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A cocharge formula for the Δ-Springer modules

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Abstract. We conjecture a simple combinatorial formula for the Schur expansion of the Frobenius series of the S_n-modules $R_{n,\lambda,s}$, which appear as the cohomology rings of the "Δ-Springer" varieties. These modules interpolate between the Garsia-Procesi modules R_μ (which are the type A Springer fiber cohomology rings) and the rings $R_{n,k}$ defined by Haglund, Rhoades, and Shimozono in the context of the Delta Conjecture.

Our formula directly generalizes the known cocharge formula for Garsia-Procesi modules and gives a new cocharge formula for the Delta Conjecture at $t = 0$, by introducing *battery-powered tableaux* that "store" extra charge in their battery. Our conjecture has been verified by computer for all $n \leq 10$ and $s \leq \ell(\lambda) + 2$, as well as for $n \leq 8$ and $s \leq \ell(\lambda) + 7$. We prove it holds for several infinite families of n, λ, s.

Keywords: Cocharge, Springer fiber, Hall-Littlewood polynomials, Delta conjecture

1 Introduction and results

The rings $R_{n,\lambda,s}$, for integers n, s and a partition λ with $|\lambda| = k \leq n$ and $s \geq \ell(\lambda)$, were first introduced in [8]. They are graded rings with an S_n-action that interpolate between the well-known Garsia-Procesi modules R_μ [6] and the rings $R_{n,k}$ introduced by Haglund, Rhoades, and Shimozono [12]. The rings R_μ, which coincide with $R_{n,\lambda,s}$ for $\lambda = \mu$, $n = |\mu|$, are isomorphic to the cohomology rings of the Springer fibers and have a natural S_n action, being a symmetric quotient of the polynomial ring $\mathbb{Q}[x_1, \ldots, x_n]$. The rings $R_{n,k}$, which coincide with $R_{n,\lambda,s}$ for $\lambda = (1^k)$ and $s = k$, arise naturally in the $t = 0$ case of the famous *Delta conjecture* [11] (part of which was proven in [2, 4]) and have two geometric interpretations [10, 14]. The common generalization $R_{n,\lambda,s}$ have been shown to have a geometric interpretation as the cohomology rings of the Δ-*Springer varieties* [10], and we therefore refer to them here as the Δ-*Springer modules*.

The decomposition of a graded S_n-module $R = \bigoplus_d R_d$ into irreducibles can be described by its *graded Frobenius character*

$$\mathrm{grFrob}(R; q) := \sum_d \mathrm{Frob}(R_d) q^d$$

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where R_d is the d-th graded piece and Frob is the additive map on representations that sends the irreducible S_n-module V_ν to the Schur function s_ν.

For the Garsia-Procesi modules R_μ, it is known (see [6]) that $\mathrm{grFrob}(R_\mu;q) = \widetilde{H}_\mu(x;q)$ where $\widetilde{H}_\mu(x;q)$ are the (transformed) symmetric *Hall-Littlewood polynomials*. Lascoux and Schutzenberger [13] provided the following explicit combinatorial formula for their Schur expansions. For a partition μ, define $\mathrm{SSYT}(\mu)$ to be the set of all (straight shape) **semistandard Young tableaux** of **content** μ, meaning that the tableau entries consist of μ_i copies of i for each i, and the entries are weakly increasing across rows and strictly increasing up columns in French notation (as in the "device" part of the tableau at left in Figure 1). Then

$$\mathrm{grFrob}(R_\mu;q) = \widetilde{H}_\mu(x;q) = \sum_{T\in \mathrm{SSYT}(\mu)} q^{\mathrm{cc}(T)} s_{\mathrm{sh}(T)}$$

where $\mathrm{sh}(T)$ is the **shape** of the tableau T, that is, the partition whose i-th part is the length of the i-th row of T, and $s_{\mathrm{sh}(T)}$ is the corresponding Schur function. Here cc is the *cocharge* statistic as defined in Section 2 below.

In this paper we provide the following full conjectural formula for $\mathrm{grFrob}(R_{n,\lambda,s};q)$ that directly generalizes the above.

Definition 1.1. For a fixed n, λ, s with $k = |\lambda| \le s$, define $\Lambda_{n,\lambda,s}$ to be the partition formed by adding an $s \times (n-k)$ rectangle at the left of the diagram of λ. In other words $\Lambda_{n,\lambda,s} = (n-k+\lambda_1, n-k+\lambda_2, \ldots, n-k+\lambda_r, n-k, \ldots, n-k)$ where there are s parts in total. As an example, for $n = 8$, $\lambda = (2,1,1)$, $s = 4$, we have $\Lambda_{n,\lambda,s} = (6,5,5,4)$.

Definition 1.2. A **battery-powered tableau** of parameters n, λ, s consists of a pair $T^+ = (D, B)$ of semistandard Young tableaux, where B is rectangular of shape $(s-1)\times(n-k)$, and the total content of D and B is $\Lambda_{n,\lambda,s}$. We call D the **device** of T^+ and B the **battery**. We define the **shape** of T^+ to be the shape of its device, that is, $\mathrm{sh}(T^+) = \mathrm{sh}(D)$.

We write $\mathcal{T}^+(n,\lambda,s)$ to denote the set of all battery-powered tableaux of parameters n, λ, s. For $T \in \mathcal{T}^+(n,\lambda,s)$, we write $\mathrm{cc}^+(T^+)$ and $\mathrm{ch}^+(T^+)$, respectively to denote the cocharge and charge of the word formed by concatenating the reading words of D and B in that order (see Section 2).

Remark 1.3. We will usually draw the battery down-and-right from the device, as in Figure 1, so that the device and the battery together form a **skew tableau** (that is, a tableau of shape θ/ρ, where θ/ρ is formed by deleting the diagram of a partition ρ from a larger partition θ). We write this tableau as $D \cdot B$ so that $\mathrm{cc}^+(T^+) = cc(D\cdot B)$.

Conjecture 1.4. *We have*

$$\widetilde{H}_{n,\lambda,s}(x;q) := \mathrm{grFrob}(R_{n,\lambda,s};q) = \frac{1}{q^{\binom{s-1}{2}(n-k)}} \sum_{T^+\in\mathcal{T}^+(n,\lambda,s)} q^{\mathrm{cc}^+(T^+)} s_{\mathrm{sh}(T^+)}.$$

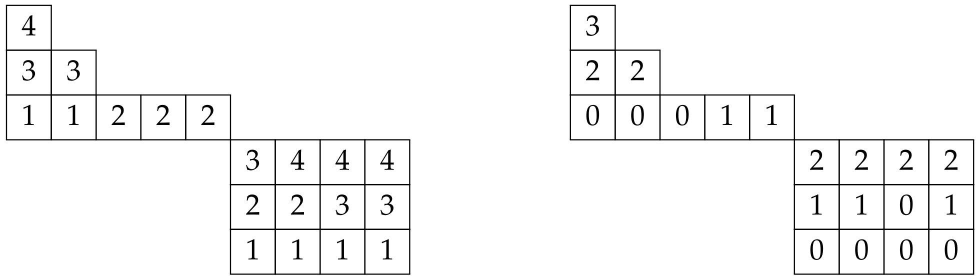

Figure 1: At left, a battery-powered tableau T^+ for $n = 8$, $\lambda = (2,1,1)$, and $s = 4$. Its shape is $(5,2,1)$. The cocharge labels are shown at right, giving $\mathrm{cc}^+(T^+) = 20$.

We think of the battery as storing extra charge for the device. The q-exponent $\binom{s-1}{2}(n-k)$ is the largest amount of cocharge that may be stored in the battery.

Example 1.5. Suppose $n = 8$, $\lambda = (2,1,1)$, and $s = 4$. Then $\Lambda_{n,\lambda,s} = (6,5,5,4)$ and an example of a battery-powered tableau is shown in Figure 1. Its cocharge is 20 and shape is $(5,2,1)$, and the normalization factor in Conjecture 1.4 is $q^{-\binom{3}{2}\cdot 4} = q^{-12}$, so one of the terms of the summation above is $q^{-12} \cdot q^{20} s_{(5,2,1)} = q^8 s_{(5,2,1)}$.

By applying rev_q (reversing the coefficients of the q polynomial by setting $q \to q^{-1}$ and multiplying by q^d where d is the degree) to Conjecture 1.4, we can obtain the following alternative simpler expansion in terms of the generalized charge statistic.

Conjecture 1.6. *We have*

$$\mathrm{rev}_q\left(\widetilde{H}_{n,\lambda,s}\right) = \mathrm{rev}_q\left(\mathrm{grFrob}(R_{n,\lambda,s})\right) = \sum_{T^+ \in \mathcal{T}^+(n,\lambda,s)} q^{\mathrm{ch}^+(T)} s_{\mathrm{sh}(T)}.$$

Using Sage [15], we have tested the above conjectures for all n, λ, s such that $n \leq 10$ and $s \leq \ell(\lambda) + 2$ (where $\ell(\lambda)$ is the number of parts of λ), as well as for $n \leq 8$ and $s \leq \ell(\lambda) + 7$. The following theorem summarizes our progress towards a proof.

Theorem 1. *Conjecture 1.4 holds for $n - k = 1$ (and any n, s), for $s = 2$ (and any n, λ), and in the $R_{n,k}$ case for the coefficient of $s_{(n)}$. Furthermore, for any n, λ, s, the highest degree terms match, which are both of q-degree $n(\lambda) + (n-k)(s-1)$.*

Note that the conjecture also agrees with the Lascoux-Schutzenberger formula for R_μ, since there is no battery when $n - k = 0$.

Below, we provide more background and terminology in Section 2. In Section 3 we prove the conjecture for $n - k = 1$. We prove that Conjectures 1.4 and 1.6 are equivalent in Section 4 and also show that the highest degree terms match. In Section 5 we give a brief sketch of our proof for $s = 2$. Finally in Section 6 we explore the connections to the *minimaj* formula [1, 11] for $\mathrm{grFrob}(R_{n,k}; q)$. Full versions of the proofs sketched here will be provided in [7].

2 Background and definitions

2.1 Cocharge and insertion

The **reading word** of a tableau is the word formed by concatenating the rows from top to bottom. For instance, the reading word of the battery-powered tableau in Figure 1 is

$$4\ 3\ 3\ 1\ 1\ 2\ 2\ 2\ 3\ 4\ 4\ 4\ 2\ 2\ 3\ 3\ 1\ 1\ 1\ 1\ .$$

The **first cocharge subword** is formed by searching right to left in the reading word for a 1, then continuing from that position to search for a 2 (wrapping around the end cyclically if necessary), and so on until we have reached the largest letter of the word:

$$\mathbf{4}\ 3\ 3\ 1\ 1\ 2\ 2\ 2\ \mathbf{3}\ 4\ 4\ 4\ 2\ \mathbf{2}\ 3\ 3\ 1\ 1\ 1\ \mathbf{1}\ .$$

The **cocharge labeling** of a permutation is computed by searching right to left cyclically as before, labeling the entries $1, 2, 3, \ldots$ in order, and starting by labeling the 1 with a 0 and incrementing the label if and only if the next entry is to the left of the previous:

$$\mathbf{4}_3 3\ 3\ 1\ 1\ 2\ 2\ 2\ \mathbf{3}_2 4\ 4\ 4\ 2\ \mathbf{2}_1 3\ 3\ 1\ 1\ 1\ \mathbf{1}_0.$$

We then similarly find and label the *second cocharge subword* among the unlabeled letters:

$$4_3 3\ \mathbf{3}_2 1\ 1\ 2\ 2\ 2\ 3_2 4\ 4\ \mathbf{4}_2 \mathbf{2}_1 2_1 3\ 3\ 1\ 1\ \mathbf{1}_0 1_0.$$

We continue to iterate this process on the unlabeled letters until all have been labeled:

$$4_3 3_2 3_2 1_0 1_0 2_0 2_1 2_1 3_2 4_2 4_2 4_2 2_1 2_1 3_0 3_1 1_0 1_0 1_0 1_0.$$

In Figure 1, the cocharge labels on the reading word elements are shown in the corresponding squares at right. The **charge labels** are placed in the same order as cocharge labels except we increment when the next element is to the **right** of the previous.

The **cocharge** (resp. **charge**) of T, written $\mathrm{cc}(T)$ and $\mathrm{ch}(T)$ respectively, is the sum of the cocharge (resp. charge) labels of its reading word. Therefore, the cocharge of the word above is $3 + 2 + 2 + 1 + 1 + 2 + 2 + 2 + 2 + 1 + 1 + 1 = 20$.

The **RSK insertion** or **bumping** of a letter i into a tableau T is the tableau T' formed by row inserting i into the bottom row of T, where it is placed at the end if i is greater than or equal to every element of the row and otherwise it replaces the leftmost entry m greater than i. Then m is inserted into the second row in the same manner, and so on until the process is complete and a new entry is added. It is well known [5] that this RSK insertion algorithm is bijective, and we call the reverse process **unbumping**. We also say the **RSK insertion** of a tableau B into a tableau D (such as in the case of a battery B and device D) is the tableau T' formed by inserting the letters of the reading word of B one at a time into D.

Cocharge and charge are invariant under bumping: we have $\mathrm{ch}(D \cdot B) = \mathrm{cc}(T')$ and $\mathrm{ch}(D \cdot B) = \mathrm{ch}(T')$ where T' is the insertion of B into D. This is because RSK insertion preserves the *Knuth equivalence class* of the reading word [5], and cocharge and charge are invariant under Knuth equivalence [13].

2.2 S_μ-invariants

A **symmetric function** is a formal power series in infinitely variables $x_1, x_2, \ldots$ with coefficients in $\mathbb{Q}(q)$. We refer to [5] for background on symmetric functions and graded Frobenius characteristic (see Section 2 for the definition), including the definition of Schur functions. We will need the following fact about the Frobenius characteristic.

Let V be an S_K-module, and let $S_\mu \subseteq S_K$ be a Young subgroup. Then the S_μ**-anti-invariants** (or **alternants**) of V are

$$V^{S_\mu\text{-anti}} := \{v \in V : \sigma \cdot v = \mathrm{sgn}(\sigma) v \text{ for all } \sigma \in S_\mu\}.$$

Suppose μ is a refinement of the composition $(1^n, K-n)$. Then $V^{S_\mu\text{-anti}}$ is an S_n-module, and it is well known (as in [12]) that

$$\mathrm{Frob}(V^{S_\mu\text{-anti}}) = e^\perp_{\mu_{n+1}} e^\perp_{\mu_{n+2}} \cdots e^\perp_{\mu_\ell} \mathrm{Frob}(V).$$

2.3 Spaltenstein and Δ-Springer varieties

Given K an integer and μ a composition of K with nonzero parts, let $\mathcal{B}_\mu$ be the partial flag variety of partial flags in $\mathbb{C}^K$ with dimension jumps recorded by the parts of μ. Given a $K \times K$ nilpotent matrix X of Jordan type ν, let $\mathcal{B}^\nu_\mu$ be the **Spaltenstein variety**,

$$\mathcal{B}^\nu_\mu := \{V_\bullet \in \mathcal{B}_\mu : XV_i \subseteq V_{i-1} \text{ for all } i \leq \ell(\mu)\}.$$

When $\mu = (1^K)$, then the Spaltenstein variety is the Springer fiber indexed by Jordan type ν. Letting $H^*(-)$ denote singular cohomology with $\mathbb{C}$ coefficients, Borho and Macpherson [3] proved the following isomorphism of graded $\mathbb{C}$-vector spaces,

$$H^*(\mathcal{B}^\nu_\mu) \cong H^*(\mathcal{B}^\nu)^{S_\mu\text{-anti}}[2d_\mu] \tag{2.1}$$

where $d_\mu = \dim(\mathcal{B}) - \dim(\mathcal{B}_\mu) = \sum_i \binom{\mu_i}{2}$ and the $[2d_\mu]$ denotes a shift in degree.

The Spaltenstein variety has the following connection with Δ-Springer varieties. Let $\nu = \Lambda$, let $K = |\Lambda|$, and let $\mu = (1^n, (s-1)^{n-k})$, which is the composition consisting of n many 1s followed by $n-k$ many $(s-1)$'s. Letting $\pi_n : \mathcal{B}_\mu \to \mathcal{B}_{(1^n, K-n)}$, be the projection map remembering only the first n many parts of the flag (and the ambient space), then the **Δ-Springer variety** $Y_{n,\lambda,s}$ [10] is

$$Y_{n,\lambda,s} = \pi_n(\mathcal{B}^\Lambda_\mu).$$

The map π_n gives us a map on cohomology

$$\pi_n^*: H^*(Y_{n,\lambda,s}) \hookrightarrow H^*(\mathcal{B}_\mu^\Lambda),$$

that is injective by [10], and the map is the one induced by sending x_i to x_i for all i. Letting S_n act on $x_1, \ldots, x_n$ by permutation of the indices, then $H^*(\mathcal{B}_\mu^\Lambda)$ is an S_n-module, and in fact $H^*(Y_{n,\lambda,s})$ is an S_n-submodule of $H^*(\mathcal{B}_\mu^\Lambda) \cong H^*(\mathcal{B}^\Lambda)^{S_\mu\text{-anti}}[2d_\mu]$.

3 Spaltenstein motivation and a special case algebraically

In this subsection, we use the geometry of Δ-Springer varieties to prove Conjecture 1.4 in the case when $n-k=1$. We then explain how this case helped us find the general case of the conjecture.

Fix $n-k=1$. Then the projection π_n from Section 2 is the identity map, and $Y_{n,\lambda,s} = \mathcal{B}_\mu^\Lambda$, where $\mu = (1^n, s-1)$. Therefore, in this case we have that $H^*(Y_{n,\lambda,s}) \cong H^*(\mathcal{B}^\Lambda)^{S_\mu\text{-anti}}[2d_\mu]$ by (2.1). Since $\mathrm{grFrob}(H^*(\mathcal{B}^\Lambda);q) = \mathrm{grFrob}(R_\Lambda;q) = \widetilde{H}_\Lambda(x;q)$, then

$$\widetilde{H}_{n,\lambda,s} = q^{-n(\mu^T)} e_{s-1}^\perp \widetilde{H}_\Lambda.$$

For the proof below, recall that a j**-vertical strip** in a partition ν is a skew shape ν/μ of size j with no two cells in the same row. Also recall a j**-horizontal strip** is a skew shape ν/μ of size j with no two cells in the same column.

Proof of Conjecture 1.4 when $n-k=1$. When $n-k=1$, then $n(\mu^T) = \binom{s-1}{2}$, so it suffices to show

$$q^{\binom{s-1}{2}} \langle s_\mu, \widetilde{H}_{n,\lambda,s}\rangle = \langle s_\mu, e_{s-1}^\perp \widetilde{H}_\Lambda\rangle = \sum_{\substack{S=(T',C)\in\mathcal{T}^+(n,\lambda,s)\\ \mathrm{sh}(T')=\mu}} q^{\mathrm{cc}^+(S)} \tag{3.1}$$

for all $\mu \vdash |\Lambda|$. Letting $N = |\Lambda|$, we have

$$\langle s_\mu, e_{s-1}^\perp \widetilde{H}_\Lambda\rangle = \langle e_{s-1}s_\mu, \widetilde{H}_\Lambda\rangle = \sum_{\substack{\nu\vdash N\\ \nu/\mu \text{ a } (s-1)\text{- vertical strip}}} \sum_{T\in\mathrm{SSYT}(\nu,\Lambda)} q^{cc(T)} \tag{3.2}$$

which follows from the Pieri rule for $e_{s-1}s_\mu$ and the cocharge formula for $\widetilde{H}_\Lambda$.

We now give a bijection between the tableaux T that we are summing over above and the battery-powered tableaux $T = (B,D) \in \mathcal{T}^+(n,\lambda,s)$ with $\mathrm{sh}(B) = \mu$ such that, if $T \mapsto (T', C)$, we have $\mathrm{cc}(T) = \mathrm{cc}^+((T',C))$. This will imply that (3.2) is equal to

$$\sum_{\substack{S=(T',C)\in\mathcal{T}^+(n,\lambda,s)\\ \mathrm{sh}(T')=\mu}} q^{\mathrm{cc}^+(S)}$$

and the result will follow.

The bijection is as follows: consider the entries of T in the squares of ν/μ, and unbump them in order from top to bottom, recording the entries that get bumped out of the tableau in a column C from bottom to top. Notice that each unbumping path P_i is weakly to the right of the previous path P_{i-1} (see [5] for proofs of these facts about unbumping). Moreover, the previous path P_{i-1} bumped out an entry w_{i-1} from the bottom row and replaced it with a larger entry (because it is not the last bumping path of the vertical strip and so we are not starting from the bottom row). It follows that the next path P_i bumps out an entry w_i that is strictly larger than w_{i-1}. This means C is increasing from bottom to top, and so we get a pair $(T', C) \in \mathcal{T}^+(n, \lambda, s)$.

This process is a bijection because given a (T', C), inserting the column C into T' adds a vertical strip to the shape μ to form a tableau T of content Λ. Finally, cocharge is constant on Knuth equivalence classes and hence is unchanged under bumping.

Thus, (3.1) holds and the result holds when $n - k = 1$. □

We now explain the motivation behind Conjecture 1.4. Recall that in general the cohomology ring $H^*(Y_{n,\lambda,s})$ is a graded S_n-submodule of $H^*(\mathcal{B}^\Lambda)^{S_\mu\text{-anti}}[2d_\mu]$, whose graded Frobenius characteristic is the symmetric function $q^{-\binom{s-1}{2}(n-k)} e^\perp_{(s-1)^{n-k}} \widetilde{H}_\Lambda$. By a similar argument as above, the s_μ coefficient of $e^\perp_{(s-1)^{n-k}} \widetilde{H}_\Lambda$ is

$$\sum_{S = T' \cdot C_1 \cdot C_2 \cdots C_{n-k}} q^{\mathrm{cc}(S)},$$

where the sum is over all $T' \in \mathrm{SSYT}(\mu)$, and $C_i \in \mathrm{SSYT}(1^{s-1})$ for all i. Therefore,

$$q^{\binom{s-1}{2}(n-k)} \langle s_\mu, \widetilde{H}_{n,\lambda,s} \rangle \le \sum_{S = T' \cdot C_1 \cdot C_2 \cdots C_{n-k}} q^{\mathrm{cc}(S)} \tag{3.3}$$

where the inequality is coefficient-wise for each power of q. Thus, Conjecture 1.4 can be interpreted as stating that the terms of the right-hand side of (3.3) that correspond to $\widetilde{H}_{n,\lambda,s}$ are those such that $C_1 \cdot C_2 \cdots C_{n-k}$ rectifies to an element of $\mathrm{SSYT}((n-k)^{s-1})$.

4 Equivalence of conjectures and highest degree terms

For any partition ν, define the statistic $n(\nu) = \sum_i (i-1)\nu_i$.

Proposition 4.1. *The maximum value of* $\mathrm{cc}^+(T^+)$ *for* $T^+ \in \mathcal{T}^+(n, \lambda, s)$ *is*

$$n(\lambda) + \binom{s}{2}(n-k).$$

Moreover, there is precisely one tableau T^+ *with this value of* cc^+ *for each device shape* ν *with* $\ell(\nu) \le s$ *and where* ν/λ *is a horizontal strip.*

Proof. The maximal cocharge among all words of a given content Λ occurs when each cocharge subword has its letters appearing in order from right to left, and in that case the cocharge is $n(\Lambda)$. For this to occur, the battery columns must be filled with $1, 2, \ldots, s-1$ from bottom to top, for otherwise some entry of the battery B would be to the right of the previous element in its cocharge subword. The subwords starting at the 1's in the bottom of B will then contain $1, 2, \ldots, s$ from right to left, with the s being in the device.

For the cocharge subwords starting at 1's in the device D to be in right to left order, D must contain the unique tableau D' of content λ and shape λ (with λ_i entries i in the i-th row for all i). So, D is formed by adding a horizontal strip of length $n-k$ labeled by s to D' such that the result is semistandard. Thus there is one tableau of maximal cocharge for each shape of height $\leq s$ formed by adding a horizontal strip to λ.

For such pairs (D, B), we have $\mathrm{cc}^+(D, B) = n(\Lambda) = n(\lambda) + \binom{s}{2}(n-k)$, as desired. □

Dividing out by the factor $q^{\binom{s-1}{2}(n-k)}$, we obtain the following corollary.

Corollary 4.2. *The top q-degree of the polynomial on the right hand side of Conjecture 1.4 is $d := n(\lambda) + (s-1)(n-k)$, and the coefficient of q^d is $\sum s_\nu$ where the sum ranges over all partitions ν of n with $\ell(\nu) \leq s$ and ν/λ a horizontal strip.*

The value d matches with the formula given for the top degree of $\mathrm{grFrob}_q(R_{n,\lambda,s})$ in [8]. In [10], it was shown that the coefficient of q^d is the *skew Schur function* $s_{\Lambda/((n-k)^{s-1})}$. A straightforward application of the Littlewood-Richardson rule shows that this agrees with our formula in Corollary 4.2, and we refer to [7] for details.

Finally, we show that Conjectures 1.4 and 1.6 are equivalent. Taking the q-reversal of both sides of Conjecture 1.4, we have

$$\mathrm{rev}_q\left(\widetilde{H}_{n,\lambda,s}\right) = \sum_{T^+ \in \mathcal{T}^+(n,\lambda,s)} q^{n(\lambda)+(n-k)(s-1)-\mathrm{cc}^+(T^+)+\binom{s-1}{2}(n-k)} s_{\mathrm{sh}(T)}.$$

Then the exponent $n(\lambda) + (n-k)(s-1) - \mathrm{cc}^+(T^+) + \binom{s-1}{2}(n-k)$ is equal to $n(\Lambda) - \mathrm{cc}^+(T)$, which is simply $\mathrm{ch}^+(T^+)$ by the definition of charge. This gives Conjecture 1.6.

5 The case $s = 2$

To prove Conjecture 1.6 in the case $s = 2$, we first outline a general strategy for a proof in all cases. Consider the expansion of $\widetilde{H}_{n,\lambda,s}$ into Hall-Littlewood polynomials [9]:

$$\mathrm{rev}_q\left(\widetilde{H}_{n,\lambda,s}(X;q)\right) = \sum_{\substack{\alpha=(\alpha_1,\ldots,\alpha_s)\models n, \\ \alpha \supset \lambda}} q^{n(\alpha/\lambda)+\mathrm{coinv}(\alpha)} H_{\mathrm{sort}(\alpha)}(x;q) \tag{5.1}$$

Above, α is a weak composition of n of length s, and $\mathrm{sort}(\alpha)$ is the partition formed by sorting the parts of α in nonincreasing order. The quantity $n(\alpha/\lambda)$ is the sum $\sum \binom{c_i}{2}$ where c_i is the number of boxes in the i-th column of the diagram of $\mathrm{sort}(\alpha)/\lambda$.

The polynomials $H_{\mathrm{sort}(\alpha)}$, written without the tilde, denote the *charge* version of Hall-Littlewood polynomials, which expand as $H_\mu(x;q) = \sum_{T \in \mathrm{SSYT}(\mu)} q^{\mathrm{ch}(T)} s_{\mathrm{sh}(T)}$. Substituting into (5.1) yields

$$\mathrm{rev}_q\left(\widetilde{H}_{n,\lambda,s}(X;q)\right) = \sum_{\substack{\alpha=(\alpha_1,\ldots,\alpha_s)\models n,\\ \alpha \supset \lambda}} \sum_{T \in \mathrm{SSYT}(\mathrm{sort}(\alpha))} q^{n(\alpha/\lambda)+\mathrm{coinv}(\alpha)+\mathrm{ch}(T)} s_{\mathrm{sh}(T)} \tag{5.2}$$

Thus, to prove Conjecture 1.6, removing both rev_q's, it suffices to show that

$$\sum_{T^+ \in \mathcal{T}^+(n,\lambda,s)} q^{\mathrm{ch}^+(T^+)} s_{\mathrm{sh}(T^+)} = \sum_{\substack{\alpha=(\alpha_1,\ldots,\alpha_s)\models n,\\ \alpha \supset \lambda}} \sum_{U \in \mathrm{SSYT}(\mathrm{sort}(\alpha))} q^{n(\alpha/\lambda)+\mathrm{coinv}(\alpha)+\mathrm{ch}(U)} s_{\mathrm{sh}(U)}.$$

To prove this, we need a shape-preserving bijection between $\mathcal{T}^+(n,\lambda,s)$ and

$$\mathcal{A}(n,\lambda,s) := \{(\alpha, U) | \alpha = (\alpha_1,\ldots,\alpha_s) \models n, \alpha \supset \lambda, U \in \mathrm{SSYT}(\mathrm{sort}(\alpha))\}$$

such that, if T^+ corresponds to (α, U), then $\mathrm{ch}^+(T^+) = \mathrm{ch}(U) + n(\alpha/\lambda) + \mathrm{coinv}(\alpha)$.

We now give such a bijection in the case $s = 2$. To do so, we begin with the following observation about the compositions α.

Definition 5.1. Let $\lambda = (\lambda_1, \lambda_2)$ be a partition with $\lambda_1 \geq \lambda_2 \geq 0$, and let $\alpha = (\alpha_1, \alpha_2)$ be a composition that contains λ. Then define $\varphi(\alpha)$ to be the composition formed by moving $n(\alpha/\lambda) + \mathrm{coinv}(\alpha)$ boxes from the bottom row of $\mathrm{sort}(\alpha)$ to the top row.

As a running example, let $n = 11$, $\lambda = (3,1)$, $s = 2$, and $\alpha = (5,6)$. Then $n(\alpha/\lambda) + \mathrm{coinv}(\alpha) = 2 + 1 = 3$. Since $\mathrm{sort}(\alpha) = (6,5)$, then $\varphi(\alpha) = (3,8)$.

Now, let $\lambda = (\lambda_1, \lambda_2)$ with $\lambda_1 + \lambda_2 = k \leq n$ and $\lambda_1 \geq \lambda_2 \geq 0$. We construct a bijection from $\mathcal{A}(n,\lambda,2)$ to $\mathcal{T}^+(n,\lambda,2)$ as follows.

Definition 5.2. Let $(\alpha, U) \in \mathcal{A}(n,\lambda,2)$. Define $\psi(U)$ to be the tableau formed by changing 1's to 2's in the bottom row of U, starting with the rightmost 1 and moving leftwards, until we obtain a tableau of content $\varphi(\alpha)$.

Continuing our running example with $\alpha = (5,6)$, letting U be the following tableau with $\mathrm{ch}(U) = 2$, then $\psi(U)$ is as below:

$$U = \begin{array}{|c|c|c|c|c|c|c|c|} \hline 2 & 2 & 2 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 \\ \hline \end{array}, \qquad \psi(U) = \begin{array}{|c|c|c|c|c|c|c|c|} \hline 2 & 2 & 2 \\ \hline 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 \\ \hline \end{array}$$

Remark 5.3. The tableau $\psi(U)$ is not necessarily semistandard; it may have columns containing two 2's.

Definition 5.4. Let $(\alpha, U) \in \mathcal{A}(n, \lambda, 2)$. Define $\Phi(\alpha, U)$ as follows. First, compute $\psi(U)$, and add 1's to the left of the bottom row and 2's to the left of the top row until the resulting tableau S has content $\Lambda_{n,\lambda,2}$ (and then left-justifying). Then, unbump a horizontal strip of size $n-k$ from S from right to left to form a tableau D of the same shape as U, and an unbumped row B of length $n-k$. We set $\Phi(\alpha, U) = (D, B)$.

For our running example, we have $\Lambda_{n,\lambda,s} = (10, 8)$ and

$$\Phi(\alpha, U) = \begin{array}{|c|c|c|c|c|c|c|c|ccccccc}
\hline
2 & 2 & 2 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & & & & & & & \\
\hline
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & & & & & & \\
\hline
\multicolumn{8}{c|}{} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{2} \\
\cline{9-15}
\end{array}$$

so that $\mathrm{ch}^+(\Phi(\alpha, U)) = 5 = \mathrm{ch}(U) + n(\alpha/\lambda) + \mathrm{coinv}(\alpha)$. We have the following result, whose proof we omit and refer to [7].

Theorem 2. *The map Φ is a bijection from $\mathcal{A}(n, \lambda, 2)$ to $\mathcal{T}^+(n, \lambda, s)$. Moreover, if $T^+ = \Phi(\alpha, U)$ then* $\mathrm{ch}^+(T^+) = \mathrm{ch}(U) + n(\alpha/\lambda) + \mathrm{coinv}(\alpha)$.

6 Bijection for coefficient of $s_{(n)}$ for $R_{n,k}$ case

We now consider the setting in which $\lambda = (1^k)$ and $s = k$, so that $R_{n,\lambda,s} = R_{n,k}$. We recall the positive Schur expansion of $\mathrm{grFrob}_q(R_{n,k})$ given in [1]. An **ordered set partition**, or OSP, of n is a partition of $\{1, 2, \ldots, n\}$ into a disjoint union of subsets called **blocks**, along with an ordering of the blocks from left to right. For instance, $(45|367|28|19)$ denotes an OSP of 9.

A **descent** of a permutation π is an index d such that $\pi_d > \pi_{d+1}$, and the **major index** of π is the sum of its descents. The **minimaj** of an OSP is the major index of the *minimaj word* formed by ordering each block's entries from least to greatest and then reading the letters in the OSP from left to right. For instance, the associated word to $(45|367|28|19)$ is 453672819, and it has descents in positions $2, 5, 7$, so the minimaj is $2 + 5 + 7 = 14$.

The **reading word** $\mathrm{rw}(P)$ of an OSP P (different from its minimaj word) is formed by reading the smallest entry of each block from right to left, and then the remaining entries from left to right. For instance, the reading word of $(45|367|28|19)$ is 123456789.

It follows from the general Schur expansion of $\mathrm{grFrob}(R_{n,k})$ in [1] that the coefficient of the Schur function $s_{(n)}$ is

$$\sum_{\substack{P \in \mathrm{OSP}(n) \\ \mathrm{rw}(P) = 123\cdots n}} q^{\mathrm{minimaj}(P)}. \tag{6.1}$$

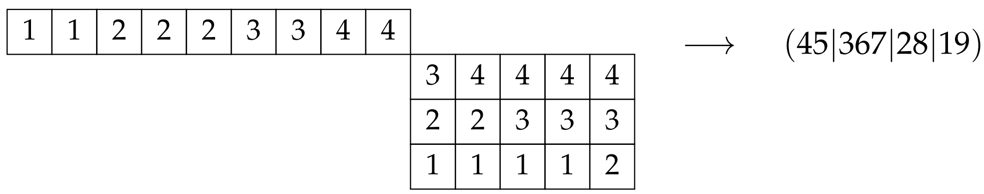

Figure 2: A battery-powered tableau T^+ of shape (9) for $\lambda = (1^4)$ and $s = 4$, and the corresponding ordered set partition P. We have $\mathrm{ch}^+(T^+) = \mathrm{minimaj}(P) = 14$.

On the other hand, the coefficient of $s_{(n)}$ in the ch^+ formula of Conjecture 1.6 is

$$\sum_{\substack{T^+ \in \mathcal{T}^+(n,(1^k),k) \\ \mathrm{sh}(T^+)=(n)}} q^{\mathrm{ch}^+(T^+)} \tag{6.2}$$

Define a bijection f from the set of T^+ tableaux in (6.2) to the OPSs in (6.1) as follows.

Definition 6.1. Given $T^+ \in \mathcal{T}^+(n,(1^k),k)$ with shape (n), define $f(T^+)$ to be the OSP P constructed as follows. Let P have exactly k blocks $B_1, \ldots, B_k$ in that order, which initially contain $k, k-1, k-2, \ldots, 1$ respectively. Then let m_i be the number of i's in the device of T^+, and place the numbers $k+1, k+2, \ldots, n$ into the blocks from left to right in the unique way so that each block B_i has size m_i for all i. The resulting OSP is P.

The map f is depicted in Figure 2. To show it is well-defined, we have $\Lambda_{n,(1^k),k} = ((n-k+1)^k)$, and so $\mathcal{T}^+$ has exactly $n-k+1$ copies of each letter from 1 through k. Thus we have $m_i \geq 1$ for all i, so P is a well-defined OSP. By its construction, the reading word of P is $123\cdots n$, and the process is reversible since there is a unique way to fill the one-row device and the battery for any sequence of block sizes m_i. Thus f is a bijection.

To see that f is weight-preserving, sending ch^+ to minimaj, we refer the reader to [7] for details. As a sketch proof, the charge labels of the battery in this case are always either 0 or 1, with the 1 labels being precisely on the entries of the battery that are larger than their row index. In the device, all of the charge labels are 0 except for those of the final charge subword, which is $123\cdots k$ in order, which has charge $\binom{k}{2}$. This is the minimaj value formed by placing $k, k-1, \ldots, 1$ in the blocks from left to right, and then placing the remaining letters in the blocks increases the minimaj by precisely the amount of charge stored in the battery.

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