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ON DECAYING PROPERTIES OF NONLINEAR SCHRÖDINGER EQUATIONS*

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Abstract. In this paper we discuss quantitative (pointwise) decay estimates for solutions to the 3D cubic defocusing nonlinear Schrödinger equation with various (deterministic and random) initial data. We show that nonlinear solutions enjoy the same decay rate as the linear ones. The regularity assumption on the initial data is much lower than in previous results (see [C. Fan and Z. Zhao, *Discrete Contin. Dyn. Syst.*, 41 (2021), pp. 3973–3984] and the references therein), and, moreover, we quantify the decay, which is another novelty of this work. Furthermore, we show that the (physical) randomization of the initial data can be used to replace the L^1-data assumption (see [C. Fan and Z. Zhao, *Proc. Amer. Math. Soc.*, 151 (2023), pp. 2527–2542] for the necessity of the L^1-data assumption).

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1. Introduction.

1.1. Background and motivations.
Linear dispersive estimates on unbounded domains play a fundamental role in the study of nonlinear dispersive PDEs. It is in some sense the starting point for studying local well-posedness of nonlinear problems. In the Schrödinger case, the dispersive estimate in $\mathbb{R}^d$ reads as

$$\|e^{it\Delta}f\|_{L_x^\infty(\mathbb{R}^d)} \lesssim |t|^{-\frac{d}{2}}\|f\|_{L_x^1(\mathbb{R}^d)}, \tag{1}$$

where d is the spatial dimension.

For the defocusing nonlinear Schrödinger equation (NLS), great progress has been made in recent years to understand the scattering behavior of solutions; see, for example, [5], [12], [15], [38]. Those results say that, given an initial datum in a certain (critical) Sobolev space $\dot{H}^{s_c}$, there exists a unique global solutions u to the NLS, and the solution scatters in the sense that there exists some $u^\pm$, so that

$$\|u(t) - e^{it\Delta}u^\pm\|_{\dot{H}^{s_c}(\mathbb{R}^d)} \to 0, \quad \text{as } t \to \pm\infty. \tag{2}$$

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Here the critical space H^{s_c} is chosen to be invariant under the scaling invariance of the NLS.

From (1) and (2) one may then conjecture for the nonlinear solutions u an estimate similar to (1), for example, for $d=3$

$$\|u(t)\|_{L_x^\infty(\mathbb{R}^3)} \leq C(u_0)|t|^{-\frac{3}{2}}. \tag{3}$$

Note that conclusion (3) is not a priori obvious since, for example, it is not known whether $u^\pm$ in (2) is in L^1, and moreover the convergence in (2) is not a priori in L^∞ and certainly the rate is also not known. Indeed, the study of a decaying estimate for nonlinear solutions is related to the asymptotic convergence rate in (2) (see the appendix of [22]).

Decay estimates for nonlinear dispersive equations have been studied widely. In [45], Lin and Strauss studied the decay of the L^∞-norm of solutions to the 3D NLS based on the Morawetz estimate. (See also Corollary 3.4 in [28] for the cubic Hartree case.) It is also possible to apply the vector field methods and use commutator-type estimates to derive decay estimates; see, for example, [42]. (See also [43], [54], [34].)

For the Schrödinger case, in particular, before the work in [5], decay estimates (3) were key steps in proving scattering results; see also [25], [26], [27]. We also refer the reader to [57], and the references therein, for results on decay estimates regarding NLSs without decay rates.

In the present article, as in previous works of the first and third authors [21], [22], the starting point is whether, given the fact that scattering behaviors have by now been studied extensively, one can further improve the understanding of (3). Conceptually, one wants to understand how quickly or how slowly scattering behaviors can happen. The answer to this question will give more quantitative estimates for $C(u_0)$ in (3) as well.

To make the question more concrete, let us focus on the defocusing cubic NLS in 3D,

$$iu_t + \Delta u = |u|^2 u, \quad u(0,x) = u_0(x). \tag{4}$$

It has been proved in [21], [22] that, for all $u_0 \in H^4 \cap L^1$, (3) holds and $C(u_0)$ only depends on the size $\|u_0\|_{H^4 \cap L^1}$ rather than the profile of u_0. This follows from some concentration compactness consideration. But it was not clear how this C depends on the $\|u_0\|_{H^4 \cap L^1}$ in a quantitative way. See also [30].

It has also been proved in [22] that to obtain estimates of the form (3) for the solution to (4), it is not enough to place the initial data only in some Sobolev-type space H^s, even if one is aiming at a weaker decay rate. To be more precise, for any $g(t) > 0$ that goes to infinity as $t \to \infty$, we can construct, for instance, H^{100} solutions to (4) such that

$$\limsup_{t\to\infty} g(t)\|u(t) - e^{it\Delta}u^+\|_{\dot{H}^{1/2}} = \infty. \tag{5}$$

Compared to [21], [22] and other previous results, to the best of our knowledge, the main three *new* points in the current paper are the following: 1. To obtain an estimate such as (9) below we can considerably *lower* the regularity assumption of the solution to almost the critical level (see Theorem 1.5). 2. We can obtain the *quantitative* decay results by characterizing the implicit constant in the decay estimates (see Theorems 1.1, 1.4, 1.5, and 1.7). 3. We discuss the *randomized* case and show that (physical) randomization of the initial data can be used to replace the L^1-data assumption (see Theorem 1.7 and [22] for the necessity of the L^1-data assumption).

One may ask why the L^1 assumption for the initial data is important in the quantitative study of decaying estimates for the NLS. One may also ask why randomization in physical space can replace such an assumption. To answer the above two questions, recall that for every scattering solution u to (4), there holds the following.
For any given $\delta > 0$, one can find $L > 0$, so that

$$\|u(t)\|_{L_t^5[L,\infty)L_x^5} < \delta. \tag{6}$$

However, it is impossible to characterize this L in any quantitative way if one considers initial data $u_0 \in H^s$, since, for any given u_0, one can evolve backwards (nonlinearly) for a long time and get a new initial data, which delays the L so that (6) holds. To be more precise, let us fix a Schwarz initial data u_0, with $\|u_0\|_{H^s} \leq 1$. Let u be the associated solution. We know that $\|u\|_{L_{t,x}^5[0,\infty)} \geq \delta_0 > 0$ for some δ_0, and $\sup_t \|u(t)\|_{H^s} \leq M$ for some $M > 0$, since such a solution scatters. Let $u_{0,n}(x) := u(-n,x)$ and let u_n be the associated solution with initial data $u_{0,n}$. Note that $\|u_{0,n}\|_{H^s} \leq M$. Now fix $\delta = \delta_0/2$ and evaluate L_n so that (6) holds. One has that $L_n \geq n$ and cannot be bounded by M. The problem lies in the time translation symmetry in the cubic NLS. Both L^1 assumption and randomization in physical space play a role in removing the time translation symmetry in cubic NLSs.

Where the topic *decay estimates for Schrödinger equations* is concerned, there is an important type of problem, that is, studying the decay estimate associated with linear Schrödinger operator $-\Delta + V$ (where V is usually called a potential). Such problems have been studied widely in recent decades. We refer the reader to [36], [53] and the references therein. In the current paper, we are mainly concerned with the difficulties that come from the nonlinear evolution, since the linear part of our model is the free evolution and well known. We remark that one can also apply the methods in the current article to study a nonlinear model of type $iu_t + (-\Delta + V)u = N(u)$.

We believe that the methods in this paper can be applied to derive decay estimates for other dispersive models with suitable modifications. See section 6 for more discussions.

1.2. Notation. Throughout this note, we use C to denote the universal constant, and C may change from line to line. Also, α, β may change from line to line. We say $A \lesssim B$ if $A \leq CB$. We say $A \sim B$ if $A \lesssim B$ and $B \lesssim A$. We also use the notation C_B to denote a constant that depends on B. We use usual L^p spaces and Sobolev spaces H^s. Since we will always work on $\mathbb{R}^3$, we will write $L^p(\mathbb{R}^3)$ as L^p and $L^p_{t,x}(I \times \mathbb{R}^3)$ as $L^p_{t,x}(I)$. We will also use S^s to denote the Strichartz space,

$$\|u\|_{S^s(I)} := \|\langle D\rangle^s u\|_{L_t^q L_x^r(I)}, \tag{7}$$

where (q,r) are admissible, $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$, $q,r \geq 2, (q,r,d) \neq (2,\infty,2)$. Here $\langle D\rangle^s$ is the usual multiplier operator in this sense: $\mathcal{F}(\langle D\rangle^s f)(\xi) = \langle\xi\rangle^s \mathcal{F}(f)(\xi)$, where $\langle\xi\rangle = (1+\xi^2)^{\frac{1}{2}}$.

1.3. Statement of main results. We are now ready to state our results. Consider the Cauchy problem for the 3D cubic defocusing NLS,

$$iu_t + \Delta u = |u|^2 u, \quad u(0,x) = u_0(x). \tag{8}$$

The purpose of the current article is to present decay estimates of the form

$$\|u(t)\|_{L_x^\infty} \leq C(\|u_0\|_X)|t|^{-\frac{3}{2}}. \tag{9}$$

Note that we are mainly interested in t large.

The constant C in (9) in this article will only depend on the size (but not the profile) of the initial data, measured by a certain norm $\|\cdot\|_X$. We will characterize how this C depends on $\|u_0\|_X$. With different choices of $\|\cdot\|$ detailed below, we will derive results involving polynomial dependence or exponential dependence.

Below, α, β, C are constants that may change from line to line. We have the following.

THEOREM 1.1. *For the initial value problem* (8) *with initial data* $u_0 \in X = H^1 \cap L^1$, *one has that* (9) *holds with*

$$C(\|u_0\|_X) = C \exp C\|u_0\|_X^\beta \tag{10}$$

for some $\beta > 0$.

We also note that if one considers the asymptotic behavior, the implicit constant in Theorem 1.1 can be improved from the exponential bound to the polynomial bound as in the following corollary.

COROLLARY 1.2. *We have*

$$\limsup_{t\to\infty} t^{\frac{3}{2}} \|u(t)\|_{L_x^\infty} \leq C(1 + \|u_0\|_X)^\beta \tag{11}$$

for some $\beta > 0$.

We will give the proof for this corollary at the end of section 3.1 (after giving the proof for Theorem 1.1). We note that the analogous statement also holds for (18) in Theorem 1.7.

Remark 1.3. We note that the quantitative decay result, Theorem 1.1, immediately gives the *quantitative scattering rate* via direct integrations and the Strichartz estimate, which describes how fast the solution scatters to the final states in the critical space. (See the appendix of [22] for more details.) To be more precise, one can obtain the following: for u satisfying (8) with initial data $u_0 \in X = H^1 \cap L^1$ and for $t > 1$, we have

$$\|u(t) - e^{it\Delta}u^+\|_{\dot{H}_x^{\frac{1}{2}}} \leq C(\|u_0\|_X)t^{-2} \tag{12}$$

and

$$\|u(t)\|_{L^5_{t,x}(t\geq s)} \leq C(\|u_0\|_X)s^{-\frac{7}{10}}, \tag{13}$$

where $C(\|u_0\|_X) = C \exp C\|u_0\|_X^\beta$.

If one further assumes $xu_0 \in L^2$ (i.e., the finite variance condition), we can improve the exponential bound to the polynomial bound in the dispersive estimate (9) in the following sense.

THEOREM 1.4. *For the initial value problem* (8) *with initial data* $u_0 \in H^1 \cap L^1$ *and* $xu_0 \in L^2$, *letting* $\|u_0\|_X = \|u_0\|_{H^1\cap L^1} + \|xu_0\|_{L^2}$, *one has that* (9) *holds for*

$$C(\|u_0\|_X) = C(1 + \|u_0\|_X)^\beta \tag{14}$$

for some $\beta > 0$.

It is often natural to consider NLSs in the H^1 space since it is corresponding to the energy conservation law. Sometimes, it is also of interest to lower the H^s regularity of the initial data, and we recall that the Schrödinger initial value problem (8) is $H^{\frac{1}{2}}$ critical. We have the $H^{\frac{1}{2}}$-type result as follows.

THEOREM 1.5. *For the initial value problem* (8) *with initial data* $u_0 \in H^s \cap L^1$ *and* $xu_0 \in L^2$, *letting* $\|u_0\|_X = \|u_0\|_{H^s \cap L^1} + \|xu_0\|_{L^2_x}$, *where* $s > \frac{1}{2}$, *one has that* (9) *holds for*

$$C(\|u_0\|_X) = C\exp(\|u_0\|_X^\beta) \tag{15}$$

for some $\beta > 0$.

Remark 1.6. We remark that reaching the end point case of Theorem 1.5, i.e., reaching the initial data $u_0 \in H^{\frac{1}{2}} \cap L^1$ and $xu_0 \in L^2$, would be as hard as reaching the (quantitative) global well-posedness for (8) with initial data in $H^{1/2}$, which is a major open problem. See, however, Dodson's recent works [16], [17]. Indeed, it is also open to prove global well-posedness of (8) with $H^s, s > \frac{1}{2}$ initial data. We will briefly point out in the proof of Theorem 1.5 why $xu_0 \in L^2$ is of help here, i.e., it is not hard to prove global well-posedness for initial data such that $\|u_0\|_{H^s} + \|xu_0\|_2 < \infty, s > 1/2$.

Sometimes, it is not favorable to have the L^1 condition. We remark below that one can remove the L^1 assumption in Theorems 1.1–1.5 by performing a randomization in physical space for the initial data. To be more precise, let $\phi_n(x) := \phi(x-n), n \in \mathbb{Z}^3$, be a partition of unity of $\mathbb{R}^3$,

$$1 = \sum_n \phi_n = \sum_n \phi(x-n). \tag{16}$$

Let $g_n(\omega)$ be i.i.d. standard Gaussian, and let

$$u_0^\omega(x) = \sum \phi_n(x) g_n(\omega) u_0(x). \tag{17}$$

We recall that randomization in physical space was also used in [49]; see also [6].

We assume $\|u_0\|_X = 1$, where $X = H^1$ or $H^1 \cap \|xu_0\|_{L^2}$ or $H^s \cap \|xu_0\|_{L^2}$. Note that the size of the data can be absorbed in the Gaussian. We then have the following.

THEOREM 1.7. *Consider the initial value problem* (8) *with randomized initial data* u_0^ω. *Let* u^ω *be the associated global solutions. Let* $u_{nl}^\omega = u^\omega - e^{it\Delta} u_0^\omega$.

- *If* $\|u_0\|_{H^1} \leq 1$, *then except for a small probability set of size* $\exp A^{-\alpha}$, *one has*

$$\|u_{nl}^\omega\|_{L_x^\infty} \leq C(\exp A^\beta)|t|^{-\frac{3}{2}} \tag{18}$$

  for some α, $\beta > 0$.
- *If* $\|u_0\|_{H^1} + \|xu_0\|_{L_x^2} \leq 1$, *then except for a small probability set of size* $\exp A^{-\alpha}$, *one has*

$$\|u_{nl}^\omega\|_{L_x^\infty} \leq C(1+A)^\beta |t|^{-\frac{3}{2}} \tag{19}$$

  for some α, $\beta > 0$.
- *If* $\|u_0\|_{H^s} + \|xu_0\|_{L^2} \leq 1$, $s > \frac{1}{2}$, *then up to small probability* $\exp A^{-\alpha}$, *one has*

$$\|u_{nl}^\omega(t)\|_{L_x^\infty} \leq C(\exp A^\beta)|t|^{-\frac{3}{2}} \tag{20}$$

  for some α, $\beta > 0$.

Remark 1.8. Regarding the linear evolution of a random data as defined above, though we cannot obtain the pointwise estimate for them, they behave in time average like a linear evolution of L^1 data. Indeed, that is why one can remove the L^1 assumptions by doing randomization in physical space.

Remark 1.9. We note that for any $u_0 \in L_x^2$ but not in L^1, one has that almost surely $\|u_0^\omega\|_{L^1} = \infty$. Indeed, $u_0 \notin L^1$ implies $\sum_{n\in\mathbb{Z}^3} \|u_0\phi_n\|_{L^1} = \infty$. And one computes $\|f^\omega\|_{L^1} = \sum_{n\in\mathbb{Z}^3} \|u_0\phi_n\|_{L^1}|g_{\mathbf{n}}|$. The desired result follows by applying Lemma A.1 in the appendix.

It is also possible to perform randomization in frequency space to lower the regularity assumption. This is a very active research field ever since the seminal work of Bourgain [2], [3] in the periodic setting; see also the recent breakthrough [14] and references therein. We do not discuss this issue here.

1.4. A technical remark. In this subsection, we briefly explain why we can lower the regularity assumption in this paper (see the proofs in the following sections for more details). For convenience, we compare Theorem 1.1 (H^1-regularity assumption) in this paper with Theorem 1.4 (H^4-regularity assumption) in [21].

As in [21], we decompose the nonlinear solution u into several parts, and we want to control all of them since we intend to use a bootstrap argument to show the decay estimate. The most nontrivial term is

$$F_3 = i\int_{t-M}^{t} e^{i(t-s)\Delta}(|u|^2u)(s)ds, \tag{21}$$

where M is a positive constant depending on the size of the initial data. This term is part of the Duhamel expression of the nonlinear solution when the integral is close to t.

We intend to control the L_x^∞-norm of F_3. However, to control this term, higher regularity (H^4-regularity) is required in [21]. The reason is that if one uses the dispersive estimate directly, one has

$$\|F_3\|_{L_x^\infty} = \left\|\int_{t-M}^{t} e^{i(t-s)\Delta}|u|^2u ds\right\|_{L_x^\infty} \tag{22}$$

$$\lesssim \int_{t-M}^{t} \left\|e^{i(t-s)\Delta}|u|^2u\right\|_{L_x^\infty} ds \tag{23}$$

$$\lesssim \int_{t-M}^{t} (t-s)^{-\frac{3}{2}}\left\||u|^2u\right\|_{L_x^1} ds \tag{24}$$

$$\lesssim \int_{t-M}^{t} (t-s)^{-\frac{3}{2}}\|u\|_{L_x^3}^3 ds. \tag{25}$$

We note that, for the integral over $[t-M, t]$, the $(t-s)^{-\frac{3}{2}}$-term is too singular (nonintegrable), and thus it is not possible to control $\|F_3\|_{L_x^\infty}$ by $ct^{-\frac{3}{2}}$ in this way for some c small. In order to solve this issue, one can control the L_x^∞-norm of F_3 by estimating its H^4-, $\dot{H}^1$-, and L^2-norms (see the proof of Theorem 1.4 in [21] for details). However, doing this will inevitably cause higher-regularity requirements.

For the current paper, we observe that one trick, which is based on classical Sobolev embedding, allows one to control the L_x^∞-norm of F_3 in the following way:

$$\begin{aligned}\|F_3\|_{L_x^\infty} &= \left\|\int_{t-M}^t e^{i(t-s)\Delta}|u|^2u ds\right\|_{L_x^\infty}\\ &\lesssim \left\|\int_{t-M}^t e^{i(t-s)\Delta}|\nabla_x|(|u|^2u)ds\right\|_{L_x^3}\\ &\lesssim \int_{t-M}^t \|e^{i(t-s)\Delta}|\nabla_x|(|u|^2u)\|_{L_x^3}ds\\ &\lesssim \int_{t-M}^t (t-s)^{-3(\frac{1}{2}-\frac{1}{3})}\||\nabla_x|(|u|^2u)\|_{L_x^{\frac{3}{2}}}ds\\ &\lesssim \left(\int_{t-M}^t ((t-s)^{-3(\frac{1}{2}-\frac{1}{3})})^{2-}ds\right)^{\frac{1}{2-}}\cdot\left(\int_{t-M}^t (\||\nabla_x|(|u|^2u)\|_{L_x^{\frac{3}{2}}})^{2+}ds\right)^{\frac{1}{2+}}.\end{aligned}$$

We can see that the advantage of performing the Sobolev inequality first is to lower the spatial exponent for the integrand from ∞ to 3; thus after using the dispersive estimate, we end up with $(t-s)^{-\frac{1}{2}}$, which is not too singular near t (i.e., it is integrable). Then, applying the Hölder inequality, it suffices to deal with the second term in the last line above, which is manageable via Strichartz-type control. Thus, one can handle the F_3-term based on the scattering result. We refer the reader to sections 3, 4, and 5 for more proof details.

This observation has proved useful in lowering the regularity requirement for many cases when one considers nonlinear decay problems. We note that assumptions with quite high regularity were needed to obtain the pointwise decay estimate for NLSs; see, for example, [21], [28], [30] and reference therein. Moreover, we note that similar ideas are also useful for studying the long time dynamics for the stochastic NLS; see the recent works [19], [20] for more information.

1.5. Structure of the paper. The rest of the article is organized as follows. In section 2, we include the basic estimates and the global results for 3D cubic NLS; in section 3, we give the proofs for Theorems 1.1 and 1.4; in section 4, we give the proof for Theorem 1.5; in section 5, we give the proof for Theorem 1.7; and in section 6, we give some further remarks on the applications of this method for other models.

2. Preliminaries.

In this section, we collect some useful results and estimates, including the standard dispersive estimate and Strichartz estimates for Schrödinger equations and the global well-posedness theory for the 3D cubic NLS.

The standard dispersive estimates and Strichartz estimates for the free Schrödinger operator $e^{it\Delta}$ when $d=3$ reads as follows. We refer the reader to [8, 55] for details.

LEMMA 2.1 (dispersive estimate). *The linear operator $e^{it\Delta}$ in $\mathbb{R}^3$ satisfies the bound*

$$\|e^{it\Delta}f\|_{L_x^\infty}\lesssim |t|^{-\frac{3}{2}}\|f\|_{L_x^1}.$$

Moreover, by interpolation with the unitary relation $\|e^{it\Delta}f\|_{L_x^2}=\|f\|_{L_x^2}$, we have

$$\|e^{it\Delta}f\|_{L_x^p}\lesssim |t|^{-d(\frac{1}{2}-\frac{1}{p})}\|f\|_{L_x^{p'}} \tag{26}$$

for every $p\geq 2$.

DEFINITION 2.2. *Let $q,r\in[2,+\infty]$. We say (q,r) is an admissible Strichartz pair in $\mathbb{R}^3$ if*

$$\frac{2}{q}+\frac{3}{r}=\frac{3}{2}.$$

LEMMA 2.3 (Strichartz estimate). *Let (q,r) be an admissible Strichartz pair in $\mathbb{R}^3$. Then we have the bound*

$$\|e^{it\Delta}f\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^3)} \lesssim \|f\|_{L^2(\mathbb{R}^3)}. \tag{27}$$

Also, for any two Strichartz pairs (q_1,r_1) and (q_2,r_2), we have

$$\left\|\int_0^t e^{i(t-s)\Delta}F(s)ds\right\|_{L_t^{q_1}L_x^{p_1}(\mathbb{R}\times\mathbb{R}^3)} \lesssim \|F\|_{L_t^{q_2'}L_x^{p_2'}(\mathbb{R}\times\mathbb{R}^3)}, \tag{28}$$

where q_2' and r_2' are conjugates of q_2 and r_2.

Then we turn to the global theory for 3D cubic NLS (8). If one works with initial data in H^1, the scattering result is indeed easier. Much stronger low regularity results holds for equation (8), [11], [37]; see also and reference therein.

As a corollary of lower regularity results [11], [37], one has the following.

PROPOSITION 2.4. *The initial value problem (8) is globally well-posed and scatters in the H^1 space. More precisely, for any u_0 with finite energy, $u_0 \in H^1$, there exists a unique global solution $u \in C_t^0(H_x^1)\cap L_{x,t}^5$ such that*

$$\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}|u(t,x)|^5 dxdt \le C(\|u_0\|_{H^1}) \tag{29}$$

for some constant $C(\|u_0\|_{H^1})$ that depends only on $\|u_0\|_{H^1}$. More precisely, $C(\|u_0\|_{H^1})$ is a polynomial of the H^1-norm of the initial data.

Remark 2.5. We note that the scattering norm $L_{x,t}^5$ can be interpolated by the interaction Morawetz bound in 3D,

$$\|u\|_{L_{t,x}^4}^4 \lesssim \|u\|_{L^2}^2 \cdot \sup_t \|\nabla^{\frac{1}{2}}u\|_{L^2}^2 \lesssim \|u\|_{H^1}^4, \tag{30}$$

and the a priori bound $\|u\|_{L_t^\infty H_x^1}$ according to the conservation of energy. Thus it can be expressed as a polynomial of the H^1-norm of the initial data.

Strictly speaking, one first performs an interpolation between $L_{t,x}^4$ and $L_t^\infty L_x^6$ to obtain a control for some $L_t^p L_x^q$, so that $\frac{2}{p}+\frac{3}{q}=1$. This a priori bound plus the local theory gives an a priori $L_t^5 L_x^5$ bound.

Remark 2.6. We also note that, for the NLS with criticality s_c larger than the criticality of the interaction Morawetz estimate, i.e., $\frac{1}{4}$, if one assumes an a priori bound higher than the critical level, $\|u\|_{L_t^\infty H_x^s} < \infty$ $(s > s_c)$, then the scattering can be obtained directly via interpolation. Thus, generally speaking, studying the NLS model with data in a critical space is highly nontrivial.

3. Proofs of Theorems 1.1 and 1.4.

3.1. Proof of Theorem 1.1. We start with Theorem 1.1. This is the most nontechnical part of the five theorems, but the proofs of the other theorems build upon this one.

We denote

$$M_1 = \|u_0\|_{L_x^1} + (\|u_0\|_{H^1}+1)^2 \tag{31}$$

and only consider M_1 large. It is enough to prove, for all $t>0$,

$$t^{\frac{3}{2}}\|u(t)\|_{L_x^\infty} \lesssim \exp M_1^\alpha \tag{32}$$

for some $\alpha > 0$.

We recall that we have scattering for such data,

$$\|u\|^p_{L^p_t L^q_x} \lesssim M_1^\beta, \tag{33}$$

and by the laws of conservation of mass and energy, we have

$$\|u(t)\|_{L^\infty_t H^1} \lesssim M_1. \tag{34}$$

We proceed with a bootstrapping argument for the quantity

$$A(t) := \sup_{0\le\tau\le t} \tau^{3/2}\|u(\tau)\|_{L^\infty_x}. \tag{35}$$

We only perform a priori estimates for $A(t)$; i.e., we will assume $A(t)$ is finite for all t and prove estimates of form, for example,

$$A(t) \le C + \epsilon A(t). \tag{36}$$

One can apply approximation and continuity arguments (bootstrapping argument) to transfer those a priori estimates to desired estimates, i.e., get rid of the assumption that $A(t)$ is finite for all t. We will need a large parameter M which will be determined later. We write down the Duhamel formula of u,

$$u(t) = e^{it\Delta}u_0 - i\int_0^t e^{i(t-s)\Delta}(|u(s)|^2u(s))ds, \tag{37}$$

and for notational simplicity, we'll denote

$$N(u) = |u|^2u. \tag{38}$$

We first prove the following lemma.

LEMMA 3.1. *For* $0 < t \ll M_1^{-100}$,

$$\|u(t)\|_{L^\infty_x} \le CM_1t^{-3/2} + Ct^{-1/2}M_1^3 + \frac{1}{2}A(t)t^{-3/2}. \tag{39}$$

This lemma, though simple looking, establishes the basis for the bootstrapping argument for $A(t)$; i.e., $A(t)$ is locally finite. But we note that this estimate is only useful for a short time.

Proof of Lemma 3.1. By a classical dispersive estimate,

$$\|e^{it\Delta}u_0\|_{L^\infty_x} \lesssim M_1t^{-3/2}, \tag{40}$$

and this handles the linear part in the Duhamel formula (37). For the nonlinear part in (37), we may split the integral into $\int_0^t = \int_0^{t/2} + \int_{t/2}^t$. We estimate the first part by

(41)
$$\int_0^{t/2}\|e^{i(t-s)\Delta}N(u)\|_{L^\infty_x}ds \lesssim t^{-3/2}\int_0^{t/2}\|N(u)\|_{L^1_x}ds \lesssim t^{-3/2}\frac{t}{2}\|u\|^3_{L^\infty_t \dot H^{1/2}} \lesssim t^{-1/2}M_1^3.$$

We use the Sobolev embedding, $W^{3+,1} \mapsto L^\infty$, and estimate via

$$\begin{aligned} &\left\|\int_{t/2}^t e^{i(t-s)\Delta}(N(u(s)))\right\|_{L^\infty_x} ds \\ &\lesssim \int_{t/2}^t \|e^{it\Delta}\langle\nabla\rangle N(u)\|_{L^{3+}_x}ds \\ &\lesssim \int_{t/2}^t (t-s)^{-(\frac{1}{2}+)}\|\langle\nabla\rangle N(u)\|_{L^{\frac{3}{2}}_x}\ ds, \end{aligned} \tag{42}$$

and we apply again a dispersive estimate in the last step.

Applying the fractional Leibniz rule[1] when $s<1$, Theorem A-12 in [39], or the usual Leibniz rule when $s=1$, we have

$$\|\langle\nabla\rangle^s N(u)\|_{L_x^p} \le \|u\|_{H^s}\|u\|_{L_x^\infty}\|u\|_{L_x^q} \quad \forall \frac{1}{p}=\frac{1}{q}+\frac{1}{2}, \quad 1<p,q<\infty. \tag{43}$$

One may now carry out the estimate (42) as

$$\begin{aligned} &\int_{t/2}^t (t-s)^{-(\frac{1}{2}+)}\|\langle\nabla\rangle N(u)\|_{L_x^{\frac{3}{2}-}} ds \\ &\lesssim \left(\int_{t/2}^t (t-s)^{-1+}\right)^{\frac{1}{2-}} \left(\int_{t/2}^t \|\langle\nabla\rangle N(u)\|_{L_x^{\frac{3}{2}}}^{2+}\right)^{\frac{1}{2+}} ds \\ &\lesssim t^{\epsilon} M_1^2 \left(\int_{t/2}^t \|u(s)\|_{L_x^\infty}^{2+}\|u\|_{L_x^{6-}}^{2+} ds\right)^{\frac{1}{2+}}. \end{aligned} \tag{44}$$

Summarizing (42), (44), and letting $(4+,6-)$ be an $\dot{H}^{1/2}$-admissible pair in the sense of $\frac{2}{4+}+\frac{3}{6-}=\frac{3}{2}-\frac{1}{2}$, we have

$$\begin{aligned} \left\|\int_{t/2}^t e^{i(t-s)\Delta}N(u(s))ds\right\|_{L_x^\infty} &\lesssim t^{\epsilon}M_1^2\left(\int_{t/2}^t \|u(s)\|_{L_x^\infty}^{2+}\|u\|_{L_x^6}^{2+}\ ds\right)^{\frac{1}{2+}} \\ &\lesssim t^{\epsilon}M_1^2(t)^{\frac{1}{4}-}\left(\int_{t/2}^t \|u\|_{L_x^\infty}^{4+}\|u\|_{L_x^{6-}}^{4+} ds\right)^{\frac{1}{4+}}. \end{aligned} \tag{45}$$

Thus, we have, by (40), (41), (45), for $t\ll M_1^{-100}$,

$$\|u(t)\|_{L_x^\infty} \le CM_1t^{-3/2}+Ct^{-1/2}M_1^2+CM_1^2t^{\frac{1}{4}-}A(t)\left(\int_{t/2}^t \|u\|_{L_x^{6-}}^{4+} ds\right)^{\frac{1}{4+}}, \tag{46}$$

which gives

$$A(t)\le CM_1+CtM_1^2+CM_1^2t^{\frac{3}{2}+\frac{1}{4}-}A(t)\left(\int_{t/2}^t \|u\|_{L_x^{6-}}^{4+} ds\right)^{\frac{1}{4+}}. \tag{47}$$

Lemma 3.1 now follows, since $t\le M_1^{-100}$ and M_1 large. ∎

Next step, we want to handle the case $t\le 2M$, and we need to refine the proof of Lemma 3.1. We state the next lemma as follows.

LEMMA 3.2. *For $t\le 2M$,*

$$\|u(t)\|_{L_x^\infty}\le CM_1t^{-3/2}+M_1^{53}+\frac{1}{2}A(t)t^{-3/2}. \tag{48}$$

Proof. We observe that conservation laws give

$$\|u(t)\|_{L_t^\infty H^1}<\infty. \tag{49}$$

[1] It is somewhat important here that the Leibniz rule covering the end point involves L_x^∞.

According to the standard argument (see Lemma 3.12 in [12], for example), one can control all the Strichartz norms by the size of energy. Thus one has

$$\|u(t)\|_{L_t^{4+}L_x^{6-}} \lesssim M_1^{-10}. \tag{50}$$

We still use the Duhamel formula, (37), and estimate the linear part as in (40). Then we split the nonlinear part into $\int_0^t = \int_0^{t-M_1^{-100}}$ and $\int_{t-M_1^{-100}}^t$. Now, for the first part we proceed as in (41) via

$$\begin{aligned}
&\left\|\int_0^{t-M_1^{-100}} e^{i(t-s)\Delta} N(u(s))ds\right\|_{L_x^\infty} \\
&\lesssim \int_0^{t-M_1^{-100}} (t-s)^{-3/2}\|u(s)\|_{L_x^3}^3 ds \\
&\lesssim M_1^{50} M_1^3,
\end{aligned} \tag{51}$$

and for the second part as in (42), (44), and we derive

$$\begin{aligned}
&\left\|\int_{t-M_1^{-100}}^t e^{i(t-s)\Delta} N(u(s))ds\right\|_{L_x^\infty} \\
&\lesssim (M_1^{-100})^{\frac{1}{2}-} M_1^2 \left(\int_{t-M_1^{-100}}^t \|u(s)\|_{L_x^\infty}^{4+} \|u\|_{L_x^{6-}}^{4+} ds\right)^{\frac{1}{4+}}.
\end{aligned} \tag{52}$$

Thus, the proof of Lemma 3.2 is now complete. ∎

Now we turn to another lemma for the case $t \geq 2M$ (M is to be decided later). The statement reads as follows.

LEMMA 3.3. *For $t \geq 2M$,*

$$\begin{aligned}
\|u(t)\|_{L_x^\infty} \leq\; & CM_1 t^{-3/2} + CM_1^7 t^{-3/2} + \frac{1}{2}A(t)t^{-\frac{3}{2}} \\
& + CM_1^{4+} t^{-3/2}\left(\int_{t-M}^t A(s)^{4+}\|u(t)\|_{L_x^{6-}}^{4+} ds\right)^{\frac{1}{4+}}.
\end{aligned} \tag{53}$$

Proof. To deal with this case, we need to, similarly as done in [21], further write (37) into

$$\begin{aligned}
u(t) &= e^{it\Delta}u_0 - i\int_0^M e^{i(t-s)\Delta}N(u)ds - i\int_M^{t-M} e^{i(t-s)\Delta}N(u(s))ds \\
&\quad - i\int_{t-M}^t e^{i(t-s)\Delta}N(u(s))ds \\
&:= e^{it\Delta}u_0 + F_1 + F_2 + F_3.
\end{aligned}$$

The estimate of F_1 will be straightforward, since $t - M \gtrsim t$, and we have, via the simple Minkowski inequality and the usual dispersive estimate,

$$\|F_1\|_{L_x^\infty} \lesssim t^{-3/2}\int_0^M \|u(s)\|_{L_x^3}^3 ds \lesssim t^{-3/2} M M_1^3. \tag{54}$$

For F_2, it is crucial that we are in dimension[2] at least 3, and we have

$$\begin{aligned}\|F_2\|_{L_x^\infty} &\le \int_M^{t-M} \|e^{i(t-s)\Delta}N(u(s))\|_{L_x^\infty} ds \\ &\lesssim \int_M^{t-M} (t-s)^{-3/2}\|u\|_{L_x^\infty}\|u\|_{L_x^2}^2 ds \\ &\lesssim \int_M^{t-M} (t-s)^{-3/2}A(s)s^{-\frac{3}{2}}M_1^2 ds.\end{aligned} \tag{55}$$

Thus, following as in [21], we have

$$\|F_2\|_{L_x^\infty} \le \frac{1}{2}A(t)t^{-3/2} \tag{56}$$

for all $M \ge CM_1^4$ for some universal C large.

For F_3, the main point here is that we use analysis similar to Lemmas 3.1 and 3.2 to refine the analysis in [21], and we gain further control via the Gronwall's argument.

Now, similar to the proof of Lemma 3.2, in particular the part for (52), we derive

$$\|F_3\|_{L_x^\infty} \lesssim \|F_3\|_{W^{3+,1}} \lesssim M^{\frac{1}{2}+}M_1^2\left(\int_{t-M}^t \|u(s)\|_{L_x^\infty}^{4+}\|u\|_{L_x^{6-}}^{4+} ds\right)^{\frac{1}{4+}}. \tag{57}$$

To summarize, using (54) and choosing $M \sim M_1^4$ so that (56) holds, and using (57), we obtain Lemma 3.2. ∎

Based on the above three lemmas, we are now ready to prove Theorem 1.1.

Concluding the proof of Theorem 1.1. By Lemma 3.2, one has, for some $\beta_1 > 0$,

$$A(t) \le CM_1^{\beta_1} \quad \forall t \le CM_1^4, \tag{58}$$

and by Lemma 3.3, one has, for some $\beta_2, \beta_3 > 0$,

$$\begin{aligned}A(t) &\le CM_1^{\beta_2} + CM_1^{\beta_3}\left(\int_{t-M}^t A(s)^{4+}\|u(t)\|_{L_x^{6-}}^{4+} ds\right)^{\frac{1}{4+}} \\ &\le CM_1^{\beta_2} + CM_1^{\beta_3}\left(\int_0^t A(s)^{4+}\|u(t)\|_{L_x^{6-}}^{4+} ds\right)^{\frac{1}{4+}}.\end{aligned} \tag{59}$$

Thus, for all t,

$$A(t)^{4+} \le CM_1^{\beta_4} + CM_1^{\beta_4}\int_0^t A(s)^{4+}\|u(s)\|_{L_x^{6-}}^{4+} ds. \tag{60}$$

Via Gronwall's inequality and (33), one has

$$A(t) \le C\exp M_1^{\beta}. \tag{61}$$

This gives Theorem 1.1. ∎

An explanation for Corollary 1.2. We now briefly explain the point of Corollary 1.2 as follows. We still define $A(t)$ in the same way. In view of the scattering result

[2]One can see that this scheme will have a log divergence if instead one considers cubic NLS in dimension 2.

(scattering norm is finite), letting t be big enough, the quantity $(\int_{t-M}^{t} \|u(t)\|_{L_x^6}^{4+} \, ds)^{\frac{1}{4+}}$ can be made arbitrarily small, say $\epsilon > 0$.

Thus, consider the first inequality in (59), letting t be big enough, we can drag $A(s)^{4+}$ out and, letting ϵ be small enough to beat other constants,

$$
\begin{aligned}
A(t) &\leq CM_1^{\beta_2} + CM_1^{\beta_3} \left(\int_{t-M}^{t} A(s)^{4+} \|u(t)\|_{L_x^{6-}}^{4+} ds \right)^{\frac{1}{4+}} \\
&\leq CM_1^{\beta_2} + CM_1^{\beta_3} A(t) \left(\int_{t-M}^{t} \|u(t)\|_{L_x^{6-}}^{4+} ds \right)^{\frac{1}{4+}} \\
&\leq CM_1^{\beta_2} + CM_1^{\beta_3} A(t) \epsilon \\
&\leq CM_1^{\beta_2} + \frac{1}{2} A(t).
\end{aligned}
\tag{62}
$$

Then moving the second term on the right-hand side to the left, this gives us a polynomial bound for $\limsup_{t\to\infty} A(t)$ as desired. In many situations one may be particularly interested in the asymptotic behavior of solutions for t large rather than estimates that are uniform with respect to t. In our case we are, for example, interested in $\sup_{t>0} t^{3/2} \|u(t)\|_{L_x^\infty}$ rather than $\limsup_{t\to\infty} t^{3/2} \|u(t)\|_{L_x^\infty}$. Thus the above observation may be useful for some problems. One may also understand this decay result in the following manner: the exponential constant dependence is caused by the finite time; for the long time, the constant dependence is essentially polynomial.

The proof of Corollary 1.2 is now complete. □

3.2. Proof of Theorem 1.4. One can obtain a polynomial-type control rather than an exponential one with the extra assumption $xu_0 \in L_x^2$. This is because in this case one can apply the pseudoconformal transformation to get a quantitative control for the decay of L_x^6. The argument below seems classical; see, for example, [48]. Indeed, letting $J(t) = x + 2it\nabla$, one has

$$\|f\|_{L_x^6} \lesssim t^{-1} \|J(t) f\|_{L_x^2}. \tag{63}$$

Furthermore, for u solving (8), the quantity

$$\frac{1}{2} \|J(t) f\|_{L_x^2}^2 + \int t^2 |u|^4 dx \tag{64}$$

is monotonically decreasing in t, and when $t = 0$, it equals $\|xu\|_{L_x^2}^2$. Thus, letting

$$M_1 := (1 + \|u_0\|_{H_1})^2 + \|xu_0\|_{L_x^2} + \|u_0\|_{L_x^1}, \tag{65}$$

one has

$$\|u(t)\|_{L_x^6} \lesssim t^{-1} M_1, \tag{66}$$

and a simple interpolation with mass conservation gives

$$\|u(t)\|_{L_x^{6-}} \lesssim t^{-(1-)} M_1. \tag{67}$$

Now we go back to (58), (59), and enhance (59) into

$$
\begin{aligned}
A(t) &\leq CM_1^{\beta_2} + CM_1^{\beta_3} \left(\int_{t-M}^{t} t^{-(4-)} A(s)^{4^+} ds \right)^{\frac{1}{4+}} \\
&\leq CM_1^{\beta_2} + CM_1^{\beta_3} \left(\int_{0}^{t} (M+s)^{-(4-)} A(s)^{4^+} ds \right)^{\frac{1}{4+}}.
\end{aligned}
\tag{68}
$$

Here we use the fact that for $s \in [t-M,t]$, $s \sim t$, since $t \geq 2M$. Thus, rather than (60), we have the estimate for A

$$A(t)^{4+} \leq C(M_1+M)^{\beta_4} + CM_1^{\beta_4} \int_0^t (M+t)^{-(4-)} A(s)^{4+} ds. \tag{69}$$

Here we need to choose M to be a large polynomial of M_1, so that

$$M_1^{\beta^4} \int_0^\infty (s+M)^{-4(-)} ds \lesssim 1, \tag{70}$$

and Gronwall's inequality for (69) gives

$$A(t) \lesssim M_1^{\beta}. \tag{71}$$

4. Proof of Theorem 1.5. We now turn to the proof of Theorem 1.5. We first note that in general it is very hard to study unconditional scattering for (8) for initial data $u_0 \in H^s, s > 1/2$, and it is a major open problem to study global well-posedness for $u_0 \in H^{1/2}$. Both are not hard if one further assumes that the initial data is in $xu_0 \in L^2$.

Assume u solves (8) with initial data

$$\|u_0\|_{H^1} + \|xu_0\|_{L_x^2} \leq M_1 < \infty. \tag{72}$$

We have, for some $\beta_0 > 0$,

$$\begin{aligned} &\|u\|_{L_{t,x}^5}^5 \leq CM_1^{\beta_0}, \\ &\|u\|_{L_t^\infty H^s} \leq Ce^{M_1^{\beta_0}}. \end{aligned} \tag{73}$$

The proof of the first estimate in (73) may be classical from the pseudoconformal symmetry (see, for example, the textbook [55] and [4]); we briefly sketch it below for the convenience of the reader. The second estimate in (73) follows from the first one by classical persistence of regularity arguments; see, for example, [55] and Lemma 3.12 in [12] (by slightly changing the value of β_0 if necessary).

Fixing $s > 1/2$, by the local theory, we have that there is a $\delta > 0$, with $\delta \sim M_1^{-\beta_1}$, so that u is well-posed, and

$$\|u\|_{L_{t,x}^5[0,\delta]} \leq 1. \tag{74}$$

Meanwhile, by the pseudoconformal transform, we may define

$$\tilde{u}(s) = \frac{1}{(-s)^{3/2}} u\left(-\frac{1}{s}, \frac{x}{-s}\right) e^{-i|x|^2 s/4}, \tag{75}$$

and note that if u solves (8) in $[0,\delta]$, then $\tilde{u}$ is well defined in $(-\infty, -1/\delta]$ and solves

$$i\partial_s \tilde{u} + \Delta \tilde{u} = -s|\tilde{u}|^2 \tilde{u}. \tag{76}$$

Consider

$$\begin{aligned} H(\tilde{u}(s)) &:= \frac{1}{2}\int |\nabla \tilde{u}(s)|^2 dx + \int \frac{1}{4}(-s)|\tilde{u}|^4 dx \\ &\equiv \frac{1}{8}\|J(t)u(t)\|_{L_x^2}^2 + \frac{1}{4}\int t^2 |u|^4 dx, \end{aligned} \tag{77}$$

where one makes the change of variable $t=-\frac{1}{s}$, and recall that we have $J(t)=x+2it\nabla$. Then we see that H is monotonically decreasing, and

$$\lim_{s\to-\infty} H(\tilde{u}(s)) = \lim_{t\to 0}\frac{1}{8}\|J(t)u(t)\|_{L_x^2}^2 + \frac{1}{4}\int t^2|u|^4 dx = \|xu_0\|_{L_x^2}^2. \tag{78}$$

Thus, we have

$$H\left(\tilde{u}\left(-\frac{1}{\delta}\right)\right) \leq CM_1^2, \tag{79}$$

and hence $\tilde{u}$ is a global solution with $\|u\|_{L_t^\infty H^1} \leq CM_1$. Moreover, the standard H^1 local theory for the usual cubic NLS gives

$$\|\tilde{u}\|_{L_{t,x}^5[-\frac{1}{\delta},0]} \leq CM_1^{\beta_2}, \tag{80}$$

since $(-s)$ is bounded by $-\frac{1}{\delta}\sim M_1^{\beta_1}$ for $s\in[-\frac{1}{\delta},0]$. Thus, one further obtains

$$\|u\|_{L_{t,x}^5[\frac{1}{\delta},0)} \leq CM_1^{\beta_3}. \tag{81}$$

This gives (73).

Furthermore, we still derive $\|J(t)u\|_{L_x^2} \leq CM_1$, and thus, by (63), we obtain

$$\|u(t)\|_{L_x^6} \leq CM_1 t^{-1}. \tag{82}$$

We can now plug (73), (82) into the scheme of the proof of Theorem 1.4. We will sketch the argument highlighting the modifications that need to be made. We still focus on a priori estimates for $A(t)$, (35), and we pose

$$(\|u_0\|_{H^1}+1)^2 + \|u_0\|_{L_x^1} + \|xu_0\|_{L_x^2} = M_1. \tag{83}$$

We start with an analogue of Lemma 3.1,

LEMMA 4.1. *There exists $\beta_4>0$, β_4 large, so that for $t\leq M_1^{-\beta_4}$, one has*

$$\|u(t)\|_{L_x^\infty} \leq CM_1 t^{-3/2} + CM_1^3 t^{-\frac{1}{2}}. \tag{84}$$

Proof of Lemma 4.1. Recall the Duhamel formula, (37); the linear part is still controlled via (40). By the local theory of (8), we have

$$\|u\|_{L_t^\infty H^s[0,M_1^{-100}]} \lesssim M_1. \tag{85}$$

For the nonlinear part, we still split $\int_0^t = \int_0^{t/2} + \int_{t/2}^0$. The first part is still controlled via

$$\left\|\int_0^{t/2} e^{i(t-s)\Delta}N(u)ds\right\|_{L_x^\infty} \lesssim t^{-3/2}\int_0^{t/2}\|u(s)\|_{L_x^3}^3 ds \lesssim t^{-1/2}M_1^3. \tag{86}$$

In the last step of (86) we applied (85). For the second term, we are in some sense at the end point case when s approaches $\frac{1}{2}$. Before we present more details, we want to mention that one will see, since we are fixing $s>\frac{1}{2}$, that we just stay away from the end point and we always have some room.[3]

[3]The easiest way to do a first check of the computation is to neglect *log* convergence and pose all the κ_i below as zero.

We now analyze the second term. We pick $0<\kappa_1,\kappa_2\ll 1$ small and fix them. We will decide their relative size later.

We may assume $\kappa_1\le \frac{1}{2}(s-\frac{1}{2})$.

We use Sobolev embedding,

$$W^{\frac{1}{2}+\kappa_1,p_1}\to L^\infty, \tag{87}$$

where $p_1=6-\kappa_2$, satisfying

$$\frac{3}{p_1}<\frac{1}{2}+\kappa_1. \tag{88}$$

(It is enough to further assume $\kappa_1>3\kappa_2$ for (88) to hold.) We now estimate

$$\begin{aligned}&\left\|\int_{\frac{t}{2}}^t e^{i(t-s)}N(u(s))ds\right\|_{L_x^\infty}\\&\le C\int_{t/2}^t \|e^{it\Delta}N(u)ds\|_{W^{\frac{1}{2}+\kappa_1,p_1}}ds\\&\le C\int_{t/2}^t (t-s)^{1+\frac{-2\kappa_2}{2(6-\kappa_2)}}\|N(u)\|_{W^{s,p_1'}}ds.\end{aligned} \tag{89}$$

In the last step we apply the dispersive estimate and recall $\|e^{it\Delta}\|_{L^{p_1'}\to L^{p_1}}\lesssim t^{-(1-\frac{\kappa_2}{2(6-\kappa_2)})}$. Now, we plug in the estimate

$$\|N(f)\|_{W^{s,p_1'}}\le C\|f\|_{H^s}\|f\|_{L_x^\infty}\|f\|_{L_x^{p_2}}, \tag{90}$$

where $\frac{1}{p_2}+\frac{1}{2}=\frac{1}{p_1'}$, and one computes $p_2=\frac{4-\kappa_2}{12-2\kappa_2}=3-$, where we have used the fractional Leibniz rule, Theorem A-12 in [39]. We now continue the estimate (89) as

$$\begin{aligned}&\le C\int_{t/2}^t (t-s)^{1-\frac{\kappa_2}{2(6-\kappa_2)}}\|u\|_{H^s}\|u\|_{L_x^\infty}\|u\|_{L_x^{p_2}}ds\\&\le CA(t)t^{-3/2}\int_{t/2}^t (t-s)^{1-\frac{\kappa_2}{2(6-\kappa_2)}}\|u(t)\|_{H^s}^2ds\le CA(t)t^{-3/2}t^{\frac{\kappa_2}{2(6-\kappa_2)}}M_1^2,\end{aligned} \tag{91}$$

since $t\le M_1^{-\beta_4}$, and thus when β_4 is large enough, one has (89) bounded by $\frac{1}{2}A(t)t^{-3/2}$. Combining this with (86) and (40), Lemma 4.1 follows. ∎

We now cover the part when $t\le 2M$, and again M is a large number which will be chosen later.

LEMMA 4.2. *For $t\le 2M$, one has, for some β_5 large,*

$$\|u(t)\|_{L_t^\infty}\le M_1t^{-\frac{3}{2}}+(M_1^{\beta_5}+\ln M)e^{3M_1\beta_0}+\frac{1}{2}A(t)t^{-\frac{3}{2}}. \tag{92}$$

Proof of Lemma 4.2. We may assume $t\ge M_1^{-\beta_4}$; otherwise we use Lemma 4.1. Recalling again (37), this time we split the nonlinear part as $\int_0^t=\int_0^{t-e^{M_1^{\beta_6}}}$ and $\int_{t-e^{M_1^{\beta_6}}}^t$.

Now, one estimates the first part as

$$
\begin{aligned}
(93) \qquad & \int_0^{t-e^{-M_1^{\beta_6}}} \|e^{i(t-s)}N(u)\|_{L_x^\infty} ds \\
& \leq C\int_0^{t-e^{-M_1^{\beta_6}}} (t-s)^{-1}\|u\|_{H^s}^3 ds \\
& \leq C(M_1^{\beta_6}+\ln M)e^{3M_1^{\beta_0}}.
\end{aligned}
$$

In the last step we have plugged in (73). For the second part, we estimate similarly as (89), (91) (choosing κ_1, κ_2 as in the proof of Lemma 4.1),

$$
\begin{aligned}
& \left\|\int_{t-e^{-M_1^{\beta_6}}}^t e^{i(t-s)\Delta}N(u(s))ds\right\|_{L_x^\infty} \\
(94) \qquad & \leq C\int_{t-e^{-M_1^{\beta_6}}}^t (t-s)^{1-\frac{\kappa_2}{2(6-\kappa_2)}}\|u\|_{H^s}^2 ds \\
& \leq Ce^{-\frac{\kappa_2}{2(6-\kappa_2)}M_1^{\beta_6}}e^{2M_1^{\beta_0}}A(t)t^{-3/2}.
\end{aligned}
$$

Thus, when β_6 is large enough, so that $Ce^{-\frac{\kappa_2}{2(6-\kappa_2)}M_1^{\beta_6}}e^{2M_1^{\beta_0}} \leq \frac{1}{2}$, then one can combine it with (93) and (40) to obtain the desired estimate. ∎

We now present an analogue of Lemma 3.3.

LEMMA 4.3. *For $t \geq 2M$, one has*

$$
(95) \qquad \|u(t)\|_{L_x^\infty} \leq Ct^{-3/2}Me^{3M_1^{\beta_0}} + \frac{1}{2}A(t)t^{-3/2} + CA(t)t^{-\frac{3}{2}}e^{M_1^{\beta_0}}M^{-(\frac{1}{2}-)}.
$$

Proof of Lemma 4.3. We write again

$$
\begin{aligned}
u(t) &= e^{it\Delta}u_0 - i\int_0^M e^{i(t-s)\Delta}N(u)ds - i\int_M^{t-M} e^{i(t-s)\Delta}N(u(s))ds \\
&\quad - i\int_{t-M}^t e^{i(t-s)\Delta}N(u(s))ds \\
&:= e^{it\Delta}u_0 + F_1 + F_2 + F_3
\end{aligned}
$$

and estimate F_1, F_2, F_3. The terms F_1, F_2 will be estimated similarly as in the proof of Lemma 3.3. We have for F_1

(96)

$$
\|F_1(t)\|_{L_x^\infty} ds \leq \int_0^M \|e^{i(t-s)\Delta}N(u(s))\|_{L_x^\infty} \leq Ct^{-3/2}M\|u\|_{L_t^\infty H^s}^3 \leq Ct^{-3/2}Me^{3M_1^{\beta_0}}.
$$

We have for F_2, exactly as in the proof of (56),

$$
(97) \qquad \|F_2\|_{L_x^\infty} \leq \frac{1}{2}A(t)t^{-3/2}.
$$

For F_3, compared to the proof of Lemmas 4.1 and 4.2, it is important now for us to also apply (82) in the estimate of F_3. Recalling $0 < \kappa_1, \kappa_2 \ll 1$ defined as in the proof of Lemma 4.1, and $p_1 = 6-\kappa_2, p_2 = \frac{4-\kappa_2}{12-2\kappa_2}$, we estimate F_3 via

$$
\begin{aligned}
&\int_{t-M}^{t} e^{i(t-s)\Delta} N(u(s))\|_{L_x^\infty} ds \\
&\le C \int_{t-M}^{t} \|e^{i(t-s)\Delta} N(u(s))\|_{W^{\frac{1}{2}+\kappa_1,3}} ds \\
&\le C \int_{t-M}^{t} (t-s)^{-(1-\frac{\kappa_2}{2(6-\kappa_2)})} \|u\|_{H^s} \|u\|_{L_x^{p_2}} \|u\|_{L_x^\infty} ds \\
&\le CA(t) t^{-3/2} t^{-(\frac{1}{2}-\kappa_3)} M_1 e^{M_1^{\beta_0}} \int_{t-M}^{t} (t-s)^{-(1-\frac{\kappa_2}{2(6-\kappa_2)})} ds \\
&\le CA(t) t^{-\frac{3}{2}} t^{-(\frac{1}{2}-\kappa_2)} M^{\frac{\kappa_2}{2(6-\kappa_2)}}.
\end{aligned} \tag{98}
$$

Above we used a simple interpolation to conclude that

$$
\|u\|_{L_x^{p_2}} = \|u\|_{L_x^{3-}} \le C \|u\|_{L_x^2}^{\frac{1}{2}+} \|u\|_{L_x^6}^{\frac{1}{2-}} \le t^{-(\frac{1}{2}-\kappa_3)} M_1
$$

for some $\kappa_1 > 0$ and we plug the result into (73).

Now, given $t \ge 2M$, one summarizes estimate (98) as

$$
\|F_3\|_{L_x^\infty} \le CA(t) t^{-\frac{3}{2}} e^{M_1^{\beta_0}} M^{-(\frac{1}{2}-)}. \tag{99}
$$

□

Summarizing Lemmas 4.1, 4.2, and 4.3, choosing $M = e^{M_1^{\beta_8}}$ so that (97) holds, we have that (98) reads as

$$
\|F_3(t)\|_{L_x^\infty} \le \frac{1}{10} A(t) t^{-3/2}. \tag{100}
$$

As a consequence we have

$$
u(t) \le CM_1 t^{-3/2} + M^{\beta_9} e^{CM_1^{\beta_0}} t^{-\frac{3}{2}} + \frac{3}{2} A(t) t^{-3/2}, \tag{101}
$$

i.e.,

$$
A(t) \le CM^{\beta_9} e^{CM_1^{\beta_0}} \lesssim e^{M_1^{\beta}} \tag{102}
$$

for some $\beta > 0$. This concludes the proof of Theorem 1.5.

5. Proof of Theorem 1.7.

What we want to present here is that one can systematically remove the L^1 assumptions in Theorems 1.1, 1.4, and 1.5 by randomizing the initial data. We will only prove the case for estimate (18); the other two cases can be generalized from Theorems 1.4 and 1.5, respectively.

We will fix a constant A, large. We will use the terminology A-certain in [13]; i.e., we say an event is A-certain if it holds up to a set with small probability e^{-A^α}. (The exact value of α may change from line to line, but at the end, one only chooses the smallest α involved.)

Note that we *cannot* conclude[4] that A-certainly

$$
\|e^{it\Delta} u_0^\omega\|_{L_x^\infty} \lesssim A^\beta t^{-(\frac{3}{2})} \quad \forall t \ge 0. \tag{103}
$$

However, we can prove, in some sense, a time average version of (103). We present more details below. We need to introduce a weight,

$$
\gamma_{p,\epsilon} = \frac{t^{100}}{1+t^{100}} t^{3(\frac{1}{2}-\frac{1}{p})-\epsilon}. \tag{104}
$$

[4]We cannot even conclude (103) if one replaces $\frac{3}{2}-$ for $\frac{3}{2}$.

LEMMA 5.1. *For all $2 < p < \infty$ and for all $\epsilon_1 > \epsilon_2$, then A-certainly*[5]

$$\|\gamma_{p,\epsilon_1} e^{it\Delta} u_0^\omega\|_{L_t^{\frac{1}{\epsilon_2}} L_x^p[0,\infty)} \leq A; \tag{105}$$

similarly, one also has

$$\|\gamma_{p,\epsilon_1} e^{it\Delta} \nabla u_0^\omega\|_{L_t^{\frac{1}{\epsilon_2}} L_x^p[0,\infty)} \leq A. \tag{106}$$

Remark 5.2. Regarding the weight in (104), one notes that if for some t $f(t) \lesssim t^{-3(\frac{1}{2}-\frac{1}{p})}$, then

$$\|\gamma_{p,\epsilon}(t) f(t)\|_{L_t^{\frac{1}{\epsilon'}}} \lesssim 1 \quad \forall \epsilon' < \epsilon. \tag{107}$$

Meanwhile, if $\|\gamma_{p,\epsilon}(t) f(t)\|_{L_t^{\frac{1}{\epsilon'}}} \lesssim 1$, then in the time average sense, $\gamma_p f(t) \lesssim t^{-\epsilon'}$, and thus in the time average sense, $f(t) \lesssim t^{-3(\frac{1}{2}-\frac{1}{p})-\epsilon+\epsilon'}$.

We now turn to the proof of Lemma 5.1, which is merely a combination of Minkowski's inequality and some standard large deviation estimate for Gaussian.

Proof. It is enough to prove, for ρ large and for $u_0^\omega = \sum_n g_n(\omega)\phi_n(x)u_0(x)$, that

$$\|\gamma_{p,\epsilon_1}(t) e^{it\Delta} u_0^\omega\|_{L_\omega^\rho L_t^{\frac{1}{\epsilon_2}} L_x^p} \lesssim \sqrt{\rho} \|u_0\|_{L_x^2}. \tag{108}$$

Then the desired A-certain claim comes from the usual Chebyshev inequality. Recall that one has (see, for example, Lemma 3.1 in [7])

$$\left\| \sum_n c_n g_n(\omega)(\omega) \right\|_{L_\omega^\rho} \lesssim \rho^{1/2} \left(\sum_n c_n^2 \right)^{1/2}. \tag{109}$$

Thus, by Minkowski's inequality, we have

$$\begin{aligned} &\|\gamma_{p,\epsilon_1}(t) e^{it\Delta} u_0^\omega\|_{L_\omega^\rho L_t^{\frac{1}{\epsilon_2}} L_x^p} \\ &\lesssim \left\| \left\| \sum_n \gamma_{p,\epsilon}(t) e^{it\Delta} \phi_n u_0 g_n(\omega) \right\|_{L_\omega^\rho} \right\|_{L_t^{\frac{1}{\epsilon_2}} L_x^p} \\ &\lesssim \rho^{1/2} \left\| \left(\sum_n |\gamma_{p,\epsilon_1}(t) e^{it\Delta}(\phi_n u_0)|^2 \right)^{1/2} \right\|_{L_t^{\frac{1}{\epsilon_2}} L_x^p}, \end{aligned} \tag{110}$$

where in the last step we apply (109). By applying Minkowski's inequality again, we have

[5]The α depend on $p, \epsilon_1, \epsilon_2$ though.

$$\left\| \left(\sum_n |\gamma_{p,\epsilon_1}(t) e^{it\Delta}(\phi_n u_0)|^2 \right)^{1/2} \right\|_{L_t^{\frac{1}{\epsilon_2}} L_x^p}$$
$$\leq \left(\sum_n \| |\gamma_{p,\epsilon_1}(t) e^{it\Delta}(\phi_n u_0)| \|_{L_t^{\frac{1}{\epsilon_2}} L_x^p} \|^2 \right)^{1/2}$$
$$\lesssim \left(\sum_n \|\phi_n u_0\|_{L_x^{p'}}^2 \right)^{1/2}$$
$$\lesssim \left(\|\sum_n \phi_n u_0\|_{L_x^2}^2 \right)^{1/2}, \tag{111}$$

where in the second inequality we use Remark 5.2 and the dispersive estimate.

Estimates (110), (111) give (108), and Lemma 5.1 thus follows. □

We also note that A-certainly

$$\|u_0^\omega\|_{H^1} \leq A. \tag{112}$$

Thus, we have that A-certainly, for some $\beta_0 > 0$,

$$\begin{aligned} &\|u^\omega\|_{L_{t,x}^5}^5 \leq A^{\beta_0}, \quad \|u^\omega\|_{L_t^\infty H^1} \leq A, \\ &\|u_{nl}^\omega\|_{L_{t,x}^5}^5 \leq A^{\beta_0}, \quad \|u_{nl}^\omega\|_{L_t^\infty H^1} \leq A, \end{aligned} \tag{113}$$

where $u_{nl}^\omega = u^\omega - e^{it\Delta} u_0^\omega$.

We will later only rely on ω-wise estimate (105), (106), (113). For notational convenience, we fix ω and omit this ω, denoting $v = e^{it\Delta} u_0^\omega$ and $w = u_{nl}^\omega$. We write down the Duhamel formula,

$$w(t) = -i \int_0^t e^{i(t-s)\Delta} |v(s)|^2 v(s) ds - i \int_0^t e^{i(t-s)\Delta} (|w+v|^2 (w+v) - |v|^2 v) ds. \tag{114}$$

At this time, we focus on the bootstrap estimate for

$$A(t) := \sup_{s \leq t} s^{-3/2} \|w(s)\|_{L_x^\infty}. \tag{115}$$

We explain the heuristic why such a generalization will work. If v satisfies the exact estimates for some linear solution $e^{it\Delta} f$, $f \in H^1 \cap L^1$, then this is exactly the same proof (for the same problem) as that of Theorem 1.1. Here, however, all v involved in (114) are in the integral. Thus, for our problem, it is enough for v to satisfy the estimates $e^{it\Delta} f$ in the time average sense; see Remark 5.2. There is some $t^{-\epsilon}$ loss but it does not matter since we are never in a critical situation in our setting. We remark that it remains a very interesting problem to understand from a quantitative viewpoint the long time dynamic for initial data randomized in physical space $H^{1/2}$.

We carry out the a priori estimate for w and $A(t)$, and later we will also need a parameter M. We write (114) as

$$w = G_1 + G_2 + G_3 + G_4, \tag{116}$$

where[6]

$$
\begin{aligned}
G_1 &= -i\int_0^t e^{i(t-s)\Delta}|v(s)|^2 v(s)ds,\\
G_2 &= -i\int_0^t e^{i(t-s)\Delta}|w|^2 w ds,\\
G_3 &= i\int_0^t e^{i(t-s)\Delta}(O(w^2 v(s)))ds,\\
G_4 &= i\int_0^t e^{i(t-s)\Delta}(O(wv^2(s)))ds.
\end{aligned}
\tag{117}
$$

The G_1 part will be addressed in Lemma 5.3. The G_2 can be estimated exactly as in the proof of Theorem 1.1, i.e., Lemmas 3.1, 3.2, and 3.3. But if one thinks more carefully, one sees that G_3 can also be estimated exactly as in the proof of Theorem 1.1. The only different part may appear in the estimate for $\int_{t-M}^t O(w^2 v)$, which involves the estimate for $\nabla(w^2 v)$, but in this part, there is still one free w left for us to apply the bootstrap control of $A(t)$, and thus one can still estimate it similarly. For the G_4 part, as discussed above, one only needs to estimate the part

$$
\int_{t-M}^t e^{i(t-s)\Delta}(O(wv^2(s)))ds, \quad t \geq 2M,
\tag{118}
$$

and more precisely, one only needs to handle the part

$$
\int_{t-M}^t (t-s)^{-(1/2+)}\|\nabla w\|_{L_x^2}\|v^2(s)\|_{L_x^6} ds,
\tag{119}
$$

and the other parts follow the same estimate as (57). The estimate for (119) will be treated in Lemma 5.4.

For G_1, one has the following lemma.

LEMMA 5.3.

$$
G_1(t) \lesssim A^3 t^{-3/2}.
\tag{120}
$$

Proof. One again splits the integral $\int_0^t = \int_0^{t/2} + \int_{t/2}^t$. For the first part, one estimates

$$
\left\|\int_0^{t/2} e^{i(t-s)\Delta}|v|^2 v(s)ds\right\|_{L_x^\infty} \lesssim t^{-3/2}\int_0^\infty \|v(s)\|_{L_x^3}^3 ds.
\tag{121}
$$

Note that locally $\|v\|_{L_x^3} \leq \|v\|_{H^1} \leq A$, and for t large, by choosing ϵ_1, ϵ_2 small, we have, thanks to (105),

$$
\int_1^t \gamma_{3,\epsilon_1}^{-3}(s)\|\gamma_{3,\epsilon_1}(s)v(s)\|_{L_x^3}^3 ds \lesssim \|\gamma_{3,\epsilon_1}^{-3}\|_{L_t^{\frac{1}{1-3\epsilon_2}}}\|\gamma_{3,\epsilon_1}(s)v(s)\|_{L_t^{1/\epsilon_2}L_x^3} \lesssim A^3.
\tag{122}
$$

[6] Here we are slightly abusing the O notation. $O(fg)$ means the term can be estimated by fg, $\bar{f}g$, $f\bar{g}$, $\bar{f}\bar{g}$, and similarly for $O(fgh)$.

This handles the $\int_0^{t/2}$ part. For the second part $\int_{t/2}^t$, when t is small, one estimates as

$$
\begin{aligned}
&\left\| \int_{t/2}^{t} e^{i(t-s)\Delta} |v(s)|^2 v(s) ds \right\|_{L_x^\infty} \\
&\int_{t/2}^{t} \| e^{i(t-s)\Delta} |v(s)|^2 v(s) \|_{L_x^\infty} ds \\
&\lesssim \int_{t/2}^{t} \| e^{i(t-s)} \Delta |v|^2 v \|_{W_x^{1,3+}} ds \\
&\lesssim \int_{t/2}^{t} (t-s)^{-(\frac{1}{2}+)} \|v\|_{H^1} \|v\|_{L_x^{12-}} \|v\|_{L_x^{12-}} ds \\
&\lesssim t^{1/2-} A \int_{t/2}^{t} \gamma_{12-,\epsilon_1}^{-2} \|\gamma_{12-,\epsilon_1} v\|^2_{L_t^{1/\epsilon_1} L_x^{12-}} ds \\
&\lesssim t^{1/2-} (1+t)^{-(\frac{5}{2}-)}.
\end{aligned} \tag{123}
$$

□

For the estimate for (119), one has the following.

LEMMA 5.4. *For M large, $t \geq 2M$, one has*

$$
\int_{t-M}^{t} (t-s)^{-(\frac{1}{2}+)} \|w\|_{H^1} \|v^2\|_{L_x^6} ds \lesssim M^{1/2+} A^2 (1+t)^{-(5/2-)}. \tag{124}
$$

Proof. This computation is a parallel of the last part of the proof of Lemma 5.3. Note that $\|f^2\|_{L_x^6} \leq \|f\|^2_{L_x^{12}}$, and we have

$$
\begin{aligned}
&\int_{t-M}^{t} (t-s)^{-(\frac{1}{2}+)} \|w\|_{H^1} \|v^2\|_{L_x^6} ds \\
&\int_{t-M}^{t} (t-s)^{-(\frac{1}{2}+)} \|w\|_{H^1} \|v\|_{L_x^{12}} \|v\|_{L_x^{12}} ds \\
&\lesssim M^{1/2-} A \int_{t/2}^{t} \gamma_{12-,\epsilon_1}^{-2} \|\gamma_{12-,\epsilon_1} v\|^2_{L_t^{1/\epsilon_1} L_x^{12-}} ds \\
&\lesssim M^{1/2+} A^2 (1+t)^{-(5/2-)}.
\end{aligned} \tag{125}
$$

□

To summarize, similarly to (60), with also Lemmas 5.3 and 5.4, choosing $M \sim A^4$, we have

$$
\begin{aligned}
A(t) \leq\; & CA^3 + CM^{1/2+} A^2 (1+t)^{-(5/2-)} t^{3/2} CA^{\beta_2} \\
&+ CA^{\beta_3} \left(\int_0^t A^{4+} (\|w\|_{L_x^{6-}} + \|v\|_{L_x^{6-}})^{4+} \right)^{1/4}.
\end{aligned} \tag{126}
$$

Note that the first two terms in (126) come from Lemmas 5.3 and 5.4. Applying Gronwall's inequality, the desired estimate (18) follows.

We briefly mention a technical point if one wants to further generalize Theorems 1.4 and 1.5. One again applies the pseudoconformal energy trick to prove

$$
\|u(t)\|_{L_x^6} \lesssim t^{-1}, \tag{127}
$$

and we only need this estimate for t large.

It should be noted that in the proof of Theorems 1.4 and 1.5, it is enough to have

$$\|u(t)\|_{L_x^6} \lesssim t^{-(1-)} \tag{128}$$

for t large, and one only needs the decay estimate (128) to hold in the time average sense. Thus, we again split $u = v + w$, and (105) will replace estimate (128) for v, and $w = u - v$ also a priori satisfies (128) in the time average sense, and this is enough. One will also need an H^s version of (112). We will prove A-certainly that

$$\left\| \sum_n g_n(\omega)\phi_n u_0 \right\|_{H^s} \leq A. \tag{129}$$

It is enough to prove the deterministic inequality

$$\|f\|_{H^s}^2 \lesssim \sum_n \|\phi_n f\|_{H^s}^2. \tag{130}$$

It is enough to prove, for all n,

$$\|\phi_n \langle \nabla \rangle^s f\|_{L_x^2}^2 \lesssim \sum_{|n-n'|\leq 10} \sum_{n'} \|\phi_{n'} f\|_{H^s}^2, \tag{131}$$

which is obvious.

6. Further remarks. In this section, we make a few more remarks on the decay results for other well-known dispersive equations. We will list some results that are expected to be obtained by using the methods in this paper with suitable modifications. We leave them for interested readers. We emphasize that these decay results are based on corresponding scattering results, and we will provide the appropriate references. Moreover, we remark here that the implicit constants appearing are dependent of the size of the initial data and the size of the scattering norm (for some models, the scattering norm can be expressed by a function of the size of the initial data according to existing results).

1. *The 3D, energy critical defocusing NLS case.* The 3D energy critical NLS is an important and well-studied model in the area of dispersive equations. We refer the reader to the seminal work [12] for the global well-posedness and scattering result for this model (see also [5], [29] for the radial case). We write the Schrödinger initial value problem as follows:

$$(i\partial_t + \Delta_{\mathbb{R}^3})u = u|u|^4, \quad u(0,x) = u_0(x) \in L^1 \cap H^{1+}(\mathbb{R}^3). \tag{132}$$

Assume ϕ is a solution to (132). We expect that one could show

$$\|\phi(t)\|_{L_x^\infty} \leq C(u_0)|t|^{-\frac{3}{2}}. \tag{133}$$

Moreover, the constant dependence is triple exponential of the initial data since the scattering norm is double exponential, and we have Gronwall's inequality to use. This result obviously improves Theorem 1.1 in [21] since the regularity requirement is much lower (from H^3 to H^{1+}). We also note that one may also consider the higher dimensional case; see [52, 56] for the global results.

We emphasize that if one only considers L_x^p-decay ($p < \infty$), in the sense of showing

$$\|\phi(t)\|_{L_x^p} \leq C(u_0)|t|^{-3(\frac{1}{2}-\frac{1}{p})}, \tag{134}$$

then assuming H^1-regularity instead of H^{1+}-regularity for the initial data is already enough, which is more natural. This ϵ-requirement is caused by a log-divergence problem when one considers the L_x^∞-decay case.

2. *The energy supercritical NLS case.* We note that this method can be modified to handle the energy supercritical case once the scattering result is known. For this case, we focus on a typical model as an example: 4D cubic defocusing NLS. The model is $\dot{H}^{\frac{3}{2}}$ critical, which is above the energy critical level. See [18] and the references therein for the corresponding scattering result and background. We consider the Schrödinger initial value problem,

$$(i\partial_t + \Delta_{\mathbb{R}^4})u = u|u|^4, \quad u(0,x) = u_0(x) \in L^1 \cap H^{\frac{3}{2}+}(\mathbb{R}^4). \tag{135}$$

Assume ϕ is the solution to (135) and the a priori bound $\limsup_{t\in I} \|u\|_{H^{\frac{3}{2}+}} = M_1 < \infty$, where I is the maximal interval of existence. We expect that one could show

$$\|\phi(t)\|_{L_x^\infty} \leq C(u_0)t^{-2}. \tag{136}$$

Here the constant depends on the size of the initial data, M_1, and the scattering norm.

As a comparison, we also note that, for certain defocusing supercritical NLS problems, one has blow-up-type results; see the recent result of Merle et al. [46] and the references therein for more information.

3. *The fourth-order NLS case.* There are many different specific fourth-order NLS models. We consider a typical case as an example: the cubic fourth-order Schrödinger equation (4NLS) on $\mathbb{R}^d$ ($5 \leq d \leq 8$). We refer the reader to [51] for the corresponding global result and the references therein for the background. We consider the Schrödinger initial value problem, for $5 \leq d \leq 8$,

$$(i\partial_t + (\Delta_{\mathbb{R}^d})^2)u = u|u|^p, \quad u(0,x) = u_0(x) \in L^1 \cap H^{2+}(\mathbb{R}^d). \tag{137}$$

Assume ϕ is the solution to (137). We expect that one could show

$$\|\phi(t)\|_{L_x^\infty} \leq C(u_0)|t|^{-\frac{d}{4}}. \tag{138}$$

Here the constant depends on the size of the initial data and the scattering norm. This result improves [59] in the sense of the regularity requirement.

4. *The fractional NLS case.* We also consider a typical model and refer the reader to [31] for the scattering result. We consider the fractional Schrödinger initial value problem for $d \geq 2$ and $\frac{d}{2d-1} < \alpha < 1$,

$$(i\partial_t + (\Delta_{\mathbb{R}^d})^\alpha)u = u|u|^{\frac{4\alpha}{d-2\alpha}}, \quad u(0,x) = u_0(x) \in L^1 \cap H^{\alpha+}. \tag{139}$$

Assume ϕ is the solution to (139). We expect that one could show

$$\|\phi(t)\|_{L_x^\infty} \leq C(u_0)|t|^{-\frac{d}{2}}. \tag{140}$$

Here the constant depends on the size of the initial data and the scattering norm. One may also consider other fractional NLS cases; see [23], [24], [44] and the references therein.

5. *Some other cases.* There are more models one may consider for an analysis similar to the one we conducted above: cubic-quintic NLS (see [50], [41] and the references therein), inhomogeneous NLS (see [47] and the references therein), NLS on waveguides (see [32], [35], [60] for examples), NLS with a partial harmonic potential (this case is similar to the waveguide case; see [1], [9], [33]), Schrödinger resonant systems (see [10], [58]), nonlinear wave equations (see [55]), the Klein Gordon equation (see [55]), and NLS with a nice potential such that the dispersive estimate and the scattering hold (see [40] for an example and the references therein).

Appendix A. A technical lemma. Though the following probabilistic lemma is probably classical, we include it with a proof for the convenience of the reader.

LEMMA A.1. *Let* $a_1, a_2, \ldots a_n, \ldots$ *be positive numbers and* $\sum a_i = \infty$*, and let* g_n *be i.i.d. Gaussian. Then almost surely one has*

$$\sum a_n |g_n| = \infty. \tag{141}$$

Proof. One may, without loss of generality, assume $a_n \to 0$; otherwise the (141) clearly holds. It will be enough to prove

$$\mathbb{E}(e^{-\sum_n a_n |g_n}|) = 0, \tag{142}$$

which follows from

$$\Pi_n(\mathbb{E}(e^{-a_n|g_n|})) = 0. \tag{143}$$

To simplify the notation, let $b_n = 1 - \mathbb{E}(e^{-a_n|g_n|})$. One may only consider n large, thus a_n small.

We will prove

$$b_n \sim a_n, \tag{144}$$

thus $b_n \geq c a_n$ for some $c > 0$,

$$\Pi_n(\mathbb{E}(e^{-a_n|g_n|})) \leq \Pi_n(1 - c a_n) = 0, \tag{145}$$

since $\sum a_n = \infty$.

To see (144), one starts with

$$\mathbb{E}(e^{-a_n|g_n|}) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}|x|^2} e^{-|x|a_n}. \tag{146}$$

One computes

$$\begin{aligned} &1 - \int_0^{\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}|x|^2 - |x|a_n} \\ &= \int_{|x| < \frac{1}{a_n^{3/4}}} e^{-\frac{1}{2}|x|^2}(1 - e^{-a_n|x|}) + O\left(\int_{|x| \geq a_n^{3/4}} e^{-\frac{1}{2}|x|^2}\right) \\ &= \int_{|x| < a_n^{3/4}} e^{-\frac{1}{2}|x|^2} a_n |x| + O(a_n^2) \sim a_n. \end{aligned} \tag{147}$$

Thus, (144) has been proved. ∎

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