (2.64). The same argument yields (2.52) which finishes the proof of Proposition 2.4. $\square$

Let us now recall the arguments that yield the bounds in Lemma 2.5. The arguments that we shall sketch are almost identical to ones in [6].

First, in view of (2.36), in order to prove (2.53), it suffices to show that

$$\begin{aligned}
(A_\nu^{\theta_0} G_{\lambda,T,N})(x,y) &= (2\pi T)^{-1} \int \beta(|t|/N) \hat{\Psi}(t/T) e^{it\lambda} (A_\nu^{\theta_0} \circ e^{-itP})(x,y) dt \\
&= (\pi T)^{-1} \int \beta(|t|/N) \hat{\Psi}(t/T) e^{it\lambda} (A_\nu^{\theta_0} \circ \cos t \sqrt{-\Delta_g})(x,y) dt \\
&\quad + (2\pi T)^{-1} A_\nu \circ \int \beta(|t|/N) \hat{\Psi}(t/T) e^{it\lambda} e^{itP}(x,y) dt \\
&= O(T^{-1} \lambda^{\frac{n-1}{2}} N^{1-\frac{n-1}{2}}), \quad N \geq 2.
\end{aligned}$$

Using (2.36) again shows via a simple integration by parts argument that the second to last term is $O(\lambda^{-m})$ for all $m \in \mathbb{N}$ since $\lambda \gg 1$ and P is a nonnegative operator. Thus, in order to establish (2.53), it suffices to show that

$$\begin{aligned}
(2.72) \quad T^{-1} \int \beta(|t|/N) \hat{\Psi}(t/T) e^{it\lambda} (A_\nu^{\theta_0} \circ \cos t \sqrt{-\Delta_g})(x,y) dt \\
= O(T^{-1} \lambda^{\frac{n-1}{2}} N^{1-\frac{n-1}{2}}), \quad N \geq 2.
\end{aligned}$$

To do this, as in Bérard [1] and many other works, we lift the calculation up to the universal cover of (M, g) using the formula (see e.g., [27, (3.6.4)])

$$(\cos t \sqrt{-\Delta_g})(x, y) = \sum_{\alpha \in \Gamma} (\cos t \sqrt{-\Delta_{\tilde{g}}}(\tilde{x}, \alpha(\tilde{y}))).$$

Here $(\mathbb{R}^n, \tilde{g})$ is the universal cover of (M, g) , with $\tilde{g}$ being the Riemannian metric on $\mathbb{R}^n$ obtained by pulling back the metric g via the covering map, also $\Gamma : \mathbb{R}^n \rightarrow \mathbb{R}^n$ are the deck transformations and we have chosen a Dirichlet domain $D \subset \mathbb{R}^n$, which we identify with $M \simeq \mathbb{R}^n/\Gamma$ and $\tilde{x} \in D$ is the lift of $x \in M$.

Thus, we can rewrite the left side of (2.72) as

$$(2.73) \quad T^{-1} \sum_{\alpha \in \Gamma} \int \beta(|t|/N) \hat{\Psi}(t/T) e^{it\lambda} (A_\nu^{\theta_0} \circ \cos t \sqrt{-\Delta_{\tilde{g}}}(\tilde{x}, \alpha(\tilde{y}))) dt.$$

By finite propagation speed of solutions to the wave equation the summand vanishes if $d_{\tilde{g}}(\tilde{x}, \alpha(\tilde{y})) > T$; however, in general there can be $\approx \exp(C_M T)$ nonzero terms due to our curvature assumptions. As exploited though in [6] one can use the Hadamard parametrix to see that the presence of microlocal operators $A_\nu^{\theta_0}$ means that there are only $O(N)$ nontrivial terms in (2.73).

To this end, we recall (see [1] and [27]) that the Hadamard parametrix tells us that we can write

$$(2.74) \quad (\cos t \sqrt{-\Delta_{\tilde{g}}}(\tilde{x}, \tilde{z})) = (2\pi)^{-n} w(\tilde{x}, \tilde{z}) \int_{\mathbb{R}^n} e^{id_{\tilde{g}}(\tilde{x}, \tilde{z})\xi_1} \cos t|\xi| d\xi + R(t; \tilde{x}, \tilde{z}),$$

where the remainder term, R , will not contribute significantly to the bounds and where the coefficient of the leading term satisfies

$$(2.75) \quad w(\tilde{x}, \tilde{z}) = O(1) \text{ if the principal curvatures of } M \text{ are nonpositive,}$$

$$\text{and } w(\tilde{x}, \tilde{z}) = O((1 + d_{\tilde{g}}(\tilde{x}, \tilde{z}))^{-m}) \forall m \text{ if the principal curvatures are all negative.}$$

Standard arguments as in [1], [6] and [27] show that if one replaces $(\cos t \sqrt{-\Delta_{\tilde{g}}}(\tilde{x}, \alpha(\tilde{y})))$ with $R(t; \tilde{x}, \alpha(\tilde{y}))$ the resulting expression is $O(\lambda^{\frac{n-1}{2}-})$ which is much better than that required for (2.72) in view of the fact that, by (2.1), we are assuming that $T = c_0 \log \lambda$ with $c_0 > 0$ allowed to be small. Also, by a simple stationary phase argument, if we replace $(\cos t \sqrt{-\Delta_{\tilde{g}}}(\tilde{x}, \alpha(\tilde{y})))$ with the first term in the right side of (2.74) with $\tilde{z} = \alpha(\tilde{y})$, then the resulting expression is always $O(T^{-1} \lambda^{\frac{n-1}{2}} (d_{\tilde{g}}(\tilde{x}, \alpha(\tilde{y}))^{-\frac{n-1}{2}}))$, and, since the amplitude in (2.72) is supported in the region where $|t| \approx N$ due to (2.7), by another simple integration by parts each of these terms is $O(\lambda^{-m})$ for all $m \in \mathbb{N}$ if $d_{\tilde{g}}(\tilde{x}, \alpha(\tilde{y})) \notin [C_0^{-1}, C_0]$ for some fixed C_0 since we are assuming that $N \geq 2$. Moreover, if we argue as in the proof of (3.8) in [6], we see that if $\alpha(D)$ is not within a fixed distance of the lift of the extension of the geodesic in M associated with the microlocal operator $A_{\nu}^{\theta_0}$ then the resulting kernel is also $O(\lambda^{-m}) \forall m$. So, arguing almost identically as in the proof of (3.8) in [6] shows that there are only $O(N)$ terms arising from the main term in the Hadamard parametrix, each of which, as we just mentioned, is $O(T^{-1} \lambda^{\frac{n-1}{2}} N^{\frac{n-1}{2}})$, while all the others, as well as the contribution of the remainder term R in (2.74) collectively contribute to a $O(\lambda^{\frac{n-1}{2}-})$ error term. This of course leads to the bounds in (2.53).

In this argument, we merely used the fact that the leading coefficient $w(\tilde{x}, \tilde{z})$ of the Hadamard parametrix is $O(1)$ if the principal curvatures of (M, g) are nonpositive. As noted in (2.75), though, it is $O((d_{\tilde{g}}(\tilde{x}, \tilde{z}))^{-m}) \forall m$ if the principal curvatures of (M, g) are all negative. Consequently, if one repeats the above argument each nontrivial term that arises must be $O(T^{-1} \lambda^{\frac{n-1}{2}} N^{-m}) \forall m$, which yields the other estimate, (2.54), in Lemma 2.5 and completes the sketch of its proof. $\square$

3. Characterizing compact space forms using log-quasimode estimates.

In this section we shall prove Theorem 1.2. We shall only prove the results for $\delta(\lambda) = (\log \lambda)^{-1}$, i.e., (1.12), since the proof of Theorem 1.1 shows that the estimates in (1.9) and (1.10) remain valid if $(\log \lambda)^{-1}$ is replaced by $\delta(\lambda)$ as in the statement of Theorem 1.2. Using this and simple modifications of the negative results to follow, one obtains the second assertion, (1.13), in Theorem 1.2.

Proving (1.12) is equivalent to proving the following three assertions for compact connected manifolds of constant sectional curvature K :

$$(3.1) \quad \limsup_{\lambda \rightarrow \infty} \lambda^{-\mu(q)} \left\| \chi_{[\lambda, \lambda + (\log \lambda)^{-1}]} \right\|_{2 \rightarrow q} \in (0, \infty) \text{ if and only if } K > 0,$$

$$(3.2) \quad \limsup_{\lambda \rightarrow \infty} (\lambda (\log \lambda)^{-1})^{-\mu(q)} \left\| \chi_{[\lambda, \lambda + (\log \lambda)^{-1}]} \right\|_{2 \rightarrow q} \in (0, \infty) \text{ if and only if } K = 0,$$

and

$$(3.3) \quad \limsup_{\lambda \rightarrow \infty} \lambda^{-\mu(q)} (\log \lambda)^{1/2} \left\| \chi_{[\lambda, \lambda + (\log \lambda)^{-1}]} \right\|_{2 \rightarrow q} \in (0, \infty) \text{ if and only if } K < 0.$$

Note that by (1.4), (1.9) and (1.10) each of these three “limsups” is finite. Thus, in order to prove (1.12) it suffices to prove that each one is nonzero.

Let us first prove that this is the case for (3.1). Without loss of generality we may assume that $K = 1$. It then is a classical theorem (see e.g. [14, 4.3 Proposition, Chapter 8] or [38]) that our compact manifold (M, g) , all of whose sectional curvatures equal one, is isometric to S^n/Γ where Γ is a subgroup of the group of isometries on the standard round S^n . Consequently, the eigenfunctions on our compact manifold M of constant curvature $K = 1$ are precisely the Γ -invariant eigenfunctions on S^n (i.e., Γ -invariant spherical harmonics), and the spectrum of our first order operator $\sqrt{-\Delta_g}$ on (M, g) must be contained in that of the round sphere.

Recall that the distinct eigenvalues of $\sqrt{-\Delta_{S^n}}$ are $(k(k+n-1))^{1/2}$, $k = 0, 1, 2, \dots$. Note that, for k is larger than a fixed constant depending on n , the gap between consecutive distinct eigenvalues of $\sqrt{-\Delta_{S^n}}$ and hence those of the operator $\sqrt{-\Delta_M}$ on our (M, g) of constant sectional curvature 1 is larger than one. So every interval $[\lambda, \lambda+1]$ with $\lambda \gg 1$ contains at most one of the distinct eigenvalues of $P = \sqrt{-\Delta_g}$. This means that if $0 \leq \delta(\lambda) \leq 1$ then $[\lambda, \lambda + \delta(\lambda)] \cap \text{Spectrum } \sqrt{-\Delta_g}$ is either empty for $\lambda \gg 1$ or is just a single point $\{\sqrt{k(k+n-1)}\}$ for some $k \in \mathbb{N}$. Thus, when the sectional curvatures all equal one we have for $q \in (2, q_c]$

$$\limsup_{\lambda \rightarrow \infty} \lambda^{-\mu(q)} \|\chi_{[\lambda, \lambda + (\log \lambda)^{-1}]} \|_{2 \rightarrow q} = \limsup_{\lambda \rightarrow \infty} \lambda^{-\mu(q)} \|\chi_{[\lambda, \lambda+1]} \|_{2 \rightarrow q}.$$

In [29] it was shown that the last “limsup” is positive on *any* (M, g) (meaning that the bounds in (1.4) are sharp), and so we conclude that the “limsup” in (3.1) must be nonzero, as desired.

It is also easy to see that this is the case for (3.3) which involves the assumption that (M, g) is of constant sectional curvature $K < 0$. Indeed, if we note that any interval $[\lambda, \lambda+1]$ can be covered by $\log \lambda + 1$ intervals of length $(\log \lambda)^{-1}$ for $\lambda \gg 1$ we can use the Cauchy-Schwarz inequality to see that for $q \in (2, q_c]$ we have

$$\|\chi_{[\lambda, \lambda+1]} \|_{2 \rightarrow q} \lesssim \sup_{\tau \in [\lambda, \lambda+1]} (\log \tau)^{1/2} \|\chi_{[\tau, \tau + (\log \tau)^{-1}]} \|_{2 \rightarrow q}.$$

Consequently, if, for such q ,

$$\limsup_{\lambda \rightarrow \infty} (\log \lambda)^{1/2} \lambda^{-\mu(q)} \|\chi_{[\lambda, \lambda + (\log \lambda)^{-1}]} \|_{2 \rightarrow q} = 0,$$

we would have $\limsup_{\lambda \rightarrow \infty} \lambda^{-\mu(q)} \|\chi_{[\lambda, \lambda+1]} \|_{2 \rightarrow q} = 0$, which, as we just mentioned is impossible on any compact manifold. So, the “limsup” in (3.3) must also be nonzero.

The proof of Theorem 1.2 would therefore be complete if we could show that whenever (M, g) is a connected compact flat manifold the “limsup” in (3.2) also must be nonzero, which is much more difficult than the two cases that we have just dealt with. To deal with the case of flat manifolds we need to construct appropriate “Knapp examples” as we shall do in the next subsection.

3.1. Characterizing flat compact manifolds.

By a classical theorem of Cartan and Hadamard, if (M, g) is a compact flat manifold, it must be of the form $\mathbb{R}^n/\Gamma$. A theorem from 1912 of Bierbach [2] (see e.g., Corollary 5.1 and Theorem 5.3 in Chapter 2 in [12] or [38]) says that the deck transformations, Γ ,

must be a Bieberbach subgroup of the group rigid motions, $E(n)$, of $\mathbb{R}^n$. This means that: i) Γ must be a discrete subgroup of $E(n)$, ii) Γ must be cocompact (i.e. $\mathbb{R}^n/\Gamma$ is compact), and iii) Γ must act freely on $\mathbb{R}^n$ (i.e., if $\alpha \in \Gamma$ and $\alpha(x) = x$ for some $x \in \mathbb{R}^n$, then α must be the identity). Subgroups of $E(n)$ satisfying these conditions are also called crystallographic subgroups. Bieberbach also showed that for each n there are only finitely many types (i.e. isomorphism classes) of Bieberbach subgroups of $E(n)$, which solved Hilbert's 18th problem. In 2-dimensions, there are only two¹—the quotients are 2-tori or Klein bottles, which are the two connected compact flat Riemannian manifolds in dimension two. In 3-dimensions it has been known since the 1930s there are ten, but the classification is incomplete in higher dimensions (see [12] and [38]).

We shall use these facts to construct our Knapp examples for our compact flat (M, g) . We recall that the rigid motions of $\mathbb{R}^n$ are of the form $\alpha(y) = my + j$, where $m \in O_n$ is an orthogonal matrix and $j \in \mathbb{R}^n$ is a translation. So, if $M \simeq \mathbb{R}^n/\Gamma$ and $\alpha \in \Gamma$, then α must be a particular element of $E(n)$ of this form, since $\Gamma \subset E(n)$ must be a Bieberbach subgroup associated with (M, g) .

To construct a Knapp example for our flat compact manifold M , we choose a periodic geodesic $\gamma_0 \subset M$. The Knapp example then will simply be the standard Knapp example for $\mathbb{R}^n$ projected to M via the covering map for $M = \mathbb{R}^n/\Gamma$. Recall that Knapp examples in $\mathbb{R}^n$ are quasimodes which are essentially supported in long thin tubes. To obtain ones for M we choose the central axis of this tube so that it projects to $\gamma_0 \subset M$ via the covering map. This leads to many windings around a thin tube about γ_0 in M , and, hence, potential concentration on a subset of M having much smaller volume than that of the Knapp tube in $\mathbb{R}^n$ from which it arises. This explains why, on compact flat manifolds, we only can have the bounds in (1.9) despite the fact that, by the Stein-Tomas theorem [37], the stronger analogues given by (1.10) with $q = q_c$ hold in $\mathbb{R}^n$. Also, to obtain concentration near γ_0 , as we shall see, we need to choose the frequencies of our quasimodes based on the length of γ_0 .

Let ℓ_0 be the length of the chosen periodic geodesic $\gamma_0 \in M$. We are not assuming that γ_0 is simply closed. It can cross itself. We can, however, pick a point $x_0 \in \gamma_0$, though, which is not a crossing point. So if $\gamma_0(t)$, $t \in [0, \ell_0)$ parameterizes the geodesic by arc length with $\gamma_0(0) = x_0$, then $\gamma_0(t) \neq x_0$ for $t \in [0, \ell_0)$. If $y \in \gamma_0$ is close to x_0 then y must also not be a crossing point. We may assume that $g_{jk}(x_0) = \delta_j^k$ in our local coordinate system about x_0 .

Next, let $p = \exp_{x_0} : \mathbb{R}^n \rightarrow M$. Then p is a covering map. If $D \subset \mathbb{R}^n$ is a Dirichlet domain containing the origin, then we identify D with M by setting $p(\tilde{x}) = x$ if $\tilde{x} \in D$. We then have $p(0) = x_0$ and the lift $\tilde{\gamma}$ of γ_0 is a straight line through the origin which we may assume is the x_1 -axis: $(t, 0, \dots, 0) = \tilde{\gamma}$. If $f \in C^\infty(M)$ and $\tilde{f}(\tilde{x}) = f(x)$ and $\Delta = \partial^2/\partial x_1^2 + \dots + \partial^2/\partial x_n^2$ is the standard Laplacian we have

$$(3.4) \quad \Delta_g f(x) = \Delta \tilde{f}(\tilde{x})$$

for $\tilde{x}$ in the interior of D .

¹When $n = 2$, up to isomorphisms, the two examples are the subgroups $\Gamma \subset E(2)$ whose generators are as follows $\{I + e_1, I + e_2\}$ and $\{(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}) + e_1, I + e_2\}$. Both have $Q = [0, 1] \times [0, 1]$ as a fundamental domain. The quotient of the first is the 2-torus and the second the Klein bottle.

Also, $\gamma_0 \subset D$ is a finite union of straight line segments some of which may cross (as in Klein bottles), but not at the origin since $p(0) = x_0$ and x_0 is not a crossing point of γ_0 . Also, since $\gamma_0(\ell_0) = \gamma_0(0) = x_0$ and its lift is $\tilde{\gamma}(t) = (t, 0, \dots, 0)$, there must be a unique $\alpha_{\gamma_0} \in \Gamma$ so that $\alpha_{\gamma_0}(0) = \tilde{\gamma}(\ell_0) = (\ell_0, 0, \dots, 0)$. Also, $\alpha_{\gamma_0}(\tilde{\gamma}) = \tilde{\gamma}$. It follows that $\{\alpha_{\gamma_0}^j\}_{j \in \mathbb{Z}}$, the stabilizers of $\tilde{\gamma}$, is a cyclic subgroup of Γ generated by α_{γ_0} . Here, by $\alpha_{\gamma_0}^j$, we mean for $j > 0$, $\alpha_{\gamma_0}^j = \alpha_{\gamma_0} \circ \dots \circ \alpha_{\gamma_0}$ (j times), $\alpha_{\gamma_0}^{-j}$ for $j > 0$ means the j -fold composition of the inverse of α_{γ_0} , and $\alpha_{\gamma_0}^j = \text{Identity}$ if $j = 0$. Thus, $\alpha_{\gamma_0}^j(0) = (j\ell_0, 0, \dots, 0)$.

Since $\alpha_{\gamma_0} \in \Gamma$, it follows that $\alpha_{\gamma_0}(y) = m_0 y + j_0$, for some $m_0 \in O_n$ and $j_0 \in \mathbb{R}^n$. Since $\alpha(0) = (\ell_0, 0, \dots, 0)$, we must have $j_0 = (\ell_0, \dots, 0)$. Also since $\alpha_{\ell_0}(\tilde{\gamma}) = \tilde{\gamma}$, m_0 must preserve the x_1 -axis. Since $m_0 \in O_n$, it follows that for some $\overline{m} \in O_{n-1}$,

$$m_0 = \begin{pmatrix} \pm 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \overline{m} & \\ 0 & & & \end{pmatrix}.$$

We cannot have “ -1 ” in the top left corner, since, in this case, we would have $\alpha_{\gamma_0}^2(0) = 0 \neq (2\ell_0, 0, \dots, 0)$. Consequently, we must have

$$(3.5) \quad \alpha_{\ell_0}(y) = m_0 y + (\ell_0, \dots, 0).$$

for some

$$(3.6) \quad m_0 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \overline{m} & \\ 0 & & & \end{pmatrix}, \quad \text{with } \overline{m} \in O_{n-1}.$$

Let us use these facts to build our Knapp example for $M \simeq D$ about our periodic geodesic γ_0 . The argument is somewhat like that in Brook [9] or Sogge [29, §5.1]. It also uses ideas from Sogge and Zelditch [33]. Our construction of quasimodes concentrating near γ_0 is a bit easier than that in [9] given the form (3.5) of the generator of the stabilizer group of our periodic geodesic γ_0 . Not surprisingly, we also are able to obtain much tighter concentration of our log-quasimodes since we are working in the flat case as opposed to the much more difficult case where $K < 0$ as in [9].

We shall use the following elementary result of Sogge and Zelditch [34, Proposition 1.3], which is valid on any compact manifold (M, g) .

Lemma 3.1. *Suppose that for $q \in (2, q_c]$ and $\delta \in (0, 1]$*

$$\|\chi_{[\tau, \tau+\delta]}\|_{L^2(M) \rightarrow L^q(M)} \leq C(\lambda, \delta), \quad \text{if } \tau \in [\lambda/2, 2\lambda].$$

Then for some uniform constant $C_0 = C_0(M)$ we have for $\lambda \gg 1$

$$\|f\|_q \leq C_0 C(\lambda, \delta) [\|f\|_2 + (\lambda\delta)^{-1} \|(\Delta_g + \lambda^2)f\|_2].$$

Thus, in order to prove that a flat manifold satisfies

$$\limsup_{\lambda \rightarrow \infty} \lambda^{-\mu(q)} (\log \lambda)^{\mu(q)} \|\chi_{[\lambda, \lambda + (\log \lambda)^{-1}]}\|_{2 \rightarrow q} \in (0, \infty), \quad \text{if } q \in (2, q_c],$$

which is the most difficult step in the proof of Theorem 1.2, in view of our positive results (1.9), it suffices to construct a sequence $\lambda_k \rightarrow \infty$ and associated “log-quasimodes” ψ_{λ_k} so that with $\delta = \delta(\lambda) = (\log \lambda)^{-1}$ above, we have

$$(3.7) \quad \|\psi_{\lambda_k}\|_{L^2(D)} + \lambda_k^{-1} \log \lambda_k \|(\Delta + \lambda^2)\psi_{\lambda_k}\|_{L^2(D)} \lesssim 1,$$

and, for uniform $c > 0$,

$$(3.8) \quad \|\psi_{\lambda_k}\|_{L^q(D)} \geq c \lambda_k^{\mu(q)} (\log \lambda_k)^{-\mu(q)}.$$

To do this by constructing a “Knapp” example for our flat compact manifold M , first fix $\eta \in \mathcal{S}(\mathbb{R})$ satisfying

$$(3.9) \quad \hat{\eta} \geq 0, \quad \hat{\eta}(0) = 1 \quad \text{and} \quad \text{supp } \hat{\eta} \subset [-c_0, c_0],$$

where $c_0 \in (0, 1)$ will be specified later. Fix also a function

$$(3.10) \quad 0 \leq a \in C_0^\infty((-1, 1)) \quad \text{with} \quad a(s) = 1, \quad |s| \leq 1/2,$$

and set

$$(3.11) \quad a_{\lambda, \delta}(\xi) = a(\lambda^{1/2} \delta^{-1/2} |e_1 - \xi/|\xi||) \beta(|\xi|/\lambda), \quad e_1 = (1, 0, \dots, 0),$$

and $\beta \in C^\infty((1/4, 4))$, satisfies $\beta(s) = 1, \quad s \in [1/2, 2]$.

Thus, $a_{\lambda, \delta}$ is supported in a dyadic region of a cone of aperture $\sim \lambda^{-1/2} \delta^{1/2}$ about the positive part of the first coordinate axis. This function satisfies the related bounds

$$(3.12) \quad \partial_{\xi_1}^j \partial_{\xi'}^\sigma a_{\lambda, \delta}(\xi) = O(\lambda^{-j} (\lambda^{-1/2} \delta^{-1/2})^{|\sigma|}), \quad \text{if } \xi' = (\xi_2, \dots, \xi_n).$$

We now define our “log-quasimodes” $\psi_\lambda(y)$, $y \in D$, as follows

$$(3.13) \quad \psi_\lambda(y) = \sum_{\alpha \in \Gamma} \lambda^{-\frac{n-1}{4}} \delta^{-\frac{n-1}{4}} \int_{\mathbb{R}^n} e^{i\alpha(y) \cdot \xi} a_{\lambda, \delta}(\xi) \eta(T(\lambda - |\xi|)) d\xi, \quad T = \delta^{-1} = \log \lambda.$$

This is analogous to the traditional Knapp example for Euclidean space that showed that the Stein-Tomas restriction theorem [37] was sharp. The amplitude in (3.13) is essentially supported in a δ by $\lambda^{1/2} \delta^{1/2}$ plate through $(\lambda, 0, \dots, 0)$, where δ is the thickness and $\lambda^{1/2} \delta^{1/2}$ is the “vertical” cross section of the plate.

In order to achieve the lower bounds in (3.8), we shall need to assume that the frequencies of the quasimode are of the form

$$(3.14) \quad \lambda = \lambda_k = 2\pi k / \ell_0, \quad \text{some } 1 \ll k \in \mathbb{N},$$

with, as above, ℓ_0 denoting the length of our periodic geodesic on which the Knapp modes in (3.13) will concentrate. As we shall see, this choice of frequencies ensures that there is minimal cancellation near the non-crossing point x_0 as the function in (3.13) wraps itself around and around γ_0 .

Let us first prove that we have (3.7). We shall use a simple argument that is based on ideas from [6]. To do so we shall need the following simple lemma about our Euclidean Knapp functions.

Lemma 3.2. *Let for $T = \delta^{-1} = \log \lambda$ as above*

$$(3.15) \quad K_\lambda(z) = \int e^{iz \cdot \xi} a_{\lambda, \delta}(\xi) \eta(T(\lambda - |\xi|)) d\xi.$$

Then if $c_0 > 0$ in (3.9) is fixed small enough and $c > 0$ is fixed we have for $\lambda \gg 1$ the uniform bounds

$$(3.16) \quad |K_\lambda(z)| + |\nabla K_\lambda(z)| \leq C_N \lambda^{-N} \quad \forall N, \quad \text{if } |z'| > c.$$

Also, assuming that $T \geq 1$,

$$(3.17) \quad |K_\lambda(z)| \leq C_N (|z| + \lambda)^{-N} \quad \forall N, \quad \text{if } |z| \geq 2T.$$

Proof. Note that

$$K_\lambda(z) = (2\pi)^{-1} \iint e^{iz \cdot \xi - it|\xi|} T^{-1} \hat{\eta}(t/T) a(\lambda^{1/2} \delta^{-1/2} |e_1 - \xi/|\xi||) \beta(|\xi|/\lambda) e^{i\lambda t} d\xi dt.$$

One gets (3.17) by integrating by parts in ξ if c_0 in (3.9) is small enough. One obtains (3.16) by integrating by parts in ξ' and using (3.12). $\square$

Proof of (3.7): Let us start by bounding the $L^2(D)$ norm of $\psi_\lambda(y)$. Note that if $\alpha, \tilde{\alpha} \in \Gamma$ and $\alpha \neq \tilde{\alpha}$, then $\alpha(D)$ and $\tilde{\alpha}(D)$ are disjoint since Γ acts freely on $\mathbb{R}^n$ and D is a fundamental domain. Also, each of these sets contains a ball of radius $r_0 > 0$ centered at the pre-image of 0 under the covering map, and is contained in a ball of radius r_0^{-1} with this center for some fixed $r_0 > 0$. Thus, by (3.17),

$$\begin{aligned} \psi_\lambda(y) = \\ (\lambda\delta)^{-\frac{n-1}{4}} \sum_{\{\alpha \in \Gamma: \text{dist}(\tilde{\gamma}, \alpha(D)) \leq 10, \text{dist}(0, \alpha(D)) \leq 10T\}} \int e^{i\alpha(y) \cdot \xi} a_{\lambda, \delta}(\xi) \eta(T(\lambda - |\xi|)) d\xi + O(\lambda^{-N}), \end{aligned}$$

and the number of terms in the sum is $O(T)$. Consequently, by the Cauchy-Schwarz inequality,

$$|\psi_\lambda(y)| \lesssim \lambda^{-\frac{n-1}{4}} \delta^{-\frac{n-1}{4}} T^{\frac{1}{2}} \left(\sum_{\alpha \in \Gamma} \left| \int e^{i\alpha(y) \cdot \xi} a_{\lambda, \delta}(\xi) \eta(T(\lambda - |\xi|)) d\xi \right|^2 \right)^{1/2} + O(\lambda^{-N}).$$

Thus, by Plancherel's theorem, modulo $O(\lambda^{-N})$

$$\begin{aligned} \int_D |\psi_\lambda(y)|^2 dy &\lesssim T \lambda^{-\frac{n-1}{2}} \delta^{-\frac{n-1}{2}} \sum_{\alpha \in \Gamma} \int_D \left| \int_{\mathbb{R}^n} e^{i\alpha(y) \cdot \xi} a_{\lambda, \delta}(\xi) \eta(T(\lambda - |\xi|)) d\xi \right|^2 dy \\ &= T \lambda^{-\frac{n-1}{2}} \delta^{-\frac{n-1}{2}} \sum_{\alpha \in \Gamma} \int_{\alpha(D)} \left| \int_{\mathbb{R}^n} e^{iy \cdot \xi} a_{\lambda, \delta}(\xi) \eta(T(\lambda - |\xi|)) d\xi \right|^2 dy \\ &= T \lambda^{-\frac{n-1}{2}} \delta^{-\frac{n-1}{2}} \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} e^{iy \cdot \xi} a_{\lambda, \delta}(\xi) \eta(T(\lambda - |\xi|)) d\xi \right|^2 dy \\ &= T \lambda^{-\frac{n-1}{2}} \delta^{-\frac{n-1}{2}} \int_{\mathbb{R}^n} |a_{\lambda, \delta}(\xi) \eta(T(\lambda - |\xi|))|^2 d\xi \\ &= O(1), \end{aligned}$$

using in the last step polar coordinates along with the fact that $\eta \in \mathcal{S}(\mathbb{R})$ and $a_{\lambda, \delta}$ is supported in a cone of aperture $\sim \lambda^{-1/2} \delta^{1/2}$.

This gives us the desired upper bound for $\|\psi_\lambda\|_{L^2(D)}$ in (3.7). We also need to see that

$$(3.18) \quad (\lambda\delta)^{-1} \|(\Delta + \lambda^2)\psi_\lambda\|_{L^2(D)} = O(1), \quad \delta = (\log \lambda)^{-1}.$$

To prove this, we shall use the fact that, as we mentioned before, by Bieberbach's theorem, each $\alpha \in \Gamma$ must be a rigid motion, i.e., $\alpha(y) = m_\alpha y + j_\alpha$ with $m_\alpha \in O_n$ and $j_\alpha \in \mathbb{R}^n$. Thus, since $|m_\alpha \xi| = |\xi|$ and the transpose of m_α is its inverse

$$\begin{aligned} \Delta_y \int e^{i\alpha(y) \cdot \xi} a_{\lambda, \delta}(\xi) \eta(T(\lambda - |\xi|)) d\xi &= \Delta_y \int e^{iy \cdot m_\alpha^{-1} \xi} e^{ij_\alpha \cdot \xi} a_{\lambda, \delta}(\xi) \eta(T(\lambda - |\xi|)) d\xi \\ &= \Delta_y \int e^{iy \cdot \xi} e^{ij_\alpha \cdot m_\alpha \xi} a_{\lambda, \delta}(m_\alpha \xi) \eta(T(\lambda - |\xi|)) d\xi \\ &= \int (-|\xi|^2) e^{iy \cdot \xi} e^{ij_\alpha \cdot m_\alpha \xi} a_{\lambda, \delta}(m_\alpha \xi) \eta(T(\lambda - |\xi|)) d\xi \\ &= \int (-|\xi|^2) \cdot e^{i\alpha(y) \cdot \xi} a_{\lambda, \delta}(\xi) \eta(T(\lambda - |\xi|)) d\xi. \end{aligned}$$

Consequently,

$$(\Delta + \lambda^2) \psi_\lambda(y) = \lambda^{-\frac{n-1}{4}} \delta^{-\frac{n-1}{4}} \sum_{\alpha \in \Gamma} \int e^{i\alpha(y) \cdot \xi} a_{\lambda, \delta}(\xi) (\lambda^2 - |\xi|^2) \eta(T(\lambda - |\xi|)) d\xi.$$

The proof of Lemma 3.2 shows that if we let

$$\tilde{K}_\lambda(z) = \int e^{iz \cdot \xi} a_{\lambda, \delta}(\xi) \cdot (\lambda^2 - |\xi|) \eta(T(\lambda - |\xi|)) d\xi,$$

then the analogs of (3.16) and (3.17) must be valid. So, if we argue as above, we find that, modulo $O(\lambda^{-N})$,

$$\begin{aligned} \int_D |(\Delta + \lambda^2) \psi_\lambda|^2 dy &\lesssim T \lambda^{-\frac{n-1}{2}} \delta^{-\frac{n-1}{2}} \int |a_{\lambda, \delta}(\xi) \cdot (\lambda^2 - |\xi|^2) \eta(T(\lambda - |\xi|))|^2 d\xi \\ &\lesssim T^{-1} \lambda^{-\frac{n-1}{2}} \delta^{-\frac{n-1}{2}} \lambda^2 \int |a_{\lambda, \delta}(\xi) \cdot T(\lambda - |\xi|) \eta(T(\lambda - |\xi|))|^2 d\xi \\ &= O(\lambda^2 T^{-2}), \end{aligned}$$

since $\tau \eta(\tau) \in \mathcal{S}$.

Thus, we have for $T = \delta^{-1} = \log \lambda$

$$\lambda^{-1} \log \lambda \|(\Delta + \lambda^2) \psi_\lambda\|_{L^2(D)} = O(1),$$

giving us (3.18), which is the remaining part of (3.7). $\square$

Proof of (3.8): To complete the proof of our results for flat compact manifolds, we must prove (3.8). Unlike the proof of (3.7), to prove this lower bound, we shall need to assume (3.14) to ensure that there is no cancellation in the nontrivial terms in the sum (3.13) defining ψ_{λ_k} .

To prove this lower bound, consider our unit speed geodesic $\tilde{\gamma}(t)$ in D , and the associated $\lambda^{-1/2} \delta^{-1/2}$ -tubes about the segment $\tilde{\gamma}(t)$, $|t| \leq \bar{c}$, where $\bar{c} \in (0, 1)$ will be fixed small enough in the ensuing calculation. In particular, it will be small enough so that there are no crossing points along $\gamma_0(t) \in M$ for $|t| \leq \bar{c}$.

As before, $\delta = (\log \lambda)^{-1}$, and so with $\lambda = \lambda_k$ as above, we let

$$(3.19) \quad \mathcal{T}_k = \{y \in D : \text{dist}(\tilde{\gamma}(t), y) \leq \lambda_k^{-1/2} (\log \lambda_k)^{1/2}, |t| \leq \bar{c}\}$$

denote a $\lambda_k^{-1/2} \delta_k^{-1/2}$ -tube about this segment. Note that $|\mathcal{T}_k| \approx \lambda_k^{-\frac{n-1}{2}} (\log \lambda_k)^{\frac{n-1}{2}}$, and so, by Hölder's inequality, if $q \in (2, q_c]$

$$\begin{aligned} \|\psi_{\lambda_k}\|_{L^2(\mathcal{T}_k)} &\lesssim (\lambda_k^{-\frac{n-1}{2}} (\log \lambda_k)^{\frac{n-1}{2}})^{(\frac{1}{2}-\frac{1}{q})} \|\psi_{\lambda_k}\|_{L^q(D)} \\ &= \lambda_k^{-\mu(q)} \delta_k^{-\mu(q)} \|\psi_{\lambda_k}\|_{L^q(D)}, \quad \delta_k = (\log \lambda_k)^{-1}, \end{aligned}$$

since $\mu(q) = \frac{n-1}{2}(\frac{1}{2} - \frac{1}{q})$ for $q \in (2, q_c]$. As a result, we would have the lower bound (3.8) and be done if we could show that for fixed $c_0 > 0$ as in (3.9) we have

$$(3.20) \quad \|\psi_{\lambda_k}\|_{L^2(\mathcal{T}_k)} \geq c_1$$

for some uniform $c_1 > 0$ when $\lambda_k \gg 1$.

To prove (3.20), by calculus, it suffices to show that we have the uniform bounds

$$(3.21) \quad |\psi_k(\tilde{\gamma}(t))| \geq c_1 \lambda_k^{\frac{n-1}{4}} \delta_k^{\frac{n-1}{4}}, \quad \text{some } c_1 > 0, \quad \text{if } |t| \leq \bar{c},$$

if $\bar{c} > 0$ as above is small enough, and also the upper bound

$$(3.22) \quad |\nabla_{y'} \psi_{\lambda_k}(y)| \leq C \lambda_k^{\frac{n-1}{4}} \delta_k^{\frac{n-1}{4}} \cdot (\lambda_k \delta_k)^{1/2}, \quad \delta_k = (\log \lambda_k)^{-1}, \quad y \in \mathcal{T}_k.$$

We can use (3.22) along with (3.20) to obtain (3.20) since $\tilde{\gamma}(t) = (t, 0, \dots, 0)$.

As we shall see, (3.22) follows from the proof of (3.18) and does not require that λ be as in (3.14). So, let us focus first on (3.21), which is the more difficult to prove.

To prove (3.21) we shall need to use the properties of the stabilizer group G_{γ_0} of our periodic geodesic of length ℓ_0 that we described before. It is a cyclic subgroup of Γ generated by $\alpha_{\gamma_0} \in \Gamma$ as in (3.6) and (3.5). Since γ_0 loops back smoothly through 0 with no other crossings there, as we mentioned before, $\alpha(0) \in \tilde{\gamma}$, the lift of γ_0 , if and only if $\alpha \in G_{\gamma_0}$.

From this we deduce that if $\alpha \notin G_{\gamma_0}$ we must have that $\text{dist}(\alpha(0), \tilde{\gamma}) \geq c$ for some uniform constant $c > 0$. This implies that if $\bar{c}$ in (3.19) is fixed small enough then for large enough λ_k we have for $c_1 = c/2$

$$(3.23) \quad \text{dist}(\alpha(\mathcal{T}_k), \tilde{\gamma}) \geq c_1, \quad \text{if } \alpha \notin G_{\gamma_0}.$$

Thus, by Lemma 3.2, if $\psi_{\lambda_k, \alpha}$ denotes the α -summand in the definition (3.13) of ψ_{λ_k} , i.e.,

$$(3.24) \quad \psi_{\lambda_k, \alpha}(y) = (\lambda_k \delta_k)^{-\frac{n-1}{4}} \int_{\mathbb{R}^n} e^{i\alpha(y) \cdot \xi} a_{\lambda_k, \delta_k}(\xi) \eta(T_k(\lambda_k - |\xi|)) d\xi, \quad T_k = \delta_k^{-1} = \log \lambda_k,$$

we must have

$$(3.25) \quad |\psi_{\lambda_k, \alpha}(y)| + |\nabla \psi_{\lambda_k, \alpha}(y)| \leq C_N (\alpha(y) + \lambda_k)^{-N},$$

if $y \in \mathcal{T}_k$ and $\alpha \notin G_{\gamma_0}$ or $\text{dist}(\alpha(y), 0) \geq 2T_k$.

Thus, if α_{γ_0} as in (3.5) is the generator of G_{γ_0} , we have for T_k and δ_k as above

$$(3.26) \quad \begin{aligned} \psi_{\lambda_k}(y) = & \sum_{\{j \in \mathbb{Z}: \text{dist}(\alpha_{\gamma_0}^j(D), 0) \leq 2T_k\}} \lambda_k^{-\frac{n-1}{4}} \delta_k^{-\frac{n-1}{4}} \int e^{i\alpha_{\gamma_0}^j(y) \cdot \xi} a_{\lambda_k, \delta_k}(\xi) \eta(T_k(\lambda_k - |\xi|)) d\xi \\ & + O(\lambda_k^{-N}) \quad \text{if } y \in \mathcal{T}_k. \end{aligned}$$

If $y = \tilde{\gamma}(t) = (t, 0, \dots, 0)$, $|t| \leq \bar{c}$, then by (3.6) and (3.5)

$$\alpha_{\gamma_0}^j(\tilde{\gamma}(t)) \cdot \xi = (t + j\ell_0)\xi_1, \quad |t| \leq \bar{c},$$

since $\alpha_{\gamma_0}^j(0) = (j\ell_0, 0, \dots, 0)$. Thus since $\beta(|\xi|/\lambda_k) = 1$ for $|\xi| \in (\lambda_k/2, 2\lambda_k)$ and $\eta \in \mathcal{S}$, by (3.11),

$$(3.27) \quad \psi_{\lambda_k}((\tilde{\gamma}(t))) = \sum_{\{j \in \mathbb{Z}: |j\ell_0| \leq 2T_k\}} (\lambda_k \delta_k)^{-\frac{n-1}{4}} \int e^{i(t+j\ell_0)\xi_1} a(\lambda_k^{1/2} \delta_k^{-1/2} |e_1 - |\xi|/|\xi||) \eta(T_k(\lambda_k - |\xi|)) d\xi + O(\lambda_k^{-N}), \quad |t| \leq \bar{c}.$$

Next, let us polar coordinates $\xi = r\omega$, $r > 0$, $\omega \in S^{n-1}$ so that $\xi/|\xi| = \omega$. We then write each summand as above as

$$\begin{aligned} & (\lambda_k \delta_k)^{-\frac{n-1}{4}} \int_{S^{n-1}} \int_0^\infty e^{i(t+j\ell_0)r\omega_1} a(\lambda_k^{1/2} \delta_k^{-1/2} |\omega - e_1|) \eta(T_k(\lambda_k - r)) r^{n-1} dr d\omega \\ &= (\lambda_k \delta_k)^{-\frac{n-1}{4}} \int_{S^{n-1}} \left(\int_{-\lambda_k}^\infty e^{i(t+j\ell_0)(\lambda_k+r)\omega_1} \eta(-T_k r) (\lambda_k + r)^{n-1} dr \right) a(\lambda_k^{1/2} \delta_k^{-1/2} |\omega - e_1|) d\omega \\ &= (\lambda_k \delta_k)^{-\frac{n-1}{4}} \int_{S^{n-1}} \left(\lambda_k^{n-1} \int_{-\infty}^\infty e^{i(t+j\ell_0)(\lambda_k-r)\omega_1} \eta(T_k r) dr \right) a(\lambda_k^{1/2} \delta_k^{-1/2} |\omega - e_1|) d\omega \\ &\quad + O(\lambda_k^{n-2} \lambda_k^{-\frac{3(n-1)}{4}} \delta_k^{\frac{n-1}{4}}), \end{aligned}$$

since $r^m \eta(r) \in \mathcal{S}(\mathbb{R})$, $0 \leq m \leq n-2$, and, by (3.10),

$$(3.28) \quad \int_{S^{n-1}} a(\lambda_k^{1/2} \delta_k^{-1/2} |\omega - e_1|) d\omega \approx \lambda_k^{-\frac{n-1}{2}} \delta_k^{\frac{n-1}{2}}.$$

Since there are $O(T_k) = O(\delta_k^{-1})$ terms in the sum in (3.27), we conclude that for $|t| \leq \bar{c}$

$$(3.29) \quad \psi_{\lambda_k}(\tilde{\gamma}(t)) = \lambda_k^{n-1} (\lambda_k \delta_k)^{-\frac{n-1}{4}} \sum_{|j\ell_0| \leq 2T_k} \int_{S^{n-1}} e^{i(t+\ell_0 j)\lambda_k \omega_1} T_k^{-1} \hat{\eta}(T_k^{-1}(t+j\ell_0)w_1) a(\lambda_k^{1/2} \delta_k^{-1/2} |\omega - e_1|) d\omega + O((\lambda_k/\delta_k)^{-1} \lambda_k^{\frac{n-1}{4}} \delta_k^{\frac{n-1}{4}}).$$

If the integrand here is nonzero, then by (3.9), (3.10) and (3.11), we must have that $\omega_1 - 1 = O(\lambda_k^{-1} \delta_k)$ and $t + j\ell_0 = O(c_0 T_k) = O(c_0 \delta_k^{-1})$, and so

$$(3.30) \quad \begin{aligned} e^{i(t+\ell_0 j)\lambda_k \omega_1} &= e^{it\lambda_k} e^{i\ell_0 j\lambda_k} + O(c_0) \\ &= e^{it\lambda_k} e^{2\pi i j k} + O(c_0) \\ &= e^{it\lambda_k} + O(c_0), \quad \text{if } |t| \leq \bar{c} < 1, \end{aligned}$$

due to our choice in (3.14) of the frequency of our log-quasimode. By (3.9) each $\hat{\eta}$ factor in the sum in (3.29) is nonnegative. So, since for $|t| \leq \bar{c}$, with $\bar{c}$ small enough, $|(t + j\ell_0)\omega_1| \approx |j|$ if $j \neq 0$ and $a(\lambda_k^{1/2} \delta_k^{-1/2} |\omega - e_1|) \neq 0$, we see by (3.9) and (3.28),

$$(3.31) \quad \sum_{|j\ell_0| \leq 2T_k} \int_{S^{n-1}} T_k^{-1} \hat{\eta}(T_k^{-1}(t+j\ell_0)w_1) a(\lambda_k^{1/2} \delta_k^{-1/2} |\omega - e_1|) d\omega \approx \lambda_k^{-\frac{n-1}{2}} \delta_k^{\frac{n-1}{2}}.$$

We obtain (3.21) by combining (3.29), (3.30) and (3.31) if c_0 and $\bar{\epsilon}$ are small enough and $\lambda_k \gg 1$.

To prove the remaining inequality (3.22), we use Lemma 3.2 and the above arguments to see that

$$(3.32) \quad \nabla_{y'} \psi_{\lambda_k}(y) = \sum_{\{j \in \mathbb{Z}: |j\ell_0| \leq 2T_k\}} \nabla_{y'} \psi_{\lambda_k, \alpha_{\gamma_0}^j}(y) + O(\lambda_k^{-N}), \quad \text{if } y \in \mathcal{T}_k,$$

where $\psi_{\lambda_k, \alpha}$ is as in (3.24). Recall that α_{γ_0} is as in (3.5). If $\bar{m} \in O_{n-1}$ is as in (3.6), then

$$\psi_{\lambda_k, \alpha_{\gamma_0}^j}(y) = \lambda_k^{-\frac{n-1}{4}} \delta_k^{-\frac{n-1}{4}} \int e^{i(y_1 + j\ell_0)\xi_1} e^{iy' \cdot (\bar{m})^j \xi'} a_{\lambda_k, \delta_k}(\xi) \eta(T_k((\lambda - |\xi|))) d\xi.$$

Since $|(\bar{m})^j \xi'| = |\xi'| = O(\lambda_k^{1/2} \delta_k^{1/2})$ on the support of the integral, we can argue as above to deduce that

$$\nabla_{y'} \psi_{\lambda_k, \alpha_{\gamma_0}^j}(y) = O(T_k^{-1} \lambda_k^{\frac{n-1}{4}} \delta_k^{\frac{n-1}{4}} (\lambda_k \delta_k)^{1/2}), \quad \text{if } y \in \mathcal{T}_k,$$

which yields (3.22) after recalling (3.32). $\square$

4. Some other problems related to the concentration of quasimodes.

Let us now see how we can use the estimates in Theorem 1.1 to make further progress on problems related to the concentration of eigenfunctions and quasimodes that were discussed before, for instance, in [17], [28] and [32].

In Sogge and Zelditch [32], lower bounds for the L^1 -norms of quasimodes,

$$(4.1) \quad \lambda^{-\frac{n-1}{4}} \|\Phi_\lambda\|_{L^2(M)} \lesssim \|\Phi_\lambda\|_{L^1(M)}, \quad \text{if } \text{Spec } \Phi_\lambda \subset [\lambda, \lambda + 1]$$

were obtained. These universal lower bounds are saturated by the Gaussian beam spherical harmonics (highest weight spherical harmonics) on S^n . The bounds in (4.1) were used in [17] and [32] to obtain progress on the problem of establishing lower bounds for the size of nodal sets of eigenfunctions, which was subsequently fully resolved by other methods by Logunov [24].

In [6] improvements were made to (4.1) under the assumption of nonpositive curvatures by including factors involving certain positive powers of $\log \lambda$ in the left. Let us now see how we can use (1.9) to obtain further improvements. If we use Hölder's inequality we see that if all of the sectional curvatures of (M, g) are nonpositive, we have for $q \in (2, q_c]$ and L^2 -normalized Φ_λ with spectrum in $[\lambda, \lambda + (\log \lambda)^{-1}]$

$$1 \leq \|\Phi_\lambda\|_1^{q-2} \|\Phi_\lambda\|_q^q \lesssim \|\Phi_\lambda\|_1^{q-2} (\lambda (\log \lambda)^{-1})^{\frac{n-1}{4}(q-2)}.$$

This of course yields the lower bound

$$(4.2) \quad \lambda^{-\frac{n-1}{4}} (\log \lambda)^{\frac{n-1}{4}} \|\Phi_\lambda\|_2 \lesssim \|\Phi_\lambda\|_1, \quad \text{if } \text{Spec } \Phi_\lambda \subset [\lambda, \lambda + (\log \lambda)^{-1}].$$

The Knapp example in §3.1 suggests that (4.2) is an optimal bound although the functions constructed there satisfied the weaker but related variant of the above spectral assumption that $\|\psi_\lambda\|_2 + (\lambda / \log \lambda) \|(\Delta_g + \lambda^2) \psi_\lambda\|_2 \approx 1$.

If one uses (1.10) with q close to 2, one can use Hölder's inequality as above to see that if all of the sectional curvatures of (M, g) are negative we have for every N that if

$\lambda \gg 1$

$$(4.3) \quad \lambda^{-\frac{n-1}{4}} (\log \lambda)^N \|\Phi_\lambda\|_2 \lesssim \|\Phi_\lambda\|_1, \quad \text{if } \text{Spec } \Phi_\lambda \subset [\lambda, \lambda + (\log \lambda)^{-1}],$$

which is a significant improvement over (4.1) as well as (4.2). It would be interesting to know to what extent this lower bound could be improved. For instance, are λ -power improvement possible?

We would also like to point out similar differences between the concentration of log-quasimodes near periodic geodesics in manifolds with negative sectional curvatures compared to flat manifolds. Recall that in the flat case the modes ψ_{λ_k} satisfying (3.7) had nontrivial L^2 -mass (3.20) in a $\lambda_k^{-1/2} (\log \lambda_k)^{1/2}$ tube about the periodic geodesic γ_0 in the flat manifold (M, g) . If one uses Lemma 3.1 along with (1.10), though, one can repeat the arguments in §3.1 to see that if all the sectional curvatures of (M, g) are negative one can never have a sequence of modes ψ_{λ_k} satisfying (3.8) and also have for a fixed periodic geodesic γ_0

$$(4.4) \quad \liminf_{\lambda_k \rightarrow \infty} \|\psi_{\lambda_k}\|_{L^2(\mathcal{T}_{\delta_{k,N}(\lambda)}(\gamma_0))} > 0, \quad \delta_{k,N}(\lambda) = \lambda_k^{-1/2} (\log \lambda_k)^N,$$

for any N , with the set in the L^2 -norm being a $\lambda_k^{-1/2} (\log \lambda_k)^N$ tube about the periodic geodesic γ_0 . It would also be interesting to show that the analog of (4.4) can never hold on manifolds of negative curvature if $\delta_{k,N}$ is replaced by $\lambda_k^{-1/2+\sigma}$ for some $\sigma > 0$. Breaking this log-power barrier, as in the analogous problem regarding (4.3), is probably difficult due to the role of the Ehrenfest time.

The fact that (4.4) can never hold when the sectional curvatures are negative seems to be somewhat related to the assumptions of Brooks [9] who showed that if $\mathcal{N}(\gamma_0)$ is any *fixed* neighborhood of γ_0 and (M, g) has constant negative sectional curvatures one can construct ψ_{λ_k} as above satisfying

$$\liminf_{\lambda_k \rightarrow \infty} \|\psi_{\lambda_k}\|_{L^2(\mathcal{N}(\gamma_0))} > c_0,$$

for some fixed constant $c_0 > 0$ depending on γ_0 (but not on $\mathcal{N}(\gamma_0)$). The log-quasimodes that Brooks constructs do not equidistribute; however, the fact that (4.4) can never hold for any N on manifolds with negative curvature quantifies that the rate at which this is manifested is much slower than exhibited by the ψ_{λ_k} constructed in §3.1 for flat manifolds, as well of course compared to much faster rate exhibited by the Gaussian beams on S^n .

We also would like to mention that it would be interesting to see to what extent one could weaken the hypotheses in Theorem 1.1 and still obtain similar improvements over the universal bounds (1.4). For quite a while, starting in Sogge and Zelditch [31], and more recently in important improvements of Canzani and Galkowski [10], [11], it has been known that for generic manifolds one can always improve the estimates in (1.4) for *supercritical* exponents by considering projection operators associated with intervals $[\lambda, \lambda + \delta(\lambda)]$ with $\delta(\lambda) \rightarrow 0$. Despite the fact that it has been over 20 years since this was proved in [31], no such results have been obtained for the critical exponent q_c or subcritical exponents $q \in (2, q_c)$. We have simplified considerably the approach to handle such exponents. Indeed, one would obtain improved bounds if one could establish pointwise kernel estimates like those in (2.19) and Lemma 2.5. These are the only places where we used the curvature assumptions in Theorem 1.1. The other estimates, which were mostly local ones, are valid for all compact manifolds.

There is one more problem suggested by our estimates (1.10) for spectral projection operators on compact manifolds all of whose sectional curvatures are *negative*. For the critical exponent $q_c = \frac{2(n+1)}{n-1}$ we conjecture that on such manifolds we have

$$(4.5) \quad \|\chi_{[\lambda, \lambda+\delta(\lambda)]}\|_{2 \rightarrow q_c} = O(1) \quad \text{if } \delta(\lambda) = \lambda^{-\frac{n-1}{n+1}}.$$

This seems very hard to verify even when (M, g) is a space form of negative curvature. For subcritical exponents on manifolds all of whose sectional curvatures are negative, we similarly conjecture that

$$(4.6) \quad \|\chi_{[\lambda, \lambda+\delta(\lambda)]}\|_{2 \rightarrow q} = O(1) \quad \text{if } \delta(\lambda) = \lambda^{-(n-1)(\frac{1}{2}-\frac{1}{q})} \quad \text{for } q \in (2, q_c),$$

with $\delta(\lambda) = \lambda^{-(n-1)(\frac{1}{2}-\frac{1}{q})}$.

These results would be optimal since such $O(1)$ bounds easily can be seen to be impossible for $\delta(\lambda) = \lambda^{-(n-1)(\frac{1}{2}-\frac{1}{q})+\sigma}$ with $\sigma > 0$ if $q \in (2, q_c]$. We also note that neither (4.5) or (4.6) can hold on flat compact manifolds.

5. Appendix: Bilinear oscillatory integrals and local harmonic analysis on manifolds.

It remains to prove Proposition 2.3. We shall first prove (2.44) and then turn to the proof of (2.45). We shall also first prove (2.44) for $n \geq 3$ and then turn to the modifications needed to handle $n = 2$.

Recall that the $A_\nu^{\theta_0}$ there are pseudo-differential cutoffs at the scale $\theta_0 = \lambda^{-1/8}$ belonging to a bounded subset of $S_{7/8, 1/8}^0$.

We first note that by (2.25)

$$(5.1) \quad \tilde{\sigma}_\lambda - \sum_\nu \tilde{\sigma}_\lambda A_\nu^{\theta_0} = R_\lambda \quad \text{where } \|R_\lambda\|_{2 \rightarrow \infty} = O(\lambda^{-N}) \forall N.$$

Thus,

$$(5.2) \quad (\tilde{\sigma}_\lambda h)^2 = \sum_{\nu, \nu'} (\tilde{\sigma}_\lambda A_\nu^{\theta_0} h) \cdot (\tilde{\sigma}_\lambda A_{\nu'}^{\theta_0} h) + O(\lambda^{-N} \|h\|_2^2).$$

Let us set

$$(5.3) \quad \Upsilon^{\text{diag}}(h) = \sum_{(\nu, \nu') \in \Xi_{\theta_0}} (\tilde{\sigma}_\lambda A_\nu^{\theta_0} h) \cdot (\tilde{\sigma}_\lambda A_{\nu'}^{\theta_0} h),$$

and

$$(5.4) \quad \Upsilon^{\text{far}}(h) = \sum_{(\nu, \nu') \notin \Xi_{\theta_0}} (\tilde{\sigma}_\lambda A_\nu^{\theta_0} h) \cdot (\tilde{\sigma}_\lambda A_{\nu'}^{\theta_0} h) + O(\lambda^{-N} \|h\|_2^2),$$

with the last term containing the error terms in (5.2). Thus,

$$(5.5) \quad (\tilde{\sigma}_\lambda h)^2 = \Upsilon^{\text{far}}(h) + \Upsilon^{\text{diag}}(h).$$

Note that the summation in $\Upsilon^{\text{diag}}(h)$ is over near diagonal pairs (ν, ν') by (2.43). In particular, for $(\nu, \nu') \in \Xi_{\theta_0} \subset \theta_0 \cdot \mathbb{Z}^{2(n-1)}$ we have $|\nu - \nu'| \leq C\theta_0$ for some uniform constant. The other term $\Upsilon^{\text{far}}(h)$ in (5.5) includes the remaining pairs, many of which are far from the diagonal, and this will contribute to the last term in (2.44).

When $n = 2$, let us further define

$$(5.6) \quad T_\nu h = \sum_{\nu': (\nu, \nu') \in \Xi_{\theta_0}} (\tilde{\sigma}_\lambda A_\nu^{\theta_0} h) (\tilde{\sigma}_\lambda A_{\nu'}^{\theta_0} h),$$

and write

$$(5.7) \quad \begin{aligned} (\Upsilon^{\text{diag}}(h))^2 &= \left(\sum_\nu T_\nu h \right)^2 \\ &= \sum_{\nu_1, \nu_2} T_{\nu_1} h T_{\nu_2} h. \end{aligned}$$

As in (2.43), if we assume that $B(x, \xi)$ has small conic support, the sum in (5.7) can be organized as

$$(5.8) \quad \begin{aligned} &\left(\sum_{\{k \in \mathbb{N}: k \geq 20 \text{ and } \theta = 2^k \theta_0 \ll 1\}} \sum_{\{(\mu_1, \mu_2): \tau_{\mu_1}^\theta \sim \tau_{\mu_2}^\theta\}} \sum_{\{(\nu_1, \nu_2) \in \tau_{\mu_1}^\theta \times \tau_{\mu_2}^\theta\}} + \sum_{(\nu_1, \nu_2) \in \Xi_{\theta_0}} \right) T_{\nu_1} h T_{\nu_2} h, \\ &= \overline{\Upsilon}^{\text{far}}(h) + \overline{\Upsilon}^{\text{diag}}(h) \end{aligned}$$

Here Ξ_{θ_0} indexes the near diagonal pairs. This is another Whitney decomposition similar to (2.43), but the diagonal set Ξ_{θ_0} is much larger than the set Ξ_{θ_0} in (2.43). More explicitly, when $n = 2$, it is not hard to check that $|\nu - \nu'| \leq 2^{11} \theta_0$ if $(\nu, \nu') \in \Xi_{\theta_0}$ while $|\nu_1 - \nu_2| \leq 2^{21} \theta_0$ if $(\nu_1, \nu_2) \in \Xi_{\theta_0}$. This will help us simplify the calculations needed for $\overline{\Upsilon}^{\text{far}}(h)$.

These terms will be treated differently as was previously done in analyzing parabolic restriction theorems or bilinear oscillatory integrals.

We can treat the terms involving $\Upsilon^{\text{diag}}(h)$ and $\overline{\Upsilon}^{\text{diag}}(h)$ as in [7] by using a variable coefficient variant of Lemma 6.1 in Tao, Vargas and Vega [35].

Lemma 5.1. *If $\Upsilon^{\text{diag}}(h)$ is as in (5.3) and $n \geq 3$, then we have the uniform bounds*

$$(5.9) \quad \|\Upsilon^{\text{diag}}(h)\|_{L^{q_c/2}} \lesssim \left(\sum_\nu \|\tilde{\sigma}_\lambda A_\nu^{\theta_0} h\|_{L^{q_c}}^{q_c} \right)^{2/q_c} + O(\lambda^{\frac{2}{q_c}-} \|h\|_2^2).$$

And for all $n \geq 2$, if $q \in (2, \frac{2(n+2)}{n}]$ and $\mu(q)$ is as in (1.4), we have

$$(5.10) \quad \|\Upsilon^{\text{diag}}(h)\|_{L^{q/2}} \lesssim \left(\sum_\nu \|\tilde{\sigma}_\lambda A_\nu^{\theta_0} h\|_{L^q}^q \right)^{2/q} + O(\lambda^{2\mu(q)-} \|h\|_2^2).$$

Also if $n = 2$ and $\overline{\Upsilon}^{\text{diag}}(h)$ is as in (5.8), we have

$$(5.11) \quad \|\overline{\Upsilon}^{\text{diag}}(h)\|_{L^{3/2}} \lesssim \left(\sum_\nu \|\tilde{\sigma}_\lambda A_\nu^{\theta_0} h\|_{L^6}^6 \right)^{2/3} + O(\lambda^{\frac{2}{3}-} \|h\|_2^4).$$

We also require the following estimates for $\Upsilon^{\text{far}}(h)$ and $\overline{\Upsilon}^{\text{far}}(h)$ which will be proved using bilinear oscillatory integral estimates of Lee [23] and slightly simplified variants of arguments in [4], [5] and [7].

Lemma 5.2. *Let $n \geq 2$. If $\Upsilon^{\text{far}}(h)$ is as in (5.4), and, as above $\theta_0 = \lambda^{-1/8}$ then for all $\varepsilon > 0$ we have*

$$(5.12) \quad \int_M |\Upsilon^{\text{far}}(h)|^{q/2} dx \leq C_\varepsilon \lambda^{1+\varepsilon} (\lambda^{7/8})^{\frac{n-1}{2}(q-q_c)} \|h\|_{L^2(M)}^q, \quad q = \frac{2(n+2)}{n},$$

assuming, as in Proposition 2.3, that the conic support of $B(x, \xi)$ in (2.8) as well as δ and δ_0 in (2.2) are sufficiently small. Similarly, for all $n \geq 2$, we have

$$(5.13) \quad \int_M |\Upsilon^{far}(h)|^{q/2} dx \leq C \lambda^{\mu(q) \cdot q^-} \|h\|_{L^2(M)}^q, \quad 2 < q < \frac{2(n+2)}{n}.$$

Also if $n = 2$ and $\overline{\Upsilon}^{far}(h)$ is as in (5.8), we have

$$(5.14) \quad \int_M |\overline{\Upsilon}^{far}(h)| dx \leq C_\varepsilon \lambda^{1+\varepsilon} \lambda^{-7/8} \|h\|_{L^2(M)}^4.$$

Here as before, $\lambda^{\mu-}$ means a factor involving an unspecified exponent smaller than μ . In Lemma 5.2 we assume that $B(x, \xi)$ has small conic support so that in (2.43) we only need to consider $\theta = 2^k \theta_0$ which are small compared to one. We want $\delta, \delta_0 > 0$ to be small in order to apply the oscillatory integral estimates in [23] for $n \geq 3$ and Hörmander [19] for $n = 2$.

Let us postpone the proofs of these lemmas for a bit and see how they can be used to prove Proposition 2.3.

Proof of (2.44) for $n \geq 3$. Let $q = \frac{2(n+2)}{n}$ as in Lemma 5.2 and note that $q < q_c$. Also,

$$|\tilde{\sigma}_\lambda h \tilde{\sigma}_\lambda h|^{q_c/2} \leq 2^{q/2} |\tilde{\sigma}_\lambda h \tilde{\sigma}_\lambda h|^{\frac{q_c-q}{2}} \cdot (|\Upsilon^{diag}(h)|^{q/2} + |\Upsilon^{far}(h)|^{q/2}).$$

As a result, taking $h = \rho_\lambda f$ and norms over A_- as in (2.44),

$$(5.15) \quad \begin{aligned} \|\tilde{\sigma}_\lambda h\|_{L^{q_c}(A_-)}^{q_c} &= \int_{A_-} |\tilde{\sigma}_\lambda h \cdot \tilde{\sigma}_\lambda h|^{q_c/2} dx \\ &\lesssim \int_{A_-} |\tilde{\sigma}_\lambda h \cdot \tilde{\sigma}_\lambda h|^{\frac{q_c-q}{2}} |\Upsilon^{diag}(h)|^{q/2} dx \\ &\quad + \int_{A_-} |\tilde{\sigma}_\lambda h \cdot \tilde{\sigma}_\lambda h|^{\frac{q_c-q}{2}} |\Upsilon^{far}(h)|^{q/2} dx = I + II. \end{aligned}$$

To estimate II we use (5.12), the ceiling for A_- in (2.17) and the fact that, by (2.11), $\tilde{\sigma}_\lambda = \tilde{\rho}_\lambda f$, to conclude that

$$\begin{aligned} II &\lesssim \|\tilde{\rho}_\lambda f\|_{L^\infty(A_-)}^{q_c-q} \cdot \lambda^{1+\varepsilon} (\lambda^{7/8})^{\frac{n-1}{2}(q-q_c)} \|h\|_2^q \\ &\lesssim \lambda^{(\frac{n-1}{4} + \frac{1}{8})(q_c-q)} \lambda^{-(q_c-q)(\frac{7}{8} \cdot \frac{n-1}{2})} \cdot \lambda^{1+\varepsilon} = O(\lambda^{1-\delta_n+\varepsilon}). \end{aligned}$$

Here $\delta_n > 0$ since $(q_c - q)(\frac{3(n-1)}{16} - \frac{1}{8}) > 0$, and we also used the fact that $\|h\|_2 = \|\rho_\lambda f\|_2 = O(1)$ by (2.15).

Since we may take $\varepsilon < \delta_n$, II^{1/q_c} is dominated by the last term in (2.44). Consequently, to finish the proof of this inequality, we just need to see that we also have suitable bounds for I^{1/q_c} . To do so we use Hölder's inequality followed by Young's inequality and (5.9)

to see that

$$\begin{aligned}
I &\leq \|\tilde{\sigma}_\lambda h \cdot \tilde{\sigma}_\lambda h\|_{L^{q_c/2}(A_-)}^{\frac{q_c-q}{2}} \cdot \|\Upsilon^{\text{diag}}(h)\|_{L^{q_c/2}}^{q/2} \\
&\leq \frac{q_c-q}{q_c} \|\tilde{\sigma}_\lambda h \cdot \tilde{\sigma}_\lambda h\|_{L^{q_c/2}(A_-)}^{q_c/2} + \frac{q}{q_c} \|\Upsilon^{\text{diag}}(h)\|_{L^{q_c/2}}^{q_c/2} \\
&\leq \frac{q_c-q}{q_c} \|\tilde{\sigma}_\lambda h\|_{L^{q_c}(A_-)}^{q_c} + C \sum_{\nu} \|\tilde{\sigma}_\lambda A_\nu^{\theta_0} h\|_{L^{q_c}}^{q_c} + O(\lambda^{1-}).
\end{aligned}$$

Since $\frac{q_c-q}{q_c} < 1$, the first term in the right can be absorbed in the left side of (5.15), and this along with the earlier estimate for II yields (2.44) when $n \geq 3$. $\square$

Proof of (2.44) for $n = 2$. If $n = 2$, we shall still use (5.15), and note that the estimate for the term II also hold for $n = 2$, so it suffices to modify the arguments for the first term I . Since $q = \frac{2(n+2)}{n} = 4$ and $q_c = 6$ if $n = 2$, by (5.8), we have

$$\begin{aligned}
(5.16) \quad I &= \int_{A_-} |\tilde{\sigma}_\lambda h \cdot \tilde{\sigma}_\lambda h| |\Upsilon^{\text{diag}}(h)|^2 dx \\
&\leq \int_{A_-} |\tilde{\sigma}_\lambda h \cdot \tilde{\sigma}_\lambda h| |\overline{\Upsilon}^{\text{diag}}(h)| dx + \int_{A_-} |\tilde{\sigma}_\lambda h \cdot \tilde{\sigma}_\lambda h| |\overline{\Upsilon}^{\text{far}}(h)| dx \\
&= A + B
\end{aligned}$$

To estimate B we use (5.14), the ceiling for A_- in (2.17) and the fact that, by (2.11), $\tilde{\sigma}_\lambda = \tilde{\rho}_\lambda f$, to conclude that

$$B \lesssim \|\tilde{\rho}_\lambda f\|_{L^\infty(A_-)}^2 \cdot \lambda^{1+\varepsilon} \lambda^{-7/8} \|h\|_2^4 \lesssim \lambda^{(\frac{1}{4}+\frac{1}{8})(2)} \lambda^{-\frac{7}{8}} \cdot \lambda^{1+\varepsilon} = O(\lambda^{1-\frac{1}{8}+\varepsilon}).$$

Since we may take $\varepsilon < \frac{1}{8}$, $B^{1/6}$ is dominated by the last term in (2.44). Thus, we just need to see that we also have suitable bounds for $A^{1/6}$. By Hölder's inequality, Young's inequality and (5.11), we have

$$\begin{aligned}
A &\leq \|\tilde{\sigma}_\lambda h \tilde{\sigma}_\lambda h\|_{L^3(A_-)} \cdot \|\overline{\Upsilon}^{\text{diag}}(h)\|_{L^{3/2}(M)} \\
&\leq \frac{1}{3} \|\tilde{\sigma}_\lambda h \tilde{\sigma}_\lambda h\|_{L^3(A_-)}^3 + \frac{2}{3} \|\overline{\Upsilon}^{\text{diag}}(h)\|_{L^{3/2}(M)}^{3/2} \\
&\leq \frac{1}{3} \|\tilde{\sigma}_\lambda h \tilde{\sigma}_\lambda h\|_{L^3(A_-)}^3 + C \sum_{\nu} \|\tilde{\sigma}_\lambda A_\nu^{\theta_0} h\|_{L^6(M)}^6 + \lambda^{1-}.
\end{aligned}$$

The first term in the right can be absorbed in the left side of (5.15), and this along with the earlier estimates yields (2.44) when $n = 2$. $\square$

Proof of (2.45). The proof of (2.45) is much simpler since we do not have to restrict to the set A_- . Since

$$|\tilde{\sigma}_\lambda h \tilde{\sigma}_\lambda h|^{q/2} \leq 2^{q/2} \cdot (|\Upsilon^{\text{diag}}(h)|^{q/2} + |\Upsilon^{\text{far}}(h)|^{q/2}),$$

we have

$$(5.17) \quad \|\tilde{\sigma}_\lambda h\|_{L^q(M)}^q = \int |\tilde{\sigma}_\lambda h \cdot \tilde{\sigma}_\lambda h|^{q/2} dx \lesssim \int |\Upsilon^{\text{diag}}(h)|^{q/2} dx + \int |\Upsilon^{\text{far}}(h)|^{q/2} dx.$$

Thus (2.45) simply follows from applying (5.10) for the first term and (5.13) for the second term on the right side. $\square$

Proof of Lemma 5.1. Let us define the wider cutoffs, after recalling (2.7) and (2.31), by setting

$$(5.18) \quad \tilde{A}_\nu^{\theta_0}(x, \xi) = \psi(x) \sum_{\{k \in \mathbb{Z}: |k| \leq C_0\}} \sum_{\{\ell \in \mathbb{Z}^{2(n-1)}: |\theta_0 \ell - \nu| \leq C_0 \theta_0\}} a_\ell^{\theta_0}(x, \xi) \beta(2^k p(x, \xi)/\lambda).$$

If C_0 is fixed large enough we then clearly have

$$(5.19) \quad \|A_\nu^{\theta_0} - A_\nu^{\theta_0} \tilde{A}_\nu^{\theta_0}\|_{p \rightarrow p} = O(\lambda^{-N}) \forall N \text{ if } 1 \leq p \leq \infty,$$

if $\tilde{A}_\nu^{\theta_0}(x, D)$ is the operator with symbol $\tilde{A}_\nu^{\theta_0}(x, \xi)$.

For later use, let us also recall that, by (2.26) and (2.27), for each fixed x the support of $\xi \rightarrow A_\nu^{\theta_0}(x, \xi)$ is contained in a cone of aperture $\lesssim \theta_0 = \lambda^{-1/8}$. So if $(\nu, \nu') \in \Xi_{\theta_0}$ then both $\xi \rightarrow A_\nu^{\theta_0}(x, \xi)$ and $\xi \rightarrow A_{\nu'}^{\theta_0}(x, \xi)$ are supported for every fixed x in a common cone of aperture $O(\lambda^{-1/8})$ since $\nu - \nu' = O(\lambda^{-1/8})$ when $(\nu, \nu') \in \Xi_{\theta_0}$. Thus, it is not difficult to check we can also fix C_0 large enough so that that we also have that

$$(5.20) \quad \text{if } (\nu, \nu') \in \Xi_{\theta_0} \text{ and } (1 - \overline{\tilde{A}_\nu^{\theta_0}(y, \zeta)}) A_\nu^{\theta_0}(y, \xi) A_{\nu'}^{\theta_0}(y, \eta) \neq 0, \\ \text{then } |\zeta - (\xi + \eta)| \geq c \theta_0 \lambda,$$

for some fixed constant $c > 0$. In what follows we fix C_0 large enough so that we have (5.19) and (5.20).

To use (5.19) we note that, since (1.4) yields $\|\tilde{\sigma}_\lambda\|_{2 \rightarrow q_c} = O(\lambda^{1/q_c})$, we conclude that, in order to prove (5.9), it suffices to prove

$$(5.21) \quad \left\| \sum_{(\nu, \nu') \in \Xi_{\theta_0}} (\tilde{\sigma}_\lambda A_\nu^{\theta_0} \tilde{A}_\nu^{\theta_0} h) (\tilde{\sigma}_\lambda A_{\nu'}^{\theta_0} \tilde{A}_{\nu'}^{\theta_0} h) \right\|_{L^{q_c/2}} \lesssim \left(\sum_\nu \|\tilde{\sigma}_\lambda A_\nu^{\theta_0} h\|_{q_c}^{q_c} \right)^{2/q_c} + O(\lambda^{\frac{2}{q_c} - \|h\|_2^2}).$$

To do this we require the following variant of (2.35)

$$(5.22) \quad \|\tilde{\sigma}_\lambda A_\nu^{\theta_0} - A_\nu^{\theta_0} \tilde{\sigma}_\lambda\|_{2 \rightarrow q_c} = O(\lambda^{\frac{1}{q_c} - \frac{1}{4}}),$$

which follows from the same argument that was used to obtain (2.35).

Since the proof of (2.33) also yields due to (5.18)

$$(5.23) \quad \sum_\nu \|\tilde{A}_\nu^{\theta_0} f\|_r^r \lesssim \|f\|_r^r, \quad 2 \leq r \leq \infty,$$

we can use this inequality for $r = 2$ along with (5.22) to see that we would obtain (5.21) if we could show that

$$\left\| \sum_{(\nu, \nu') \in \Xi_{\theta_0}} (A_\nu^{\theta_0} \tilde{\sigma}_\lambda \tilde{A}_\nu^{\theta_0} h) \cdot (A_{\nu'}^{\theta_0} \tilde{\sigma}_\lambda \tilde{A}_{\nu'}^{\theta_0} h) \right\|_{L^{q_c/2}} \leq C \left(\sum_\nu \|\tilde{\sigma}_\lambda A_\nu^{\theta_0} h\|_{q_c}^{q_c} \right)^{2/q_c} + O(\lambda^{\frac{2}{q_c} - \|h\|_2^2}).$$

Next, if we take $r = (q_c/2)'$ so that r is the conjugate exponent, we conclude that suffices to show that

$$(5.24) \quad \left| \sum_{(\nu, \nu') \in \Xi_{\theta_0}} \int (A_\nu^{\theta_0} \tilde{\sigma}_\lambda \tilde{A}_\nu^{\theta_0} h) \cdot (A_{\nu'}^{\theta_0} \tilde{\sigma}_\lambda \tilde{A}_{\nu'}^{\theta_0} h) \cdot \bar{f} \, dx \right| \\ \leq C \left(\sum_\nu \|\tilde{\sigma}_\lambda A_\nu^{\theta_0} h\|_{q_c}^{q_c} \right)^{2/q_c} + O(\lambda^{\frac{2}{q_c} - \|h\|_2^2}), \quad \text{if } \|f\|_r = 1.$$

To do this, we note that by (5.20) and a simple integration by parts argument we have

$$\|(I - \tilde{A}_\nu^{\theta_0})^*(A_\nu^{\theta_0} h_1 \cdot A_{\nu'}^{\theta_0} h_2)\|_{L^\infty} \leq C_n \lambda^{-N} \|h_1\|_1 \cdot \|h_2\|_1 \quad \forall N, \quad \text{if } (\nu, \nu') \in \Xi_{\theta_0}.$$

Thus, modulo $O(\lambda^{-N} \|h\|_2^2)$ errors, the left side of (5.24) is dominated by

$$\begin{aligned} (5.25) \quad & \left| \sum_{(\nu, \nu') \in \Xi_{\theta_0}} \int (A_\nu^{\theta_0} \tilde{\sigma}_\lambda \tilde{A}_\nu^{\theta_0} h) \cdot (A_{\nu'}^{\theta_0} \tilde{\sigma}_\lambda \tilde{A}_{\nu'}^{\theta_0} h) \cdot \overline{\tilde{A}_{\nu'}^{\theta_0} f} dx \right| \\ & \leq \left(\sum_{(\nu, \nu') \in \Xi_{\theta_0}} \|(A_\nu^{\theta_0} \tilde{\sigma}_\lambda \tilde{A}_\nu^{\theta_0} h) \cdot (A_{\nu'}^{\theta_0} \tilde{\sigma}_\lambda \tilde{A}_{\nu'}^{\theta_0} h)\|_{L^{q_c/2}}^{q_c/2} \right)^{2/q_c} \cdot \left(\sum_{(\nu, \nu') \in \Xi_{\theta_0}} \|\tilde{A}_{\nu'}^{\theta_0} f\|_r^r \right)^{1/r} \\ & \lesssim \left(\sum_\nu \|A_\nu^{\theta_0} \tilde{\sigma}_\lambda \tilde{A}_\nu^{\theta_0} h\|_{q_c}^{q_c} \right)^{2/q_c} \cdot \left(\sum_\nu \|\tilde{A}_\nu^{\theta_0} f\|_r^r \right)^{1/r} \\ & \lesssim \left(\sum_\nu \|A_\nu^{\theta_0} \tilde{\sigma}_\lambda \tilde{A}_\nu^{\theta_0} h\|_{q_c}^{q_c} \right)^{2/q_c}, \end{aligned}$$

using Hölder's inequality, the fact that if ν is fixed there are just $O(1)$ indices ν' with $(\nu, \nu') \in \Xi_{\theta_0}$, followed by (5.23) and the fact that $q_c \leq 4$ if $n \geq 3$ and so $r \geq 2$. Based on this, modulo $O(\lambda^{\frac{2}{q_c}-} \|h\|_2^2)$, the left side of (5.24) is dominated by $(\sum_\nu \|A_\nu^{\theta_0} \tilde{\sigma}_\lambda \tilde{A}_\nu^{\theta_0} h\|_{q_c}^{q_c})^{2/q_c}$. So, if we repeat the earlier arguments and use (5.19) again we conclude that this last expression is dominated by $(\sum_\nu \|\tilde{\sigma}_\lambda A_\nu^{\theta_0} h\|_{q_c}^{q_c})^{2/q_c} + O(\lambda^{\frac{2}{q_c}-} \|h\|_2^2)$, which yields (5.11).

The proof of (5.10) is exactly the same, one can just repeat the arguments and use the following variant of (5.22),

$$(5.26) \quad \|\tilde{\sigma}_\lambda A_\nu^{\theta_0} - A_{\nu'}^{\theta_0} \tilde{\sigma}_\lambda\|_{2 \rightarrow q} = O(\lambda^{\mu(q)-}),$$

which is a consequence of interpolation between (5.22) and the trivial $L^2 \rightarrow L^2$ estimates. Also note that in (5.10) holds for all $n \geq 2$, since we are assuming $q \leq \frac{2(n+2)}{n}$, which implies that $q \leq 4$ for all $n \geq 2$, and thus $r \geq 2$ in (5.25).

Now we shall prove (5.11), we have to treat the $n = 2$ case separately due to the failure of (5.25) when $q_c = 6$. If we repeat the arguments in (5.18)-(5.23), it suffices to show that

$$\begin{aligned} (5.27) \quad & \left| \sum_{(\nu_1, \nu_2) \in \Xi_{\theta_0}} \int (A_{\nu_1}^{\theta_0} \tilde{\sigma}_\lambda \tilde{A}_{\nu_1}^{\theta_0} h) (A_{\nu_1'}^{\theta_0} \tilde{\sigma}_\lambda \tilde{A}_{\nu_1'}^{\theta_0} h) (A_{\nu_2}^{\theta_0} \tilde{\sigma}_\lambda \tilde{A}_{\nu_2}^{\theta_0} h) (A_{\nu_2'}^{\theta_0} \tilde{\sigma}_\lambda \tilde{A}_{\nu_2'}^{\theta_0} h) \cdot \bar{f} dx \right| \\ & \leq C \left(\sum_\nu \|\tilde{\sigma}_\lambda A_\nu^{\theta_0} h\|_6^6 \right)^{2/3} + O(\lambda^{\frac{2}{3}-} \|h\|_2^4), \quad \text{if } \|f\|_3 = 1. \end{aligned}$$

Here $(\nu_1, \nu_1') \in \Xi_{\theta_0}$, $(\nu_2, \nu_2') \in \Xi_{\theta_0}$, and the set Ξ_{θ_0} is as in (5.8). So $\nu_1, \nu_1', \nu_2, \nu_2'$ in (5.27) satisfy $|\nu_1 - \nu_1'| + |\nu_1 - \nu_2| + |\nu_1 - \nu_2'| = O(\lambda^{-1/8})$. Thus, if we choose C_0 in (5.18) large enough, by a simple integration by parts argument we have

$$\begin{aligned} & \|(I - \tilde{A}_{\nu_1}^{\theta_0})^*(A_{\nu_1}^{\theta_0} h_1 \cdot A_{\nu_1'}^{\theta_0} h_2 A_{\nu_2}^{\theta_0} h_3 \cdot A_{\nu_2'}^{\theta_0} h_4)\|_{L^\infty} \\ & \leq C_n \lambda^{-N} \|h_1\|_1 \cdot \|h_2\|_1 \cdot \|h_3\|_1 \cdot \|h_4\|_1 \quad \forall N. \end{aligned}$$

Thus, modulo $O(\lambda^{-N} \|h\|_2^4)$ errors, the left side of (5.27) is dominated by

$$\begin{aligned}
& \left| \sum_{(\nu_1, \nu_2) \in \Xi_{\theta_0}} \int (A_{\nu_1}^{\theta_0} \tilde{\sigma}_\lambda \tilde{A}_{\nu_1}^{\theta_0} h) (A_{\nu_1'}^{\theta_0} \tilde{\sigma}_\lambda \tilde{A}_{\nu_1'}^{\theta_0} h) (A_{\nu_2}^{\theta_0} \tilde{\sigma}_\lambda \tilde{A}_{\nu_2}^{\theta_0} h) (A_{\nu_2'}^{\theta_0} \tilde{\sigma}_\lambda \tilde{A}_{\nu_2'}^{\theta_0} h) \cdot \overline{\tilde{A}_{\nu_1}^{\theta_0} f} dx \right| \\
& \leq \left(\sum_{(\nu_1, \nu_2) \in \Xi_{\theta_0}} \| (A_{\nu_1}^{\theta_0} \tilde{\sigma}_\lambda \tilde{A}_{\nu_1}^{\theta_0} h) (A_{\nu_1'}^{\theta_0} \tilde{\sigma}_\lambda \tilde{A}_{\nu_1'}^{\theta_0} h) (A_{\nu_2}^{\theta_0} \tilde{\sigma}_\lambda \tilde{A}_{\nu_2}^{\theta_0} h) (A_{\nu_2'}^{\theta_0} \tilde{\sigma}_\lambda \tilde{A}_{\nu_2'}^{\theta_0} h) \|_{L^{3/2}}^{3/2} \right)^{2/3} \\
(5.28) \quad & \cdot \left(\sum_{(\nu_1, \nu_2) \in \Xi_{\theta_0}} \|\tilde{A}_{\nu_1}^{\theta_0} f\|_3^3 \right)^{1/3} \\
& \lesssim \left(\sum_\nu \|A_\nu^{\theta_0} \tilde{\sigma}_\lambda \tilde{A}_\nu^{\theta_0} h\|_6^6 \right)^{2/3} \cdot \left(\sum_\nu \|\tilde{A}_\nu^{\theta_0} f\|_3^3 \right)^{1/3} \\
& \lesssim \left(\sum_\nu \|A_\nu^{\theta_0} \tilde{\sigma}_\lambda \tilde{A}_\nu^{\theta_0} h\|_6^6 \right)^{2/3},
\end{aligned}$$

using Hölder's inequality, the fact that if ν_1 is fixed there are just $O(1)$ indices ν_1', ν_2 and ν_2' with $(\nu_1, \nu_2) \in \Xi_{\theta_0}$, $(\nu_1, \nu_1') \in \Xi_{\theta_0}$ and $(\nu_2, \nu_2') \in \Xi_{\theta_0}$, followed by (5.23) with $r = 3$. Based on this, modulo $O(\lambda^{\frac{2}{3}-} \|h\|_2^4)$, the left side of (5.27) is dominated by $(\sum_\nu \|A_\nu^{\theta_0} \tilde{\sigma}_\lambda \tilde{A}_\nu^{\theta_0} h\|_6^6)^{2/3}$. So, if we repeat the earlier arguments and use (5.19) again we conclude that this last expression is dominated by $(\sum_\nu \|\tilde{\sigma}_\lambda A_\nu^{\theta_0} h\|_6^6)^{2/3} + O(\lambda^{\frac{2}{3}-} \|h\|_2^4)$, which yields (5.11) and completes the proof of Lemma 5.1. $\square$

5.1. Proof of Lemma 5.2.

In this subsection we shall start the proof the other lemma, Lemma 5.2, which is a bit more difficult. We shall see that it is a consequence of the bilinear estimates of Lee [23].

To prove (5.12) we recall (2.43) and (5.4) and note that for a given $\theta = 2^k \theta_0$, $k \geq 10$, we have for each fixed $c_0 > 0$

$$(5.29) \quad \tilde{\sigma} A_\nu^{\theta_0} h = \sum_{\tilde{\mu} \in (c_0 \theta) \cdot \mathbb{Z}^{2(n-1)}} \tilde{\sigma}_\lambda A_{\tilde{\mu}}^{c_0 \theta} A_\nu^{\theta_0} h + O(\lambda^{-N} \|h\|_2).$$

We are only considering $k \geq 10$ due to the organization of the sum in the left side of (2.43). As in [4], we shall choose $c_0 = 2^{-m_0} < 1$ to be specified later to ensure that we have the separation needed to apply bilinear oscillatory integral estimates.

Keeping this in mind fix $k \geq 10$ in the first sum in (2.43). We then have for a given c_0 as above and pairs of dyadic cubes $\tau_\mu^\theta, \tau_{\mu'}^\theta$ with $\tau_\mu^\theta \sim \tau_{\mu'}^\theta$,

$$\begin{aligned}
(5.30) \quad & \sum_{(\nu, \nu') \in \tau_\mu^\theta \times \tau_{\mu'}^\theta} (\tilde{\sigma}_\lambda A_\nu^{\theta_0} h) (\tilde{\sigma}_\lambda A_{\nu'}^{\theta_0} h) \\
& = \sum_{(\nu, \nu') \in \tau_\mu^\theta \times \tau_{\mu'}^\theta} \sum_{\substack{\tau_{\tilde{\mu}}^{c_0 \theta} \cap \tau_\mu^\theta \neq \emptyset \\ \tau_{\tilde{\mu}'}^{c_0 \theta} \cap \tau_{\mu'}^\theta \neq \emptyset}} (\tilde{\sigma}_\lambda A_{\tilde{\mu}}^{c_0 \theta} A_\nu^{\theta_0} h) (\tilde{\sigma}_\lambda A_{\tilde{\mu}'}^{c_0 \theta} A_{\nu'}^{\theta_0} h) + O(\lambda^{-N} \|h\|_2^2),
\end{aligned}$$

if τ_μ^θ and $\tau_{\mu'}^\theta$ are cubes with the same centers but $11/10$ times the side length of τ_μ^θ and $\tau_{\mu'}^\theta$, respectively, so that we have $\text{dist}(\tau_\mu^\theta, \tau_{\mu'}^\theta) \geq \theta/2$ when $\tau_\mu^\theta \sim \tau_{\mu'}^\theta$. We obtain (5.30) from the fact that the product of the symbol of $A_{\tilde{\mu}}^{c_0 \theta}$ and $A_{\nu'}^{\theta_0}$ vanishes if $\tau_{\tilde{\mu}}^{c_0 \theta} \cap \tau_{\mu'}^\theta = \emptyset$

and $\nu \in \tau_\mu^\theta$ since $\theta = 2^k \theta_0$ with $k \geq 10$. Also note that we then have for fixed $c_0 = 2^{-m_0}$ small enough

$$(5.31) \quad \text{dist}(\tau_{\tilde{\mu}}^{c_0\theta}, \tau_{\tilde{\mu}'}^{c_0\theta}) \in [4^{-1}\theta, 4^n\theta], \text{ if } \tau_\mu^\theta \sim \tau_{\mu'}^\theta, \tau_{\tilde{\mu}}^{c_0\theta} \cap \overline{\tau}_\mu^\theta \neq \emptyset \text{ and } \tau_{\tilde{\mu}'}^{c_0\theta} \cap \overline{\tau}_{\mu'}^\theta \neq \emptyset.$$

Also, of course, for each μ there are $O(1)$ indices $\tilde{\mu}$ with $\tau_{\tilde{\mu}}^{c_0\theta} \cap \overline{\tau}_\mu^\theta \neq \emptyset$ with $c_0 > 0$ fixed. Note also that if we fix c_0 then for our pair $\tau_\mu^\theta \sim \tau_{\mu'}^\theta$ of θ -cubes there are only $O(1)$ summands involving $\tilde{\mu}$ and $\tilde{\mu}'$ in the right side of (5.30).

Based on this we claim that we would have favorable bounds for the $L^{q/2}$ -norm, $q = \frac{2(n+2)}{n}$, of the first term in (2.44) and hence $\Upsilon^{\text{far}}(h)$ if we could prove the following key result.

Proposition 5.3. *Let $\theta = 2^k \theta_0 = 2^k \lambda^{-1/8} \ll 1$ with $k \in \mathbb{N}$. Then we can fix $c_0 = 2^{-m_0}$ small enough so that whenever*

$$(5.32) \quad \text{dist}(\tau_\nu^{c_0\theta}, \tau_{\nu'}^{c_0\theta}) \in [4^{-1}\theta, 4^n\theta]$$

one has the uniform bounds for each $\varepsilon > 0$

$$(5.33) \quad \int |(\tilde{\sigma}_\lambda A_\nu^{c_0\theta} h_1)(\tilde{\sigma}_\lambda A_{\nu'}^{c_0\theta} h_2)|^{q/2} dx \leq C_\varepsilon \lambda^{1+\varepsilon} (2^k \lambda^{7/8})^{\frac{n-1}{2}(q-q_c)} \|h_1\|_{L^2}^{q/2} \|h_2\|_{L^2}^{q/2},$$

with, as in (5.12), $q = \frac{2(n+2)}{n}$.

The proof of this proposition is based on the bilinear oscillatory integral estimates of Lee [23], we shall postpone the proof to the next section. Now let us verify the above claim. We first note that if $h_1 = \sum_{\nu \in \tau_\mu^\theta} A_\nu^{\theta_0} h$ and $h_2 = \sum_{\nu' \in \tau_{\mu'}^\theta} A_{\nu'}^{\theta_0} h$, then by the almost orthogonality of the $A_\nu^{\theta_0}$ operators,

$$\|h_1\|_2^2 \lesssim \sum_{\nu \in \tau_\mu^\theta} \|A_\nu^{\theta_0} h\|_2^2 \quad \text{and} \quad \|h_2\|_2^2 \lesssim \sum_{\nu' \in \tau_{\mu'}^\theta} \|A_{\nu'}^{\theta_0} h\|_2^2.$$

Thus, (5.29), (5.31), (5.33) and Minkowski's inequality yield the following estimates for the k -summand in (2.43) with $k \geq 10$, $\theta = 2^k \theta_0$ and $q = \frac{2(n+2)}{n}$:

$$\begin{aligned} (5.34) \quad & \left\| \sum_{(\mu, \mu') : \tau_\mu^\theta \sim \tau_{\mu'}^\theta} \sum_{(\nu, \nu') \in \tau_\mu^\theta \times \tau_{\mu'}^\theta} (\tilde{\sigma} A_\nu^{\theta_0} h)(\tilde{\sigma} A_{\nu'}^{\theta_0} h) \right\|_{L^{q/2}} \\ & \leq \sum_{(\mu, \mu') : \tau_\mu^\theta \sim \tau_{\mu'}^\theta} \left\| \sum_{\substack{\tau_{\tilde{\mu}}^{c_0\theta} \cap \overline{\tau}_\mu^\theta \neq \emptyset \\ \tau_{\tilde{\mu}'}^{c_0\theta} \cap \overline{\tau}_{\mu'}^\theta \neq \emptyset}} (\tilde{\sigma}_\lambda A_{\tilde{\mu}}^{c_0\theta} (\sum_{\nu \in \tau_\mu^\theta} A_\nu^{\theta_0} h)) \cdot (\tilde{\sigma}_\lambda A_{\tilde{\mu}'}^{c_0\theta} (\sum_{\nu' \in \tau_{\mu'}^\theta} A_{\nu'}^{\theta_0} h)) \right\|_{L^{q/2}} + O(\lambda^{-N} \|h\|_2^2) \\ & \lesssim_\varepsilon \lambda^{(1+\varepsilon)\frac{2}{q}} (2^k \lambda^{7/8})^{\frac{n-1}{q}(q-q_c)} \sum_{(\mu, \mu') : \tau_\mu^\theta \sim \tau_{\mu'}^\theta} \left(\sum_{\nu \in \tau_\mu^\theta} \|A_\nu^{\theta_0} h\|_2^2 \right)^{1/2} \left(\sum_{\nu' \in \tau_{\mu'}^\theta} \|A_{\nu'}^{\theta_0} h\|_2^2 \right)^{1/2} + O(\lambda^{-N} \|h\|_2^2) \\ & \lesssim \lambda^{(1+\varepsilon)\frac{2}{q}} (2^k \lambda^{7/8})^{\frac{n-1}{q}(q-q_c)} \sum_{\mu} \sum_{\nu \in \tau_\mu^\theta} \|A_\nu^{\theta_0} h\|_2^2 + O(\lambda^{-N} \|h\|_2^2) \\ & \lesssim \lambda^{(1+\varepsilon)\frac{2}{q}} (2^k \lambda^{7/8})^{\frac{n-1}{q}(q-q_c)} \|h\|_2^2 + O(\lambda^{-N} \|h\|_2^2). \end{aligned}$$

In the above we used the fact that for each τ_μ^θ there are $O(1)$ cubes $\tau_{\tilde{\mu}}^{c_0\theta}$ with $\tau_{\tilde{\mu}}^{c_0\theta} \cap \overline{\tau}_\mu^\theta \neq \emptyset$ and $O(1)$ $\tau_{\mu'}^\theta$ with $\tau_\mu^\theta \sim \tau_{\mu'}^\theta$, and we also used (2.33).

Since $q - q_c < 0$, we conclude from this that if we replace $\Upsilon^{\text{far}}(h)$ by the first term in (2.43) then the resulting expression satisfies the bounds in (5.12). Since, by (5.4), the additional part of $\Upsilon^{\text{far}}(h)$ is pointwise bounded by $O(\lambda^{-N} \|h\|_2^2)$, the proof of (5.12) is complete.

To prove (5.13), note that (5.12) implies that (5.13) is valid if $q = \frac{2(n+2)}{n}$, since $1 + \varepsilon + \frac{7}{8} \frac{n-1}{2} (q - q_c) < \mu(q) \cdot q$ if we choose ε to be small enough. By interpolation, it suffices to show that for the other endpoint $q = 2$, we have

$$(5.35) \quad \int_M |\Upsilon^{\text{far}}(h)| dx \leq C \|h\|_{L^2(M)}^2.$$

Recall that as in (5.5), $\Upsilon^{\text{far}}(h) = (\tilde{\sigma}_\lambda h)^2 - \Upsilon^{\text{diag}}(h)$. By (2.33) and triangle inequality, it is not hard to see that

$$(5.36) \quad \int_M \left| \sum_{(\nu, \nu') \in \Xi_{\theta_0}} (\tilde{\sigma}_\lambda A_\nu^{\theta_0}) \cdot (\tilde{\sigma}_\lambda A_{\nu'}^{\theta_0} h) \right| dx \leq C \|h\|_{L^2(M)}^2.$$

Thus (5.35) just follow from (5.36) and the fact that $\tilde{\sigma}_\lambda$ is a bounded operator on L^2 .

Now we shall see how we can use Proposition 5.3 to prove (5.14). Note that by (5.8) and triangle inequality, it suffices to show that for fixed $\theta = 2^k \theta_0$ with $k \geq 20$,

$$(5.37) \quad \sum_{\{(\mu_1, \mu_2): \tau_{\mu_1}^\theta \sim \tau_{\mu_2}^\theta\}} \sum_{\{(\nu_1, \nu_2) \in \tau_{\mu_1}^\theta \times \tau_{\mu_2}^\theta\}} \int |T_{\nu_1} h T_{\nu_2} h| dx \leq C_\varepsilon \lambda^{1+\varepsilon} (\lambda^{\frac{7}{8}} 2^k)^{-1} \|h\|_{L^2(M)}^4.$$

To see this, since $|\nu - \nu'| \leq 2^{11} \theta_0$ if $(\nu, \nu') \in \Xi_{\theta_0}$, by the definition of T_ν in (5.6) and the Schwarz inequality, we have

$$|T_{\nu_1} h T_{\nu_2} h| \leq C \left(\sum_{\nu'_1: |\nu'_1 - \nu_1| \leq 2^{11} \theta_0} |\tilde{\sigma}_\lambda A_{\nu'_1}^{\theta_0} h|^2 \right) \left(\sum_{\nu'_2: |\nu'_2 - \nu_2| \leq 2^{11} \theta_0} |\tilde{\sigma}_\lambda A_{\nu'_2}^{\theta_0} h|^2 \right).$$

Thus the integrand in the left side of (5.37) is dominated by

$$(5.38) \quad \sum_{\{(\mu_1, \mu_2): \tau_{\mu_1}^\theta \sim \tau_{\mu_2}^\theta\}} \sum_{\{(\nu_1, \nu_2) \in \tilde{\tau}_{\mu_1}^\theta \times \tilde{\tau}_{\mu_2}^\theta\}} |\tilde{\sigma}_\lambda A_{\nu_1}^{\theta_0} h|^2 \cdot |\tilde{\sigma}_\lambda A_{\nu_2}^{\theta_0} h|^2.$$

Here $\tilde{\tau}_{\mu_1}^\theta$ and $\tilde{\tau}_{\mu_2}^\theta$ are the cubes with the same centers but $11/10$ times the side length of $\tau_{\mu_1}^\theta$ and $\tau_{\mu_2}^\theta$, respectively, we used the fact that the side length of $\tau_{\mu_1}^\theta$ is $\geq 2^{20} \theta_0$, so $0.1 * \text{side length} \gg 2^{10} \theta_0$.

Furthermore, if we use (5.29) again, for a given fixed $c_0 = 2^{-m_0}$, with $m_0 \in \mathbb{N}$ small enough, and pair of dyadic cubes $\tau_{\mu_1}^\theta, \tau_{\mu_2}^\theta$ with $\tau_{\mu_1}^\theta \sim \tau_{\mu_2}^\theta$ and $\theta = 2^k \theta_0$, we have the following analog of (5.30)

$$(5.39) \quad \begin{aligned} & \sum_{(\nu_1, \nu_2) \in \tilde{\tau}_{\mu_1}^\theta \times \tilde{\tau}_{\mu_2}^\theta} |\tilde{\sigma}_\lambda A_{\nu_1}^{\theta_0} h|^2 \cdot |\tilde{\sigma}_\lambda A_{\nu_2}^{\theta_0} h|^2 \\ &= \sum_{(\nu_1, \nu_2) \in \tilde{\tau}_{\mu_1}^\theta \times \tilde{\tau}_{\mu_2}^\theta} \sum_{\substack{\tau_{\mu_1}^{c_0 \theta} \cap \tilde{\tau}_{\mu_1}^\theta \neq \emptyset \\ \tau_{\mu_2}^{c_0 \theta} \cap \tilde{\tau}_{\mu_2}^\theta \neq \emptyset}} |\tilde{\sigma}_\lambda A_{\mu_1}^{c_0 \theta} A_{\nu_1}^{\theta_0} h|^2 \cdot |\tilde{\sigma}_\lambda A_{\mu_2}^{c_0 \theta} A_{\nu_2}^{\theta_0} h|^2 + O(\lambda^{-N} \|h\|_2^4), \end{aligned}$$

if $\tilde{\tau}_{\mu_1}^\theta$ and $\tilde{\tau}_{\mu_2}^\theta$ the cubes with the same centers but $12/10$ times the side length of $\tau_{\mu_1}^\theta$ and $\tau_{\mu_2}^\theta$, respectively, so that we have $\text{dist}(\tilde{\tau}_{\mu_1}^\theta, \tilde{\tau}_{\mu_2}^\theta) \geq \theta/2$ when $\tau_{\mu_1}^\theta \sim \tau_{\mu_2}^\theta$. This follows from

the fact that for c_0 small enough the product of the symbol of $A_{\tilde{\mu}_1}^{c_0\theta}$ and $A_{\nu_1}^{\theta_0}$ vanishes identically if $\tau_{\tilde{\mu}_1}^{c_0\theta} \cap \bar{\tau}_{\mu_1}^\theta = \emptyset$ and $\nu_1 \in \tilde{\tau}_{\mu_1}^\theta$, since $\theta = 2^k\theta_0$ with $k \geq 20$. And we also have for fixed c_0 small enough

$$(5.40) \quad \text{dist}(\tau_{\tilde{\mu}_1}^{c_0\theta}, \tau_{\tilde{\mu}_2}^{c_0\theta}) \in [4^{-1}\theta, 4^2\theta], \quad \text{if } \tau_{\mu_1}^\theta \sim \tau_{\mu_2}^\theta, \tau_{\tilde{\mu}_1}^{c_0\theta} \cap \bar{\tau}_{\mu_1}^\theta \neq \emptyset, \text{ and } \tau_{\tilde{\mu}_2}^{c_0\theta} \cap \bar{\tau}_{\mu_2}^\theta \neq \emptyset.$$

By applying Proposition 5.3 for $n = 2$ and repeating the arguments in (5.34), we have

$$(5.41) \quad \begin{aligned} & \sum_{\{(\mu_1, \mu_2): \tau_{\mu_1}^\theta \sim \tau_{\mu_2}^\theta\}} \sum_{\{(\nu_1, \nu_2) \in \tilde{\tau}_{\mu_1}^\theta \times \tilde{\tau}_{\mu_2}^\theta\}} \int |\tilde{\sigma}_\lambda A_{\nu_1}^{\theta_0} h|^2 \cdot |\tilde{\sigma}_\lambda A_{\nu_2}^{\theta_0} h|^2 dx \\ & \leq \sum_{(\mu_1, \mu_2): \tau_{\mu_1}^\theta \sim \tau_{\mu_2}^\theta} \sum_{\substack{\tau_{\tilde{\mu}_1}^{c_0\theta} \cap \bar{\tau}_{\mu_1}^\theta \neq \emptyset \\ \tau_{\tilde{\mu}_2}^{c_0\theta} \cap \bar{\tau}_{\mu_2}^\theta \neq \emptyset}} \sum_{\{(\nu_1, \nu_2) \in \tilde{\tau}_{\mu_1}^\theta \times \tilde{\tau}_{\mu_2}^\theta\}} \int |\tilde{\sigma}_\lambda A_{\tilde{\mu}_1}^{c_0\theta} A_{\nu_1}^{\theta_0} h|^2 \cdot |\tilde{\sigma}_\lambda A_{\tilde{\mu}_2}^{c_0\theta} A_{\nu_2}^{\theta_0} h|^2 dx \\ & \quad + O(\lambda^{-N} \|h\|_{L_x^2}^4) \\ & \leq C_\varepsilon \lambda^{1+\varepsilon} (2^k \lambda^{7/8})^{-1} \sum_{(\mu_1, \mu_2): \tau_{\mu_1}^\theta \sim \tau_{\mu_2}^\theta} \left(\sum_{\nu_1 \in \tilde{\tau}_{\mu_1}^\theta} \|A_{\nu_1}^{\theta_0} h\|_{L_x^2}^2 \right) \left(\sum_{\nu_2 \in \tilde{\tau}_{\mu_2}^\theta} \|A_{\nu_2}^{\theta_0} h\|_{L_x^2}^2 \right) + O(\lambda^{-N} \|h\|_{L_x^2}^4) \\ & \leq C_\varepsilon \lambda^{1+\varepsilon} (2^k \lambda^{7/8})^{-1} \sum_{\mu} \sum_{\nu \in \tau_\mu^\theta} \|A_\nu^{\theta_0} h\|_{L_x^2}^4 + O(\lambda^{-N} \|h\|_{L_x^2}^4) \\ & \leq C_\varepsilon \lambda^{1+\varepsilon} (2^k \lambda^{7/8})^{-1} \|h\|_{L_x^2}^4 + O(\lambda^{-N} \|h\|_{L_x^2}^4). \end{aligned}$$

In the above we used the fact that for each $\tau_{\mu_1}^\theta$ there are $O(1)$ cubes $\tau_{\tilde{\mu}_1}^{c_0\theta}$ with $\tau_{\tilde{\mu}_1}^{c_0\theta} \cap \bar{\tau}_{\mu_1}^\theta \neq \emptyset$ and $O(1)$ $\tau_{\mu_2}^\theta$ with $\tau_{\mu_1}^\theta \sim \tau_{\mu_2}^\theta$ and we also used (2.33).

This completes the proof of (5.14). So we conclude that we have reduced the proof of Lemma 5.2 to proving Proposition 5.3.

5.2. Proof of Proposition 5.3.

Let us collect some facts about the kernels of the operators $\tilde{\sigma}_\lambda A_\nu^{c_0\theta}$ in (5.33) that we shall use. As we shall shortly see they are highly concentrated near certain geodesics in M . Recall that $A_\nu^{c_0\theta}(x, D)$ is a “directional operator” with $\nu \in c_0\theta \cdot \mathbb{Z}^{2(n-1)}$ and, by (2.29), symbol $A_\nu^{c_0\theta}(x, \xi)$ highly concentrated near a unit speed geodesic

$$(5.42) \quad \gamma_\nu(s) = (x_\nu(s), \xi_\nu(s)) \in S^*\Omega.$$

Since γ_ν is of unit speed, we have $d_g(x_\nu(s), x_\nu(s')) = |s - s'|$.

To state the properties of the kernels $K_\nu^{c_0\theta}(x, y)$ of the operators $\tilde{\sigma}_\lambda A_\nu^{c_0\theta}$, as in earlier works, it is convenient to work in Fermi normal coordinates about the spatial geodesic $\bar{\gamma}_\nu = \{x_\nu(s)\}$. In these coordinates the geodesic becomes part of the last coordinate axis, i.e., $(0, \dots, 0, s)$ in $\mathbb{R}^n$, with, as in the earlier construction of the symbols of the $A_\nu^{c_0\theta}$, s being close to 0. For the remainder of this section we shall let $x = (x_1, \dots, x_n)$ denote these Fermi normal coordinates about our geodesic $\bar{\gamma}_\nu$ associated with $A_\nu^{c_0\theta}$. We then have

$$(5.43) \quad d_g((0, \dots, 0, x_n), (0, \dots, 0, y_n)) = |x_n - y_n|,$$

and, moreover, on $\overline{\gamma}_\nu$ we have that the metric is just $g_{jk}(x) = \delta_j^k$ if $x = (0, \dots, 0, x_n)$, and, additionally, all of the Christoffel symbols vanish there as well.

It also follows that the symbols $A_\mu^{c_0\theta}(x, \xi)$ of $A_\mu^{c_0\theta}$, $\mu = \nu, \nu'$ satisfy for some fixed C_1

$$(5.44) \quad |\partial_{x_n}^j \partial_{x'}^\alpha \partial_{x_n}^k \partial_{\xi_n}^\ell \partial_{\xi'}^\beta A_\mu^{c_0\theta}(x, \xi)| \lesssim_{c_0} \theta^{-|\alpha|-|\beta|} \lambda^{-|\beta|-\ell}, \quad \text{and} \quad A_\mu^{c_0\theta}(x, \xi) = 0$$

if $d_g(x, \overline{\gamma}_\mu) \geq C_1 c_0 \theta$, $\xi_n < 0$, $|\xi'/|\xi|| \geq C_1 \theta$, or $|\xi/\lambda| \notin [C_1^{-1}, C_1]$, $\mu = \nu, \nu'$,

with, as before, $\xi' = (\xi_1, \dots, \xi_{n-1})$. Additionally,

$$(5.45) \quad A_\nu^{c_0\theta}(x, \xi) = 0 \quad \text{if} \quad |\xi'/|\xi|| \geq C_1 c_0 \theta, \quad \text{and} \quad \Phi_t(0, \eta) = (t\eta, \eta),$$

if $\eta = (0, \dots, 0, 1)$, with, as before, Φ_t being geodesic flow in $S^*\Omega$.

In what follows $c_0 > 0$ will be fixed later small enough, depending on (M, g) , so that we can apply Lee's [23] bilinear oscillatory integral estimates. As in (5.44), various constants in the inequalities we shall state depend on the constant c_0 that we shall eventually specify. Also, as before θ will always be taken to be larger than $\lambda^{-1/8}$; however, we may assume it is small compared to one by choosing the cutoff B in the definition of $\tilde{\sigma}_\lambda$ to have small support. Also, as above, $x' = (x_1, \dots, x_{n-1})$ refers to the first $(n-1)$ coordinates.

We can now formulate the properties of the kernels which we shall require.

Lemma 5.4. *Fix $0 < \delta \ll \frac{1}{2} \text{Inj } M$. Assume further that $\mu = \nu, \nu'$ are as in (5.32) and let $K_\mu^{c_0\theta}$ be the kernel of $\tilde{\sigma}_\lambda A_\mu^{c_0\theta}$. In the above coordinates if $c_0 \ll 1$ we have*

$$(5.46) \quad K_{\lambda, \mu}^{c_0\theta}(x, y) = \lambda^{\frac{n-1}{2}} e^{i\lambda d_g(x, y)} a_\mu(\lambda; x, y) + O(\lambda^{-N}), \quad \mu = \nu, \nu',$$

where

$$(5.47) \quad \left| \left(\frac{\partial}{\partial x_n} \right)^{m_1} \left(\frac{\partial}{\partial y_n} \right)^{m_2} D_{x, y}^\beta a_\mu \right| \leq C_{m_1, m_2, \beta} \theta^{-|\beta|}, \quad \mu = \nu, \nu'.$$

Furthermore, for small θ and c_0 there is a constant C_0 so that the above $O(\lambda^{-N})$ errors can be chosen so that the amplitudes have the following support properties: First, if $\overline{\gamma}_\nu$ denotes the projection onto M of the geodesic in (5.42) and $\overline{\gamma}_{\nu'}$ the one corresponding to ν'

$$(5.48) \quad a_\mu(\lambda; x, y) = 0 \quad \text{if} \quad d_g(x, \overline{\gamma}_\mu) + d_g(y, \overline{\gamma}_\mu) \geq C_0 c_0 \theta, \quad \mu = \nu, \nu',$$

and

$$(5.49) \quad a_\mu(\lambda; x, y) = 0 \quad \text{if} \quad |x'| + |y'| \geq C_0 \theta, \quad \mu = \nu, \nu'.$$

As well as, for small $\delta, \delta_0 > 0$ as in (2.2)

$$(5.50) \quad a_\mu(\lambda; x, y) = 0 \quad \text{if} \quad |d_g(x, y) - \delta| \geq 2\delta_0 \delta, \quad \text{or} \quad x_n - y_n < 0, \quad \mu = \nu, \nu'.$$

This lemma is just a small variation of Lemma 4.3 in [29] (see also Lemma 3.2 in [4]). We shall postpone its proof until the end of this section.

Let us describe some properties of the phase function

$$(5.51) \quad \varphi(x, y) = d_g(x, y)$$

of our kernels in (5.46). First, in addition to (5.43), since we are working in the above Fermi normal coordinates we have

$$(5.52) \quad \partial\varphi/\partial x_j, \partial\varphi/\partial y_j = 0, \quad j = 1, \dots, n-1, \quad \text{if} \quad x' = y' = 0.$$

Consequently, by the last part of (5.50)

$$(5.53) \quad \tilde{\varphi}(x, y) = \varphi(x, y) - (x_n - y_n)$$

vanishes to second order when $x' = y' = 0$ and the amplitude is nonzero. This means that if we use the parabolic scaling $(x', x_n) \rightarrow (\theta x', x_n)$ we have

$$(5.54) \quad D_{x,y}^\beta (\theta^{-2} \tilde{\varphi}(\theta x', x_n, \theta y', y_n)) = O_\beta(1) \text{ if } |x'|, |y'| = O(1).$$

By (5.47) we also have

$$(5.55) \quad D_{x,y}^\beta a_\mu(\lambda; \theta x', x_n, \theta y', y_n) = O_\beta(1) \text{ if } |x'|, |y'| = O(1).$$

It also follows from Lemma 5.4 and a straightforward calculation that, in order to prove (5.33), it suffices to show that

$$(5.56) \quad \|(T_1 f_1)(T_2 f_2)\|_{L^{q/2}} \lesssim_\varepsilon \lambda^{-\frac{2n}{q} + \varepsilon} \theta^{-\frac{2}{n+2}} \|f_1\|_2 \|f_2\|_2, \quad q = \frac{2(n+2)}{n},$$

where

$$\begin{aligned} (T_1 f_1)(x) &= \int e^{i\lambda \tilde{\varphi}(x,y)} a_\nu(\lambda; x, y) f_1(y) dy \\ (T_2 f_2)(x) &= \int e^{i\lambda \tilde{\varphi}(x,z)} a_{\nu'}(\lambda; x, z) f_2(z) dz. \end{aligned}$$

As we may, in (5.56) we are neglecting the $O(\lambda^{-N})$ error terms in Lemma 5.4. Also, as above, we clearly may replace φ by $\tilde{\varphi}$ since, by (5.53), the difference is linear in the last variable. Note also that by (5.49) we have

$$(T_1 f_1)(x) = (T_2 f_2)(x) = 0 \quad \text{if } |x'| \geq C_0 \theta.$$

Next, we note that in order to prove (5.56), by Minkowski's inequality and the Schwarz inequality, if we define the “frozen” bilinear oscillatory integral operators

$$(5.57) \quad (B_{\lambda, \nu, \nu'}^{y_n, z_n})(h_1, h_2)(x) = \iint e^{i\lambda(\tilde{\varphi}(x, y', y_n) + \tilde{\varphi}(x, z', z_n))} a_\nu(\lambda; x, y', y_n) a_{\nu'}(\lambda; x, z', z_n) h_1(y') h_2(z') dy' dz',$$

then it suffices to prove that

$$(5.58) \quad \|B_{\lambda, \nu, \nu'}^{y_n, z_n}(h_1, h_2)\|_{L^{q/2}(\{x: |x'| \leq C_0 \theta\})} \lesssim_\varepsilon \lambda^{-\frac{2n}{q} + \varepsilon} \theta^{-\frac{2}{n+2}} \|h_1\|_2 \|h_2\|_2.$$

We note that $B_{\lambda, \nu, \nu'}^{y_n, z_n}(h_1, h_2)$ factors as the product of two oscillatory integral operators involving the (x, y') variables. The two phase functions are

$$(5.59) \quad \phi_{y_n}(x, y') = \tilde{\varphi}(x, y', y_n) \quad \text{and} \quad \phi_{z_n}(x, z') = \tilde{\varphi}(x, z', z_n).$$

In order to apply Lee's [23] bilinear oscillatory integral estimates when $n \geq 3$ or Hörmander's [19] when $n = 2$ we need another simple consequence of Lemma 5.4 which gives us key separation properties of the supports of the amplitudes.

Lemma 5.5. *Let $\delta < 1/8$ in (2.2) be given. Then we can fix c_0 as in (5.30) so that there are constants $c_\delta, C_\delta \in (0, \infty)$ so that for sufficiently small θ and $|x'| \leq C_0 \theta$, with C_0 as in (5.49) we have*

$$(5.60) \quad \text{if } a_\nu(\lambda; x, y) \cdot a_{\nu'}(\lambda; x, z) \neq 0 \text{ then } |y'|, |z'| \leq C_\delta \theta \quad \text{and} \quad |y' - z'| \geq c_\delta \theta.$$

Additionally, for sufficiently small θ we have

$$(5.61) \quad \text{if } a_\mu(\lambda, x, y) \neq 0 \text{ then } |\delta - (x_n - y_n)| \leq 4\delta_0\delta, \mu = \nu, \nu'.$$

Proof. The first assertion in (5.60) follows trivially from (5.49). To see the other part, we note that by (5.48) if the product of the amplitudes in (5.60) is nonzero then we must have, for a fixed constant C_1 , $x \in \mathcal{T}_{C_1 c_0 \theta}(\overline{\gamma}_\nu) \cap \mathcal{T}_{C_1 c_0 \theta}(\overline{\gamma}_{\nu'})$, $y \in \mathcal{T}_{C_1 c_0 \theta}(\overline{\gamma}_\nu)$ and $z \in \mathcal{T}_{C_1 c_0 \theta}(\overline{\gamma}_{\nu'})$. By (5.50), we must also have that $d_g(x, y), d_g(x, z) \in [\delta - 2\delta_0\delta, \delta + 2\delta_0\delta]$ for our small $\delta_0 > 0$. Since we are assuming (5.32) the tubes of width $\approx c_0\theta$ intersect at angle $\approx \theta$, which implies that $|y' - z'| \approx \theta$ if the product in (5.60) is nonzero and c_0 and θ are small.

The other, assertion, (5.61) just follows from (5.49) and (5.50) if θ is small enough. $\square$

We have collected the main ingredients that will allow us to prove the bilinear oscillatory integral estimates (5.58), which will complete the proof of Proposition 5.3.

To prove (5.58), in addition to following the proof of [23][Theorem 1.3], we shall also follow the related arguments in [4] which proved analogous bilinear estimates for $n = 2$ using the simpler classical bilinear oscillatory integral estimates implicit in Hörmander [19].

Just as in [23] we first perform a parabolic scaling as in (5.54) and (5.55) to be able to apply the main estimate, Theorem 1 in Lee [23]. So, for small $\lambda^{-1/8} \leq \theta \ll 1$, we let

$$(5.62) \quad \phi_{y_n}^\theta(x', x_n, y') = \theta^{-2} \tilde{\varphi}(\theta x', x_n, \theta y', y_n) \quad \text{and} \quad \phi_{z_n}^\theta = \theta^{-2} \tilde{\varphi}(\theta x, x_n, \theta z', z_n),$$

and corresponding amplitudes

$$(5.63) \quad a_\nu^\theta(\lambda; x, y) = a_\nu(\theta x', x_n, \theta y', y_n) \quad \text{and} \quad a_{\nu'}^\theta(\lambda; x, z) = a_{\nu'}(\theta x', x_n, \theta z', z_n).$$

Then, as we noted before

$$D_{x,y}^\beta a_\mu^\theta = O_\beta(1), \mu = \nu, \nu' \quad \text{and} \quad D_{x,y}^\beta \phi_j = O_\beta(1), \phi_1 = \phi_{y_n}^\theta, \phi_2 = \phi_{z_n}^\theta.$$

By Lemma 5.5 we also have the key separation properties for small enough θ

$$(5.64) \quad \text{if } a_\nu^\theta(\lambda; x, y) a_{\nu'}^\theta(\lambda; x, z) \neq 0 \\ \text{then } |y'|, |z'| = O(1), |y' - z'| \geq c_\delta \quad \text{and} \quad |y_n - z_n| \leq 8\delta_0\delta,$$

with δ and δ_0 as in (2.2).

Additionally, by a simple scaling argument, our remaining task, (5.58) is equivalent to the following bounds for small enough θ :

$$(5.65) \quad \|B_{\lambda, \nu, \nu'}^{\theta, y_n, z_n}(h_1, h_2)\|_{L^{q/2}(\{x: |x'| \leq C_0\})} \lesssim_\varepsilon (\lambda \theta^2)^{-\frac{2n}{q} + \varepsilon} \|h_1\|_2 \|h_2\|_2, \quad q = \frac{2(n+2)}{n},$$

where we have the scaled version of (5.58)

$$(5.66) \quad B_{\lambda, \nu, \nu'}^{\theta, y_n, z_n}(h_1, h_2)(x) = \\ \iint e^{i(\lambda \theta^2)[\phi_{y_n}^\theta(x, y') + \phi_{z_n}^\theta(x, z')]} a_\nu^\theta(\lambda; x, y) a_{\nu'}^\theta(\lambda; x, z) h_1(y') h_2(z') dy' dz'.$$

To prove this, let us see how we can use our earlier observation based on (5.52) and (5.50) that $\tilde{\varphi}$ vanishes to second order when $(x', y') = (0, 0)$ to see that the scaled phase functions in (5.66) closely resemble Euclidean ones if θ is small which will allow us to

verify the hypotheses in Lee's bilinear oscillatory integral theorem [23][Theorem 1.3] if $\delta, \delta_0 > 0$ in (2.2) are fixed small enough.

To do this, consider the following $(n-1) \times (n-1)$ Hessians

$$(5.67) \quad A(x_n, y_n) = \frac{\partial^2 \tilde{\varphi}}{\partial y'_j \partial y'_k}(0, x_n, 0, y_n), \quad B(x_n, y_n) = \frac{\partial^2 \tilde{\varphi}}{\partial x'_j \partial y'_k}(0, x_n, 0, y_n),$$

$$\text{and } C(x_n, y_n) = \frac{\partial^2 \tilde{\varphi}}{\partial x'_j \partial x'_k}(0, x_n, 0, y_n).$$

Then the Taylor expansion about $(x', y') = (0, 0)$ is

$$(5.68) \quad \tilde{\varphi}(x', x_n, y', y_n) = \frac{1}{2}(y')^t A(x_n, y_n) y' + (x')^t B(x_n, y_n) y' + \frac{1}{2}(x')^t C(x_n, y_n) x' + r(x', x_n, y', y_n),$$

where $r(x', x_n, y', y_n)$ vanishes to third order at $(x', y') = (0, 0)$ and so

$$(5.69) \quad D_{x,y}^\beta r^\theta(x, y) = O(\theta), \quad \text{if } r^\theta(x', x_n, y', y_n) = \theta^{-2} r(\theta x', x_n, \theta y', y_n).$$

This means that $r^\theta \rightarrow 0$ in the C^∞ topology as $\theta \rightarrow 0$.

To utilize (5.68) we shall use parabolic scaling and the following standard lemma (c.f. [29, §5.1]) saying that the phase functions that arise satisfy the Carleson-Sjölin condition.

Lemma 5.6. *Let $A(x_n, y_n)$ and $B(x_n, y_n)$ be as in (5.67). Then if $\delta, \delta_0 > 0$ in (2.2) are small enough*

$$(5.70) \quad \det B(x_n, y_n) = \det \frac{\partial^2 \tilde{\varphi}(0, x_n, 0, y_n)}{\partial x'_j \partial y'_k} \neq 0 \quad \text{if } a_\nu^\theta \cdot a_{\nu'}^\theta \neq 0.$$

Also, on the support of $a_\nu^\theta \cdot a_{\nu'}^\theta$, $-(\frac{\partial}{\partial x_n} A(x_n, y_n))^{-1} = -(\frac{\partial}{\partial x_n} \frac{\partial^2 \varphi}{\partial y'_j \partial y'_k}(0, x_n, 0, y_n))^{-1}$ is positive definite, i.e.,

$$(5.71) \quad \xi^t \left(-\frac{\partial}{\partial x_n} A(x_n, y_n) \right)^{-1} \xi, \quad \xi^t \left(-\frac{\partial}{\partial x_n} A(x_n, z_n) \right)^{-1} \xi \geq c_\delta |\xi|^2 \quad \text{if } a_\nu^\theta \cdot a_{\nu'}^\theta \neq 0,$$

and also

$$(5.72) \quad \left| \frac{\partial}{\partial x_n} A(x_n, y_n) \xi \right| \geq c_\delta |\xi|, \quad \left| \frac{\partial}{\partial x_n} A(x_n, z_n) \xi \right| \geq c_\delta |\xi|,$$

for some $c_\delta > 0$.

Proof. Recall that by (5.51) and (5.53) $\tilde{\varphi}(x, y) = d_g(x, y) - (x_n - y_n)$. As a result,

$$A(x_n, y_n) = \frac{\partial^2}{\partial y'_j \partial y'_k} d_g(0, x_n, 0, y_n) \quad \text{and} \quad B(x_n, y_n) = \frac{\partial^2}{\partial x'_j \partial y'_k} d_g(0, x_n, 0, y_n).$$

Since we are working in Fermi normal coordinates we have $d_g(x, y) = |x - y| + O(|x - y|^2)$ if $x' = 0$. From this we deduce that

$$B(x_n, y_n) = -(x_n - y_n)^{-1} I_{n-1} + O(1),$$

which yields (5.70) if $\delta, \delta_0 > 0$ in (2.2) are small since then $d_g(x, y) \approx \delta$ on the support of the amplitudes. Since we similarly have

$$\frac{\partial}{\partial x_n} A(x_n, y_n) = -(x_n - y_n)^{-2} I_{n-1} + O(|x_n - y_n|^{-1}),$$

we similarly obtain (5.71) and (5.72) if δ, δ_0 are small. $\square$

Let us use (5.68) and (5.69) and this lemma to prove our remaining estimate (5.65) using the estimate [23][Theorem 1.1] of Lee. As we shall see, it is crucial for us that $-\frac{\partial}{\partial x_n}A(x_n, y_n)$ is positive definite.

Note that, in addition to the θ parameter, (5.65) also involves the (y_n, z_n) parameters. For simplicity, let us first see how Lee's result yields (5.65) in the case where these two parameters agree, i.e., $y_n = z_n$. We then will argue that if δ_0 in (2.2) and hence (5.64) is fixed small enough we can also handle the case where $y_n \neq z_n$ due to the fact that Lee's estimates are valid under small perturbations.

To do this, we first note that the parabolic scaling in (5.69), which agrees with that in (5.62), preserves the first three terms in the right side of (5.68) as they are quadratic. Also, in proving (5.65), we may subtract $\frac{1}{2}(x')^t C(x_n, y_n)x'$ from $\phi_{y_n}^\theta$ and $\frac{1}{2}(x')^t C(x_n, z_n)x'$ from $\phi_{z_n}^\theta$ as these quadratic terms do not involve y' . We point out that this trivial reduction also works if $y_n \neq z_n$.

Next, note that by (5.70) and our temporary assumption that $y_n = z_n$, after making a linear change of variables depending on (x_n, y_n) , we may reduce to the case where $B(x_n, y_n) = I_{n-1}$, the $(n-1) \times (n-1)$ identity matrix. This means that for the special case where $y_n = z_n$ we have reduced to verifying that (5.65) is valid where now

$$(5.73) \quad \begin{aligned} \phi_{y_n}^\theta(x', x_n; y') &= \langle x', y' \rangle + \frac{1}{2} \sum_{j,k=1}^{n-1} \frac{\partial^2 \tilde{\varphi}}{\partial y'_j \partial y'_k}(0, x_n, 0, y_n) y'_j y'_k + \tilde{r}^\theta(x', x_n, y', y_n) \\ &= \langle x', y' \rangle + (y')^t A(x_n, y_n) y' + \tilde{r}^\theta(x', x_n, y', y_n), \end{aligned}$$

with $\tilde{r}^\theta$ denoting r^θ written in the new x variables coming from $B(x_n, y_n)$. For later use, note that if we change variables according to y_n as above, then for z_n near y_n if

$$(5.74) \quad B(x_n, y_n, z_n) = (B(x_n, z_n))^t ((B(x_n, y_n))^{-1})^t = I_{n-1} + O(|y_n - z_n|),$$

then

$$(5.75) \quad \begin{aligned} \phi_{z_n}^\theta(x, z') &= \langle x', B(x_n, y_n, z_n) y' \rangle + \frac{1}{2} \sum_{j,k=1}^{n-1} \frac{\partial^2 \tilde{\varphi}}{\partial y'_j \partial y'_k}(0, x_n, 0, z_n) + \tilde{r}^\theta(x, z) \\ &= \phi_{y_n}^\theta(x, z') + O(|y_n - z_n|). \end{aligned}$$

We fix δ and δ_0 in (2.2) so that the conclusions of Lemma 5.4 and 5.6 are valid. We can also finally fix c_0 so that the results in Lemma 5.5 are valid. If we only needed to handle the case where $y_n = z_n$ then the above choice of δ_0 would work; however, as we shall see, to handle the case where $y_n \neq z_n$ we shall need to choose δ_0 small enough to exploit the last part of (5.64).

Let us now verify that we can apply [23, Theorem 1.1] to obtain (5.65) for sufficiently small θ . This would complete the proof of Proposition 5.3.

We recall that we are assuming for the moment that $y_n = z_n$ and that we have reduced matters to the case where $B(x_n, y_n) = I_{n-1}$ and $C(x_n, y_n) = 0$ in (5.68) and so

$$(5.76) \quad \phi_{y_n}^\theta(x, y') = \langle x', y' \rangle + \frac{1}{2}(y')^t A(x_n, y_n) y' + \tilde{r}^\theta(x, y),$$

with $\tilde{r}^\theta$ satisfying (5.69).

By (5.69) and (5.76) we have

$$(5.77) \quad \frac{\phi_{y_n}^\theta}{\partial x'}(x, y') = y' + \frac{\partial \tilde{r}^\theta}{\partial x'} = y' + \varepsilon(\theta, x, y),$$

where $y' \rightarrow \varepsilon(\cdot)$ and its derivatives are $O(\theta)$. Thus, for small enough θ , the inverse function also satisfies

$$(5.78) \quad y' \rightarrow \left(\frac{\partial \phi_{y_n}^\theta}{\partial x'}(x', x_n, \cdot) \right)^{-1}(y') = y' + \tilde{\varepsilon}(\theta, x, y),$$

where

$$(5.79) \quad D_{y'}^\beta \tilde{\varepsilon}(\theta, x, y) = O_\beta(\theta).$$

Next, define in the notation of [23]

$$(5.80) \quad q_s^\theta(x', x_n, y') = \frac{\partial}{\partial x_n} \phi_s^\theta(x', x_n; \left(\frac{\partial \phi_{y_n}^\theta}{\partial x'}(x', x_n, \cdot) \right)^{-1}(y')) \\ = \frac{\partial}{\partial x_n} \phi_s^\theta(x', x_n, y' + \tilde{\varepsilon}(\theta, x, y', s)), \quad s = y_n, z_n,$$

as well as

$$(5.81) \quad \delta_{y_n, z_n}^\theta(x', x_n; y', z') = \\ \partial_{y'} q_{y_n}^\theta(x', x_n; \partial_{x'} \phi_{y_n}^\theta(x', x_n, y')) - \partial_{y'} q_{z_n}^\theta(x', x_n; \partial_{x'} \phi_{z_n}^\theta(x', x_n, z')).$$

Even though we are assuming for now that $y_n = z_n$ these two quantities will be needed for $y_n \neq z_n$ as well to be able to allow us to use [23, Theorem 1.1] to obtain (5.65). The conditions [23, (1.4)] needed to ensure these bounds are

$$(5.82) \quad |\langle \partial_{x' y'}^2 \phi_{y_n}^\theta(x, y') \delta_{y_n, z_n}^\theta, [\partial_{x' y'}^2 \phi_{y_n}^\theta(x, y')]^{-1} [\partial_{y' y'}^2 q_{y_n}^\theta(x; \partial_{x'} \phi_{y_n}^\theta(x, y'))]^{-1} \delta_{y_n, z_n}^\theta \rangle| > 0, \\ \delta_{y_n, z_n}^\theta = \delta_{y_n, z_n}^\theta(x', x_n; y', z') \text{ on } \text{supp}(a_\nu^\theta \cdot a_{\nu'}^\theta),$$

as well as

$$(5.83) \quad |\langle \partial_{x' y'}^2 \phi_{z_n}^\theta(x, z') \delta_{y_n, z_n}^\theta, [\partial_{x' y'}^2 \phi_{z_n}^\theta(x, z')]^{-1} [\partial_{y' y'}^2 q_{z_n}^\theta(x; \partial_{x'} \phi_{z_n}^\theta(x, z'))]^{-1} \delta_{y_n, z_n}^\theta \rangle| > 0, \\ \delta_{y_n, z_n}^\theta = \delta_{y_n, z_n}^\theta(x', x_n; y', z') \text{ on } \text{supp}(a_\nu^\theta \cdot a_{\nu'}^\theta),$$

By (5.69), (5.73), (5.78), (5.79) and (5.80) for small θ we have

$$(5.84) \quad (\partial_{y' y'}^2 q_{y_n}^\theta(x', x_n; y'))^{-1} = \left(\frac{\partial A}{\partial x_n}(x_n, y_n) \right)^{-1} + O(\theta),$$

and also by (5.77), (5.78) and (5.79)

$$(5.85) \quad \partial_{x' y'}^2 \phi_{y_n}^\theta(x', x_n, y') = I_{n-1} + O(\theta),$$

as well as

$$(5.86) \quad (\partial_{x' y'}^2 \phi_{y_n}^\theta(x', x_n, y'))^{-1} = I_{n-1} + O(\theta),$$

By (5.73), (5.72), (5.80) and the separation condition in (5.60) if $y_n = z_n$ we have

$$(5.87) \quad |\delta_{y_n, z_n}^\theta(x', x_n; y', z')| > 0 \text{ on } \text{supp}(a_\nu^\theta \cdot a_{\nu'}^\theta),$$

if θ is small. Thus, in this case the quantities inside the absolute values in (5.82) and (5.83) both equal (5.88)

$$\langle \delta_{y_n, y_n}^\theta(x', x_n; y', z'), (\frac{\partial A}{\partial x_n}(x_n, y_n))^{-1} \delta_{y_n, y_n}^\theta(x', x_n; y', z') \rangle + O(\theta) \text{ on supp } (a_\nu^\theta \cdot a_{\nu'}^\theta).$$

Therefore, by (5.71) and (5.87) the conditions (5.82) and (5.83) are valid when $y_n = z_n$. Thus by [23, Theorem 1.1], we obtain (5.58) in this case.

If $y_n \neq z_n$ in (5.58), we must replace δ_{y_n, y_n}^θ by δ_{y_n, z_n}^θ . In order to accommodate this, we first need to use that, by the last part of (5.61)

$$\delta_{y_n, z_n}^\theta(x', x_n; y', z') = \delta_{y_n, y_n}^\theta(x', x_n; y', z') + O(\delta_0) \text{ on supp } (a_\nu^\theta \cdot a_{\nu'}^\theta).$$

This means that if we replace $O(\theta)$ by $O(\theta + \delta_0)$ in (5.88), then the quantity in (5.82) is of this form.

The other condition, (5.83) involves the phase function $\phi_{z_n}^\theta$ and the corresponding $q_{z_n}^\theta$. However, if $B = B(x_n, y_n, z_n)$ is as in (5.74), then we have the analog of (5.77) where we replace the first term the right by By' and the analog of (5.78) where we replace the first term in the right side by $B^{-1}y'$. Also, clearly $\frac{\partial A}{\partial x_n}(x_n, z_n) = \frac{\partial A}{\partial x_n}(x_n, y_n) + O(|y_n - z_n|)$. As a result $q_{z_n}^\theta = q_{y_n}^\theta + O(|y_n - z_n|) = q_{y_n}^\theta + O(\delta_0)$ if $a_\nu^\theta \cdot a_{\nu'}^\theta \neq 0$. Also, by (5.74) the analogs of (5.85) and (5.86) remain valid if y_n is replaced by z_n provided that $O(\theta)$ there is replaced by $O(\theta + \delta_0)$. So, like (5.82), if we replace $O(\theta)$ by $O(\theta + \delta_0)$ in (5.88), then the quantity in (5.83) is of this form.

Consequently, if δ_0 in (2.2) is fixed small enough, and, as above, θ is small, we conclude that the condition (1.4) in [23] is valid, which yields (5.57) and thus completes the proof of Proposition 5.3.

5.3. Proof of Lemma 5.4. To finish matters we need to prove the properties of the microlocalized kernels that we used.

Proof of Lemma 5.4. The straightforward proof is almost identical to that of Lemma 3.2 in [4] or Lemma 4.3 in [7]; however, we shall present it for the sake of completeness. Note note that when $\theta \approx 1$, this result is standard. See, e.g., Lemma 4.3 in [29], and the proof of our results are just a small variation on that of this standard one.

We recall that

$$(5.89) \quad \tilde{\sigma}_\lambda = (2\pi)^{-1} \int e^{i\lambda t} (B \circ e^{-itP}) \hat{\rho}(t) dt.$$

Since $(Be^{-itP})(x, y)$ is smooth near (x, y, t) if $d_g(x, y) \neq |t|$ and $\hat{\rho}(t) = 0$ for $|t - \delta| > \delta_0$, we clearly have the first part of (5.50). The second part similarly comes from the fact that by (5.44) the symbol of $A_\mu^{c_0\theta}$, $\mu = \nu, \nu'$, vanishes if ξ is not in a small conic neighborhood of $(0, \dots, 0, 1)$.

Recall $B \in S_{1,0}^0$ has symbol $B(x, \xi)$ vanishing when $|\xi|$ is not comparable to λ or when (x, ξ) is not in a small conic neighborhood of $(0, (0, \dots, 0, 1))$ if δ is small. Therefore, using the calculus of Fourier integrals for $|t| < 2\delta$ with δ as in (2.2), modulo smoothing

errors,

$$(5.90) \quad (Be^{-itP})(x, y) = (2\pi)^{-n} \int e^{iS(t, x, \xi) - iy \cdot \xi} \alpha(t, x, \xi) d\xi \\ = (2\pi)^{-n} \lambda^n \int e^{i\lambda(S(t, x, \xi) - y \cdot \xi)} \alpha(t, x, \lambda\xi) d\xi,$$

where $\alpha \in S_{1,0}^0$ also vanishes when $|\xi|$ is not comparable to λ or (x, ξ) is not in a small conic neighborhood of $(0, (0, \dots, 0, 1))$. Also, the phase function here S is homogeneous of degree one in ξ and is a generating function for the half-wave group e^{-itP} . Thus, S solves the eikonal equation,

$$(5.91) \quad \partial_t S(t, x, \xi) = -p(x, \nabla_x S(t, x, \xi)), \quad S(0, x, \xi) = x \cdot \xi,$$

and, if Φ_t here denotes the Hamilton flow in $T^*M \setminus 0$ associated with $p(x, \xi)$,

$$(5.92) \quad \Phi_{-t}(x, \nabla_x S) = (\nabla_\xi S, \xi),$$

and

$$(5.93) \quad \det \left(\frac{\partial^2 S}{\partial x \partial \xi} \right) \neq 0.$$

Additionally, by the above facts regarding $\alpha \in S_{1,0}^0$, for $|t| < 2\delta$ we have

$$(5.94) \quad \partial_t^j \partial_{x, \xi}^\beta \alpha(t, x, \lambda\xi) = O(1) \quad \text{and} \quad \alpha(t, x, \lambda\xi) = 0 \quad \text{if} \quad |\xi| \notin [C^{-1}, C]$$

for some uniform constant C .

By (5.90) and (5.91), we have

$$(5.95) \quad \tilde{\sigma}_\lambda(x, y) = (2\pi)^{-n-1} \lambda^n \iint e^{i\lambda[t + S(t, x, \xi) - y \cdot \xi]} \hat{\rho}(t) \alpha(t, x, \lambda\xi) d\xi dt + O(\lambda^{-N}).$$

This implies that

$$(5.96) \quad K_{\lambda, \mu}^{c_0 \theta}(x, y) \\ = (2\pi)^{-2n-1} \lambda^{2n} \iint e^{i\lambda[t + S(t, x, \xi) - z \cdot \xi + (z - y) \cdot \eta]} \hat{\rho}(t) \alpha(t, x, \lambda\xi) A_\mu^{c_0 \theta}(z, \lambda\eta) dz d\xi d\eta dt \\ + O(\lambda^{-N}), \quad \mu = \nu, \nu'.$$

By (5.45) and a simple integration by parts argument we have $K_{\lambda, \mu}^{c_0 \theta}(x, y) = O(\lambda^{-N})$ if $d_g(y, \overline{\gamma}_\mu) \geq C_1 c_0 \theta$, $\mu = \nu, \nu'$.

If $\mu = \nu$, let us prove that this is the case also if $d_g(x, \overline{\gamma}_\nu) \geq C_1 c_0 \theta$, for large enough C_1 . To so, we note that by (5.92) and the fact that we are working in Fermi normal coordinates about $\overline{\gamma}_\nu$, we have

$$\nabla_\xi(S(t_0, x_0, \xi) - z_0 \cdot \xi) = 0,$$

$$\text{if } \xi = (0, \dots, 0, \xi_n), \quad \xi_n > 0, \quad x_0, z_0 \in \overline{\gamma}_\nu \quad \text{and} \quad x_0 - z_0 = t_0(0, \dots, 0, 1), \quad t_0 \approx \delta.$$

Note that for such x_0, z_0 we have $t_0 = d_g(x_0, z_0)$. By (5.93) we have for $t_0 \approx \delta$ and $z_0 \in \overline{\gamma}_\nu$ and x near x_0

$$|\nabla_\xi(S(t_0, x, \xi) - z_0 \cdot \xi)| \approx d_g(x, x_0), \quad \text{if } \xi = (0, \dots, 0, \xi_n), \quad \xi_n > 0.$$

By (5.44) and (5.45) this implies that we must have for $t_0 \approx \delta$

$$(5.97) \quad |\nabla_{\xi, \eta, z}(S(t_0, x, \xi) - z \cdot \xi + (z - y) \cdot \eta)| \geq c\theta, \\ \text{if } d_g(x, \bar{\gamma}_\nu) \geq C_1 c_0 \theta, \text{ and } \alpha(t, x, \lambda \xi) A_\nu^{c_0 \theta}(z, \lambda \eta) \neq 0,$$

for some constant $c > 0$ if C_1 is fixed large enough. Since $\theta \geq \lambda^{-1/8}$, we obtain (5.48) for $\mu = \nu$ from this and (5.96) via a simple integration by parts argument. We similarly obtain (5.48) for $\mu = \nu'$ if we work in Fermi normal coordinates about $\bar{\gamma}_{\nu'}$. We obtain (5.49) from (5.48) since we are assuming that $\nu - \nu' = O(1)$ which forces the $O(c_0 \theta)$ tubes described in (5.48) about $\bar{\gamma}_\nu$ and $\bar{\gamma}_{\nu'}$ to be a $O(\theta)$ distance apart, and $(x_1, \dots, x_{n-1}) = 0$ on $\bar{\gamma}_\nu$.

It remains to prove that the kernels are as in (5.46) with amplitudes satisfying (5.47). Let

$$\Psi(t, x, y, z, \xi, \eta) = t + S(t, x, \xi) - z \cdot \xi + (z - y) \cdot \eta$$

be the phase function in the oscillatory integral in (5.96). Then, at a stationary point where

$$\nabla_{z, \xi, \eta, t} \Psi = 0,$$

we must have $y = z$ and hence $\Psi = d_g(x, y)$, due to the fact that $S(t, x, \xi) - z \cdot \xi = 0$ and, as we just pointed out, $t = d_g(x, y)$ at points where the ξ -gradient vanishes. Additionally, it is straightforward to see that

$$\det \frac{\partial^2 \Psi}{\partial(\xi, t) \partial(\xi, t)} \neq 0.$$

This follows from the proof of Lemma 5.1.3 in [29]. It also clearly implies that the $(3n+1) \times (3n+1)$ Hessian of the phase function in (5.96) satisfies

$$\det \frac{\partial^2 \Psi}{\partial(z, \xi, \eta, t) \partial(z, \xi, \eta, t)} \neq 0.$$

Also, considering the z -gradient, we have $\xi = \eta$ at stationary points. Thus, by (5.92), the oscillatory integral in (5.96) has an expansion (see Hörmander [20, Theorem 7.7.5]) where the leading term is a dimensional constant times

$$(5.98) \quad \lambda^{\frac{n-1}{2}} e^{i\lambda t} \hat{\rho}(t) \alpha(t, x, \lambda \xi) A_\mu(y, \lambda \xi), \quad \text{if } t = d_g(x, y), \quad p(x, \nabla_x S(t, x, \xi)) = 1, \\ \text{and } \Phi_{-t}(x, \zeta) = (y, \xi), \text{ with } \zeta = \nabla_x S(t, x, \xi) \text{ and } y = \nabla_\xi S(t, x, \xi).$$

Thus, here $\xi = \xi(x, y) \in S_y^* \Omega$ is the unit covector over y of the unit-speed geodesic in $S^* \Omega$ which passes through (x, ζ) at time $t = d_g(x, y)$, $t \in \text{supp } \hat{\rho}$, and starts at (y, ξ) .

Consequently, since we are working in Fermi normal coordinates about $\bar{\gamma}_\nu$, it follows that $\xi((0, \dots, 0, x_n), (0, \dots, 0, y_n)) \equiv (0, \dots, 0, 1)$ when $(x_1, \dots, x_{n-1}) = (y_1, \dots, y_{n-1}) = 0$. Consequently, we have $\partial_{x_n}^j \partial_{y_n}^k \xi_\ell(x, y) = O(\theta)$, $\ell = 1, \dots, n-1$, if the kernels are not $O(\lambda^{-N})$. Therefore, it follows from (5.44) that

$$\hat{\rho}(d_g(x, y)) \alpha(d_g(x, y), x, \lambda \xi(x, y)) A_\mu^{c_0 \theta}(y, \lambda \xi(x, y))$$

satisfies the bounds in (5.47) if (x, y) are not in the regions described in (5.48), (5.49) or (5.50) where the kernels are $O(\lambda^{-N})$. Thus, the leading term in the stationary phase expansions for the oscillatory integrals in (5.96) have the desired form. The same will be

true for the other terms which involve increasing powers of $\lambda^{-3/4}$ by a straightforward variant of [20, (7.7.1)]. $\square$

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