REGULARIZATION ESTIMATES OF THE LANDAU-COULOMB DIFFUSION

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ABSTRACT. At the present moment, it remains uncertain whether the Landau-Coulomb equation possesses a unique smooth solution for arbitrarily large times. Alongside the diffusion term, this equation includes a reaction term that could rapidly transform nice configurations into singularities. In this manuscript we show that the diffusion operator in the Landau-Coulomb equation provides much stronger $L^1 \to L^\infty$ regularization effects than its linear counterpart, the Laplace operator. Our novel quantification suggests that when seeking a proof of global well-posedness, the nonlinear diffusion term could play a pivotal role.

1. INTRODUCTION AND MAIN RESULT

We consider the diffusion equation

$$u_t = \mathrm{div}(A[u]\nabla u), \quad x \in \mathbb{R}^d, \quad t > 0, \tag{1.1}$$

where $A[u]$ is a $d \times d$ matrix defined as

$$A[u](x,t) := c_d \int_{\mathbb{R}^d} \frac{\mathbb{P}(x-y)}{|x-y|} u(y,t)\, dy,$$

and $\mathbb{P}(z)$ is the projection matrix over the space perpendicular to z, defined as

$$\mathbb{P}(z) := \left(\mathbb{I} - \frac{z \otimes z}{|z|^2} \right),$$

Hereafter the dimension d is greater or equal 3. Equation (1.1) represents the diffusion part of the homogeneous Landau-Coulomb equation, which reads as

$$u_t = \mathrm{div}(A[u]\nabla u - u\, \mathrm{div}A[u]), \tag{1.2}$$

or alternatively in the non-divergence form

$$u_t = \mathrm{Tr}(A[u]D^2 u) + u^2.$$

Equation (1.2) has attracted a lot of attention in the past two decades. Although the proof of global-well-posedness for general initial data seems elusive at this moment, the collective investigation for (1.2) of the past years has advanced forward the knowledge in several directions: (i) The existence and uniqueness of smooth solutions for short times have been extensively explored. Notably, Golding and Loher's work [7] has established the most comprehensive result for initial data in $L^p(\mathbb{R}^3)$ with $p > \frac{3}{2}$, while the case $p \leq \frac{3}{2}$ remains an open question. (ii) Global existence and uniqueness of smooth solutions with initial data close to equilibrium has been addressed across various contexts. For initial data small in Sobolev spaces, contributions can be

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found in [14, 2, 4] and and related references. The first work that considers initial data in L^∞ is the one by Kim, Guo and Hwang [15]. Recently, Golding, Gualdani and Loher [6] encompassed the problem in all L^p spaces with $p > \frac{3}{2}$, leaving the case $p \leq \frac{3}{2}$ still unexplored. (iii) Conditional regularity. This line of research concerns the investigations of conditions that guarantee global well-posedness of solutions for arbitrarily large times. Silvestre [17] and Gualdani, Guillen [12] showed that if the function $u(x,t) \in L^p(\mathbb{R}^3)$ with $p > \frac{3}{2}$ uniformly in time, then it is smooth. Recently, Alonso, Bagland, Desvillettes, and Lods [1] showed that if $u(x,t) \in L^q(0,T,L^p(\mathbb{R}^3)$ for a certain range of q and p, then it is automatically in $L^p(\mathbb{R}^3)$ with $p > \frac{3}{2}$ and therefore smooth. Regarding conditional uniqueness, Fournier in [5] showed that solutions which have L^∞ norm integrable in time are unique. Chern and Gualdani [3] showed that uniqueness holds in the class of high integrable functions. (iv) Partial regularity. This line of research for the Landau equation started with Golse, Gualdani, Imbert and Vasseur [8]. In this work it is shown that, if singularities occur, they are concentrated in a time interval that has Hausdorff measure at most $\frac{1}{2}$. Most recently, Golse, Imbert and Vasseur showed that the spatial and temporal domain for singularities to happen has Hausdorff measure $1 + \frac{5}{2}$ [9]. (v) Study of modified models that pertain the same difficulties of the Landau equation but seem analytically more tractable. This line of research started with the work of Gressmann, Krieger and Strain [10, 16] and their analysis of an isotropic version of (1.2),

$$u_t = a[u]\Delta u + \alpha u^2, \quad \alpha > 0. \tag{1.3}$$

In [10, 16] they show that (1.3) is globally well-posed if initial data are radially symmetric and monotonically decreasing and $\alpha \in (0, \frac{74}{75})$. Later, Gualdani and Guillen [11] proved global well-posedness for $\alpha = 1$ also in the case when initial data are radially symmetric and monotonically decreasing. These works proved the conjecture that, unlike what happens in the semilinear heat or porous media equations, the nonlinear diffusion $a[u]\Delta u$ is strong enough to overcome the reaction u^2. Later, Gualdani and Guillen [13] showed the the isotropic Landau equation with less singular potentials ($\gamma \in (-2.5, -2]$) is also globally well-posed.

These findings lead us to the motivation behind the present manuscript. The proofs in groups (i), (iii) and (iv) primarily rely on the ellipticity estimates provided by the lower bound of $A[u]$. Specifically, if the function u has mass, second moment and entropy bounded, the matrix $A[u]$ is uniformly bounded from below by

$$A[u] \geq \frac{c}{1+|x|^d}\mathbb{I}\mathrm{d}, \quad x \in \mathbb{R}^d,$$

where c only depends on mass, second moment and entropy of u. While a weighted Laplacian operator is analytically more tractable then the full nonlinear nonlocal diffusion $\mathrm{div}(A[u]\nabla u)$, one might argue that by using the lower bound on $A[u]$ we discard an important element that could actually prevent singularities following the intuition that when u is big so is the diffusion coefficient $A[u]$ and this strength could prevent formation of singularities. However, this intuition has been very difficult to apply in practice. Interestingly, however, is the fact that all the global-well-posedness results for general data mentioned above use the full power of the diffusion operator. These include the results in [10, 16], which are a consequence of a novel weighted Poincare inequality

$$\int_{\mathbb{R}^d} u^{p+1}\,dx \leq \left(\frac{p+1}{p}\right)^2 \int_{\mathbb{R}^d} A[u]|\nabla u^{\frac{p}{2}}|^2\,dx,$$

the ones in [11], which use a geometric argument in which the coefficient $a[u]$ plays a pivotal rule, and lastly, the ones in [13], proven via new weighted Hardy inequalities of the form

$$\int_{\mathbb{R}^d} (u * |x|^{\gamma}) u^p \, dx \le c_{d,\gamma,p} \int_{\mathbb{R}^d} (u * |x|^{\gamma+2}) |\nabla u^{\frac{p}{2}}|^2 \, dx, \quad \gamma > -d.$$

Lastly, it was already noted in [12] that conditional regularity result for (1.2) shows a rate of regularization much stronger than what is usually expected for regular parabolic equations.

In this manuscript we provide a new and precise quantification of the regularization power of the Landau diffusion operator. Notably, this regularization exhibits a significantly faster rate than that achieved by the Laplacian operator.

Theorem 1.1. *Let $u(t,x)$ be a solution to (1.1) with initial data $0 \le u_{in}$ that belongs to $L^1_m(\mathbb{R}^d)$ for some $m > 3d(d-2)$ and $d \ge 3$. Then we have*

$$\sup_{t>0} \|u\|_{L^\infty(\mathbb{R}^d)} \le \frac{c}{t^{1+\varepsilon}}, \tag{1.4}$$

for ε as small as one wishes and $c > 0$ a constant that only depends on the L^1_m-norm of the initial data.

Hereafter, we denote by $L^1_m(\mathbb{R}^d)$ the space of all $L^1(\mathbb{R}^d)$ functions such that

$$\int_{\mathbb{R}^d} |f|(1+|x|^2)^{m/2} \, dx < +\infty.$$

The proof of Theorem 1.1 follows from two steps. In the first one we show a $L^1 \to L^p$ gain of integrability for u, solution to (1.1). The second step includes a De Giorgi iteration that covers the $L^p \to L^\infty$ jump. The combination of these two steps yields (1.4).

2. Some technical lemmas

We first recall well-known results on the bounds of the diffusion matrix $A[u]$:

Lemma 2.1. *There exist positive constants C_0 and c_0 depending on the dimension $d \ge 3$ such that*

$$\|A[u]\|_{L^\infty(\mathbb{R}^d)} \ \le \ C_0 \|u\|_{L^p(\mathbb{R}^d)}^{\frac{p(d-2)}{d(p-1)}} \|u\|_{L^1(\mathbb{R}^d)}^{\frac{2p-d}{d(p-1)}}, \quad p > \frac{d}{2},$$

and

$$\|div A[u]\|_{L^\infty(\mathbb{R}^d)} \ \le \ c_0 \|u\|_{L^p(\mathbb{R}^d)}^{\frac{p(d-1)}{d(p-1)}} \|u\|_{L^1(\mathbb{R}^d)}^{\frac{p-d}{d(p-1)}}, \quad p > d.$$

We will also use the following weighted Sobolev inequality: for f smooth enough and any $1 \le s \le \frac{2d}{d-2}$ we have

$$\left(\int_{\mathbb{R}^d} |f|^{\frac{2d}{d-2}} \langle x \rangle^{-3d} \, dx \right)^{\frac{d-2}{d}} \le c_1 \int_{\mathbb{R}^d} |\nabla f|^2 \langle x \rangle^{-d} \, dx + c_2 \left(\int_{\mathbb{R}^d} |f|^s \, dx \right)^{2/s}. \tag{2.1}$$

Here, $\langle x \rangle := (1+|x|^2)^{1/2}$. The derivation of (2.1) for $d = 3$ can be found in [6]. We use (2.1) to prove the following interpolation inequality:

Lemma 2.2. *Let $p>1$ and q such that $p+\frac{2}{d}<q<p\left(1+\frac{2}{d}\right)$. Let m be defined as*

$$m:=\frac{3d(d-2)(p-1)}{(d+2)p-dq}.$$

For any g smooth function the following bound holds:

$$\|g\|^q_{L^q(\mathbb{R}^d)}\le C\left(\|\langle\cdot\rangle^{-\frac{d}{2}}\nabla g^{\frac{p}{2}}\|^2_{L^2(\mathbb{R}^d)}+\|g\|^p_{L^p(\mathbb{R}^d)}\right)\|g\|_{L^p(\mathbb{R}^d)}^{p\left(\frac{q-p-\frac{2}{d}}{p-1}\right)}\|g\langle\cdot\rangle^m\|_{L^1(\mathbb{R}^d)}^{\frac{(d+2)p-dq}{d(p-1)}}. \tag{2.2}$$

Proof. We first establish the following interpolation inequality

$$\|g\|^q_{L^q}\le\|\langle\cdot\rangle^{-\frac{3(d-2)}{p}}\,g\|^p_{L^{\frac{dp}{d-2}}}\,\|g\|_{L^p}^{p\left(\frac{q-p-\frac{2}{d}}{p-1}\right)}\|g\,\langle\cdot\rangle^m\|_{L^1}^{\frac{(d+2)p-dq}{d(p-1)}}, \tag{2.3}$$

that holds for any $p+\frac{2}{d}<q<p\left(1+\frac{2}{d}\right)$ and $m=\frac{3d(d-2)(p-1)}{(d+2)p-dq}$. The lemma follows easily once we prove (2.3): use (2.1) with $f=g^{\frac{p}{2}}$ and $s=2$ to bound the first term on the right hand side of (2.3) and get

$$\|g\|^q_{L^q}\le C\left(\|\langle\cdot\rangle^{-\frac{d}{2}}\nabla g^{\frac{p}{2}}\|^2_{L^2}+\|g\|^p_{L^p}\right)\|g\|_{L^p}^{p\left(\frac{q-p-\frac{2}{d}}{p-1}\right)}\|g\langle\cdot\rangle^m\|_{L^1}^{\frac{(d+2)p-dq}{d(p-1)}}.$$

Next, to show (2.3) we start with a weighted interpolation

$$\|g\|^q_{L^q(\mathbb{R}^d)}\le\left(\int_{\mathbb{R}^d}g^{\frac{dp}{d-2}}\langle x\rangle^{-\frac{dp\alpha}{\theta q(d-2)}}\,dx\right)^{\frac{\theta q(d-2)}{dp}}\left(\int_{\mathbb{R}^d}g^r\langle x\rangle^{\frac{r\alpha}{(1-\theta)q}}\,dx\right)^{\frac{(1-\theta)q}{r}},$$

with α, θ and r satisfy

$$\begin{cases}\frac{\theta q(d-2)}{dp}+\frac{(1-\theta)q}{r}=1,\\ \frac{\theta q(d-2)}{dp}=\frac{d-2}{d},\\ \frac{dp\alpha}{\theta q(d-2)}=3d.\end{cases}$$

The above system has solutions $\alpha=3(d-2)$, $\frac{(1-\theta)q}{r}=\frac{2}{d}$, $r=\frac{d}{2}(q-p)$, and $\theta=\frac{p}{q}$, which yield

$$\int g^q\,dx\le\left(\int g^{\frac{dp}{d-2}}\langle\cdot\rangle^{-3d}\,dx\right)^{\frac{d-2}{d}}\left(\int g^{\frac{d}{2}(q-p)}\langle\cdot\rangle^{\frac{3d(d-2)}{2}}\,dx\right)^{\frac{2}{d}}.$$

Let us focus on the last term: once more, use Hölder's inequality and get

$$\begin{aligned}\left(\int g^{\frac{d}{2}(q-p)}\langle\cdot\rangle^{\frac{3d(d-2)}{2}}dx\right)^{\frac{2}{d}}&=\left(\int g^{\alpha}g^{\frac{d}{2}(q-p)-\alpha}\,\langle\cdot\rangle^{\frac{3d(d-2)}{2}}dx\right)^{\frac{2}{d}}\\&\le\left[\left(\int g^p dx\right)^{\frac{\alpha}{p}}\left(\int g^{(\frac{d}{2}(q-p)-\alpha)\beta}\langle\cdot\rangle^{\frac{3d(d-2)}{2}\beta}dx\right)^{\frac{1}{\beta}}\right]^{\frac{2}{d}},\end{aligned}\tag{2.4}$$

where $\beta:=\frac{p}{p-\alpha}$. We chose α such that $(\frac{d}{2}(q-p)-\alpha)\beta=1$, which implies

$$\alpha=\left(\frac{d}{2}(q-p)-1\right)\frac{p}{p-1}.$$

Note that $\alpha > 0$ requires $q > p + \frac{2}{d}$. Since also β has to be positive, and

$$\beta = \frac{2(p-1)}{(d+2)p - dq},$$

we require $q < \left(\frac{d+2}{d}\right) p$. Substitution of α and β in (2.4) yields

$$\left(\int g^{\frac{d}{2}(q-p)} \langle \cdot \rangle^{\frac{3d(d-2)}{2}} dx \right)^{\frac{2}{d}} \leq \left(\int g^p \, dx \right)^{\frac{q-p-\frac{2}{d}}{p-1}} \left(\int g \langle x \rangle^m \right)^{\frac{(d+2)p-dq}{d(p-1)}}, \tag{2.5}$$

with $m = \frac{3d(d-2)(p-1)}{(d+2)p-dq}$. This proves (2.3) and finishes the proof. $\square$

Remark 2.3. The condition $p + \frac{2}{d} < q < \frac{d+2}{d} p$ indicates that $m := \frac{3d(d-2)(p-1)}{(d+2)p-dq}$ is such that $m > \frac{3d(d-2)}{2}$.

3. $L^1 \to L^p$ gain of integrability

The next theorem shows a $L^1 \to L^p$ gain of integrability for solutions to (1.1). The proof follows almost directly from the weighted Poincare's inequality

$$\int_{\mathbb{R}^d} u^{p+1} \, dx \leq \left(\frac{p+1}{p} \right)^2 \int_{\mathbb{R}^d} A[u] |\nabla u^{p/2}|^2 \, dx, \tag{3.1}$$

first proven in [10]. The gain of integrability we obtain is much faster that the one of the solution to the heat equation, which is of the order of $\frac{1}{t^{\frac{d}{2}\left(1-\frac{1}{p}\right)}}$.

Theorem 3.1. *Let $u(t,x)$ be a solution to (1.1). For any $p > 1$ we have*

$$\sup_{t>0} \|u\|_{L^p(\mathbb{R}^d)} \leq \frac{c}{t^{1-\frac{1}{p}}},$$

with c a constant depending only on p and $\|u_{in}\|_{L^1(\mathbb{R}^d)}$.

Proof. Multiply (1.1) by $\varphi := u^{p-1}$ and integrate the resulting equation in $\mathbb{R}^d$. Integration by parts yields

$$\partial_t \int u^p \, dx = -\frac{4(p-1)}{p} \int \left\langle A[u] \nabla u^{\frac{p}{2}}, \nabla u^{\frac{p}{2}} \right\rangle \, dx.$$

Inequality (3.1) imply

$$\partial_t \int u^p(x) \, dx \leq -\frac{4p(p-1)}{(p+1)^2} \int u^{p+1}(x) \, dx. \tag{3.2}$$

Combining the interpolation inequality

$$\|u\|_{L^p} \leq \|u\|_{L^1}^{\theta} \|u\|_{L^{p+1}}^{1-\theta}, \quad \theta = \frac{1}{p^2},$$

with (3.2) yields

$$\partial_t \|u\|_{L^p}^p \leq -C \|u\|_{L^p}^{\frac{p^2}{(p-1)}},$$

with $C = \frac{4p(p-1)}{(p+1)^2} \|u_{\text{in}}\|_{L^1}^{-\frac{1}{p-1}}$. Define $y := \|u\|_{L^p}^p$; the solution to the differential inequality

$$y' \leq -C y^{\frac{p}{p-1}},$$

has the bound

$$y \leq \frac{1}{\left(y_0^{-\frac{1}{p-1}} + \frac{C}{p-1} t\right)^{p-1}}.$$

This implies that

$$\sup_{t>0} \|u\|_{L^p} \leq \left(\frac{(p-1)}{C}\right)^{1-\frac{1}{p}} \frac{1}{t^{1-\frac{1}{p}}},$$

and this finishes the proof. □

We also have the following moment estimate:

Lemma 3.2. *Let $u(t,x)$ be a smooth solution to (1.1) in the time interval $[0,T]$ with initial datum $u_{in} \in L^1_m(\mathbb{R}^d)$ for some $m \geq 2$. Then there exists a constant c that only depends on T and the L^1_m-norm of u_{in} such that*

$$\sup_{t\in[0,T]} \|u(t,x)\|_{L^1_m(\mathbb{R}^d)} \leq c.$$

Proof. We start with $m = 2$. Testing with $\phi = (1 + |x|^2)$ and integrating by parts yield

$$\begin{aligned} \partial_t \int u(1+|x|^2)dx &\leq 4\int u\ |x||\nabla A[u]|\ dx + 4d\int u\ \mathrm{Tr}(A[u])\ dx \\ &=: \mathcal{J}_1 + \mathcal{J}_2. \end{aligned}$$

Let us first estimate $\mathcal{J}_2$. Applying the first estimate of Lemma 2.1 to $\mathcal{J}_2$, we get

$$\mathcal{J}_2 \leq C_0 \|u\|_{L^p(\mathbb{R}^d)}^{\frac{p(d-2)}{(p-1)d}} \|u\|_{L^1(\mathbb{R}^d)}^{\frac{2p-d}{d(p-1)}+1}. \tag{3.3}$$

Then an application of Theorem 3.1 to the L^p-norm of (3.3), gives

$$\mathcal{J}_2 \lesssim \frac{1}{t^{1-\frac{2}{d}}} \|u\|_{L^1(\mathbb{R}^d)}^{\frac{2p-d}{d(p-1)}+1}. \tag{3.4}$$

Next, we estimate $\mathcal{J}_1$. We have

$$\mathcal{J}_1 \leq \|\nabla A[u]\|_{L^\infty} \int u\ (1+|x|^2)\ dx.$$

Apply once more Lemma 2.1 and Theorem 3.1 to get

$$\mathcal{J}_1 \lesssim \frac{1}{t^{1-\frac{1}{d}}} \int u(1+|x|^2)\ dx.$$

Gathering the estimates of $\mathcal{J}_1$ and $\mathcal{J}_2$ together, we acquire the bound

$$\begin{aligned} \partial_t \int u(1+|x|^2)dx &= \mathcal{J}_1 + \mathcal{J}_2 \\ &\leq \frac{c}{t^{\frac{d-1}{d}}} \int u(1+|x|^2)\ dx + \frac{C}{t^{\frac{d-2}{d}}}, \end{aligned}$$

where $c := \|u\|_{L^1(\mathbb{R}^d)}^{\frac{p-d}{d(p-1)}}$ and $C := \|u\|_{L^1}^{\frac{2p-d}{d(p-1)}+1}$. The last inequality is equivalent to the differential inequality

$$y' \leq \frac{c}{t^{\frac{d-1}{d}}} y + \frac{C}{t^{\frac{d-2}{d}}}, \tag{3.5}$$

which, after multiplying by $\mu(s) = e^{-ds^{\frac{1}{d}}}$, reduces to

$$(y\mu(s))' \leq \mu(s) s^{-\frac{d-2}{d}},$$

and has solution:

$$y(t) = e^{dt^{1/d}} \left\{ \int_0^t e^{-ds^{1/d}} s^{\frac{2-d}{d}} ds + y_0 e^{-dt^{\frac{1}{d}}} \right\}.$$

Applying the same argument iteratively, we can get the estimate for any $m > 2$. □

Remark 3.3. Thanks to the bound on the second moments from Lemma 3.2, the conservation of mass and the decay of entropy, the matrix $A[u]$ satisfies the following ellipticity condition:

$$A[u](x,t) \geq \frac{c(T)}{\langle x \rangle^d} \quad \text{for any} \quad x \in \mathbb{R}^d,\ t \in [0,T]. \tag{3.6}$$

4. $L^1 \to L^\infty$ gain of integrability

In this section we first show the $L^p \to L^\infty$ gain of integrability for solutions to (1.1). This, combined with the estimate of Theorem 3.1 will conclude the proof of Theorem 1.1. We follow a modification of the De Giorgi iteration previously used in [6] and [7]. We start with a technical lemma. Let $M > 0$ and $t > 0$; for each $k \in \mathbb{N}$, define

$$C_k := M(1 - 2^{-k}), \quad T_k := \frac{t}{2}\left(1 - \frac{1}{2^k}\right).$$

We denote with $(u-c)_+$ the maximum between 0 and $(u-c)$.

Lemma 4.1. *Let $p > d/2$, $\gamma > 0$ defined as*

$$\gamma = -1 + \frac{2}{d}p - \frac{3}{m}(d-2)(p-1),$$

and $m \geq 2$ such that

$$m > \frac{3d(d-2)}{2} \max\left\{1, \frac{p-1}{p-\frac{d}{2}}\right\}.$$

For each $k \geq 1$ we have the bound

$$\int_{\mathbb{R}^d} (u - C_k)_+^p dx \leq \left(\frac{c_0 2^k}{M}\right)^{1+\gamma} \left(\|\langle\cdot\rangle^{-d/2} \nabla (u - C_{k-1})_+^{\frac{p}{2}}\|_{L^2}^2 + \|(u - C_{k-1})_+\|_{L^p}^p \right)$$
$$\cdot \|(u - C_{k-1})_+\|_{L^p}^{p\left(\frac{2}{d} - \frac{3}{m}(d-2)\right)} \|(u - C_{k-1})_+\|_{L^1_m}^{\frac{3}{m}(d-2)},$$

with c_0 dimensionless constant.

Proof. Observe that $0 \leq C_{k-1} < C_k$. From this we have

$$0 \leq (u - C_k)_+ \leq (u - C_{k-1})_+. \tag{4.1}$$

Moreover $u - C_{k-1} = u - C_k + C_k - C_{k-1}$. Dividing by $C_k - C_{k-1}$ we acquire

$$\frac{u - C_{k-1}}{C_k - C_{k-1}} = \frac{u - C_k}{C_k - C_{k-1}} + 1 \geq 1.$$

This tells us that

$$\mathbb{1}_{\{u-C_k\geq 0\}} \leq \frac{(u-C_{k-1})_+}{C_k - C_{k-1}}.$$

Hence, for any $a>0$ we have

$$\mathbb{1}_{\{u-C_k\geq 0\}} \leq \left(\frac{(u-C_{k-1})_+}{C_k - C_{k-1}}\right)^a.$$

Multiplying the above inequality by $(u-C_k)_+$ and using (4.1), we deduce

$$(u-C_k)_+ \leq \frac{(u-C_{k-1})_+^{1+a}}{(C_k-C_{k-1})^a} \quad \text{for any} \quad a>0. \tag{4.2}$$

Chose $a=\frac{1+\gamma}{p}$ for some $\gamma>0$ to be defined later. Inequality (4.2) implies

$$\int_{\mathbb{R}^d} (u-C_k)_+^p \, dx \leq \left(\frac{2^k}{M}\right)^{1+\gamma} \int_{\mathbb{R}^d} (u-C_{k-1})_+^{p+1+\gamma} \, dx.$$

Lemma 2.2 with $q=1+\gamma+p$ yields

$$\int_{\mathbb{R}^d} (u-C_k)_+^p \, dx \leq c_0 \left(\frac{2^k}{M}\right)^{1+\gamma} \left(\left\| \nabla (u-C_{k-1})_+^{\frac{p}{2}} \langle\cdot\rangle^{-\frac{d}{2}} \right\|_{L^2}^2 + \|(u-C_{k-1})_+\|_{L^p}^p \right)$$
$$\cdot \|(u-C_{k-1})_+\|_{L^2}^{p\left(\frac{\frac{d-2}{d}+\gamma}{p-1}\right)} \|(u-C_{k-1})_+\|_{L^1_m}^{\frac{2p-d-d\gamma}{d(p-1)}},$$

with c_0 dimensionless constant and $m=\frac{3d(d-2)(p-1)}{(d+2)p-d(1+\gamma+p)}$. Next, we express γ in terms of m, and get $\gamma=\frac{2}{d}p-1-\frac{3(d-2)(p-1)}{m}$, which implies, after substitution in the norms,

$$\int_{\mathbb{R}^d} (u-C_k)_+^p \, dx \leq c_0 \left(\frac{2^k}{M}\right)^{1+\gamma} \left(\left\| \langle\cdot\rangle^{-d/2} \nabla (u-C_{k-1})_+^{\frac{p}{2}} \right\|_{L^2}^2 + \|(u-C_{k-1})_+\|_{L^p}^p \right)$$
$$\cdot \|(u-C_{k-1})_+\|_{L^p}^{p\left(\frac{2}{d}-\frac{3}{m}(d-2)\right)} \|(u-C_{k-1})_+\|_{L^1_m}^{\frac{3}{m}(d-2)}.$$

The constraint $\gamma>0$ implies $m>\frac{3d(d-2)}{2}\frac{(p-1)}{p-\frac{d}{2}}$. The proof of the lemma is complete after recalling Remark 2.3. $\square$

We are now ready to start the De Giorgi iteration. For any $k\geq 1$ let us define the energy $\mathcal{E}_k$ as

$$\mathcal{E}_k(T_{k+1},t) := \sup_{\tau\in(T_{k+1},t)} \int (u-C_k)_+^p(\tau,x)dx + C(p)\int_{T_{k+1}}^t \int \langle x\rangle^{-d} \left|\nabla (u-C_k)_+^{\frac{p}{2}}\right|^2 dxd\tau,$$

and $\mathcal{E}_0$ as

$$\mathcal{E}_0 := \sup_{(t/4,t)} \int_{\mathbb{R}^d} u^p \, dx + C(p)\int_{t/4}^t \int_{\mathbb{R}^d} \langle x\rangle^{-d} \left|\nabla u^{\frac{p}{2}}\right|^2 \, dx \, d\tau. \tag{4.3}$$

Lemma 4.2. *Given $p>\frac{d}{2}$, $\gamma=-1+\frac{2}{d}p-\frac{3(d-2)(p-1)}{m}$ and $m>\frac{3d(d-2)}{2}\max\left\{1,\frac{p-1}{p-\frac{d}{2}}\right\}$. For all $k\geq 1$ we have*

$$\mathcal{E}_k(T_{k+1},t) \lesssim \frac{1}{tM^{1+\gamma}}\, \mathcal{E}_{k-1}(T_k,t)^{\left(1+\frac{2}{d}-\frac{3}{m}(d-2)\right)}.$$

Proof. We test (1.1) with $(u-C_k)_+^{p-1}$, integrate in $\mathbb{R}^d \times (s,\tau)$ with $0 \le T_k \le s \le T_{k+1} \le \tau$. After averaging on s between T_k and T_{k+1}, and taking the supremum of τ in (T_{k+1},t) we get

$$\sup_{\tau\in(T_{k+1},t)} \int (u-C_k)_+^p(\tau,x)dx + C(p)\int_{T_{k+1}}^t \int A[u]\left|\nabla(u-C_k)_+^{\frac{p}{2}}\right|^2 dxds \\ \le \frac{1}{T_{k+1}-T_k}\int_{T_k}^t \int (u-C_k)_+^p dx\, ds,$$

which can be also written as

$$\mathcal{E}_k(T_{k+1},T) \le \frac{1}{T_{k+1}-T_k}\int_{T_k}^t \int (u-C_k)_+^p dx\, ds. \tag{4.4}$$

Since $(T_{k+1}-T_k) = \frac{t}{2^{k+2}}$, we apply the integral bound of Lemma 4.1 to get

$$\begin{aligned}\mathcal{E}_k &\lesssim \frac{2^{k+1}}{t}\left(\frac{2^k}{M}\right)^{1+\gamma} \sup_{(T_k,t)} \|(u-C_{k-1})_+\|_{L^1_m}^{\frac{3(d-2)}{m}} \sup_{(T_k,t)} \|(u-C_{k-1})_+\|_{L^p}^{p\left(\frac{2}{d}-\frac{3}{m}(d-2)\right)} \\ &\quad\cdot\left[\sup_{(T_k,t)} \|(u-C_{k-1})_+\|_{L^p}^p + \int_{T_k}^t \|\langle\cdot\rangle^{-d/2}\nabla(u-C_{k-1})_+^{\frac{p}{2}}\|_{L^2}^2\, ds\right] \\ &\le \frac{2^{k+1}}{t}\left(\frac{2^k}{M}\right)^{1+\gamma} \sup_{(T_k,t)} \|(u-C_{k-1})_+\|_{L^1_m}^{\frac{3(d-2)}{m}} \\ &\quad\cdot\left[\sup_{(T_k,t)} \|(u-C_{k-1})_+\|_{L^p}^p + \int_{T_k}^t \|\langle\cdot\rangle^{-d/2}\nabla(u-C_{k-1})_+^{\frac{p}{2}}\|_{L^2}^2\, ds\right]^{1+\frac{2}{d}-\frac{3}{m}(d-2)} \\ &= \frac{C^k C_0}{tM^{1+\gamma}}\mathcal{E}_{k-1}^{\left(1+\frac{2}{d}-\frac{3}{m}(d-2)\right)},\end{aligned}$$

with $C_0 := \sup_{(0,T)} \|u\|_{L^1_m}^{\frac{3}{m}(d-2)}$. □

For simplicity in the notation, we define $\beta_1 := \frac{2}{d}-\frac{3}{m}(d-2)$. The inequality of the previous lemma shows that, iteratively,

$$\mathcal{E}_k \lesssim \left(\frac{c_0}{t^{\frac{1}{\beta_1}}M^{\frac{(1+\gamma)}{\beta_1}}}\mathcal{E}_0\right)^{(1+\beta_1)^k}. \tag{4.5}$$

Recall the definition of $\mathcal{E}_0$:

$$\mathcal{E}_0 = \sup_{(t/4,t)} \int u^p(s,x)\, dx + C(p)\int_{t/4}^t \int A[u]\left|\nabla u^{\frac{p}{2}}\right|^2\, dx\, ds.$$

Since

$$\mathcal{E}_0 \le \sup_{(0,t)} \int u^p(s,x)\, dx,$$

Theorem 3.1 implies

$$\mathcal{E}_0 \le \frac{c}{t^{p-1}},$$

where c only depends on p and on the L^1-norm of the initial data. Passing to the limit $k \to +\infty$ in (4.5) we obtain

$$u \leq M,$$

provided

$$M \lesssim \frac{c_0}{t^{1+\varepsilon}},$$

with $\varepsilon = \frac{1-\frac{2}{d}}{1+\gamma}$ and c_0 only dependent on the L^1_m-norm of the initial data. Note that $\varepsilon > 0$ can be as small as one wishes by choosing p arbitrarily large. To see this, first note that if p is greater than $d-1$ then $1 < \max\left\{1, \frac{p-1}{p-\frac{d}{2}}\right\} \leq 2$. Then, taking $m > 3d(d-2)$, we get

$$\varepsilon = \frac{1-\frac{2}{d}}{1+\gamma} = \frac{1-\frac{2}{d}}{\frac{2p}{d} - \frac{3(d-2)(p-1)}{m}} \leq \frac{d-2}{p+1}.$$

This finishes the proof of Theorem 1.1.

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