How Using Visual Representations May Provide Teacher Leaders with a Tool for Supporting Sustained Teacher Learning

Abstract

This paper highlights two teachers that participated in two different professional development (PD) experiences who sustained new teaching practices and learning five years after participating. Both PD projects focused on visual representations (VRs) and encouraged and modeled ambitious teaching practices. Teachers provided video clips and participated in interviews to illustrate and describe changes that took place in their learning and practice. Our qualitative analysis showed that (1) the teachers' use of VRs appears to be strongly connected to teachers' own active learning of VRs in PD, (2) VRs appears to be a key factor that supported the teachers' use of other ambitious teaching practices in their classroom and (3) that the two teachers remembered and continued to use ambitious practices and VRs in their classrooms in ways that not only aligned to the goals and intention of the PD, but also adapted and extended representations to different mathematical domains and settings. Implications for mathematics education leaders suggest that a focus on VRs may be one tool to anchor learning to deepen teachers' abilities to engage in ambitious teaching practices.

Keywords: professional development, mathematics education, teacher education, professional learning, representations

How Using Visual Representations May Provide Teacher Leaders with a Tool for Supporting Sustained Teacher Learning

Understanding what teachers take up and use from professional development (PD) years after their participation is of great interest to those who lead and study PD. One central challenge for the field is how to design interventions that target teacher knowledge, while also maintaining a focus on instructional practice and student learning (Jacobs et al., 2020). Researchers have worked to address this challenge and there is now a strong research base delineating critical design aspects of effective PD (Borko et al., 2010; Darling-Hammond et al., 2017; Desimone et al., 2002; Heck et al., 2019; Hill et al, 2013). Effective PD contains some agreed upon qualities: a focus on subject matter content, teacher's active learning, collective participation, coherence, and adequate duration (Desimone, 2009; Garet et al., 2001; Putnam & Borko 2000). PD also needs to be connected to practice and enable participants to develop their pedagogical content knowledge and implement new strategies in their settings (Ball & Bass 2003; Ball & Even 2009; Kennedy 2016). However, these are necessary but not sufficient conditions as studies of PD outcomes yield a mixed picture. Although some PD programs that adhere to design recommendations by the literature have produced encouraging results (Franke et al., 2001; Kutaka et al., 2017; Taylor et al., 2017), others have proven much less successful (Jacob et al., 2017; Santagata et al., 2010).

These mixed empirical results have led to the call for more research to better understand how teacher PD translates into effective practice (Desimone & Garet, 2015). One hypothesis that could account for these varying results is that most studies about the effects of PD in mathematics education focus primarily on the period immediately after the program activities. Relatively few studies have been conducted on their longer-term effectiveness (Brendefur et al., 2013; Franke et al., 2001). In her review of the literature on the impact of PD, Kennedy (2016) concluded, “the ultimate effects of PD are likely not completely visible at the end of the program year” (p. 960). As a result, it is important to investigate what practices are sustained over time as well as factors that may contribute to long-term change in teacher learning and practice.

To fully understand the impact of PD on teacher learning, it is important to look at both short-term changes in teacher’ knowledge and practice immediately after the PD and long-term generative change that occurs several years out. For high-quality PD to have a lasting impact on mathematics instruction, gains from PD need to be sustained after the support ends. Furthermore, more research is needed to better understand the aspects of PD that have the potential to impact students and their learning. For instance, does providing resources and materials play a role? Does the degree to which one learns the content have a lasting effect? Or is a pedagogical strategy an impetus for long term change? Studying the impact of PD on teaching is a complex endeavor, intermingling the constructs of what is the nature of the impact (if there is one) and why there may or may not be an impact. More needs to be known about the ways in which teachers sustain their learning and how the learning unfolds several years after the PD. This study sought to examine what aspects of ambitious mathematics teaching were related to PD and sustained over time and why. After a cursory data analysis, we hypothesized that the use of visual representations (VRs) in PD may play a role and sought to better examine that. While there is evidence that VRs can improve student learning (Boonen, et al., 2014), less is known about the role they play in teacher learning and professional development. This study explored the following research questions: In what ways are ambitious mathematics practices advocated in PD sustained over time? In what ways do VRs play a role in teachers’ learning and instructional practice?

Literature Review

Ambitious Mathematics Teaching

Mathematics instruction that aims to develop all students' conceptual understanding, procedural fluency, reasoning, and problem solving is often referred to as ambitious mathematics teaching (Lampert & Graziani, 2009; Lampert et al., 2010). Ambitious teaching requires viewing students as sense-makers; eliciting and responding to students' thinking; and providing equitable access to learning mathematics. Ambitious teaching is complex and demanding as it requires continual learning about many things– the mathematics content, how to facilitate student understanding, how to foster engagement and how to make learning meaningful for students.

NCTM's Principles to Action (2014) provides additional insight into ambitious teaching. Principles to Action (PtA) identified eight Mathematics Teaching Practices that "represent a core set of high-leverage practices and essential teaching skills necessary to promote deep learning of mathematics" (NCTM, 2014, p. 9). These practices include (1) establish mathematics goals to focus learning, (2) implement tasks that promote reasoning and problem solving, (3) use and connect mathematical representations, (4) facilitate meaningful mathematical discourse, (5) pose purposeful questions, (6) build procedural fluency from conceptual understanding, (7) support productive struggle in learning mathematics, and (8) elicit and use evidence of student thinking. This framework offers a lens to examine instruction that supports successful mathematics learning.

Understanding how to support teachers' development of these ambitious teaching practices is of interest to mathematics coaches, PD providers, and teacher educators. A growing body of research has examined how PD can assist teachers in developing these ambitious teaching practices, such as noticing and analyzing students' mathematical thinking and understanding (e.g., Kazemi et al., 2009; van Es & Sherin, 2008). What is less known is how these ambitious teaching practices change and are sustained years after the support ends.

Teacher Learning

Overarching goals for mathematics teachers' learning include improving knowledge of the content they teach, better understanding of student thinking and learning, and improving instructional practices to meet the needs of diverse learners (NCTM, 2014). Ball and colleagues define mathematical knowledge for teaching (MKT) as the complex set of knowledge needed to effectively teach mathematics to learners (Ball & Bass, 2000; Ball et al., 2005; Ball, et al., 2008; Hill & Ball, 2004). MKT is multi-faceted and includes both content and pedagogical knowledge and provides the field with a framework to focus on in professional development (Jacob et al., 2017).

While substantial research has been conducted to examine the effectiveness of mathematics PD programs on developing teachers' MKT (Copur-Gencturk et al., 2019; Hill & Ball, 2004; Polly et al., 2014), less is known about whether this effect persists or continues to grow after the completion of the programs. Some studies that have examined long term uptake have found that after the completion of the PD, teacher knowledge and practice is sustained or continues to improve. For example, one study reported that mathematics and science teachers' use of inquiry- based teaching practices was sustained during the three years following their PD experience (Supovitz et al., 2000) and another showed that teachers' use of students' mathematical thinking in classroom observations was maintained or continued to grow four years after PD ended (Franke et al, 2001). However, others found that not all teachers sustain what they learned from PD and that initial changes in practice fade over time (Boston & Smith, 2011). Some helpful insights come from Copur-Gencturk and Papakonstantinou's (2016) longitudinal study of a Math and Science Partnership Program for high school mathematics teachers, an effort intentionally designed to incorporate key features of high-quality PD. The researchers followed participants for four years and documented linear instructional growth in several of the targeted areas. While teachers made statistically significant changes in some areas of their instruction over time, teachers were less likely to incorporate multiple representations in their classrooms despite the PD's focus on this. Given these mixed results in what is taken up and used over time from PD, more needs to be known about why this variation exists.

Visual Representations

Visual representations (VRs) are graphic creations such as diagrams or drawings that illustrate quantities, quantitative relationships, or geometric relationships (DePiper & Driscoll, 2018). Using models or representations is an important component of doing mathematics as they support students to make sense of problems by identifying quantities and the relationships between quantities to use VRs to reason with and ultimately justify mathematical solutions (Ng & Lee, 2009). When students learn to represent mathematical ideas and make connections between them, they demonstrate deeper conceptual understanding and problem-solving capabilities (Fuson et al. 2005; Lesh et al.1987). The use of VRs in the classroom also helps students reason mathematically and engage in mathematical discourse (Arcavi, 2003; Fuson & Murata, 2007; Stylainou & Silver, 2004).

The use of VRs by mathematics teachers is complex and requires challenging skills including a strong grasp of the content, anticipation of students' thinking, and selecting the most appropriate VRs to use with their students (DePiper & Driscoll, 2018). Teachers need to recognize what is involved in using particular representations and when they are appropriate to use (Ball et al., 2008). PtA outlines specific teacher actions that can support students in using and connecting representations such as: (1) introducing forms of representations that can be useful to students; (2) asking students to make math drawings or use other visual supports to explain and justify their reasoning; and (3) designing ways to elicit and assess students' abilities to use representations meaningfully to solve problems.

Research on teacher's knowledge and use of visual representations in the classroom has been limited (Dreher and Kuntze 2015; Stylianou, 2010). Researchers have found that teachers struggle with their own use of VRs to solve problems as well as their ability to use and interpret them in the classroom (Dreher & Kuntze, 2015; Orrill et al., 2008). Many teachers are also unaware of key instructional issues when using representations (Bossé et al, 2011; Dreher & Kuntze, 2015).

Both professional development programs in this study included a focus on VRs. Participants learned about mathematical content using VRs and were exposed to different pedagogical strategies that involved the use of these representations. We were curious how this focus on VRs was related to teachers' uptake of not only their use of VRs, but also other ambitious teaching practices four to five years after the PD ended.

Overview of the XXXX project

This paper highlights a project that is part of a large three-year impact study, XXXX[1], that collects qualitative data from three large U.S. National Science Foundation (NSF) PD projects to understand what teachers take up and use as well as the factors that influence uptake five years after the PD experience. This paper focuses on two of the PD projects, Project 1[2] and Project 2[3], that aligned with recommended effective practice and were designed to support

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teachers in ambitious mathematics teaching. Table 1 provides a summary of the two projects, and they are then described in detail below.

Table 1

Overview of PD projects

	Project 1	Project 2
Sample size	90 participants	120 participants
# of Hours of PD Intervention	54 hours of in person PD	30 hours of in person PD & 32 hours of online PD
Content Focus	Similarity Transformations-based geometry	Rational numbers in the middle grades
Pedagogical Focus	Classroom Discourse Representations	Visual Representations Support for multilingual learners

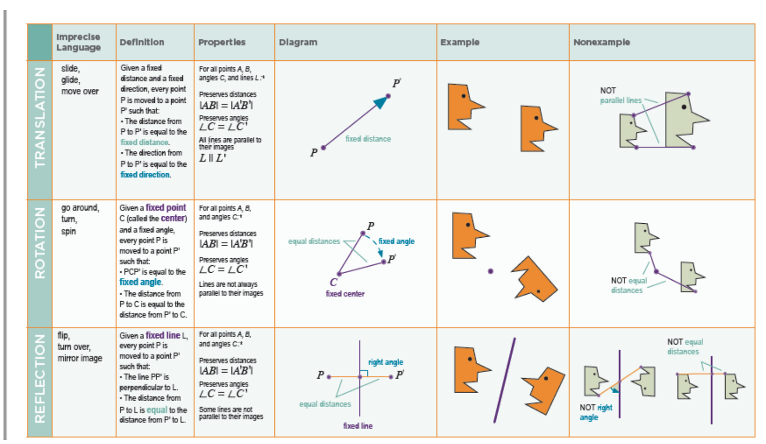
Project 1

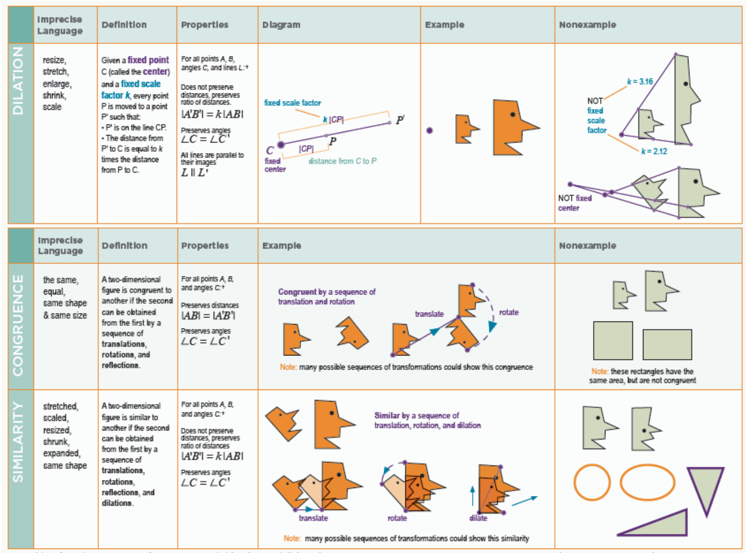
The first NSF project sought to study the impact of the PD on teacher's knowledge and instructional practices. The goal of the PD was not only to improve teachers' conceptual content knowledge and increase their ability to engage students in mathematical practices but to also increase students' conceptual understanding of transformations-based geometry. Project 1 consisted of 54 hours of video-based PD that was grounded in modules focused on dynamic transformations-based geometry which is aligned with the Common Core State Standards in mathematics (CCSSM). Through video analysis, teachers worked together to solve problems and further their knowledge in mathematics teaching in the domain of geometry. The PD allowed teachers to better support students in their attempt to gain a deeper understanding of transformations-based geometry through activities like rate of change on a graph, scaling activities, and similarity tools. The material strongly connects to other critical domains including similarity, proportional reasoning, slope, and linear functions.

The PD highlighted various representations of geometric transformations, congruence, and similarity. All participating teachers received an illustrated glossary called the field guide at the beginning of the PD, which provided definitions, properties, corresponding diagrams (with examples and nonexamples), and imprecise language examples for terms *translation*, *rotation*, *reflection*, *dilation*, *congruence*, and *similarity* (see Figure 1). In addition, applets were used in the PD to display important mathematical content within an interactive, dynamic representation. In addition to a focus on representations, the PD also addressed additional ambitious mathematical teaching practices as defined by NCTM's PtA as teachers discussed facilitation of the tasks and watched classroom videos.

Figure 1

Project 1's Field Guide





Project 2

The second NSF project, Project 2, was a "60-hour blended, face to face and online course to build teachers' knowledge of and self-efficacy about linguistically responsive teaching (LRT) strategies to strengthen multilingual learners' problem solving and discourse in middle grades" (Neumayer-De Piper et al., 2021 p. 491). The goals and intentions of the project were to cultivate in teachers the fluent use of representations, anticipation of students' strategies, the ability to interpret and construct various mathematical solutions, and to reason within and across representations. Teachers learned how to strategically select and align VRs with their instructional goals, anticipate student thinking and misconceptions, and then implement lessons using these strategies in their classrooms. Once implemented they would share experiences and student work, and collaboratively and independently reflect on the teaching cycle in the PD's online workshops.

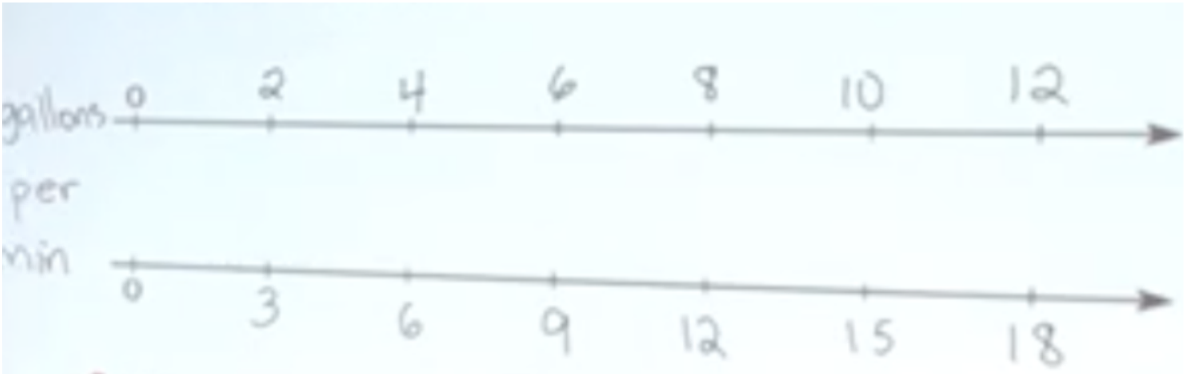
Project 2 focused on two VRs, the double number line (DNL) and tape diagrams (See Figure 2). Both VRs are effective tools that have the potential to foster students' understanding of proportional reasoning and reinforce students' conceptual understanding of rational numbers

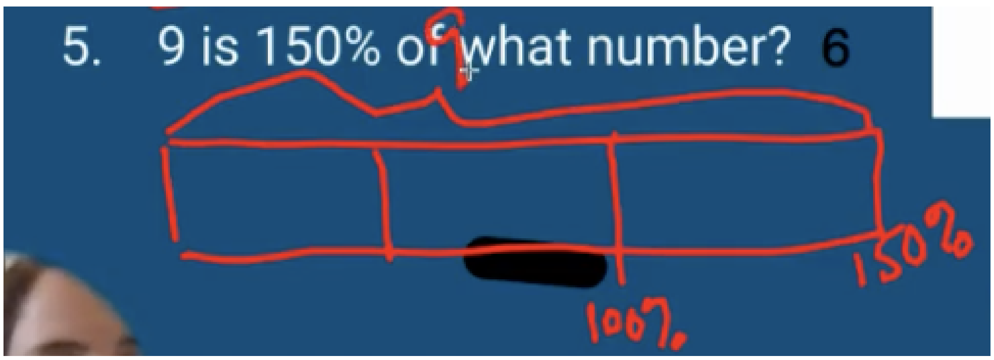
(DePiper & Driscoll, 2018). The DNL is a representation that uses a pair of parallel lines to represent equivalent ratios. Tape diagrams, also referred to as bar diagrams, are rectangular representations that illustrate number relationships. Both diagrams represent quantities and the relationships between quantities. These diagrams allow students to think more additively to “see” multiplicative relationships and examine the relationships between quantities with the representation. Project 2 focused on problem solving with rational number tasks that were easily represented on a DNL or tape diagram. Subsequently, these VRs were used as a communication tool to show and explain students’ mathematical thinking in a very concrete and conceptual manner.

Project 2 also focused on additional ambitious mathematical teaching practices as participants engaged in tasks, planned lessons, and reviewed student work. In particular, the PD focused on LRT strategies to facilitate multilingual learners’ mathematical problem solving and discourse. Participants learned about and experienced these different strategies and their implementation, planned the use of the strategies, and then analyzed and reflected on their implementation using student work.

Figure 2

DNL and Tape Diagram





Methods

Participants

Four to five years after their participation in the two PDs, project investigators from the initial two NSF projects reached out to all participants from their respective projects to support our recruitment efforts. There was no PD support provided after the two projects ended and the only contact the PD providers had with the teachers was to support our recruitment. Subsequently we used a survey to better understand participants' experiences (AUTHOR et al., 2022) and to select case study teachers. Eighteen case study teachers were initially selected based on self-reported levels of uptake (high, medium, low) and were asked to videotape their classroom approximately once a month and identify clips in which they believed they were using content, pedagogy and/or resources from the PD they participated in. The research team also conducted think aloud interviews along with the videos that is the basis of this paper and will be described in greater detail in the following section.

As we analyzed the interviews and classroom videos, the use of representations by the participants emerged as a salient theme. To further investigate this, we focused the study that is the aim of this paper on the results of two participants—one from Project 1 and one from Project 2. They were chosen as they reported and demonstrated high levels of uptake from two different PDs and provided evidence that teachers continued to hone their craft, modify, and expand their use of VRs using different mathematical problems, domains and contexts. We wanted to further examine the ways in which these teachers used both VRs and other ambitious teaching practices five years after their participation in the PD.

Data Collection

For the purposes of this study, qualitative data (interviews and classroom video) were used. Each participant videotaped six lessons between January 2021- December 2021. The teachers were not observed live in their classrooms. The XXXX project research team conducted four semi-structured interviews with each case study participant (two in spring 2021 and two in fall 2021). Each interview took approximately one hour. The first part of these interviews asked teachers to reflect on their experiences with the PD, what they remembered related to the goals and intentions of the PD and what strategies, content, and resources they used from the PD in the past and continue to use currently in their classrooms. The second part of these interviews followed a think aloud protocol (Charters, 2003), where teachers and researchers watched video clips that the teachers selected. The teachers then described their own interpretation of uptake and implementation of content, pedagogy, and resources from the PD. The use of video allowed teachers to reflect on their practice and describe how they perceived their uptake in specific contexts and why they attributed specific learning to the PD. Moreover, they explained whether the learning started with the PD or whether the PD supported something they learned in another PD or something they were currently working towards. These interviews were recorded on Zoom and transcribed.

Data Analysis

A multiple-case study design (Merriam, 2002) was used to analyze the ways in which ambitious mathematical teaching practices were taken up and used in each of the teacher's individual contexts and how the teachers attributed this use to the PDs they attended. After initial cursory analyses the research team recognized that representations learned in their respective PDs were playing an important role in the case study teachers' classrooms. Thus, in year two, we intentionally asked probing questions related to their uptake and use of representations that were originally learned in their respective PDs to get a deeper understanding of the relationship between representations and their long term learning. At least two of the research team members reviewed and took detailed notes on the interview transcripts and video data several times to create a profile for each teacher. We recognize that these teachers have continued to hone their practice, attend different PD workshops, and attend to different goals of the school or district. We tried to account for these ongoing learning experiences to understand whether they have supported similar learning goals and objectives of the original PD as well as how the totality of experiences has supported productive teacher learning. We did this by asking specific questions about the other PDs they have attended and then asked them to pinpoint the origin of the content, pedagogy, or resource in the interviews.

After the profiles were created, they were then analyzed for segments that related to how participants took up and used VRs and the ways in which they attributed this use to the PD they attended. We also noticed that representations were playing an important role in the implementation of other mathematical practices, so we went back and further analyzed the segments that included representations for the other practices, such as implementing tasks that promote reasoning and problem-solving. At least two researchers independently coded these segments for evidence of the NCTM PtA Mathematics Teaching Practices. For example, a researcher might have coded a DNL task used in a classroom video and discussed in an interview that the DNL task promoted reasoning and problem solving or a researcher might have coded an example where the teacher facilitated meaningful discourse or encouraged productive struggle in a lesson using a ratio table. Triangulation of multiple data sources across time, multiple methodologies (interviews and classroom data), and multiple researchers was used to address issues of validity and credibility (Cresswell & Miller, 2000).

After examining these examples, the members of the research team met to discuss salient themes and patterns that emerged. Following this, narratives were written for each teacher. Findings are reported as a case for each teacher. Each case includes specific examples to demonstrate how each teacher used VRs connected to the NCTM PtA Mathematics Teaching Practices which we believe are examples of ambitious mathematics teaching. Additionally we highlight how these examples are related to their experiences in PD.

Teacher Case Studies

Teacher #1: Brianna

Brianna took part in the Project 1 PD. She had taught for 11 years at the start of our study and has taught grades 3-8 mathematics. Throughout this research project, Brianna taught 6th and 8th grade mathematics. She attended the PD because she was teaching geometry and since she had recently moved to teaching middle school mathematics she was interested in learning and teaching transformations-based geometry. Brianna's experience in the PD is described below and four examples are provided that illustrate what we identified as robust uses of representation in her classroom that she attributes to participation in the PD. The examples show her use of VRs and how they sometimes cut across different mathematical domains and tasks compared to those used within the PD. Each example also illustrates how the use of the representations is connected to other ambitious teaching as defined by the teaching practices outlined in PtA.

Experience in PD

Brianna discussed how participating in the PD allowed her to deepen her conceptual understanding of the mathematics content that the PD focused on (geometry). She also felt that it was helpful to work through the math problems as students would during the PD. She mentioned in one of her interviews:

> It was really helpful for me to gain a better insight of the math that I was teaching…to be a student, to learn how to better understand all of these ideas in a way that's more conceptual than what I learned as a student. So that's how I felt like it was most helpful to me.

She also mentioned that in addition to learning new mathematics content this way, it also exposed her to new pedagogical skills to teach the content. She felt an important component of the PD was the opportunity to view a lesson from a different perspective and put herself "in the shoes of my students."

Example #1: Number Lines

In her first interview, Brianna was teaching remotely due to the COVID-19 pandemic. One of the video clips she chose to share with us was a synchronous lesson about negative and positive integers. In the lesson, students were shown vertical number lines and encouraged to use them to think about changes in temperature. She then used an online platform (PearDeck) to have students solve various problems with the vertical number line. The online platform allowed her to choose different student strategies all using the vertical number line and then facilitate a discussion with the class. Her interview and classroom video show many of the teaching practices from PtA but especially highlighted are the use of tasks that promote reasoning, use mathematical representations, facilitate discourse, and elicit student thinking.

After she shared a video clip of the vertical number line lesson with us in the interview, she discussed how this lesson connected to her learning from the PD:

> One of the things that was really important in the PD was the use of models. And while a lot of those models were on coordinate grids and graphing and shapes and scaling and whatnot, these that I used in this lesson were number lines that we were using to be able to go from negative to positive numbers. And the problems and the number lines also required students to be really precise with how they were measuring and how they were representing temperature change in their number lines.

Her reflection on the clip indicates that the use of models or representations was something she felt was an important component of the PD. She also explained to us how she utilized VRs in her classroom practice regularly which was confirmed by each of her classroom videos. She felt strongly that representations supported access and student learning. In her video about the vertical number line, we saw her engage in several mathematical practices related to the use of representations. She introduced a representation that can be useful to students (number lines) and she asked students to use their number lines to explain and justify their reasoning. We also saw how the use of VRs helped her facilitate meaningful discourse by having students examine and discuss each other's representations. In addition, although the PD focused on representations in transformation-based geometry, she generalized this use of VRs and the need for precision in a representation to a topic that wasn't discussed in the PD: operations with rational numbers.

Briana noted how the representations allowed her to have students to view one another's work and learn from one another which was a teaching practice that was important to her:

> What I did was, I could select a couple and then I could show some exemplars. So kids could see at the beginning what some exemplar work looked like. So rather than me showing them how to do it again they could see from their peers and then the rest of the problems that they did more independently - they had a good starting point for how to do those.

Her decision to use a number line representation was related to several ambitious teaching practices she felt were important for instruction and related to the PD: selecting tasks that promote reasoning, using mathematical representations, facilitating discourse by engaging students in purposeful sharing of varied representation and eliciting and using evidence of student thinking.

Example #2: Dot Images

In another example, Briana shared a clip of how she used representations to begin a unit on writing equations. Students were given the following task: *I'm going to show you an image made up of dots, but only for three seconds. You need to find out how many dots there are and be prepared to explain how you saw them*. Students then discussed what they saw in the image and later connected this to writing equations for growing patterns.

Briana explained to us how she felt this clip demonstrated her implementation of ideas she learned in the PD. She connected this task to her learning from a video used in the PD that included VRs of transformations where students shared what they noticed about the different images. That video from PD resonated with her years later and although she was teaching different content, she utilized a representation and similar teaching practices to facilitate discussion and allow students to share their thinking. Below is her description of how this example connected to the PD:

> I chose this one because I remember when I took the geometry class (PD), I feel like they were on an overhead projector, but the teacher's teaching in, I think it's Hawaii. They're looking at something and all of these kids just keep sharing their answers, and it's this very open classroom environment where kids just get to talk and share and they're oohing and aahing about what they're noticing. And so, this was the opening to a lesson on we're talking about factors and multiples, and so they're subitizing. They're looking at different dot structures to figure out how much they are or how many are there.

We see in her comments and in the videos that she viewed the dot image as a vehicle for her to facilitate meaningful discourse and allow students to present and explain their ideas.

She also discussed another important idea from the PD about connecting mathematical concepts and using a warm-up task to help create an entry point for learners:

> This goes back to when I was talking about being thoughtful about the type of warmup that I'm doing, that it's this fun task that relates to what we're going to do later. And it goes from just, "How many dots do you see?" to being able to write an equation. So there's this very easy (task), all the way to something that's more difficult. But if I just started off by saying, ‘Hey, write an equation for this,’ they'd be like…
> ‘How do I do that? What does that even mean?’ So getting them to that point of something that's much higher-level thinking through, something that's kind of fun.

In this example, Briana highlights how her use of a visual representation is connected to the PtA Mathematics Teaching Practice of implementing tasks that promote reasoning and problem solving. Her choice of the dot image task provided students with multiple entry points to the mathematical concepts, allowed students to build on and extend their current mathematical understanding and encouraged students to use a variety of approaches and solutions. Finally, we see that while this task was not one used in the geometry PD, she generalized the ideas she saw in the PD and applied them to the content she was currently teaching.

Example 3: Ratio Table

Briana also chose to share several clips of her teaching with ratio tables with us. Again, we were curious how she felt this connected to the geometry PD. She explained to us how this was related to the scale factor content that had been addressed in the PD:

> This school year we did so much work with ratio tables. And while it's different from the scaling up and down work that we did in the PD, it really helped me to say, okay, these are the foundations that I want to set for my kids to be able to do that work later on in two years. So that was another piece that was really helpful as well.

Again, we see how Briana takes ideas from the PD and transfers them to other content areas. She also described how she felt that the introduction of these ratio tables in sixth grade would prepare students for the content that had been discussed in the PD when they entered eighth grade. This reflects her understanding of mathematical learning progressions and points to her ability to establish mathematical goals to focus learning (PtA Mathematics Teaching Practice #1) by identifying how the goals of a lesson fit within a mathematical learning progression. She viewed the ratio table representation as a tool for her to build important mathematical understanding for both the current content and the content students would be exposed to later.

In her classroom video clips she shared multiple lessons where students used ratio tables to compare rates and ratios. In her think aloud protocol during the interview, she talked about how she viewed this representation, ratio tables, as a tool that allows students an entry point to solve challenging problems. She viewed the ratio tables a tool to help students develop conceptual understanding of ratios and indicated that she saw the ratio table as having multiple uses across different types of problems.

> The sixth-grade teacher and I do many things with ratio tables. We're very much like, we want them to have that tool to be able to use, so we do so much work with our ratio tables. By sixth grade, our goal is not really for them to have an algorithm, but to be able to just be super efficient with their ratio tables. And we talk to kids a lot about how the whole goal is that all of your work is on your ratio table. This is the tool that you're using. So you're not doing all this work off to the side, but this is the tool that you use. And then kids get really good at it, and it also helps their mental math, so that then they'll start to solve other problems like, "Oh, I can just think about that in my head this way."

In this example, Briana highlights how she used a visual representation as a tool to implement tasks that promote reasoning and problem solving (PtA Mathematics Teaching Practice #2). The use of the ratio table provided students with multiple entry points to the mathematical concepts and allowed students to build on and extend their current mathematical understanding.

Example 4: Polygon Sort

In this final example, Briana shared a video of a sorting task she used in which students explored examples and non-examples of various polygons. Students sorted shapes into categories with a partner. They discussed the characteristics of the different figures and Briana supported them with using precise vocabulary to describe the shapes and creating definitions. When asked how this lesson connects to her learning from the PD, she explained how she modified transformation tasks from the PD to this content:

> (In the PD) ..they would say things like, here's your original and here's your image. And we would talk about which one of these would work. So it would have examples of, 'Well, this could be a translation.' These could all be translations, but here's an example of something that is not a translation maybe because it's rotated too. And so to get kids to, not only am I going to notice one of these that's different, but I also have to support that with my thinking. So I have to go one step further to be able to explain what I observed and why I feel like it fits this set of rules or why I feel like it does not fit the set of rules.

We see that Briana took the format of these transformation tasks and the visuals that they used and applied the ideas to a lesson on polygons. One idea from the task that she found salient was that it elicited students' thinking as they had to notice differences and that it encouraged them to justify their thinking to develop their understanding. This is consistent with PtA Mathematics Teaching Practice #8: Elicit and use evidence of student thinking. The sort with the VRs of polygons allowed her to elicit and gather evidence of student thinking at strategic points in the lesson. We also saw her engaging in PtA Mathematics Teaching Practice #2: Implement tasks that promote problem solving and reasoning. The sorting task allowed students to use representations to make sense of the mathematics and develop a conceptual understanding of what a polygon is. Finally, we see through these examples how she once again applied learning from the PD to a different content area with her sixth-grade students.

She also discussed a resource from the PD that she found very helpful and explained how she modified it for her students for polygons. As mentioned earlier, a visual one-pager called a "field guide", was given to teachers in the PD to support their understanding of transformations. She created a similar one for polygons and used this visual support for students in her classroom.

> Yes. That field guide is, every teacher who teaches geometry should have that. And we've tried to emulate that with some of our other things that we teach. To have some sort of reference guide that kids can use. Because I kept this on the board for a few days and we continued to talk about polygons and you could see them, they would start to talk about polygons and then their eyes would look over. I know there's a reference sheet that is on board that I can use to help me, support my thinking.

Briana took the idea of this field guide resource from the PD and had students co-create a graphic organizer for a different content area. She also highlighted the importance of visuals in both the guide from PD and in this resource the students create.

> What we've made is more of a graphic organizer that has the things we want them to know, but then we fill it out together. So it's not just a blank sheet where they have to write down all the notes, but it has visual examples which were big in the field guide, but also opportunities we'll go through and we'll annotate different aspects of whatever we're thinking about…So the page it's about parallelograms has a picture of a parallelogram. It shows how a parallelogram can be rearranged into a rectangle. So there's a visual example at the top and then there's a couple practice problems for them to do. So I would say the top part of each page is a lot of the field guide.

In these examples, we see how Brianna uses the representations to implement tasks that promote reasoning, facilitate discourse, and elicit and use evidence of student thinking. The sorting task incorporated VRs, provided entry points for students, and helped develop their justification skills. The adaptation and creation of the graphic organizer provided visual support for students that they could use as they worked on tasks. She anticipated what students might struggle with and provided a tool that would support them.

Teacher #2: Rachel

Rachel took part in Project 2's PD. She had taught for 11 years at the start of our study and has taught grades 5-8. Throughout this research project, Rachel taught 7th grade mathematics. She attended the PD because she wanted to learn new strategies and obtain new resources. Rachel's experience in the PD is described below and three examples are provided that illustrate her extensive use of representations in the classroom years after she participated in the PD. Like Brianna, her use of VRs cuts across different contexts and domains than those used within the PD. The examples presented also demonstrate how her use of these VRs was connected to other ambitious teaching practices, such as implementing tasks that promote reasoning and problem solving and facilitating meaningful mathematical discourse.

Experience in PD

Rachel described having a positive experience in the PD. When asked about what she remembered learning and continues to use, her answers focused on the importance of teaching with VRs. Prior to participating in the PD, she was unfamiliar with using the DNL and tape diagrams to solve ratio and proportion problems and had not used them in her classrooms. She mentioned in her interviews:

> I didn't know a lot of the representations that they were teaching us [in the PD]. I had been teaching middle school math for 8 years and I had never used a DNL. I was solving these problems using proportions or equations and I never knew this thing existed.

Prior to the PD, Rachel mentioned that she used equations to solve proportions, as she did not know about the VR options. This aligns with the research indicating that when teachers are unfamiliar with how to use these tools, they typically rely on algorithmic thinking to solve these types of ratio and proportional reasoning problems (Orrill & Brown, 2012).

She also described how the PD helped her develop a conceptual understanding of topics, as well as helped her think about how these VRs could be used with her students to create access for different learners.

> …it was really more just pushing me out of a procedural way of thinking about math. One example of that is one of the first days they asked us to make a visual model to represent some situation. I didn't know what that was so I solved it using algebra. Other people in the PD had experience using tape diagrams and double number lines and I didn't understand why they would do it that way. I thought the way I did it was so much easier. They told us to represent this using a tape diagram and a double number line which sort of forced me into thinking about how do I represent this same problem using this method and what are the benefits of presenting it this way and how can that help struggling learners or students that are not really able to access the curriculum because of language.

After the PD, Rachel began to incorporate these representations in her teaching and continues to use them five years later. She explains how one representation, the DNL, transformed her teaching.

> I started to take the DNL and completely change the way I teach ratio and proportion and percent and I started to use the tape diagrams and the DNL for everything. I still am using the materials from Project 2 for those units.

Rachel's exposure to and practice with different VRs in the PD seemed to have impacted her own understanding of how to solve ratio and proportion problems with representations, as well as provided her with new pedagogical strategies to use with her students. Three examples that demonstrate this uptake of these ideas are presented in the following sections.

Example #1: Unit Rate

Rachel shared a video with us related to unit rate and described how she used the DNL to provide access for students. The videos were collected during a synchronous zoom class. Students had previously worked on unit rate and in this lesson, she introduced them to the DNL as another way to think about the rate. They had worked with a warm-up that involved them figuring out the clicks per second of a robot. The students then created a DNL for this situation along with her and they discussed how to label it and use it to solve problems.

> We had started unit rate. The day before this we did an activity in Desmos and it's robots clicking per second and that is how we got into the unit rate, that way they had an idea of what unit rate is. We had a way to jump into double number lines with something they were already kinda familiar with. That's kinda part of the PD because they talked about giving students experiences to relate to. But the biggest PD thing here is the double number line here because I would have never used it otherwise. Figuring out how to relate this double number with what we did for the warmup was kind of a Project 2 PD thing too, making a connection between what is going on in this situation and the visual representation of it. I can't remember what the specific warmup problem was but they were finding the unit rate of the clicks per second, the clickbot they called it. We were then showing the unit rate on the double number line of the clicks per second.

She also discussed how the DNL representation allowed students to access the problem in different ways which is connected to implementing tasks that promote reasoning and problem solving. She described how the task and in particular the use of the DNL allowed for multiple entry points and helped students make sense of the problem posed. She explicitly connected this to a learning from the PD: “That is also part of the PD, being able to get every kid a way into a problem so that way everyone feels like they have a way to solve it.”

The use of the DNL also allowed her to facilitate meaningful discourse with her students. She shared a clip of the students in a breakout room discussing their solutions to a problem where they had used the DNLs to solve.

> When I got into their breakout room, they were yelling at each other. Not yelling angry but they were both arguing what I thought was the same argument. The first kid that was talking couldn't put into words how many minutes he thought it was because he was getting tripped up by the fractions. The other kid knew it was a fraction or a decimal number but he called it a half because he didn't really know what else to call it either. Using the double number for this helped them to be able to put a value on what that fractional value could be. This is one of those things that using the double number line makes it so much easier for the kids to kind of think about that instead of getting stuck. It means more when they are able to break it down and look between. It makes for a much richer problem solving experience rather than just going back and thinking about it as an equation or proportion to solve it. It is helping them break down the relationship between the two units, where the rates are.

In this example, she discusses how the introduction of the DNL representation was a useful way to help students develop their conceptual understanding and problem-solving skills. She also focused on the role it plays in allowing students to communicate their thinking. Rachel noted throughout her interviews that one of the ideas that resonated with her from the PD was the use of VRs as communication tools to engage students in purposeful sharing of their mathematical ideas

(PtA Mathematics Teaching Practice #3). We saw several other classroom examples of her engaging in this ambitious teaching practice of facilitating mathematical discourse by asking the class to compare student created double number representations and discuss what they noticed and what they would change about the representations (PtA Mathematics Teaching Practice #4).

Example #2: Rational Numbers

Rachel shared with us an example of students using visuals for a lesson related to operations with rational numbers. Students had previously worked on a word problem about the temperature dropping a certain rate and many of them had gotten the problem incorrect. She began her lesson the next day by asking them to draw a picture of the following scenario: *The temperature is 24 degrees; it falls three degrees every hour for six hours. Draw me a picture of it.* Students then drew pictures and she strategically selected students to share their representations and discuss them. Rachel talked about how this was an example of how she continually encourages students to make representations as they work through problems:

> Every day, I'm saying, 'Can you draw something to represent this? Draw me a picture of what's happening? Can you show me on a number line? What is happening?' And every time I asked them to draw it out, all of a sudden, their question disappears. So, I wanted to make the point that if you draw it out you've answered the question before you even read it, and so I kept the question out of the warm up…so um the pictures all had the minus 18 and they all had the you know falling and they all knew where it was ending up, and so, then I was like, I didn't even ask you a question and you've answered like two different questions, right now, and this is why, when I when you ask me a question about a math problem or something I say draw a picture because chances are it's going to solve

> the problem before you even know what you're supposed to be answering so when you can model it out that helps you.

Although this was not a specific representation discussed in the PD, Rachel chose to share this clip as an example of how she took up and used something from Project 2. We see that encouraging students to make representations of a situation is a way in which she helps students engage in reasoning and problem solving. It also connects to the teaching practice of supporting productive struggle (PtA Mathematics Teaching Practice #7) as she anticipates what students might struggle with and encourages them to use representations as a scaffold for accessing the tasks.

Example #3: Expressions and Equations

In the final example, we see how Rachel takes a task from a different curriculum and applies what she learned about representations and ambitious teaching practices from Project 2 to it. Students were working on an activity where they had cards and had to first sort and notice things about the algebraic expressions, equations and VRs and then match the equation or expression to the correct VRs. She talks about how she felt this video related to the PD:

> We didn't get this from Project 2, I believe it is from Open Middle Math curriculum…. Not from Project 2 but a Project 2-y type problem where it is very open-ended at the beginning where they are sorting and noticing different things about the equation. Then they have to commit and say this matches this visual because of this and this. Drawing the connections between the equation notation and a visual model of what is actually happening.

We see again how she selected and implemented a task that aligned with ambitious teaching practices she felt were important from the PD. In this case, she chose a task that promoted reasoning and problem solving (PtA Mathematics Teaching Practice #2) and had students first explore the VRs and what they noticed to provide an entry point for students. She then extended the task and provided them with the opportunity to connect algebraic expressions and equations to visual models, like tape diagrams. Although the task was not from the PD, she refers to it as a "Project2-y" type problem which demonstrates how she has generalized aspects of the tasks used in the PD and selected a task that aligns with these principles, such as providing access for students.

She also talked about how the visuals helped them develop a conceptual understanding of the equations:

> For the most part they did pretty awesome, they got really tripped up with the 19/2=x+5. One of the equations was 2(x+5) and they got that really quick. The working backwards part they got all sorts of confused because 19 is not easily divisible by 2. Their little seventh grade brains were like "Can't do it! It's a decimal." We talked about how if it's 19/2 that means we are cutting nineteen in half. We don't need to think about it as 9.5. Think about it as taking half of the tape diagram. So, which of our choices is half the tape diagram equal to x+5. Once they saw it like that there were a lot of light bulb moments.

In this case, she used the visual diagrams to help students make sense of the quantitative relationships in the equation. She also articulated other ways that the task allowed them to better understand the expressions and equations and stressed how this related to her goals of having them make sense of different ways to write equivalent expressions and equations. This example also demonstrates her ability to take up the learning about representations and ambitious teaching practices and apply them to a content area that had not been discussed in the PD: algebraic equations and expressions.

Summary of Case Studies

The two case studies demonstrate that there is promising evidence of strong residual learning and uptake related to content and pedagogy from PDs that included a focus on VRs. Although the format and implementation of the two PD projects differed, similarities emerged between the uptake and residual learning related to ambitious practices and representations. These similarities are detailed below.

The prevalence of the use of representations in the teachers' classrooms appeared to be strongly connected to teachers' own learning about VRs in PD, both in terms of content knowledge of how to use them to solve problems and pedagogical knowledge related to how to use them with students. Both teachers' use of representations and their explanations related to their choice to use representations is nontrivial. They were able to discuss the complex mathematical content they learned using a representation as well as make connections to other mathematical domains. They also discussed the ways students might use the representation and anticipated student strategies, both correct and incorrect using the representation. The representation seems to be the catalyst in many or most instances. For example, Rachel talks about how her understanding of solving ratio problems changed because of seeing other participants in the PD use tape diagrams and DNLs to solve the problems and how this in turn caused her to think about how the use of VRs could provide more access for her students. This learning about new representations in Project 2's PD changed the ways in which she taught her ratio unit as evidenced in her classroom videos. We saw similar changes in Brianna as she mentioned how learning about geometry through representations allowed her to put herself in the shoes of students and then changed the ways in which she provided access through representations in her lessons. Thus, we hypothesize that a representation can be an important catalyst to teachers' mathematical learning, which also supports their pedagogical practice-typically using the representation in problem solving situations similar to their experience in PD whereby the representation becomes an important mediator for teachers' learning as well as their implementation of ambitious mathematical practices to support student learning.

Both teachers discussed how they used representations in their classroom and how they provided access for students and they both highlighted how selecting different ways students used representations to solve the problem helped them to facilitate communication and discussions in their classrooms. Thus, this encouraged and supported their use of the mathematical practices or ambitious teaching practices. More specifically, some of the practices evidenced in video and discussed during interviews included using rich tasks that promote reasoning and problem solving, facilitating meaningful student discourse, promoting productive struggle and eliciting student thinking. One example from Brianna's video was when she used a dot image task to provide an entry point into writing algebraic equations. Although this was not a task from the PD, she was inspired by the way the PD facilitator used VRs to provide entry points and facilitate discussion. This led to students sharing strategies, making connections, debating solutions, and using reasoning to justify their responses. It also allowed her to make connections to more complex content. Again, these results may suggest that learning about the VRs appears to be a key factor that supported teachers to use other ambitious teaching practices in their classroom.

The two teachers remembered and continued to use ambitious practices and VRs in their classrooms in ways that not only aligned to the goals and intention of the PD, but also adapted and extended representations to different mathematical domains and settings. They attributed their use of VRs to the PD itself and sometimes the PD helped to solidify their learning and support changes in their instructional practice to include more ambitious approaches. They also designed tasks for new content areas that incorporated VRs and other ambitious practices and adapted tasks and strategies from the PDs to their new online settings during the pandemic.

Discussion

In order to understand the impact of PD on teacher learning, this study examined long-term learning, changes in pedagogy and potential generative change four to five years post PD experience. The findings add new insights for teachers, teacher leaders and administrators related to mathematics teacher learning from PD, the importance of VRs for both teacher and student learning of mathematics, the design and development of effective PD and the importance of studying teacher learning and the impact of PD over time. This study contributes to the literature by providing examples of how learning about a specific pedagogical tool, in this case VRs, along with specified content, can have an impact on teacher learning and pedagogy over time.

Our analysis indicated three main findings as described above: (1) the teachers' use of VRs appears to be strongly connected to teachers' own active learning of VRs and content in PD, (2) VRs appears to be a key factor that supported the teachers' use of other ambitious teaching practices in their classroom and (3) the two teachers remembered and continued to use and hone ambitious practices and VRs in their classrooms in ways that not only aligned to the goals and intention of the PD, but also adapted and extended representations to different mathematical domains and settings.

The sustained use of VRs appeared to be strongly connected to teachers' own active learning in PD and their development of content and pedagogical knowledge which have been cited as components of effective PD (Desimone & Garet, 2015). Teachers, like students, used the VRs to make sense of new mathematical concepts, make connections among concepts and as a tool to communicate and share their thinking. This study provides some evidence that there is some similarity between teacher learning using VRs and that of students' improved learning when using VRs (Boonen, et al., 2014). Furthermore, in both projects, the VRs used in the PD were intentionally selected to teach specified mathematical content. We hypothesize that PDs designed in this way, where teachers are learning new mathematical content with the use of a specified representation, have the potential to create robust learning which is translated to practice and sustained use. Additionally, the use of VRs does not appear to be contained to the content area that was the focus of the PD as we have evidence that the case study teachers continued to use VRs and ambitious practices not only with the content that aligned with the PD they attended but also across contexts and domains.

This is in contrast to PD programs where the use of multiple representations is introduced and encouraged but perhaps lacks the specificity of when and how VRs would best be used and with what aspects of the curricula. We wonder if this might be related to the fact that the case study teachers presented here learned both relatively new content to them with specific VRs to support their conceptual understanding. And in turn, we hypothesize that this also supports the sustained use of VRs to teach mathematics. This may also explain results from Copur-Gencturk and Papakonstantinou's (2016) study where the PD was a large endeavor focused on comprehensive content domains including geometry, linear algebra, and statistics and probability and also featured a vast array of pedagogical techniques including a focus on discourse, formative assessment, habits of mind and multiple representations. Thus, the goals and intentions of this large effort where the mathematical practices and the learning standards at the national, state and district level became the foci for all high school teachers might actually be well intended and have success in more broad pedagogical areas such as discourse but perhaps might not provide enough specificity for the selection and implementation of specific VRs to support both teacher and student mathematics learning. This might provide insight into why the researchers found statistically significant changes in mathematical discourse, instructional clarity, and the development of students' mathematical habit of mind over time, but not in the use of multiple representations.

This study also sought to examine if ambitious mathematics teaching practices that were related to the PDs were picked up and sustained over time and why. This study provides evidence of two case study teachers that continued to use ambitious mathematics practices. We reported on the importance of these practices to reach more learners, but what we noticed in particular was that much of the time these ambitious mathematical practices were used when they were tied to the use of VRs. The particular VR was the focus of many students' strategies and therefore the basis of important and rich mathematical discussions. Students used the mathematical representations to justify their thinking and make their point when explaining their solution to complex problems. The VR essentially became a mediator between the student and their mathematical thinking and the way in which they conveyed their understandings to the teacher and classmates.

This is somewhat similar to research on designed instructional activities (Lampert & Graziani, 2009; Lampert et al., 2010). Lampert and colleagues found that creating design activities supported and encouraged novice teachers to implement ambitious mathematics teaching. Similar to their finding that the use of instructional routines may reduce the cognitive load of ambitious teaching, the use of VRs may also have the potential to serve a similar role. One difference might be the fact that teachers need to develop pedagogical content knowledge related to the content and the VR such as how and when to use VRs whereas routines are more prescribed. However the case study teachers presented here seemed well equipped to implement

VRs in different contexts possibly because of the relationship between their own learning of content with the support of the VR. Additionally, perhaps more research is needed to investigate routines or design activities that might support teachers to use specified VRs effectively or perhaps more research is needed to understand the differences between designed instructional activities and the use of dynamic representations and their alignment to ambitious practices.

This study also showed that studying teacher learning over time was important in our study as it allowed us to assess and learn that teachers continued to hone their pedagogical practices learned in PD over time. As Kennedy (2016) noted in a review of the literature, the impact of PD isn't always visible immediately following participation. These cases demonstrate more evidence of ways teachers used learnings from the PD, what they attended to, and the different and generative ways they implemented mathematics instruction in other mathematical domains, with different problems, and with different grade levels.

Lastly these findings provide insight for mathematics education leaders into the ways in which teachers continue to take up ideas that they learned in PD, years after their participation and the ways in which they adapt and apply them to novel contexts. Implications for PD providers, teacher educators and school leaders suggest that a focus on VRs may be one tool to anchor learning in a PD to deepen teachers' abilities to engage their students in ambitious teaching practices. Some suggestions include:

- Exposing teachers to new VRs in professional learning. Teachers may benefit from learning about new VRs and using them to solve mathematical problems in PD to learn new content or to deepen their conceptual knowledge.
- The use of VRs in professional learning can be used as a tool to highlight mathematical practices and as a model to support their implementation of pedagogy to support

ambitious teaching. PD facilitators can help teachers explore how representations can be used to facilitate meaningful discussion. For instance as teachers grapple with how to select, sequence, and have students share different strategies in their class.

- The use of VRs in PD can also facilitate discussion with teachers about how the use of representations can provide multiple entry points to tasks that promote reasoning and problem solving as well as how representations can be used to support productive struggle.
- Even when content of PD is specific, including a focus on representations may allow participants to generalize to other topics and contexts. Both participants applied the use of representations to online contexts and to different mathematical domains that were not discussed during their time in the PD.

This study focused on two teachers with high levels of uptake and provided some understanding into what these teachers took up and used. While we do not suggest that the interpretations found are generalizable for all teachers, these patterns begin to illuminate the important utility and the role representations may play in the enactment of ambitious teaching practices. In addition, more needs to be understood about teachers that participate in PD and do not attain the same level of uptake. Future case study research is planned to begin to understand the differences in uptake among participants.

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