Rate-Limited Optimal Transport for Quantum Gaussian Observables

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***Abstract*—The rate-limited optimal transport problem is introduced for the continuous-variable quantum measurement systems in the form of output-constrained rate-distortion coding. The main coding theorem provides a single-letter characterization of the achievable rate region for lossy quantum-to-classical source coding that transforms a sufficiently large tensor product of IID continuous-variable quantum states from a quantum source to a sequence of IID samples from a classical continuous destination distribution with a prescribed distortion level. The evaluation of rate region is performed for the systems with quantum Gaussian source and Gaussian destination distribution. We establish a Gaussian observable optimality theorem for such systems and provide an analytical formulation of the rate-limited quantum-classical Wasserstein distance in the case of isotropic and one-mode Gaussian quantum systems.**[1]

I. Introduction

The goal of optimal transport is to map a source probability measure into a destination one with the minimum possible cost [2], [3]. Let X be a random variable in the source probability space $(\mathcal{X}, \mathcal{F}_{\mathcal{X}}, P_X)$, where $\mathcal{X}$ is the support, $\mathcal{F}_{\mathcal{X}}$ is the event space defined by the σ-algebra of sets on $\mathcal{X}$, and P_X is the probability measure. Let Y be a random variable in the target probability space $(\mathcal{Y}, \mathcal{F}_{\mathcal{Y}}, P_Y)$. The optimal transport problem aims at finding an optimal mapping $f : \mathcal{X} \to \mathcal{Y}$ that converts X to Y and minimizes the expectation of the transportation cost $c(x, y)$, i.e., $\mathbb{E}[c(X, f(X))]$ [4]. However, as such deterministic mappings do not exist in many cases, one has to resort to stochastic channels to transform the source distribution to the target distribution. Thus the problem boils down to finding the optimal *coupling* $\pi^* \in \mathcal{P}(\mathcal{X} \times \mathcal{Y})$ of marginal distributions P_X and P_Y that minimizes the expected transportation cost [5]. For a metric space $(\mathcal{X} \times \mathcal{Y}, d)$, and the cost given by d^p, for some $p \geq 1$, Wasserstein p-distance between the two probability measures P_X and P_Y is defined as the p-th power of the optimal transportation cost [3]–[6]. This problem has been studied extensively in the literature with applications in many areas such as information theory, machine learning and statistical inference [2], [7].

In [8], the authors formally introduced the problem of information-constrained optimal transport by imposing an additional constraint on coupling π in the form of a threshold on the mutual information between X and Y, and established an upper bound on the information-constrained Wasserstein distance by generalizing Talagrand's transportation inequality. It is worth noting that the information-cost function in [8] is equivalent to the rate-distortion function of Output-Constrained (OC) lossy source coding with unlimited common randomness [9]. The protocol associated with this source coding problem provides an operational meaning to the mutual information constraint in [8], as described in [10]. We refer to this problem as *Rate-Limited Optimal Transport* (RLOT).

The quantum version of the optimal transport problem has also been investigated in recent years [11]–[15]. In [13], the authors proposed a generalization of the quantum Wasserstein distance of order 2 and proved that it satisfies the triangle inequality. In an earlier work [10], we introduced Quantum-to-Classical (QC) RLOT along with its corresponding OC lossy source coding problem for the case of finite-dimensional quantum systems.

In this work we consider the QC RLOT problem for Continuous-Variable (CV) quantum systems and characterize the associated performance limits with the development of a novel continuous measurement coding protocol (see Theorem 1). We further employ a novel Gaussian observable optimality theorem (Theorem 2) to perform an evaluation of the optimal performance limit for the Gaussian QC systems with unlimited common randomness using a quadratic distortion observable constructed from canonical quadrature operators (Theorem 3, 4).

II. Continuous-Variable QC Optimal Transport

In this section, we consider measurement lossy source coding for CV quantum systems with separable Hilbert space $\mathcal{H}_A = \mathcal{L}^2(\mathbb{R})$ [16, Chapters 11, 12]. Note that that the proof of the achievability part of the discrete coding theorem [10] does not directly apply to the continuous quantum systems. The first reason is that the operator Chernoff bound as defined in [17] is only applicable to a finite-dimensional Hilbert space. Secondly, in infinite-dimensional systems, an observable with a non-discrete set of outcomes cannot define a quantum channel. Therefore, it is not possible to represent the outcome space using quantum registers defined on separable Hilbert spaces [16].

[1]The proofs of the theorems and the details of the results are provided in the extended online version [1] available at https://arxiv.org/abs/2305.10004 for further reference. This work was supported in part by NSF grants CCF 2007878 and CCF 2132815.

A. Generalized Definitions of Continuous Quantum Systems

We first appeal to the generalized definition for the ensemble [16, Definition 11.22] and generalized definition of POVM [16, Definition 11.29] for quantum systems with continuous outcome space. In contrast to the finite-dimensional Hilbert space for which the POVM is defined for all possible outcomes, in the continuous quantum measurement systems, the generalized POVM is defined over the subset of σ-algebra of Borel subsets $\mathcal{B}$.

An observable M acting on a CV quantum state ρ, with outcomes in measurable space $\mathcal{X}$, results in the following probability measure $\pi_\rho^M(B) = \mathrm{Tr}\{\rho M(B)\}$ for all $B \in \mathcal{B}(\mathcal{X})$. It is also necessary to have a proper definition of post-measurement states. The a posteriori average density operator for a subset $B \in \mathcal{B}$ is defined in [18] for a general POVM M as $\rho_B = \frac{\sqrt{M(B)}\rho\sqrt{M(B)}}{\mathrm{Tr}\{\rho M(B)\}}$.

Based on that, Ozawa proves in [18, Theorem 3.1] that for any continuous source ρ and observable M, there exists a family of a posteriori density operators $\{\rho_x; x \in \mathcal{X}\}$, where the mapping $x \mapsto \rho_x$ is strongly Borel measurable and where $\pi(B) = \mathrm{Tr}\{M(B)\rho\}$, $B \in \mathcal{B}$, is the probability measure of the outcome space.

If the M POVM is such that the measure $M(B)$ is absolutely continuous with respect to some measure μ on $\mathcal{X}$, then there exists a weakly measurable function $y \mapsto m(y)$ with values in the cone of bounded positive operators of $\mathcal{H}$ (Radon-Nikodym derivative) such that [19, Section IV]:

$$\mathcal{E}': \qquad \pi(B) = \mathrm{Tr}\{\rho M(B)\}, \quad \rho_y = \frac{\rho^{1/2} m(y) \rho^{1/2}}{\mathrm{Tr}\{\rho m(y)\}},$$
$$M(B) = \int_B m(y)\mu(dy).$$

Using the definition of post-measured ensemble one can define the information gain for an input state ρ and output ensemble $\{\rho_B, \pi_\rho^M(B)\}_{B\in\mathcal{B}}$ as [16], [20]:

$$I_g(\rho, X) := H(\rho) - \int_{\mathcal{X}} H(\rho_x)\pi_\rho^M(dx). \tag{1}$$

Definition 1. *An (n, R, R_c) source coding scheme for the continuous QC system is comprised of an encoder $\mathcal{E}_n$ on Alice's side and a decoder $\mathcal{D}_n$ on Bob's side. The encoder $\mathcal{E}_n$ is a set of $|\mathcal{M}| = 2^{nR_c}$ collective n-letter measurement POVMs $\Upsilon^{(m)} \equiv \{\Upsilon_l^{(m)}\}_{l\in\mathcal{L}}$, each comprised of $|\mathcal{L}| = 2^{nR}$ outcomes and the common randomness value m, which determines the specific POVM that will be applied to the source state. Bob receives the outcome L of the measurement through a classical channel and applies the randomized decoder $\mathcal{D}_n(B|l, m)$ to this input pair (L, M) to obtain the final output sequence X^n in the output space $\mathcal{X}^n$ with the probability measure $\{\pi_{X^n}(B),\ B \in \mathcal{B}(\mathcal{X}^n)\}$. Therefore, the average Post-Measured Reference (PMR) state $\{\hat{\rho}_B^{R^n}\}_{B\in\mathcal{B}(\mathcal{X}^n)}$, along with the probability measure, forms the output ensemble over their corresponding Borel subset $\mathcal{B}(\mathcal{X})$, where*

$$\hat{\rho}_B^{R^n} = \frac{1}{|\mathcal{M}|P_{X^n}(B)} \sum_{m,l} \mathrm{Tr}_{A^n}\Big\{(id \otimes \Upsilon_l^{(m)})[\Psi_{RA}^{\rho}]^{\otimes n}\Big\}\mathcal{D}_n(B|l, m),$$

$$\pi_{X^n}(B) = \sum_{m,l} \frac{1}{|\mathcal{M}|} \mathrm{Tr}\Big\{\Upsilon_l^{(m)}\rho^{\otimes n}\Big\}\mathcal{D}_n(B|l, m).$$

We further define the average n-letter distortion for the source coding system with encoder-decoder pair $(\mathcal{E}_n, \mathcal{D}_n)$ as the average single-letter distortion of the local PMR states, given the set of distortion observable operators $\Delta(x), x \in \mathcal{X}$, and a continuous memoryless source state $\rho^{\otimes n}$ as

$$d_n(\rho^{\otimes n}, \mathcal{D}_n \circ \mathcal{E}_n) = \frac{1}{n}\sum_{i=1}^n \int_{x\in\mathbb{R}} \mathrm{Tr}_{R_i}\left[\hat{\rho}_x^{R_i}\,\Delta_R(x)\right]\pi_{X_i}(dx),$$

where $\hat{\rho}_{x_i}^{R_i} := \mathbb{E}_{x^{n\setminus[i]}}\left[\mathrm{Tr}_{n\setminus[i]}\{\hat{\rho}_{X^n}^{R^n}\}\right]$ is the i-th local PMR state, conditioned on local outcome $x_i \in \mathcal{X}$, and π_{X_i} is the marginal probability measure of the i-th local outcome.

Definition 2. *Consider a QC system with a distortion observable Δ_{RX} with operator mapping $x \mapsto \Delta_R(x), x \in \mathcal{X}$, and an input quantum state ρ forming (Δ_{RX}, ρ). The pair is called uniformly integrable if for any $\epsilon > 0$ there exists a $\delta > 0$ such that*

$$\sup_{\Pi}\sup_{\Lambda} \mathbb{E}_X\left[\mathrm{Tr}_R\{\Pi_X\rho_X^R\Pi_X\Delta_R(X)\}\right] \le \epsilon, \tag{2}$$

where the supremum is over all POVMs $\Lambda \equiv \{\Lambda_x\}_{x\in\mathcal{X}}$ and all projectors of the form $\Pi = \sum_x \Pi_x \otimes |x\rangle\langle x|$ such that $\mathbb{E}_X[\mathrm{Tr}(\rho_X\Pi_X)] \le \delta$, and ρ_x^R is the PMR state of ρ given the outcome x with respect to Λ.

Remark 1. *We further make the following assumptions. (I) The density operator ρ belongs to a compact subset K of $\mathcal{G}(\mathcal{H})$ (see [16, Definition 11.2]). (II) The system is uniformly integrable. (III) the operator mapping of the distortion $x \mapsto \Delta_R(x)$ is uniformly continuous with respect to the trace norm.*

B. Main Results: Achievable Rate Region for CV Quantum Systems

Using the above definitions, we define the achievability of a rate pair for the given coding system as follows.

Definition 3. *Given a probability measure π_X on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ and a distortion level D, and assuming a product input state of $\rho^{\otimes n}$ of infinite-dimensional separable Hilbert space, a rate pair (R, R_c) is said to be achievable if, for any sufficiently large n and any positive value $\epsilon > 0$, there exists an (n, R, R_c) coding scheme comprising of a measurement encoder $\mathcal{E}_n$ and a decoder $\mathcal{D}_n$ as described in Definition 1 that satisfy the following conditions,*

$$X^n \sim \pi_X^n, \quad d_n(\rho^{\otimes n}, \mathcal{D}_n \circ \mathcal{E}_n) \le D + \epsilon. \tag{3}$$

The closure of the set of all such achievable rate pairs is referred to as the achievable rate region with respect to distortion level D.

The following theorem provides a single-letter characterization of the achievable rate region for the continuous quantum system.

Theorem 1. *Given a pair (π_X, D) and having a product input state $\rho^{\otimes n}$ of continuous infinite-dimensional Hilbert space* *with limited von Neumann entropy, a rate pair (R, R_c) is inside the achievable rate region in accordance with Definition 3 if and only if there exists an intermediate state W with a corresponding measurement POVM $M_W = \{M_W(B), B \in \mathcal{B}_\mathcal{W}\}$ where $\mathcal{B}_\mathcal{W}$ is the σ-algebra of the Borel sets of $\mathcal{W}$, and randomized post-processing channel $P_{X|W}$ which satisfies the rate inequalities*

$$R \geq I_g(R; W), \tag{4}$$

$$R + R_c \geq I(W; X), \tag{5}$$

where W constructs a quantum Markov chain of the form $R - W - X$, generating the ensemble $E_w := \{\hat{\rho}_w, \pi_W(w)\}_{w \in \mathcal{W}}$ on the intermediate space and the ensemble $E_x := \{\hat{\rho}_x, \pi_X(x)\}_{x \in \mathcal{X}}$ on the output space, (as described by [18, Theorem 3.1]) according to the following set:

$$\mathcal{M}_c(\mathcal{D}) = \left\{ (E_w, E_x) \,\middle|\, \begin{array}{r} \int_w P_{X|W}(A|w)\pi_W(dw) = \pi_X(A), \\ \text{for } A \in \mathcal{B}(\mathcal{X}), \\ \pi_W(B) = \mathrm{Tr}\{M(B)\rho\}, \\ \text{for } B \in \mathcal{B}(\mathcal{W}), \\ \int_{x \in \mathbb{R}} \mathrm{Tr}_R[\rho_x \, \Delta_R(x)]\pi_X(dx) \leq D \end{array} \right\}. \tag{6}$$

Remark 2. *The classical channel $P_{X|W} : \mathcal{W} \times \mathcal{B}(\mathcal{X}) \rightarrow \mathcal{X}$ is a mapping such that for every $w \in \mathcal{W}$, $P_{X|W}(.|w)$ is a probability measure on $\mathcal{B}(\mathcal{X})$ and for every $B \in \mathcal{B}(\mathcal{X})$, $P_{X|W}(B|.)$ is a Borel-measurable function.*

Remark 3. *The converse of the finite quantum system in [10] directly applies to the CV quantum systems, except for the cardinality bound.*

III. Evaluation of the Gaussian Quantum States

A. A Brief Overview of Gaussian Quantum Systems

Before providing the evaluation of the Gaussian systems, we first introduce the principal definitions and provide a brief overview of the Gaussian quantum systems.

1) Gaussian Quantum Systems: Let $\mathcal{H}_A \equiv \mathcal{L}^2(\mathbb{R}^s)$ be the separable infinite-dimensional Hilbert space of the square-integrable functions $\psi(\eta)$ for $\eta \in \mathbb{R}^s$ corresponding to s Harmonic oscillators. The canonical observable operators of this system are formed in a $2s$-vector operator $R_1 = Q_1$, $R_2 = P_1$, $R_3 = Q_2$, $R_4 = P_2, ..., R_{2s-1} = Q_s$, $R_{2s} = P_s$ with continuous eigenspectra, where Q_i, P_i are the position and momentum quadrature operators of the i-th harmonic oscillator. In this section, we consider the energy-bounded harmonic oscillator system as described by [13], [16].

The canonical observables satisfy the Canonical Commutation Relation (CCR) $[R_i, R_j] = i\Delta_{ij} I_\mathcal{H}$, for $i, j = 1, ..., 2m$, with Δ being a non-degenerate skew-symmetric symplectic matrix defined as $\Delta = \bigoplus_{k=1}^{s} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

The Weyl operator is further defined by $W(z) = \exp\{iRz\}$ where $z := [q_1, p_1, ..., q_s, p_s], z \in \mathbb{R}^{2s}$ is a vector of values in some phase space corresponding to the eigenvalues of the quadrature operators. The displacement operator is also defined by $D(m) := W(\Delta^{-1}m)$. The *Wigner characteristic function* is further given by $\phi_\rho(z) = \mathrm{Tr}\{\rho W(z)\}$. The domain of the Wigner function $\mathcal{Z} = \mathbb{R}^{2s}$, together with the symplectic matrix Δ, forms a *symplectic space* $\mathcal{K} := (\mathcal{Z}, \Delta)$, which is called the phase space.

For a density operator ρ, the mean vector and covariance matrix are given by $m = \mathrm{Tr}\{\rho R\}$ and $\alpha - \frac{i}{2}\Delta = \mathrm{Tr}\big\{(R - m)^T \rho \, (R - m)\big\}$, respectively, where α is the covariance matrix of the Wigner quasi-probability distribution.

The quantum Gaussian state is defined on $\mathcal{H}$ as a state whose Wigner characteristic function is Gaussian:

$$\phi_\rho(z) = \exp\left[-\frac{1}{2} z^T \alpha z + i m^T z\right].$$

2) Gaussian Observables: The measurement POVM M has a general form as defined in [16]. For any observable POVM M, the information gain can be obtained by substituting its corresponding PMR ensemble $\{\rho_z, \pi_Z(dz)\}_{z \in \mathcal{Z}}$ into (1). An important group of observables is the general form of the covariant Gaussian observable as provided in [19]:

$$\tilde{M}(d^{2s}z) = D(Kz)\rho_G D(Kz)^\dagger \frac{|\det K|^2 d^{2s}z}{\pi^s}, \tag{7}$$

where ρ_G is a density operator. The von-Neumann entropy of the general Gaussian state with covariance matrix α is given by [21]:

$$H(\rho_N) = H_G(\alpha) := \frac{1}{2}\mathrm{Sp} g\left(\left|\Delta^{-1}\alpha\right| - \frac{I}{2}\right), \tag{8}$$

where $\mathrm{Sp}(.)$ is the matrix trace as opposed to the $\mathrm{Tr}(.)$ the trace in Hilbert space and $g(x) = (x+1)\log(x+1) - x\log x$, for $x > 0$, with $g(0) = 0$ is the Gordon function.

3) Distortion Observable: Let $\mathcal{H}^*$ be the space of continuous linear functionals on Hilbert space $\mathcal{H}$ [13]. We consider the following definition of the transposition.

Definition 4. *For any operator $X \in \mathcal{L}(\mathcal{H})$, let X^T be the linear operator on $\mathcal{H}^*$ given by*

$$X^T \langle\phi| = \langle\phi| X \quad \text{for all } \langle\phi| \in \mathcal{H}^*. \tag{9}$$

Remark 4. *Assume having a source state $\rho \in \mathcal{H}_A$ and a measurement POVM with outcomes $z \in \mathcal{Z}$. Then the conditional PMR states $(\hat{\rho}_z^\mathcal{T})^R$ and the unrevealed PMR state $(\rho^\mathcal{T})^R$ live in $\mathcal{H}^*$ [13], [17]. Furthermore, the entropic quantities are invariant under the transposition.*

The following distortion observable operator was introduced by [13] using the quadrature operators:

$$C = \frac{1}{2s}\sum_{i=1}^{2s}(R_i^\mathcal{T} \otimes I_\mathcal{H} - I_{\mathcal{H}^*} \otimes R_i)^2. \tag{10}$$

The significance of this cost operator is that it acts on the composite state of the reference and the destination system in the $\mathcal{H}^* \otimes \mathcal{H}$ space. As a result, the optimal coupling that minimizes this distortion measure has a unique correspondence to the physical quantum channels.

We modify this distortion observable in a way that fits our QC Gaussian measurement system. Consider a POVM Λ with outcomes in $\mathcal{Z}$. Let $\pi_Z(B) = \mathrm{Tr}\{\rho^A \Lambda(B)\}$ for all $B \in \mathcal{B}(\mathcal{Z})$. Define the distortion of the PMR ensemble $\{\{\hat{\rho}_z\}_{z\in\mathcal{Z}}, \pi_Z\}$ as

$$\begin{aligned} d(\rho_A, \Lambda) &:= \frac{1}{2s}\sum_{i=1}^{2s}\int_{\mathcal{Z}}\Big[\mathrm{Tr}\{\hat{\rho}_z^R(R_i - \bar{m}_i(z))^2\} \\ &\qquad + \langle z|(R_i-\bar{m}_i(z))^2|z\rangle\Big]\pi_Z(dz) \quad (11) \\ &= \frac{1}{2s}\int_{\mathcal{Z}}\Big(\mathrm{Sp}\{\Sigma(z)\} + \|z-\bar{m}(z)\|_2^2\Big)\pi_Z(dz) := d(\rho_Z, \pi_Z), \end{aligned}$$

where $\bar{m}_i(z) = \mathrm{Tr}\{\hat{\rho}_z^R R_i\} = \mathrm{Tr}\Big\{(\hat{\rho}_z^{\mathcal{T}})^R R_i^{\mathcal{T}}\Big\}$ is the first moment and $\Sigma(z)$ is the covariance matrix of the state $\hat{\rho}_z^R$. Further, note that a Gaussian measurement system with the above distortion measure satisfies the uniform integrability property in Definition 2. For a proof see [1, Appendix A].

B. Problem Formulation

Suppose having a QC optimal transport system with Gaussian source states and Gaussian output distribution. Recall the single-letter conditions of the rate pair in Theorem 1. We further restrict our evaluation to the systems with unlimited common randomness ($R_c = \infty$). Thus, with no loss of generality we can assume $W = Z$. Moreover, due to the ensemble-observable duality [19], instead of searching for the optimal measurement itself, we can search for the optimum measurement outcome ensembles $\{\{\hat{\rho}_z\}_{z\in\mathcal{Z}}, \pi_Z\}$. In that case, the *OC rate-distortion function* [9], [10] is defined by the following optimization problem:

$$R(D; R_c = \infty, \rho||\pi_Z) := \min_{\{\hat{\rho}_z:\ z\in\mathcal{Z}\}}\Big[H(\rho) - \int_{\mathcal{Z}} H(\hat{\rho}_z)\pi_Z(dz)\Big],$$

subject to,

$$\int_{\mathcal{Z}}\hat{\rho}_z\pi_Z(dz) = \rho,$$

$$\frac{1}{2s}\int_{\mathcal{Z}}\Big(\mathrm{Sp}\{\Sigma(z)\} + \|z-\bar{m}(z)\|_2^2\Big)\pi_Z(dz) \le D.$$

Definition 5. *The rate-limited Wasserstein distance $W_2(R,\rho||\pi_Z)$ of the system is defined as the square root of the inverse of the above OC rate-distortion function within the proper range. The QC Wasserstein distance of order 2 is defined by $W_2(\rho||\pi_Z) := W_2(\infty,\rho||\pi_Z)$. Then the rate of QC Wasserstein distance is defined as the lowest rate achieving the Wasserstein distance:*

$$R_{W_2}(\rho||\pi_Z) := \inf\Big\{R : W_2(R,\rho||\pi_Z) = W_2(\rho||\pi_Z)\Big\}.$$

C. Main Result: Optimality of Gaussian Observables

We establish a Gaussian measurement optimality theorem which shows that for the aforementioned OC rate-distortion quantum measurement system with Gaussian source state and Gaussian output distribution and unlimited common randomness, the Gaussian observables achieve the optimal rate. The proof employs a QC version of the *law of total variance*. Consider a measurement that is applied on a source state ρ and results in the PMR ensemble $\{\{\hat{\rho}_z^{\mathcal{T}}\}_{z\in\mathcal{Z}}, \pi_Z\}$. In this setting, $\bar{m}(Z) = \mathrm{Tr}\{\hat{\rho}_Z R\}$ can be interpreted as the classical estimator of the source state ρ. [2] The following lemma provides the relation for the covariance matrix of this estimator.

Lemma 1. ***(QC Law of Total Variance)*** *For a source quantum state ρ with first moment $\bar{m}_\rho$ and covariance Σ_ρ and a measurement POVM with the corresponding PMR ensemble $\{\{\hat{\rho}_z^{\mathcal{T}}\}_{z\in\mathcal{Z}}, \pi_Z\}$, we have, $\Sigma_\rho = \int\hat{\Sigma}(z)\pi_Z(dz) + \tilde{\Sigma}$, where*

$$\begin{aligned} \Sigma_\rho - \frac{i}{2}\Delta &= \mathrm{Tr}\{\rho(R-\bar{m}_\rho)(R-\bar{m}_\rho)^T\}, \\ \hat{\Sigma}(z) - \frac{i}{2}\Delta &= \mathrm{Tr}\{\hat{\rho}_z(R-\bar{m}(z))(R-\bar{m}(z))^T\}, \quad \forall z\in\mathcal{Z}, \\ \tilde{\Sigma} &:= Cov(\bar{m}(Z)) = \mathbb{E}_Z\big[\bar{m}(Z)\bar{m}(Z)^T\big] - \bar{m}_\rho\bar{m}_\rho^T, \end{aligned}$$

are the covariance matrix of the source, the conditional covariance of PMR state $\hat{\rho}_z$ and the covariance of the classical estimator $\bar{m}(Z)$ respectively.

Using the above lemma, we obtain the following Gaussian observable optimality theorem.

Theorem 2. *In a quantum measurement system, suppose having a Gaussian quantum source state $\rho \sim \mathcal{QN}(\bar{m}_\rho, \Sigma_\rho)$ and a classical Gaussian output distribution $\pi_Z \equiv \mathcal{N}(\mu_Z, \Sigma_Z)$. Also, suppose using the quadratic distortion observable operator* (10) *as the distortion measure. Then, for any feasible distortion value D, the information gain $I_g(R;Z)$ is minimized by a Gaussian observable. This optimal Gaussian observable has the output ensemble representation $\{\tilde{\rho}_{N_G,z}, \pi_Z(dz)\}_{z\in\mathcal{Z}}$ with post-measured states*

$$\tilde{\rho}_{N_G,z} := D(Kz)\rho_{N_G}D(Kz)^\dagger, \qquad (12)$$

where $\rho_{N_G} \sim \mathcal{QN}(0,\Sigma_N)$ is a zero-mean Gaussian state representing the measurement noise with $\Sigma_N \le \Sigma_\rho$ and K is a transformation matrix of the form $K = \Sigma_Z^{-1/2}\Big(\Sigma_Z^{1/2}(\Sigma_\rho - \Sigma_N)\Sigma_Z^{1/2}\Big)^{1/2}\Sigma_Z^{-1/2}$.

Corollary 1. *To characterize the rate-distortion $R(D; R_c = \infty, \rho||\pi_Z)$, it suffices to consider Gaussian PMR ensembles with*

$$\begin{aligned} I_g(R;X) &= H_G(\Sigma_\rho) - H_G(\Sigma_N), \\ d(\tilde{\rho}_{N_G,z}, \pi_Z) &= \frac{1}{2s}\Big(\mathrm{Sp}(\Sigma_N) + \mathbb{E}_{\bar{\pi}_Z}[Z - KZ] + \|\bar{m}_\rho - \mu_Z\|_2^2\Big), \end{aligned}$$

where the expectation in the second term is with respect to $\bar{\pi}_Z(B) = \pi_Z(B - \mu_Z)$ for $B \in \mathcal{B}(\mathcal{Z})$.

D. The QC Gaussian Optimal Transport

By using the above Gaussian optimality theorem, it suffices to find the optimal noise covariance matrix Σ_N that minimizes the information gain given the distortion constraint. By further simplifying the distortion constraint, it reduces to,

$$\begin{aligned} &R(D; R_c = \infty, \rho||\pi_Z) \\ &:= \min_{\Sigma_N}\frac{1}{2}\mathrm{Sp}\ g\Big(\big|\Delta^{-1}\Sigma_\rho\big| - \frac{I}{2}\Big) - \frac{1}{2}\mathrm{Sp}\ g\Big(\big|\Delta^{-1}\Sigma_N\big| - \frac{I}{2}\Big), \end{aligned}$$

[2] This is analogous to the estimator $\tilde{X} = \mathbb{E}[X|Y]$ for an observation Y of a source X in a fully classical system.

subject to $\frac{1}{s}\operatorname{Sp}\Big((\Sigma_Z^{1/2}(\Sigma_\rho - \Sigma_N)\Sigma_Z^{1/2})^{1/2}\Big) \geq D_{\max} - D$ and $\pm\frac{i}{2}\Delta \leq \Sigma_N \leq \Sigma_\rho$, where $|A| := \sqrt{AA^T}$ is the absolute value of the matrix A, and where

$$D_{\max} := \frac{1}{2s}\operatorname{Sp}(\Sigma_\rho + \Sigma_Z) + \frac{1}{2s}\|\bar{m}_\rho - \mu_Z\|_2^2 \tag{13}$$

is the zero-crossing point. Next, we evaluate this OC rate-distortion function for the following example systems.

1) The isotropic Gaussian source and destination: In this part, we formulate the RLOT problem for the isotropic Gaussian Observable system with mean vectors $\bar{m}_\rho = m_{\rho,1}\mathbf{1}_{2s}$ and $\mu_Z = \mu_{Z,1}\mathbf{1}_{2s}$, and covariance matrices $\Sigma_\rho = \sigma_\rho^2 I_{2s\times 2s}$ and $\Sigma_Z = \sigma_Z^2 I_{2s\times 2s}$, respectively. Then the following theorem directly follows.

Theorem 3. *The OC rate-distortion function of the QC Gaussian isotropic system is given by*

$$\begin{aligned} R\Big(D;\infty,\mathcal{QN}(m_{\rho,1},\sigma_\rho^2)||\mathcal{N}(\mu_{Z,1},\sigma_Z^2)\Big) \\ = s\cdot\Big[g\Big(\sigma_\rho^2 - 1/2\Big) - g\Big(n^*\sigma_\rho^2 - 1/2\Big)\Big], \end{aligned}$$

where $n^*(D)$ is the noise-to-signal power ratio of the Gaussian observable

$$n^*(D) = \begin{cases} 1 - \left(\frac{D_{max}-D}{2\sigma_\rho\sigma_Z}\right)^2 & D_{OT} \leq D \leq D_{max}, \\ 1 & D_{max} < D. \end{cases} \tag{14}$$

The upper threshold value $D_{\max}$ is given by (13) *and the lower threshold value D_{OT} is the 2-Wasserstein distance of the QC Gaussian isotropic system given by*

$$W_2^2\Big(\mathcal{QN}(m_{\rho,1},\sigma_\rho^2)\,,\,\mathcal{N}(\mu_{Z,1},\sigma_Z^2)\Big) = D_{max} - 2\sigma_Z\sqrt{(\sigma_\rho^2 - \frac{1}{2}}).$$

2) The one-mode Gaussian case: We first define a few parameters that are necessary for the theorem. Define the matrix $\Sigma = \Sigma_Z^{1/2}\Sigma_\rho\Sigma_Z^{1/2}$ with eignen-decomposition $\Sigma = U\text{diag}[\sigma_1^2,\sigma_2^2]U^T$. Also define $X = U\text{diag}[x_1,x_2]U^T$, where x_1 and x_2 satisfy

$$\frac{x_1}{\sigma_1^2 - x_1^2} = \frac{x_2}{\sigma_2^2 - x_2^2}, \quad x_1 + x_2 = D_{\max} - D.$$

Next define $X_{OT} = U\text{diag}[y_1,y_2]U^T$, where y_1 and y_2 satisfy

$$\frac{y_1}{\sigma_1^2 - y_1^2} = \frac{y_2}{\sigma_2^2 - y_2^2}, \quad (y_1^2 - \sigma_1^2)(y_2^2 - \sigma_2^2) = \frac{1}{4}\det\{\Sigma_Z\}.$$

Theorem 4. *For a one-mode Gaussian quantum measurement system ($s = 1$), the OC rate-distortion function for the interval $D_{OT} \leq D \leq D_{max}$ is given by*

$$\begin{aligned} &R(D;\infty,\mathcal{QN}(0,\Sigma_\rho)||\mathcal{N}(0,\Sigma_Z)) \\ &= g\Big(\sqrt{\det\{\Sigma_\rho\}} - 1/2\Big) - g\Big(\sqrt{\det\{\Sigma_N\}} - 1/2\Big), \end{aligned}$$

where the noise covariance matrix is $\Sigma_N = \Sigma_\rho - \Sigma_Z^{-1/2}X^2\Sigma_Z^{-1/2}$. Furthermore, the QC Wasserstein distance D_{OT} for the one-mode case is obtained by

$$W_2^2\Big(\mathcal{QN}(0,\Sigma_\rho)\,,\,\mathcal{N}(0,\Sigma_Z)\Big) = D_{OT} = D_{max} - \operatorname{Sp}(X_{OT}).$$

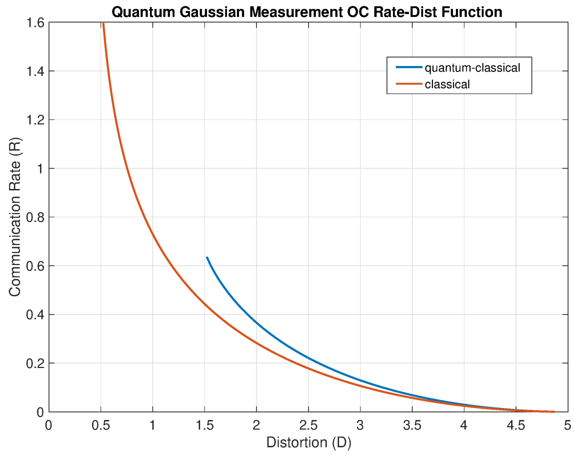

Fig. 1. Plot of the OC rate-distortion function of (Blue): the QC Gaussian measurement system with zero mean and given marginal covariance matrices, and (Red): the fully classical system for Gaussian marginal distributions with zero mean and same given covariance matrices.

Remark 5. *In the classical systems, the Wasserstein 2-distance between Gaussian distributions occurs with an infinite communication rate ($R = \infty$). In contrast, the required communication rate for QC Wasserstein distance is bounded, and is given by $R_{W_2}(\rho||\pi_Z) = g\Big(\sqrt{\det\{\Sigma_\rho\}} - 1/2\Big)$. It is achieved when $\big|\Delta^{-1}\Sigma_N\big| = I/2$, which is a restatement of the Heisenberg uncertainty principle.*

E. Numerical Example

We consider an example of the above system when the input state is a Gaussian quantum state and the output has a classical Gaussian distribution with covariance matrices

$$\Sigma_\rho = \begin{bmatrix} 1.0783 & -0.8976 \\ -0.8976 & 1.3155 \end{bmatrix}, \quad \Sigma_Z = \begin{bmatrix} 1.7471 & -1.2224 \\ -1.2224 & 0.8583 \end{bmatrix}.$$

The displacement is assumed to be zero for both source and destination. For this system, the optimal OC rate-distortion function is given in Figure 1. The rate-distortion function is compared with the fully classical Gaussian system with same covariance matrices. The observation of limited maximum rate in the QC case illustrates the rate-limit imposed by the noise of Heisenberg uncertainty principle in Remark 5.

IV. Conclusion

The problem of rate-limited QC optimal transport was introduced and an operational interpretation was provided in the form of OC lossy source coding for CV quantum systems. Further, a detailed analysis of the RLOT for the Gaussian quantum systems was provided along with a Gaussian Observable optimality theorem. Future works may include studying RLOT for the fully quantum systems and the fully quantum Gaussian states.

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