

PRODUCT MANIFOLDS WITH IMPROVED SPECTRAL CLUSTER AND WEYL REMAINDER ESTIMATES

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ABSTRACT. We show that if Y is a compact Riemannian manifold with improved L^q eigenfunction estimates then, at least for large enough exponents, one always obtains improved L^q bounds on the product manifold $X \times Y$ if X is another compact manifold. Similarly, improved Weyl remainder term bounds on the spectral counting function of Y lead to corresponding improvements on $X \times Y$. The latter results partly generalize recent ones of Iosevich and Wyman [14] involving products of spheres. Also, if Y is a product of five or more spheres, we are able to obtain optimal $L^q(Y)$ and $L^q(X \times Y)$ eigenfunction and spectral cluster estimates for large q , which partly addresses a conjecture from [14] and is related to (and is partly based on) classical bounds for the number of integer lattice point on $\lambda \cdot S^{n-1}$ for $n \geq 5$.

In memoriam: Robert Strichartz (1943-2021)

1. Introduction.

Spectral cluster estimates are operator norm estimates from L^2 to L^q of spectral projectors for the Laplace operator on a compact Riemannian manifold. In more detail, if X is a compact Riemannian manifold of dimension d_X , and Laplace-Beltrami operator Δ_X , the universal estimate of [18] has the form

$$(1.1) \quad \left\| \mathbb{1}_{[\lambda-1, \lambda+1]}(P_X) \right\|_{L^2(X) \rightarrow L^q(X)} = O(\lambda^{\alpha(q)}), \quad P_X = \sqrt{-\Delta_X},$$

where, with $n = d_X$,

$$(1.2) \quad \begin{aligned} \alpha(q) &= \alpha(q, n) = \max\left(n\left(\frac{1}{2} - \frac{1}{q}\right) - \frac{1}{2}, \frac{n-1}{2}\left(\frac{1}{2} - \frac{1}{q}\right)\right) \\ &= \begin{cases} n\left(\frac{1}{2} - \frac{1}{q}\right) - \frac{1}{2}, & \text{if } q \geq q_c(n) = \frac{2(n+1)}{n-1}, \\ \frac{n-1}{2}\left(\frac{1}{2} - \frac{1}{q}\right), & \text{if } 2 \leq q \leq q_c(n). \end{cases} \end{aligned}$$

Much work has been done on the study of special classes of compact Riemannian manifolds for which stronger estimates hold. One of our goals here is to show that if Y is a compact Riemannian manifold, of dimension d_Y , with Laplace-Beltrami operator Δ_Y , for which such stronger results (cf. (1.3)) hold, and if X is an arbitrary compact Riemannian manifold, as described above, then, under broad circumstances, the product manifold $X \times Y$, of dimension $d = d_X + d_Y$, with product metric tensor and Laplace operator $\Delta = \Delta_X + \Delta_Y$, also has improved spectral cluster estimates.

2010 *Mathematics Subject Classification.* 58J50, 35P15.

Key words and phrases. Eigenfunctions, Weyl formula, spectrum.

The second author was supported in part by the NSF (DMS-1665373).

We formulate the improved spectral cluster estimates on Y as follows:

$$(1.3) \quad \left\| \mathbb{1}_{[\lambda-\varepsilon(\lambda), \lambda+\varepsilon(\lambda)]}(P_Y) \right\|_{L^2(Y) \rightarrow L^q(Y)} \lesssim \sqrt{\varepsilon(\lambda)} B(\lambda),$$

where $P_Y = \sqrt{-\Delta_Y}$, $B(\lambda) = B(\lambda, q, Y)$, and $2 < q \leq \infty$. Typically $B(\lambda)$ is λ raised to a power (see e.g. [22], [5], [11], and [14]) and possibly also involving $\log \lambda$ -powers (see e.g. [2], [4], [7], [10], [20]). So it is natural to assume that

$$(1.4) \quad B(\theta\lambda) \leq C_0 B(\lambda) \quad \text{if } \lambda^{-1} \leq \theta \leq 2.$$

As we shall see later in (2.17), the improved bound in (1.3) requires

$$(1.4') \quad B(\lambda) \gtrsim \lambda^{\alpha(q)}, \quad \forall \lambda \geq 1,$$

with $\alpha(q) = \alpha(q, d_Y)$ being the exponent in (1.2) depending on the dimension d_Y and q .

If $B(\lambda) = \lambda^{\alpha(q)}$, with $\alpha(q)$ being the exponent in the universal bounds that are due to the second author [16], [18] (see (1.2)), then (1.3) would represent a $\sqrt{\varepsilon(\lambda)}$ improvement over these bounds by projecting onto bands of size $\approx \varepsilon(\lambda)$ (as opposed to unit sized bands). We shall always assume that $\varepsilon(\lambda)$ decreases to 0 as $\lambda \rightarrow \infty$.

We prefer to express improvements including the factor $\sqrt{\varepsilon(\lambda)}$ since this would match with the bounds for the Stein-Tomas extension operators for $q \geq q_c$ (see §4). If $B(\lambda) = \lambda^{\alpha(q)}$ as in (1.2) and if $\varepsilon(\lambda)$ is very small, then they are only possible for certain q larger than the critical exponent $q_c = \frac{2(d_Y+1)}{d_Y-1}$. For instance, if $\varepsilon(\lambda) = \lambda^{-1}$ then one must have $q \geq \frac{2d_Y}{d_Y-2}$, since otherwise $\sqrt{\varepsilon(\lambda)} \lambda^{\alpha(q)} \rightarrow 0$ as $\lambda \rightarrow \infty$.

We shall also assume that $\varepsilon(\lambda)$ does not go to zero faster than the wavelength of eigenfunctions of frequency λ . More precisely, we shall assume that

$$(1.5) \quad t \rightarrow t\varepsilon(t) \quad \text{is non-decreasing for } t \geq 1.$$

We note that this is equivalent to the condition that

$$(1.5') \quad \theta \varepsilon(\theta\lambda) \leq \varepsilon(\lambda) \quad \text{if } \lambda \geq 1 \text{ and } \lambda^{-1} \lesssim \theta \leq 1.$$

Typically $\varepsilon(\lambda) = \lambda^{-\delta}$ for some $\delta \in (0, 1]$ or $\varepsilon(\lambda) = (\log \lambda)^{-\delta}$ for some $\delta > 0$.

Here is our first main result.

Theorem 1.1. *Assume that (1.3) is valid with $B(\lambda)$ and $\varepsilon(\lambda)$ satisfying (1.4)–(1.4') and (1.5), respectively. Then if $\Delta = \Delta_X + \Delta_Y$ is the Laplace-Beltrami operator on the product manifold and $P = \sqrt{-\Delta}$ we have for all $q > 2$*

$$(1.6) \quad \left\| \mathbb{1}_{[\lambda-\varepsilon(\lambda), \lambda+\varepsilon(\lambda)]}(P) \right\|_{L^2(X \times Y) \rightarrow L^q(X \times Y)} \lesssim \sqrt{\varepsilon(\lambda)} B(\lambda) \lambda^{\alpha(q, d_X)+1/2}, \quad \forall \lambda \geq 1.$$

As we shall see at the end of the next section, if (1.3) is an improvement on Y over the universal bounds in [18], then, at least for sufficiently large exponents, (1.6) says that there are improved L^q spectral projection estimates on $X \times Y$.

Our second main result deals with the Weyl law, for the spectral counting function of the Laplace operator. Recall that the universal Weyl formula of Avakumovic [1], Levitan [15] and Hörmander [12] states that if $N(X, \lambda)$ denotes the number of eigenvalues, counted with multiplicity, of P_X which are $\leq \lambda$ then

$$(1.7) \quad N(X, \lambda) = (2\pi)^{-d_X} \omega_{d_X} (\text{Vol } X) \lambda^{d_X} + O(\lambda^{d_X-1}),$$

with ω_n denoting the volume of the unit ball in $\mathbb{R}^n$.

This result cannot be improved if X is the round sphere of dimension d_X . On the other hand, there are a number of results that do yield improved Weyl remainder estimates. In [8] it is shown that one can improve $O(\lambda^{d_X-1})$ to $o(\lambda^{d_X-1})$ in case the set of periodic geodesics has measure zero. The paper [2] shows that under certain geometrical conditions, such as nonpositive curvature, one can improve the remainder estimate to $O(\lambda^{d_X-1}/\log \lambda)$. Recently, Canzani and Galkowski [6] obtained such an improved remainder estimate for a much broader class of Riemannian manifolds. Among the results obtained there is that one gets this $(\log \lambda)^{-1}$ improvement on each Cartesian product manifold, with the product metric. Iosevich and Wyman [14] showed there are power improvements for products of spheres.

As with the L^q improvements in Theorem 1.1, we shall assume that there are $\varepsilon(\lambda)$ improvements, with $\varepsilon(\lambda)$ as in (1.5)–(1.5'), for Y , and show that these carry over for $X \times Y$. So, we shall assume that

$$(1.8) \quad N(Y, \lambda) = (2\pi)^{-d_Y} \omega_{d_Y} (\text{Vol } Y) \lambda^{d_Y} + R_Y(\lambda), \quad \text{with } R_Y(\lambda) = O(\varepsilon(\lambda) \lambda^{d_Y-1}).$$

Here is our second main result.

Theorem 1.2. *Assume that (1.8) is valid. Then, for each compact Riemannian manifold, of dimension d_X ,*

$$(1.9) \quad N(X \times Y, \lambda) = (2\pi)^{-d} \omega_d \text{Vol}(X \times Y) \lambda^d + O(\varepsilon(\lambda) \lambda^{d-1}), \quad d = d_X + d_Y.$$

The structure of the rest of this paper is the following. In §2 we prove Theorem 1.1 and give applications. In §3 we prove Theorem 1.2. Section 4 presents some further results, including a study of multiple products of spheres.

2. Proof of L^q -improvements and some applications.

Let us now turn to the proof of Theorem 1.1. We first choose an orthonormal basis, $\{e_{\mu_i}^X\}$, of eigenfunctions of P_X with eigenvalues μ_i and $\{e_{\nu_j}^Y\}$ of P_Y with eigenvalues ν_j . Thus,

$$(2.1) \quad -\Delta_X e_{\mu_i}^X = \mu_i^2 e_{\mu_i}^X \quad \text{and} \quad -\Delta_Y e_{\nu_j}^Y = \nu_j^2 e_{\nu_j}^Y.$$

Then $\{e_{\mu_i}^X(x) \cdot e_{\nu_j}^Y(y)\}_{i,j}$ is an orthonormal basis of eigenfunctions of $P = \sqrt{-\Delta}$, where $\Delta = \Delta_X + \Delta_Y$, with eigenvalues $\sqrt{\mu_i^2 + \nu_j^2}$, i.e.,

$$(2.2) \quad -\Delta(e_{\mu_i}^X e_{\nu_j}^Y) = (\mu_i^2 + \nu_j^2) e_{\mu_i}^X e_{\nu_j}^Y.$$

Consequently, the first inequality, (1.6), is equivalent to the following

$$(2.3) \quad \left\| \sum_{(\mu_i, \nu_j) \in A_{\lambda, \varepsilon}} a_{ij} e_{\mu_i}^X e_{\nu_j}^Y \right\|_{L^q(X \times Y)} \lesssim B(\lambda) \lambda^{\alpha(q)} \sqrt{\lambda \varepsilon(\lambda)} \|a\|_{\ell^2}$$

with $\|a\|_{\ell^2} = (\sum_{i,j} |a_{ij}|^2)^{1/2}$ and $A_{\lambda, \varepsilon}$ denoting the $\varepsilon(\lambda)$ -annulus about $\lambda \cdot S^2$, i.e.,

$$(2.4) \quad A_{\lambda, \varepsilon} = \{(\mu, \nu) : |\lambda - \sqrt{\mu^2 + \nu^2}| \leq \varepsilon(\lambda)\}.$$

To be able to use our assumptions (1.3) and (1.5') and the universal bounds (1.1) it is natural to break up the annulus into several pieces. Specifically, let

$$(2.5) \quad \Omega_{high} = \{(\mu, \nu) \in A_{\lambda, \varepsilon} : |\nu| \geq \lambda/2\},$$

denote the portion of $A_{\lambda, \varepsilon}$ where $|\nu|$ is relatively large and

$$(2.6) \quad \Omega_{low} = \{(\mu, \nu) \in A_{\lambda, \varepsilon} : |\nu| \leq 1\},$$

be the portion where it is relatively small. We shall also break up the remaining region into the following disjoint dyadic pieces

$$(2.7) \quad \Omega_\ell = \{(\mu, \nu) \in A_{\lambda, \varepsilon} \setminus \Omega_{high} : |\nu| \in (\lambda 2^{-\ell}, \lambda 2^{-\ell+1}]\}.$$

Thus,

$$(2.8) \quad A_{\lambda, \varepsilon} = \Omega_{high} \cup \Omega_{low} \cup \left(\bigcup_{2^{-\ell} \in [\lambda^{-1}, \frac{1}{4}]} \Omega_\ell \right).$$

We shall use the following simple lemma describing the geometry of each of these pieces.

Lemma 2.1. *There is a uniform constant C_0 so that*

$$(2.9) \quad |\sqrt{\lambda^2 - \mu^2} - \nu| \leq C_0 \varepsilon(\lambda) \quad \text{if } (\mu, \nu) \in \Omega_{high} \text{ and } \mu, \nu \geq 0.$$

Also if $\Omega_\ell \neq \emptyset$ then

$$(2.10) \quad |\sqrt{\lambda^2 - \mu^2} - \nu| \leq C_0 2^\ell \varepsilon(\lambda) \quad \text{if } (\mu, \nu) \in \Omega_\ell \text{ and } \mu, \nu \geq 0,$$

and if

$$(2.11) \quad I_\ell = \{\mu : (\mu, \nu) \in \Omega_\ell, \mu, \nu \geq 0\},$$

then for fixed ℓ with $2^{-\ell} \geq \lambda^{-\frac{1}{2}}$,

$$(2.12) \quad I_\ell \text{ is an interval in } [0, \lambda] \text{ of length } |I_\ell| \leq C_0 \lambda 2^{-2\ell},$$

and also

$$(2.13) \quad |\sqrt{\lambda^2 - \nu^2} - \mu| \leq C_0 \quad \text{if } \mu, \nu \geq 0,$$

and $(\mu, \nu) \in \Omega_\ell$ with $2^{-\ell} \leq \lambda^{-\frac{1}{2}}$, or $(\mu, \nu) \in \Omega_{low}$.

The bound in (2.9) is straightforward since $\nu \geq \lambda/2$ for the (μ, ν) there. One obtains (2.13) similarly and indeed can replace C_0 there by $C_0 \varepsilon(\lambda)$, although this will not be needed. One obtains (2.10) and (2.12) by noting that the (μ, ν) there must be of the form

$$(\mu, \nu) = r(\cos \theta, \sin \theta) \quad \text{with } r \in [\lambda - \varepsilon(\lambda), \lambda + \varepsilon(\lambda)] \text{ and } \theta \approx 2^{-\ell}.$$

We also require the following estimates which are a simple consequence of our main assumption (1.3) and the universal bounds (1.1).

Lemma 2.2. *There is a universal constant C_0 so that*

$$(2.14) \quad \left\| \mathbb{1}_{[\lambda-\rho, \lambda+\rho]}(P_X) \right\|_{L^2(X) \rightarrow L^q(X)} \leq C_0 \rho^{1/2} \lambda^{\alpha(q)}, \quad \text{if } \rho \in [1, \lambda],$$

and

$$(2.15) \quad \left\| \mathbb{1}_{[\lambda-\rho, \lambda+\rho]}(P_Y) \right\|_{L^2(Y) \rightarrow L^q(Y)} \leq C_0 \rho^{1/2} B(\lambda), \quad \text{if } \rho \in [\varepsilon(\lambda), \lambda],$$

and also

$$(2.16) \quad \left\| \mathbb{1}_{[2^k, 2^{k+1}]}(P_Y) \right\|_{L^2(Y) \rightarrow L^q(Y)} \leq C_0 2^{d_Y(\frac{1}{2} - \frac{1}{q})k}.$$

Note that if we let $\rho = 1$ in (2.15), we have

$$(2.17) \quad \begin{aligned} B(\lambda) &\gtrsim \left\| \mathbb{1}_{[\lambda-1, \lambda+1]}(P_Y) \right\|_{L^2(Y) \rightarrow L^q(Y)} \\ &\gtrsim \lambda^{\alpha(q)}, \end{aligned}$$

where in the second inequality we used the lower bounds on the spectral projection operator, which holds in the general case (see [19]).

Proof. The proofs are well known. One obtains (2.14) from (1.1) by writing $[\lambda - \rho, \lambda + \rho]$ as the union of $O(\rho)$ disjoint intervals I_k of length 1 or less, each contained in $[0, 2\lambda]$. By the Cauchy-Schwarz inequality one then has

$$\begin{aligned} \left\| \mathbb{1}_{[\lambda-\rho, \lambda+\rho]}(P_X)f \right\|_{L^q(X)} &\lesssim \rho^{1/2} \left(\sum_k \left\| \mathbb{1}_{I_k}(P_X)f \right\|_{L^q(X)}^2 \right)^{1/2} \\ &\lesssim \rho^{1/2} \lambda^{\alpha(q)} \left(\sum_k \left\| \mathbb{1}_{I_k}(P_X)f \right\|_{L^2(X)}^2 \right)^{1/2} \\ &\lesssim \rho^{1/2} \lambda^{\alpha(q)} \|f\|_{L^2(X)}, \end{aligned}$$

using (1.1) in the second inequality and orthogonality in the last one.

To prove (2.15) we note that our assumptions (1.4) and (1.5') imply that $B(\lambda_1) \approx B(\lambda_2)$ and $\varepsilon(\lambda_1) \approx \varepsilon(\lambda_2)$ if $\lambda_1 \approx \lambda_2$. Taking this into account, if $\rho \in [\varepsilon(\lambda), 1]$ one proves (2.15) by a similar argument used for (2.14) if one covers $[\lambda - \rho, \lambda + \rho]$ by $O(\rho/\varepsilon(\lambda))$ disjoint intervals of length $\varepsilon(\lambda)$ or less and uses the fact that (1.3) includes a $(\varepsilon(\lambda))^{1/2}$ factor in the right. Similarly, if $I_k = [k, k+1)$ with $k \leq \lambda$ then one concludes that

$$\left\| \mathbb{1}_{I_k} \right\|_{L^2(Y) \rightarrow L^q(Y)} \lesssim B(k) \lesssim B(\lambda),$$

which can be used with the above argument involving the use of the Cauchy-Schwarz inequality to handle the case where $\rho \in [1, \lambda]$.

The remaining inequality, (2.16), is a standard Bernstein estimate (see [19]). $\square$

Having collected the tools we need, let us state the bounds associated with the various regions (2.5)–(2.7) of $A_{\lambda, \varepsilon}$ that will give us (2.3).

First for all $\varepsilon(\lambda)$ satisfying the conditions in Theorem 1.1 we claim that we have

$$(2.18) \quad \left\| \sum_{(\mu_i, \nu_j) \in \Omega_{high}} a_{ij} e_{\mu_i}^X e_{\nu_j}^Y \right\|_{L^q(X \times Y)} \lesssim \sqrt{\varepsilon(\lambda)} B(\lambda) \cdot \lambda^{\alpha(q)} \sqrt{\lambda} \|a\|_{\ell^2},$$

$$(2.19) \quad \left\| \sum_{(\mu_i, \nu_j) \in \Omega_{low}} a_{ij} e_{\mu_i}^X e_{\nu_j}^Y \right\|_{L^q(X \times Y)} \lesssim \lambda^{\alpha(q)} \|a\|_{\ell^2},$$

as well as

$$(2.20) \quad \left\| \sum_{(\mu_i, \nu_j) \in \Omega_\ell} a_{ij} e_{\mu_i}^X e_{\nu_j}^Y \right\|_{L^q(X \times Y)} \lesssim \sqrt{2^\ell \varepsilon(\lambda)} B(2^{-\ell} \lambda) \cdot \lambda^{\alpha(q)} \sqrt{\lambda 2^{-2\ell}} \|a\|_{\ell^2},$$

if $2^{-\ell} \geq \lambda^{-\frac{1}{2}}$.

For remaining pieces $2^{-\ell} \leq \lambda^{-\frac{1}{2}}$, we need to obtain estimates to handle the two cases where $\varepsilon(\lambda) \leq \frac{1}{4} \cdot \lambda 2^{-2\ell}$ and $\varepsilon(\lambda) \geq \frac{1}{4} \cdot \lambda 2^{-2\ell}$. For the first case, we shall use the following estimate which is valid for all $\varepsilon(\lambda)$ satisfying (1.5')

$$(2.21) \quad \left\| \sum_{(\mu_i, \nu_j) \in \Omega_\ell} a_{ij} e_{\mu_i}^X e_{\nu_j}^Y \right\|_{L^q(X \times Y)} \lesssim \sqrt{2^\ell \varepsilon(\lambda)} B(2^{-\ell} \lambda) \cdot \lambda^{\alpha(q)} \|a\|_{\ell^2},$$

while for the other case we shall use the fact that we also always have

$$(2.22) \quad \left\| \sum_{(\mu_i, \nu_j) \in \Omega_\ell} a_{ij} e_{\mu_i}^X e_{\nu_j}^Y \right\|_{L^q(X \times Y)} \lesssim \lambda^{\alpha(q)} (\lambda 2^{-\ell})^{d_Y(\frac{1}{2} - \frac{1}{q})} \|a\|_{\ell^2}.$$

Let us now see how the bounds in (2.18)–(2.22) yield those in Theorem 1.1. To use (2.20) we note that by (1.4) with $\theta = 2^{-\ell}$

$$\sqrt{2^\ell \varepsilon(\lambda)} B(2^{-\ell} \lambda) \lambda^{\alpha(q)} \sqrt{\lambda 2^{-2\ell}} \lesssim 2^{-\ell/2} B(\lambda) \lambda^{\alpha(q)} \sqrt{\lambda \varepsilon(\lambda)},$$

and so

$$(2.23) \quad \sum_{2^{-\ell} \in [\lambda^{-\frac{1}{2}}, \frac{1}{4}]} \left\| \sum_{(\mu_i, \nu_j) \in \Omega_\ell} a_{ij} e_{\mu_i}^X e_{\nu_j}^Y \right\|_{L^q(X \times Y)} \lesssim B(\lambda) \lambda^{\alpha(q)} \sqrt{\lambda \varepsilon(\lambda)} \|a\|_{\ell^2}.$$

To use (2.21) we note that by (1.4) with $\theta = 2^{-\ell}$,

$$\sqrt{2^\ell \varepsilon(\lambda)} B(2^{-\ell} \lambda) \lambda^{\alpha(q)} \lesssim 2^{\ell/2} B(\lambda) \lambda^{\alpha(q)} \sqrt{\varepsilon(\lambda)},$$

and so

$$(2.24) \quad \sum_{2^{-\ell} \in [2\lambda^{-\frac{1}{2}} \varepsilon(\lambda)^{\frac{1}{2}}, \lambda^{-\frac{1}{2}}]} \left\| \sum_{(\mu_i, \nu_j) \in \Omega_\ell} a_{ij} e_{\mu_i}^X e_{\nu_j}^Y \right\|_{L^q(X \times Y)} \lesssim B(\lambda) \lambda^{\alpha(q)} (\lambda \varepsilon(\lambda))^{1/4} \|a\|_{\ell^2},$$

which is better than desired since we are assuming $\varepsilon(\lambda) \geq \lambda^{-1}$.

Finally, if $2^{-\ell} \in [\lambda^{-1}, 2\lambda^{-\frac{1}{2}} \varepsilon(\lambda)^{\frac{1}{2}}]$, by (2.22), we have

$$(2.25) \quad \sum_{2^{-\ell} \in [\lambda^{-1}, 2\lambda^{-\frac{1}{2}} \varepsilon(\lambda)^{\frac{1}{2}}]} \left\| \sum_{(\mu_i, \nu_j) \in \Omega_\ell} a_{ij} e_{\mu_i}^X e_{\nu_j}^Y \right\|_{L^q(X \times Y)} \lesssim \lambda^{\alpha(q)} (\lambda \varepsilon(\lambda))^{\frac{d_Y}{2}(\frac{1}{2} - \frac{1}{q})} \|a\|_{\ell^2}.$$

It is straightforward to check that for all $q \geq 2$,

$$(\lambda \varepsilon(\lambda))^{\frac{d_Y}{2}(\frac{1}{2} - \frac{1}{q})} \lesssim B(\lambda) \sqrt{\lambda \varepsilon(\lambda)},$$

given the fact that $B(\lambda) \geq \lambda^{\alpha(q)}$ and $\lambda^{-1} \leq \varepsilon(\lambda) \leq 1$. Thus, our proof would be complete if we could establish (2.18)–(2.22).

To prove the first one, (2.18), we note that if $y \in Y$ is fixed, since $\mu_i \leq \lambda$ if $(\mu_i, \nu_j) \in \Omega_{high}$, by (2.14) with $\rho = \lambda$ and orthogonality

$$\begin{aligned} \left\| \sum_{(\mu_i, \nu_j) \in \Omega_{high}} a_{ij} e_{\mu_i}^X(\cdot) e_{\nu_j}^Y(y) \right\|_{L^q(X)} &\lesssim \lambda^{\alpha(q)+1/2} \left\| \sum_{(\mu_i, \nu_j) \in \Omega_{high}} a_{ij} e_{\mu_i}^X(\cdot) e_{\nu_j}^Y(y) \right\|_{L^2(X)} \\ &= \lambda^{\alpha(q)+1/2} \left(\sum_i \left| \sum_{\{j: (\mu_i, \nu_j) \in \Omega_{high}\}} a_{ij} e_{\nu_j}^Y(y) \right|^2 \right)^{1/2}. \end{aligned}$$

If we take the $L^q(Y)$ norm of the left side and use this inequality along with Minkowski's inequality we conclude that

$$(2.26) \quad \begin{aligned} & \left\| \sum_{(\mu_i, \nu_j) \in \Omega_{high}} a_{ij} e_{\mu_i}^X e_{\nu_j}^Y \right\|_{L^q(X \times Y)} \\ & \lesssim \lambda^{\alpha(q)+1/2} \left(\sum_i \left\| \sum_{\{j: (\mu_i, \nu_j) \in \Omega_{high}\}} a_{ij} e_{\nu_j}^Y \right\|_{L^q(Y)}^2 \right)^{1/2}. \end{aligned}$$

Since $\nu_j \approx \lambda$ if $(\mu_i, \nu_j) \in \Omega_{high}$, by (2.9) and (2.15) with $\rho = C_0 \varepsilon(\lambda)$, we have for each fixed i

$$(2.27) \quad \begin{aligned} \left\| \sum_{\{j: (\mu_i, \nu_j) \in \Omega_{high}\}} a_{ij} e_{\nu_j}^Y \right\|_{L^q(Y)} & \lesssim \sqrt{\varepsilon(\lambda)} B(\lambda) \left\| \sum_{\{j: (\mu_i, \nu_j) \in \Omega_{high}\}} a_{ij} e_{\nu_j}^Y \right\|_{L^2(Y)} \\ & = \sqrt{\varepsilon(\lambda)} B(\lambda) \left(\sum_{\{j: (\mu_i, \nu_j) \in \Omega_{high}\}} |a_{ij}|^2 \right)^{1/2}. \end{aligned}$$

Clearly (2.26) and (2.27) imply (2.18).

The proof of (2.20) is similar. Recall that the nonzero terms involve $\mu_i \in I_\ell$, if $2^{-\ell} \geq \lambda^{-\frac{1}{2}}$, then I_ℓ is an interval of length $\rho \leq C_0 \lambda 2^{-2\ell}$ as in (2.11). So, if we use the analog of (2.14) with this value of ρ and with λ replaced by the center of I_ℓ , we can repeat the proof of (2.26) to conclude that

$$(2.28) \quad \begin{aligned} & \left\| \sum_{(\mu_i, \nu_j) \in \Omega_\ell} a_{ij} e_{\mu_i}^X e_{\nu_j}^Y \right\|_{L^q(X \times Y)} \\ & \lesssim \lambda^{\alpha(q)} \sqrt{\lambda 2^{-2\ell}} \left(\sum_i \left\| \sum_{\{j: (\mu_i, \nu_j) \in \Omega_\ell\}} a_{ij} e_{\nu_j}^Y \right\|_{L^q(Y)}^2 \right)^{1/2}. \end{aligned}$$

Since $\nu_j \approx 2^{-\ell} \lambda$ if $(\mu_i, \nu_j) \in \Omega_\ell$, and by (1.5') together with the fact that $2^{-\ell} \geq \lambda^{-\frac{1}{2}}$, we have

$$(2.29) \quad \varepsilon(2^{-\ell} \lambda) \leq 2^\ell \varepsilon(\lambda) \leq 2^{-\ell} \lambda.$$

By (2.10) and (2.15) with $\rho = 2^\ell \varepsilon(\lambda)$, we can argue as above to see that for each fixed i we have

$$(2.30) \quad \left\| \sum_{\{j: (\mu_i, \nu_j) \in \Omega_\ell\}} a_{ij} e_{\nu_j}^Y \right\|_{L^q(Y)} \lesssim (2^\ell \varepsilon(\lambda))^{1/2} B(2^{-\ell} \lambda) \left(\sum_{\{j: (\mu_i, \nu_j) \in \Omega_\ell\}} |a_{ij}|^2 \right)^{1/2}.$$

By combining (2.28) and (2.30) we obtain (2.20).

Next, we turn to (2.21). We note that if $2^{-\ell} \leq \lambda^{-\frac{1}{2}}$, there is a uniform constant C_0 so that

$$(2.31) \quad \mu_i \in [\lambda - C_0, \lambda + C_0], \quad \text{if } (\mu_i, \nu_j) \in \Omega_\ell \text{ for some } j.$$

This just follows from the fact that if $(\mu_i, \nu_j) \in \Omega_{low}$ then we can write $(\mu_i, \nu_j) = r(\cos \theta, \sin \theta)$ with $0 \leq \theta \lesssim \lambda^{-1/2}$ and $r \in [\lambda - \varepsilon(\lambda), \lambda + \varepsilon(\lambda)] \subset [\lambda - 1, \lambda + 1]$. If we use (2.31) and (1.1) we can argue as above to see that

$$(2.32) \quad \left\| \sum_{(\mu_i, \nu_j) \in \Omega_\ell} a_{ij} e_{\mu_i}^X e_{\nu_j}^Y \right\|_{L^q(X \times Y)} \lesssim \lambda^{\alpha(q)} \left(\sum_i \left\| \sum_{\{j: (\mu_i, \nu_j) \in \Omega_\ell\}} a_{ij} e_{\nu_j}^Y \right\|_{L^q(Y)}^2 \right)^{1/2}.$$

For fixed μ_i , if $(\mu_i, \nu_j) \in \Omega_\ell$, we have $\nu_j \approx 2^{-\ell}\lambda$, and by (2.10), ν_j lie in an interval of length $\varepsilon(\lambda)2^\ell$. Now using (1.5') again together with the fact that $\varepsilon(\lambda) \leq \frac{1}{4}\lambda 2^{-2\ell}$, one can see that (2.29) still hold in this case. Thus by (2.15) with $\rho = 2^\ell \varepsilon(\lambda)$, we can argue as above to see that for each fixed i , we have the analogous inequality as in (2.30), which, combined with (2.32), implies (2.21).

To prove (2.22), if one uses (2.16) then we find we can replace (2.30) with

$$\left\| \sum_{\{j: (\mu_i, \nu_j) \in \Omega_\ell\}} a_{ij} e_{\nu_j}^Y \right\|_{L^q(Y)} \lesssim (\lambda 2^{-\ell})^{d_Y(\frac{1}{2} - \frac{1}{q})} \left(\sum_{\{j: (\mu_i, \nu_j) \in \Omega_\ell\}} |a_{ij}|^2 \right)^{1/2},$$

this along with (2.32) yields (2.22).

The proof of (2.19) is similar. Since in this case, there is a uniform constant C_0 so that

$$(2.33) \quad \mu_i \in [\lambda - C_0, \lambda + C_0], \quad \text{if } (\mu_i, \nu_j) \in \Omega_{low} \text{ for some } j.$$

Thus (2.32) still holds in this case, and by (2.16), we can replace (2.30) with

$$(2.34) \quad \left\| \sum_{\{j: (\mu_i, \nu_j) \in \Omega_{low}\}} a_{ij} e_{\nu_j}^Y \right\|_{L^q(Y)} \lesssim \left(\sum_{\{j: (\mu_i, \nu_j) \in \Omega_{low}\}} |a_{ij}|^2 \right)^{1/2}.$$

this along with (2.32) yields (2.19).

2.1. Some applications. Let us now show that for products of round spheres $S^{d_1} \times S^{d_2}$ one can obtain power improvements over the universal bounds in [18] for *all* exponents $2 < q \leq \infty$. This generalizes the L^∞ improvements of Iosevich and Wyman [14]. Using our improved L^q -estimates we can also obtain improved bounds for large exponents for products of the form $S^{d_1} \times S^{d_2} \times M^n$ where M^n is an arbitrary compact manifold of dimension n . If M^n is a product of spheres and $q = \infty$ the bounds agree with the ones that are implicit in Iosevich and Wyman [14].

Theorem 2.3. *Suppose that $d_1, d_2 \geq 1$. Then for all $\varepsilon > 0$ we have the following estimates for eigenfunctions on $S^{d_1} \times S^{d_2}$*

$$(2.35) \quad \|e_\lambda\|_{L^q(S^{d_1} \times S^{d_2})} \leq C_\varepsilon \lambda^{\alpha(q, d_1) + \alpha(q, d_2) + \varepsilon} \|e_\lambda\|_{L^2(S^{d_1} \times S^{d_2})}, \quad 2 < q \leq \infty,$$

where

$$(2.36) \quad \alpha(q, d) = \max\left(d\left(\frac{1}{2} - \frac{1}{q}\right) - \frac{1}{2}, \frac{d-1}{2}\left(\frac{1}{2} - \frac{1}{q}\right)\right)$$

is the λ -exponent in the d -dimensional universal bounds.

In order to use Theorem 1.1 to obtain bounds for product manifolds involving $S^{d_1} \times S^{d_2}$ we note that the distinct eigenvalues of $P = \sqrt{-\Delta_{S^{d_1} \times S^{d_2}}}$ are of the form

$$\sqrt{(k + (d_1 - 1)/2)^2 + (\ell + (d_2 - 1)/2)^2} \quad \text{with } k, \ell = 0, 1, 2, \dots$$

Consequently the gap between successive distinct eigenvalues which are comparable to λ must be larger than a fixed multiple of λ^{-1} . So, (2.35) implies that we also have

the corresponding bounds for the spectral projection operators onto windows of length $\varepsilon(\lambda) = \lambda^{-1}$:

$$(2.35') \quad \begin{aligned} & \left\| \mathbb{1}_{[\lambda-\lambda^{-1}, \lambda+\lambda^{-1}]}(P) \right\|_{L^2(S^{d_1} \times S^{d_2}) \rightarrow L^q(S^{d_1} \times S^{d_2})} \leq C_\varepsilon \lambda^{\alpha(q, d_1) + \alpha(q, d_2) + \varepsilon}, \quad \forall \varepsilon > 0, \\ & \text{if } 2 < q \leq \infty \text{ and } P = \sqrt{-\Delta_{S^{d_1} \times S^{d_2}}}. \end{aligned}$$

A calculation shows that for *all* $q > 2$

$$\alpha(q, d_1) + \alpha(q, d_2) < \alpha(q, d_1 + d_2).$$

Thus (2.35') says that on $S^{d_1} \times S^{d_2}$ one has power improvements over the universal bounds for manifolds of dimension $d = d_1 + d_2$ (but with $\varepsilon(\lambda) \equiv 1$). Also, by considering tensor products of spherical harmonics that saturate the $L^q(S^{d_j})$, $j = 1, 2$, bounds (see e.g. [16]) one sees that (2.35) and hence (2.35') are optimal (up to possibly the λ^ε factor).

Let us now single out a couple of special cases of (2.35').

First, we have

$$(2.37) \quad \begin{aligned} & \left\| \mathbb{1}_{[\lambda-\lambda^{-1}, \lambda+\lambda^{-1}]}(P) \right\|_{L^2(S^{d_1} \times S^{d_2}) \rightarrow L^q(S^{d_1} \times S^{d_2})} \leq C_\varepsilon \lambda^{d(\frac{1}{2} - \frac{1}{q}) - \frac{1}{2}} \lambda^{-\frac{1}{2} + \varepsilon}, \quad \forall \varepsilon > 0, \\ & \text{if } d = d_1 + d_2 \text{ and } q \geq \max\left(\frac{2(d_1+1)}{d_1-1}, \frac{2(d_2+1)}{d_2-1}\right), \end{aligned}$$

which is a $\lambda^{-\frac{1}{2} + \varepsilon}$ improvement for this range of exponents in dimension d versus the universal bounds. This is optimal in the sense that no bounds of this type may hold on *any* manifold of dimension d with $\lambda^{-\frac{1}{2} + \varepsilon}$ replaced by $\lambda^{-\frac{1}{2} - \delta}$ for some $\delta > 0$. For, by Bernstein inequalities, such an estimate would imply that the above spectral projection operators map $L^2 \rightarrow L^\infty$ with norm $O(\lambda^{\frac{d}{2} - 1 - \delta})$. This cannot hold in dimension d since it would imply that the number of eigenvalues of the square root of minus the Laplacian counted with multiplicity which are in subintervals of length λ^{-1} in $[\lambda/2, \lambda]$ would be $O(\lambda^{d-2-\delta})$, and this would contradict the Weyl formula for P .

Second, we note that if $d_1 = d_2 = 1$ and $q = \infty$, then (2.37) is just

$$(2.38) \quad \left\| \mathbb{1}_{[\lambda-\lambda^{-1}, \lambda+\lambda^{-1}]}(\sqrt{-\Delta_{\mathbb{T}^2}}) \right\|_{L^2(\mathbb{T}^2) \rightarrow L^\infty(\mathbb{T}^2)} = O(\lambda^\varepsilon), \quad \forall \varepsilon > 0.$$

This is equivalent to the classical fact that the number of integer lattice points on $\lambda \cdot S^1$ is $O(\lambda^\varepsilon)$, i.e.,

$$(2.38') \quad \#\{j \in \mathbb{Z}^2 : |j| = \lambda\} = O(\lambda^\varepsilon) \quad \forall \varepsilon > 0.$$

We shall use this bound in our proof of Theorem 2.3.

Before proving this result, let us show how we can use the bounds in Theorem 1.1 to obtain a couple of corollaries.

The first says that sufficiently large exponents we can obtain power improvements of the universal bounds for products involving $S^{d_1} \times S^{d_2}$.

Corollary 2.4. *Let M^n be a compact manifold of dimension $n \geq 1$ and consider the product manifold $S^{d_1} \times S^{d_2} \times M^n$ where $d_1, d_2 \geq 1$. Then if*

$$P = \sqrt{-(\Delta_{S^{d_1}} + \Delta_{S^{d_2}} + \Delta_{M^n})},$$

we have

$$(2.39) \quad \left\| \mathbb{1}_{[\lambda-\lambda^{-1}, \lambda+\lambda^{-1}]}(P) \right\|_{L^2(S^{d_1} \times S^{d_2} \times M^n) \rightarrow L^q(S^{d_1} \times S^{d_2} \times M^n)} \leq C_\varepsilon \lambda^{d(\frac{1}{2}-\frac{1}{q})-\frac{1}{2}} \lambda^{-\frac{1}{2}+\varepsilon},$$

$$\forall \varepsilon > 0, \text{ if } d = d_1 + d_2 + n \text{ and } q \geq \max\left(\frac{2(d_1+1)}{d_1-1}, \frac{2(d_2+1)}{d_2-1}, \frac{2(n+1)}{n-1}\right).$$

Furthermore, we have

$$(2.40) \quad \left\| \mathbb{1}_{[\lambda-\lambda^{-1}, \lambda+\lambda^{-1}]}(P) \right\|_{L^2(S^{d_1} \times S^{d_2} \times M^n) \rightarrow L^q(S^{d_1} \times S^{d_2} \times M^n)} \leq C_q \lambda^{d(\frac{1}{2}-\frac{1}{q})-\frac{1}{2}-\delta},$$

for some $\delta = \delta(q, d_1, d_2, n) > 0$ if $q > \frac{2(d+1)}{d-1}$, $d = d_1 + d_2 + n$.

To prove these bounds we note that for q as in (2.39) we have

$$\alpha(q, d_1) + \alpha(q, d_2) + \alpha(q, n) + \frac{1}{2} = d\left(\frac{1}{2} - \frac{1}{q}\right) - 1.$$

Consequently, (2.39) follows immediately from (2.35) and (1.6) with $\varepsilon(\lambda) = \lambda^{-1}$ and

$$B(\lambda) = \lambda^{(d_1+d_2)(\frac{1}{2}-\frac{1}{q})-\frac{1}{2}+\varepsilon}.$$

Since (2.39) is a power improvement over $O(\lambda^{d(\frac{1}{2}-\frac{1}{q})-\frac{1}{2}})$ bounds for large exponents and the universal bounds imply that (2.40) is valid when $\delta = 0$ and $q = q_c(d) = \frac{2(d+1)}{d-1}$, one obtains (2.40) via a simple interpolation argument.

A calculation show that we cannot use Theorem 1.1 to obtain improvements over the universal bounds when $q \in (2, q_c(d)]$ with $q_c(d)$ as above being the critical exponent. We should point out that Canzani and Galkowski [7] recently obtained $\sqrt{\log \lambda}$ -improvements over the universal bounds (with $\varepsilon(\lambda) = (\log \lambda)^{-1}$) for arbitrary products of manifolds and $q > q_c(d)$. They as well as Iosevich and Wyman [14] conjectured that for such manifolds appropriate power improvements over the universal bounds should always be possible. Obtaining any improvements for $q \in (2, q_c(d)]$, though, appears difficult except in special cases such as for products involving products of spheres as above. Perhaps, though, the Kakeya-Nikodym approach that was used in [3] and [4] to obtain log-power improvements of eigenfunction estimates for manifolds of nonpositive sectional curvature could be used to handle critical and subcritical exponents.

Let us also state one more corollary which generalizes the well known higher dimensional version of (2.38'):

$$(2.41) \quad \#\{j \in \mathbb{Z}^n : |j| = \lambda\} = O(\lambda^{n-2+\varepsilon}) \quad \forall \varepsilon > 0 \text{ if } n \geq 3.$$

Just as for the special case where $n = 2$ discussed above, this is easily seen to be equivalent to the following sup-norm bounds

$$(2.41') \quad \left\| \mathbb{1}_{[\lambda-\lambda^{-1}, \lambda+\lambda^{-1}]}(\sqrt{-\Delta_{\mathbb{T}^n}}) \right\|_{L^2(\mathbb{T}^n) \rightarrow L^q(\mathbb{T}^n)} = O(\lambda^{\frac{n-1}{2}} \lambda^{-\frac{1}{2}+\varepsilon}) \quad \forall \varepsilon > 0 \text{ if } n \geq 3.$$

If we use Theorem 1.1 for $q = \infty$ with $\varepsilon(\lambda) = \lambda^{-1}$ and $B(\lambda) = \lambda^{\frac{n-1}{2}+\varepsilon}$ we can argue as above to obtain the following generalization of (2.41').

Corollary 2.5. *Let M^{n-2} be a compact Riemannian manifold of dimension $n-2$ where $n \geq 3$. Then if*

$$P = \sqrt{-(\Delta_{\mathbb{T}^2} + \Delta_{M^{n-2}})}$$

is the square root of minus the Laplacian on the n -dimensional product manifold $\mathbb{T}^2 \times M^{n-2}$ we have

$$(2.42) \quad \left\| \mathbb{1}_{[\lambda-\lambda^{-1}, \lambda+\lambda^{-1}]}(P) \right\|_{L^2(\mathbb{T}^2 \times M^{n-2}) \rightarrow L^q(\mathbb{T}^2 \times M^{n-2})} = O(\lambda^{\frac{n-1}{2}} \lambda^{-\frac{1}{2}+\varepsilon}) \quad \forall \varepsilon > 0.$$

Consequently if $0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \dots$ are the eigenvalues of P

$$(2.43) \quad \#\{\lambda_j \in [\lambda - \lambda^{-1}, \lambda + \lambda^{-1}]\} = O(\lambda^{n-2+\varepsilon}), \quad \forall \varepsilon > 0.$$

The first estimate, (2.42), follows from (2.38) and Theorem 1.1. As is well known (see e.g., [20]) it implies the counting bounds (2.43). These are optimal, since as we discussed before, $O(\lambda^{n-2-\delta})$ with $\delta > 0$ bounds cannot hold due to the Weyl formula.

One can also obtain power improvements for products $X \times \mathbb{T}^n$ using the following “discrete restriction theorem” of Bourgain and Demeter [5] (toral eigenfunction bounds):

$$(2.44) \quad \left\| \mathbb{1}_{[\lambda-\lambda^{-1}, \lambda+\lambda^{-1}]}(\sqrt{-\Delta_{\mathbb{T}^n}}) \right\|_{L^2(\mathbb{T}^n) \rightarrow L^{\frac{2(n+1)}{n-1}}(\mathbb{T}^n)} \lesssim \lambda^\varepsilon, \quad \forall \varepsilon > 0.$$

This represents a $1/q_c$ -power improvement over the universal estimates [18] with $q_c = \frac{2(n+1)}{n-1}$. Similar to the case above in Corollary 2.4 if we use (1.6) with $B(\lambda) = \lambda^{n(\frac{1}{2}-\frac{1}{q})-\frac{1}{q_c}+\varepsilon}$, we obtain from (2.44) that if $P = \sqrt{-(\Delta_X + \Delta_{\mathbb{T}^n})}$ then

$$(2.45) \quad \left\| \mathbb{1}_{[\lambda-\lambda^{-1}, \lambda+\lambda^{-1}]}(P) \right\|_{L^2(X \times \mathbb{T}^n) \rightarrow L^q(X \times \mathbb{T}^n)} \lesssim \lambda^{d(\frac{1}{2}-\frac{1}{q})-1/2-1/q_c+\varepsilon}, \quad \forall \varepsilon > 0,$$

if $q_c = \frac{2(n+1)}{n-1}$, and $d = d_X + n$,

and

$$(2.46) \quad q \geq \max\left(\frac{2(n+1)}{n-1}, \frac{2(d_X+1)}{d_X-1}\right).$$

It is conjectured that (2.44) should also be valid when $\frac{2(n+1)}{n-1}$ is replaced by the larger exponent $\frac{2n}{n-2}$, which would represent the optimal $\lambda^{-1/2+\varepsilon}$ improvement of the universal bounds in [18]. If this result held, then one would obtain the optimal bounds where in the exponent in (2.45) $-1/q_c$ is replaced by $-1/2$, which would be optimal, as well as the range of exponents in (2.46).

Also, using results of Hickman [11] and Germain and Myerson [9] one can also obtain improved spectral projection bounds when $\varepsilon(\lambda) = \lambda^{-\sigma}$ with $\sigma \in (0, 1)$.

Let us now present the proof of Theorem 2.3 which in the case of $q = \infty$ strengthens the bounds that are implicit in Iosevich and Wyman [14].

Proof of Theorem 2.3. Let $\{e_k^\mu\}_\mu$ be an orthonormal basis for spherical harmonics of degree k on S^{d_1} and $\{e_\ell^\nu\}_\nu$ be an orthonormal basis of spherical harmonics of degree ℓ on S^{d_2} . Then an orthonormal basis of eigenfunctions on $S^{d_1} \times S^{d_2}$ is of the form

$$(2.47) \quad e_k e_\ell$$

where $e_k = e_k^\nu$ for some ν and $e_\ell = e_\ell^\mu$ for some μ . So,

$$(-\Delta_{d_1} + (\frac{d_1-1}{2})^2)e_k = (k + \frac{d_1-1}{2})^2 e_k$$

and

$$(-\Delta_{d_2} + (\frac{d_2-1}{2})^2)e_\ell = (\ell + \frac{d_2-1}{2})^2 e_\ell$$

so that for the Laplacian on $S^{d_1} \times S^{d_2}$, $\Delta = \Delta_{d_1} + \Delta_{d_2}$, we have

$$(-\Delta + (\frac{d_1-1}{2})^2 + (\frac{d_2-1}{2})^2)e_k e_\ell = ((k + \frac{d_1-1}{2})^2 + (\ell + \frac{d_1-1}{2})^2)e_k e_\ell.$$

Thus if $P = \sqrt{-\Delta + (\frac{d_1-1}{2})^2 + (\frac{d_2-1}{2})^2}$ its eigenvalues are

$$(2.48) \quad \lambda = \lambda_{k,\ell} = \sqrt{((k + \frac{d_1-1}{2})^2 + (\ell + \frac{d_1-1}{2})^2)}.$$

Thus, if e_λ as in Theorem 2.3 is an eigenfunction of P with this eigenvalue we must have

$$(2.49) \quad e_\lambda(x, y) = \sum_{\{(k,\ell): \lambda_{k,\ell}=\lambda\}} \left(\sum_{\mu,\nu} a_{k,\ell}^{\mu,\nu} e_k^\mu(x) e_\ell^\nu(y) \right), \quad (x, y) \in S^{d_1} \times S^{d_2}.$$

Let us now prove (2.35). We note that if (k, ℓ) are fixed then, for every fixed $y \in S^{d_2}$, the function on S^{d_1}

$$x \rightarrow \sum_{\mu,\nu} a_{k,\ell}^{\mu,\nu} e_k^\mu(x) e_\ell^\nu(y)$$

is a spherical harmonic of degree k . Thus by [16] or [18]

$$\left\| \sum_{\mu,\nu} a_{k,\ell}^{\mu,\nu} e_k^\mu(\cdot) e_\ell^\nu(y) \right\|_{L^q(S^{d_1})} \leq C \lambda^{\alpha(q,d_1)} \left\| \sum_{\mu,\nu} a_{k,\ell}^{\mu,\nu} e_k^\mu(\cdot) e_\ell^\nu(y) \right\|_{L^2(S^{d_1})}.$$

Next, by Minkowski's inequality and another application of the universal bounds, we obtain from this

$$(2.50) \quad \begin{aligned} \left\| \sum_{\mu,\nu} a_{k,\ell}^{\mu,\nu} e_k^\mu e_\ell^\nu \right\|_{L^q(S^{d_1} \times S^{d_2})} &\leq C \lambda^{\alpha(q,d_1)} \left\| \sum_{\mu,\nu} a_{k,\ell}^{\mu,\nu} e_k^\mu(x) e_\ell^\nu(y) \right\|_{L_x^2 L_y^q(S^{d_1} \times S^{d_2})} \\ &\lesssim \lambda^{\alpha(q,d_1)+\alpha(q,d_2)} \left\| \sum_{\mu,\nu} a_{k,\ell}^{\mu,\nu} e_k^\mu e_\ell^\nu \right\|_{L^2(S^{d_1} \times S^{d_2})}. \end{aligned}$$

Since, by (2.38'), the number of $\{(k, \ell) : \lambda_{k,\ell} = \lambda\}$ is $O(\lambda^\varepsilon)$ we also have by the Cauchy-Schwarz inequality that if e_λ is as in (2.49)

$$(2.51) \quad |e_\lambda(x, y)| \lesssim \lambda^\varepsilon \left(\sum_{\{(k,\ell): \lambda_{k,\ell}=\lambda\}} \left| \left(\sum_{\mu,\nu} a_{k,\ell}^{\mu,\nu} e_k^\mu(x) e_\ell^\nu(y) \right) \right|^2 \right)^{1/2}.$$

Thus, by (2.50)–(2.51)

$$(2.52) \quad \begin{aligned} &\|e_\lambda\|_{L^q(S^{d_1} \times S^{d_2})} \\ &\lesssim \lambda^{\alpha(q,d_1)+\alpha(q,d_2)+\varepsilon} \cdot \left(\sum_{\{(k,\ell): \lambda_{k,\ell}=\lambda\}} \left\| \sum_{\mu,\nu} a_{k,\ell}^{\mu,\nu} e_k^\mu e_\ell^\nu \right\|_{L^2(S^{d_1} \times S^{d_2})}^2 \right)^{1/2}, \end{aligned}$$

which leads to (2.35) since, by orthogonality, the last factor in (2.52) is $\|e_\lambda\|_{L^2}$. $\square$

3. Improved Weyl formulae.

To prove Theorem 1.2, we first observe that if as above μ_i^2 are the eigenvalues of $-\Delta_X$ then by (1.8) we have

$$(3.1) \quad N(X \times Y, \lambda) = \sum_{\mu_i \leq \lambda} \left[(2\pi)^{-d_Y} \omega_{d_Y} (\text{Vol } Y) (\lambda^2 - \mu_i^2)^{d_Y/2} + R_Y(\sqrt{\lambda^2 - \mu_i^2}) \right].$$

We can estimate the last sum using (1.8) and (1.5'):

$$\begin{aligned}
R_\lambda &= \sum_{\mu_i \leq \lambda} R_Y(\sqrt{\lambda^2 - \mu_i^2}) \\
&\lesssim \sum_{\mu_i \leq \lambda} \varepsilon(\sqrt{\lambda^2 - \mu_i^2}) \cdot (\lambda^2 - \mu_i^2)^{\frac{d_Y-1}{2}} \\
&\lesssim \lambda^{d_Y-1} \sum_{\mu_i \leq \lambda} \varepsilon(\lambda \cdot \sqrt{1 - \mu_i^2/\lambda^2}) \cdot \sqrt{1 - \mu_i^2/\lambda^2} \times (1 - \mu_i^2/\lambda^2)^{\frac{d_Y-2}{2}}.
\end{aligned}$$

Since $d_Y \geq 2$, we have

$$(1 - \mu_i^2/\lambda^2)^{\frac{d_Y-2}{2}} \leq 1, \quad \text{if } \mu_i \leq \lambda.$$

Thus, if we use (1.5') with $\theta = \sqrt{1 - \mu_i^2/\lambda^2}$ to estimate the terms with $\sqrt{\lambda^2 - \mu_i^2} \geq 1$, we get

$$\varepsilon(\lambda \cdot \sqrt{1 - \mu_i^2/\lambda^2}) \cdot \sqrt{1 - \mu_i^2/\lambda^2} \times (1 - \mu_i^2/\lambda^2)^{\frac{d_Y-2}{2}} \leq \varepsilon(\lambda) \quad \text{if } \sqrt{\lambda^2 - \mu_i^2} \geq 1.$$

Thus, by (1.7).

$$\sum_{\mu_i \leq \lambda, \sqrt{\lambda^2 - \mu_i^2} \geq 1} R_Y(\sqrt{\lambda^2 - \mu_i^2}) \lesssim \lambda^{d_Y-1} \varepsilon(\lambda) \sum_{\mu_i \leq \lambda} 1 \lesssim \varepsilon(\lambda) \lambda^{d_Y-1} \cdot \lambda^{d_X} = \varepsilon(\lambda) \lambda^{d-1},$$

with, as before, $d = d_X + d_Y$. We also need to estimate the terms where $\sqrt{\lambda^2 - \mu_i^2} \leq 1$. In this case we just use that $R_Y(\theta) = O(1)$ if $\theta \leq 1$ and so

$$\sum_{\mu_i \leq \lambda, \sqrt{\lambda^2 - \mu_i^2} \leq 1} R_Y(\sqrt{\lambda^2 - \mu_i^2}) \leq \sum_{\mu_i \leq \lambda} 1 \lesssim \lambda^{d_X} = \lambda^{d-1} \cdot \lambda^{1-d_Y},$$

and since $d_Y \geq 2$,

$$\lambda^{1-d_Y} \leq \lambda^{-1} \leq \varepsilon(\lambda).$$

By combining these two estimates we conclude that, if as above, R_λ is the second sum in the right side of (3.1) then

$$R_\lambda = O(\varepsilon(\lambda) \lambda^{d-1}), \quad d = d_X + d_Y,$$

as desired.

Based on this, we conclude that the improved Weyl formula (1.9) would be a consequence of the following

$$\begin{aligned}
(3.2) \quad & (2\pi)^{-d_Y} \omega_{d_Y}(\text{Vol } Y) \lambda^{d_Y} \cdot \sum_{\mu_i \leq \lambda} (1 - \mu_i^2/\lambda^2)^{d_Y/2} \\
&= (2\pi)^{-d} \omega_d(\text{Vol } Y \cdot \text{Vol } X) \lambda^d + O(\lambda^{d-2}).
\end{aligned}$$

Note that $\sum_{\mu_i \leq \lambda} (1 - \mu_i^2/\lambda^2)^{d_Y/2}$ is the trace of the kernel of $(1 - (P_X)^2/\lambda^2)_+^{d_Y/2}$, i.e.,

$$(3.3) \quad \sum_{\mu_i \leq \lambda} (1 - \mu_i^2/\lambda^2)^{d_Y/2} = \int_M \sum_{\mu_i \leq \lambda} (1 - \mu_i^2/\lambda^2)^{d_Y/2} e_{\mu_i}^X(x) \overline{e_{\mu_i}^X(x)} dV(x),$$

with dV denoting the volume element on X . For $\delta \geq 0$

$$(3.4) \quad S_\lambda^\delta(x, y) = \sum_{\mu_i \leq \lambda} (1 - \mu_i^2/\lambda^2)^\delta e_{\mu_i}^X(x) \overline{e_{\mu_i}^X(y)},$$

denotes the kernel of the Bochner-Riesz operators $(1 - (P_X)^2/\lambda^2)_+^\delta$ (see, e.g. [19]). Keeping (3.3) in mind, we claim that (3.2) (and hence Theorem 1.2) would be a consequence of the following pointwise estimates for these kernels restricted to the diagonal in $X \times X$.

Proposition 3.1. *Let S_λ^δ be as in (3.4). Then if $\delta \geq 1$ we have*

$$(3.5) \quad S_\lambda^\delta(x, x) = (2\pi)^{-d_X} |S^{d_X-1}| \times \frac{1}{2} B(\delta + 1, d_X/2) \lambda^{d_X} + O(\lambda^{d_X-2}),$$

with $|S^{d_X-1}|$ denoting the area of the unit sphere in $\mathbb{R}^{d_X}$ and

$$B(s, t) = \int_0^1 (1-u)^{s-1} u^{t-1} du$$

being the beta function.

To see that (3.3)–(3.5) imply (3.2) we recall the formulae

$$\omega_n = \frac{1}{n} |S^{n-1}| = \frac{\pi^{n/2}}{\Gamma(n/2 + 1)},$$

and

$$B(s, t) = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}.$$

Thus,

$$\begin{aligned} |S^{d_X-1}| \cdot \frac{1}{2} B(d_Y/2 + 1, d_X/2) &= \frac{d_X \pi^{d_X/2}}{\Gamma(d_X/2 + 1)} \cdot \frac{\Gamma(d_Y/2 + 1)\Gamma(d_X/2)}{2\Gamma(d/2 + 1)} \\ &= \frac{\pi^{d_X/2} \Gamma(d_Y/2 + 1)}{\Gamma(d/2 + 1)}, \end{aligned}$$

and so

$$\begin{aligned} (3.6) \quad \omega_{d_Y} \cdot |S^{d_X-1}| \cdot \frac{1}{2} B(d_Y/2 + 2, d_X/2) &= \frac{\pi^{d_Y/2}}{\Gamma(d_Y/2 + 1)} \cdot \frac{\pi^{d_X/2} \Gamma(d_Y/2 + 1)}{\Gamma(d/2 + 1)}, \\ &= \frac{\pi^{d/2}}{\Gamma(d/2 + 1)} = \omega_d. \end{aligned}$$

Thus, since $d_Y/2 \geq 1$, if (3.5) were valid, we would have

$$\begin{aligned} &(2\pi)^{-d_Y} \omega_{d_Y} (\text{Vol } Y) \lambda^{d_Y} \times \int_X S_\lambda^{d_Y/2}(x, x) dV(x) \\ &= (2\pi)^{-d} \lambda^{d_Y} \omega_d (\text{Vol } Y \cdot \text{Vol } X) \cdot \lambda^{d_X} + O(\lambda^{d_Y} \cdot \lambda^{d_X-2}) \\ &= (2\pi)^{-d} \omega_d \text{Vol}(X \times Y) \lambda^d + O(\lambda^{d-2}), \end{aligned}$$

Since, by (3.3) and (3.4), this yields (3.2), we conclude that the proof of Theorem 1.2 would be complete if we could establish Proposition 3.1.

Proof of Proposition 3.1. The proof of kernel estimates for Bochner-Riesz estimates are well known. See, e.g., [13], [17], [19] and [20]. We shall adapt the argument in the latter reference, which is based on arguments that exploit the Hadamard parametrix and go back to Avakumovic [1] and Levitan [15]. These dealt with the analog of (3.5) where $\delta = 0$ and then the error bounds in (3.5) must be replaced by $O(\lambda^{d_X-1})$.

To proceed, let $m_\delta(\tau)$, $\tau \in \mathbb{R}$, denote the even function

$$\tau \rightarrow m_\delta(\tau) = (1 - \tau^2)_+^\delta.$$

Then, if $\hat{m}_\delta$ denotes its Fourier transform, we have by Fourier's inversion theorem

$$S_\lambda^\delta f(x) = m_\delta(P_X) f(x) = \frac{1}{\pi} \int_0^\infty \lambda \hat{m}_\delta(\lambda t) (\cos t P_X) f(x) dt.$$

Thus,

$$\begin{aligned} S_\lambda^\delta(x, y) &= \frac{1}{\pi} \int_0^\infty \lambda \hat{m}_\delta(\lambda t) (\cos t P_X)(x, y) dt \\ (3.7) \quad &= \frac{1}{\pi} \sum_i \int_0^\infty \lambda \hat{m}_\delta(\lambda t) \cos t \mu_i e_{\mu_i}^X(x) \overline{e_{\mu_i}^X(y)} dt. \end{aligned}$$

To be able to exploit this, we require a couple of facts about m_δ . First, we can write its Fourier transform as follows

$$\begin{aligned} \hat{m}_\delta(t) &= a_0^\delta(t) + a_+^\delta(t) e^{it} + a_-^\delta(t) e^{-it}, \quad \text{where} \\ (3.8) \quad |\partial_t^j a(t)| &\lesssim O((1 + |t|)^{-1-\delta-j}) \quad \forall j = 0, 1, 2, \dots, \quad \text{if } a = a_0, a_+, a_-. \end{aligned}$$

Also,

$$(3.9) \quad \int_0^\infty m_\delta(r) r^{d_X-1} dr = \frac{1}{2} B(\delta + 1, d_X/2).$$

Let us postpone the simple proofs of these two facts for a moment and see how they can be used, along with the Hadamard parametrix, to prove Proposition 3.1.

Let us first fix an even function $\rho(t) \in C_0^\infty(\mathbb{R})$ satisfying the following

$$(3.10) \quad \text{supp } \rho \subset (-c/2, c/2) \quad \text{and} \quad \rho \equiv 1 \quad \text{on} \quad [-c/4, c/4],$$

where we assume that

$$c = \min\{1, \text{Inj } X/2\},$$

with $\text{Inj } X$ denoting the injectivity radius of (X, g_X) . It follows from (3.8) that

$$\begin{aligned} r_\lambda^\delta(\mu) &= \frac{1}{\pi} \int_0^\infty (1 - \rho(t)) \lambda \hat{m}_\delta(\lambda t) \cos t \mu dt \\ (3.11) \quad &= O(\lambda^{-\delta} (1 + |\lambda - \mu|)^{-N}), \quad \forall N, \quad \text{if } \mu \geq 0. \end{aligned}$$

Thus, if we modify the kernels in (3.7) as follows

$$\begin{aligned} \tilde{S}_\lambda^\delta(x, y) &= \frac{1}{\pi} \int_0^\infty \rho(t) \lambda \hat{m}_\delta(\lambda t) (\cos t P_X)(x, y) dt \\ (3.12) \quad &= \frac{1}{\pi} \sum_i \int_0^\infty \rho(t) \lambda \hat{m}_\delta(\lambda t) \cos t \mu_i e_{\mu_i}^X(x) \overline{e_{\mu_i}^X(y)} dt, \end{aligned}$$

and let

$$R_\lambda^\delta(x, y) = S_\lambda^\delta(x, y) - \tilde{S}_\lambda^\delta(x, y).$$

It follows that

$$(3.13) \quad R_\lambda^\delta(x, y) = \sum_i r_\lambda(\mu_i) e_{\mu_i}^X(x) \overline{e_{\mu_i}^X(y)},$$

satisfies

$$(3.14) \quad |R_\lambda^\delta(x, y)| \lesssim \lambda^{-\delta} \sum_i (1 + |\lambda - \mu_i|)^{-N} |e_{\mu_i}^X(x)| |e_{\mu_i}^X(y)|, \quad N = 1, 2, \dots$$

As is well known, the pointwise Weyl formula of Avakumovic [1] and Levitan [15] (see also [12], [20]) yields the uniform bounds

$$\sum_{\mu_i \in [\tau, \tau+1)} |e_{\mu_i}^X(x)|^2 \lesssim (1 + \tau)^{d_X - 1}, \quad \tau \geq 0,$$

which in turn give us

$$(3.15) \quad R_\lambda^\delta(x, y) = O(\lambda^{d_X - 1 - \delta}) = O(\lambda^{d_X - 2}),$$

since we are assuming in Proposition 3.1 that $\delta \geq 1$.

Consequently, it suffices to show that $\tilde{S}_\lambda^\delta(x, x)$ equals the first term in the right side of (3.5) up to error terms which are $O(\lambda^{d_X - 2})$.

To do this, if $d_X \geq 2$, we recall that the Hadamard parametrix implies that for $|t|$ smaller than half the injectivity radius of X we can write

$$(3.16) \quad (\cos t P_X)(x, x) = (2\pi)^{-d_X} \int_{\mathbb{R}^{d_X}} \cos t |\xi| d\xi + \alpha_0(x) \int_{\mathbb{R}^2} t \frac{\sin t |\xi|}{|\xi|} d\xi \\ + \int_{\mathbb{R}^{d_X}} \cos t |\xi| \alpha_1(t, x, \xi) d\xi + \int_{\mathbb{R}^{d_X}} \sin t |\xi| \alpha_2(t, x, \xi) d\xi + O(1),$$

where α_0 is a smooth function, and the α_j are symbols of order -3 , so, in particular

$$|\partial_\xi^\gamma \alpha_j| \lesssim (1 + |\xi|)^{-3 - |\gamma|}.$$

We shall first deal with the second term on the right side. If we take $x = y$ in (3.12) and replace $(\cos t P_X)(x, x)$ by the second term in the right side of (3.16), our goal is to show that

$$(3.17) \quad \int_0^\infty \int_{\mathbb{R}^{d_X}} t \frac{\sin t |\xi|}{|\xi|} \lambda \hat{m}_\delta(\lambda t) \rho(t) d\xi dt \lesssim \lambda^{d_X - 2}.$$

To handle the part of the integral where $|\xi| \geq 2\lambda$, note that since

$$\hat{m}_\delta(\lambda t) = \frac{1}{\pi} \int_{-1}^1 e^{-it\lambda\tau} m_\delta(\tau) d\tau$$

is an even function in t , it suffices to show that

$$(3.18) \quad \int_{|\xi| \geq 2\lambda} \int_{-\infty}^\infty \int_{-1}^1 t \frac{\sin t |\xi|}{|\xi|} \lambda e^{-it\lambda\tau} m_\delta(\tau) \rho(t) dt d\tau d\xi \lesssim \lambda^{d_X - 2}.$$

However, by integrating by parts in t , if $|\xi| \geq 2\lambda$, it is easy to see that

$$(3.19) \quad \int_{-\infty}^{\infty} \int_{-1}^1 t \frac{\sin t|\xi|}{|\xi|} \lambda e^{-it\lambda\tau} m_{\delta}(\tau) \rho(t) dt d\tau \lesssim O(1 + |\xi|)^{-N}, \quad \forall N,$$

which clearly implies (3.18).

For the part of integral where $|\xi| \leq 2\lambda$, let us fix $\eta \in C_0^{\infty}$ satisfying

$$\text{supp } \eta \subset (1/2, \infty) \quad \text{and} \quad \eta \equiv 1 \text{ on } [1, +\infty),$$

and write

$$(3.20) \quad \begin{aligned} & \int_0^{\infty} \int_{|\xi| \leq 2\lambda} t \frac{\sin t|\xi|}{|\xi|} \lambda \hat{m}_{\delta}(\lambda t) \rho(t) d\xi dt \\ &= \int_0^{\infty} \int_{|\xi| \leq 2\lambda} (1 - \eta(t|\xi|)) t \frac{\sin t|\xi|}{|\xi|} \lambda \hat{m}_{\delta}(\lambda t) \rho(t) d\xi dt \\ & \quad + \int_0^{\infty} \int_{|\xi| \leq 2\lambda} \eta(t|\xi|) t \frac{\sin t|\xi|}{|\xi|} \lambda \hat{m}_{\delta}(\lambda t) \rho(t) d\xi dt \\ &= I + II. \end{aligned}$$

For the first term on the right, note that $|\xi| \leq \min\{t^{-1}, 2\lambda\}$, and so by (3.8)

$$(3.21) \quad \begin{aligned} I &\lesssim \int_{t \geq (2\lambda)^{-1}} \int_{|\xi| \leq t^{-1}} \frac{1}{(t\lambda)^{\delta} |\xi|} d\xi dt \\ &\quad + \int_{t \leq (2\lambda)^{-1}} \int_{|\xi| \leq 2\lambda} \frac{1}{|\xi|} d\xi dt \\ &\lesssim \int_{t \geq (2\lambda)^{-1}} t^{1-d_X-\delta} \lambda^{-\delta} dt + \int_{t \leq (2\lambda)^{-1}} \lambda^{d_X-1} dt \\ &\lesssim \lambda^{d_X-2}. \end{aligned}$$

To bound II , by integrating by parts in t , we rewrite it as

$$(3.22) \quad \begin{aligned} & \int_{|\xi| \leq 2\lambda} \int_0^{\infty} \eta(t|\xi|) \frac{\sin t|\xi|}{|\xi|} t \lambda \hat{m}_{\delta}(\lambda t) \rho(t) d\xi dt \\ &= \int_{|\xi| \leq 2\lambda} \int_0^{\infty} \eta(t|\xi|) \frac{\cos t|\xi|}{|\xi|^2} t \lambda \hat{m}_{\delta}(\lambda t) \rho'(t) d\xi dt \\ & \quad + \int_{|\xi| \leq 2\lambda} \int_0^{\infty} \eta'(t|\xi|) \frac{\cos t|\xi|}{|\xi|} t \lambda \hat{m}_{\delta}(\lambda t) \rho(t) d\xi dt \\ & \quad + \int_{|\xi| \leq 2\lambda} \int_0^{\infty} \eta(t|\xi|) \frac{\cos t|\xi|}{|\xi|^2} (t \lambda \hat{m}_{\delta}(\lambda t))' \rho(t) d\xi dt \\ &= I + II + III. \end{aligned}$$

For the first term, since $\rho'(t)$ is supported where $t \approx 1$, by (3.8), we have

$$\begin{aligned} I &\lesssim \int_{|\xi| \leq 2\lambda} \frac{1}{|\xi|^2} \lambda^{-\delta} d\xi \\ &\lesssim \lambda^{d_X-2}, \quad \text{if } \delta > 0. \end{aligned}$$

For the second term, since $\eta'(t|\xi|)$ is supported where $t \approx |\xi|$, by (3.8), we have

$$\begin{aligned} II &\lesssim \int_{|\xi| \leq 2\lambda} \int_{t \approx |\xi|^{-1}} \frac{1}{(t\lambda)^\delta |\xi|} dt d\xi \\ &\lesssim \int_{|\xi| \leq 2\lambda} \frac{1}{|\xi|^{2-\delta}} \lambda^{-\delta} d\xi \\ &\lesssim \lambda^{d_X-2}. \end{aligned}$$

For the third term, we use the fact that by (3.8), $(t\lambda\hat{m}_\delta(\lambda t))' \lesssim \lambda^{-\delta} t^{-1-\delta}$, which implies

$$\begin{aligned} III &\lesssim \int_{|\xi| \leq 2\lambda} \int \eta(t|\xi|) \frac{1}{|\xi|^2} \lambda^{-\delta} t^{-1-\delta} \rho(t) dt d\xi \\ &\lesssim \int_{|\xi| \leq 2\lambda} \frac{1}{|\xi|^{2-\delta}} \lambda^{-\delta} d\xi \\ &\lesssim \lambda^{d_X-2}. \end{aligned}$$

Thus the proof of (3.17) is complete.

On the other hand, if we take $x = y$ in (3.12) and replace $(\cos tP_X)(x, x)$ by the third or fourth terms in the right side of (3.16), one can use (3.8) to see that the resulting expression must be bounded by

$$\begin{aligned} &\int_{\{\xi \in \mathbb{R}^{d_X} : |\xi| \leq 2\lambda\}} (1 + |\xi|)^{-3} d\xi + \int_{\{\xi \in \mathbb{R}^{d_X} : |\xi| \geq 2\lambda\}} (1 + |\lambda - |\xi||)^{-N} d\xi \\ &= \begin{cases} O(\lambda^{d_X-3}), & d_X > 3, \\ O(\log \lambda), & d_X = 3, \\ O(1), & d_X = 2, \end{cases} \end{aligned}$$

which is better than desired. Clearly, if we also do this for the last term in the right side of (3.16), the resulting term will be $O(1)$.

Based on this, we would have the bounds in Proposition 3.1 for $d_X \geq 2$ if we could show that

$$\begin{aligned} (3.23) \quad &(2\pi)^{-d_X} \int_0^\infty \int_{\mathbb{R}^{d_X}} \frac{1}{\pi} \rho(t) \lambda \hat{m}_\delta(t) \cos t|\xi| d\xi dt \\ &= (2\pi)^{-d_X} |S^{d_X-1}| \times \frac{1}{2} B(\delta+1, d_X/2) \lambda^{d_X} + O(\lambda^{d_X-2}). \end{aligned}$$

If we repeat the argument that lead to (3.14), we find if we replace $\rho(t)$ here by one then the difference between this expression and the left side of (3.23) is bounded by

$$\lambda^{-\delta} \int_{\mathbb{R}^{d_X}} (1 + |\lambda - |\xi||)^{-N} d\xi$$

for any N and hence $O(\lambda^{d_X-1-\delta}) = O(\lambda^{d_X-2})$.

Thus, by Fourier's inversion formula, up to these errors, the expression in the left side of (3.23) is

$$(3.24) \quad (2\pi)^{-d_X} \int_{\mathbb{R}^{d_X}} m_\delta(|\xi|/\lambda) d\xi = (2\pi)^{-d_X} |S^{d_X-1}| \left(\int_0^1 m_\delta(r) r^{d_X-1} dr \right) \cdot \lambda^{d_X}.$$

Since by (3.9) the integral in the right side is equal to $\frac{1}{2} B(\delta+1, d_X/2)$, we obtain (3.23).

For the remaining case $d_X = 1$, we shall use the fact that for $|t|$ smaller than a fixed constant c , by choosing coordinates such that the metric equals dx^2 , we have

$$(3.25) \quad (\cos t P_X)(x, y) = (2\pi)^{-1} \int_{-\infty}^{\infty} \cos t\tau e^{i\tau(x-y)} d\tau.$$

Moreover, one can simply repeat the argument in (3.17)-(3.18) to see that,

$$(3.26) \quad (2\pi)^{-1} \int_0^{\infty} \int_{-\infty}^{\infty} \frac{1}{\pi} \rho(t) \lambda \hat{m}_{\delta}(t) \cos t\tau d\tau dt = (2\pi)^{-1} B(\delta + 1, 1/2) \lambda + O(\lambda^{-1}).$$

By (3.25), (3.26) and the arguments in (3.10)-(3.15), we obtain (3.5) when $d_X = 1$.

To finish, we still need to prove the facts (3.8) and (3.9) about m_{δ} that we used above. The latter just follows from the standard formula for the beta function stated above and a change of variables. To prove the former, (3.8) we note that if ρ is as in (3.10) then $\hat{m}_{\delta}(t)$ can be written as

$$\int \rho(1-\tau) (1-\tau)_{+}^{\delta} (1+\tau)^{\delta} e^{-it\tau} d\tau + \int \rho(1+\tau) (1+\tau)_{+}^{\delta} (1-\tau)^{\delta} e^{-it\tau} d\tau + a_0^{\delta}(t),$$

where $a_0^{\delta} \in \mathcal{S}(\mathbb{R})$ and hence satisfies the bounds in (3.8). A simple argument shows that the first two terms in the right can be written as $a_{+}^{\delta}(t)e^{it}$ and $a_{-}^{\delta}(t)e^{-it}$, respectively, with $a_{\pm}^{\delta}$ as in (3.8), which finishes the proof. $\square$

4. Further results and remarks.

In our main results, Theorem 1.1, Theorem 1.2 and Theorem 2.3, we focused on products of length two, as was the case of some of the earlier results, e.g., [6] and [7]. On the other hand, Iosevich and Wyman [14] obtained further improved Weyl error bounds for products of spheres $S^{d_1} \times S^{d_2} \times \cdots \times S^{d_n}$ as the length $n = 2, 3, \dots$ increased, and their $O(\lambda^{d-1-\delta_n})$ bounds, $d = d_1 + \cdots + d_n$, have $\delta_n \rightarrow 1$ as $n \rightarrow \infty$. As we noted earlier, such bounds are impossible for $\delta_n > 1$. Iosevich and Wyman conjectured that for such products of length $n \geq 5$ one should be able to take $\delta_n = 1 - \varepsilon$ for all $\varepsilon > 0$ or even $\delta_n = 1$, which would agree with the classical error term bounds for the n -torus (i.e., $d_1 = \cdots = d_n = 1$ and $n \geq 5$). See e.g. Walfisz [21].

Let us now show that the proof of Theorem 2.3 yields optimal L^q estimates for such products with q large. The particular case where $q = \infty$ can be thought of as a weaker version of the conjecture of Iosevich and Wyman [14] in the sense that it would follow from the somewhat stronger pointwise Weyl remainder variant of their conjecture.

The improved variant of Theorem 2.3 that follows from its proof and the aforementioned optimal bounds for $\mathbb{T}^n$ for $n = 5$, is the following.

Theorem 4.1. *Let $Y = S^{d_1} \times \cdots \times S^{d_5}$ be a product of 5 round spheres and let $-\Delta_Y = -(\Delta_{S^{d_1}} + \cdots + \Delta_{S^{d_5}})$ and $P_Y = \sqrt{-\Delta_Y}$. We then have for $\lambda \geq 1$*

$$(4.1) \quad \begin{aligned} & \|\mathbb{1}_{[\lambda^{-1}, \lambda]}(P_Y)\|_{L^2(Y) \rightarrow L^q(Y)} = O(\lambda^{d(\frac{1}{2} - \frac{1}{q}) - 1}), \\ & \text{if } d = d_1 + \cdots + d_5 \text{ and } q \geq \max\left\{\frac{2(d_j+1)}{d_j-1} : 1 \leq j \leq 5\right\}. \end{aligned}$$

Additionally, if $X = M^n$ is an n -dimensional, $n \geq 1$, compact Riemannian manifold and $P = \sqrt{-(\Delta_Y + \Delta_X)}$ then for $\lambda \geq 1$

$$(4.2) \quad \begin{aligned} & \left\| \mathbb{1}_{[\lambda-\lambda^{-1}, \lambda+\lambda^{-1}]}(P) \right\|_{L^2(X \times Y) \rightarrow L^q(X \times Y)} = O(\lambda^{(d+n)(\frac{1}{2}-\frac{1}{q})-1}), \\ & \text{if } q \geq \max\left\{\frac{2(d_1+1)}{d_1-1}, \frac{2(d_2+1)}{d_2-1}, \dots, \frac{2(d_5+1)}{d_5-1}, \frac{2(n+1)}{n-1}\right\}. \end{aligned}$$

Both estimates represents a $\lambda^{-1/2}$ improvement over the universal bounds in [18], and, as mentioned before, this is optimal.

To prove the Theorem, we first note that the second estimate, (4.2), is a simple consequence of the first one, (4.1), and Theorem 1.1 after noting that $\alpha(q, n) = n(\frac{1}{2} - \frac{1}{q}) - \frac{1}{2}$ for q as in (4.2).

Let us now see how we can use the proof of Theorem 2.3 and the classical improved lattice point counting bounds in dimension 5 to obtain (4.1).

Just as we did before for products of length 2, we first note that if $\{e_{j,k}^\nu\}_\nu$ is an orthonormal basis of spherical harmonics of degree k on S^{d_j} then an orthonormal basis of eigenfunctions on $Y = S^{d_1} \times \dots \times S^{d_5}$ is of the form

$$e_{1,k_1} \cdot e_{2,k_2} \cdot e_{3,k_3} \cdot e_{4,k_4} \cdot e_{5,k_5},$$

where $e_{j,k_j} = e_{j,k_j}^{\nu_j}$ for some ν_j , with $j = 1, 2, 3, 4, 5$. Consequently,

$$\begin{aligned} & \left(-\Delta_Y + \left(\frac{d_1-1}{2}\right)^2 + \dots + \left(\frac{d_5-1}{2}\right)^2\right) [e_{1,k_1} \cdots e_{5,k_5}] \\ & = \left(\left(k_1 + \frac{d_1-1}{2}\right)^2 + \dots + \left(k_5 + \frac{d_5-1}{2}\right)^2\right) e_{1,k_1} \cdots e_{5,k_5}, \end{aligned}$$

and so, analogous to (2.48), the eigenvalues of P_Y are

$$(4.3) \quad \lambda = \lambda_{k_1, \dots, k_5} = \sqrt{\left(k_1 + \frac{d_1-1}{2}\right)^2 + \dots + \left(k_5 + \frac{d_5-1}{2}\right)^2}, \quad k_j = 0, 1, 2, \dots, \quad 1 \leq j \leq 5.$$

Also, analogous to before, an eigenfunction with this eigenvalue must be of the form

$$(4.4) \quad e_\lambda(x_1, \dots, x_5) = \sum_{\{(k_1, \dots, k_5): \lambda_{k_1, \dots, k_5} = \lambda\}} \left(\sum_{\nu_{j_1}^1, \dots, \nu_{j_5}^5} a_{k_1, \dots, k_5}^{\nu_{j_1}^1, \dots, \nu_{j_5}^5} e_{k_1}^{\nu_{j_1}^1}(x_1) \cdots e_{k_5}^{\nu_{j_5}^5}(x_5) \right).$$

Here $\{e_{k_\ell}^{\nu_{j_\ell}^\ell}\}_{j_\ell}$ is the orthonormal basis of spherical harmonics of degree k_ℓ on S^{d_ℓ} , $\ell = 1, \dots, 5$.

Next, we note that for q as in (4.1), we have that if, as in (1.2), $\alpha(q, d_j)$ denotes the λ -power in the universal L^q -estimates then

$$(4.5) \quad \alpha(q, d_j) = d_j\left(\frac{1}{2} - \frac{1}{q}\right) - \frac{1}{2},$$

if q is as in (4.1). Thus, if we inductively use the universal bounds from [18] (or the earlier bounds for spherical harmonics [16]), we find that if $k_1, \dots, k_5$ are fixed and $\lambda_{k_1, \dots, k_5} = \lambda$

$$\begin{aligned}
 (4.6) \quad & \left\| \sum_{\nu_{j_1}^1, \dots, \nu_{j_5}^5} a_{k_1, \dots, k_5}^{\nu_{j_1}^1, \dots, \nu_{j_5}^5} e_{k_1}^{\nu_{j_1}^1} \dots e_{k_5}^{\nu_{j_5}^5} \right\|_{L^q(Y)} \\
 & \leq C \left(\prod_{j=1}^5 \lambda^{d_j(\frac{1}{2} - \frac{1}{q}) - \frac{1}{2}} \right) \cdot \left(\sum_{\nu_{j_1}^1, \dots, \nu_{j_5}^5} |a_{k_1, \dots, k_5}^{\nu_{j_1}^1, \dots, \nu_{j_5}^5}|^2 \right)^{1/2} \\
 & = C \lambda^{d(\frac{1}{2} - \frac{1}{q}) - \frac{5}{2}} \left(\sum_{\nu_{j_1}^1, \dots, \nu_{j_5}^5} |a_{k_1, \dots, k_5}^{\nu_{j_1}^1, \dots, \nu_{j_5}^5}|^2 \right)^{1/2}.
 \end{aligned}$$

To use this we recall that when $n \geq 5$ we have the following improvement of (2.41)

$$(4.7) \quad \#\{j \in \mathbb{Z}^n : |j| = \lambda\} = O(\lambda^{n-2}), \quad \text{if } n \geq 5.$$

Indeed, this is a consequence of the stronger result for the problem of counting the number of integer lattice points inside λ -balls centered at the origin (e.g. [21, p. 45]).

If we use (4.7) along with Cauchy-Schwarz inequality we deduce that (4.6) implies that if e_λ is as in (4.4)

$$\begin{aligned}
 (4.1') \quad & \|e_\lambda\|_{L^q(Y)} \leq C \lambda^{d(\frac{1}{2} - \frac{1}{q}) - \frac{5}{2}} \sqrt{\lambda^3} \left(\sum_{\nu_{j_1}^1, \dots, \nu_{j_5}^5} |a_{k_1, \dots, k_5}^{\nu_{j_1}^1, \dots, \nu_{j_5}^5}|^2 \right)^{1/2} \\
 & = C \lambda^{d(\frac{1}{2} - \frac{1}{q}) - 1} \|e_\lambda\|_{L^2(Y)}.
 \end{aligned}$$

As before, this estimate for eigenfunctions implies the spectral projection bounds due to the fact that successive distinct eigenvalues of P_Y which are comparable to λ have gaps that are bounded below by $c_0 \lambda^{-1}$ for some uniform $c_0 > 0$. This completes the proof of Theorem 4.1.

Remark. It would be interesting to investigate other situations involving product manifolds where one is able to obtain L^q estimates that improve ones that follow from Theorem 1.1. For instance Canzani and Galkowski [7] showed that if Y is a product manifold then one has $\sqrt{\log \lambda}$ improvements over the universal bounds for large q (i.e., $\varepsilon(\lambda) = (\log \lambda)^{-1}$ in Theorem 1.1). In this case, $X \times Y$ in Theorem 1.1 would be a product of three manifolds, yet our results do not give further improvements over the results coming from [7]. Similarly, if *both* X and Y have improved eigenfunction bounds are there situations where $X \times Y$ can inherit both improvements, as opposed to the better of the two improvements for X and Y as guaranteed by Theorem 1.1? Our proof does not seem to yield such a result. Moreover, in many cases one cannot obtain $\sqrt{\varepsilon_X(\lambda)} \cdot \sqrt{\varepsilon_Y(\lambda)}$ improvements if X and Y , respectively, have $\sqrt{\varepsilon_X(\lambda)}$ and $\sqrt{\varepsilon_Y(\lambda)}$ improvements. This is the case, for instance, when they both involve power improvements $\sqrt{\varepsilon_X(\lambda)} = \lambda^{-\delta_X}$ and $\sqrt{\varepsilon_Y(\lambda)} = \lambda^{-\delta_Y}$ with $\delta_X + \delta_Y > 1/2$, for reasons mentioned before.

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