AN ALMGREN MONOTONICITY FORMULA FOR DISCRETE HARMONIC FUNCTIONS

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Abstract. The celebrated Almgren monotonicity formula for harmonic functions $u : \mathbb{R}^n \to \mathbb{R}$ says that its L^2-energy concentrated on a sphere of radius r, when measured in a suitable sense, is non-decreasing: if u oscillates at a certain scale, it has even larger oscillations at a larger scale. We prove a discrete analogue of the Almgren monotonicity formula for harmonic functions on infinite combinatorial graphs $G = (V, E)$. Some applications are discussed.

1. Introduction and Results

1.1. **Introduction.** We will show that the classical Almgren monotonicity formula for harmonic functions $u : \mathbb{R}^n \to \mathbb{R}$ has an extension to the setting of discrete harmonic functions on combinatorial graphs. The celebrated Almgren monotonicity formula [1, 2] is a cornerstone in the study of harmonic functions. It also plays a crucial role in studying unique continuation, and has been used extensively in free boundary problems [3, 8, 9]. It is usually stated as follows: let $u : \mathbb{R}^n \to \mathbb{R}$ be a harmonic function and let B_r denote a ball of radius r, then

$$N(r) = \frac{r \int_{B_r} |\nabla u|^2 dx}{\int_{\partial B_r} u(x)^2 d\sigma} \qquad \text{is non-decreasing.}$$

One often lets $r \to 0$ to deduce information from the limit. Conversely, the presence of oscillation implies the existence of larger oscillations at a larger scale. Using integration by parts, the Dirichlet energy can be written as

$$\begin{aligned}\int_{B_r} |\nabla u|^2 dx &= \int_{\partial B_r} u \frac{\partial u}{\partial n} d\sigma \\ &= \frac{1}{2} \frac{d}{dr} \int_{\partial B_r} u(x)^2 d\sigma - \frac{n-1}{r} \int_{\partial B_r} u(x)^2 d\sigma.\end{aligned}$$

Therefore, Almgren's monotonicity can be equivalently written as saying that

$$N(r) = r \frac{d}{dr} \log \left(\int_{\partial B_r} u(x)^2 d\sigma \right) \quad \text{is non-decreasing.}$$

This implies that if a harmonic function is large in L^2 on a sphere, then it will be even larger on spheres with the same center and larger radius. After suitably adapting the functional to the setting of combinatorial graphs, we will obtain an analogous identity with the same consequences.

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1.2. **Discrete Almgren Monotonicity.** We can now introduce our discrete analogue. Recall that a function $u : V \to \mathbb{R}$ is harmonic if, for each vertex $x \in V$,

$$\sum_{(x,y) \in E} (u(x) - u(y)) = 0 \qquad \text{or, equivalently} \qquad u(x) = \frac{1}{\deg(x)} \sum_{(x,y) \in E} u(y).$$

Such functions have been intensively studied and have a rich theory, see, for example [4, 5, 6, 7]. We assume we are given a connected graph $G = (V, E)$. The graph may be finite (in which case our result will be trivial: harmonic functions are constant). If the graph is infinite, we require that it is locally finite: each vertex has finite degree. We interpret Almgren's functional quite literally and regard

$$\int_{\partial B_r} u(x)^2 dx \qquad \text{as} \qquad \sum_{d(x,y)=k} u(y)^2,$$

where $x \in V$ is an arbitrary vertex. The differentiation in r is replaced by a discrete difference between $k+1$ and k. There is an important conceptual change: vertices with a higher degree have a bigger impact on their neighborhood and this has to be accounted for. Fixing the base vertex $x \in V$ induces a partition of the vertex set by distance from x (see Fig. 1). We introduce the in-degree of a vertex $v \in V$

$$d_{\text{in}}(v) = \# \left\{ w \in V : (w,v) \in E \quad \text{and} \quad d(w,x) = d(v,x) - 1 \right\}$$

and, completely analogously, the out-degree as

$$d_{\text{out}}(v) = \# \left\{ w \in V : (w,v) \in E \quad \text{and} \quad d(w,x) = d(v,x) + 1 \right\}.$$

We always have $d_{\text{in}}(v) \geq 1$ as well as $d_{\text{in}}(v) + d_{\text{out}}(v) \leq \deg(v)$.

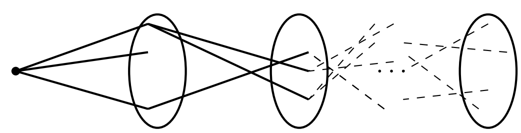

FIGURE 1. Ordering vertices by distance from a fixed vertex.

Theorem (Discrete Almgren Monotonicity Formula)**.** *Let $G = (V, E)$ be locally finite, let $x \in V$ and let $u : V \to \mathbb{R}$ be harmonic. Then, for $k \geq 0$,*

$$N(k) = \sum_{d(x,y)=k+1} d_{\text{in}}(y) \cdot u(y)^2 - \sum_{d(x,y)=k} d_{\text{out}}(y) \cdot u(y)^2$$

is non-negative and satisfies $N(k+1) \geq N(k)$.

The result has an immediate extension to weighted graphs, see §2.5. If there is a dramatic change in the (suitably normalized) ℓ^2-energy on a sphere, then this is indicative of even larger fluctuations at larger spheres. When $u \equiv 1$ is constant, we have $N \equiv 0$ since the number of incoming edges in $\{y : d(x,y) = k+1\}$ is exactly the same as the number of outgoing edges from $\{y : d(x,y) = k\}$. The inequality is sharp in the sense that $N(k+1) - N(k)$ may tend to 0, see §2.4.

1.3. **Doubling estimates.** The continuous Almgren monotonicity formula immediately leads to doubling estimates. Something similar is true in the discrete setting. Let $0 \leq a \leq b < \infty$ be two integers, then the quantities $N(a)$ and $N(b)$ are related to the growth of the harmonic function. We say that a graph is *locally expansive* in $\{v \in V : a \leq d(x,v) \leq b\}$ if for all vertices satisfying $a \leq d(v,x) \leq b$, we have

$$d_{\text{out}}(v) \geq d_{\text{in}}(v).$$

Likewise, we say that the graph is *locally contractive* if $d_{\text{out}}(v) \leq d_{\text{in}}(v)$ for every vertex in that region. In either of these cases, we can obtain bounds on the growth in terms of $N(k)$ via a simple telescoping argument.

Corollary 1. *Let $G = (V,E)$ be locally finite, let $x \in V$ and let $u : V \to \mathbb{R}$ be a harmonic function. If G is locally expansive in $\{v \in V : a \leq d(x,v) \leq b\}$, then*

$$\sum_{d(x,y)=b+1} d_{\text{in}}(y) \cdot u(y)^2 \geq (b-a+1)N(a) + \sum_{d(x,y)=a} d_{\text{out}}(y) \cdot u(y)^2.$$

If G is locally contractive in the same region, then

$$\sum_{d(x,y)=b+1} d_{\text{in}}(y) \cdot u(y)^2 \leq (b-a+1)N(b) + \sum_{d(x,y)=a} d_{\text{out}}(y) \cdot u(y)^2$$

One way of phrasing the corollary is that in locally expansive regions, the size of $N(a)$ can be seen as a lower bound on the guaranteed ℓ^2-growth of the harmonic function. In locally contractive regions, the size of $N(b)$ provides an upper bound on how much growth can have happened in that region. We note that, in contrast to Euclidean space, these estimates are weaker insofar as they provide only additive (as opposed to multiplicative) control; this reflects the fact that harmonic functions on graphs are more versatile than they are in Euclidean space (see §2.4 for an example).

1.4. **A continuous application.** The purpose of this section is to consider a particularly natural example, the lattice graph, and to deduce a continuous result from it. We consider the lattice graph $V = \mathbb{Z}^d$ where $(x,y) \in E$ if and only if $\|x-y\|_{\ell^1} = 1$. This is a $2d-$regular graph (see Fig. 2 for an example in $\mathbb{Z}^2$).

		2		
	2	1	2	
2	1	0	1	2
	2	1	2	
		2		

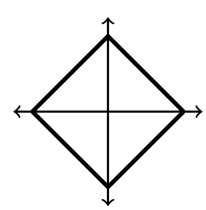

FIGURE 2. Left: elements of a grid graph enumerated by distance from a fixed element. Right: the ℓ^1-unit ball in $\mathbb{R}^2$.

Fixing a vertex $x \in V$, one sees that once we are far away from that vertex and consider $V_k = \{y \in V : d(x,y) = k\}$, 'most' vertices in V_k will satisfy $d_{\text{in}}(y) = d = d_{\text{out}}(y)$. There are exceptions (the 'tips' of the diamond) but the set of exceptional vertices is 'small' as k becomes large. This suggests an interesting continuous analogue in the plane since V_k, in a suitable limit, is simply an ℓ^1-ball in $\mathbb{R}^2$. An

ℓ^1-ball in $\mathbb{R}^2$ is simply a rotated cube. Finally, in the setting where vertices satisfy $d_{\text{in}}(y) = d = d_{\text{out}}(y)$ the discrete Almgren monotonicity formula simplifies to

$$\sum_{d(x,y)=k+1} u(y)^2 - 2 \sum_{d(x,y)=k+1} u(y)^2 + \sum_{d(x,y)=k-1} u(y)^2 \geq 0$$

which is the discrete analogue of convexity. We will prove that this purely discrete result has a 'continuous limit' in the form of an analogous convexity result in $\mathbb{R}^d$.

Corollary 2. *Let $u : \mathbb{R}^d \to \mathbb{R}$ be harmonic. Then*

$$E(t) = \int_{\|x\|_{\ell^\infty}=t} u(x)^2 \, d\mathcal{H}^{n-1} \qquad \textit{is convex on} \quad [0, \infty].$$

Since harmonic functions remain harmonic under rescaling and under orthogonal transformations and rotations, the result is naturally also true for 'tilted' cubes. When $d = 2$ and $u \equiv 1$ is a constant function, then $E(t)$ is a linear function and $E''(t) \geq 0$ cannot be improved. The proof is not literally a Corollary of our main result (though philosophically inspired). We give a self-contained argument in §2.3. Corollary 2 suggests a couple of natural questions. It is clear that the proof can be (slightly) extended to other shapes such as the ℓ^1-unit ball. It would be interesting to understand under which assumptions on the underlying geometry such a convexity statement holds. Another natural question is dictated by scaling: the set $\{\|x\|_{\ell^\infty} = t\}$ grows like $\sim c_d t^{d-1}$ and one might be inclined to believe that the average of a harmonic function on the boundary should be increasing, which would suggest that the first d derivatives might be positive. This would coincide with Corollary 2 when $d = 2$. It might be interesting to see whether the Discrete Almgren Monotonicity Formula, when applied to other types of graphs, might suggest other continuous analogues.

2. Proofs

2.1. **Proof of the Theorem.**

Proof. For simplicity of exposition, we give the proof when the graph is simple, which means no self-loops and the edges are unweighted. The argument generalizes easily once the notion of in-degree and out-degree are refined to account for weights, see §2.5. We fix $x \in V$ and start by showing that $N(0) \geq 0$. Note that

$$N(0) = \left(\sum_{(x,y) \in E} u(y)^2 \right) - \deg(x) \cdot u(x)^2.$$

Since the function is harmonic,

$$u(x) = \frac{1}{\deg(x)} \sum_{(x,y) \in E} u(y),$$

which simplifies the problem to showing that

$$\sum_{(x,y) \in E} u(y)^2 \geq \frac{1}{\deg(x)} \left(\sum_{(x,y) \in E} u(y) \right)^2.$$

This follows from the Cauchy-Schwarz inequality. It remains to prove the monotonicity. We will work with the decomposition

$$V = \bigcup_{k=0}^{\infty} V_k \qquad \text{where} \qquad V_k = \{w \in V : d(x,w) = k\}.$$

We will also introduce an abbreviation for edges that run between V_k and V_{k+1},

$$E_k = \{e \in E : e \text{ runs between } V_k \text{ and } V_{k+1}\}.$$

For each vertex $y \in V_{k+1}$ there exist $y_1, \dots, y_\ell \subset V_k$ that are connected to y by an edge. We have $\ell \geq 1$ because there exists a shortest path from y to x. Then

$$\begin{aligned} d_{\text{in}}(y) \cdot u(y)^2 &= \sum_{i=1}^{\ell} (u(y_i) + (u(y) - u(y_i))^2 \\ &\geq \sum_{i=1}^{\ell} u(y_i)^2 + 2u(y_i)(u(y) - u(y_i)). \end{aligned}$$

We sum this inequality over all vertices $y \in V_{k+1}$ and obtain

$$\sum_{y \in V_{k+1}} d_{\text{in}}(y) \cdot u(y)^2 \geq \sum_{z \in V_k} d_{\text{out}}(z) u(z)^2 + \sum_{\substack{e=(z,y) \\ e \in E_k}} 2u(z)(u(y) - u(z)).$$

We rewrite the second sum as

$$\begin{aligned} \sum_{\substack{e=(z,y) \\ e \in E_k}} 2u(z)(u(y) - u(z)) &= \sum_{z \in V_k} \sum_{\substack{e=(z,y) \\ y \in V_{k+1}}} 2u(z)(u(y) - u(z)) \\ &= \sum_{z \in V_k} 2u(z) \sum_{\substack{e=(z,y) \\ y \in V_{k+1}}} (u(y) - u(z)). \end{aligned}$$

At this point, we use that u is harmonic. For each $z \in V_k$, we have

$$\sum_{(z,w) \in E} (u(w) - u(z)) = 0.$$

All the neighbors of $z \in V_k$ are either in V_{k-1}, in V_k or in V_{k+1} and thus

$$\sum_{\substack{(z,w) \in E \\ w \in V_{k+1}}} (u(w) - u(z)) = \sum_{\substack{(z,w) \in E \\ w \in V_{k-1}}} (u(z) - u(w)) + \sum_{\substack{(z,w) \in E \\ w \in V_k}} (u(z) - u(w)).$$

We first argue that, when summing over all $z \in V_k$, the second sum on the right-hand side can be simplified since we sum over each edge twice and

$$\begin{aligned} \sum_{z \in V_k} \sum_{\substack{(z,w) \in E \\ w \in V_k}} u(z)(u(z) - u(w)) &= \sum_{\substack{e=(a,b) \in E \\ a \in V_k, b \in V_k}} u(a)(u(a) - u(b)) + u(b)(u(b) - u(a)) \\ &= \sum_{\substack{e=(a,b) \in E \\ a \in V_k, b \in V_k}} (u(a) - u(b))^2 \geq 0. \end{aligned}$$

This implies

$$\sum_{z\in V_k} 2u(z) \sum_{\substack{e=(z,y)\\ y\in V_{k+1}}} (u(y)-u(z)) \geq \sum_{z\in V_k} 2u(z) \sum_{\substack{e=(z,w)\\ w\in V_{k-1}}} (u(z)-u(w))$$
$$= \sum_{z\in V_k} \sum_{\substack{e=(z,w)\\ w\in V_{k-1}}} 2u(z)(u(z)-u(w)).$$

Using

$$2u(z)^2 - 2u(z)u(w) \geq u(z)^2 - u(w)^2,$$

we have

$$\sum_{z\in V_k} \sum_{\substack{e=(z,w)\\ w\in V_{k-1}}} 2u(z)(u(z)-u(w)) \geq \sum_{z\in V_k} \sum_{\substack{e=(z,w)\\ w\in V_{k-1}}} u(z)^2 - u(w)^2.$$

This sum, in turn, is merely the sum over all edges E_{k-1} and can be rewritten as

$$\sum_{z\in V_k} \sum_{\substack{e=(z,w)\\ w\in V_{k-1}}} u(z)^2 - u(w)^2 = \sum_{z\in V_k} d_{\text{in}}(z)u(z)^2 - \sum_{w\in V_{k-1}} d_{\text{out}}(w)u(w)^2.$$

Altogether, we see that

$$\sum_{y\in V_{k+1}} d_{\text{in}}(y)u(y)^2 - \sum_{z\in V_k} d_{\text{out}}(z)u(z)^2 \geq \sum_{z\in V_k} d_{\text{in}}(z)u(z)^2 - \sum_{w\in V_{k-1}} d_{\text{out}}(w)u(w)^2.$$

This is the desired statement. □

2.2. **Proof of Corollary 1.**

Proof. The argument follows from telescoping the monotonicity formula. We have

$$\sum_{k=a}^{b} N(k) = \sum_{k=a}^{b} \left[\sum_{d(x,y)=k+1} d_{\text{in}}(y)\cdot u(y)^2 - \sum_{d(x,y)=k} d_{\text{out}}(y)\cdot u(y)^2 \right].$$

If the Graph is locally expansive, meaning $d_{\text{out}}(y) \geq d_{\text{in}}(y)$, then

$$(b-a+1)N(a) \leq \sum_{k=a}^{b} N(k) \leq \sum_{d(x,y)=b+1} d_{\text{in}}(y)\cdot u(y)^2 - \sum_{d(x,y)=a} d_{\text{out}}(y)\cdot u(y)^2.$$

Likewise, if the graph is locally contractive, meaning $d_{\text{out}}(y) \leq d_{\text{in}}(y)$, then

$$(b-a+1)N(b) \geq \sum_{k=a}^{b} N(k) \geq \sum_{d(x,y)=b+1} d_{\text{in}}(y)\cdot u(y)^2 - \sum_{d(x,y)=a} d_{\text{out}}(y)\cdot u(y)^2.$$

□

2.3. **Proof of Corollary 2.**

Proof. We give a proof when $d = 2$, where the proof can be nicely visualized; the higher-dimensional case is completely analogous. We consider a square Q_t centered at a point and define the energy

$$E(t) = \int_{\partial Q_t} u(x)^2 d\sigma.$$

For comparison, we also consider a slightly larger square $Q_{t+\varepsilon}$ centered in the same point. We also consider the 'reduced' larger cube $Q_{t+\varepsilon}^*$ that is obtained by simply translating part of Q_t a distance ε along a coordinate axis (see Fig. 3 for a sketch). We note that $\partial Q_{t+\varepsilon}$ and $\partial Q_{t+\varepsilon}^*$ differ by line segments of size $\sim \varepsilon^2$ and thus

$$\int_{\partial Q_{t+\varepsilon}} u(x)^2 d\sigma = \int_{\partial Q_{t+\varepsilon}^*} u(x)^2 d\sigma + \mathcal{O}(\varepsilon^2).$$

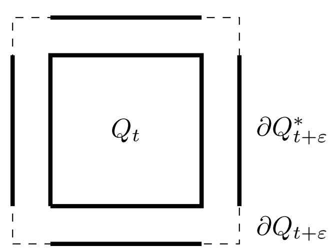

FIGURE 3. The cube Q_t, the boundary $\partial Q_{t+\varepsilon}$ of the slighly larger cube and the reduced boundary of the larger cube $\partial Q_{t+\varepsilon}^*$.

At the same time, we see that we can control the change of $u(x)^2$ from $Q_{t+\varepsilon}$ to $Q_{t+\varepsilon}^*$ by simply differentiating in the direction of x and the direction of y, respectively, and these directions correspond to the normal directions on Q_t. Therefore

$$\int_{\partial Q_{t+\varepsilon}^*} u(x)^2 d\sigma = \int_{\partial Q_t} u(x)^2 d\sigma + \varepsilon \int_{\partial Q_t} 2u \frac{\partial u}{\partial n} d\sigma + \mathcal{O}(\varepsilon^2).$$

This implies that

$$E'(t) = 2 \int_{\partial Q_t} u \frac{\partial u}{\partial n} d\sigma.$$

Rewriting this using the Green's identity, we obtain

$$E'(t) = 2 \int_{\partial Q_t} u \frac{\partial u}{\partial n} d\sigma = 2 \int_{Q_t} |\nabla u|^2 dx \geq 0.$$

As t gets larger, we simply integrate a non-negative function over a larger domain and therefore $E'(t)$ is also non-decreasing and $E(t)$ is convex. $\square$

2.4. **An example.** The geometry of infinite graphs can be quite different from that of the Euclidean space. Liouville's theorem can fail and bounded harmonic functions can exist, which leads to several other interesting consequences. The purpose of this section is to construct an explicit example of a harmonic function to illustrate our Theorem. We will work on the infinite 3-regular tree. Note that, except in the root, $d_{\text{in}}(v) = 1$ while $d_{\text{out}}(v) = 2$. The function is sketched in Fig. 4. We fix a vertex to be the root and set the function to be 0 there and in an entire branch leading away from it. We moreover set the harmonic function to be a function that is 'symmetric' around the middle and merely a function of the distance to the origin. This leads to the sequence

$$a_0 = 0, a_1 = 1 \qquad \text{and} \qquad a_{k+1} = \frac{3a_k - a_{k-1}}{2}.$$

On easily sees that, for $n \geq 1$

$$a_n = 2 - \frac{1}{2^{n-1}}.$$

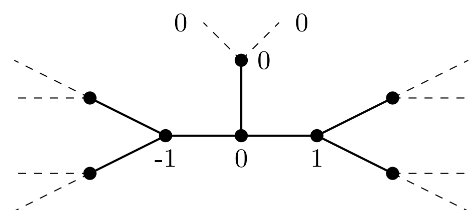

FIGURE 4. Sketch of the function: a symmetric function around the root at 0, the third branch (going up) is set to always be 0.

On the 'left' branch, the values are simply $-a_n$. This leads to a non-constant harmonic function with values between -2 and 2. We can now illustrate the discrete Almgren Monotonicity Formula for this function. It states that

$$N(k) = \sum_{d(x,y)=k+1} u(y)^2 - 2 \sum_{d(x,y)=k} u(y)^2$$

is non-negative and that $N(k)$ is monotonically increasing in k. There are 2^k points at distance k (counting both branches) and thus

$$\sum_{d(x,y)=k} u(y)^2 = 2^k \left(2 - \frac{1}{2^{k-1}}\right)^2.$$

A short computation shows that

$$N(k) = 2^{k+1} \left(2 - \frac{1}{2^k}\right)^2 - 2^{k+1} \left(2 - \frac{1}{2^{k-1}}\right)^2 = 8 - \frac{3}{2^{k-1}}.$$

$N(k)$ non-negative and monotonically increasing and converges to 8.

2.5. **Weighted Graphs.** We briefly comment on the setting of weighted graphs $G = (V, E, w)$ where every edge $e \in E$ is additionally equipped with a weight $w_e > 0$. Harmonic functions are now functions $u : V \to \mathbb{R}$ satisfying

$$\sum_{(x,y)\in E} w_{xy}(u(y) - u(x)) = 0.$$

We note that, in principle, we could even allow for self-loops $(x, x) \in E$ since they automatically disappear when considering harmonic functions. In the weighted setting, we can adapt the notion of in-degree and out-degree for $v \in V_k$ as

$$d_{\text{in}}(v) = \sum_{\substack{e=(x,v)\in E \\ x\in V_{k-1}}} w_{xv} \qquad \text{and} \qquad d_{\text{out}}(v) = \sum_{\substack{e=(v,x)\in E \\ x\in V_{k+1}}} w_{vx}.$$

Note that if the edges all have weight 1, then these definitions reduce themselves to the definitions already used above. Having these definitions in place, we will conclude just as above that

$$N(k) = \sum_{d(x,y)=k+1} d_{\text{in}}(y) \cdot u(y)^2 - \sum_{d(x,y)=k} d_{\text{out}}(y) \cdot u(y)^2$$

is non-negative and satisfies $N(k + 1) \geq N(k)$. Showing that $N(1) \geq N(0)$ then amounts to showing that

$$\sum_{d(x,y)=1} w_{xy}u(y)^2 \geq \left(\sum_{d(x,y)=1} w_{xy} \right) u(x)^2 = \frac{1}{d_{\text{out}}(x)} \left(\sum_{d(x,y)=1} w_{xy}u(y) \right)^2$$

which follows from applying Cauchy-Schwarz to

$$\left(\sum_{d(x,y)=1} w_{xy}u(y) \right)^2 = \left(\sum_{d(x,y)=1} \left(\sqrt{w_{xy}}u(y)\right) \cdot \sqrt{w_{xy}} \right)^2.$$

As for the remainder of the argument, we argue in parallel and obtain

$$\sum_{y\in V_{k+1}} d_{\text{in}}(y) \cdot u(y)^2 \geq \sum_{z\in V_k} d_{\text{out}}(z)u(z)^2 + \sum_{\substack{e=(z,y) \\ e\in E_k}} 2w_e u(z)(u(y) - u(z)).$$

The weight w_e can then be carried through the rest of the argument without having any further impact and the result follows.

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